02 - Math Preliminaries

Dr. Robert Lowe

Division of Mathematics and Computer Science
Maryville College

Outline

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- We often write relations this: aUb where aUb is true if and only if $(a, b) \in U$

Digraphs

A **digraph** is a graphical representation of a relation.

Example 2.3.1
The relationship 'is a town in' between S = the set of towns and T = the set of countries looks like this:

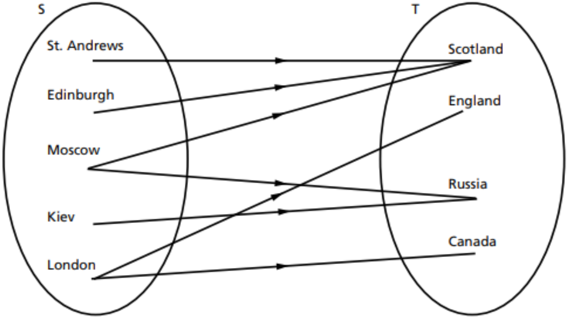

Fig. 2.1

From Davie and Morrison page 26

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Exercise

For each of the above properties, define a relation that exhibits this property.

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  - $A^{n+1} = A^n A$ for $n > 0$

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- Finite closures exist when A^+ converges.

Reflexive Transitive Closure

Adding $I = A^0$ to A^+ yields the reflexive transitive closure of A.

$$A^* = \sum_{i=0}^{\infty} A^i$$

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- This relation's digraph is:

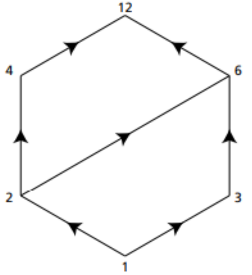

Fig. 2.4

(Davie and Morrison page 30)

Example: Binary Matrix of Direct Divisor Relation

$$M(A) = \begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 6 & 12 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 6 \\ 12 \end{array} & \begin{bmatrix} & * & * & & & \\ & & & * & * & \\ & & & & * & \\ & & & & & * \\ & & & & & * \\ & & & & & \end{bmatrix} \end{array}$$

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- This gives us A^+, to get A^* we just make all diagonal elements true.

Computing transitive closure in S-Algol

```
!Read or calculate bool matrix A(nxn)
for i=1 to n do      !Each time round, add a new node
                     !(number i) to the transitive closure graph
  for j=1 to n do    !Find all arrows leading into node i
  if A(j,i) do       !If there is an arrow from node j to node i,
                     !find all those out of node i
    for k=1 to n do  !make a direct path arrow between nodes j
        A(j,k):=A(j,k) or A(i,k)
                     !Now A contains the transitive closure matrix
```

From Davie and Morrison page 33