Meaning of P.D.E. Partial differential equations arise in geometry, physics and applied mathematics when the number of independent variables in the problem under consideration is two or more.

under such a situation, any dependent variables will be a function of more than one variable and hence it possesses not ordinary derivatives with respect to a single variable but partial derivatives with respect to several independent variables.

Definition: An equation containing one or more partial derivatives of an unknown function of two or more independent variables is known as a P.D.E.

Ex: (I) $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z + xy$

(II) $\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial^3 z}{\partial y^3} = 2x \frac{\partial z}{\partial x}$

(III) $z\left(\frac{\partial z}{\partial x}\right) + \frac{\partial z}{\partial y} = x$

(IV) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = xyz$

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(V) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \left(1 + \frac{\partial z}{\partial y}\right)^{1/2}$

(VI) $y\left\{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right\} = z\left(\frac{\partial z}{\partial y}\right)$

<u>Order of a P.D.E</u>: The order of a P.D.E is defined as the order of the highest Partial derivatives occuring in the P.D.E.

Exp (1), (3), (4) & (6) → order 1

(5) → order 2

(2) → order 3

<u>Degree of a P.D.E</u>: The degree of a P.D.E is the degree of the highest order derivative which occurs in it after the equation has been rationalised ie made free from radicals and fractions so far as derivatives are concerned.

Exp (1), (2), (3), (4) — degree 1

(5) & (6) → degree 2

<u>Linear and Non-linear P.D.E</u>: A P.D.E is said to be linear if the dependent variable and its partial derivatives occur only

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in the first degree and not multiplied

A p.d.e which is not linear is called a non-linear P.D.E.

Exp (1) & (4) — linear

(2), (3), (5) & (6) → non-linear.

<u>Notations</u>: if $z = z(x, y)$

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}$$

$$t = \frac{\partial^2 z}{\partial y^2}$$

if $z = z(x_1, x_2, \ldots x_n)$

$$p_1 = \frac{\partial z}{\partial x_1}, \quad p_2 = \frac{\partial z}{\partial x_2} \quad \ldots \quad p_n = \frac{\partial z}{\partial x_n}$$

Some times the partial diff eqns are also denoted by making use of suffixes.

$$u_x = \frac{\partial u}{\partial x}, \quad u_y = \frac{\partial u}{\partial y}, \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}$$

$$u_{xy} = \frac{\partial^2 u}{\partial x \partial y}$$ and so on.

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Classification of first order P.D.E.

Linear equation: A first order equation $f(x, y, z, p, q) = 0$ is known as linear if it is linear in p, q and z.

ie if given equation is of the form

$$P(x,y)p + Q(x,y)q = R(x,y)z + S(x,y)$$

Ex (i) $x^2 y p + x y^2 q = x y z + x^2 y^3$

(ii) $p + q = z + xy$

Semi-linear equation: A first order P.D.E $f(x, y, z, p, q) = 0$ is known as a semi-linear equation, if it is linear in p and q and the coefficients of p and q are functions of x and y only

ie if the given equation is of the form

$$P(x,y)p + Q(x,y)q = R(x,y,z)$$

Ex (i) $xyp + x^2 y q = x^2 y^2 z^2$

(ii) $yp + xq = \frac{x^2 z^2}{y^2}$

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Quasi-linear equation: A first order p.d.e $f(x, y, z, p, q) = 0$ is known as quasi linear equation, if it is linear in p and q ie if the given equation is of the form

$$P(x, y, z)\, p + Q(x, y, z)\, q = R(x, y, z)$$

Ex (i) $x^2 z p + y^2 z q = xy$

(ii) $(x^2 - yz) p + (y^2 - zx) q = (z^2 - xy)$

Non-linear equation: A first order p.d.e $f(x, y, z, p, q) = 0$ which does not come under the above three types, is known as a non-linear equation.

Ex: (i) $p^2 + q^2 = 1$ (ii) $pq = z$

(iii) $x^2 p^2 + y^2 q^2 = z^2$.

Origin of P.D.Es: We shall now examine How partial differential equations arise. We show that such equations can be formed by the elimination of arbitrary constants or arbitrary functions.

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Rule-1: Derivation of a P.D.E by the elimination of arbitrary constants.

Consider the eqn:

$$F(x, y, z, a, b) = 0 \quad \text{--- (1)}$$

diff. eqn (1) w.r.t. x and y partially

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x} = 0$$

or $$\frac{\partial F}{\partial x} + p \frac{\partial F}{\partial z} = 0 \quad \text{--- (2)}$$

and $$\frac{\partial F}{\partial y} + q \frac{\partial F}{\partial z} = 0 \quad \text{--- (3)}$$

eliminating two constants a and b from (1), (2) and (3), we shall obtain an eqn of the form

$$f(x, y, z, p, q) = 0$$

which is a P.D.E of the first order.

Working Rule for solving Problems:

If $F(x, y, z, a, b) = 0$

Then differentiated partially with respect to x and y. Finally arbitrary constants a and b are eliminated from the relations

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$F(x, y, z, a, b)$, $\frac{\partial F}{\partial a} = 0$ and $\frac{\partial F}{\partial y} = 0$

The equation free from a and b will be the required P.D.E.

<u>Situation 1</u> - When the no of arbitrary consts is less than the no of independent variables, then the elimination of arbitrary constants usually give rise to more than one P.D.E of order one.

Ex) ① $z = ax + y$ — ①

Diff (1) partially w.r.t. x, we get

$$\frac{\partial z}{\partial x} = a \quad \text{— ②}$$

Diff (1) partially w.r.t y, we get

$$\frac{\partial z}{\partial y} = 1 \quad \text{— ③}$$

eliminate ~~(ex~~ a from ① and ②

$$z = x\left(\frac{\partial z}{\partial x}\right) + y \quad \text{— ④}$$

Hence ③ and ④ are two p.d.e's.

<u>Situation 2</u>: When the no of arbitrary contants is equal to the no of independent variables, then the elimination of arbitrary contants shall give rise to a unique P.D.E. of order one.

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Exp $\quad az + b = a^2x + y \quad$ — (1)

Differentiate (1) partially w.r.t x & y

$$\left.\begin{aligned} a\frac{\partial z}{\partial x} &= a^2 \\ a\frac{\partial z}{\partial y} &= 1 \end{aligned}\right\} \quad (2)$$

Eliminate a from (2), we have

$$\frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} = 1$$

which is the unique p.d.e. of order one.

Situation 3: When the no of arbitrary const is greater than the no of independent variable. then the elimination of arbitrary consts leads to a p.d.e of order usually greater than one.

Exp $\quad \cancel{az + b =} \quad z = ax + by + cxy \quad$ — (1)

Diff (1) partially w.r.t x and y, we have

$$\frac{\partial z}{\partial x} = a + cy \quad \text{— (2)} \qquad \frac{\partial z}{\partial y} = b + cx \quad \text{— (3)}$$

from (2) and (3)

$$\frac{\partial^2 z}{\partial x^2} = 0 \quad \text{— (4)} \quad \text{and} \quad \frac{\partial^2 z}{\partial y^2} = 0 \quad \text{— (5)}$$

and $\quad \dfrac{\partial^2 z}{\partial x \partial y} = c \quad$ — (6)

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Now (2) and (3)

$$x\frac{\partial z}{\partial x} = ax + cxy \quad \text{and} \quad y\frac{\partial z}{\partial y} = by + cxy$$

$$\therefore x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = ax + by + cxy + cxy$$

$$\text{or } x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z + xy\left(\frac{\partial^2 z}{\partial x \partial y}\right) \quad \text{—(7)}$$

Thus, we get three p.d.e given by (4), (5) and (7), which are all of order two.

<u>Exp</u> (1) Form the diff. eqn by eliminating a and b from $z = (x^2+a)(y^2+b)$ —(1)

<u>Soln</u>: Diff (1) partially w.r.t x and y.

$$\frac{\partial z}{\partial x} = 2x(y^2+b) \quad \text{or } (y^2+b) = \frac{1}{2x}\frac{\partial z}{\partial x} \quad (2)$$

$$\frac{\partial z}{\partial y} = 2y(x^2+a) \quad \text{or } (x^2+a) = \frac{1}{2y}\frac{\partial z}{\partial y} \quad (3)$$

Now from (1), (2) and (3)

$$z = \frac{1}{4xy}\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)$$

$$\text{or } 4xyz = \left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)$$

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Exp (2) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ — (1)

Soln. Diff (1) partially w.r.t. x & y.

$$\frac{2x}{a^2} + \frac{2z}{c^2}\frac{\partial z}{\partial x} = 0 \quad \text{or} \quad c^2 x + a^2 z \frac{\partial z}{\partial x} = 0 \quad (2)$$

$$\frac{2y}{b^2} + \frac{2z}{c^2}\frac{\partial z}{\partial y} = 0 \quad \text{or} \quad c^2 y + b^2 z \frac{\partial z}{\partial y} = 0 \quad (3)$$

Diff. (2) w.r.t. x and (3) w.r.t. y,

$$c^2 + a^2\left(\frac{\partial z}{\partial x}\right)^2 + a^2 z \frac{\partial^2 z}{\partial x^2} = 0 \quad (4)$$

$$c^2 + b^2\left(\frac{\partial z}{\partial y}\right)^2 + b^2 z \frac{\partial^2 z}{\partial y^2} = 0 \quad (5)$$

from (2) $c^2 = -\frac{a^2 z}{x}\left(\frac{\partial z}{\partial x}\right)$ — (6)

Putting this value of c^2 in (4) and dividing by a^2, we obtain

$$-\frac{z}{x}\frac{\partial z}{\partial x} + \left(\frac{\partial z}{\partial x}\right)^2 + z\frac{\partial^2 z}{\partial x^2} = 0$$

$$\text{or} \quad zx\frac{\partial^2 z}{\partial x^2} + x\left(\frac{\partial z}{\partial x}\right)^2 - z\frac{\partial z}{\partial x} = 0 \quad (7)$$

Similarly for (3) and (5)

$$zy\frac{\partial^2 z}{\partial y^2} + y\left(\frac{\partial z}{\partial y}\right)^2 - z\frac{\partial z}{\partial y} = 0 \quad (8)$$

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Diff. (2) Partially w.r.t y,

$$0 + a^2\left\{\left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial z}{\partial x}\right) + z\left(\frac{\partial^2 z}{\partial x \partial y}\right)\right\} = 0$$

or $$\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) + z\left(\frac{\partial^2 z}{\partial x \partial y}\right) = 0 \quad \text{—(5)}$$

so (5), (8) and (9) are three possible form of the required diff eqn.

<u>Rule-2</u>: Derivation of P.D.E by eliminating of arbitrary function ϕ from the eqn $\phi(u, v) = 0$, where u & v are functions of x, y, z

<u>Proof</u>: $$\phi(u, v) = 0 \quad \text{—(1)}$$

Treat $z = z(x, y)$

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial y}{\partial x} = 0 \ \& \ \frac{\partial x}{\partial y} = 0$$

Diff (1) partially w.r.to x

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right) = 0$$

or $$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0$$

or $$\frac{\partial \phi}{\partial u} \Big/ \frac{\partial \phi}{\partial v} = -\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) \Big/ \left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) \quad \text{—(2)}$$

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similarly, diff ① p.w.r.t. z, we get

$$\frac{\partial \phi}{\partial u} \Big/ \frac{\partial \phi}{\partial v} = -\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) \Big/ \left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right)$$

Eliminate ϕ from ② & ③

$$\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) \Big/ \left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) = \left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) \Big/ \left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right)$$

or $$\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = \left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right)$$

or $$Pp + Qq = R \quad \text{— ④}$$

where
$$P = \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}$$

$$Q = \frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}$$

$$R = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

Thus we obtain a linear PDE of first order and of first degree in p and q.

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Exp $\phi(x+y+z,\ x^2+y^2-z^2) = 0$ — (1)

eliminate the ϕ and what is the order of this PDE.

Soln: Let $u = x+y+z$ & $v = x^2+y^2-z^2$ — (2)

$\phi(u, v) = 0$ — (3)

Diff (3) p.w.r.t. x, we get,

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0 \quad \text{— (4)}$$

from (2) $\frac{\partial u}{\partial z} = 1,\ \frac{\partial u}{\partial x} = 1,\ \frac{\partial u}{\partial y} = 1$

$\frac{\partial v}{\partial x} = 2x,\ \frac{\partial v}{\partial y} = 2y,\ \frac{\partial v}{\partial z} = -2z$ (5)

from (4) and (5)

$$\left(\frac{\partial \phi}{\partial u}\right)(1+1) + \frac{\partial \phi}{\partial v}(2x - 2zp) = 0$$

$$\text{or } \left(\frac{\partial \phi}{\partial u}\right) \Big/ \left(\frac{\partial \phi}{\partial v}\right) = -2(x - pz)/(1+p) \quad \text{— (6)}$$

Again diff (3) p.w.r.t. y, we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) = 0 \quad \text{— (7)}$$

from (4) & (7)

$$\frac{\partial \phi}{\partial u}(1+q) + \frac{\partial \phi}{\partial v}(2y - q2z) = 0$$

$$\Rightarrow \frac{\partial \phi}{\partial u} \Big/ \frac{\partial \phi}{\partial v} = -\frac{2(y - qz)}{(1+q)} \quad \text{— (8)}$$

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from (6) & (8)

$$\frac{(x-pz)}{(1+p)} = \frac{(y-qz)}{(1+q)}$$

or $(1+q)(x-pz) = (1+p)(y-qz)$

or $(y+z)p - (x+z)q = (x-y)$.

Exp $z = f(x^2-y^2)$ — (1)

Diff. (1) p.w.r.t. x and y, we get

$$\frac{\partial z}{\partial x} = f'(x^2-y^2) \times 2x$$

$$\Rightarrow f'(x^2-y^2) = \frac{1}{2x}\cdot\frac{\partial z}{\partial x} \quad \text{— (2)}$$

$$\frac{\partial z}{\partial y} = f'(x^2-y^2) \times -2y$$

$$\Rightarrow f'(x^2-y^2) = -\frac{1}{2y}\frac{\partial z}{\partial y} \quad \text{— (3)}$$

Eliminate $f'(x^2-y^2)$ from Eqn (3)

we have $\frac{1}{2x}\frac{\partial z}{\partial x} = -\frac{1}{2y}\frac{\partial z}{\partial y}$

or $y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0$

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Lagrange's Equation :

A quasi-linear p.d.e. of order one is of the form $Pp + Qq = R$, where P, Q, R are functions of x, y, z. Such a P.D.E is known as Lagrange eqn.

Exp $\quad xyp + yzq = zx$

Lagrange's Method of solving $Pp + Qq = R$

The general soln of Lagrange equation

$$Pp + Qq = R \quad — (1)$$

is

$$\phi(u, v) = 0 \quad — (2)$$

where ϕ is an arbitrary function and

$$u(x, y, z) = c_1 \quad \text{and} \quad v(x, y, z) = c_2 \quad (3)$$

are two independent solutions of

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad — (4)$$

Here c_1, c_2 are two const. and at least one of u and v must contain z.

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Working Rule for solving $Pp + Qq = R$ by Lagrange's Method

Step-1: Put the given P.D.E of the first order in the standard form

$$Pp + Qq = R \quad \text{--- (1)}$$

Step-2: write down Lagrange's auxiliary eqn for (1)

$$\frac{dx}{P} = \frac{dy}{q} = \frac{dz}{R} \quad \text{--- (2)}$$

Step 3 Solve (2) by using the well known methods.

Let $u(x, y, z) = C_1$ & $v(x, y, z) = C_2$ be two independent soln of (2)

Step (4). The general soln of (1) is then written in one of the following three equivalent forms:

$$\phi(u, v) = 0 \quad , \text{or} \quad u = \phi(v)$$

$$\text{or} \quad v = \phi(u)$$

where ϕ - being an arbitrary function

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Type 1 based on Rule 1 for solving $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Suppose that one of the variables is either absent or cancels out from any two fractions of given equations (1). Then an integral can be obtained by the usual method. The same procedure can be repeated with another set of two fractions of the given eqns (1).

Ex ① $\left(\frac{y^2 z}{x}\right) p + xzq = y^2$ — ①

Soln The Lagrange's Auxiliary eqns for ① are

$$\frac{dx}{\left(\frac{y^2 z}{x}\right)} = \frac{dy}{xz} = \frac{dz}{y^2} \quad \text{— ②}$$

taking the first two fractions of (2), we have

$$x^2 z\, dx = y^2 z\, dy$$

or $x^2 dx = y^2 dy$ or $3x^2 dx = 3y^2 dy$ — ③

on integrate ③ $x^3 - y^3 = c_1$

Net taking first and the last fraction

$$xy^2 dx = y^2 z\, dz \quad \text{or} \quad 2x\,dx = 2z\,dz \quad \text{— ④}$$

on integrat ④ $x^2 - z^2 = c_2$

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Thu the required general solⁿ (Integral is

$$\phi(x^3-y^3,\ x^2-z^2)=0$$

Ex $y^2p - xyq = x(z-2y)$

Solⁿ

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)} \quad (1)$$

first two

$$2x\,dx + 2y\,dy = 0 \Rightarrow x^2+y^2 = c_1 \quad (2)$$

Last two

$$\frac{dz}{dy} = -\frac{z-2y}{y} =$$

$$\Rightarrow \frac{dz}{dy} + \frac{1}{y}z = 2 \quad (3)$$

$$IF = e^{\int \frac{1}{y}dy} = e^{\log y} = y$$

$$zxy = \int 2y\,dy + c_2$$

$$zy = y^2 + c_2$$

$$zy - y^2 = c_2$$

Hence $\phi(x^2+y^2,\ zy-y^2)=0$

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Type 2 based on Rule - II for solving

$$\frac{dx}{P} = \frac{dy}{q} = \frac{dz}{R} \quad \text{--- (1)}$$

suppose that one integral of (1) is known by using rule 1 and suppose also that another integral cannot be obtained by using Rule 1. Then one integral known to us is used to find another integral as shown in the following exp.

Exp: solve $p + 3q = 5z + \tan(y-3x)$ --- (1)

Soln By Lagrange's eqns.

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z + \tan(y-3x)} \quad \text{--- (2)}$$

taking first two fractions,

$$\frac{dx}{1} = \frac{dy}{3} \Rightarrow dy = 3dx \quad \text{--- (3)}$$

on integrating $y - 3x = c_1$ --- (4)

using (4), (2) reduce.

$$\frac{dx}{1} = \frac{dz}{5z + \tan c_1} \quad \text{--- (5)}$$

$$\Rightarrow x - \frac{1}{5}\log(5z + \tan c_1) = \frac{1}{5}c_2$$

or $5x - \log[5z + \tan(y-3x)] = c_2$ --- (6)

from (4) & (6) are required ~~eqn~~ general intg

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$5x - \log(\log z) + \tan(y-3x)] = \phi(y-3x)$

Ex ② $xzp + yzq = xy$ — ①

$\Rightarrow \dfrac{dx}{xz} = \dfrac{dy}{yz} = \dfrac{dz}{xy}$ — ②

taking first two fraction of ②

$\dfrac{dx}{x} = \dfrac{dy}{y} \Rightarrow \log x = \log y + \log c_1$

$\Rightarrow \dfrac{x}{y} = c_1$ — ③

from ③ $x = c_1 y$ and hence from ②

$\dfrac{dy}{yz} = \dfrac{dz}{c_1 y \cdot y}$

$\Rightarrow c_1 y\,dy = z\,dz$

$\Rightarrow c_1 y^2 - z^2 = c_2$

$\Rightarrow \dfrac{x}{y} y^2 - z^2 = c_2$

$\Rightarrow xy - z^2 = c_2$ — ④

from ③ and ④ The required soln is

$\phi(xy - z^2, x/y) = 0$

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Type 3 based on Rule - III for solving

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \text{--- (1)}$$

Let P_1, Q_1 and R_1 be functions of x, y, z. Then each fractions of (1) will be equal to

$$\frac{P_1\,dx + Q_1\,dy + R_1\,dz}{P_1P + Q_1Q + R_1R} \quad \text{--- (2)}$$

If $P_1P + Q_1Q + R_1R = 0$, then we know that the numerator of (2) is also zero. This gives $P_1\,dx + Q_1\,dy + R_1\,dz = 0$ which can be integrate to give $u_1(x, y, z) = c_1$

This method may be repeated to get another integral $u_2(x, y, z) = c_2$. P_1, Q_1, R_1 are called multipliers. Some times only one integral is possible by use of multipliers. In such cases second integral should be obtained by Rule-I or Rule-II.

Ex ① $(mz - ny)p + (nx - lz)q = ly - mx$

Soln: The Lagrange's Auxiliary eqn for the given eqn are

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx} \quad \text{--- (1)}$$

Choosing x, y, z as multipliers, each fraction of (1):

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$$= \frac{x\,dx + y\,dy + z\,dz}{x(mz-ny)+y(nx-lz)+z(ly-mx)} = \frac{x\,dx+y\,dy+z\,dz}{0}$$

$$\Rightarrow \quad x\,dx + y\,dy + z\,dz = 0$$

$$\Rightarrow \quad 2x\,dx + 2y\,dy + 2z\,dz = 0$$

$$\Rightarrow \quad x^2+y^2+z^2 = c_1 \quad \text{--- (2)}$$

Again, choosing l, m, n as multipliers.

$$= \frac{l\,dx + m\,dy + n\,dz}{l(mx-ny)+m(nx-lz)+n(ly-mx)}$$

$$= \frac{l\,dx + m\,dy + n\,dz}{0}$$

$$\therefore \quad l\,dx + m\,dy + n\,dz = 0$$

$$\Rightarrow \quad lx + my + nz = c_2 \quad \text{--- (3)}$$

from (2) and (3), the required general soln is given by

$$\phi(x^2+y^2+z^2,\ lx+my+nz) = 0$$

<u>Exp</u> (2) $x(y^2+z)p - y(x^2+z)q = z(x^2-y^2)$

<u>Soln</u>

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)} \quad \text{--- (1)}$$

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choosing $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers, each fraction of (1)

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y^2+z-(x^2+z)+x^2-yz} = \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$$

$$\Rightarrow \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

$$\Rightarrow \log x + \log y + \lg z = \log c_1$$

$$\Rightarrow xyz = c_1 \quad \text{——— (2)}$$

choosing $x, y, -1$ as multipliers, each fraction of (1)

$$= \frac{x\,dx + y\,dy - dz}{x^2(y^2+z) - y^2(x^2+z) - z(x^2-y^2)}$$

$$= \frac{x\,dx + y\,dy - dz}{0}$$

$$\Rightarrow x\,dx + y\,dy - dz = 0$$

$$\Rightarrow x^2 + y^2 - 2z = c_2 \quad \text{——— (3)}$$

$\therefore$ from (2) and (3) soln is

$$\phi(xyz,\ x^2+y^2-2z) = 0$$

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Type 4 based on Rule-IV for solving

$$\longrightarrow \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \text{--- (1)}$$

Let P_1, Q_1 and R_1 be functions of x, y and z. Then by well-known principle of algebra, each fraction of (1) will be equal to

$$\frac{P_1\,dx + Q_1\,dy + R_1\,dz}{P_1 P + Q_1 Q + R_1 R} \quad \text{--- (2)}$$

suppose the numerator of (2) is exact diff. eqn. of the denominator of (2) then (2) can be combined with a suitable fraction in (1) to given an integral. However, in some problems another set of multipliers P_2, Q_2, and R_2 are so chosen that the fraction

$$\frac{P_2\,dx + Q_2\,dy + R_2\,dz}{P_2 P + Q_2 Q + R_2 R} \quad \text{--- (3)}$$

is such that its numerator is exact diff of denominator. fractions (2) and (3) are then combined to give an integral.

Exp (1) $(y+z)p + (z+x)q = x+y$

Soln: L.E.S are

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} \quad \text{--- (1)}$$

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choosing 1, -1, 0 as multipliers, each fraction of (1)

$$= \frac{dx - dy}{(y+z) - (z+x)} = \frac{d(x-y)}{-(x-y)} \quad \text{--- (2)}$$

Again choosing 0, 1, -1 as multipliers of (1)

$$= \frac{dy - dz}{(z+x) - (x+y)} = \frac{d(y-z)}{-(y-z)} \quad \text{--- (3)}$$

Finally choosing 1, 1, 1 as multipliers, each fraction of (1)

$$= \frac{dx + dy + dz}{(y+z) + (z+x) + (x+y)} = \frac{d(x+y+z)}{2(x+y+z)} \quad \text{--- (4)}$$

(2), (3) and (4)

$$\Rightarrow \frac{d(x-y)}{-(x-y)} = \frac{d(y-z)}{-(y-z)} = \frac{d(x+y+z)}{2(x+y+z)} \quad \text{--- (5)}$$

taking the first two fraction of (5).

$$\frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}$$

$$\Rightarrow \lg(x-y) = \lg(y-z) + \lg c_1$$

$$\frac{x-y}{y-z} = c_1 \quad \text{--- (6)}$$

taking first and third of (5)

$$2\frac{d(x-y)}{x-y} + \frac{d(x+y+z)}{(x+y+z)} = 0$$

$$2\lg(x-y) + \lg(x+y+z) = \lg c_2$$

$$(x-y)^2(x+y+z) = c_2 \quad \text{--- (7)}$$

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from (6) and (7) soln is

$$\phi\left[(x-y)^2(x+y+z),\ (x-y)/(y-z)\right]=0$$

Eg (2) $(x^2-yz)p+(y^2-zx)q=z^2-xy$

Soln

$$\frac{dx}{(x^2-yz)}=\frac{dy}{(y^2-zx)}=\frac{dz}{z^2-xy} \quad \text{--- (1)}$$

choose 1, -1, 0 and 0, 1, -1 as multr

$$=\frac{dx-dy}{x^2-y^2+z(x-y)}=\frac{dy-dz}{(y-z)(y+z+x)}$$

so that $\dfrac{dx-dy}{(x-y)(x+y+z)}=\dfrac{dy-dz}{(y-z)(x+y+z)}$

$$\Rightarrow \frac{dx-dy}{x-y}=\frac{dy-dz}{y-z} \quad \text{--- (2)}$$

or $\dfrac{x-y}{y-z}=c_1$ --- (2)

choose x, y, z as multr.

$$=\frac{xdx+ydy+zdz}{x^3+y^3+z^3-3xyz}=\frac{xdx+ydy+zdz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} \quad \text{--- (3)}$$

choose 1, 1, 1 as multr of (1)

$$=\frac{dx+dy+dz}{(x^2+y^2+z^2-xy-yz-zx)} \quad \text{--- (4)}$$

for (3) and (4)

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$$\frac{x\,dx + y\,dy + z\,dz}{x + y + z} = dx + dy + dz$$

or $2(x+y+z)\,d(x+y+z) - (2x\,dx + 2y\,dy + 2z\,dz) = 0$

$\Rightarrow (x+y+z)^2 - (x^2+y^2+z^2) = 2c_2$

$\Rightarrow 2xy + 2yz + 2zx = 2c_2$

$\Rightarrow xy + yz + zx = c_2$ — (5)

for (2) and (5)

$$\phi\left[xy + yz + zx,\ \frac{x-y}{y-z}\right] = 0$$

Non-linear P.D.E of order one.

→ **Complete Integral:** A complete integral, being a relation between x, y, and z represents equation of a surface. since it involves two arbitrary parameters, it belongs to a double infinite system of surfaces or to a single infinite system of family of surfaces.

→ **General integral:** Let a complete soln of $f(x, y, z, p, q) = 0$ be $\phi(x, y, z, a, b) = 0$ — (1)

A general soln is obtained by eliminating a between (1) and the eqns

$$b = \psi(a) \quad \text{— (2)}$$

$$\frac{\partial \phi}{\partial a} + \frac{\partial \phi}{\partial b} \cdot \psi'(a) = 0 \quad \text{— (3)}$$

where ψ is an arbitrary function.

→ **Singular Integral:** The singular integral is obtained by eliminating 'a' and 'b' between eqn (1)

$$\frac{\partial \phi}{\partial a} = 0 \quad \text{— (4)}$$

$$\frac{\partial \phi}{\partial b} = 0 \quad \text{— (5)}$$

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Compatible system of first-order Equations

Consider the first order P.D.Es

$$f(x, y, z, p, q) = 0 \quad \text{--- (1)}$$

and

$$g(x, y, z, p, q) = 0 \quad \text{--- (2)}$$

Equations (1) and (2) are known as compatible when every soln of one is also a solution of the other.

To find condition for (1) and (2) to be compatible

Let

$$J = J(f, g) = \frac{\partial(f, g)}{\partial(p, q)} \neq 0 \quad \text{--- (3)}$$

Then (1) and (2) can be solved to obtain the explicit expressions for p and q given by

$$p = \phi(x, y, z) \quad \text{and} \quad q = \psi(x, y, z) \quad \text{--- (4)}$$

The conditions that the pair of equations (1) and (2) should be compatible reduces then to the condition that the system of equations (4) should be completely integrable.

ie that the equation

$$dz = p\,dx + q\,dy \quad \text{or} \quad \phi\,dx + \psi\,dy - dz = 0 \quad \text{--- (5)}$$

should be integrable.

Eqn (5) is integrable if

$\{ P\,dx + Q\,dy + R\,dz = 0$ is integrable if $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$

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$$\phi\left(\frac{\partial\psi}{\partial z}-0\right)+\psi\left(0-\frac{\partial\phi}{\partial z}\right)+(-1)\left(\frac{\partial\phi}{\partial y}-\frac{\partial\psi}{\partial x}\right)$$

which is equivalent to

$$\frac{\partial\psi}{\partial x}+\phi\frac{\partial\psi}{\partial z}=\frac{\partial\phi}{\partial y}+\psi\frac{\partial\phi}{\partial z} \quad \text{—(6)}$$

Substituting from eqns (4) in (1) and differentiating w.r.t. x and z resp., we get

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial p}\frac{\partial\phi}{\partial x}+\frac{\partial f}{\partial q}\cdot\frac{\partial\psi}{\partial x}=0 \quad \text{—(7)}$$

and $$\frac{\partial f}{\partial z}+\frac{\partial f}{\partial p}\frac{\partial\phi}{\partial z}+\frac{\partial f}{\partial q}\frac{\partial\psi}{\partial z}=0 \quad \text{—(8)}$$

From (7) and (8);

$$\frac{\partial f}{\partial x}+\phi\frac{\partial f}{\partial z}+\frac{\partial f}{\partial p}\left(\frac{\partial\phi}{\partial x}+\phi\frac{\partial\phi}{\partial z}\right)+\frac{\partial f}{\partial q}\left(\frac{\partial\psi}{\partial x}+\phi\frac{\partial\psi}{\partial z}\right)=0 \quad \text{—(9)}$$

similarly (2) yields

$$\frac{\partial g}{\partial x}+\phi\frac{\partial g}{\partial z}+\frac{\partial g}{\partial p}\left(\frac{\partial\phi}{\partial x}+\phi\frac{\partial\phi}{\partial z}\right)+\frac{\partial g}{\partial q}\left(\frac{\partial\psi}{\partial x}+\phi\frac{\partial\psi}{\partial z}\right)=0 \quad \text{—(10)}$$

solving (9) and (10),

$$\frac{\partial\psi}{\partial x}+\phi\frac{\partial\psi}{\partial z}=\frac{1}{J}\left\{\frac{\partial(f,g)}{\partial(x,p)}+\phi\frac{\partial(f,g)}{\partial(z,p)}\right\} \quad \text{—(11)}$$

Again, substituting from eqs (4) in (1) and diff. w.r.t. y and z resp and proceeding as before, we have

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$$\frac{\partial \phi}{\partial y} + \psi \frac{\partial \phi}{\partial z} = -\frac{1}{J}\left\{ \frac{\partial(f,g)}{\partial(y,q)} + \psi \frac{\partial(f,g)}{\partial(z,q)} \right\} \quad (12)$$

substituting from equation (11) and (12) in (1) and replacing ϕ, ψ by p, q resp., we get

$$\frac{1}{J}\left\{ \frac{\partial(f,g)}{\partial(x,p)} + p \frac{\partial(f,g)}{\partial(z,p)} \right\} = -\frac{1}{J}\left\{ \frac{\partial(f,g)}{\partial(y,q)} + q \frac{\partial(f,g)}{\partial(z,q)} \right\}$$

or $$[f, g] = 0 \quad \text{--- (13)}$$

where $$[f, g] = \frac{\partial(f,g)}{\partial(x,p)} + p \frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q \frac{\partial(f,g)}{\partial(z,q)} \quad \text{--- (14)}$$

$= 0$

→ To show that first order PDEs $p = P(x,y)$ and $q = Q(x,y)$ are compatible iff.

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

<u>Proof:</u> $\frac{\partial z}{\partial x} = p = P(x,y)$ & $\frac{\partial z}{\partial y} = q = Q(x,y)$ –

Since $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = p\,dx + q\,dy$ –

it follows that the given PDEs (1) are compatible iff the single diff eqn

$$dz = P\,dx + Q\,dy \quad \text{--- (3)}$$

is integrable.

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since P and Q are functions of two variables x and y, hence $P\,dx + Q\,dy$ is an exact diff. eqn iff $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

$\therefore$ eqn (3) is integrable iff $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

Note (1) If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, then the system of two given P.D.Es (1) is compatible and hence they will possess a common soln.

Note (2) If $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, then the system of two given P.D.Es (1) is not compatible and hence these eqns possess no soln.

Exp (1) $\frac{\partial z}{\partial x} = 5x - 7y$, $\frac{\partial z}{\partial y} = 6x + 8y$ — (1)

Soln Here, by $p = P(x, y)$ and $q = Q(x, y)$

$P = 5x - 7y$ $\quad Q = 6x + 8y$

Now $\frac{\partial P}{\partial y} = -7$ and $\frac{\partial Q}{\partial x} = 6$

Since $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$

$\Rightarrow$ eqn (1) is not compatible

(2) $p = x^2 - ay$, $q = y^2 - ax$ — (1)

Here $P = x^2 - ay$ & $Q = y^2 - ax$

Now $\frac{\partial P}{\partial y} = -a$ and $\frac{\partial Q}{\partial x} = -a$

$\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = -a$

so eqn (1) is compatible

Common soln: The common soln of eqn (1) is given by

$$dz = P\,dx + Q\,dy.$$

$$dz = (x^2 - ay)dx + (y^2 - ax)dy.$$

$$dz = x^2 dx + y^2 dy - a\,d(xy)$$

on integrate:

$$z = \frac{x^3}{3} + \frac{y^3}{3} - axy + c$$

Exp (3) show that the eqns $xp - yq = x$ and $x^2 p + q = xz$ are compatible and find their soln.

Soln: $f(x, y, z, p, q) = xp - yq - x = 0$ — (1)

$g(x, y, z, p, q) = x^2 p + q - xz = 0$ — (2)

$$\therefore \frac{\partial(f, g)}{\partial(x, p)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} p-1 & x \\ 2xp - z & x^2 \end{vmatrix}$$

$$= (p-1)x^2 - x(2xp - z)$$

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Similarly $\frac{\partial(f,g)}{\partial(z,p)} = x^2$, $\frac{\partial(f,g)}{\partial(y,q)} = -q$

and $\frac{\partial(f,g)}{\partial(z,q)} = -xy$

$$\therefore [f,g] = \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)}$$

$$= (p-1)x^2 - x(2px-2) - px^2 - q - xyq$$

$$= -x^2 + 2x - q - xyq$$

$$= -x^2 + x^2p + \cancel{q} - \cancel{q} - xyq \quad (\text{using } (2))$$

$$= x(-x + xp - yq) = 0 \quad (\text{using } (1))$$

$$\Rightarrow [f,g] = 0$$

Hence (1) and (2) are compatible.

<u>Solution</u>: from (1) and (2)

$$p = \frac{(1+yz)}{(1+xy)}, \quad q = \frac{x(z-x)}{(1+xy)} \quad (3)$$

$$\therefore dz = p\,dx + q\,dy$$

$$dz = \frac{(1+yz)}{(1+xy)}dx + \frac{x(z-x)}{(1+xy)}dy$$

$$(1+xy)dz = (1+yz)dx + x(z-x)dy.$$

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or $(1+xy)dz - z(ydx + xdy) = dx - x^2 dy$

$$\Rightarrow \frac{(1+xy)dz - z\,d(xy)}{(1+xy)^2} = \frac{dx - x^2 dy}{(1+xy)^2}$$

$$\Rightarrow \frac{(1+xy)dz - z\,d(xy)}{(1+xy)^2} = \frac{\frac{dx}{x^2} - dy}{\left(y+\frac{1}{x}\right)^2}$$

$$\Rightarrow d\left(\frac{z}{1+xy}\right) = \frac{-d\left(y+\frac{1}{x}\right)}{\left(y+\frac{1}{x}\right)^2}$$

on integrating, $\dfrac{z}{1+xy} = \dfrac{1}{\left(y+\frac{1}{x}\right)} + C$

or $\dfrac{z}{1+xy} = \dfrac{x}{1+xy} + C$

or $(z-x) = C(1+xy)$

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Working Rule while using Charpit's Meth

Step ①: Transfer all terms of the given eqn. to L.H.S and denote the entire expression By f.

Step ②: Write down the charpit's Auxiliary eqn

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{df}{0}$$

Step ③: Using the value of f in step ① write down the values of $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ ... etc. and put these in charpit's eqn.

Step ④: After simplifying the step ③, select two proper fractions so that the resulting integral may come out to be the simplest relation involving at least one of p and q.

Step ⑤: The simplest relation of step ④ is solved along with the given equation to determine p and q. Put these value of p and q in $dz = pdx + qdy$ which on integration gives the complete integral of the given equation.

→ The general and singular integrals may be obtained in the usual manner.

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Note: If charbit's equations give rise to $p = a$ and $q = b$, in such cases, putting $p = a$ and $q = b$ in the given equation will give the required complete integral.

Exp ① Find the complete integral of

$$z = px + qy + p^2 + q^2$$

Solⁿ: Let $f(x, y, z, p, q) = z - px - qy - p^2 - q^2 = 0$ — ①

Charbit's Auxiliary eqns are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} \quad ②$$

from ①, $f_x = -p$, $f_y = -q$, $f_z = 1$, $f_p = -x - 2p$

and $f_q = -y - 2q$ — ③

using ③, ② reduces to

$$\frac{dp}{0} = \frac{dq}{0} = \frac{dz}{p(x+2p) + q(y+2q)} = \frac{dx}{x+2p} = \frac{dy}{y+2q} \quad ④$$

from ④ $dp = 0 \Rightarrow p = a$

& $dq = 0 \Rightarrow q = b$

Now putting $p = a$ and $q = b$ in (1), the requi complete integral is

$$\boxed{z = ax + by + a^2 + b^2}$$

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Exp ② Find the complete, singular and general integrals of $(p^2+q^2)y = qz$

Solⁿ: Here the given eqⁿ is

$$f(x, y, z, p, q) = (p^2+q^2)y - qz = 0 \quad (1)$$

charpit's auxiliary equations are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

$$\text{or } \frac{dp}{-pq} = \frac{dq}{p^2-q^2} = \frac{dz}{-2p^2y + qz - 2q^2y} = \frac{dx}{-2py} = \frac{dy}{-2qy+z} \quad (2)$$

taking first two fractions

$$\frac{dp}{-pq} = \frac{dq}{p^2} \Rightarrow pdp + qdq = 0$$

$$\Rightarrow p^2 + q^2 = a^2 \quad \text{—— (3)}$$

using (3), (1) gives $ay = qz$

$$\text{or } \boxed{q = \frac{ay}{z}}$$

Putting this value of q in (3), we get

$$p = \sqrt{(a^2-q^2)} = \sqrt{a^2 - \frac{a^4y^2}{z^2}}$$

$$\boxed{p = \frac{a}{z}\sqrt{z^2 - a^2y^2}}$$

Now putting these values of p and q in

$dz = pdx + qdy$, we have

$$dz = \frac{a}{z}\sqrt{(z^2 - a^2y^2)}\,dx + \frac{a^2y\,dy}{z}$$

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$$\Rightarrow \frac{z\,dz - a^2 y\,dy}{\sqrt{(z^2 - a^2 y^2)}} = a\,dx$$

on integrating, $\sqrt{z^2 - a^2 y^2} = ax + b$

or $z^2 - a^2 y^2 = (ax+b)^2$ — (4)

which is a required complete integral, a, b being arbitrary constants.

<u>singular Integral</u>: Diff. (4) partially w.r.t a and b, we have

$$0 - 2ay^2 = 2(ax+b)\cdot x \quad \text{— (5)}$$

$$0 = 2(ax+b) \quad \text{— (6)}$$

eliminating a, b, between (4), (5) and (6), we get $z = 0$, which satisfies (1) and hence it is the singular solⁿ

<u>General Integral</u>: Replacing b by $\phi(a)$ in (4), we have

$$z^2 - a^2 y^2 = [ax + \phi(a)]^2 \quad \text{— (7)}$$

Diff. (7) partially w.r.t. a,

$$-2ay^2 = 2[ax + \phi(a)][x + \phi'(a)] \quad \text{— (8)}$$

General ~~Sol~~ integral is obtained by eliminating a from (7) and (8).

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Exp ③ Find a complete and singular integral of $2xz - px^2 - 2qxy + pq = 0$

Soln: Here given equation is

$$f(x, y, z, p, q) = 2xz - px^2 - 2qxy + pq = 0 \quad (1)$$

Charpit's auxiliary equations are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

$$\Rightarrow \frac{dp}{2z - 2qy} = \frac{dq}{0} = \frac{dx}{x^2 - q} = \frac{dy}{2xy - p} = \frac{dz}{px^2 + 2xyq - 2pq}$$

The second fraction gives $dq = 0$ so $q = a$

Putting $q = a$ in (1), we get $p = \frac{2x(z - ay)}{(x^2 - a)}$

Putting values p and q in $dz = p\,dx + q\,dy$, we get

$$dz = \frac{2x(z - ay)}{(x^2 - a)}\,dx + a\,dy$$

or $$\frac{dz - a\,dy}{z - ay} = \frac{2x\,dx}{x^2 - a}$$

Integrating $\log(z - ay) = \log(x^2 - a) + \log b$

or $z - ay = b(x^2 - a)$

or $z = ay + b(x^2 - a)$ —— ②

which is the complete integral.

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Singular Integral.

Diff. (2) P.w.r.t a and b we, get

$$0 = y - b \quad \text{and} \quad 0 = x^2 - a \quad \text{--- (3)}$$

solving (3) for a and b

$$x^2 = a \quad \text{and} \quad b = y \quad \text{--- (4)}$$

substituting the values of a and b given by (4) in (2), we get $z = x^2 y$ which is the required singular integral.

Jacobi's Method.

This method is used for solving P.D.Es involving three or more independent variables. The central idea of Jacobi's method is almost the same as that of Charpit's method for two independent variables.

Working Rule.

Step ①: Suppose the given eqn with three independent variable is

$$f(x_1, x_2, x_3, p_1, p_2, p_3) = 0 \quad \text{--- (1)}$$

Step ②: Write the Jacobi's auxiliary eqn

$$\frac{dp_1}{f_{x_1}} = \frac{dx_1}{-f_{p_1}} = \frac{dp_2}{f_{x_2}} = \frac{dx_2}{-f_{p_2}} = \frac{dp_3}{f_{x_3}} = \frac{dx_3}{-f_{p_3}}$$

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solving these eqn we obtain two additional eqns.

$$F_1(x_1, x_2, x_3, p_1, p_2, p_3) = a_1 \quad - (2)$$

$$F_2(x_1, x_2, x_3, p_1, p_2, p_3) = a_2 \quad - (3)$$

while obtaining (2) and (3), try to select simple eqns so that later on sol^ns of (1), (2) and (3) may be as easy as possible

<u>step (3)</u>: Verify that relations (2) and (3) satisfy the condition

$$(F_1, F_2) = \sum_{r=1}^{3} \left(\frac{\partial F_1}{\partial x_r} \frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r} \frac{\partial F_2}{\partial x_r} \right) = 0 \quad (4)$$

If (4) is satisfied then solve (1), (2) and (3) for p_1, p_2, p_3 in terms of x_1, x_2, x_3. their substitution in

$$dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$$

and subsequent integration leads to a complete integral of the given eqn.

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Note: while solving a P.D.E with four variables, we have

Step ①: suppose the given eqn with 4 ind. variables is

$$f(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = 0 \quad (1)$$

Step ②: write Jacobi's A-ES

$$\frac{dp_1}{f_{x_1}} = \frac{dx_1}{-f_{p_1}} = \frac{dp_2}{f_{x_2}} = \frac{dx_2}{-f_{p_2}} = \frac{dp_3}{f_{x_3}} = \frac{dx_3}{-f_{p_3}} = \frac{dp_4}{f_{x_4}}$$

$$= \frac{dx_4}{-f_{p_4}}$$

solving these we obtain three additional eqns

$$F_1(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = a_1 \quad (2)$$

$$F_2(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = a_2 \quad (3)$$

$$F_3(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = a_3 \quad (4)$$

Step ③: verify that relation (2), (3) and (4) satisfy following three condition

$$(F_1, F_2) = \sum_{r=1}^{4} \left(\frac{\partial F_1}{\partial x_r} \frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r} \frac{\partial F_2}{\partial x_r} \right) = 0 \quad (5)$$

$$(F_2, F_3) = \sum_{r=1}^{4} \left(\frac{\partial F_2}{\partial x_r} \frac{\partial F_3}{\partial p_r} - \frac{\partial F_2}{\partial p_r} \frac{\partial F_3}{\partial x_r} \right) = 0 \quad (6)$$

$$(F_3, F_1) = \sum_{r=1}^{4} \left(\frac{\partial F_3}{\partial x_r} \frac{\partial F_1}{\partial p_r} - \frac{\partial F_3}{\partial p_r} \frac{\partial F_1}{\partial x_r} \right) = 0 \quad (7)$$

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If (5), (6) and (7) are satisfied, then solve (1), (2), (3) and (4) for p_1, p_2, p_3 and p_4 in terms of x_1, x_2, x_3 and x_4.

Then solve $dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3 + p_4 dx_4$ and subsequent integration leads to a complete integral of the given eqn.

<u>Exp ①</u> Find a complete integral of

$$p_1^3 + p_2^2 + p_3 = 1$$

<u>Soln.</u> Let $f(x_1, x_2, x_3, p_1, p_2, p_3) = p_1^3 + p_2^2 + p_3 - 1 = 0$ — ①

$\therefore$ Jacobi's auxiliary eqns are.

$$\frac{dp_1}{f_{x_1}} = \frac{dx_1}{-f_{p_1}} = \frac{dp_2}{f_{x_2}} = \frac{dx_2}{-f_{p_2}} = \frac{dp_3}{f_{x_3}} = \frac{dx_3}{-f_{p_3}}$$

using ①, we have

$$\Rightarrow \frac{dp_1}{0} = \frac{dx_1}{-3p_1^2} = \frac{dp_2}{0} = \frac{dx_2}{-2p_2} = \frac{dp_3}{0} = \frac{dx_3}{-1}$$

from first and third fractions.

$$dp_1 = 0 \text{ and } dp_2 = 0$$

so $p_1 = a_1$ & $p_2 = a_2$

$\therefore$ Here $F_1(x_1, x_2, x_3, p_1, p_2, p_3) = p_1 = a_1$ — ②

$F_2(x_1, x_2, x_3, p_1, p_2, p_3) = p_2 = a_2$ — ③

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Now $(F_1 F_2) = \left(\frac{\partial F_1}{\partial x_1}\frac{\partial F_2}{\partial p_1} - \frac{\partial F_1}{\partial p_1}\frac{\partial F_2}{\partial x_1}\right) + \left(\frac{\partial F_1}{\partial x_2}\frac{\partial F_2}{\partial p_2} - \frac{\partial F_1}{\partial p_2}\frac{\partial F_2}{\partial x_2}\right)$

$$+ \left(\frac{\partial F_1}{\partial x_3}\frac{\partial F_2}{\partial p_3} - \frac{\partial F_1}{\partial p_3}\frac{\partial F_2}{\partial x_3}\right)$$

$$(F_1, F_2) = 0\times 0 - 1\times 0 + 0\times 1 - 0\cdot 0 + 0\cdot 0 - 0\cdot 0 = 0$$

$$\Rightarrow (F_1, F_2) = 0$$

Hence (2) and (3) may be taken as additional eqns.

solving (1), (2) and (3) for p_1, p_2, p_3

$$p_1 = a_1, \quad p_2 = a_2, \quad p_3 = 1 - a_1^3 - a_2^2$$

Putting these value in

$$dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$$

$$dz = a_1 dx_1 + a_2 dx_2 + (1 - a_1^3 - a_2^2) dx_3$$

on integrating

$$z = a_1 x_1 + a_2 x_2 + (1 - a_1^3 - a_2^2) x_3 + a_3$$

which is the complete integral with three constants (arbitrary) a_1, a_2 and a_3.

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Ex (2) Find complete integral of

$$2p_1x_1x_3 + 3p_2x_3^2 + p_2^2p_3 = 0$$

Solⁿ $f(x_1, x_2, x_3, p_1, p_2, p_3) = 2p_1x_1x_3 + 3p_2x_3^2 + p_2^2p_3 = 0$ (1)

$$\frac{dp_1}{f_{x_1}} = \frac{dx_1}{-f_{p_1}} = \frac{dp_2}{f_{x_2}} = \frac{dx_2}{-f_{p_2}} = \frac{dp_3}{dx_3} = \frac{dx_3}{-f_{p_3}}$$

or $$\frac{dp_1}{2p_1x_3} = \frac{dx_1}{-2x_1x_3} = \frac{dp_2}{0} = \frac{dx_2}{-3x_3^2 - 2p_1p_3} = \frac{dp_3}{2p_1x_1 + 6p_2x_3}$$

$$= \frac{dx_3}{-p_2^2} \quad \text{— (2)}$$

taking first two fraction of (2)

$$\frac{dp_1}{p_1} = -\frac{dx_1}{x_1} \Rightarrow p_1x_1 = a_1$$

Let $F_1(x_1, x_2, x_3, p_1, p_2, p_3) = p_1x_1 = a_1$ — (3)

taking the third fraction of (2)

$$dp_2 = 0 \Rightarrow p_2 = a_2$$

Let $F_2(x_1, x_2, x_3, p_1, p_2, p_3) = p_2 = a_2$ — (4)

varify that $(F_1 F_2) = 0$

Now, solving (1), (3) and (4) for p_1, p_2, p_3

$$p_1 = \frac{a_1}{x_1}, \quad p_2 = a_2, \quad p_3 = -\frac{(2a_1x_3 + 3a_2x_3)}{a_2^2}$$

Putting these values in $dz = p_1dx_1 + p_2dx_2 + p_3dx_3$

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$$dz = a_1 \frac{dx_1}{x_1} + a_2 dx_2 - \left\{ \frac{(2a_1 x_3 + 3a_2 x_3^2)}{a^2} \right\} dx_3$$

$$\Rightarrow z = a_1 \log x_1 + a_2 x_2 - \frac{(a_1 x_3^2 + a_2 x_3^3)}{a_2^2} + a_3$$

Jacobi's method for solving a non-linear first order PDE in two independent variables

Let $F(x, y, z, p, q) = 0$ — (1)
be the non-linear first order eqn in two indep variables x, y.

Then we know that a soln of (1) is of the form $u(x, y, z) = 0$ — (2)
showing that u can be treated as a dependent variable and x, y, z as three independent variables.

Diff. (2) partially w.r.t. x and y resp. we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial y} = 0$$

or $P_1 + P_3 P = 0$ and $P_2 + P_3 q = 0$ — (3)

where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, $P_1 = \frac{\partial u}{\partial x}$, $P_2 = \frac{\partial u}{\partial y}$, $P_3 = \frac{\partial u}{\partial z}$

$P_1 = \frac{\partial u}{\partial x_1}$, $P_2 = \frac{\partial u}{\partial x_2}$ $P_3 = \frac{\partial u}{\partial x_3}$

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by taking $x = x_1$, $y = x_2$, $z = x_3$ — (4)

from (3), $p = -\frac{P_1}{P_3}$ and $q = -\frac{P_2}{P_3}$ — (5)

using (4) and (5), (1) reduces to

$$f(x_1, x_2, x_3, P_1, P_2, P_3) = 0 \quad \text{— (6)}$$

we now solve (6) by Jacobi's method as usual to get the complete integral of (6). Finally putting $x = x_1$, $y = x_2$, $x_3 = z$, we obtain soln of (6) containing original variables x, y, z and new dependent variable u. The solution so obtained will contain three arbitrary constants a_1, a_2, a_3 (say). However, for the given equation in the form (1), we need only two arbitrary consts. in the final soln. The required soln $u = 0$ of (1) is obtained by making different choice of our third arbitrary constant.

<u>Exp (1)</u> $p^2x + q^2y = z$, solve by Jacobi's method.

<u>Soln</u>: Given $p^2x + q^2y = z$ — (1)

Let a soln of (1) be of the form:

$$u(x, y, z) = a \quad \text{— (2)}$$

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so treating u as dependent variable and x, y, z as three indep. variables, diff (2) partially w.r.t x and y resp. gives

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y} = 0$$

i.e. $p_1 + p_3 p = 0$ and $p_2 + p_3 q = 0$

so $p = -p_1/p_3$ and $q = -p_2/p_3$ — (3)

where $p_1 = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial x_1}$, $p_2 = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial x_2}$

$p_3 = \frac{\partial u}{\partial z} = \frac{\partial u}{\partial x_3}$, $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$

by taking $x = x_1$, $y = x_2$ and $z = x_3$ — (4)

using (3) and (4), (1) $\Rightarrow$

$$x_1\left(\frac{p_1}{p_3}\right)^2 + x_2\left(\frac{p_2}{p_3}\right)^2 = x_3$$

$$\Rightarrow \quad x_1 p_1^2 + x_2 p_2^2 - x_3 p_3^2 = 0$$

Let $f(x_1, x_2, x_3, p_1, p_2, p_3) = x_1 p_1^2 + x_2 p_2^2 - x_3 p_3^2$ — (5)

Now, the Jacobi's A Eq are

$$\frac{dp_1}{f_{x_1}} = \frac{dx_1}{-f_{p_1}} = \frac{dp_2}{f_{x_2}} = \frac{dx_2}{-f_{p_2}} = \frac{dp_3}{f_{x_3}} = \frac{dx_3}{-f_{p_3}}$$

or $$\frac{dp_1}{p_1^2} = \frac{dx_1}{-2p_1x_1} = \frac{dp_2}{p_2^2} = \frac{dx_2}{-2p_2x_2} = \frac{dp_3}{-p_3^2} = \frac{dx_3}{2p_3x_3}$$

taking first two fractions

$$2\frac{dp_1}{p_1} + \frac{1}{x_1}dx_1 = 0 \quad \Rightarrow \quad 2\log p_1 + \log x_1 = \log a_1$$

$$\Rightarrow \quad x_1 p_1^2 = a_1$$

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or $p_1 = \left(\frac{a_1}{x_1}\right)^{1/2}$

Similarly, by third and forth factor

$$p_2 = \left(\frac{a_2}{x_2}\right)^{1/2}$$

Substituting these values of p_1 and p_2 in (5),

we get

$$p_3 = \left\{\frac{(a_1 + a_2)}{x_3}\right\}^{1/2}$$

Putting these values of p_1 p_2 and p_3 in

$$du = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$$

$$du = a_1^{1/2} x_1^{-1/2} dx_1 + a_2^{1/2} x_2^{-1/2} dx_2 + (a_1+a_2)^{1/2} x_3^{-1/2} dx_3$$

$$\Rightarrow u = 2(a_1 x_1)^{1/2} + 2(a_2 x_2)^{1/2} + 2\{(a_1+a_2)x_3\}^{1/2} + a_3 \quad \text{— (6)}$$

Taking $a_2 = 1$ and using (4), the required soln $u = 0$ is given by

$$2(a_1 x)^{1/2} + 2y^{1/2} + 2(a_1+1)^{1/2} z^{1/2} + a_3 = 0$$

which is complete integral

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Integral surfaces passing through a given curve

Here, we shall now discussed two methods of using the general soln of $Pp + Qq = R$ for getting the integral surface which passes through a given curve.

Method - 1. Let $Pp + Qq = R$ — (1)
be the given eqn. Let its AEs give the two independent solutions

$$u(x, y, z) = c_1 \quad \text{and} \quad v(x, y, z) = c_2 \quad \text{— (2)}$$

x_3 Suppose we wish to obtain the integral surface
y_3 which passes through the curve whose eqn in parametric form is given by

$$x = x(t), \quad y = y(t), \quad z = z(t) \quad \text{— (3)}$$

where t is a parameter.
Then (2) may be expressed as

$$\left.\begin{aligned} u[x(t), y(t), z(t)] &= c_1 \quad \text{and} \\ v[x(t), y(t), z(t)] &= c_2 \end{aligned}\right\} \quad (4)$$

We eliminate single parameter t from the eqns of (4) and get a relation involving c_1 and c_2. Finally, we replace c_1, c_2 with help of (2) and obtain the required integral surface.

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Exp Find the integral surface of the linear PDE

$$x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$$

which contains the straight line $x+y=0,\ z=1$

Soln $x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$ — (1)

Lagrange's AES are

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{(x^2-y^2)z} \quad \text{— (2)}$$

choosing $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multiplier

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y^2+z-x^2-z+x^2-y^2} = \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz$$

$\Rightarrow \log x + \log y + \log z = \log c_1$

$\Rightarrow xyz = c_1$

choosing $x, y, -1$ as multiplier

$\Rightarrow x\,dx + y\,dy - dz = 0$

$\Rightarrow x^2 + y^2 - 2z = c_2,\ xyz = c_1$ — (3)

taking t as parameter, the given eqn of the straight line $x+y=0,\ z=1$ can be put in parametric form

$x = t \qquad y = -t \qquad z = 1$ — (4)

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using (4), (3) $\Rightarrow$

$$-t^2 = c_1 \quad \text{and} \quad 2t^2 - 2 = c_2 \quad (5)$$

eliminate t from the eqn (5), we have

$$2(-c_1) - 2 = c_2 \quad \text{or} \quad 2c_1 + c_2 + 2 = 0 \quad (6)$$

Putting the values of c_1 and c_2 from (1) and (6) the desired integral surface is

$$2xyz + x^2 + y^2 - 2z + 2 = 0$$

Exp (2) $2y(z-3)p + (2x-2)q = y(2x-3)$ — (1)

pass through the circle $z=0$, $x^2+y^2=2x$ (2)

Soln: $$\frac{dx}{2y(z-3)} = \frac{dy}{2x-2} = \frac{dz}{y(2x-3)} \quad (3)$$

taking first and third

$$(2x-3)\,dx - 2(z-3)\,dz = 0$$

$$x^2 - 3x - z^2 + 6z = c_1 \quad (4)$$

choosing $\frac{1}{2}$, y, -1 as multipliers

$$\frac{1}{2}dx + y\,dy - dz = 0 \Rightarrow x + y^2 - 2z = c_2 \quad (5)$$

Now, the parametric eqns of (2) are

$$x = t \qquad y = (2t - t^2)^{1/2} \qquad z = 0 \quad (6)$$

Putting these value in (4) and (5), we have

$$t^2 - 3t = c_1 \quad \text{and} \quad t + 2t - t^2 = c_2$$

$$\text{or} \quad t^2 - 3t = c_1 \quad \text{and} \quad 3t - t^2 = c_2 \quad (7)$$

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eliminate f from above eqn (7), we

$$c_1 + c_2 = 0 \quad -(8)$$

Put the value of c_1 and c_2

$$x^2 - 3x - z^2 + 6z + x + y^2 - 2z = 0$$

$$\Rightarrow x^2 + y^2 - z^2 - 2x + 4z = 0$$

<u>Method-2</u>: Let $Pp + Qq = R \quad -(1)$ be the given eqn. Let its L.A.E are give the two independent ~~soln~~ integrals

$$u(x, y, z) = c_1 \quad \text{and} \quad v(x, y, z) = c_2 \quad -(2)$$

suppose we wish to obtain the integral surface passing through the curve which is determined by the two eqns

$$\phi(x, y, z) = 0 \quad \text{and} \quad \psi(x, y, z) = 0 \quad -(3)$$

we eliminate, x, y, z from four eqns of (2) and (3) and obtain a relation between c_1 and c_2. Finally replace c_1 by $u(x, y, z)$ and c_2 by $v(x, y, z)$ in that relation and obtain the desired integral surface.

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Exp ① $(2xy-1)p + (z-2x^2)q = 2(x-yz)$ — ①

Pass through the line $x=1 \quad y=0$ — ②

Soln: L.A.E.s are

$$\frac{dx}{2xy-1} = \frac{dy}{z-2x^2} = \frac{dz}{2(x-yz)} \quad \text{— ③}$$

taking $z, 1, x$ as multipliers

$$= \frac{(z\,dx + dy + x\,dz)}{0} \Rightarrow z\,dx + dy + x\,dz = 0$$

$$\Rightarrow d(zx) + dy = 0 \Rightarrow xz + y = c_1 \quad \text{— ④}$$

Again choosing $x, y, \frac{1}{2}$ as multiplier

$$\Rightarrow x\,dx + y\,dy + \frac{1}{2}dz = 0$$

$$\Rightarrow x^2 + y^2 + z = c_2 \quad \text{— ⑤}$$

Since the required curve given by ④ and ⑤ passes through the line (2). So put $x=1$ and $y=0$ in ④ and ⑤, we get

$$z = c_1 \quad \text{and} \quad 1+z = c_2$$

$$\Rightarrow 1 + c_1 = c_2 \quad \text{— ⑥}$$

Putting value of c_1 & c_2 in ⑥ we get the required eqn of integral surface.

$$1 + xz + y = x^2 + y^2 + z \text{ or}$$

$$\boxed{x^2 + y^2 + z - xz - y = 1}$$

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Exp $x^2p + y^2q = z^2$

Pass through the hyperbola $xy = x+y$, $z=$

Ans $\frac{1}{x} + \frac{1}{y} + \frac{2}{z} = 3$ or $yz + 2xy + xz = 3xyz$

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{-z^2}$$

taking first and third fraction

$$-\frac{1}{x} - \frac{1}{z} = -c_1 \quad \text{or} \quad \frac{1}{x} + \frac{1}{z} = c_1 \quad \text{—(1)}$$

taking second and third fraction

$$-\frac{1}{y} - \frac{1}{z} = -c_2 \quad \text{or} \quad \frac{1}{y} + \frac{1}{z} = c_2 \quad \text{—(2)}$$

adding (1) and (2)

$$\frac{1}{x} + \frac{1}{y} + \frac{2}{z} = c_1 + c_2$$

$$\text{or} \quad \frac{x+y}{xy} + \frac{2}{z} = c_1 + c_2 \quad \text{— (3)}$$

Using $x+y = xy$ and $z = 1$ in (3), we have

$$1 + 2 = c_1 + c_2 \quad \text{or} \quad c_1 + c_2 = 3 \quad \text{—(4)}$$

Now substituting values of c_1 and c_2, we get

$$\frac{1}{x} + \frac{1}{y} + \frac{2}{z} = 3 \quad \text{or} \quad yz + 2xy + xz = 3xyz$$

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Cauchy's Problem.

The aim of an existence theorem is to establish conditions under which we can decide whether or not a given P.D.E has a soln at all. the next step of providing that the solution, when it exists, is unique requires a uniqueness theorem. The cauchy's problem for two independent variables can be stated as follows

If (a) $x_0(u)$, $y_0(u)$ and $z_0(u)$ are functions which, together with their first derivatives, are continuous in the interval I defined by $u_1 < u < u_2$

(b) And if $f(x, y, z, p, q)$ is a continuous function of x, y, z, p and q in a certain region U of the $xyzpq$ space. then it is required to establish the existence of a function $\phi(x, y)$ with the following properties:

(1) $\phi(x, y)$ and its partial derivatives with respect to x and y are continuous functions of x and y in a region R of the xy space.

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(II) For all values of x and y lying in R, the point $\{x, y, \phi(x,y), \phi_x(x,y), \phi_y(x,y)\}$ lies in $\frac{U}{x}$ and

$$f[x, y, \phi(x,y), \phi_x(x,y), \phi_y(x,y)] = 0$$

(III) For all u belonging to the interval I, the point $\{x_0(u), y_0(u)\}$ belongs to the region R, and

$$\phi\{x_0(u), y_0(u)\} = z_0$$

Stated geometrically, what we wish to prove is that there exists a surface $z = \phi(x, y)$ which passes through the curve C whose parametric equations are given by $x = x_0(u)$, $y = y_0(u)$, $z = z_0(u)$ and at every point of which the *direction $(p, q, -1)$ of the normals is such that

$$f(x, y, z, p, q) = 0$$

* Let $z = \phi(x, y)$ — (1)

be the eqn of the given surface.

Let $F(x, y, z) = \phi(x, y) - z$ — (2)

from (1) and (2),

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$$\frac{\partial F}{\partial x} = \frac{\partial \phi}{\partial x} = \frac{\partial z}{\partial x} = p$$

$$\frac{\partial F}{\partial y} = \frac{\partial \phi}{\partial y} = \frac{\partial z}{\partial y} = q$$

$$\frac{\partial F}{\partial z} = -1$$

since ∇F is normal to the surface $F(x, y, z) = 0$

$\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$ ie $p, q, -1$ are direction ratios of the normal of $F(x, y, z) = 0$

or $z = \phi(x, y)$

<u>Exp</u> ① Solve the Cauchy's Problem for $zp + q = 1$, when the initial data curve is $x_0 = u$, $y_0 = u$, $z_0 = u/2$ $\quad 0 \le u \le 1$

<u>Sol^n</u>: Given $f(x, y, z, p, q) = zp + q - 1 = 0$ ①

Given initial data curve

$$x_0 = u, \quad y_0 = u, \quad z_0 = \frac{u}{2}, \quad 0 \le u \le 1 \quad ②$$

from ① $\frac{\partial f}{\partial p} = z$, $\quad \frac{\partial f}{\partial q} = 1$

and $\frac{\partial f}{\partial q} \cdot \frac{dx_0}{du} - \frac{\partial f}{\partial p} \frac{dy_0}{du} = 1 \times 1 - z \times 1 = 1 - \frac{u}{2}$

$= 1 - \frac{u}{2} \neq 0$ for $0 \le u \le 1$

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Now, we have the following ordinary diff eqns:

$$\frac{dx}{dt} = \frac{\partial f}{\partial p} \quad \text{and} \quad \frac{dy}{dt} = \frac{\partial f}{\partial q}$$

and $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

or $$\frac{dx}{dt} = 2, \quad \frac{dy}{dt} = 1, \quad \frac{dz}{dt} = p\frac{dx}{dt} + q\frac{dy}{dt} \quad (3)$$

$$\frac{dz}{dt} = pz + q = 1 \quad (4)$$

From (3) and (4) $y = t + c_1$ and $z = t + c_2$ — (5)

From (2), at $t = 0$

$$x(\mu, 0) = \mu, \quad y(\mu, 0) = \mu, \quad z(\mu, 0) = \frac{\mu}{2} \quad (6)$$

using (6), (5) reduce to

$$y = t + \mu \quad \text{and} \quad z = t + \frac{\mu}{2} \quad (7)$$

Then, from (3) and (7),

$$\frac{dx}{dt} = t + \frac{\mu}{2} \quad \text{so} \quad x = \frac{t^2}{2} + \frac{1}{2}\mu t + c_3 \quad (8)$$

using (6), (8) reduce to

$$x = \frac{1}{2}t^2 + \frac{1}{2}\mu t + \mu \quad (9)$$

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Solving $y = l + u$ with (3) for u and l in terms of x and y, we get

$$l = \frac{y - x}{1 - \frac{y}{2}} \quad \text{and} \quad u = \frac{x - \frac{y^2}{2}}{1 - \frac{y}{2}}$$

Putting these values in b

$$z = l + \frac{u}{2}$$

The required soln passing through the initial data curve is

(5) $$z = \left\{ 2(y - x) + x - \frac{y^2}{2} \right\} / (2 - y)$$

(6) Cauchy's Method of characteristics for solving non-linear P.D.E

(7) $$f(x, y, z, p, q) = 0 \quad \text{——} \quad (1)$$

we know that the plane passing through the Point $P(x_0, y_0, z_0)$ with its normal

(8) parallel to the direction $\vec{n}$ whose direction ratios are $p_0, q_0, -1$ is uniquely given by the set of five numbers $D(x_0, y_0, z_0, p_0, q_0)$ and conversely any such set of five numbers

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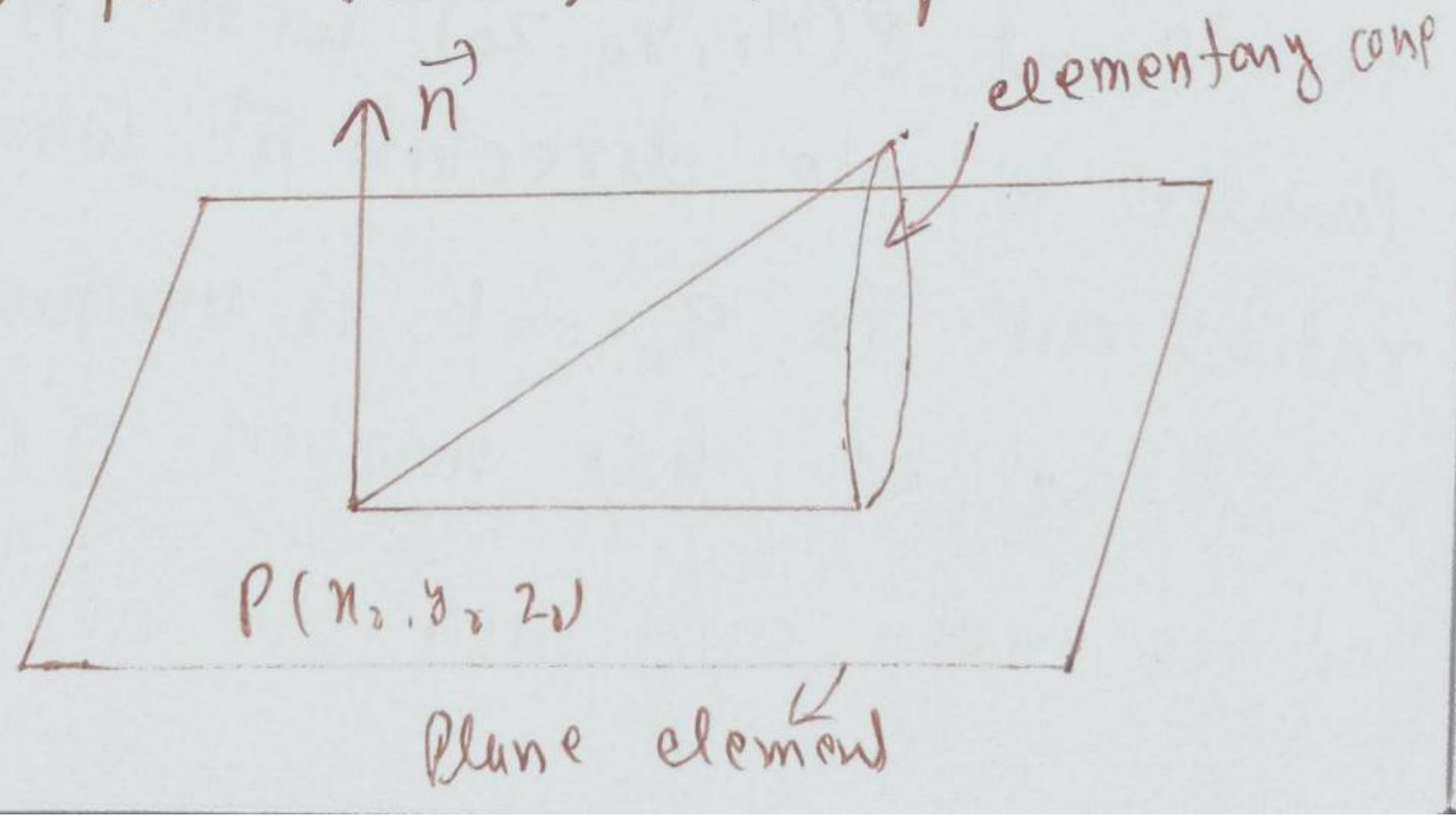
$D(x, y, z, p, q)$ is known as a plane element of a three dimensional space. As a special case a plane element $(x_0, y_0, z_0, p_0, q_0)$ whose components satisfy (1) is known as an integral element of (1) at P. Solving (1) for q, suppose we get

$$q = F(x, y, z, p)$$

which gives a value of q corresponding to known values of x, y, z and p. Then, keeping x_0, y_0 and z_0 fixed and varying p, we shall arrive at a set of plane element $\{x_0, y_0, z_0, p_0, F(x_0, y_0, z_0, p)\}$ which depend on the single parameter p. As p varies, we get a set of plane elements all of which pass through the point P.



Hence the above mentioned set of plane elements envelop a cone with vertex P.

2) The cone thus obtained is known as the <u>elementary cone</u> of (1) at the point P.

Consider a surface S with eqn

$$z = g(x, y) \quad — \quad (2)$$

If the function $g(x, y)$ and its partial derivatives $g_x(x, y)$ and $g_y(x, y)$ are continuous in a certain region R of the x-y plane, then the tangent plane at each point of S determines a plan element of the form $\{x_0, y_0, g(x_0, y_0), g_x(x_0, y_0), g_y(x_0, y_0)\}$ which will be referred as the <u>tangent element</u> of the surface S at the point $\{x_0, y_0, g(x_0, y_0)\}$.

Consider a curve C with parametric eqns

$$x = x(t), \quad y = y(t), \quad z = z(t) \quad — \quad (3)$$

Then curve C lies on (2) provided

$$z(t) = g\{x(t), y(t)\} \quad — \quad (4)$$

holds good for all values of t in the appropriate interval I.

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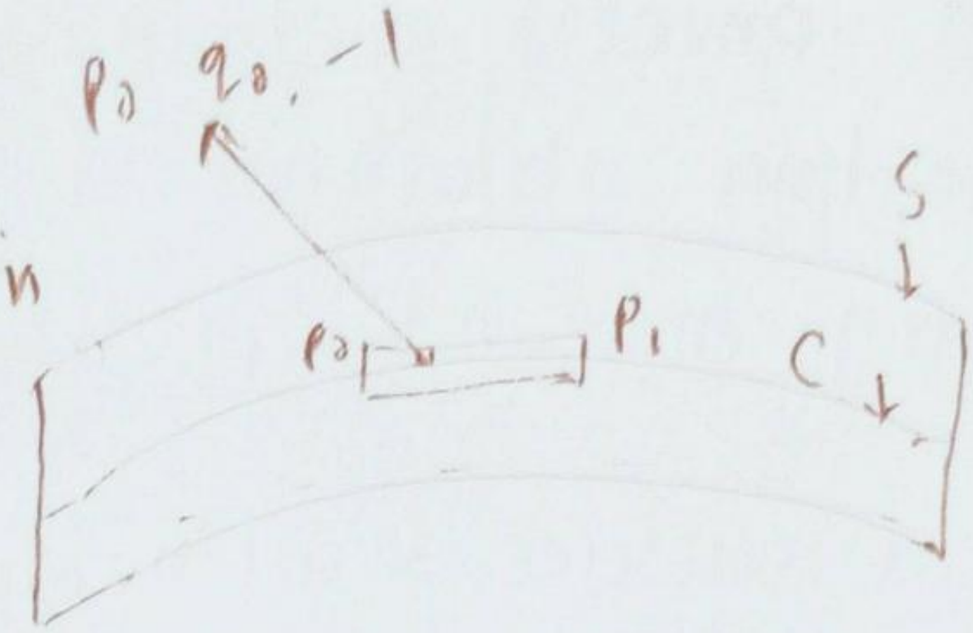
Let P_0 be a point on curve C corresponding to $t = t_0$.



Now, the direction ratios of the tangent line P_0P_1 are $x'(t_0)$, $y'(t_0)$, $z'(t_0)$.

This direction will be $\perp^r$ to direction of normal $\vec{n}$ (with direction ratios $p_0, q_0, -1$)

If

$$p_0 x'(t_0) + q_0 y'(t_0) + (-1) z'(t_0) = 0$$

or $$z'(t_0) = p_0 x'(t_0) + q_0 y'(t_0)$$

it follows that any set

$$\{ x(t), y(t), z(t), p(t), q(t) \} \quad -(5)$$

of five real function satisfying the condition that

$$z'(t) = p(t) x'(t) + q(t) y'(t) \quad -(4)$$

define a strip at the point (x, y, z) of the curve C. When such a strip is also an integral element of (1), then the strip under consideration is known as an <u>integral strip</u> of (1).

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In other words, the set of functions (5) is known as an integral strip of (1) provided these satisfy (6) and the following additional condition

$$f\{x(t), y(t), z(t), p(t), q(t)\} = 0 \quad \forall t \in I$$

If at each point of the curve (3) touches a generator of the elementary cone, then the corresponding strip is known as a characteristic strip.

<u>Derivation of the equations determining a characteristic strip:</u>

Clearly, the point $(x+dx, y+dy, z+dz)$ lies in the tangent plane to the elementary cone at P if

$$dz = p\,dx + q\,dy \qquad -(7)$$

where p, q satisfy (1).

Diff. (7) w.r.t. p, we get

$$0 = dx + \frac{dq}{dp} \cdot dy \qquad -(8)$$

Again, diff (1). Partially w.r.t p, we get

$$\frac{\partial f}{\partial p} + \frac{\partial f}{\partial q} \cdot \frac{dq}{dp} = 0 \quad \text{or} \quad f_p + f_q \cdot \frac{dq}{dp} = 0 \qquad -(9)$$

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Here $\partial_p = f_p$ and $\partial_q = f_q$

solving eqn (7), (8) and (9) for the ratios dy, dz to dx, we get

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} \quad \text{— (10)}$$

Hence along a characteristic strip $x'(t)$, $y'(t)$, $z'(t)$ will be proportional to f_p, f_q, $pf_p + qf_p$ resp. If the parameter t be selected satisfying the relations

$$x'(t) = f_p \quad \text{and} \quad y'(t) = f_q$$

then, we have $z'(t) = pf_p + qf_q$

since along a characteristic strip p is a function of t, hence

$$p'(t) = \frac{\partial p}{\partial x}\frac{dx}{dt} + \frac{\partial p}{\partial y}\frac{dy}{dt}$$

$$= \frac{\partial p}{\partial x}\frac{\partial f}{\partial p} + \frac{\partial p}{\partial y}\cdot\frac{\partial f}{\partial q}.$$

$$p'(t) = \frac{\partial p}{\partial x}\frac{\partial f}{\partial p} + \frac{\partial q}{\partial x}\frac{\partial f}{\partial q} \quad \text{— (12)}$$

$$\left\{ \because \frac{\partial p}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial q}{\partial x} \right\}$$

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Now, differentiating (1) partially w.r.t x gives

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p + \frac{\partial f}{\partial p} \frac{\partial p}{\partial x} + \frac{\partial f}{\partial q} \cdot \frac{\partial q}{\partial x} = 0$$

or $f_x + p f_z + p'(t) = 0$ using (12)

Hence on a ~~sho~~ characteristic strip,

$$p'(t) = -f_x - p f_z \quad \text{— (13)}$$

similarly $q'(t) = -f_y - q f_z$ — (14)

Here $f_x = \frac{\partial f}{\partial x}$, $f_y = \frac{\partial f}{\partial y}$, $f_z = \frac{\partial f}{\partial z}$

From (11), (13) and (14), we get the following system of five ordinary diff eqns for the determination of the characteristic strip

$$x'(t) = f_p \quad , \quad y'(t) = f_q$$

$$z'(t) = p f_p + q f_q \quad , \quad p'(t) = -f_x - p f_z$$

and $q'(t) = -f_y - q f_z$ — (12)

The above eqns are called the characteristic eqns of (1).

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Working Rule for solving Cauchy's Problem

Suppose we wish to find the integral surface of $f(x, y, z, p, q) = 0$ which passes through a given curve with parametric eqn (1)

$$x = f_1(\alpha), \quad y = f_2(\alpha), \quad z = f_3(\alpha) \quad (16)$$

where α is the parameter.

then in the soln

$$x = x(p_0, q_0, x_0, y_0, z_0, t) \text{ etc} \quad (17)$$

of the characteristic eqns (15), we shall assume that

$$x_0 = f_1(\alpha), \quad y_0 = f_2(\alpha), \quad z_0 = f_3(\alpha)$$

are the initial values of x, y, z, respectively the corresponding values of p_0, q_0 can be obtained by the following relation

$$f_3'(\alpha) = p_0 f_1'(\alpha) + q_0 f_2'(\alpha)$$

and $f\{f_1(\alpha), f_2(\alpha), f_3(\alpha), p_0, q_0\} = 0$

when the above values of x_0, y_0, z_0, p_0, q_0 and the appropriate value of t_0 is substituted

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in (17), we shall be able to express x, y, z involving the two parameters l and d of the form

$$x = \phi_1(l, d), \quad y = \phi_2(l, d), \quad z = \phi_3(l, d) \quad (18)$$

which are known as characteristics of (1).

Finally, by eliminating λ and t from (18) we arrive at a relation of the form

$$G(x, y, z) = 0.$$

(19) which is the required eqn of the integral surface of (1) passing through the given curve (16).

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Solution of semi-linear equations

consider a semi-linear equation

$$P(x,y)\,z_x + Q(x,y)\,z_y = R(x,y,z) \quad \text{--- (1)}$$

where P, Q and R are continuous diff. functions and both are $P(x,y)$ and $Q(x,y)$ do not vanish simultaneously.

Since the left hand side of (1) is the directional derivative of $z(x,y)$ in the direction $(P(x,y), Q(x,y))$ at the point (x,y)

therefore, we consider a one-parameter family of curves in the x-y plane defined by the ordinary diff. eqn

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}$$

or the system of ordinary diff. eqns

$$\frac{dx}{dt} = P(x,y), \quad \frac{dy}{dt} = Q(x,y) \quad \text{--- (2)}$$

these curves have the property that along $z(x,y)$ will satisfy the ordinary diff. eqn

$$\frac{dz}{dx} = z_x + z_y \frac{dy}{dx} = \frac{P(x,y)\,z_x + Q(x,y)\,z_y}{P(x,y)}$$

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or $$\frac{dz}{dt} = z_x \frac{dx}{dt} + z_y \frac{dy}{dt}$$

$$= z_x P(x,y) + z_y Q(x,y) = R(x,y,z) \quad \text{--- (3)}$$

The one-parameter family of curves defined by eqn (2) are called the characteristic curves of the P.D.E (1).

Let (x_0, y_0) be a point in xy-plane. By the existence and uniqueness of the solution of the initial value problem for the ODEs. eqn (2) will define a unique characteristic curve

$$x(t) = x(x_0, y_0, t), \quad y(t) = y(x_0, y_0, t) \quad \text{--- (4)}$$

such that $x(0) = x_0$, and $y(0) = y_0$. suppose we now assign a value z_0 for $z(x,y)$ at (x_0, y_0)

Then eqn (4) determines a unique soln z as

$$z = z(x_0, y_0, t) \quad \text{--- (5)}$$

such that $z(x_0, y_0, 0) = z_0$.

ie $z(x,y)$ is uniquely determined along the whole characteristic passing through (x_0, y_0) If we assign a value for z at (x_0, y_0)

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Hence if $z(x,y)$ is known on a curve Γ_0 in the xy-plane and if this curve is such that it intersects the one-parameter family of characteristic curves C_b, we can determine $z(x,y)$ uniquely in the region covered by C_{b_0}.

Ex ① Solve $xz_y - yz_x = z$ with the initial condition $z(x,0) = f(x)$ $x \geq 0$

Soln: The characteristic curves are given by

$$\frac{dy}{dx} = -\frac{x}{y}$$

having soln $x^2 + y^2 = c^2$.

Along this curve, z satisfies the ODE

$$\frac{dz}{dx} = z_x + z_y \frac{dy}{dx} = -\frac{z}{y}$$

$$\frac{dz}{dx} = -\frac{z}{\sqrt{c^2 - x^2}}$$

ie $$\frac{dz}{z} = -\frac{dx}{\sqrt{c^2 - x^2}}$$

$$\log z = -\sin^{-1}\frac{x}{c} + \log k(c)$$

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$$z = k(c)\, e^{-\sin^{-1} x/c}$$

where the arbitrary function k may depend on c. Hence the general soln is

$$z = k(x^2+y^2)\, e^{-\sin^{-1}\frac{x}{c}}$$

using initial conditions, we get

$$f(x) = k(x^2)\, e^{-\pi/2}$$

$$k(x) = f(\sqrt{x})\, e^{\pi/2}$$

Hence the required soln is

$$\boxed{z(x,y) = f(\sqrt{x^2+y^2})\, e^{\pi/2 - \sin^{-1}\frac{x}{c}}}$$

Solution of Quasi-linear equations.

consider a quasi-linear eqn

$$P(x,y,z)\, z_x + Q(x,y,z)\, z_y = R(x,y,z) \quad -(1)$$

where P, Q, R are continuous diff. functions of x, y and z.

Its solution $z(x,y)$ defines an integral surface $z = z(x,y)$ in the xyz-space.

The normal to this surface has direction ratios $(z_x, z_y, -1)$.

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Hence eqn (1) states the condition that the integral surface is such that at each point, the line with direction ratios (P, Q, R) is tangent to the surface at that point.

We consider below the integral curves of this field called the characteristic curves. They are a family of space curves whose tangent at each point coincides with the characteristic directions (P, Q, R) at that point. and are given by the following system of ODEs

$$\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)} = dt \text{ (say)} \quad \text{--- (2)}$$

or

$$\frac{dx}{dt} = P(x,y,z), \quad \frac{dy}{dt} = Q(x,y,z), \quad \frac{dz}{dt} = R(x,y,z) \quad \text{--- (3)}$$

By the existence and uniqueness of the solution of the initial v.p. for a system of ODE, there passes a characteristic curve.

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$$\left.\begin{aligned} x &= x(x_0, y_0, z_0, t) \\ y &= y(x_0, y_0, z_0, t) \\ z &= z(x_0, y_0, z_0, t) \end{aligned}\right\} \quad (4)$$

The system of ODEs (4) will be a two parameter family of integral curves.

Hence there is a two parameter family of characteristic curves. Every surface generated by one-parameter family of char. is an integral surface. For, if we consider any point on such a surface, then the tangent to the char. curve passing through that point lies on the tangent plane to the surface. Thus the tangent plane to the surface at each point contains the line with direction ratios (P, Q, R).

(2) Let us consider an integral surface S given by $z = z(x, y)$.

(3) Consider any point (x_0, y_0, z_0) on S.

Let the soln of

$$\frac{dx}{dt} = P(x, y, z(x, y))$$

$$\frac{dy}{dt} = Q(x, y, z(x, y))$$

with initial condition $x = x_0$, $y = y_0$ at $t = 0$

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be given by

$$x = x(t) \quad y = y(t).$$

consider the corresponding curve in three dimension

$$x = x(t). \quad y = y(t) \quad z = z(x(t), y(t))$$

this curve lies on the given ~~curve~~ integral surface. Moreover

$$\frac{dz}{dt} = z_x \frac{dx}{dt} + z_y \frac{dy}{dt}$$

$$= P(x,y,z)\, z_x + Q(x,y,z)\, z_y$$

$$\frac{dz}{dt} = R(x,y,z)$$

Therefore the curve satisfies eqn ③ for characteristic curves and it passes through the point (x_0, y_0, z_0). Therefore S is generated by the char. curves as it contains the char. curve through each point (x_0, y_0, z_0) on S.

Further, if a char. curve has one point in common with the integral surface, it lies entirely on the integral surface since only one char. curve passes through a given point.

In addition, if two integral surfaces intersect

at a point, then they intersect along the entire characteristic curve through this point. Hence the curve of intersection of two integral surfaces must be a characteristic curve.

<u>Theorem</u>: Consider the first order quasi-linear PDE

$$P(x,y,z)z_x + Q(x,y,z)z_y = R(x,y,z) \quad \text{—(1)}$$

where P, Q, and R have continuous partial derivatives with respect to x, y and z and they do not vanish simultaneously. Let the value $z = z_0(s)$ be prescribed along the initial curve Γ_0: $x = x_0(s)$, $y = y_0(s)$, x_0, y_0, z_0 being continuous diff. functions $(a \le s \le b)$.

Further, for $a \le s \le b$, if

$$\frac{dy_0}{ds} P(x_0(s), y_0(s), z_0(s)) - \frac{dx_0}{ds} Q(x_0(s), y_0(s), z_0(s)) \neq 0.$$

then there exists a unique solution $z(x,y)$ defined in some neighbourhood of the Γ_0 which satisfies the p.d.e. and the initial condition

$$z(x_0(s), y_0(s)) = z_0(s)$$

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Exp ① Solve the initial value Problem for the quasi-linear eqn $z z_x + z_y = 1$ containing the initial data curve $C: x_0 = s,\ y_0 = s,\ z_0 = \frac{s}{2}$ for $0 \le s \le 1$

Soln: $f(x, y, z, p, q) = z z_x + z_y - 1 = 0$ (1)

Given initial data curve

$$x_0 = s, \quad y_0 = s, \quad z_0 = \frac{s}{2} \quad 0 \le s \le 1 \quad -(2)$$

Since $\frac{dy_0}{ds} P - \frac{dx_0}{ds} Q = 1 \times 2 - 1 \times 1 = 2 - 1$

$$= \frac{s}{2} - 1 \neq 0 \text{ for } 0 \le s \le 1$$

Now, we have the following ODEs:

$$\frac{dx}{dt} = z, \quad \frac{dy}{dt} = 1 \quad \text{and} \quad \frac{dz}{dt} = 1 \quad -(3)$$

with initial conditions

$$x(s, 0) = s, \quad y(s, 0) = s, \quad z(s, 0) = \frac{s}{2} \quad -(4)$$

The family of characteristics through the initial data curve is found to be

$$x = \frac{1}{2} t^2 + \frac{1}{2} s t + s$$

$$y = t + s$$

$$z = t + \frac{s}{2}$$

Solving for s and t in terms of x and y, we obtain

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$$s = \frac{x - \frac{y^2}{2}}{1 - \frac{y}{2}}, \quad t = \frac{y - x}{1 - \frac{y}{2}}$$

Putting these values in $z = t + \frac{s}{2}$

$$z = \frac{2(y-x) + (x - \frac{y^2}{2})}{2-y} = \frac{2y - x - \frac{y}{2}}{2-y}$$

$$\boxed{z = \frac{4y - 2x - y^2}{2(2-y)}}$$

<u>Exp</u> ② Find the solution of the initial value problem for the quasi-linear equation $z_x - z z_y + z = 0 \quad \forall\, x, y > 0$, for the initial data curve C: $x_0 = 0$, $y_0 = s$, $z_0 = -2s$, $-\infty < s < \infty$.

<u>Solution</u>: Here $z_x - z z_y + z = 0$ — ①

initial data curve

$$x_0 = 0, \quad y_0 = s, \quad z_0 = -2s \quad \text{— ②}$$

since $\frac{dy_0}{ds} P - \frac{dx_0}{ds} Q = 1 \times 1 - 0 \times z = 1 \neq 0 \quad \forall\, s$

The characteristic curves which generate the surface are obtained form the sol^ns of

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = -z, \quad \frac{dz}{dt} = -z$$

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satisfying the following condition

$x(s,0) = 0$, $y(s,0) = s$, $z(s,0) = -2s$

The solutions are found to be

$$x = t \quad , \quad y = -2se^{-t} + 3s \quad , \quad z = -2se^{-t}$$

The parameters s and t in terms of x and y are ~~same~~ found to be

$$s = -\frac{y}{2e^{-x} - 3} \quad , \quad t = x$$

Therefore

$$z = \frac{2y}{2e^{-x} - 3} e^{-x}$$

$$z = -\frac{2y}{3e^{x} - 2} \quad , \quad \log\left(\frac{2}{3}\right) > x \geq 0$$

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Pfaffian Differential Equations

By a Pfaffian differential equation, we mean a differential equation of the form

$$F_1(x_1, x_2 \cdots x_n)dx_1 + F_2(x_1, x_2 \cdots x_n)dx_2 + \cdots + F_n(x_1, x_2 \cdots x_n)dx_n = 0 \quad (1)$$

where F_i's $(i = 1, 2 \cdots n)$ are continuous functions. The expression on the left hand side of eq^n (1) is called a Pfaffian differential form.

<u>Definition</u>: A Pfaffian differential form is said to be exact if we can find a continuously differentiable function $u(x_1, x_2 \cdots x_n)$ such that

$$du = F_1(x_1 \cdots x_n)dx_1 + F_2(x_1 \cdots x_n)dx_2 + \cdots F_n(x_1 \cdots x_n)dx_n$$

<u>Definition</u>: A Pfaffian differential equation is said to be integrable if there exists a non-zero differentiable function $u(x_1 \cdots x_n)$ such that the Pfaffian differential form

$$u[F_1(x_1 \cdots x_n)dx_1 + \cdots + F_n(x_1 \cdots x_n)dx_n],$$

is exact. The function $u(x_1 \cdots x_n)$ is called

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the integrating factor and $u(x_1, \ldots, x_n) = c$ where c is an arbitrary constant, is called the integral of the corresponding Pfaffian differential equation.

<u>Note</u>: A Pfaffian differential equation is said to be exact if the Pfaffian differential form on the hand side of eqn (1) is exact.

<u>Theorem</u>: There always exists an integrating factor for a Pfaffian differential equation in two variables.

<u>Proof</u>: consider $P(x,y)dx + Q(x,y)dy = 0$

If $Q(x,y) \neq 0$, then

$$\frac{dy}{dx} = -\frac{P(x,y)}{Q(x,y)}$$

From the existence theorem for a first order ordinary differential equation, the above equation has a solution $F(x,y) = c$

then $$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0$$

This exact Pfaffian form differs from $Pdx + Qdy = 0$ by a factor as

they have the same solution. The factor is nothing but the integrating factor.

<u>Lemma</u> ① Let $u(x,y)$ and $v(x,y)$ be two functions of x and y such that

$$\frac{\partial v}{\partial y} \neq 0 \quad \text{--- (2)}$$

If, further

$$\frac{\partial(u,v)}{\partial(x,y)} = 0 \quad \text{--- (3)}$$

Then there exists a relation

$$F(u,v) = 0 \quad \text{--- (4)}$$

between u and v not involving x and y explicitly.

<u>Proof</u>: Since $\frac{\partial v}{\partial y} \neq 0$, we can eliminate y between $u = u(x,y)$ and $v = v(x,y)$ and obtain the relation

$$F(u,v,x) = 0 \quad \text{--- (5)}$$

We will now show that F does not depend on x.

On differentiating (5) w.r.t. x and y, we get

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\cdot\frac{\partial v}{\partial x} = 0 \quad \text{--- (6)}$$

$$\frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} = 0 \quad \text{--- (7)}$$

On eliminating $\frac{\partial F}{\partial v}$ from these equations,

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which is possible if $\frac{\partial v}{\partial x} \neq 0$, we find that

$$\frac{\partial f}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial(u\,v)}{\partial(x,y)}\frac{\partial f}{\partial u} = 0 \quad — (8)$$

ie $$\frac{\partial f}{\partial x}\cdot\frac{\partial v}{\partial y} = 0$$

since $\frac{\partial v}{\partial y} \neq 0 \Rightarrow \frac{\partial f}{\partial x} = 0$.

Suppose $\frac{\partial v}{\partial x} = 0$, then $\frac{\partial u}{\partial x} = 0$

In this case also, eqn (8) implies that $\frac{\partial F}{\partial x} = 0$. Hence F is independent of x

Lemma (2): If $\vec{X}\cdot \text{curl}\,\vec{X} = 0$ where $\vec{X} = (P, Q, R)$ and μ is an arbitrary differentiable function of x, y and z, then

$$\mu\vec{X}\cdot \text{curl}(\mu\vec{X}) = 0$$

Proof: consider

$$\mu\vec{X}\cdot \text{curl}(\mu\vec{X}) = \sum_{x,y,z}(\mu P)\left[\frac{\partial(\mu R)}{\partial y} - \frac{\partial(\mu Q)}{\partial z}\right]$$

$$= \mu^2\sum_{x,y,z} P\left[\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right] - \mu\sum_{x,y,z}\left[PQ\frac{\partial \mu}{\partial z} - PR\frac{\partial \mu}{\partial y}\right]$$

$$= \mu^2\sum_{x,y,z} P\left[\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right]$$

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$$= \mu^2 (\vec{X} \cdot \text{curl}\, \vec{X}) = 0$$

Conversely, if $\mu \vec{X} \cdot \text{curl}(\mu \vec{X}) = 0$ then

$\vec{X} \cdot \text{curl}\, \vec{X} = 0$ for $\mu \neq 0$

<u>Theorem</u>: A necessary and sufficient condition that the Pfaffian differential equation

$$\vec{X} \cdot d\vec{r} = P(x, y, z)\,dx + Q(x, y, z)\,dy + R(x, y, z)\,dz = 0 \quad \text{—— (9)}$$

be integrable is that

$$(\vec{X} \cdot \text{curl}\, \vec{X}) = 0 \quad \text{—— (10)}$$

<u>Proof</u>: We first show that the <u>condition is necessary</u>.

For, if eqn (9) is integrable, then there exist differentiable function $\mu(x, y, z)$ and $u(x, y, z)$ such that

$$du = \mu(x, y, z)\,[P(x, y, z)\,dx + Q(x, y, z)\,dy + R(x, y, z)\,dz] \quad \text{—— (11)}$$

where $\mu(x, y, z) \neq 0$.

[$\mu R \frac{\partial u}{\partial y}$] Since $du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$ —— (12)

comparing eqn (11) and (12), we get

$$\mu P = \frac{\partial u}{\partial x}, \quad \mu Q = \frac{\partial u}{\partial y}, \quad \mu R = \frac{\partial u}{\partial z}$$

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ie $\mu \vec{X} = \nabla u$

Since $\text{curl}(\nabla u) = 0 \Rightarrow \text{curl}(\mu \vec{X}) = 0$

Hence $\mu \vec{X} \cdot \text{curl}(\mu \vec{X}) = 0$

$\Rightarrow \vec{X} \cdot \text{curl}(\vec{X}) = 0$

Now we shall prove that the condition is sufficient. suppose z is treated as constant, the diff. eqn (9) becomes

$$P(x, y, z)\,dx + Q(x, y, z)\,dy = 0$$

Since a Pfaffian differential equation in two variables is always integrable, this has a soln of the form $U(x, y, z) = c_1$, where c_1 may involve z.

In addition, there must exist a non-zero differentiable function μ such that

$$\frac{\partial U}{\partial x} = \mu P \qquad \frac{\partial U}{\partial y} = \mu Q$$

On multiplying (9) by μ and substituting the above, we get

$$\frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz + \left(\mu R - \frac{\partial U}{\partial z}\right) dz = 0$$

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$\Rightarrow \quad dU + K\,dz = 0 \qquad \text{--- (13)}$

where $K = \left(\mu R - \frac{\partial U}{\partial z}\right)$

we are given that $\vec{X} \cdot \text{curl}\,\vec{X} = 0$

$\Rightarrow \quad \mu \vec{X} \cdot \text{curl}(\mu \vec{X}) = 0$

so

$$\mu \vec{X} = (\mu P, \mu R, \mu R)$$

$$= \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} + K\right)$$

$$= \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right) + (0, 0, K)$$

$$= \nabla U + (0, 0, K)$$

Hence

$$\mu \vec{X} \cdot \text{curl}(\mu \vec{X}) = \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} + K\right) \cdot \left(\frac{\partial K}{\partial y}, -\frac{\partial K}{\partial x}, 0\right)$$

$$= \frac{\partial U}{\partial x} \cdot \frac{\partial K}{\partial y} - \frac{\partial U}{\partial y} \frac{\partial K}{\partial x}$$

It follows that $\dfrac{\partial(U, K)}{\partial(x, y)} = 0 \qquad \text{--- (14)}$

Hence K can be expressed as a function of U and z alone as eqn (14) indicates

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that there is a relation between U and K independent of x and y, but not necessarily of z (from Lemma ①). Therefore eqn (13) becomes

$$\frac{dU}{dz} + K(U, z) = 0$$

It possesses a soln $\phi(U, z) = c$.
where c is an arbitrary constant.
This soln can now be expressed in the form $u(x, y, z) = c$ by using the expression for U in terms of x, y and z.

Therefore the Pfaffian differential eqn (9) is integrable

<u>Note</u>: The Pfaffian differential eqn

$$\vec{X} \cdot d\vec{r} = P(x, y, z)\,dx + Q(x, y, z)\,dy + R(x, y, z)\,dz = 0$$

is exact iff $\text{curl}\,\vec{X} = 0$.

<u>Exp</u> ① Show that the following Pf. Diff eqn is integrable and find its integral.

$$y\,dx + x\,dy + 2z\,dz = 0$$

<u>Soln</u>. Here $\vec{X} = (y, x, 2z)$

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so $\text{curl}\,\vec{x} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & 2z \end{vmatrix}$

$\text{curl}\,\vec{x} = (0, 0, 0) = 0$

$\Rightarrow \quad \vec{x} \cdot \text{curl}\,\vec{x} = 0$

Hence Pfaffian diff eqn is exact and integrable.

Also the given eqn can be written as

$y\,dx + x\,dy + 2z\,dz = d(xy + z^2)$

$\Rightarrow \boxed{xy + z^2 = c}$

<u>Exp ②</u>: Find the integral of

$yz\,dx + 2xz\,dy - 3xy\,dz = 0$

<u>Soln</u>: Here $\vec{x} = (yz, 2xz, -3xy)$

$\text{curl}\,\vec{x} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & 2xz & -3xy \end{vmatrix}$

$\text{curl}\,\vec{x} = i(-3x - 2x) + j(y + 3y) + k(2z - z)$

$\text{curl}\,\vec{x} = -5xi + 4yj + zk$

$\Rightarrow \vec{x} \cdot \text{curl}\,\vec{x} = -5xyz + 8xyz - 3xyz = 0$

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therefore given eqn is integrable

treating z as a constant, given eqn becomes

$$y\,dx + 2x\,dy = 0$$

its soln $U = xy^2 = c_1$

Then by $\frac{\partial U}{\partial x} = \mu P$ or $\frac{\partial U}{\partial y} = \mu Q$

$$\Rightarrow \quad \mu = \frac{y}{z}.$$

Further $K = \mu R - \frac{\partial U}{\partial z}$

$$= \frac{y}{z}x - 3xy - 0$$

$$K = -\frac{3U}{z}$$

Therefore by $\frac{dU}{dz} + K(U, z) = 0$

$$\Rightarrow \quad \frac{dU}{dz} - \frac{3}{z}U = 0$$

its soln is

$$\Rightarrow \quad U = cz^3$$

Therefore the integral of the given Pfaffian form is

$$u = xy^2 = cz^3$$

$$u = \frac{xy^2}{z^3} = c$$

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Non-linear First order P.D.E.

Consider a non-linear first order pde.

$$f(x, y, z, p, q) = 0 \quad \text{——} \quad (1)$$

We assume that f has continuous second order derivatives with respect to its variables x, y, z, p, and q and either f_p or f_q is non-zero at every point.

Suppose that $f_q \neq 0$, so that we can solve eqn (1) at each point (x, y, z) for q as $q = q(x, y, z, p)$.

Monge Cone: Let (x_0, y_0, z_0) be some point in space. We consider a possible integral surface $z = z(x, y)$ and the direction ratios $(p, q, -1)$ of the normal to the tangent plane.

$$z - z_0 = p(x - x_0) + q(y - y_0) \quad \text{——} \quad (2)$$

to the integral surface at that point.

The equation (1) which is equivalent to

$$q = q(x_0, y_0, z_0, p) \quad \text{——} \quad (3)$$

indicates that p and q are not independent at (x_0, y_0, z_0).

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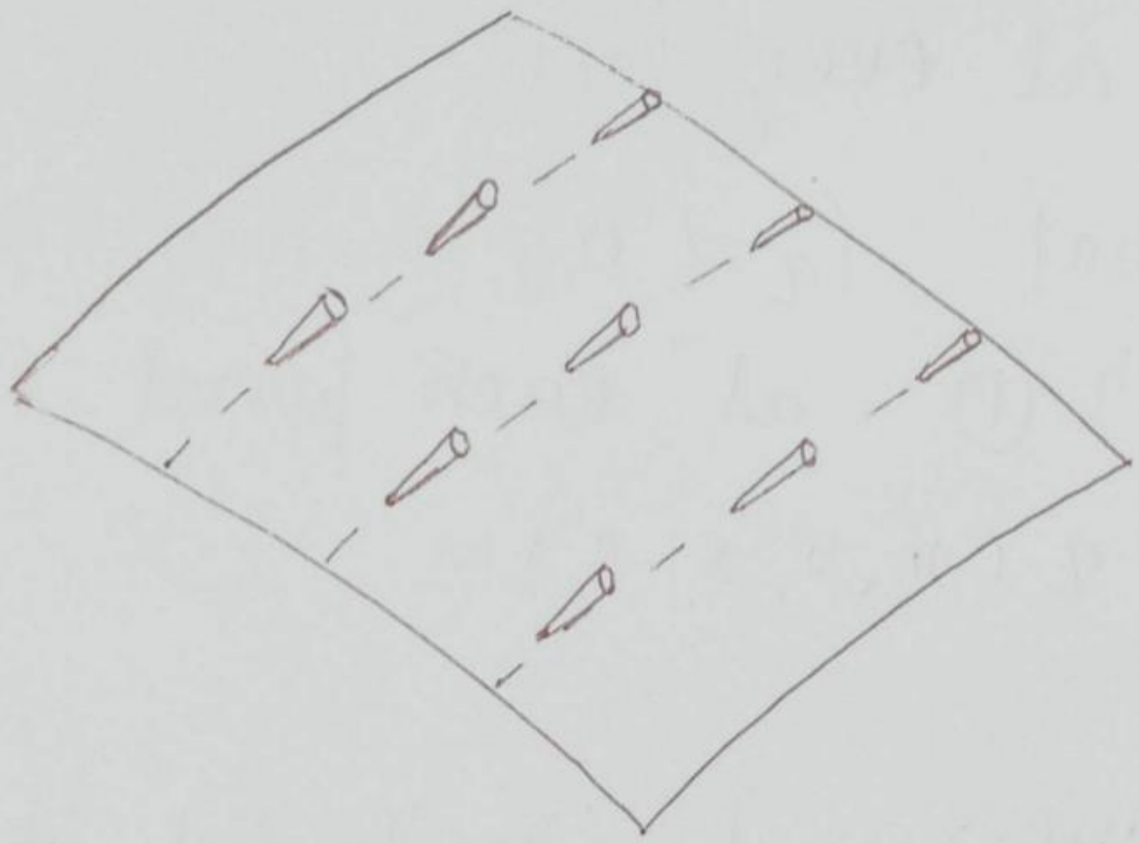
This implies that the integral surfaces are those surfaces having tangent planes belonging to a one-parameter (p) family.



Consider the family of planes given by eqn (2) passing through (x_0, y_0, z_0) as p varies and q is determined by eqn (3). These planes envelope a cone called the Monge cone having its vertex at (x_0, y_0, z_0).

Thus if $z = z(x, y)$ is a soln of (1) it must be tangent to the Monge cone at each point (x, y, z) on the surface. Thus the diff eqn (1) characterizes a field of cones

such that a surface will be an integral surface iff it is tangent to the Monge cone at each point. as shown in fig (1).

The line of contact of the Monge cone with the tangent plane to the integral surface at each point gives a field of direction. These directions are called the characteristic directions at that point and lie along the generators of the Monge cone. The integral curves of this field of directions on the surface define a family of curves called the characteristic curves.

Analytic expression for the Monge cone at (x_0, y_0, z_0).

As the Monge cone at (x_0, y_0, z_0) is the envelope of the one-parameter family of planes

$$z - z_0 = p(x - x_0) + q(y - y_0) \quad \text{--- (1)}$$

where q is given by

$$q = q(x_0, y_0, z_0, p) \quad \text{--- (2)}$$

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it can be obtained by eliminating p from the following equations

$$z - z_0 = p(x - x_0) + q(x_0, y_0, z_0, p)(y - y_0) \quad \text{—(3)}$$

and

$$0 = (x - x_0) + (y - y_0)\frac{dq}{dp} \quad \text{— (4)}$$

on diff. eqn $f(x, y, z, p, q) = 0$ with respect to p, we get

$$\frac{df}{dp} = f_p + f_q \frac{dq}{dp} = 0$$

as q is a function of p. If one eliminates $\frac{dq}{dp}$ from (4) and the above equation, then the equations describing the Monge cone can be written as

$$\left.\begin{aligned} q &= q(x_0, y_0, z_0, p) \\ z - z_0 &= p(x - x_0) + q(y - y_0) \\ \frac{x - x_0}{f_p} &= \frac{(y - y_0)}{f_q} \end{aligned}\right\} \quad (5)$$

$=$

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Let us now find the equations which give the characteristic curves. As the line of contact between the tangent plane to an integral surface and the Monge cone at that point is the generator of the Monge cone. it can be written as

$$\frac{x-x_0}{f_p} = \frac{y-y_0}{f_q} = \frac{z-z_0}{pf_p+qf_q} \quad -(1)$$

Hence the characteristic directions which lie along the generator of the Monge cone are given by (f_p, f_q, pf_p+qf_q).

The characteristic curves are thus the solutions of

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p+qf_q} = dt \text{ (say)}$$

or $\frac{dx}{dt} = f_p, \quad \frac{dy}{dt} = f_q, \quad \frac{dz}{dt} = pf_p+qf_q \quad -(2)$

Since these three ODEs for determining the characteristic curves $(x(t), y(t), z(t))$ cannot be solved because they involve the functions p and q, we required more information regarding the variations of p and q along a char. curve. Along such a curve on the given integral surface.

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$$\left.\begin{aligned}\frac{dp}{dt} &= p_x \frac{dx}{dt} + p_y \frac{dy}{dt} = p_x f_p + p_y f_q \\ \frac{dq}{dt} &= q_x \frac{dx}{dt} + q_y \frac{dy}{dt} = q_x f_p + q_y f_q\end{aligned}\right\} \quad (3)$$

Further, by diff. eq^n $f(x, y, z, p, q) = 0$ w.r.t x as well as y, we get

$$f_x + f_z p + f_p p_x + f_q q_x = 0$$

$$f_y + f_z q + f_p p_y + f_q q_y = 0$$

Since $p_y = q_x$, eq^{ns} (3) may now be rewritten as

$$\left.\begin{aligned}\frac{dp}{dt} &= -f_x - f_z p \\ \frac{dq}{dt} &= -f_y - f_z q\end{aligned}\right\} \quad (4)$$

Thus for a given integral surface $z = z(x, y)$, there corresponds a family of char. curves on the surface associated with it such that the co-ordinates of the curve $(x(t), y(t), z(t))$ and the numbers $p(t)$ and $q(t)$ along the curve are related by a system of five ODEs given by (3) and (4). These five ODEs are called char. diff. eqs. related to the given p.d.f

$$f(x, y, z, p, q) = 0$$

In order to determine the integral surface, we consider the p.d.e. $f(x, y, z, p, q) = 0$ along with the system of char. eqns. (3) and (4) as a system of six equations.

$$
\left.
\begin{aligned}
f(x, y, z, p, q) &= 0 \\
\frac{dx}{dt} &= f_p \\
\frac{dy}{dt} &= f_q \\
\frac{dz}{dt} &= p f_p + q f_q \\
\frac{dp}{dt} &= -f_x - f_z p \\
\frac{dq}{dt} &= -f_y - f_z q
\end{aligned}
\right\} \quad (5)
$$

for the five unknown functions $x(t)$, $y(t)$, $z(t)$, $p(t)$ & $q(t)$. This system is not over-determined. Consider a soln of the last five eqns. of (5). Along such soln we find that

$$
\begin{aligned}
\frac{df}{dt} &= f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt} + f_p \frac{dp}{dt} \\
&\quad + f_q \frac{dq}{dt} \\
&= f_x f_p + f_y f_q + p f_z f_p + q f_z f_q - f_p f_x \\
&\quad - f_p f_z p - f_q f_y - f_q f_z q
\end{aligned}
$$

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$= 0$

Hence $f =$ constant, is an integral of the system of ODEs. Thus we can conclude that the eqn $f(x, y, z, p, q) = 0$ is not much of a restriction. Suppose that $f = 0$ at $(x_0, y_0, z_0, p_0, q_0)$ at $t = 0$. Then the unique solution of the char. eqns passing through this point is such that $f = 0$ will be satisfied for all t - along the solution.

Characteristic strip.

A solution $(x(t), y(t), z(t), p(t), q(t))$ of the system of ODEs (5) can be interpreted as a strip.

The first three functions $(x(t), y(t), z(t))$ determine a space curve. At each point of the space curve, $p(t)$ and $q(t)$ defined a tangent plane with $(p, q, -1)$ as the normal vector. The curve along with these tangent planes at each point is called

the characteristic strip and the curve is called the char curve.

For a fixed t given by t_0, $(x_0, y_0, z_0, p_0, q_0)$ is said to define an element of the strip. i.e. the point (x_0, y_0, z_0) on the curve and the corresponding tangent plane whose normal is given by $(p_0, q_0, -1)$ at that point.

<u>Note</u>: Any set of five functions $x(t)$, $y(t)$, $z(t)$, $p(t)$ and $q(t)$ can be interpreted as strip only if the following condition called the strip condition

$$\frac{dz}{dt}(t) = p(t)\frac{dx}{dt}(t) + q(t)\frac{dy}{dt}(t)$$

is satisfied. This is the condition that (6) at each point on the space curve $x = x(t)$, $y = y(t)$, $z = z(t)$, the tangent to the space curve lies on the plane whose normal has direction ratios $(p(t), q(t), -1)$.

any set $(x(t), y(t), z(t), p(t), q(t))$ also satisfies the strip condition.

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To solving the Cauchy problem for the non linear p.d.e. $f(x, y, z, p, q) = 0$. The method involves the following steps.

Step ①:

Initial strip: Let $x = x_0(s)$, $y = y_0(s)$, $z = z_0(s)$ be some arbitrary initial data curve. suppose that along this curve we can specify the functions $p_0(s)$ and $q_0(s)$ such that together with the initial data curve $(x_0(s), y_0(s), z_0(s))$, they satisfy the equation

$$f(x_0(s), y_0(s), z_0(s), p_0(s), q_0(s)) = 0 \quad — ⑦$$

and the strip condition

$$\frac{dz_0}{ds} = p_0 \frac{dx_0}{ds} + q_0 \frac{dy_0}{ds} \quad — ⑧$$

Then such an initial eliment $x_0(s)$, $y_0(s)$, $z_0(s)$, $p_0(s)$, $q_0(s)$ is said to define an initial strip for the initial data curve $(x_0(s), y_0(s), z_0(s))$. The integral surface may be constructed with the help of the characteristic strips issuing from each point of the initial data curve by piecing up the char. curves together to from a smooth surface.

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Method of solving the Cauchy problem for the non-linear p.d.e. $f(x, y, z, p, q) = 0$ — (1)

The method involves the following steps.

<u>Step 1:</u> Given the p.d.e. $f(x, y, z, p, q) = 0$ and an initial data curve

$x = x_0(s)$, $y = y_0(s)$, $z = z_0(s)$, determine the functions $p_0(s)$ and $q_0(s)$ such that the five functions $x_0(s)$, $y_0(s)$, $z_0(s)$, $p_0(s)$ and $q_0(s)$ satisfy eqns (7) and (8). There could be several choice for $p_0(s)$ and $q_0(s)$. One may expect to find a unique solution for each such choice.

<u>Step (2)</u>: Once a choice is made for $p_0(s)$ and $q_0(s)$ ie the initial strip is chosen, we can solve the last five equations of eqn (5) subject to the initial conditions $x = x_0(s)$, $y = y_0(s)$ $z = z_0(s)$, $p = p_0(s)$ and $q = q_0(s)$ at $t = 0$.

Hence we have the charac. strips issuing from each point of the initial data curve. The corresponding char. curves generate the required integral surface.

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Exp ① Find the solution of the eqn

$$z = \frac{1}{2}(p^2+q^2) + (p-x)(q-y) \quad \text{—①}$$

which passes through the x-axis.

Soln: The initial data curve is

$$x_0 = s, \quad y_0 = 0, \quad z_0 = 0 \quad \text{—②}$$

and the eqns ⑦ and ⑧ respectively are

$$0 = \frac{1}{2}(p_0^2+q_0^2) + (p_0 - s)q_0$$

$$0 = p_0 \times 1 + q_0 \times 0$$

$$\Rightarrow p_0 = 0 \quad \Rightarrow \quad \frac{1}{2}q_0^2 - sq_0 = 0$$

$$\Rightarrow q_0 = 0 \text{ or } q_0 = 2s$$

Therefore there are two initial strips

Case (i) $x_0 = s, \; y_0 = 0, \; z_0 = 0, \; p_0 = 0, \; q_0 = 2s$ —③

Case (ii) $x_0 = s, \; y_0 = 0, \; z_0 = 0, \; p_0 = 0, \; q_0 = 0$ —④

the characteristic eqns of this p.d.e are given by

$$\frac{dx}{dt} = p + q - y$$

$$\frac{dy}{dt} = p + q - x$$

$$\frac{dz}{dt} = p(p+q-y) + q(p+q-x)$$

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$$\frac{dp}{dt} = p + q - y$$

$$\frac{dq}{dt} = p + q - x$$

Therefore $\frac{dx}{dt} - \frac{dp}{dt} = 0 \Rightarrow \frac{d}{dt}(x-p) = 0$

and $\frac{dy}{dt} - \frac{dq}{dt} = 0 \Rightarrow \frac{d}{dt}(y-q) = 0$ — (5)

<u>In case (1)</u>. eqns (5) gives

$$x = v + p \quad , \quad y = q - 2v$$

observe that

$$\frac{d}{dt}(p+q-x) = p+q-x$$

$$\Rightarrow \quad p+q-x = ve^t$$

similarly $\frac{d}{dt}(p+q-y) = p+q-y$

$$\Rightarrow \quad p+q-y = 2ve^t$$

Hence.

$$x = v(2e^t - 1) \qquad y = v(e^t + 1) \quad \text{— (6)}$$

$$p = 2v(e^t - 1) \quad , \quad q = v(e^t + 1) \quad \text{— (7)}$$

on substituting in the eqn for z

$$\frac{dz}{dt} = 2v(e^t - 1) \times 2ve^t + v(e^t + 1) \times ve^t$$

$$\frac{dz}{dt} = 4v^2(e^{2t} - e^t) + v^2(e^{2t} + e^t)$$

$$z = 4v^2\left[\frac{e^{2t}}{2} - e^t\right] + v^2\left[\frac{e^{2t}}{2} + e^t\right] + c$$

$$0 = -\frac{4v^2}{2} + \frac{5v^2}{2} + c \Rightarrow c = \frac{v^2}{2}$$

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$$z = 4v^2(e^{2t} - e^t) + v^2(e^{2t} - e^t) + \frac{v^2}{2}$$

$$z = \frac{5}{2}v^2(e^{2t} - 1) - 3v^2(e^t - 1) \quad \text{— (8)}$$

On solving for t and v from (6), we get

$$e^t = \frac{y-x}{2y-x}, \quad v = x - 2y$$

On substituting for t and v in terms of x and y in eqn 8

$$\boxed{z = \frac{1}{2}y(4x - 3y)}$$

<u>Case (II)</u> eqns (5) gives

$$x = s + p \qquad y = q$$

observe that $\frac{d}{dt}(p + q - x) = p + q - x$

$$\Rightarrow p + q - x = -se^t$$

and $p + q - y = 0$

Therefore $p = 0$, $q = y$, $x = s$

and $q = y = s - se^t$

Hence $s = x$ $\quad e^t = \frac{x-y}{x}$

$$\therefore \frac{dz}{dt} = y(y-x) = s^2(e^{2t} - e^t)$$

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On integrating and using initial conditions we get

$$z = s^2\left(\frac{e^{2t}}{2} - e^t\right) + c$$

$$0 = -\frac{s^2}{2} + c \implies c = \frac{s^2}{2}$$

$$z = \frac{s^2}{2}\left(e^{2t} - 2e^t + 1\right) = \frac{s^2}{2}\left(e^t - 1\right)^2$$

On eliminating s and t, we get

$$z = \frac{x^2}{2}\left(\frac{x-y}{x} - 1\right)^2 = \frac{y^2}{2}$$

<u>Exp</u> Find by the method of char., the integral surface of $pq = xy$ which passes through the line $z = x$, $y = 0$

<u>Soln</u>: The initial data curve is

$$x_0(s) = s, \quad y_0(s) = 0, \quad z_0(s) = s$$

The eqn (7) and (8) resp become

$$p_0(s) \cdot q_0(s) = 0$$

and $1 = p_0 \times 1 + q_0 \times 0$

$\implies p_0 = 1 \quad q_0 = 0$ (unique initial strip)

ie $x_0 = s \quad y_0 = 0 \quad z_0 = s \quad p_0 = 1, \quad q_0 = 0$

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The char eqns are

$$\frac{dx}{dt} = q \qquad \frac{dy}{dt} = p \qquad \frac{dz}{dt} = 2pq$$

$$\frac{dp}{dt} = y \qquad \frac{dq}{dt} = x$$

$$\frac{dx}{dt} = q \Rightarrow \frac{d^2x}{dt^2} = \frac{dq}{dt} = x$$

$$\Rightarrow D^2x - x = 0 \Rightarrow D^2 - 1 = 0$$

$$\Rightarrow D = \pm 1$$

$$x = Ae^{t} + Be^{-t}, \quad q = Ae^{t} - Be^{-t}$$

$$y = Ce^{t} + De^{-t} \qquad p = Ce^{t} - De^{-t}$$

$$\frac{dz}{dt} = 2(Ce^{t} + De^{-t})(Ae^{t} - Be^{-t})$$

$$\frac{dz}{dt} = 2ACe^{2t} + 2BDe^{-2t} - 2(BC + AD)$$

$$\Rightarrow z = ACe^{2t} - BDe^{-2t} - 2(BC + AD)t + E$$

using initial conditions

$$A + B = s \qquad A - B = 0 \Rightarrow A = B = \frac{s}{2}$$

$$C + D = 0 \qquad C - D = 1 \Rightarrow C = -D = \frac{1}{2}$$

$$AC - BD + E = s$$

$$\Rightarrow E = \frac{s}{2}$$

$$x = s\left(\frac{e^{t} + e^{-t}}{2}\right) = s\cosh t$$

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$$y = \frac{e^{t} - e^{t}}{2} = \sinh t$$

$$z = \frac{8}{4} e^{2t} + \frac{8}{4} e^{2t} + \frac{8}{2}$$

$$= \frac{8}{2}\left[\cosh 2t + 1\right] = \frac{8}{2}\left[2\cosh^2 t \cancel{-1} \cancel{+1}\right]$$

$$\boxed{z = 8\cosh^2 t}$$

$$p = \cosh t, \quad q = 8\sinh t$$

Hence the surface passing through the initial data curve is given by $z^2 = x^2(1+y^2)$

P.D.E. of order two.

An equation is said to be of order two, if it involves at least one of the differential coefficients $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$, $t = \frac{\partial^2 z}{\partial y^2}$, but none of higher order; the quantities p and q may also enter into the equation.

Thus, the general form of a second order p.d.e. is

$$f(x, y, z, p, q, r, s, t) = 0 \quad \text{—(1)}$$

The most general linear p.d.e. of order two in two ind. variables x and y with variable coeff. is of the form

$$Rr + Ss + Tt + Pp + Qq + Zz = F \quad \text{—(2)}$$

where R, S, T, P, Q, Z, F are continuous functions of x and y only and not all R, S, T are zero.

Note that x and y, being ind. variables, are const. w.r.t. each other in diff. and integration.

ie If $s = 2x + 2y$. —(3)

$$\frac{\partial^2 z}{\partial x \partial y} = 2x + 2y \quad \text{or} \quad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 2x + 2y \quad \text{(4)}$$

Integrating (4) w.r.t. x

$$\frac{\partial z}{\partial y} = x^2 + 2xy + f(y), \quad \text{—(5)}$$

$f(y)$ is an arb. function of y

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Int. (5) w.r.t y,

$$z = x^2y + xy^2 + F(y) + g(x),$$

<u>Type - I</u> : consider

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{F}{R} = f_1(x,y), \quad t = \frac{\partial^2 z}{\partial y^2} = \frac{F}{T} = f_2(x,y)$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{F}{S} = f_3(x,y)$$

we will solve by direct method.

<u>Ex</u> (1) $r = 6x \Rightarrow \frac{\partial^2 z}{\partial x^2} = 6x$ — (1)

Int (1) w.r.t. x, $\frac{\partial z}{\partial x} = 3x^2 + \phi_1(y)$ — (2)

Int. (2) w.r.t x, $z = x^3 + x\phi_1(y) + \phi_2(y)$

(2) $xys = 1$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{1}{xy} \quad \text{— (1)}$$

Int (1) w.r.t x, $\frac{\partial z}{\partial y} = \frac{1}{y}\log x + \phi_1(y)$ — (2)

Int. (2) w.r.t y, $z = \log x \cdot \log y + \int \phi_1(y)\,dy + \psi_2(x)$

$$\Rightarrow z = \log x \cdot \log y + \varphi_1(y) + \varphi_2(x)$$

<u>Type - II</u> : Consider

$Rr + Pp = F$; $R\frac{\partial p}{\partial x} + Pp = F$ $\quad Ss + Pp = F$; $S\frac{\partial p}{\partial y} + Pp = F$

$Ss + Qq = F$; $S\frac{\partial q}{\partial x} + Qq = F$ $\quad Tt + Qq = F$ $\quad T\frac{\partial q}{\partial y} + Qq = F$

These eqn will be treated as ODE of order one in which p (or q) is the dependent variable.

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Ex ① $t - xq = x^2$

ie $\frac{\partial q}{\partial y} - xq = x^2$ — (1)

which is linear diff eqn in variable q and y regarding x as const.

$$I.F = e^{\int -x\,dy} = e^{-xy}$$

$$q \times e^{-xy} = \int x^2 \times e^{-xy}\,dy + \phi_1(x)$$

$$q e^{-xy} = -\frac{x^2}{x} e^{-xy} + \phi_1(x)$$

$$q = \frac{\partial z}{\partial y} = -x + e^{xy}\phi_1(x)$$

Int. it w.r.t y

$$z = -xy + \frac{1}{x} e^{xy}\phi_1(x) + \psi_2(x)$$

$$z = -xy + \psi_1(x)\, e^{xy} + \psi_2(x)$$

② $xr + p = 9x^2y^3$

$$x\frac{\partial p}{\partial x} + p = 9x^2y^3 \Rightarrow \frac{\partial p}{\partial x} + \frac{1}{x}p = 9xy^3$$

$$I.F = e^{\int \frac{1}{x}dx} = x$$

$$px = \int x \times 9xy^3\,dx + \phi_1(y)$$

$$px = 3x^3y^3 + \phi_1(y)$$

$$\frac{\partial z}{\partial x} = 3x^2y^3 + \frac{1}{x}\phi_1(y)$$

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Int. w.r.t x $\quad z = x^3y^3 + \phi_1(x)\log x + \phi_2(x)$

Type. III : Consider

$$Rr + Ss + Pp = V \quad \text{or} \quad P\frac{\partial p}{\partial x} + S\frac{\partial p}{\partial y} = V - Pp$$

and $$Ss + Tt + Qq = V \quad \text{or} \quad S\frac{\partial q}{\partial x} + T\frac{\partial q}{\partial y} = V - Qq$$

These are Linear P.d.e of order one in p or q, as dep variable and x, y as ind variables. In such situations we shall apply well known Lagrange's method.

Ex $\quad t + s + q = 0$

$$\frac{\partial q}{\partial y} + \frac{\partial p}{\partial y} + \frac{\partial z}{\partial y} = 0$$

Int. w.r.t y $\Rightarrow$ $q + p + z = f(x)$

or $\quad p + q = f(x) - z \quad$ — (1)

which is in Lagrange's form $Pp + Qq = R$

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{f(x) - z} \quad \text{— (2)}$$

$\Rightarrow dx - dy = 0 \Rightarrow x - y = c_1 \quad$ — (3)

from (1) + (3) $\quad \frac{dx}{1} = \frac{dz}{f(x) - z} \Rightarrow \frac{dz}{dx} + z = f(x)$

$$z \times e^x = \int e^x f(x)\,dx + c_2$$

or $\quad ze^x = \phi(x) + c_2 \quad$ — (4) where $\phi(x) = \int e^x f(x)\,dx$

from (3) and (4), the required general soln —

$$ze^x - \phi(x) = \psi(x - y)$$

$\Rightarrow ze^x = \phi(x) + \psi(x - y)$ $\quad$ ψ & ϕ - arbitrary functions

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Type-IV. consider

$$Rr + Pp + Zz = F \quad \text{or} \quad R\frac{\partial^2 z}{\partial x^2} + P\frac{\partial z}{\partial x} + Zz = F \quad (1)$$

$$Tt + Qq + Zz = F \quad \text{or} \quad T\frac{\partial^2 z}{\partial y^2} + Q\frac{\partial z}{\partial y} + Zz = F \quad (2)$$

which are linear O.D.Es of order two with x as indep. vari in (1) and y as indep. vari in (2)

Ex ① $t - 2xq + x^2 z = (x-2)e^{3x+2y}$

$$\text{or} \quad \frac{\partial^2 z}{\partial y^2} - 2x\frac{\partial z}{\partial y} + x^2 z = (x-2)e^{3x+2y}$$

taking $D = \frac{\partial}{\partial y}$

$$\Rightarrow (D^2 - 2xD + x^2)z = (x-2)e^{3x+2y}$$

$$\Rightarrow (D-x)^2 z = (x-2)e^{3x+2y} \quad (1)$$

$\therefore$ Complementary function of (1)

$$= e^{xy}\{\phi_1(x) + x\phi_2(x)\}.$$

Particular integral of (1)

$$= \frac{1}{(D-x)^2}(x-2)e^{3x+2y}$$

$$= \frac{(x-2)e^{3x+2y}}{(2-x)^2}$$

$$= \frac{e^{3x+2y}}{x-2}$$

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Required soln is $z = e^{xy}\{\phi_1(x) + x\phi_2(x)\}$

$+ \dfrac{e^{3x+2y}}{x-2}$ ϕ_1, ϕ_2 bary cnts

Classification of P.D.E. Reduction to canonical or Normal Forms.

Consider a general P.D.E. of second order for a function of two ind. variables x and y in the form:

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \quad -(1)$$

where R, S, and T are continuous functions of x and y only. possessing partial derivatives defined in some domain D on the xy-plane. Then (1) is said to be -

(I) Hyperbolic at a point (x, y) in domain D if $S^2 - 4RT > 0$

(II) Parabolic at a point (x, y) in domain D if $S^2 - 4RT = 0$

(III) Elliptic at a point (x, y) in domain D if $S^2 - 4RT < 0$

Exp (1) One dim. wave eqn $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial^2 z}{\partial y^2}$ ie $r - t = 0$

Soln: Here $R = 1$, $S = 0$ and $T = -1$

Hence $S^2 - 4RT = 0 + 4 = 4 > 0$

So the given eqn is hyperbolic.

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(II) Consider one-dim heat (diffusion) eqn

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial y} \quad \text{ie} \quad r - q = 0$$

Here $R = 1, \quad S = 0, \quad T = 0$

Hence $S^2 - 4RT = 0 - 0 = 0$

So the given eqn is parabolic.

(III) Consider two-dim Laplace's eqn.

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \quad \text{ie.} \quad r + t = 0$$

Here $R = 1, \quad S = 0, \quad T = -1$

Hence $S^2 - 4RT = 0 - 4 = -4 < 0$

So the given eqn is elliptic

(IV) $xyr - (x^2 - y^2)s - xyt + py - qx = 2(x^2 - y^2)$

Here $R = xy, \quad S = -(x^2 - y^2), \quad T = -xy$

Hence $S^2 - 4RT = (x^2 - y^2)^2 + 4x^2y^2$

$= (x^2 + y^2)^2 > 0$

So the given eqn is Hyperbolic at all points.

(V) $u_{xx} + 2\sin x\, u_{xy} - \cos^2 x\, u_{xy} - \cos x\, u_y = 0$

Here $R = 1, \quad S = 2\sin x, \quad T = -\cos^2 x$

Hence $S^2 - 4RT = 4\sin^2 x + 4\cos^2 x$

$= 4 \times 1 = 4 > 0$

So the given eqn is Hyperbolic

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Cauchy's Problem for 2nd order p.d.e.

Characteristic eqn and char. curves of 2nd order p.d.e.

Consider the 2nd order p.d.e.

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \quad -(1)$$

in which R, S, T are functions of x and y only. The Cauchy's Prob. consists of the problem of determining the soln of (1) such that on a given space curve C, z takes on prescribed values of z and $\frac{\partial z}{\partial n}$, where n is the distance measured along the normal to the curve

characteristic equations and char. curves.

corresponding to (1). consider the λ-quadratic

$$R\lambda^2 + S\lambda + T = 0 \quad -(2)$$

where $S^2 - 4RT \geq 0$, (2) has real roots. Then, the ODEs

$$\frac{dy}{dx} + \lambda(x, y) = 0 \quad -(3)$$

are called the char. equations.

The soln's of (3) are known as char. curves or simply the characteristics of the 2nd order p.d.e (1).

Now, consider the following three cases:

Case (i): If $S^2 - 4RT > 0$ i.e (1) is Hyperbolic then (2) has two distinct real roots λ_1, λ_2 (say) so that we have two char. eqns

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$\frac{dy}{dx} + \lambda_1(x,y) = 0$ and $\frac{dy}{dx} + \lambda_2(x,y) = 0$

solving these we get two distinct families of chars

<u>Case-(ii)</u>: If $S^2 - 4RT = 0$ ie (1) is Parabolic

then (2) has two equal real roots λ, λ so the we get only one char. eqn (3). solving it, we get only one family of characteristics.

<u>Case-(iii)</u>: If $S^2 - 4RT < 0$ ie (1) is elliptic

then (2) has complex roots.

Hence there are no real characteristics

Thus we get two families of complex char. when (1) is elliptic.

<u>Exp ①</u> Find the chars of $y^2 r - x^2 t = 0$ — ①

<u>soln</u>: Here $R = y^2$. $S = 0$, $T = -x^2$

Then $S^2 - 4RT = 0 + 4x^2y^2 > 0$.

Hence (1) is hyperbolic everywhere except $x=0$ and $y=0$

The λ-quadratic is $R\lambda^2 + S\lambda + T = 0$

or $y^2\lambda^2 - x^2 = 0$ — (2)

solving (2) $\lambda = \frac{x}{y}, -\frac{x}{y}$ (two distinct real roots)

corresponding char. eqns are

$\frac{dy}{dx} + \frac{x}{y} = 0$ and $\frac{dy}{dx} - \frac{x}{y} = 0$

$\Rightarrow x\,dx + y\,dy = 0$ $\qquad$ $x\,dx - y\,dy = 0$

Int. $x^2 + y^2 = c_1$ and $x^2 - y^2 = c_2$

which are the required families of chars.

Exp (2) Find the chars of $x^2 r + 2xy s + y^2 t = 0$ —(1)

Soln Here $R = x^2$, $S = 2xy$, $T = y^2$

then $S^2 - 4RT = 4x^2y^2 - 4x^2y^2 = 0$

Hence (1) is parabolic everywhere.

The λ-quadratic is $R\lambda^2 + S\lambda + T = 0$

or $x^2\lambda^2 + 2xy\lambda + y^2 = 0$ —(2)

solving (2) $(x\lambda + y)^2 = 0$ so that $\lambda = -\frac{y}{x}, -\frac{y}{x}$.

(equal roots)

The char. eqn is $\frac{dy}{dx} - \frac{y}{x} = 0$

or $\frac{1}{y}dy - \frac{1}{x}dx = 0 \Rightarrow \frac{y}{x} = c_1$

or $y = c_1 x$

which is the required family of chars.

Exp (3): Find the chars of $4r + 5s + t + p + q - z = 0$ —(1)

Soln. Here $R = 4$, $S = 5$, $T = 1$

then $S^2 - 4RT = 25 - 16 = 9 > 0$

Hence (1) is Hyp.

λ-quadratic is $4\lambda^2 + 5\lambda + 1 = 0$

$\lambda = -1, \quad \lambda = -4$

$\frac{dy}{dx} - 1 = 0$ $\qquad$ $\frac{dy}{dx} - 4 = 0$

$y - x = c_1$ and $y - 4x = c_2$ //

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Reduction to canonical (or normal) forms

Consider a p.d.e. of the type

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \quad \text{--- (1)}$$

where R, S, T are continuous functions of x and y possessing continuous partial derivatives.

Laplace transformation on (1) consists of changing the independent variables x, y to new set of continuous differentiable independent variables u, v, where

$$u = u(x, y) \quad \text{and} \quad v = v(x, y) \quad \text{--- (2)}$$

are to be chosen so that the resulting eqn in independent variables u, v is transformed into one of three canonical form.

from (2), we have

$$\left.\begin{aligned} p &= \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} \\ q &= \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} \end{aligned}\right\} \quad (3)$$

$$(3) \Rightarrow \left.\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v} \\ \& \quad \frac{\partial}{\partial y} &= \frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v} \end{aligned}\right\} \quad (4)$$

$$\therefore r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

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$$= \left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\cdot\frac{\partial}{\partial v}\right)\left(\frac{\partial u}{\partial x}\cdot\frac{\partial z}{\partial u} + \frac{\partial v}{\partial x}\cdot\frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial u}{\partial x}\cdot\frac{\partial}{\partial u}\left(\frac{\partial u}{\partial x}\cdot\frac{\partial z}{\partial u} + \frac{\partial v}{\partial x}\cdot\frac{\partial z}{\partial v}\right) + \frac{\partial v}{\partial x}\cdot\frac{\partial}{\partial v}\left(\frac{\partial u}{\partial x}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial x}\cdot\frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + 2\frac{\partial^2 z}{\partial u\partial v}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2$$

$$+ \frac{\partial z}{\partial u}\cdot\frac{\partial^2 u}{\partial x^2} + \frac{\partial z}{\partial v}\cdot\frac{\partial^2 v}{\partial x^2}$$

$$s = \frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$= \left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\cdot\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}\right)$$

$$= \frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial u\partial v}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}\right)$$

$$+ \frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial y} + \frac{\partial z}{\partial u}\cdot\frac{\partial^2 u}{\partial y\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial^2 v}{\partial y\partial x}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

$$= \left(\frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v}\right)\left(\frac{\partial u}{\partial y}\cdot\frac{\partial z}{\partial u} + \frac{\partial v}{\partial y}\cdot\frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial u}{\partial y}\cdot\frac{\partial}{\partial u}\left(\frac{\partial u}{\partial y}\cdot\frac{\partial z}{\partial u} + \frac{\partial v}{\partial y}\cdot\frac{\partial z}{\partial v}\right) + \frac{\partial v}{\partial y}\frac{\partial}{\partial v}\left(\frac{\partial u}{\partial y}\cdot\frac{\partial z}{\partial u} + \frac{\partial v}{\partial y}\cdot\frac{\partial z}{\partial v}\right)$$

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$$t = \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial y}\right)^2 + 2\frac{\partial^2 z}{\partial u \partial v}\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y} + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2$$

$$+ \frac{\partial z}{\partial u}\cdot\frac{\partial^2 u}{\partial y^2} + \frac{\partial z}{\partial v}\cdot\frac{\partial^2 v}{\partial y^2}$$

Putting the above values of p, q, r, s, t in (1) and simplifying, we get

$$A\frac{\partial^2 z}{\partial u^2} + 2B\frac{\partial^2 z}{\partial u \partial v} + C\frac{\partial^2 z}{\partial v^2} + F\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right) = 0 \quad \text{--- (5)}$$

where

$$A = R\left(\frac{\partial u}{\partial x}\right)^2 + S\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + T\left(\frac{\partial u}{\partial y}\right)^2 \quad \text{--- (6)}$$

$$B = R\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{1}{2}S\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial y}\right) + T\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y} \quad \text{--- (7)}$$

$$C = R\left(\frac{\partial v}{\partial x}\right)^2 + S\frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial y} + T\left(\frac{\partial v}{\partial y}\right)^2 \quad \text{--- (8)}$$

and $F\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right)$ is the transformed form of $f(x, y, z, p, q)$.

Now we shall find out u and v so that (5) reduces to simplest possible form. The method of evaluation of desired values of u and v becomes easy when the discriminant $S^2 - 4RT$ of the quadratic eqn.

$$Rd^2 + Sd + T = 0 \quad \text{--- (9)}$$

is everywhere either positive, negative or zero.

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Case-I: Let $S^2 - 4RT > 0$

then λ_1, λ_2 are real and distinct roots of eqn (5). The coefficients of $\frac{\partial^2 z}{\partial u^2}$ and $\frac{\partial^2 z}{\partial v^2}$ in eqn (9) will vanish if we choose u and v such that

$$\frac{\partial u}{\partial x} = \lambda_1 \left(\frac{\partial u}{\partial y}\right) \quad \text{— (10)}$$

and

$$\frac{\partial v}{\partial x} = \lambda_2 \left(\frac{\partial v}{\partial y}\right) \quad \text{— (11)}$$

Since λ_1 is a root of (9), we have

$$R\lambda_1^2 + S\lambda_1 + T = 0 \quad \text{— (12)}$$

using (10), (6) gives

$$A = (R\lambda_1^2 + S\lambda_1 + T)\left(\frac{\partial u}{\partial y}\right)^2 = 0 \quad \text{— (13)}$$

Again since λ_2 is a root of (9), we have

$$R\lambda_2^2 + S\lambda_2 + T = 0 \quad \text{— (14)}$$

using (11), (8) gives

$$C = (R\lambda_2^2 + S\lambda_2 + T)\left(\frac{\partial v}{\partial y}\right)^2 = 0 \quad \text{— (15)}$$

Re-writing (10), we have

$$\frac{\partial u}{\partial x} - \lambda_1 \frac{\partial u}{\partial y} = 0 \quad \text{— (16)}$$

Lagrange's aux. eqn for (16) are

$$\frac{dx}{1} = \frac{dy}{-\lambda_1} = \frac{du}{0} \quad \text{— (17)}$$

taking third fraction $du = 0 \Rightarrow u = c_1$

taking first & second fraction of (17)

$$\frac{dy}{dx} + \lambda_1 = 0 \quad \text{— (19)}$$

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Let the soln of (19) be $f_1(x, y) = c_2$

from (18) & (20), the general soln of (16) is

$$u = f_1(x, y) \quad \text{--- (21)}$$

Similarly, the general soln of (11) can be taken as

$$v = f_2(x, y) \quad \text{--- (22)}$$

Here f_1 & f_2 are arbitrary functions

Now $AC - B^2 = \frac{1}{4}(4RT - S^2)\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)^2$

As $A = C = 0$

or $B^2 = \frac{1}{4}(S^2 - 4RT)\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)^2$ --- (23)

$\Rightarrow B^2 > 0$

Hence we may divide both sides of (5) by B^2. Then noting that $A = C = 0$, (5) transform to the form

$$\frac{\partial^2 z}{\partial u \partial v} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right) \quad \text{--- (24)}$$

which is the canonical form of (1) in this case.

<u>Case – II</u> Let $S^2 - 4RT = 0$

Then the roots r_1, r_2 of (9) are real and equal. We now take u exactly as in case I and take v be any function of x & y which is ind. of u.

So $A = 0$ (like in case - I)

$\because S^2 - 4RT = 0 \Rightarrow B^2 = 0 \Rightarrow B = 0$

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in this case $C \neq 0$ otherwise v would be a function of u but we already assume u is ind. of u

Putting $A = 0$, $B = 0$ and dividing by C, (5) transforms to the form

$$\frac{\partial^2 z}{\partial u \partial v} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right) \quad -(25)$$

which is the canonical form of (1) in this case.

Case-III Let $S^2 - 4RT < 0$

Here $u = \alpha + i\beta$ $\quad v = \alpha - i\beta$

so $\alpha = \frac{u+v}{2}$, $\beta = \frac{i(v-u)}{2}$ $\quad -(26)$

Now $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial u} + \frac{\partial z}{\partial \beta} \cdot \frac{\partial \beta}{\partial u} = \frac{1}{2}\left(\frac{\partial z}{\partial \alpha} - i\frac{\partial z}{\partial \beta}\right)$ (27)

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial v} + \frac{\partial z}{\partial \beta} \cdot \frac{\partial \beta}{\partial v} = \frac{1}{2}\left(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}\right) \quad -(28)$$

$$\therefore \frac{\partial^2 z}{\partial u \partial v} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right) = \frac{1}{2}\left(\frac{\partial}{\partial \alpha} - i\frac{\partial}{\partial \beta}\right) \times \frac{1}{2}\left(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}\right)$$

$$= \frac{1}{4}\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}\right) - i\frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}\right)\right]$$

$$= \frac{1}{4}\left[\frac{\partial^2 z}{\partial \alpha^2} + \cancel{i\frac{\partial^2 z}{\partial \alpha \partial \beta}} - \cancel{i\frac{\partial^2 z}{\partial \beta \partial \alpha}} + \frac{\partial^2 z}{\partial \beta^2}\right]$$

$$\text{or } \frac{\partial^2 z}{\partial u \partial v} = \frac{1}{4}\left(\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2}\right) \quad -(29)$$

Putting $u = \alpha + i\beta$, $v = \alpha - i\beta$ and using (27), (28) and (29), (24) reduces to

$$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \psi\left(\alpha, \beta, z, \frac{\partial z}{\partial \alpha}, \frac{\partial z}{\partial \beta}\right) \quad (3)$$

which is canonical form of (1) in this case

Working Rule for reducing a hyperbolic eqn to its canonical form

Step (1) Let the given eqn $Rr + Ss + Tt + f(x, y, z, p, q) = 0$ — (1)

be hyperbolic so that $S^2 - 4RT > 0$

Step (2) write the λ-quadratic eqn

$$R\lambda^2 + S\lambda + T = 0 \quad \text{— (2)}$$

Let λ_1 and λ_2 be its two distinct roots of (2)

Step (3): Then corr. char. eqns are

$$\frac{dy}{dx} + \lambda_1 = 0 \quad \text{and} \quad \frac{dy}{dx} + \lambda_2 = 0 \quad \text{—}$$

solving these, we get $f_1(x, y) = c_1$ & $f_2(x, y) = c_2$ (3)

Step (4): We select u, v such that

$$u = f_1(x, y) \quad \text{and} \quad v = f_2(x, y) \quad \text{— (4)}$$

Step (5): transforms p, q, r, s and t in terms of u, and v

Step (6): substituting the values of p, q, r, s, t obtained in step (5) in eqn (1) and simplifying we get the following canonical form of (1)

$$\frac{\partial^2 z}{\partial u \partial v} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right) \quad //$$

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Ex ① write canonical form of $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$

Sol^n: Re-write eq^n, we get $r - t = 0$ — ①

Here $R = 1$, $S = 0$ and $T = -1$

so that $S^2 - 4RT = 4 > 0$, so eq^n ① is hyperbolic

The λ-quadratic eq^n $R\lambda^2 + S\lambda + T = 0$ reduce to

$$\lambda^2 - 1 = 0$$

Hence $\lambda = 1, -1$ so $\lambda_1 = 1$, $\lambda_2 = -1$ (real and distinct)

Then the char. eq^{n}s $\frac{dy}{dx} + \lambda_1 = 0$ & $\frac{dy}{dx} + \lambda_2 = 0$

reduce to

$$\frac{dy}{dx} + 1 = 0 \quad \text{and} \quad \frac{dy}{dx} - 1 = 0$$

$$\Rightarrow \quad y + x = c_1 \quad \text{and} \quad y - x = c_2$$

In order to reduce ① to its canonical form, we choose

$$u = y + x \quad \text{and} \quad v = y - x \quad \text{— ②}$$

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \quad \text{— ③}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \quad \text{— ④}$$

from ③ and ④ $\frac{\partial}{\partial x} = \frac{\partial}{\partial u} - \frac{\partial}{\partial v}$ & $\frac{\partial}{\partial y} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$ ⑤

$$\therefore \quad r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) - \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

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$$r = \frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \quad - (6)$$

and $t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$

$$t = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$t = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \quad - (7)$$

using (6) and (7) in (1), the required canonical form is

$$\left(\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) - \left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right)$$

$$\Rightarrow \boxed{\frac{\partial^2 z}{\partial u \partial v} = 0}$$

$$\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right) = 0$$

Int. w.r.t u $\quad \frac{\partial z}{\partial v} = \phi_1(v)$

Again int. w.r.t v $\quad z = \int \phi_1(v)\,dv + \psi_2(u)$

$$\boxed{z = \psi_1(v) + \psi_2(u)}$$

Exp (2) Reduce to canonical form of $r - x^2 t = 0$ —(1)

Here $R = 1$, $S = 0$, $T = -x^2$

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λ quadratic $R\lambda^2 + S\lambda + T = 0$ gives $\lambda^2 - x^2 = 0$

so that $\lambda = \pm x$

Hence cha. eqns

$$\frac{dy}{dx} + x = 0 \quad , \quad \frac{dy}{dx} - x = 0$$

$$\Rightarrow \quad y + \frac{x^2}{2} = c_1 \qquad y - \frac{x^2}{2} = c_2$$

choose

$$u = y + \frac{x^2}{2} \quad , \quad v = y - \frac{x^2}{2} \quad -(2)$$

Now

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$p = x\frac{\partial z}{\partial u} - x\frac{\partial z}{\partial v} \quad -(3)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$q = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \quad -(4)$$

$$\therefore \quad r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left\{x\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right\}$$

$$= x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + 1\cdot\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$= x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\cdot\frac{\partial v}{\partial x}\right] + \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$$

$$r = x^2\left(\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) + \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$$

and $t = \dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right) = \left(\dfrac{\partial}{\partial u} + \dfrac{\partial}{\partial v}\right)\left(\dfrac{\partial z}{\partial u} + \dfrac{\partial z}{\partial v}\right)$

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$$t = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

Putting the values of r and t in (1), we get

$$x^2\left(\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) + \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} - x^2\left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) = 0$$

$$\frac{\partial^2 z}{\partial u \partial v} = \frac{1}{4x^2}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

or $$\frac{\partial^2 z}{\partial u \partial v} = \frac{1}{4(u-v)}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

which is the required canonical form of the given eqn.

Working rule for reducing a parabolic eqn to its canonical form:

Step (1). Let the given eqn $Rr + Ss + Tt + f(x, y, z, p, q) = 0$ be parabolic so that $S^2 - 4RT = 0$

Step (2): write λ-quadratic eqn $R\lambda^2 + S\lambda + T = 0$ (2)

Let λ_1, λ_1 be two equal roots of (2)

Step (3): write the char. eqn. corr. to $\lambda = \lambda_1$

$$\frac{dy}{dx} + \lambda_1 = 0$$

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solving it, we get $f_1(x, y) = c_1$ — (3)

Step (4): Choose $u = f_1(x, y)$, $v = f_2(x, y)$ (4)

where $f_2(x, y)$ is an arbi function of x and y and is ind. of $f_1(x, y)$. for this verify that Jacobian J of u and v given by (4) is non-zero

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \neq 0 \quad -(5)$$

Step (5): Using relation (4), find p, q, r, s and t in terms of u and v

Step (6): Substituting the values of p, q, r, s and t obtained in step (1) and simplifying we get the following canonical forms of (1)

$$\frac{\partial^2 z}{\partial u^2} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right)$$

or $$\frac{\partial^2 z}{\partial v^2} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right).$$

Exp: Reduce the eqn $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$ to canonical form and hence solve it.

Soln: Re-writing the given eqn, we get

$$r + 2s + t = 0 \quad -(1)$$

Here $R = 1$, $S = 2$, $T = 1$

so $S^2 - 4RT = 4 - 4 = 0$

ie (1) is Parabolic

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The λ-quadratic eq^n reduces to $\lambda^2 + 2\lambda + 1 = 0$

so that $\lambda = -1, -1$ (equal roots)

The corr. char eq^n is $\frac{dy}{dx} - 1 = 0$

or $dx - dy = 0 \Rightarrow x - y = c_1$.

Choose $u = x - y$ and $v = x + y$ — (2)

ie $J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2 \neq 0$

Now

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \quad (3)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = -\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \quad (4)$$

from (3) and (4) $\frac{\partial}{\partial x} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$ & $\frac{\partial}{\partial y} = -\frac{\partial}{\partial u} + \frac{\partial}{\partial v}$ (5)

$$\therefore r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$r = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \quad (6)$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(-\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= -\frac{\partial}{\partial u}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$t = \frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \quad (7)$$

and $s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$

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$$= \frac{\partial}{\partial u}\left(-\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)+\frac{\partial}{\partial v}\left(-\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)$$

$$= -\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2} \quad \text{--- (8)}$$

using (6), (7) and (8) in (1), the required canonical form is

$$\frac{\partial^2 z}{\partial v^2}=0 \quad \text{or} \quad \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)=0 \quad \text{--- (9)}$$

Int. w.r.t v, we get

$$\frac{\partial z}{\partial v}=\phi(u)$$

Again int. w.r.t v: $z=\int \phi(u)\,dv+\psi(u)$

$$z = v\,\phi(u)+\psi(u)$$

or $z=(x+y)\,\phi(x-y)+\psi(x-y)$.

Ex. $y^2\frac{\partial^2 z}{\partial x^2}-2xy\frac{\partial^2 z}{\partial x\partial y}+x^2\frac{\partial^2 z}{\partial x^2}=\frac{y^2}{x}\frac{\partial z}{\partial x}+\frac{x^2}{y}\frac{\partial z}{\partial y}$

Ans $\frac{\partial^2 z}{\partial v^2}=0$ and $z=\left(\frac{x^2-y^2}{2}\right)\phi\left(\frac{x^2+y^2}{2}\right)+\psi\left(\frac{x^2+y^2}{2}\right)$

Working Rule for reducing an elliptic eqn to its canonical form.

Step (1): Let the given eqn $Rr+Ss+Tt+f(x,y,z,p,q)=0$ (1) be an elliptic so that $S^2-4RT<0$ --- (2)

Step (2): write λ-quadratic eqn $R\lambda^2+S\lambda+T=0$ (3)

Let roots λ_1, λ_2 of (2) be complex conjugates.

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Step (3): Then (0)x8 char eqns are

$$\frac{dy}{dx} + \lambda_1 = 0 \quad \text{and} \quad \frac{dy}{dx} + \lambda_2 = 0$$

solving these, we shall obtain solns of the eqn

$$f_1(x,y) + i f_2(x,y) = c_1 \quad \text{and} \quad f_1(x,y) - i f_2(x,y) = c_2 \quad (3)$$

Step (4): choose

$$u = f_1(x,y) + i f_2(x,y) \quad \& \quad v = f_1(x,y) - i f_2(x,y)$$

Let α, and β be two new real ind. variables such that $u = \alpha + i\beta$ and $v = \alpha - i\beta$ so that $\alpha = f_1(x,y)$ and $\beta = f_2(x,y)$ (4)

Step (5): Using relations (4) find p, q, r, s and t in terms of α and β.

Step (6): substituting the values of p, q, r, s, t and relations (4) in (1) and simplifying we shall get the following canonical form of (1)

$$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \phi\left(\alpha, \beta, z, \frac{\partial z}{\partial \alpha}, \frac{\partial z}{\partial \beta}\right)$$

Exp (1): Reduce the eqn $\frac{\partial^2 z}{\partial x^2} + x^2 \frac{\partial^2 z}{\partial y^2} = 0$ to canonical form.

Soln: Re-writing the given eqn we get $r + x^2 t = 0$ (1)

Here $R = 1$, $S = 0$, $T = x^2$ so that

$S^2 - 4RT = -4x^2 < 0$, $x \neq 0$, showing that (1) is elliptic

The λ-quadratic $R\lambda^2 + S\lambda + T = 0$ reduce to

$\lambda^2 + x^2 = 0$ gives $\lambda = ix, -ix$

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The corr. char. eqns are given by

$$\frac{dy}{dx} + ix = 0 \quad \text{and} \quad \frac{dy}{dx} - ix = 0$$

or $y + i\frac{x^2}{2} = c_1$ and $y - i\frac{x^2}{2} = c_2$

Choose $u = y + i\frac{x^2}{2}$ and $v = y - i\frac{x^2}{2}$

where $\alpha = y$ and $\beta = \frac{x^2}{2}$ — (2)

are two new ind variables

Now $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}\cdot\frac{\partial \beta}{\partial x} = x\frac{\partial z}{\partial \beta}$ (3)

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta}\cdot\frac{\partial \beta}{\partial y} = \frac{\partial z}{\partial \alpha} \quad \text{— (4)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial \beta}\right) = \frac{\partial z}{\partial \beta} + x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \beta}\right)$$

$$= \frac{\partial z}{\partial \beta} + x\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\cdot\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\cdot\frac{\partial \beta}{\partial x}\right]$$

$$r = \frac{\partial z}{\partial \beta} + x^2\frac{\partial^2 z}{\partial \beta^2} \quad \text{— (5)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right) = \frac{\partial^2 z}{\partial \alpha^2} \quad (6)$$

Using (5) and (6) in (1) the required canonical form is

$$\frac{\partial z}{\partial \beta} + x^2\frac{\partial^2 z}{\partial \beta^2} + x^2\frac{\partial^2 z}{\partial \alpha^2} = 0$$

or $$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = -\frac{1}{x^2}\left(\frac{\partial z}{\partial \beta}\right)$$

$\beta = \frac{x^2}{2}$ so $x^2 = 2\beta$

$$\Rightarrow \frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = -\frac{1}{2\beta}\frac{\partial z}{\partial \beta} \quad //$$

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Ex: $y^2\left(\frac{\partial^2 z}{\partial y^2}\right) + \frac{\partial^2 z}{\partial x^2} = 0$

$$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = -\frac{1}{2\alpha}\frac{\partial z}{\partial \alpha}, \quad \text{when } \alpha = \frac{y^2}{2}, \ \beta = x$$

Vibrations of an infinite string:

Consider the following one-dim wave eqn

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \quad -\infty < x < \infty, \ t > 0 \qquad \text{— (1)}$$

If we introduce the characteristic variables

$$\xi = x - ct, \quad \eta = x + ct,$$

Eqn (1) transforms to $\frac{\partial^2 y}{\partial \xi \partial \eta} = 0$

Therefore:

$$y(x,t) = F(\xi) + G(\eta)$$

or $y(x,t) = F(x-ct) + G(x+ct)$.

We need the arbitrary functions F and G to be C^2 functions so that $y(x,t)$ is a regular soln of (1). The above solution is indeed the general soln of (1).

Let the initial conditions be given as follows

$$y(x,0) = f(x), \quad y_t(x,0) = g(x), \quad -\infty < x < \infty$$

where $y = f(x)$ is the initial position of the string and $g(x)$ is the initial velocity at the

point x. Hence

$$F(x) + G(x) = f(x)$$

$$-cF'(x) + cG'(x) = g(x)$$

$$-cF(x) + cG(x) = \int_{x_0}^{x} g(x)\,dx$$

solving for $F(x)$ and $G(x)$, we get

$$F(x) = \frac{1}{2c}\left[c f(x) - \int_{x_0}^{x} g(x)\,dx\right]$$

& $$G(x) = \frac{1}{2c}\left[c f(x) + \int_{x_0}^{x} g(x)\,dx\right].$$

Therefore

$$y(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

where we require $f \in C^2$ and $g \in C^1$ —(2)
so that $y(x,t)$ is a C^2 function.
This is called d'Alembert's solution which describes the vibrations of an infinite string.

Vibrations of a semi-infinite string:

Consider the motion of a semi-infinite string (ie one end $x=0$ is kept fixed). The eqn. governing the motion of the string is

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \qquad 0 < x < \infty,\ t > 0$$

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The initial conditions are

$$y(x,0) = u(x) \quad , \quad y_t(x,0) = v(x) \qquad x \geq 0$$

The boundary conditions is

$$y(0,t) = 0 \qquad t \geq 0$$ (ie the end $x=0$ is fix

observe that the above condition implies

$$y_t(0,t) = 0$$

However, d'Alembert's sol^n cannot be used for the initial value problem since $u(x-ct)$ has no meaning for the values $t > \frac{x}{c}$.

Instead we consider the modified problem for an infinite string as follows

$$y(x,0) = U(x) \quad , \quad y_t(x,0) = V(x) \qquad -\infty < x < \infty$$

where

$$U(x) = \begin{cases} u(x) & \text{if } x \geq 0 \\ -u(-x) & \text{if } x \leq 0 \end{cases}$$

and

$$V(x) = \begin{cases} v(x) & \text{if } x \geq 0 \\ -v(-x) & \text{if } x \leq 0 \end{cases}$$

Observe that $U(x)$ and $V(x)$ are defined as odd function of x so as to satisfy the b.c's

Then d'Alembert's sol^n becomes

$$y(x,t) = \frac{1}{2}\left[U(x-ct) + U(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} V(s)\,ds$$

when $x = 0$

$$y(0,t) = \frac{1}{2}\left[U(-ct) + U(ct)\right] + \frac{1}{2c}\int_{-ct}^{ct} V(s)\,ds$$

$$y_t(0,t) = \frac{1}{2}\left[-cU'(-ct) + cU'(ct)\right] + \frac{1}{2}\left[V(ct) + V(-ct)\right]$$

Hence, by making use of the fact that U and V are odd functions, we get

$$y(0,t) = \frac{1}{2}\left[-u(ct) + u(ct)\right] = 0$$

$$y_t(0,t) = \frac{c}{2}\left[u'(ct) - u'(ct)\right] + \frac{1}{2}\left[v(ct) - v(ct)\right]$$

$$y_t(0,t) = 0$$

At $t = 0$ and $x > 0$

$$y(x,0) = \frac{1}{2}\left[U(x) + U(x)\right] = \frac{1}{2}\left[u(x) + u(x)\right] = u(x)$$

$$y_t(x,0) = \frac{1}{2}\left[cU'(x) - cU'(x)\right] + \frac{1}{2}\left[V(x) + V(x)\right] = v(x)$$

Thus the initial and boundary conditions are satisfied. If the string is released from rest, ie if $v \equiv 0$ the solution becomes.

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$$y(x,t) = \begin{cases} \frac{1}{2}\left[u(x-ct) + u(x+ct)\right] & x \geq ct \\ \frac{1}{2}\left[u(x+ct) - u(x-ct)\right] & x \leq ct \end{cases}$$

Vibrations of a string of finite length

We now consider the vibrations of a string of finite length l. The initial conditions are

$$y(x,0) = u(x), \quad y_t(x,0) = v(x), \quad 0 \leq x \leq l$$

The boundary conditions are

$$\left.\begin{aligned} y(0,t) &= y_t(0,t) = 0 \\ y(l,t) &= y_t(l,t) = 0 \end{aligned}\right\} \quad t > 0$$

This case can also be deduced from d'Alembert's solution by converting this problem into a problem of vibrations of an infinite string. For this purpose, we extend the given data as follows. define

$$U(x) = \begin{cases} u(x) & \text{if } 0 \leq x \leq l \\ -u(-x) & \text{if } -l \leq x \leq 0 \end{cases}$$

$$u(x + 2rl) = U(x) \quad \text{if } -l \leq x \leq l$$

$$r = \pm 1, \pm 2 \ldots$$

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observe that

$$y(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi c t}{l} + b_n \sin \frac{n\pi c t}{l} \right] \sin\left(\frac{n\pi x}{l}\right),$$

$$\Rightarrow \frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} \left(-\frac{n\pi c}{l} a_n \sin \frac{n\pi c t}{l} + \frac{n\pi c}{l} b_n \cos \frac{n\pi c t}{l} \right) \sin \frac{n\pi x}{l}$$

using $y(x,0) = f(x)$ & $y_t(x,0) = g(x)$

so $f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l}$, $g(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{l} b_n \sin \frac{n\pi x}{l}$

There are Fourier sine series.

so

$$a_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx$$

$$\frac{n\pi c}{l} b_n = \frac{2}{l} \int_0^l g(x) \sin \frac{n\pi x}{l} \, dx$$

or $$b_n = \frac{2}{n\pi c} \int_0^l g(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

Particular case (I): If the initial velocity $= y_t(x,0) = g(x) = 0$

Then $b_n = 0$ Thus $y(x,t) = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi c t}{l} \sin \frac{n\pi x}{l}$

Particular case (II): If the initial displacement

$y(x,0) = f(x) = 0$ then $a_n = 0$

Thus $y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi c t}{l} \sin \frac{n\pi x}{l}$

Ex (1) A string of length l has its ends $x = 0$ and $x = l$ fixed. It is released from rest in the position $y = \frac{4Ax(l-x)}{l^2}$. Find an expression for the displacement of the string at any subsequent time.

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Soln: In this case

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \quad -(1)$$

with b.c. $y(0,t) = y(l,t) = 0 \quad \forall\ t \geq 0$ (2)

and the initial cond. Initial veloc. $y_t(x,0) = 0 \quad \forall\ 0 \leq x \leq l$ -(3a)

Initial displacement $y(x,0) = f(x) = \frac{4\lambda x(l-x)}{l^2}$ (3b)

The solution of (1) satisfying the above boundary and initial conditions is given by

$$y(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l} \cos\frac{n\pi c t}{l} \quad -(4)$$

where

$$a_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\, dx \quad -(5)$$

using (3b)

$$a_n = \frac{2}{l} \times \frac{4\lambda}{l^2}\int_0^l \underbrace{(lx - x^2)}_{I} \underbrace{\sin\frac{n\pi x}{l}}_{II}\, dx$$

$$a_n = \frac{8\lambda}{l^3}\left[(lx-x^2)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - (l-2x)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right) + (-2)\left(\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{8\lambda}{l^3}\left[-\frac{l^3}{n^2\pi^2}\sin n\pi - \frac{2l^3}{n^3\pi^3}\cos n\pi + \frac{2l^3}{n^3\pi^3}\right]$$

$$a_n = \begin{cases} 0, & \text{if } n = 2m \text{ (even) with } m = 1, 2, 3 \ldots \\ \dfrac{32\lambda}{(2m-1)^3\pi^3} & \text{if } n = 2m-1 \text{ (odd) with } m = 1, 2, \ldots \end{cases}$$

Thus

$$y(x,t) = \frac{32\lambda}{\pi^3}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^3}\sin\frac{(2m-1)\pi x}{l}\cos\frac{(2m-1)\pi c t}{l}$$

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Prob (2) $\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$ $0 \le x \le 2\pi$ $t \geq 0$ — (1)

with

$y(x,0) = \sin^3 x$; $y_t(x,0) = 0$ $0 \le x \le 2\pi$ — (2)

$y(0,t) = y(2\pi,t) = 0$ $\forall\, t \geq 0$ — (3)

Soln. In this case the soln of wave eqn is

$$y(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{2\pi} \cos\frac{n\pi c t}{2\pi}$$

or $$y(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{nx}{2} \cos\frac{nct}{2} \quad \text{— (4)}$$

Putting $t=0$ in (4) and using $y(x,0) = \sin^3 x$

$$\sin^3 x = \sum_{n=1}^{\infty} a_n \sin\frac{nx}{2}$$

or $$a_1 \sin\frac{x}{2} + a_2 \sin x + a_3 \sin\frac{3x}{2} + a_4 \sin 4x + a_5 \sin\frac{5x}{2} + a_6 \sin 3x + \cdots = \frac{1}{4}(3\sin x - \sin 3x)$$

$\Rightarrow$ $a_2 = \frac{3}{4}$, $a_6 = -\frac{1}{4}$ and $a_n = 0$ $\forall\, n \neq 2$ or $n \neq 6$

Thus

$$y(x,t) = \frac{3}{4}\sin x \cos ct - \frac{1}{4}\sin 3x \cos 3ct$$

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Riemann method of solution of general linear hyperbolic eqn of the second order:

Assume that the given linear hyperbolic eqn is reducible to canonical form

$$L(u) = u_{xy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u$$

$$L(u) = f(x,y) \quad -(1)$$

where a, b, c and f are continuously diff. functions with respect to x and y..

The characteristics are $x = \text{const.}$ and $y = \text{const.}$

Let v be another function with continuous second order partial derivatives. Again, let M be another operator defined by the relation.

$$Mv = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial(av)}{\partial x} - \frac{\partial(bv)}{\partial y} + cv \quad -(3)$$

The operator M defined by (4) is called the adjoint operator to the operator L.

$$\therefore \quad vLu - uMv = v\left(\frac{\partial^2 u}{\partial x \partial y} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + cu\right)$$

$$- u\left(\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial(av)}{\partial x} - \frac{\partial(bv)}{\partial y} + cv\right)$$

$$= \left(v\frac{\partial^2 u}{\partial x \partial y} - u\frac{\partial^2 v}{\partial x \partial y}\right) + \left(va\frac{\partial u}{\partial x} + u\frac{\partial(av)}{\partial x}\right)$$

$$+ \left(vb\frac{\partial u}{\partial y} + u\frac{\partial(bv)}{\partial y}\right) + (\cancel{cuv} - \cancel{cuv})$$

$$= \frac{\partial}{\partial y}\left(v\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right) + \frac{\partial(avu)}{\partial x} + \frac{\partial(bvu)}{\partial y}$$

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$$= \frac{\partial}{\partial x}\left(auv - u\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(buv + v\frac{\partial u}{\partial x}\right)$$

$$= \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \qquad \text{——— (4)}$$

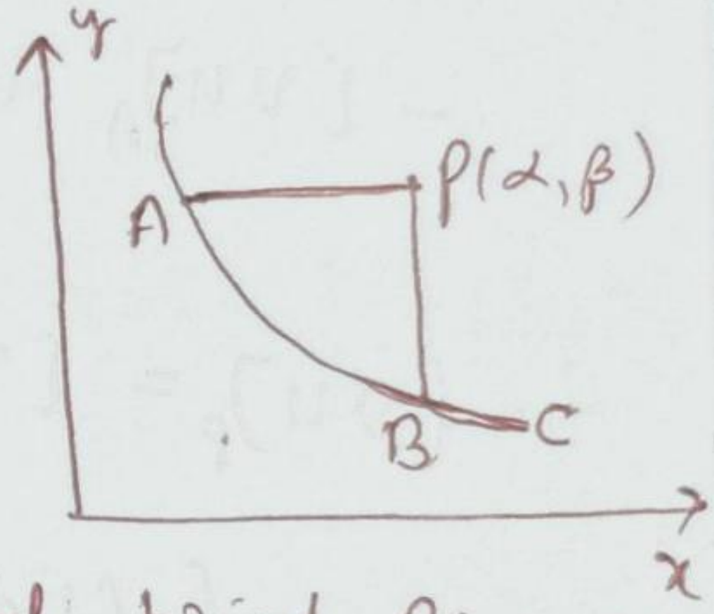
where

$$\left.\begin{aligned} U &= auv - uv_y \\ V &= buv + vu_x \end{aligned}\right\} \quad (5)$$

Now if C' is a closed curve enclosing an area S then

$$\iint_S (vLu - uMv)\,dx\,dy = \iint_S \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right) dx\,dy$$

$$= \int_{C'} (U\,dy - V\,dx) \qquad \text{——— (6)}$$



Assume that the values of u and $\frac{\partial u}{\partial x}$ (or $\frac{\partial u}{\partial y}$) are prescribed along a curve C in the x-y plane and further assume that we are required to find the soln of (1) at point $P(\alpha, \beta)$ agreeing with these b.c.s. draw PA and PB parallel to x-axis and y-axis and cutting the curve C in the point A and B resp. The closed circuit $PABP$ can be taken as the closed curve C'. Then (6) reduces to

$$\iint_S (vLu - uMv)\,dx\,dy = \int_{AB} (U\,dy - V\,dx) + \int_{BP} (U\,dy - V\,dx)$$

$$+ \int_{PA} (U\,dy - V\,dx)$$

$$= \int_{AB} (U\,dy - V\,dx) + \int_{BP} (U\,dy) - \int_{PA} V\,dx \qquad \text{——— (7)}$$

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Now $\int_{PA} v\,dx = \int_{PA} (buv + v u_x)\,dx$

$$= \int_{PA} buv\,dx + \int_{PA} v u_x\,dx$$

$$= \int_{PA} buv\,dx + (vu)_P^A - \int_{PA} u \frac{\partial v}{\partial x}\,dx$$

$$= [vu]_A - [vu]_P + \int_{PA} u\left(bv - \frac{\partial v}{\partial x}\right)dx \quad (8)$$

using (5) and (8), (7) becomes

$$\iint (vLu - uMv)\,dx\,dy = \int_{AB} (U\,dy - V\,dx) + \int_{BP} \left(av - \frac{\partial v}{\partial y}\right)u\,dy$$

$$- [vu]_A + [vu]_P - \int_{PA} u\left(bv - \frac{\partial v}{\partial x}\right)dx$$

$$\therefore (vu)_P = [vu]_A + \int_{PA} u\left(bv - \frac{\partial v}{\partial x}\right)dx - \int_{BP} u\left(av - \frac{\partial v}{\partial y}\right)dy$$

$$- \int_{AB} (U\,dy - V\,dx) + \iint_S (vLu - uMv)\,dx\,dy \quad (9)$$

So far we have treated v as an arbitrary function. Now, we choose a function $v(x, y; \alpha, \beta)$ which has the following four properties, namely:

(i) $Mv = 0$ (ii) $v = 1$ at $x = \alpha$, $y = \beta$

(iii) $\frac{\partial v}{\partial x} = bv$ on $y = \beta$ (iv) $\frac{\partial v}{\partial y} = av$ on $x = \alpha$

such a function $v(x, y; \alpha, \beta)$ is known as Green's function for the problem or Riemann-Green function. Using the above four properties of v, (9) may be re-written as

$$[u]_P = [uv]_A - \int_{AB} (v\,dy - v\,dx) + \iint_S v\,L u\,dx\,dy$$

$$= [uv]_A - \int_{AB} \left(auv - u\frac{\partial v}{\partial y}\right) dy + \int_{AB} \left(buv + v\frac{\partial u}{\partial x}\right) dx$$

$$+ \iint_S f v\,(f)\,dx\,dy$$

$$[u]_P = [uv]_A - \int_{AB} vu\,(a\,dy - b\,dx) + \int_{AB} \left(u\frac{\partial v}{\partial y}dy + v\frac{\partial u}{\partial x}dx\right)$$

$$+ \iint_S f v\,f\,dx\,dy \qquad \text{—(10)}$$

eqn (10) may be used to determine the value of u at the point P when $\frac{\partial u}{\partial x}$ is prescribed along the curve C.

Suppose, in place of the prescribed value of $\frac{\partial u}{\partial x}$, we are now given a prescribed value of $\frac{\partial u}{\partial y}$. Then, we make use of the following relation

$$\int_{AB} d(uv) = \int_{AB} \left(\frac{\partial (vu)}{\partial x}dx + \frac{\partial (vu)}{\partial y}dy\right)$$

$$\Rightarrow 0 = [uv]_B - [uv]_A - \int_{AB} \left(\frac{\partial}{\partial x}(uv)\,dx + \frac{\partial}{\partial y}(uv)\,dy\right) \qquad \text{—(11)}$$

Adding the corresponding sides of (10) and (11), we get

$$[u]_P = [uv]_B - \int_{AB} uv\,(a\,dy - b\,dx) + \int_{AB} \left(u\frac{\partial v}{\partial y}dy + v\frac{\partial u}{\partial x}dx\right)$$

$$- \int_{AB} \left(\frac{\partial}{\partial x}(uv)\,dx + \frac{\partial}{\partial y}(uv)\,dy\right) + \iint_S f\cdot v\,dx\,dy. \qquad \text{—(12)}$$

$$= [uv]_B - \int_{AB} uv\,(a\,dy - b\,dx) - \int_{AB} \left(u\frac{\partial v}{\partial x}dx + v\frac{\partial u}{\partial y}dy\right)$$

$$+ \iint_S f\cdot v\,dx\,dy \qquad \text{—(12)}$$

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Eqn (12) may be used to determine the ~~[illegible]~~ z at the point P when $\frac{\partial u}{\partial y}$ is prescribed along C

Finally, by adding (10) and (12), we get the following symmetrical result which can be used to find value of z at point P when both $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are prescribed along the curve C.

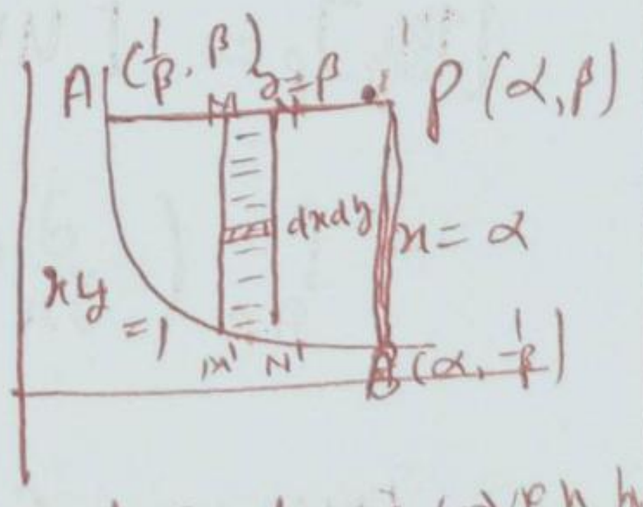
$$[z]_P = \frac{1}{2}\left\{ [uv]_A + [uv]_B \right\} - \int_{AB} uv\,(a\,dy - b\,dx)$$

$$+ \iint_S (f \cdot v)\,dx\,dy - \frac{1}{2}\int_{AB} v\left(\frac{\partial u}{\partial y}dy - \frac{\partial u}{\partial x}dx\right)$$

$$- \frac{1}{2}\int_{AB} u\left(\frac{\partial v}{\partial x}dx - \frac{\partial v}{\partial y}dy\right) \quad \text{--- (13)}$$

By means of whichever of the formulas (10), (12) and (13) is suitable, we may determine the soln of (1) at any point in terms of the prescribed values of u, $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ along the given curve.

<u>Ex</u>: Find the soln, valid when $x, y > 0$, $xy > 1$ of the eqn $\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{x+y}$ st. $z = 0$, $P = \frac{2x}{x+y}$ on the hyperbola $xy = 1$.



<u>Soln</u>: Here $L(z) = f(x, y) = \frac{1}{x+y}$

and $a = b = c = 0$

Hence the adjoint operator M of the operator L is given by

$$M = \frac{\partial^2}{\partial x \partial y}.$$

So the Riemann function can taken as $v = 1$ (1)

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In this case, the values of z and $\frac{\partial z}{\partial x} = p$ are given by

$$z = 0 \quad , \quad \frac{\partial z}{\partial x} = \frac{2y}{x+y} \qquad \text{— (2)}$$

along the curve C, which is hyperbola.

$$xy = 1 \qquad \text{— (3)}$$

Then we wish to find the soln of given eqn at the point $P(\alpha, \beta)$ agreeing with these b.cs. Through P we draw PA $\parallel l$ to the x-axis and cutting $xy=1$ in the point A and PB $\parallel l$ to the y-axis and cutting $xy=1$ in B. Then region enclosed by $xy=1$ $x=\alpha$, $y=\beta$ is denoted by S. We know that

$$[z]_P = [vz]_A - \int_{AB} z\,(a\,dy - b\,dx) + \int_{AB} \left(z\frac{dv}{dy}dy + v\frac{dz}{dx}dx\right) + \iint_S v f\,dx\,dy$$

using (1) and (2) in above

$$\text{or } [z]_P = \int_{AB} \frac{2y}{x+y}\,dx + \iint_S \frac{1}{x+y}\,dx\,dy \qquad \text{— (4)}$$

Now: $$\int_{AB} \frac{2y}{x+y}\,dx = 2\int_A^B \frac{xy}{x^2+xy}\,dx = 2\int_{1/\beta}^{\alpha} \frac{1}{1+x^2}\,dx$$

$$= 2\left\{\tan^{-1}\alpha - \tan^{-1}\frac{1}{\beta}\right\} \qquad \text{— (5)}$$

and $$\iint_S \frac{1}{1+x}\,dx\,dy = \int_{x=\frac{1}{\beta}}^{\alpha} \left\{\int_{y=\frac{1}{x}}^{\beta} \frac{1}{x+y}\,dy\right\} dx \qquad \text{— (6)}$$

$$= \int_{\frac{1}{\beta}}^{\alpha} \left[\log(x+y)\right]_{\frac{1}{x}}^{\beta} dx = \int_{\frac{1}{\beta}}^{\alpha} \left[\log(x+\beta) - \log\left(x+\frac{1}{x}\right)\right] dx$$

$$= \int_{\frac{1}{\beta}}^{\alpha} \left\{\log(x+\beta) - \log(1+x^2) + \log x\right\} dx$$

$$= \left[\left\{\log(x+\beta) - \log(1+x^2) + \log x\right\} x\right]_{\frac{1}{\beta}}^{\alpha} - \int_{\frac{1}{\beta}}^{\alpha} x\left(\frac{1}{x+\beta} - \frac{2x}{1+x^2} + \frac{1}{x}\right) dx$$

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$$= \alpha\left\{\log(\alpha+\beta) - \log(1+\alpha^2) + \log\alpha\right\} - \frac{1}{\beta}\left\{\log\left(\beta+\frac{1}{\beta}\right) - \log\left(1+\frac{1}{\beta^2}\right) + \log\frac{1}{\beta}\right\}$$

$$- \int_{\frac{1}{\beta}}^{\alpha} \left(\frac{2}{1+x^2} - \frac{\beta}{x+\beta}\right) dx$$

$$= \alpha \log\frac{\alpha(\alpha+\beta)}{1+\alpha^2} - \left[2\tan^{-1}x - \beta\log(x+\beta)\right]_{\frac{1}{\beta}}^{\alpha}$$

$$= \alpha\log\frac{\alpha(\alpha+\beta)}{1+\alpha^2} - 2\left(\tan^{-1}\alpha - \tan^{-1}\frac{1}{\beta}\right) + \beta\log\frac{\beta(\alpha+\beta)}{1+\beta^2} \quad -(7)$$

using (5) and (7), (4) reduces to

$$[Z]_P = \alpha\log\frac{\alpha(\alpha+\beta)}{1+\alpha^2} + \beta\log\frac{\beta(\alpha+\beta)}{1+\beta^2} \quad (8)$$

Replacing α and β by x and y resp. in (8), the value of z (i.e. soln of given eqn (1)) at any point (x, y) is given by

$$z = x\log\frac{x(x+y)}{1+x^2} + y\log\frac{y(x+y)}{1+y^2}$$

$$x\left(\frac{1}{x+\beta} - \frac{2x}{1+x^2} + \frac{1}{x}\right) = \frac{x}{x+\beta} - \frac{2x^2}{1+x^2} + 1$$

$$= \frac{(x+\beta)-\beta}{x+\beta} - \frac{(2(1+x^2)-1)}{1+x^2} + 1$$

$$= -\frac{\beta}{x+\beta} + \frac{2}{1+x^2}$$

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$$z\, e^{-3/x^2} = \frac{2}{3}(y+a)^{3/2} e^{-3/x^2} - 2\int \frac{1}{x^2} e^{-3/x^2} \frac{1}{x^3} dx$$

$$z\, e^{-3/x^3} = \frac{2}{3}(y+a)^{3/2} e^{-3/x^2} - \frac{1}{9}\int u e^u du \qquad u = -\frac{3}{x^2}$$

$$z\, e^{-3/x^3} = \frac{2}{3}(y+a)^{3/2} e^{-3/x^2} - \frac{1}{9}\left(u e^u - e^u\right) + b$$

$$z\, e^{-3/x^3} = \frac{2}{3}(y+a)^{3/2} e^{-3/x^2} - \frac{1}{9} \times -\frac{3}{x^2} e^{-3/x^2} + \frac{1}{9} e^{-3/x^2} + b$$

$$\underline{\underline{z}} = \frac{2}{3}(y+a)^{3/2} + \frac{1}{3x^2} + \frac{1}{9} + \underline{\underline{b e^{3/x^2}}}$$

(2) $p^2 x + q^2 y = z$

$$f(x, y, z, p, q) = p^2 x + q^2 y - z = 0 \qquad -(1)$$

$$\frac{dp}{f_x + p f_z} = \frac{dq}{f_y + q f_z} = \frac{dz}{-p f_p - q f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

$$\frac{dp}{-p+p^2} = \frac{dq}{-q+q^2} = \frac{dz}{-2(p^2x+q^2y)} = \frac{dx}{-2px} = \frac{dy}{-2qy} \qquad -(2)$$

$$\Rightarrow \frac{2px\,dp + p^2 dx}{2px(-p+p^2) + p^2(-2px)} = \frac{2qy\,dq + q^2 dy}{2qy(-q+q^2) + q^2(-2qy)}$$

$$\Rightarrow \frac{d(p^2x)}{-2p^2x} = \frac{d(q^2y)}{-2q^2y} \Rightarrow \frac{d(p^2x)}{p^2x} = \frac{d(q^2y)}{q^2y}$$

$$\Rightarrow \log(p^2x) = \log(q^2y) + \log a \Rightarrow p^2x = q^2y \cdot a \qquad -(3)$$

from (1) and (3) $\quad a q^2 y + q^2 y = z \quad$ or $\quad q = \left\{\frac{z}{(1+a)}\right\}^{1/2} \qquad -(4)$

$$\therefore dz = p\,dx + q\,dy \qquad \Rightarrow p = \left(\frac{zq}{(1+a)x}\right)^{1/2}$$

$$dz = \left\{\frac{zq}{(1+a)x}\right\}^{1/2} dx + \left\{\frac{z}{(1+a)y}\right\}^{1/2} dy$$

$$(1+a)^{1/2} z^{-1/2} dz = \sqrt{a}\, x^{-1/2} dx + y^{-1/2} dy$$

$$(1+a)^{1/2} \sqrt{z} = \sqrt{a}\sqrt{x} + \sqrt{y} + b \;//$$

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$f(x, y, z, p, q) = px^5 - 4q^3x^2 + 6x^2z - 2 = 0$ (1)

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

$$\frac{dp}{5px^4 - 8q^3x + 12xz + 6px^2} = \frac{dq}{6qx^2} = \frac{dz}{-px^5 + 12q^3x^2} = \frac{dx}{-x^5} = \frac{dy}{12q^2x^2}$$

taking the second and last fraction,

$$2q\,dq = dy \Rightarrow q^2 = y + a$$

$$q = (y+a)^{1/2} \quad -(2)$$

$$px^5 - 4x^2(y+a)^{3/2} + 6x^2z - 2 = 0$$

$$p = \left(4x^2(y+a)^{3/2} - 6x^2z + 2\right)/x^5 \quad -(3)$$

$\therefore dz = p\,dx + q\,dy$

$$dz = \frac{4}{x^3}(y+a)^{3/2}dx - \frac{6z}{x^3}dx + \frac{2}{x^5}dx + (y+a)^{1/2}dy$$

$$\frac{6z}{x^3}dx + dz = \left[\frac{4}{x^3}(y+a)^{3/2}dx + (y+a)^{1/2}dy\right] + \frac{2}{x^5}dx \quad -(4)$$

Total diff eqn (1) is always integrable. To solve (4), we first proceed to find the integrating factor of the LHS of (4) compare with $M\,dx + N\,dz$.

Here $M = 6z/x^3$ $\quad N = 1$

$$\frac{1}{N}\left(\frac{\partial M}{\partial z} - \frac{\partial N}{\partial x}\right) = \frac{6}{x^3} \qquad IF = e^{\int \frac{6}{x^3}dx} = e^{-3/x^2}$$

Multiply (1) by e^{-3/x^2}

$$\frac{6z}{x^3}\cdot e^{-3/x^2}dx + e^{-3/x^2}dz = \frac{2}{3}\left\{\frac{6}{x^3}(y+a)^{3/2}e^{-3/x^2}dx + \frac{3}{2}(y+a)^{1/2}e^{-3/x^2}dy\right\} + \frac{2}{x^5}e^{-3/x^2}dx$$

$$\therefore d\left(z e^{-3/x^2}\right) = \frac{2}{3}d\left\{(y+a)^{3/2}e^{-3/x^2}\right\} + \frac{2}{x^5}e^{-3/x^2}dx$$

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$\log(xp) = \log(yq) + \lg a^2$

$\Rightarrow xp = a^2 yq$

Solving ① and ② for p, q.

$p = \frac{az}{x}$ and $q = \frac{z}{ay}$

$dz = \frac{az}{x}dx + \frac{z}{ay}dy$

$\frac{1}{z}dz = \frac{a}{x}dx + \frac{1}{ay}dy \Rightarrow \log z = a\lg x + \frac{1}{a}\lg y + \lg m$

$\Rightarrow z = x^a y^{1/a} b.$

Exp $f(p, q, z) = 0 = z^2(p^2 + a^2 + 1) - 1 = 0$

Let $u = x + ay$ $\quad p = \frac{dz}{du}$ $\quad q = a\frac{dz}{du}$

$z^2\left[\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2 + 1\right] = 1$

or $(1+a^2)\left(\frac{dz}{du}\right)^2 = \frac{1-z^2}{z^2}$

or $\pm(1+a^2)^{1/2}\left(\frac{z}{1-z^2}\right)^{1/2} dz = du$

or $\pm\frac{1}{2}(1+a^2)^{1/2}(1-z^2)^{-1/2}(-2z\,dz) = du$

$\Rightarrow \pm(1+a^2)^{1/2}(1-z^2)^{\frac{1}{2}} = u + b$

or $(1+a^2)(1-z^2) = (u+b)^2$

or $(1+a^2)(1-z^2) = (x+ay+b)^2$

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$$\frac{dx}{x^3+3xy^2} = \frac{dy}{y^3+3x^2y} = \frac{dz}{2(x^2+y^2)z}$$

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial x} = 0$$

$$\frac{1}{x} \quad \frac{1}{y}$$

$$2x^2 + 6y^2 - 2y^2 - 6x^2$$

$$\frac{1}{2}(\log(x^2-y^2)) = \log R C$$

$$\frac{1}{x^2-y^2}(2x\,dx - 2y\,dy) = dz$$

$x = u + v$

$y = v - 2u$

$v = \frac{x-y}{2}$

$u = \frac{x+y}{2}$

(1) $$\frac{x\,dx - y\,dy}{x^4+3x^2y^2-y^4-3x^2y^2} = \frac{dz}{2(x^2+y^2)z}$$

$$\frac{x\,dx - y\,dy}{(x^2+y^2)(x^2-y^2)} = \frac{dz}{2(x^2+y^2)z}$$

$$\frac{2x\,dx - 2y\,dy}{x^2-y^2} = \frac{dz}{z}$$

$$\log(x^2-y^2) = \log z + \log c_1$$

1.7.1 $f = z^2 - pqxy = 0$ (1)

$$\frac{dp}{-pqy+2pz} = \frac{dq}{-pqx+2qz} = \frac{dz}{-p(-qxy)-q(-pxy)} = \frac{dx}{qxy} = \frac{dy}{pxy} \quad (2)$$

$$\Rightarrow \frac{x\,dp + p\,dx}{x(-pqy+2pz)+pqxy} = \frac{y\,dq + q\,dy}{y(-pqx+2qz)+pqxy}$$

$$\Rightarrow \frac{x\,dp + p\,dx}{2pxz} = \frac{y\,dq + q\,dy}{2qyz} \Rightarrow \frac{d(xp)}{xp} = \frac{d(yq)}{yq}$$

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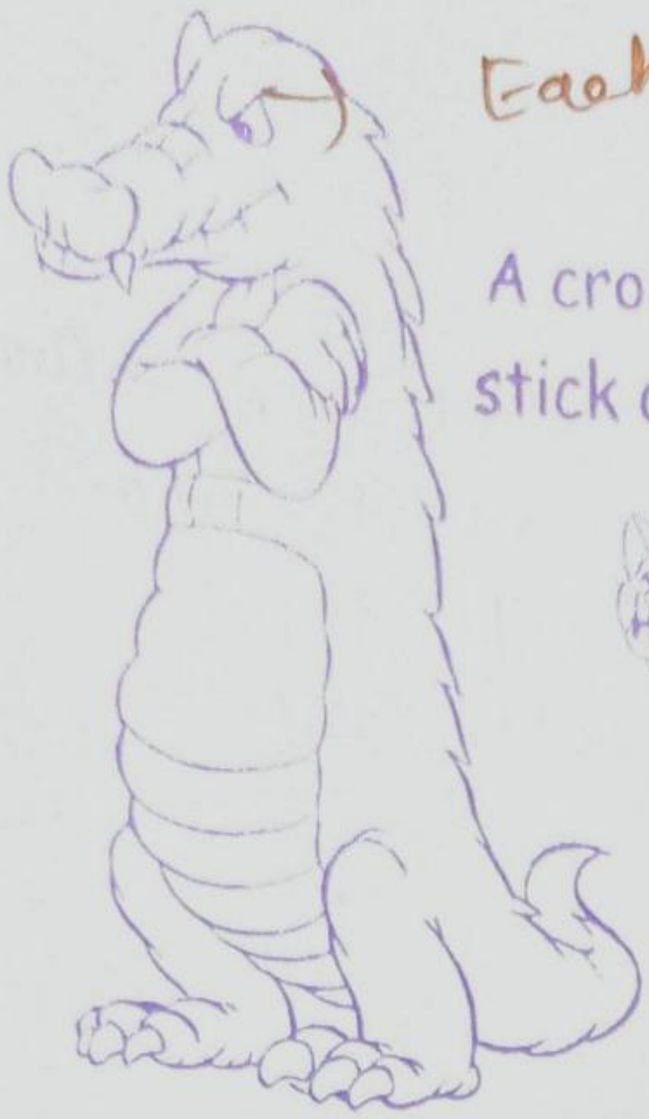
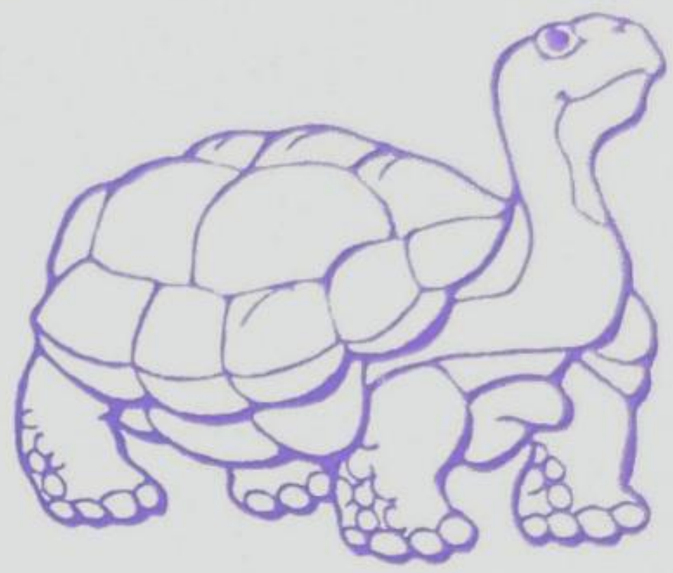
Each finite subset are L-I

DID YOU KNOW?



A crocodile cannot stick out its tongue

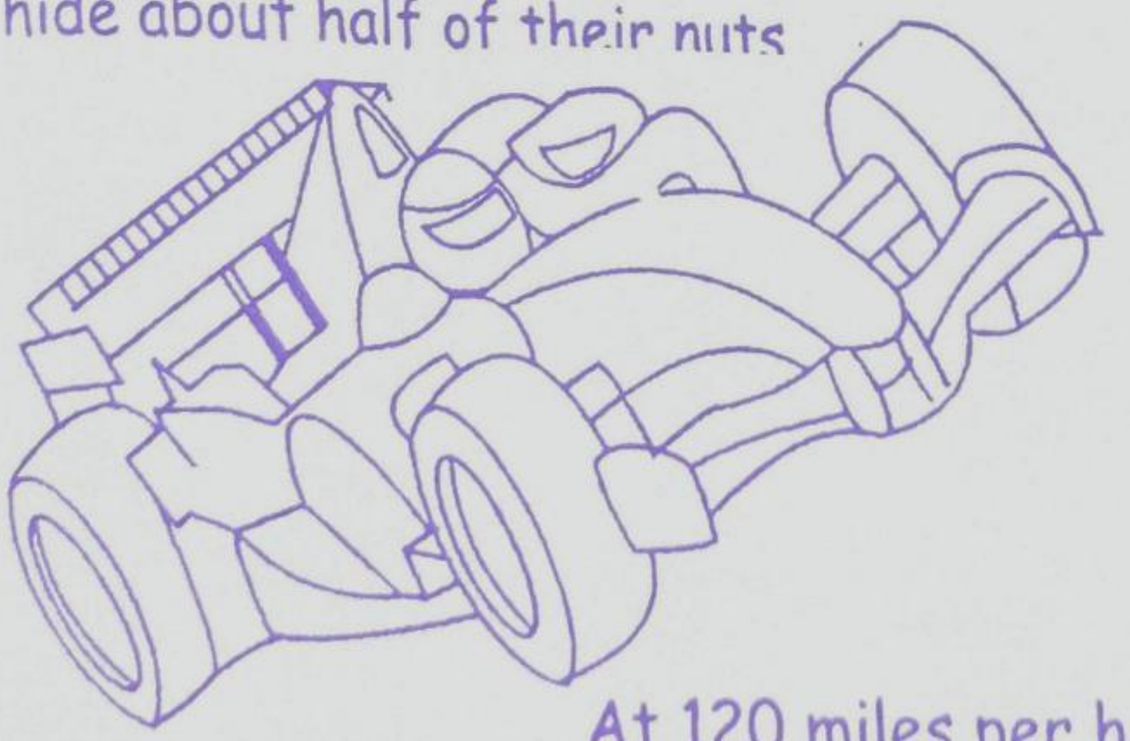
Cat families usually play best in even numbers. Cats and kittens should be acquired in pairs whenever possible!



Turtles can breathe through their butts

Squirrels forget where they hide about half of their nuts

Tigers can see 6 times better at nigh than humans.



At 120 miles per hour, a Formula One car generates so much downforce that it can drive upside down on the roof of a tunnel.

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