

# Annuities and Loans

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## 1 Annuities

- The compound interest formula assumes that we will deposit or borrow money and then never make payments or further deposits.
- Most of us cannot do this!
- A compound interest account into which we deposit regularly is called an **annuity**.
- Example annuities are savings accounts, retirement accounts (such as a 401k), and IRAs.
- Recall that the recursive formula for compound interest is:

$$P_m = \left(1 + \frac{r}{k}\right) P_{m-1}$$

- If, in addition to interest, we make a monthly deposit of  $d$ , this becomes:

$$P_m = \left(1 + \frac{r}{k}\right) P_{m-1} + d$$

- Suppose I deposit \$100.00 each month into a savings account which has 2.02% APY (Annual Percent Yield) compounded monthly. Use the recursive formula to find the amount in the account over the first 3 months.
- Solving this recursive formula in the general case gives us the explicit annuity formula:

$$P_N = \frac{d \left( \left(1 + \frac{r}{k}\right)^{Nk} - 1 \right)}{\left(\frac{r}{k}\right)}$$

Where:

$N$  - Number of Years

$k$  - Compounding Periods per Year

$d$  - Deposit per Compounding Period

$r$  - APY or APR

Note that the standard for most accounts is to compound monthly. (If  $k$  is not given, it can be assumed to be 12).

- In my above scheme, how much will I have in my savings account after 10 years?

## 2 Payout Annuities

- A payout annuity is one in which we make a regular monthly withdrawal of  $d$ .
- For instance, suppose we retire. We stop paying into our retirement account, and it becomes a payout annuity.
- The formula for a payout annuity is as follows:

$$P_0 = \frac{d \left( 1 - \left( 1 + \frac{r}{k} \right)^{-Nk} \right)}{\left( \frac{r}{k} \right)}$$

Where  $P_0$  is our starting balance.

- Suppose our retirement goal is to have a retirement income of \$50,000.00 per year. Assuming our retirement account earns 6% interest, we retire at 65, and we die at age 80, how much must we have in our account when we retire?
- Assuming we begin saving for retirement at age 22, how much must we save each month to reach our goal?
- Assuming that our employer matches our retirement savings up to 5% of our gross income, how much must we demand in salary to reach our goal with our employer paying half of our retirement contributions?

## 3 Compound Interest Loans

- A compound interest loan is effectively a payout annuity, so it uses the same formula.
- The payout annuity formula also tells us the remaining balance in a loan, if we know the monthly payment.
- The deposits are your payment. Solving the annuity formula for  $d$ , we can work out the payment for the loan:

$$d = \frac{\left( \frac{r}{k} \right) P_0}{\left( 1 - \left( 1 + \frac{r}{k} \right)^{-Nk} \right)}$$

- The amortization schedule of a loan is the listing of how much interest is paid each payment period and how much principal remains.
- The full amortization of a loan tells us how much money will actually be repaid over the course of a loan.
- The difference between the value of the purchased asset and the remaining principal of the loan is called the **equity** of the asset. Generally, only real-estate gains equity.
- Suppose we purchase a home at \$100,000.00 with a down payment of \$10,000.00 and finance the rest with a fixed-rate 30 year mortgage at 5% interest. The loan compounds monthly. What will our monthly payment be?
- How much, in total, will we have paid for this house at the end of our loan period?