A. Bayesian deconvolution method

Avec le in order to deconvolve ... we implement ... la mettre au dbut le implement deconvolution method.

In order to account for the convolution of $\Delta W_{S,n}^{(e)}$ with g_n [mettre la relation de convolution avec les variables], the experimental data $\widetilde{\Delta W}_{S,n}^{(e)}$ is deconvoluted using a Bayesian framework. [Given our experimental situation, supprimer] where g_n is well known but $\widetilde{\Delta W}_{S,n}^{(e)}$ is spoiled by noise, the deconvolution is an ill-posed problem. It cannot be solved by a simple division in Fourier space. Bayesian framework allows to add [prior informations, remplacer par correct ou appropriate information] which are necessary to find a solution [with correct physical properties]. In signal processing, image deconvolution problems have been widely investigated [refs sur deconvolution], [and recent related works aims for example to tackle high dimensionality problem and real-time performances [refs suplec] supprimer]. [In this experiment we implement a specifically designed algorithm for a Wigner function restoration of electrical currents. dj la premire phrase]

[The convolution which links $\Delta W_{S,n}^{(e)}$ and $\widetilde{\Delta W}_{S,n}^{(e)}$ is a linear relation.   supprimer] And the values are naturally discretized by measurement points. So it can be expressed as a matrix product: [dire directement aprs discrtisation on peut exprimer par matrix product]

$$\underline{\widetilde{\Delta W}}_{S,n}^{(e)} = H_n.\underline{\Delta W}_{S,n}^{(e)} + \underline{N}_n \tag{1}$$

[Mettre ici la discrtization naturelle]. where H_n is the convolution matrix. and [the error term Nn represents the experimental noise and the discretization error term.]. the [noise, error term] $\underline{N}_n$ is modeled as Gaussian with measured variance $\underline{V}_e$. The quantity we are looking for is the more likely $\underline{\Delta W}_{S,n}^{(e)}$ knowing $\underline{\widetilde{\Delta W}}_{S,n}^{(e)}$ and $\underline{V}_e$. So we want to find $\underline{\Delta W}_{S,n}^{(e)}$ which maximizes $p\left(\underline{\Delta W}_{S,n}^{(e)}|\underline{\widetilde{\Delta W}}_{S,n}^{(e)}, \underline{V}_e\right)$. This probability can be expressed thanks to Bayes formula as:

$$p\left(\underline{\Delta W}_{S,n}^{(e)}|\underline{\widetilde{\Delta W}}_{S,n}^{(e)}, \underline{V}_e\right) = \frac{1}{p\left(\underline{\widetilde{\Delta W}}_{S,n}^{(e)}\right)} p\left(\underline{\widetilde{\Delta W}}_{S,n}^{(e)}|\underline{\Delta W}_{S,n}^{(e)}, \underline{V}_e\right) p\left(\underline{\Delta W}_{S,n}^{(e)}\right) \tag{2}$$

In this equation $p\left(\underline{\widetilde{\Delta W}}_{S,n}^{(e)}\right)$ is just a normalization constant, it does not depend on $\underline{\Delta W}_{S,n}^{(e)}$. The likelihood term is $p\left(\underline{\widetilde{\Delta W}}_{S,n}^{(e)}|\underline{\Delta W}_{S,n}^{(e)}, \underline{V}_e\right) \propto \exp\left(-\frac{1}{2}\left|\left|\underline{\widetilde{\Delta W}}_{S,n}^{(e)} - H_n.\underline{\Delta W}_{S,n}^{(e)}\right|\right|_{\underline{V}_e}^2\right)$ given by the Gaussian noise of variance $\underline{V}_e$. The last term $p\left(\underline{\Delta W}_{S,n}^{(e)}\right)$ express all prior knowledge on the solution. The information a priori added are that $\Delta W_{S,n}^{(e)}(\omega)$ should take finite finite values and tends to zero when $|\omega|$ increase. To do so we define an envelope function V_f of amplitude v_f and width w such that $V_f\left(\omega\right) = v_f \exp\left(-\frac{\omega^2}{2w^2}\right)$. We finally end up with $p\left(\underline{\Delta W}^{(e)}_{S,n}\right) \propto \exp\left(-\frac{1}{2}\left|\left|\underline{\Delta W}^{(e)}_{S,n}\right|\right|^2_{V_f}\right)$. Maximizing $p\left(\underline{\Delta W}^{(e)}_{S,n}|\underline{\widetilde{\Delta W}}^{(e)}_{S,n}, \underline{V}_e\right)$ is then equivalent to minimizing the criteria $J(\underline{\Delta W}^{(e)}_{S,n}) = \left|\left|\underline{\widetilde{\Delta W}}^{(e)}_{S,n} - H_n.\underline{\Delta W}^{(e)}_{S,n}\right|\right|^2_{\underline{V}_e} + \left|\left|\underline{\Delta W}^{(e)}_{S,n}\right|\right|^2_{V_f}$. Assuming that V_f follow an Inverse-Gamma distribution of parameters α_f and β_f, the algorithm [ref JMAP] can update its values at each minimization step and optimize jointly V_f and $\underline{\Delta W}^{(e)}_{S,n}$. The Wigner function estimated should also verify the box-constraint given by Pauli exclusion principle and Cauchy-Schwartz inequality [refs papier lyon]:

$$0 \leq f_{\mathrm{Fermi-Dirac}}(\omega) + \underline{\Delta W}^{(e)}_{S,0}(\omega) \leq 1 \tag{3}$$

$$\left|\underline{\Delta W}^{(e)}_{S,n}(\omega)\right|^2 \leq \left|\underline{W}^{(e)}_{S,0}(\omega - n\pi f)\underline{W}^{(e)}_{S,0}(\omega + n\pi f)\right| \tag{4}$$

$$\left|\underline{\Delta W}^{(e)}_{S,n}(\omega)\right|^2 \leq \left|\left(1 - \underline{W}^{(e)}_{S,0}(\omega - n\pi f)\right)\left(1 - \underline{W}^{(e)}_{S,0}(\omega + n\pi f)\right)\right| \tag{5}$$

The implemented algorithm looks for a minimum of the criteria $J(\underline{\Delta W}^{(e)}_{S,n})$ inside the box-constraint thanks to a Gradient Projected Descent method [refs gradient projected]. It consists of first initializing the solution inside the allowed solution ensemble. And than descent along the least gradient direction. While performing descent gradient steps, we enforce the solution to stay in the box by projected the gradient and we update V_f.

The detailed algorithm is presented in the following.

Algorithm

1. compute Cauchy-Schwartz bounds $\mathrm{MAX}\underline{\Delta W}^{(e)}_{S,n}(\omega)$

2. choose amplitude v_f and width w of $V_f(\omega)$ envelope.

3. compute $\underline{\Delta W}^{(e)}_{S,n} = (H'V_e^{-1}H + V_f^{-1})^{-1}H'V_e^{-1}\underline{\widetilde{\Delta W}}^{(e)}_{S,n}$

4. project under Cauchy-Schwartz bounds $\underline{\Delta W}^{(e)}_{S,n}(\omega) := \min(\underline{\Delta W}^{(e)}_{S,n}(\omega), \mathrm{MAX}\underline{\Delta W}^{(e)}_{S,n}(\omega))$ and $\underline{\Delta W}^{(e)}_{S,n}(\omega) := \max(\underline{\Delta W}^{(e)}_{S,n}(\omega), -\mathrm{MAX}\underline{\Delta W}^{(e)}_{S,n}(\omega))$

5. repeat

5.1. compute gradient $\nabla\underline{\Delta W}^{(e)}_{S,n} = -H'V_e^{-1}(\widetilde{\underline{\Delta W}}^{(e)}_{S,n} - H\underline{\Delta W}^{(e)}_{S,n}) + V_f^{-1}\underline{\Delta W}^{(e)}_{S,n}$

5.2. project gradient $\mathrm{P}\nabla\underline{\Delta W}^{(e)}_{S,n}(\omega)$

if $|\underline{\Delta W}^{(e)}_{S,n}(\omega)| \geq \mathrm{MAX}\underline{\Delta W}^{(e)}_{S,n}(\omega)$ and $\underline{\Delta W}^{(e)}_{S,n}(\omega) * \nabla\underline{\Delta W}^{(e)}_{S,n} \leq 0$

than $\mathrm{P}\nabla\underline{\Delta W}^{(e)}_{S,n}(\omega) = 0$

else $\mathrm{P}\nabla\underline{\Delta W}^{(e)}_{S,n}(\omega) = \nabla\underline{\Delta W}^{(e)}_{S,n}(\omega)$

5.3. compute furthest accessible displacement d_∞ along $\mathrm{P}\nabla\underline{\Delta W}^{(e)}_{S,n}$ direction

for all $\mathrm{P}\nabla\underline{\Delta W}^{(e)}_{S,n}(\omega) \neq 0$, $d_\infty = \min\left(\frac{\mathrm{MAX}\underline{\Delta W}^{(e)}_{S,n}(\omega) - \underline{\Delta W}^{(e)}_{S,n}(\omega)}{\mathrm{P}\nabla\underline{\Delta W}^{(e)}_{S,n}(\omega)}\right)$

5.4. compute the optimum displacement d_0 along $\mathrm{P}\nabla\underline{\Delta W}^{(e)}_{S,n}$ direction

$d_0 = (\mathrm{P}\nabla\underline{\Delta W}^{(e)\prime}_{S,n}\mathrm{P}\nabla\underline{\Delta W}^{(e)-1}_{S,n})\left(\mathrm{P}\nabla\underline{\Delta W}^{(e)\prime}_{S,n}H'V_e^{-1}H\mathrm{P}\nabla\underline{\Delta W}^{(e)}_{S,n} + \mathrm{P}\nabla\underline{\Delta W}^{(e)\prime}_{S,n}V_f^{-1}\mathrm{P}\nabla\underline{\Delta W}^{(e)}_{S,n}\right)$

5.5. compute the descent gradient $\underline{\Delta W}^{(e)}_{S,n}(\omega) := \underline{\Delta W}^{(e)}_{S,n}(\omega) + \min(d_0, d_\infty)\mathrm{P}\nabla\underline{\Delta W}^{(e)}_{S,n}(\omega)$

5.6. **until** criteria is minimized