5A ModIA

Report on Poisson processes and application

Modelling the Ruin of Trees under Climate Hazards

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Contents

List of Figures

1 Introduction

Our ecosystem is being jeopardized and risks collapsing due to natural or human-made hazards. Thus, it is extremely important to examine the current situation and take action accordingly. In this report, we focus on forestry, where climate variations can impact tree growth in diverse ways. Instances of heatwaves and droughts have the potential to modify tree reserves, affecting their overall growth and elevating the risk of mortality. The recent accumulation of stress in forests from droughts and heatwaves may reduce their resilience to future extreme events. The resilience and capacity to withstand adverse conditions in forests have been linked to their ability to accumulate carbohydrate reserves and possess a branching structure that rapidly generates an expansive leaf surface, thereby enhancing potential productivity[10].

Existing literature on ecosystem tipping points is useful in raising awareness of the dangers of climate change but often lacks quantitative estimates[7], [8], [2], which makes these results difficult to include in decision chains.

This project aims to provide precise answers to "when" and "how intense" are the hazards affecting trees. Particularly, how many days of drought during heatwaves and how often they both recur can cause trees to die. To achieve this, we develop growth models with existing observations, to obtain quantitative information on the carbohydrate reserves according to damages caused by droughts and heatwaves. To do so, we get inspiration from the existing work in the econometrics literature [4], [5], that describes ruin models used in insurance and finance to establish premium rates. These papers provide all the tools for decision-making but has never been transposed to environmental sciences.

In Section 2 we introduce a growth/ruin model for trees based on a Cramér-Lundberg ruin model, and discuss the interpretation of its parameters. In Section 3 we compute the theoretical ruin probability of our model. In Section 4 we evaluate the net profit condition. The numerical results, interpretations and comparison with theoretical results are developed in Section 5.

2 Model

2.1 Cramér-Lundberg ruin model

The insurance sector employs statistical models to establish premium rates. The aim is to strike a balance among rival companies by proposing the cheapest premium while reducing the risk of financial collapse. These statistical models derive from the basic Cramér-Lundberg model, which can be expressed as

$$R(t) = R_0 + pt - S(t) \tag{1}$$

where $R(t)$ is the capital at time t, $R_0 > 0$ is the initial capital, $p > 0$ is the premium rate per year t, and $S(t) \geq 0$ are the losses to hazards that occur up to time t.
The losses $S(t)$ are represented as a random sum of random variables:

$$S(t) = \sum_{k=1}^{N(t)} X_k \tag{2}$$

where $N(t)$ is a Poisson random variable that accounts for the number of hazards occurring up to time t, and X_k are independent and identically distributed non-negative random variables that account for the cost of each hazard.

$$N(t) \sim \mathcal{P}(\lambda)$$

with
 $\lambda > 0$: the average rate of occurrence of the hazards.

2.2 A ruin model for trees

This section explain the model presented in the paper [1].
Where, to create a tree growth model that explicitly incorporates a climate hazard $S(t)$, they modify the Cramér-Lundberg model presented in equation Eq. (1). In our context, $R(t)$ represents non-structural carbohydrates (referred to as reserves) crucial for tree growth during the vegetative period.

They assume that tree resources are bounded by an optimal value R_{max}. Also, depending on the tree species, the impact of a drought hazard on tree growth extends beyond the immediate year of the hazard. They also use the term "legacy effects" to refer to this phenomenon, because it suggests that the consequences persist into the subsequent year. The work in [3] provides evidence that supports this observation. The Net Primary Production (NPP) that represents the amount of organic matter that plants produce through photosynthesis, will decrease due to the "legacy effect", which they assume influence how trees allocate resources to carbohydrate reserves in the year following the hazard. Hence, the yearly NPP allocated to reserves $p(t)$ depends on the climate hazard that occurred during the previous year $S(t-1)$:

$$p(t) = p_0 - B \cdot S(t-1) \tag{3}$$

with
p_0: the optimum average yearly NPP
$B \leq 0$: a memory factor.

Base on these observations, a ruin model for trees is introduced:

$$R(t) = \min\left((1-b) \cdot R(t-1) + p(t) - S(t),\ R_{\max}\right) \tag{4}$$

Where $b \leq 0$ is the fraction of resources at time $t-1$ devoted to growth. We assume that ruin is reached when $R(t) = 0$.

The damages $S(t)$ are defined with

$$S(t) = A_h \cdot \sum_{k=1}^{N(t)} X_k \tag{5}$$

A_h is a constant that interprets how much damage the climate hazard intensity X_k will cause to a specific tree species.

$X_k \sim \text{Pareto}(\sigma, \xi)$
with
$\sigma > 0$: scale parameter
ξ: shape parameter (states how fast extremes grow)

$N(t)$ is the number of hot/dry days during year t; it follows a Poisson distribution with parameter λ.

They also had to take into account that hazards may not manifest annually. By considering that their occurrences follow an exponential distribution with a parameter Λ. Consequently, the intervals between arrivals follow a Poisson distribution, with a mean value θ equal to $1/\Lambda$, representing the average time between hazard occurrences.

So, equation Eq. (5), is a representation of the damages when climate hazards occur.

3 Ruin probability

In this Section, we compute the ruin probability of the model which is the probability that a tree dies after a certain time T.

3.1 Ruin probability of a very simplified model

First, to ensure the calculation of ruin probability, we make further assumptions and slightly modify the model. Thus, we suppose that $B = 0$, $R(t-1)$ is already known and $\forall k, X_k \sim \mathcal{E}(\theta)$ with θ a known parameter.

Then, the ruin probability of the new model is given by

$$\begin{aligned}
\mathbb{P}(R(t) \leq 0) &= \mathbb{P}((1-b)\cdot R(t-1) + p(t) - S(t) \leq 0) \\
&= \mathbb{P}(r_{t-1} \leq S(t)) \\
&= \mathbb{P}\left(r_{t-1} \leq A_h \cdot \sum_{k=1}^{N(t)} X_k\right) \\
&= \mathbb{P}\left(\frac{r_{t-1}}{A_h} \leq \sum_{k=1}^{N(t)} X_k\right) \\
&= \mathbb{P}\left(\frac{r_{t-1}}{A_h} \leq \sum_{k=1}^{n} X_k | N(t) = n\right) \cdot \mathbb{P}(N(t) = n)
\end{aligned}$$

Yet, $N(t) \perp\!\!\!\perp X_k, \quad \forall k \in [1, n]$
Then,

$$\begin{aligned}
\mathbb{P}(R(t) \leq 0) &= \mathbb{P}\left(\frac{r_{t-1}}{A_h} \leq \sum_{k=1}^{n} X_k\right) \cdot \mathbb{P}(N(t) = n) \\
&= \left(1 - \mathbb{P}\left(\sum_{k=1}^{n} X_k \leq \frac{r_{t-1}}{A_h}\right)\right) \cdot \mathbb{P}(N(t) = n)
\end{aligned}$$

However, $\sum_{k=1}^{n} X_k = \mathcal{Z} \sim \Gamma(n, \theta)$.
Hence,

$$\begin{aligned}
\mathbb{P}(R(t) \leq 0) &= \frac{e^{-\lambda} \cdot \lambda^n}{n!} \cdot \left(1 - \mathbb{P}\left(\mathcal{Z} \leq \frac{r_{t-1}}{A_h}\right)\right) \\
&= \frac{e^{-\lambda} \cdot \lambda^n}{n!} \cdot \left(1 - F_{\mathcal{Z}}\left(\frac{r_{t-1}}{A_h}\right)\right)
\end{aligned}$$

with $F_{\mathcal{Z}}$ the distribution function of a $\Gamma(n, \theta)$.

Eventually,

$$\mathbb{P}(R(t) \leq 0) \sim \frac{e^{-\lambda} \cdot \lambda^n}{n!} \quad \text{when } t \to +\infty$$

3.2 Ruin probability of the complete model

Here, we compute the ruin probability of the full trees model. We assume that $b = 0$ because b is always near to zero and that $R(t) \leq R_{max}$.

Then, the ruin probability of the model is given by:

$$\begin{aligned}\mathbb{P}(R(t) \leq 0) &= \mathbb{P}(R(t-1) + p(t) - S(t) \leq 0) \\ &= \mathbb{P}(R(t-2) + p(t) - S(t) + p(t-1) - S(t-1) \leq 0) \\ &= \mathbb{P}\left(R_0 + \sum_{i=1}^{t}(p(i) - S(i)) \leq 0\right)\end{aligned}$$

Yet,

$$p(t) - S(t) = p_0 - B \cdot S(t-1) - A_h \cdot \sum_{k=1}^{N(t)} X_k, \quad \forall t \in [0, T]$$

Then,

$$\begin{aligned}\mathbb{P}(R(t) \leq 0) &= \mathbb{P}\left(R_0 + p_0 \cdot t - \sum_{i=1}^{t}(S(i) + B \cdot S(i-1)) \leq 0\right) \\ &= \mathbb{P}\left(R_0 + p_0 \cdot t - A_h \cdot \sum_{i=1}^{t}\left(\sum_{k=1}^{N(i)} X_k + B \cdot \sum_{k'=1}^{N(i-1)} X_{k'}\right) \leq 0\right) \\ &= \mathbb{P}\left(R_0 + p_0 \cdot t - A_h \cdot \left(\sum_{k=1}^{N(t)}\sum_{i=1}^{t} X_k + B \cdot \sum_{k'=1}^{N(t-1)}\sum_{i=1}^{t} X_{k'}\right) \leq 0\right) \\ &= \mathbb{P}\left(R_0 + p_0 \cdot t - A_h \cdot \left((1+B) \cdot \sum_{k=1}^{N(t-1)}\sum_{i=1}^{t} X_k + \sum_{i=1}^{t} X_{N(t)}\right) \leq 0\right)\end{aligned}$$

If $B = 0$ and $Y_k(t) = A_h \cdot \sum_{i=1}^{t} X_k$ then,

$$\mathbb{P}(R(t) \leq 0) = \mathbb{P}(R_0 + p_0 \cdot t - \sum_{k=1}^{N(t)} Y_k(t) \leq 0) \tag{6}$$

Yet, even if the $Y_k(t)$ are $\perp\!\!\!\perp$, we do not know their law. Then, we assume that $\forall k, Y_k(t) \sim Pareto(\beta, \alpha)$ with β and α known parameters.

Hence, the ruin probability is given by

$$\phi(R_0) \sim \left(\frac{\beta}{R_0}\right)^{\alpha-1} \quad \text{when } t \to +\infty[1] \tag{7}$$

4 Net Profit Condition

In this Section, we take the same assumptions that allowed us to define the ruin probability in Section 3.1: $B = 0$ and $Y_k(t) = \sum_{i=1}^{t} X_k$:

$$R(t) = R_0 + p_0 \cdot t - \sum_{k=1}^{N(t)} Y_k(t)$$

The expected value of $R(t)$ is given by

$$\mathbb{E}[R(t)] = R_0 + p_0 \cdot t - \mathbb{E}[\sum_{k=1}^{N(t)} Y_k(t)]$$

Yet, using Wald identity

$$\mathbb{E}[R(t)] = R_0 + p_0 \cdot t - \mathbb{E}[N(t)]\mathbb{E}[Y_1(t)]$$

We also know that $\mathbb{E}[N(t)] = \lambda.t$
Thus,

$$\mathbb{E}[R(t)] = R_0 + p_0 \cdot t - \lambda t(\mathbb{E}[Y_1(t)])$$

However, $Y_1(t) \sim \text{GPD}(\sigma, \xi)$

$$\begin{aligned}\mathbb{E}[R(t)] &= R_0 + p_0 \cdot t - \lambda t(\mu + \frac{\sigma}{\xi - 1}) \\ &= R_0 + (p_0 - \lambda(\mu + \frac{\sigma}{\xi - 1})) \cdot t\end{aligned}$$

In this case, to assure a profit, we must have,

$$p_0 - \lambda(\mu + \frac{\sigma}{\xi - 1}) > 0$$

Thus,

$$p_0 > \lambda(\mu + \frac{\sigma}{\xi - 1}) \tag{8}$$

This is what we refer to as "the net profit condition".

5 Experiments

In this Section we create our own synthesized data by simulating trajectories with defined parameters. This data is next used to test our model Eq. (4). Then, we add some suggestions that help make our results more realistic.

5.1 Synthesized data

We simulate 10^4 trajectories and illustrate the time variations fluctuations of our model Eq. (4) using:

- Initial tree parameters: $S_0 = 0$ (initial damages) and R_0 (initial reserves).
- Growth tree parameters: $p_0 = 5$, $b = 0.05$, $A_h = 0.6$ and $R_{max} = 100$.
- GPD distribution parameters: $\sigma = 0.1$, $\xi = -0.2$ and $u = 1$ (threshold).
- Poisson distribution parameters: $\lambda = 10$ days and $\Lambda = 5$ years (hazard arrivals).

The model is tested with two different values of the memory factor: $B = 0$ (cash) and $B = 1.5$ (credit).

5.1.1 Results

Figure 1 shows the 0th, 5th, 50th and 95th quantile of the average of R.

For $B = 0$, none of the trees die. For $B = 1.5$ more than 5% of the trees die.
However, we notice that the trees experience the same amount of damage which is not realistic because trees that suffer all the damage at the current year of drought ("cash") are supposed to incur more than the others ("credit").

This behavior is well represented in article [1]. Thus we have to change our damage modelling to a more realistic one.

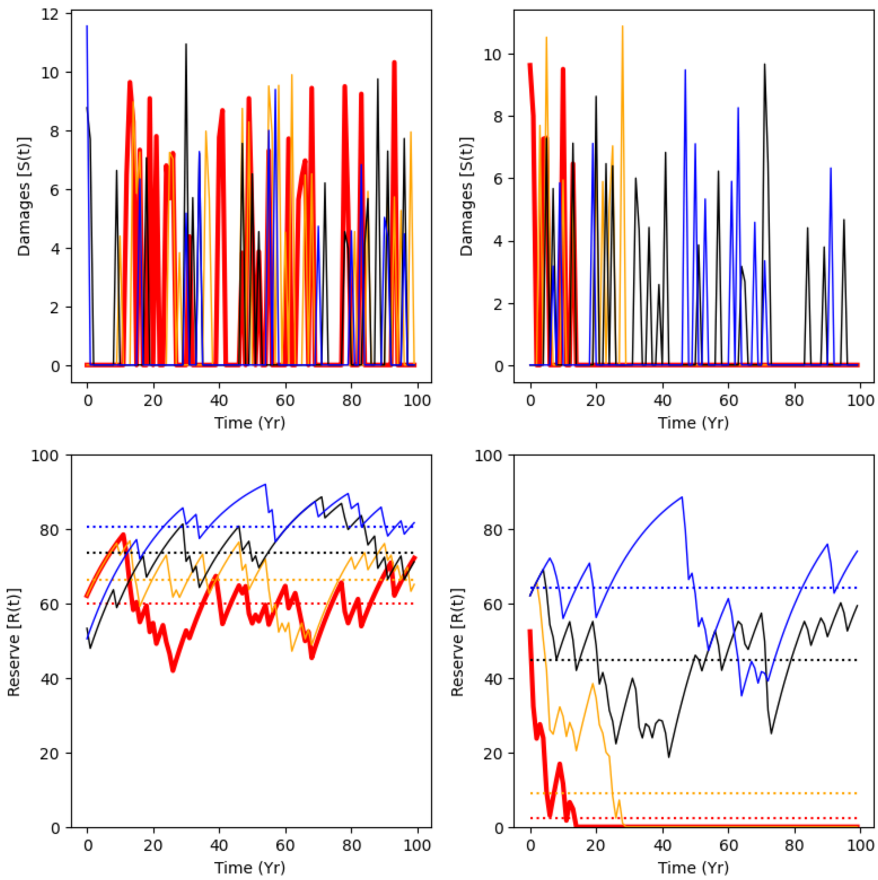

Figure 1: Sample of four simulations of S(t) (upper panels) and R(t) (lower panels) for $B = 0$ ("cash": left panels) and $B = 1.5$ ("credit": right panels) and $A_h = 0.6$. The black lines are the trajectories of S and R that achieve the median of the average of R (among 10^4 simulations). The blue lines are for the trajectories of the 95th quantile of R. The orange lines are for the 5th quantile trajectories of R. The red lines are for trajectories with the lowest average of R. The horizontal dotted lines indicate the mean R(t) values of the trajectories.

5.2 More realistic damages

In reality, trees do not behave in the same way in response to drought. We can simplify the concept by drawing a parallel with an insurance system. If the parameter B equals to 0, trees release a "cash" reserve, resulting in a large reserve for the current year. However, if B is greater than zero, trees provide a "credit" towards the next year, leading to a reduced reserve for the current year, which affects the ability to use the reserve in the following year.
Hence, the parameter A_h has to be related to the parameter B.

5.2.1 Changes in the model

According to the article [1], "the values of B are chosen so that the average value of damages (i.e., the expected value of $(1+B)S(t)$) is a constant. This constant gives the scale of the impact parameter A_h." Thus, having a constant parameter A_h for different values B is a nonsense and the parameter A_h can be found as follow:

$$(1+B) \cdot S = K$$

And,

$$S = A_h \cdot L$$

Where K, S and L are constants (S refers to damages and L refers to $\sum X_k$).

Yet, we know according to the article and our results that for $B = 1$, $A_h = 0.6$.

Then,

$$K = (1+B) \cdot A_h \cdot L = 2 \cdot 0.6 \cdot L = 1.2 \cdot L$$

Thus,

$$A_h = \frac{K}{L \cdot (1+B)} = \frac{1.2 \cdot L}{L \cdot (1+B)} = \frac{1.2}{1+B} \tag{9}$$

5.2.2 Results

With this modified model, we obtain results (Figure 2) similar to those of the article [1]. As expected the damages for $B = 0$ are significantly higher than those for $B = 1.5$. This is more realistic because trees employing a 'cash' strategy incur all damages in the current year, while trees following a 'credit' strategy defer and spread the damage into the subsequent year. We can also observe that the average of R for $B = 0$ and $B = 1.5$ is roughly the same for each displayed tree strategy.

A description of the influence of each parameter from this model can be found in Appendix A.

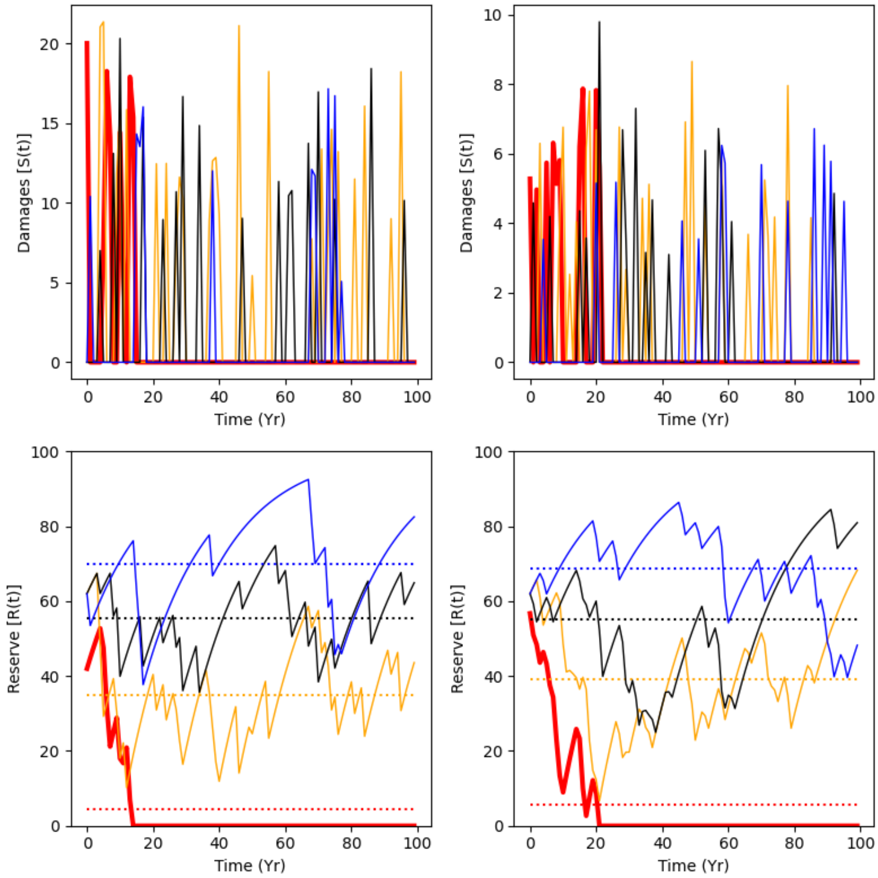

Figure 2: Sample of four simulations of $S(t)$ (upper panels) and $R(t)$ (lower panels) for $B = 0$ ("cash": left panels) and $B = 1.5$ ("credit": right panels) and A_h function of B. The black lines are the trajectories of S and R that achieve the median of the average of R (among 10^4 simulations). The blue lines are for the trajectories of the 95th quantile of R. The orange lines are for the 5th quantile trajectories of R. The red lines are for trajectories with the lowest average of R. The horizontal dotted lines indicate the mean R(t) values of the trajectories.

5.3 A climate change-informed model

The previous adjustments have made the model more realistic. However, to make the model even more consistent with the current context, climate change must be taken into account.

5.3.1 Taking climate change into account

In order to make the model realistic even further, we initially modify the previous model to incorporate climate change within the current year. Then, we adjust the number of drought days based on the years (as the years progress, the annual number of drought days increase). In our model, this modification is represented by an increasing of λ over time, unlike the previously constant λ.

5.3.2 Results

In Figure 3 we chose λ based on Figure 6. Our choices were even slightly more optimistic in comparison to the one in Appendix B: 2.5 days of droughts per decade instead of 2.7. We started with $\lambda = 0$, and then we increased by 2.5 days every 10 years, which make a slope of $\frac{2.5*A_h}{10}$. We can observe that approximately 70% of trees die before 100 years.

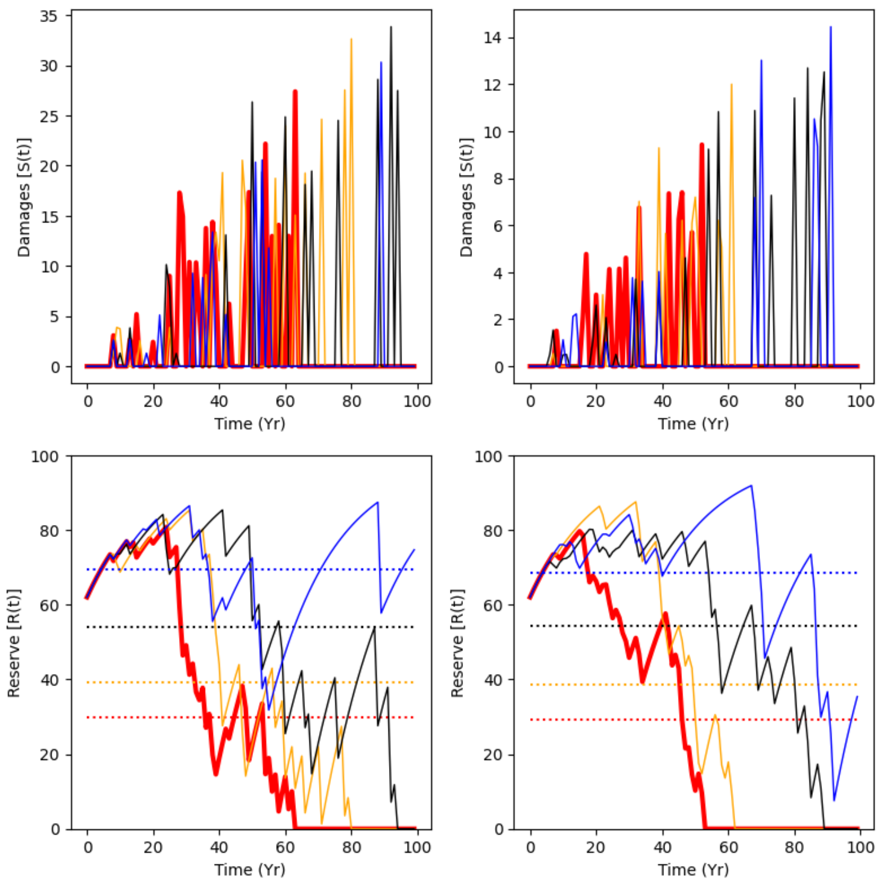

Figure 3: Sample of four simulations of S(t) (upper panels) and R(t) (lower panels) for $B = 0$ ("cash": left panels) and $B = 1.5$ ("credit": right panels), A_h function of B and λ is 0 at $t = 0$ and then increases by 2.5 every 10 years. The black lines are the trajectories of S and R that achieve the median of the average of R (among 10^4 simulations). The blue lines are for the trajectories of the 95th quantile of R. The orange lines are for the 5th quantile trajectories of R. The red lines are for trajectories with the lowest average of R. The horizontal dotted lines indicate the mean R(t) values of the trajectories.

5.3.3 A model that fully takes into account climate change

However, climate change doesn't just impact the drought within the current year. It also has an effect on the frequency of drought years, increasing the frequency over time. To accommodate this change, we adjust the Λ based on time. As time progresses, drought years become more frequent. Therefore, drought years follow an inhomogeneous Poisson process with a Λ parameter.

5.3.4 Results

In Figure 4, Λ starts equal to 15 and decreases by 2 every 10 years. We can clearly observe that damages occur more frequently which is due to more frequent heatwaves. We can also notice that over 95% of trees die before 80 years. It is worth noting that these figures are underestimated, considering that since 2000, there have been 26 [6] heatwaves in France (22 since 2010 [9]).

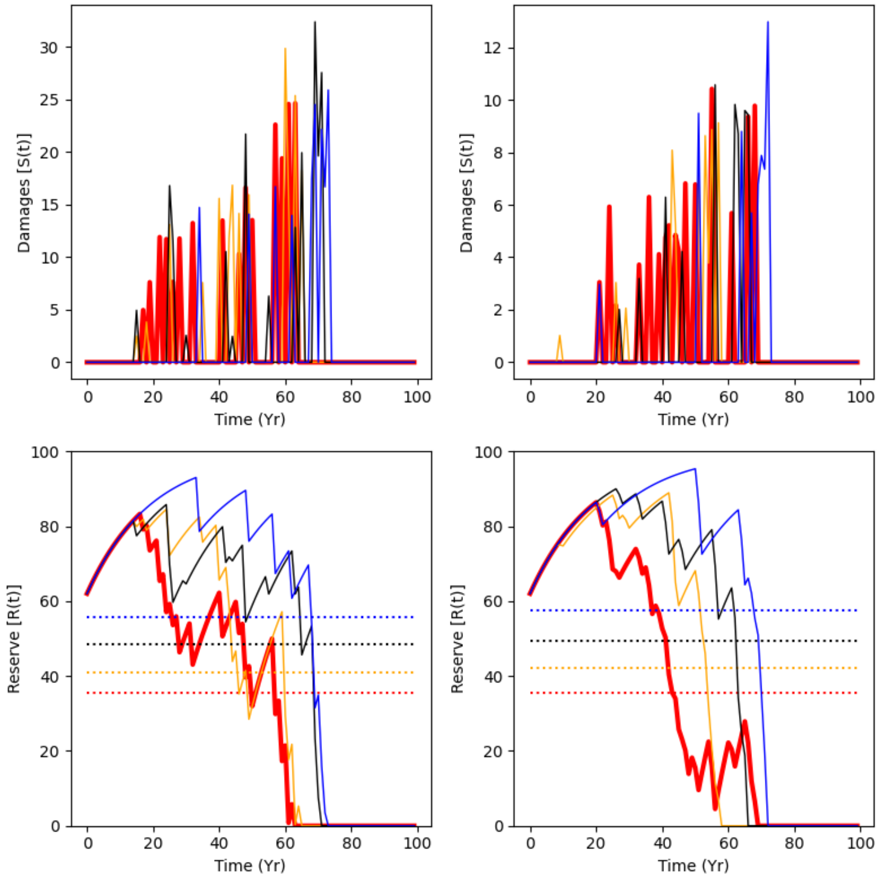

Figure 4: Sample of four simulations of S(t) (upper panels) and R(t) (lower panels) for $B = 0$ ("cash": left panels) and $B = 1.5$ ("credit": right panels), A_h function of B, λ is 0 at $t = 0$ and then increases by 2.5 every 10 years and Λ is 15 at $t = 0$ and then decreases by 2 every 10 years. The black lines are the trajectories of S and R that achieve the median of the average of R (among 10^4 simulations). The blue lines are for the trajectories of the 95th quantile of R. The orange lines are for the 5th quantile trajectories of R. The red lines are for trajectories with the lowest average of R. The horizontal dotted lines indicate the mean R(t) values of the trajectories.

6 Conclusion

This report presents ruin models to assess the susceptibility of ecosystems, focusing on the potential collapse of trees due to extreme events induced by climate change: droughts and heatwaves.

All of these models were inspired by the Cramér-Lundberg model, while making different changes and assumptions such as the dependence of $R(t)$ and $S(t-1)$. The model presented in [1] helps evaluate the risks and uncertainties related to how well ecosystems can survive for a long period. Yet, the models presented in Section 5.3 are more realistic and take into account the climate changes conditions that our planet has been suffering from over the last few decades.

Regardless the numerous natural/human made hazards that were not taken into consideration by our models, the results shown in Section 5 confirm that we are witnessing a major danger caused by climate change hazards as we obtain a ruin probability of 100% with parameters that simulate the climate change phenomenon, meaning that all trees die before they reach 80 years. Hence, accepting climate change seems to be a too risky gamble and we have to act as soon as possible to reduce climate change as much as we can. After all, even if people are starting to be aware of the outsize danger that we are facing, we still have to work more for the sake of our planet.

7 Appendices

A Influence of each parameter on the ruin time

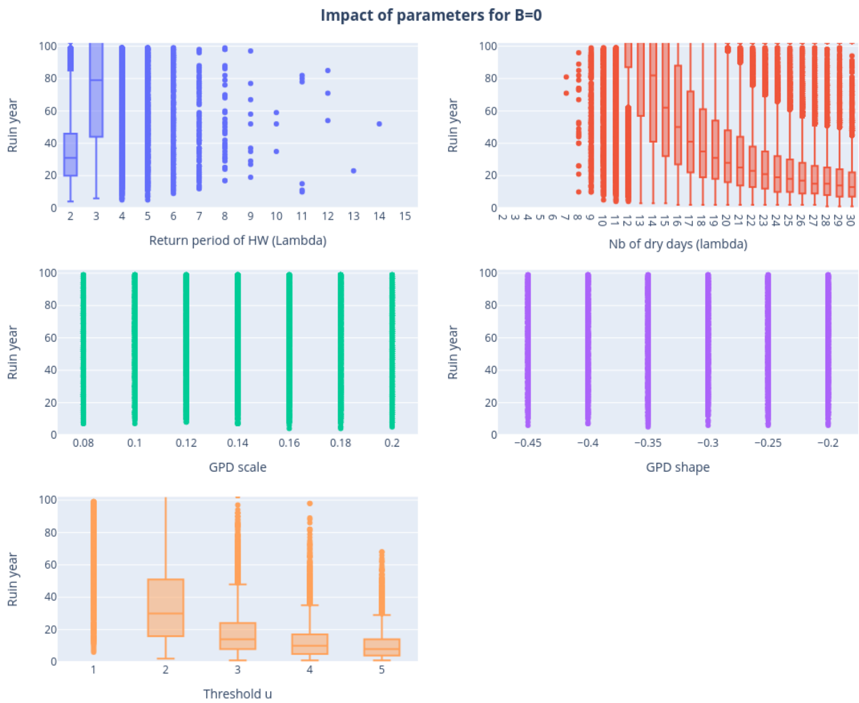

Figure 5: Influence of parameters on ruin time with $B = 0$ with 10^4 simulations per value of each parameter

Figure 5 emphasizes that the system can transition from a state of "no ruin" to probable ruin within a century, with relatively minor changes. The influence of the scale parameter σ and the shape parameter ξ of the Pareto law appears to be relatively weak, as depicted in the second line of Figure 5. The GPD threshold u, which represents the intensity of hot days, the number of dry days λ and the return period of heatwave Λ significantly affect the ruin time of trees. To save the trees, there should be fewer than 10 days of drought per heatwave, and a heatwave should occur with a return period of at least 5 years, without an increase in the intensity of hot days.

The results are the same with $B = 1.5$.

B Number of anormaly hot days across years

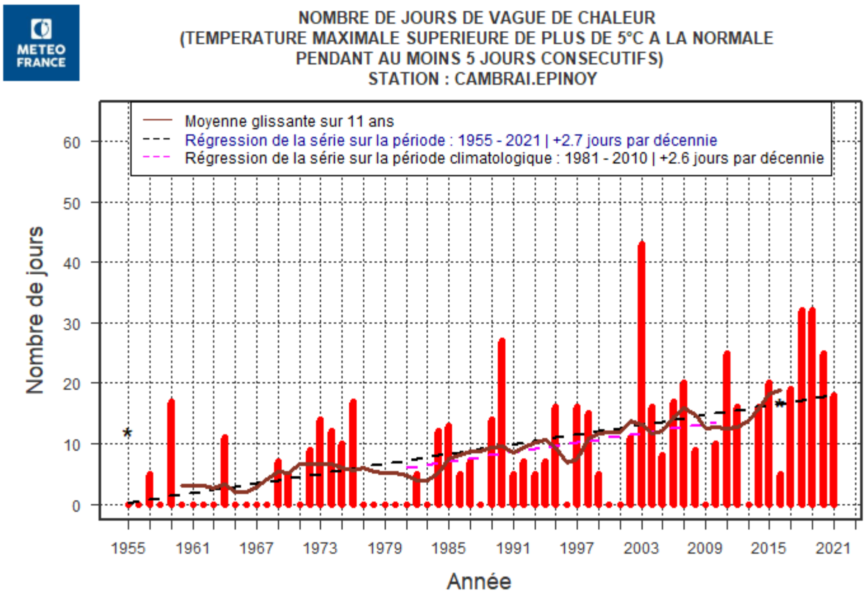

Figure 6: Number of anormaly hot days across years. Maximum temperature more than 5 degrees above normal 1981 – 2010 for Beauvais.Tille station

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