

# Méthodes Itératives

Carola Kruse et Ronan Guivarch

Cours 1, 03/10/2023  
Méthodes Multigrilles

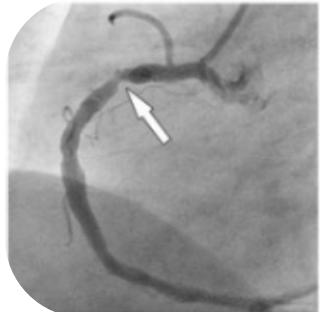
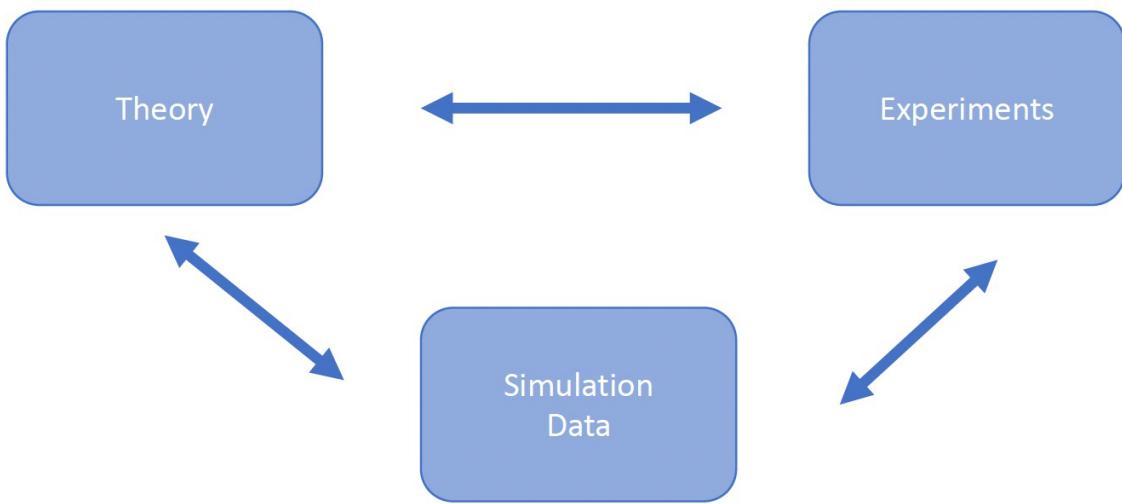
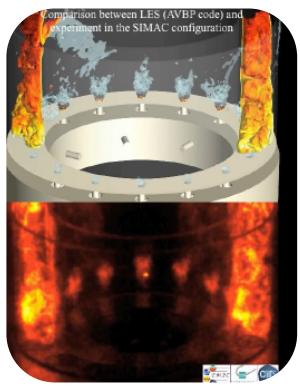
# Organisation

---

- 3 séances de Cours « Méthodes Multigrilles »
- 2 séances de Cours « Résolution des problèmes de point-selle»
- 2 TP qui vont mettre en pratique la méthode multigrilles

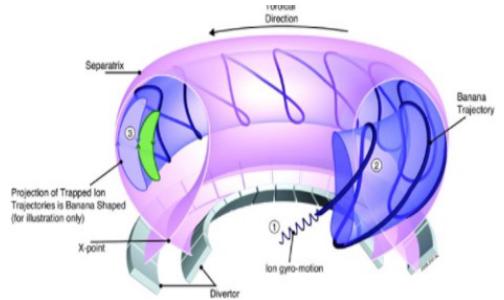
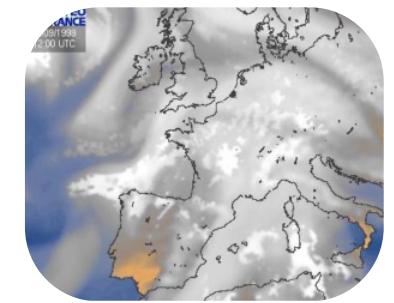
# Numerical simulation

Third pillar for the development of scientific discoveries at the same level as theory and experiments.



Many problems come down to **solving** a

- **Linear system** of type  $Ax = b$
- An eigenvalue problem of type  $Ax = \lambda b$



# What kind of solvers exists?

---

## Direct methods

- Accurate to machine precision
- High computational cost
- Efficient, parallel sparse direct solvers exist

## Iterative methods

- Classical relaxation schemes
- Preconditioning
- Multigrid methods
- Accurate to a predefined precision
- (Generally) lower computational cost



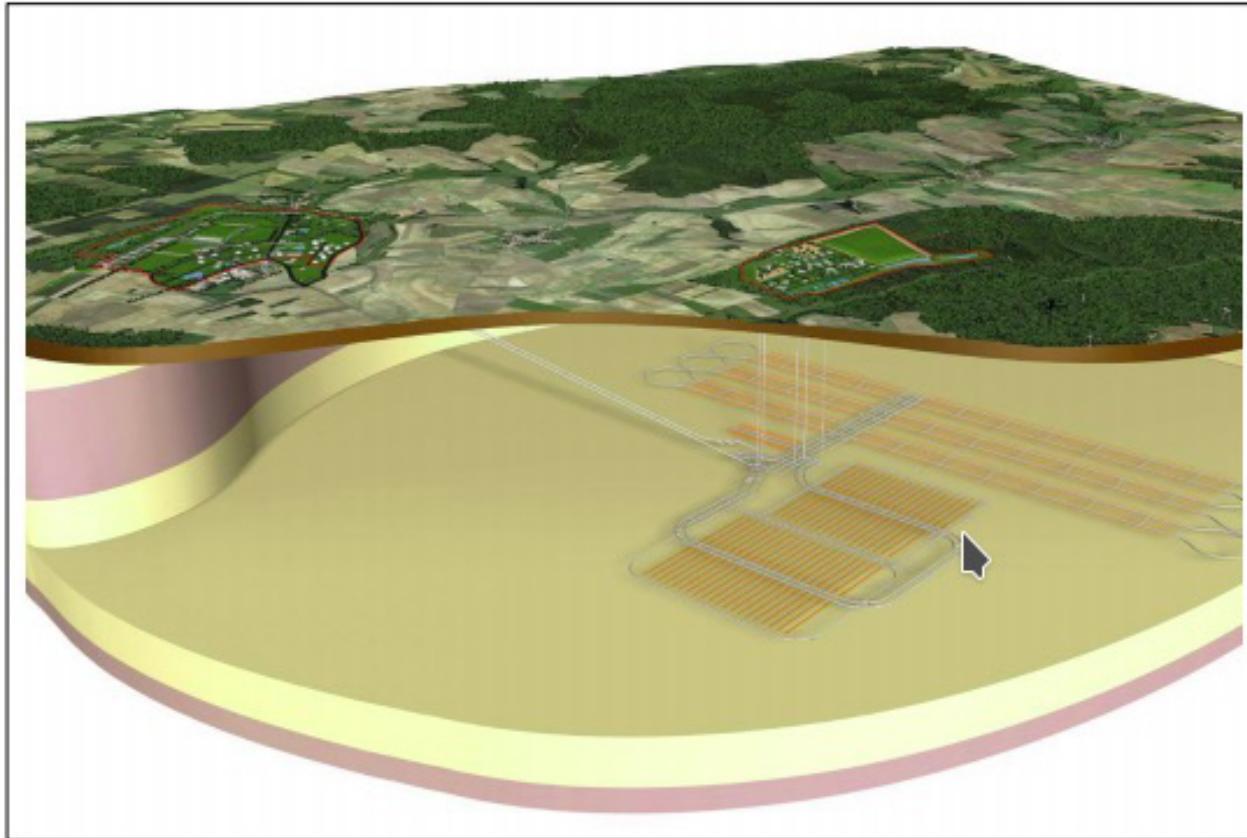
We start with a reminder



This will be the first part of the lecture

# Examples – Discretization of PDE

---

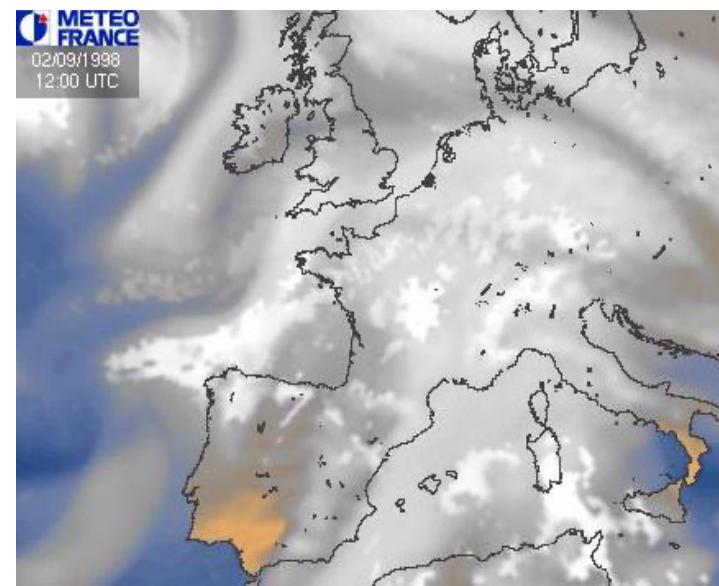
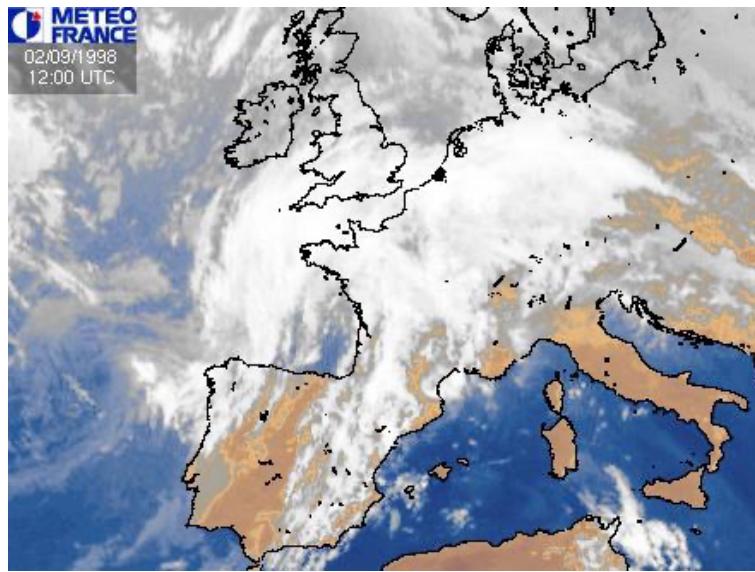


- Gallery crossing model
- 3D thermo-hydro-mechanics modelling (Biot's consolidation problem)
- Discretization of PDE leads to non-symmetric systems with about 100 million unknowns.
- We need fast iterative solvers!

# Examples – Weather forecasting

---

- Time constraints: The weather forecast has to be done for 'soonish'



# What will we look at

---

- Solving linear systems of the form  $Ax = b$
- Matrix A shall be sparse (and large)
- Solve this problem **iteratively** (e.g. stationary methods, Krylov methods, **multigrid**)
- Do this in an **inexpensive** and **fast** way.

# Large scale problems – mind experiment

---

- Let us take a matrix of size  $N \times N$ .
- Let us suppose that we have three algorithms with different computational complexities
  - **Algorithm 1:**  $10^6 N$  operations
  - **Algorithm 2:**  $10^3 N^2$  operations
  - **Algorithm 3:**  $N^3$  operations
- Suppose that the problem size  $N$  is chosen, such that Algorithm 1 requires 1 second to compute the result.
- How long do the others need?

# Speed of algorithms

---

N	Algorithm 1 $10^6 N$ operations	Algorithm 2 $10^3 N^2$ operations	Algorithm 1 $N^3$ operations	Computer Speed (ops/sec)
1	1 sec	0.001 sec	0.000001 sec	1M ( $\sim 10^6$ ) (1980's)
$10^3$	1 sec	1 sec	1 sec	1G ( $\sim 10^9$ ) (1990's)
$10^6$	1 sec	17 min	12 days	1T ( $\sim 10^{12}$ ) (2000's)
$10^9$	1 sec	12 days	31710 years	1P ( $\sim 10^{15}$ ) (2010's)

Stronger computers 

Greater advantage (necessity) of efficient algorithms!

# Message

---

- Computational complexity is important!
- For very large problems, an algorithm with optimal complexity (i.e., the work grows only linearly with problem size) wins dramatically against an algorithm whose complexity is quadratic.
- Not seen, but: Also the energy consumption will become a bottleneck.

# Classic schemes

# Classical iterative methods - basic theory

---

- Let  $A \in \mathbb{R}^{n \times n}$  be a **non-singular** matrix and  $\mathbf{f} \in \mathbb{R}^n$ .

**Problem:** Find  $\mathbf{u} \in \mathbb{R}^n$ , such that

$$A\mathbf{u} = \mathbf{f} \quad (1)$$

---

- Decompose the matrix into a **non-singular** matrix  $M$  and a matrix  $N$  as

$$A = M - N$$

- Equation (1) can then be written as

$$M\mathbf{u} = N\mathbf{u} + \mathbf{f}$$

or

$$\mathbf{u} = \underbrace{M^{-1}N\mathbf{u}}_{:=S} + M^{-1}\mathbf{f}$$

# Classical iterative schemes – fixed point iterations

---

Let  $\mathbf{u}^0$  be an [initial guess](#). Then a fixed point iteration can be applied to the equation

$$\mathbf{u}^{(m+1)} = S\mathbf{u}^{(m)} + M^{-1}\mathbf{f}$$

For  $\mathbf{u}$  being a solution,  $\mathbf{u}$  must be a [fixed point](#).

This basic iterative approach might also be damped.

$$\mathbf{u}^\star = S\mathbf{u}^{(m)} + M^{-1}\mathbf{f}$$

$$\mathbf{u}^{(m+1)} = \omega\mathbf{u}^\star + (1 - \omega)\mathbf{u}^{(m)}$$

Then we get

$$\mathbf{u}^{(m+1)} = (\omega S + (1 - \omega)\mathbf{I})\mathbf{u}^{(m)} + \omega M^{-1}\mathbf{f}$$

# The error and residual equation

---

- Let  $\mathbf{u}^{(m)}$  be the approximation at step  $m$ . The **error** is defined by

$$\mathbf{e}^{(m)} = \mathbf{u} - \mathbf{u}^{(m)}$$

- The **residual** at step  $m$  is defined by

$$\mathbf{r}^{(m)} = \mathbf{f} - A\mathbf{u}^{(m)}$$

- For both the weighted and the non-weighted iterative schemes above, the **residual equation** has the form

$$A\mathbf{e}^{(m)} = \mathbf{r}^{(m)}$$

- In some methods (e.g. multigrid), this residual equation is used to update the new  $\mathbf{u}^{(m+1)}$  by computing an approximation  $\tilde{\mathbf{e}}^{(m)}$  and then the new iterate

$$\mathbf{u}^{(m+1)} = \mathbf{u}^{(m)} + \tilde{\mathbf{e}}^{(m)}.$$

# The error equation

---

- Let  $\mathbf{u}$  be the solution of  $A\mathbf{u} = \mathbf{f}$  and  $\mathbf{u}^{(m)}$  an approximation. We have

$$\mathbf{u}^{(1)} = S\mathbf{u}^{(0)} + M^{-1}\mathbf{f}$$

- The solution  $\mathbf{u}$  is a fixed point

$$\mathbf{u} = S\mathbf{u} + M^{-1}\mathbf{f}$$

- Subtraction of the initial error

$$\mathbf{e}^{(1)} = S\mathbf{u} - S\mathbf{u}^{(0)} = S\mathbf{e}^{(0)}$$

- Repeating this argument, we have

$$\mathbf{e}^{(m)} = S\mathbf{u} - S\mathbf{u}^{(m)} = S\mathbf{e}^{(m-1)}$$

- Recursively, we then get

$$\mathbf{e}^{(m)} = S^m \mathbf{e}^{(0)}$$

# Convergence

---

From the previous equation follows

$$\|\mathbf{e}^{(m)}\| \leq \|S^m\| \|\mathbf{e}^{(0)}\|$$

The iteration is called convergent, if

$$\lim_{m \rightarrow \infty} \|S^m\| = 0$$

$\|S\|$  is called **contraction number** of the fixed point iteration.

## Theorem

Let  $\rho(S) = \max(\lambda(S))$  be the spectral radius of  $S$ . The iteration associated with the matrix  $S$  converges for **all** initial guesses, **if and only if**

$$\rho(S) < 1.$$

# Convergence – Error reduction

---

- Suppose the matrix is symmetric positive definite (spd). We have

$$\frac{\|\mathbf{e}^{(m)}\|}{\|\mathbf{e}^{(0)}\|} \leq \|S^m\| \approx \rho(S^m) = \rho(S)^m \approx 10^{-1}$$

- How many iterations do we need to guarantee the reduction of the error by a factor of 10?

$$m \geq -\frac{1}{\log_{10} |\rho(S)|}$$

- Convergence factor

$$\rho(S)$$

Loosely speaking: The worst factor for the Reduction of the error in each step.

- Convergence rate

$$-\log_{10} |\rho(S)|$$

The convergence becomes the faster, the higher the convergence rate.

# Jacobi method

---

- Correct  $x_i^{old}$  by  $x_i^{new}$  such that the  $i$ -th equation of  $Ax = f$  is correct.

$$a_{11} \cdot x_1 + \cdots + a_{1n} \cdot x_n = b_1$$

$$a_{21} \cdot x_1 + \cdots + a_{2n} \cdot x_n = b_2$$

⋮

$$a_{n1} \cdot x_1 + \cdots + a_{nn} \cdot x_n = b_n$$

- Then the **Jacobi method** is given by

$$x_i^{(m+1)} := \frac{1}{a_{ii}} \left( b_i - \sum_{i \neq j} a_{ij} x_j^{(m)} \right), \quad i = 1, \dots, n$$

# Jacobi Method

---

- Let  $D$  be the diagonal part and  $-L$  and  $-U$  the strictly lower and upper triangular parts of  $A$ , thus

$$A = D - L - U$$

## Jacobi Method

- Calculate the relaxations simultaneously for all  $i = 1, \dots, n$ . In matrix notation

$$\mathbf{x}^{(m+1)} = D^{-1}(L + U)\mathbf{x}^{(m)} + D^{-1}\mathbf{f}$$

Define

$$S_J := D^{-1}(L + U)$$

## Weighted Jacobi Method

- Let  $\omega \in \mathbb{R}$  be a weighting factor. Then

$$\mathbf{x}^{(m+1)} = [(1 - \omega)\mathbf{I} + \omega S_J] \mathbf{x}^{(m)} + \omega D^{-1} \mathbf{r}^{(m)}$$

Define

$$S_\omega := (1 - \omega)I + \omega S_J$$

# Gauss-Seidel Method

---

- The Gauss-Seidel method is given by

$$x_k^{(m+1)} := \frac{1}{a_{kk}} \left( b_k - \sum_{i=1}^{k-1} a_{ki} x_i^{(m+1)} - \sum_{i=k+1}^n a_{ki} x_i^{(m)} \right)$$

$$\begin{aligned} a_{11} \cdot x_1 + \cdots + a_{1n} \cdot x_n &= b_1 \\ a_{21} \cdot x_1 + \cdots + a_{2n} \cdot x_n &= b_2 \\ &\vdots \\ a_{n1} \cdot x_1 + \cdots + a_{nn} \cdot x_n &= b_n \end{aligned}$$

- Components of the new approximation are used as soon as they are computed
- G-S method is sequential, as to compute a new entry, we first need to compute all previous entries.

# Relaxation schemes

---

## Gauss-Seidel method

- Calculate an entry  $x_k$  of the new iteration and use it in the computation of  $x_i, i = k + 1, \dots, n$ .

$$\mathbf{x}^{(m+1)} = (\mathbf{D} - \mathbf{L})^{-1} \mathbf{U} \mathbf{x}^{(m)} + (\mathbf{D} - \mathbf{L})^{-1} \mathbf{f}$$

Define  
 $S_{GS} := (D - L)^{-1} U$

## SOR method

$$\mathbf{x}^{(m+1)} = (\mathbf{D} - \omega \mathbf{L})^{-1} (\omega \mathbf{U} - (\omega - 1) \mathbf{D}) \mathbf{x}^{(m)} + \left(\frac{1}{\omega} \mathbf{D} - \mathbf{L}\right)^{-1} \mathbf{f}$$

Define  
 $S_{GS\omega} := (D - \omega L)^{-1} (-\omega U + (\omega - 1) D)$

- If  $\omega = 1$ , then the Gauss-Seidel method is recovered.

# Properties

---

## Weighted Jacobi

- For  $A \in \mathbb{R}^{n \times n}$ , weighted Jacobi **converges** for all initial  $x^0$ , if  $\omega \in (0, 2/(\lambda_{\max}(D^{-1}A)))$ .
- The **rate** of convergence **depends** on  $\omega$ .
- The convergence does **not depend** on the **numbering** of the unknowns.
- Entries can be computed all at once.

$$\begin{aligned} a_{11} \cdot x_1 + \cdots + a_{1n} \cdot x_n &= b_1 \\ a_{21} \cdot x_1 + \cdots + a_{2n} \cdot x_n &= b_2 \\ &\vdots \\ a_{n1} \cdot x_1 + \cdots + a_{nn} \cdot x_n &= b_n \end{aligned}$$

## SOR

- For  $A \in \mathbb{R}^{n \times n}$ , SOR **converges** for all initial  $x^0$ , if  $\omega \in (0, 2)$ .
- The **rate** of convergence **depends** on  $\omega$ .
- The convergence **depends** on the **numbering** of the unknowns.
- We could also first solve for  $x_{nn}$  instead of  $x_{11}$
- We could also **alternate** between ascending and descending order (**symmetric** Gauss-Seidel)

# Example: 2D Poisson equation

---

$$\begin{aligned} -\Delta u &= f, && \text{in } \Omega = [0, \pi] \times [0, \pi] \\ u &= u_0, && \text{on } \Gamma = \partial\Omega \end{aligned}$$

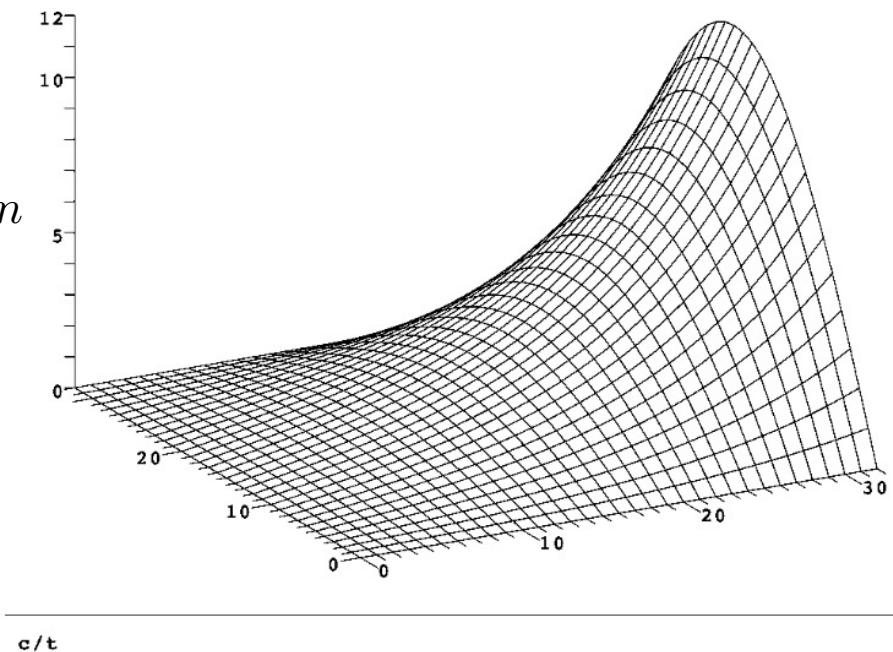
- Discretize  $\Omega$  into a regular grid with  $N$  intervals in  $x$  and  $y$
- We discretize using a 5-point stencil

$$\Delta u_{ij} \approx \frac{1}{h^2} (u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{ij}), \quad \text{for } i, j = 1, \dots, n$$

- Then, we choose as exact solution

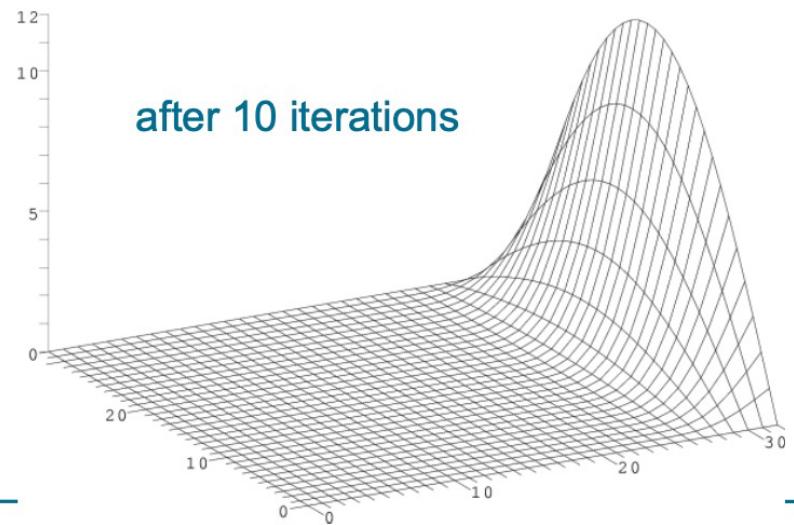
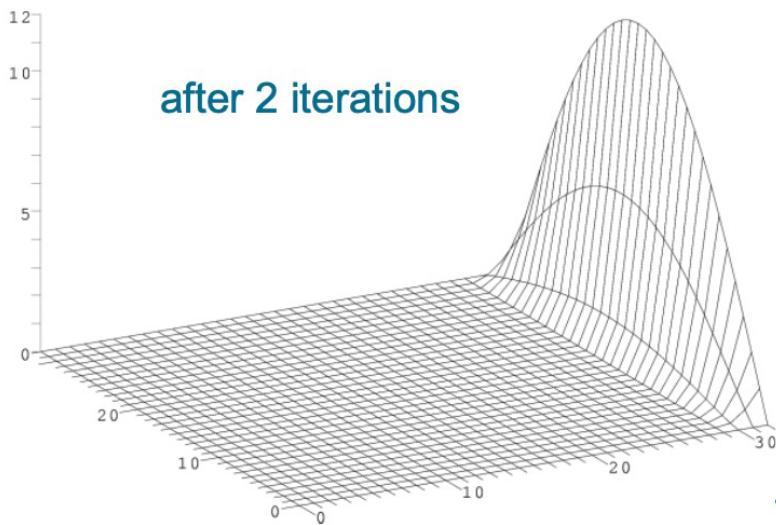
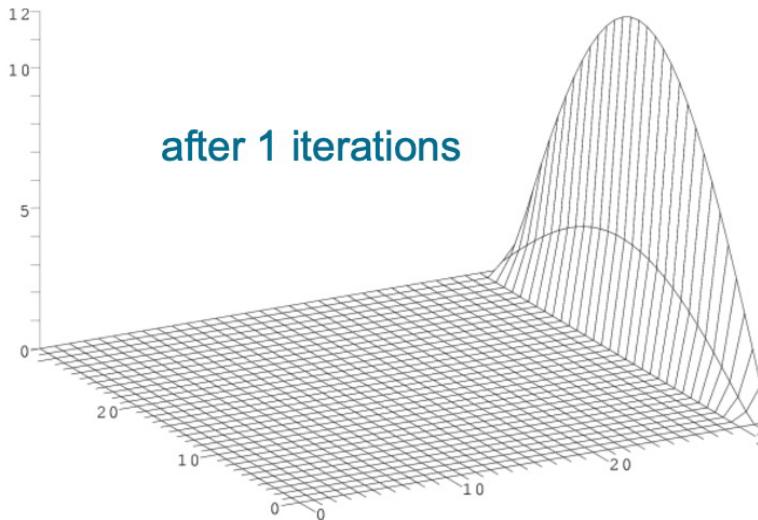
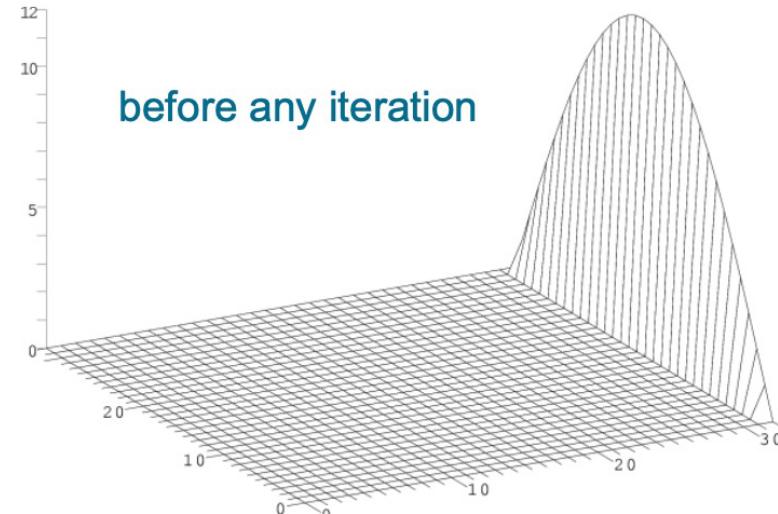
$$u = \sinh(x) \sin(y)$$

and obtain the right-hand side  $f = 0$ .



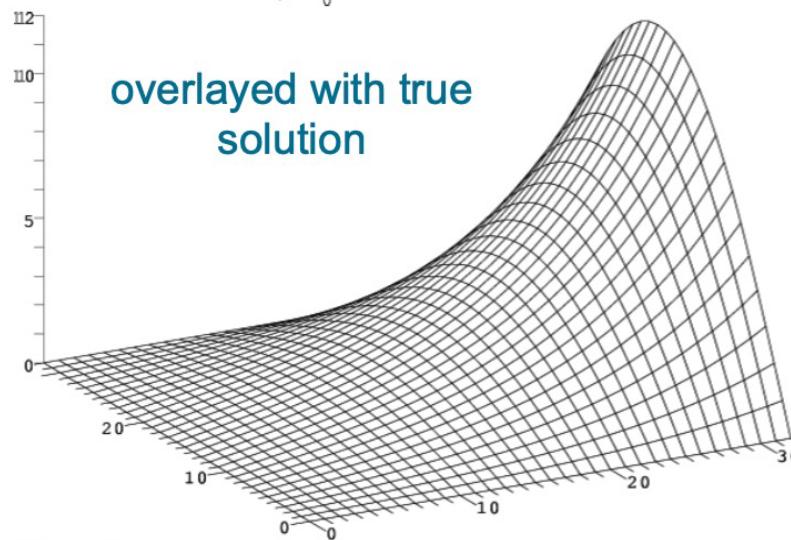
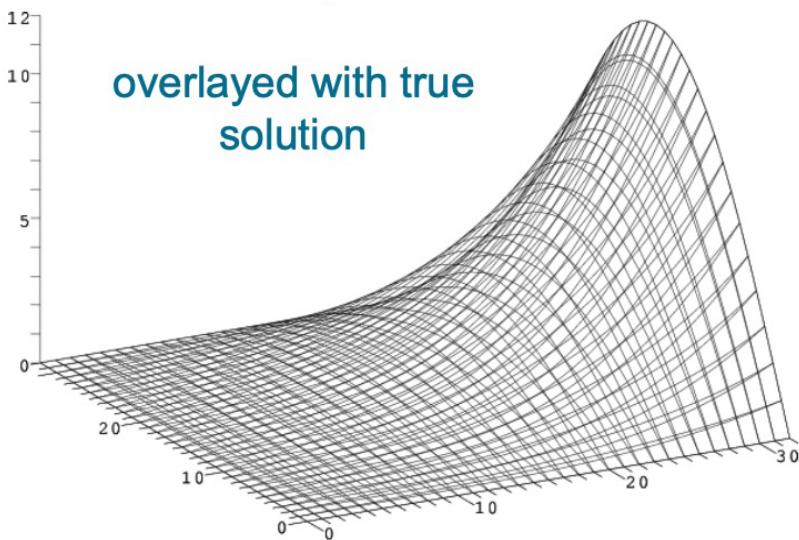
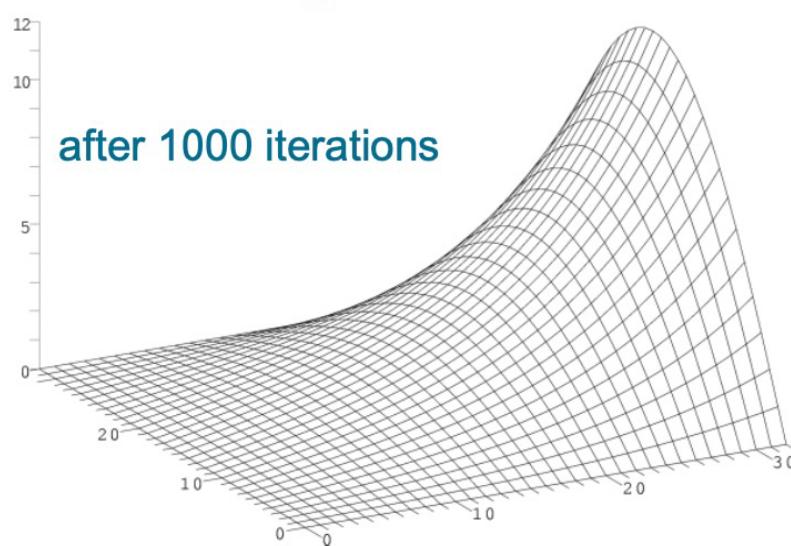
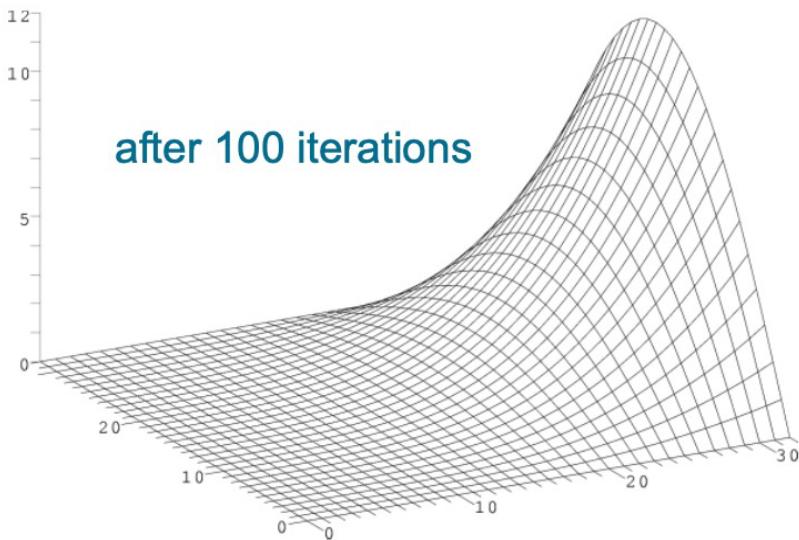
# Visualization of Gauss-Seidel iterations

---

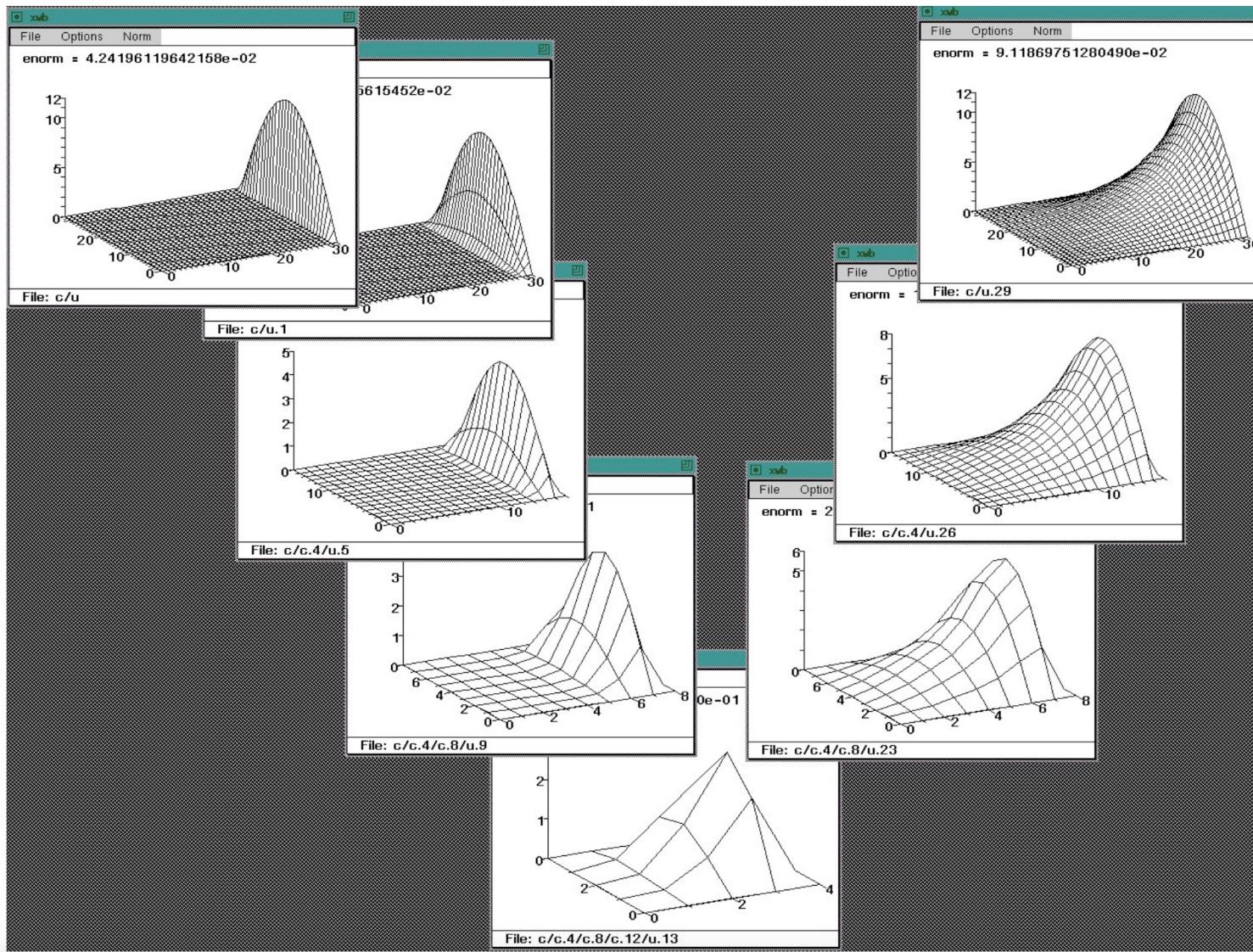


# Visualization of Gauss-Seidel iterations

---



# Multigrid



# Trying out algorithms and computational cost

---

- Let us now apply some algorithms to the 2D Poisson problem.

level	$h$	d.o.f.	SSOR		PCG		MG		FGMRES+MG		UMFPACK	
			ite	time	ite	time	ite	time	ite	time	ite	time
1	1/4	25	49	0	3	0	11	0	6	0	1	0
2	1/8	81	164	0	9	0	13	0	8	0	1	0
3	1/16	289	543	0	31	0	13	0	8	0	1	0
4	1/32	1089	2065	0.07	66	0.01	14	0.03	8	0.01	1	0.01
5	1/64	4225	7998	0.92	132	0.02	14	0.11	8	0.03	1	0.03
6	1/128	16641	31054	14.61	263	0.16	13	0.35	8	0.21	1	0.12
7	1/256	66049	> 100000		524	1.79	13	1.55	8	1.06	1	0.75
8	1/512	263169			1038	16.55	12	6.09	8	3.90	1	5.40
9	1/1024	1050625			1986	127.76	12	27.46	7	18.32	1	46.46
10	1/2048	4198401			3944	1041.68	12	111.03	7	68.38		
factor $\approx$			4		4	16	2	8	1	4	1	4

Table taken from the lecture notes 'Multigrid Methods by Volker John, Winter Semester 2013/2014.'

# Multigrid

---

- Framework: common concepts, different methods
- Efficient: usually  $O(N)$  or  $O(N \log N)$  operations
- The importance of efficient methods becomes greater, as computers grow stronger!
- Iterative: Most nontrivial problems in our field cannot be solved directly
- Many variables: the larger the number of variables, the greater the gain of efficient methods

A framework of efficient iterative methods for solving problems with many variables and many scales.

# This lecture is based on:

---

- A Multigrid Tutorial – W.L. Briggs, V.E. Henson, S.F. McCormick
- Why Multigrid Methods Are So Efficient - Irad Yavneh, Computing in Science and Engineering, 8(6):12 – 22
- Multigrid Methods, the basics - Luke Olson, Copper Mountain Multigrid Conference tutorial, 2021
- Multi-Grid Methods and Applications – W. Hackbusch

Some nice illustrations in jupyter notebooks:

- <https://github.com/lukeolson/imperial-multigrid/tree/master/lecture-1-mg-basics>

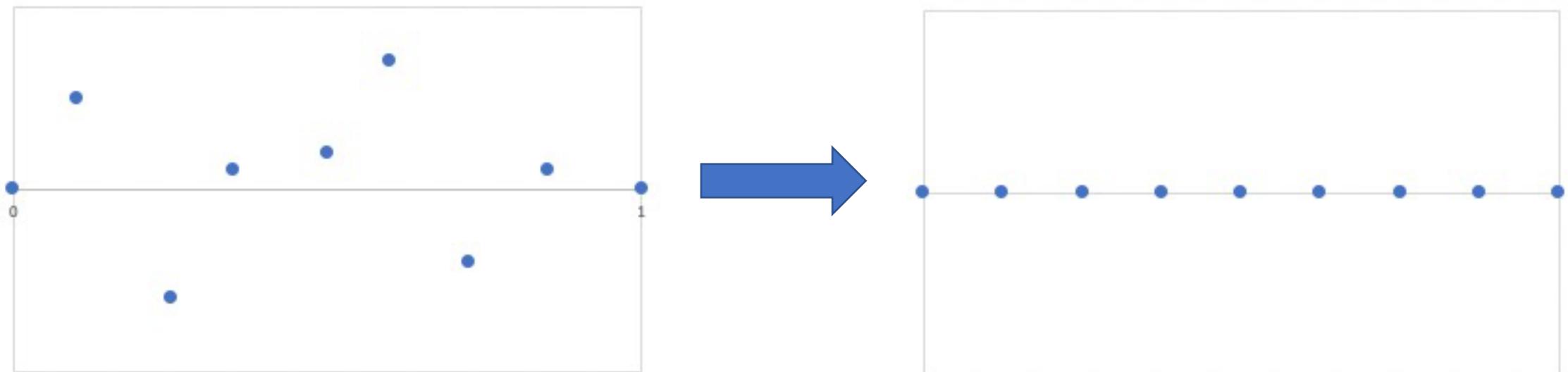
# A multiscale approach

---

Following Irad Yaneh (*Why Multigrid Methods Are So Efficient*)

Imagine:

- A football coach wants to line up his players in equal distances on the goal line.
- Player  $0$  and player  $N$  shall stand on the left and right goal post.
- The remaining players stand in between them. He now moves them according to some rule.



# 1. possibility: Global processing

---

- The coach numbers its players from  $0$  to  $N$ .
- He tells player  $j$  to move to the goal line connecting player  $0$  to player  $N$  and then at a distance  $j \frac{L}{N}$  from player  $0$ .
- This solves the problem directly.

But

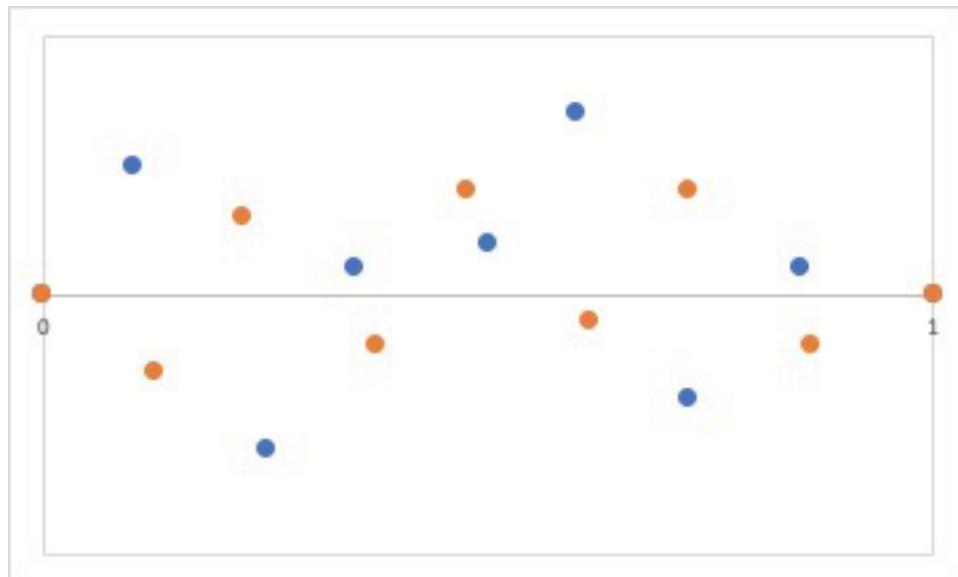
- Each player has to recognize the extreme players to find the ‘goal line’.
- He has to do some ‘fancy’ arithmetic  $\left(\frac{L}{N}\right)$ , and has to move  $j$  times of this.

Let us see how we could achieve the same goal with some local processing.

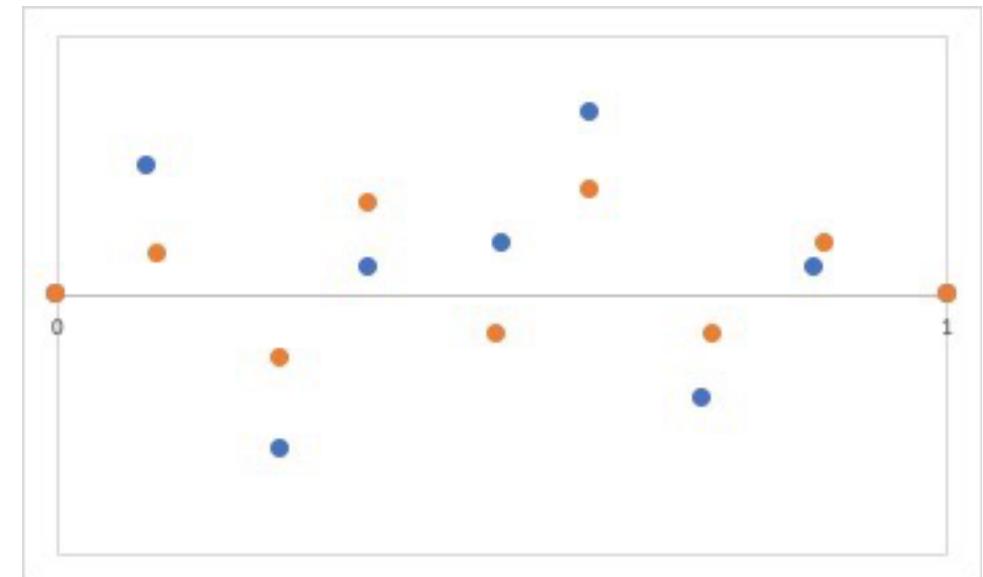
# Local movements

---

- We now require that each player detects the two players next to him and moves to the midpoint of the straight line connecting them.
- This is done at the whistle blow of the coach.
- Blue: initial position, Orange: new position



After one iteration



After two iterations

# Local movements

---

- Blue: initial position, black: final position
- orange: after 15 iterations, green: after 30 iterations, black: after 64 iterations ( $err \approx 5 \cdot 10^{-3}$ )



# Some slow convergence

---

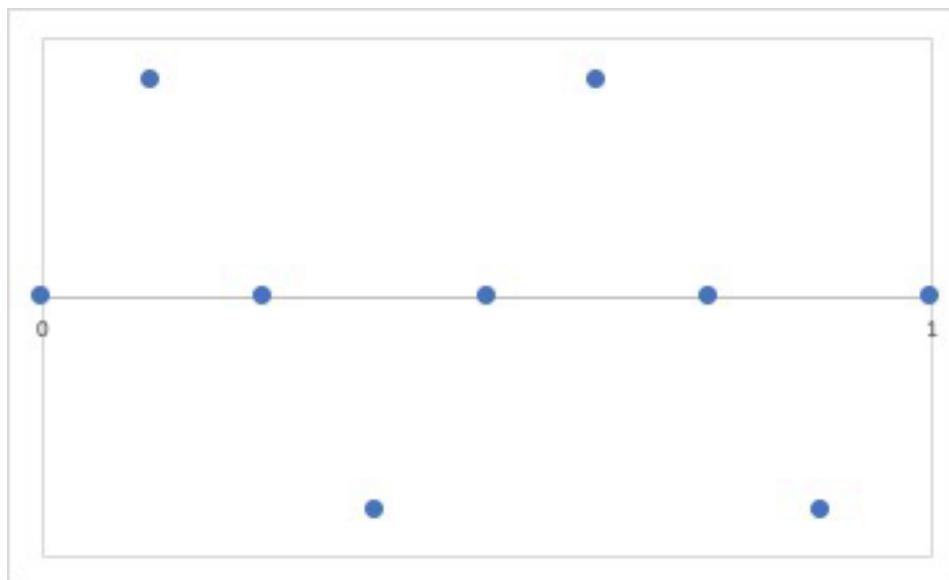
- Blue: initial position,
- Orange: new position after the whistle blow



# Fast convergence

---

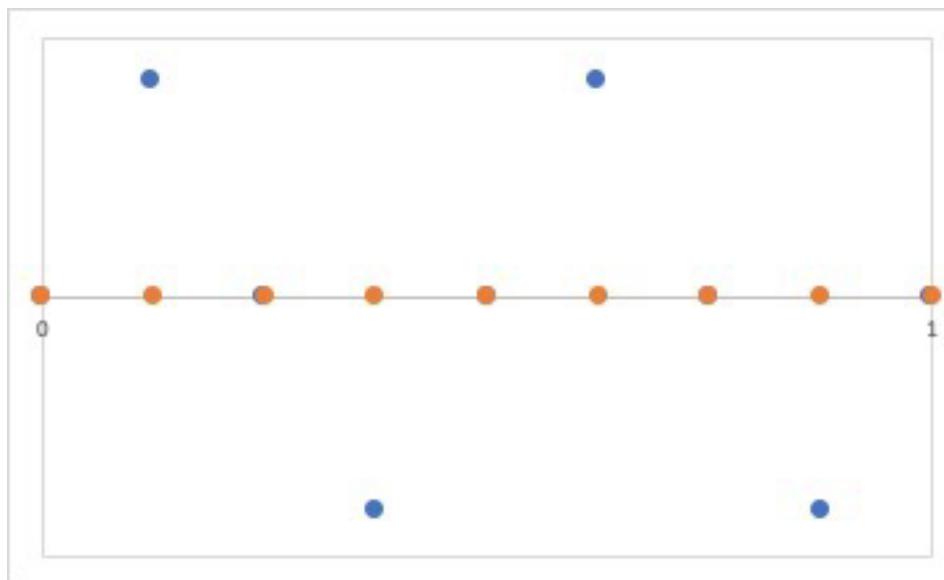
- Blue: initial position,
- Orange: new position after the whistle blow



# Fast convergence

---

- Blue: initial position,
- Orange: new position after the whistle blow



# Slow convergence

---

- Blue: initial position,
- Orange: position after one whistle blow
- Green: position after two whistle blows



# Slow convergence

---

- Blue: initial position,
- Orange: position after one whistle blow
- Green: position after two whistle blows



# Slow convergence

---

- Blue: initial position,
- Orange: position after one whistle blow
- Green: position after two whistle blows



# Slow convergence - improvement through damping

---

- Blue: initial position,
- Orange: position after one whistle blow
- Green: position after two whistle blows



# Slow convergence - improvement through damping

---

- Blue: initial position,
- Orange: position after one whistle blow
- Green: position after two whistle blows



# Slow convergence - improvement through damping

---

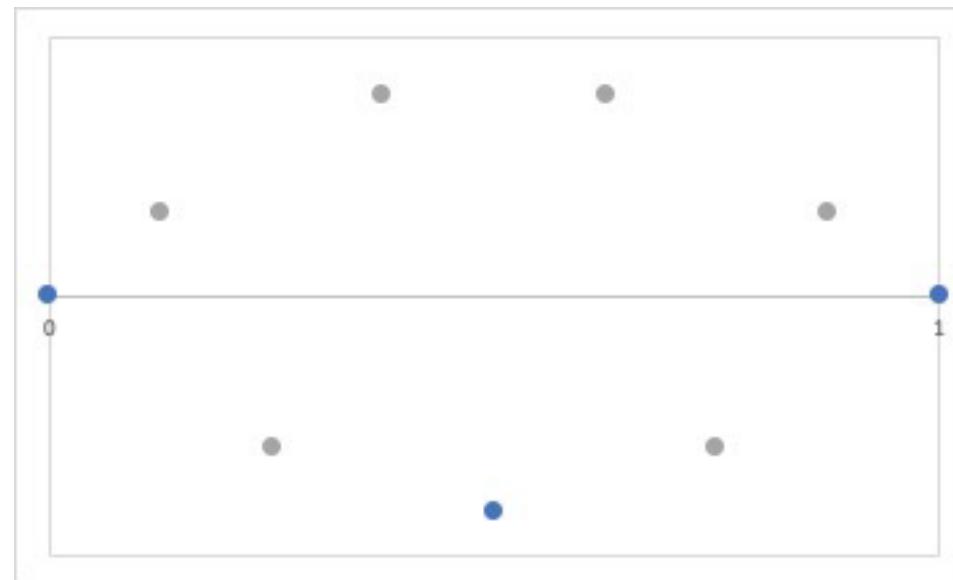
- Blue: initial position,
- Orange: position after one whistle blow
- Green: position after two whistle blows



# The Multi-scale approach

---

- Employ the local processing with simple arithmetic.
- Do this on different scales.

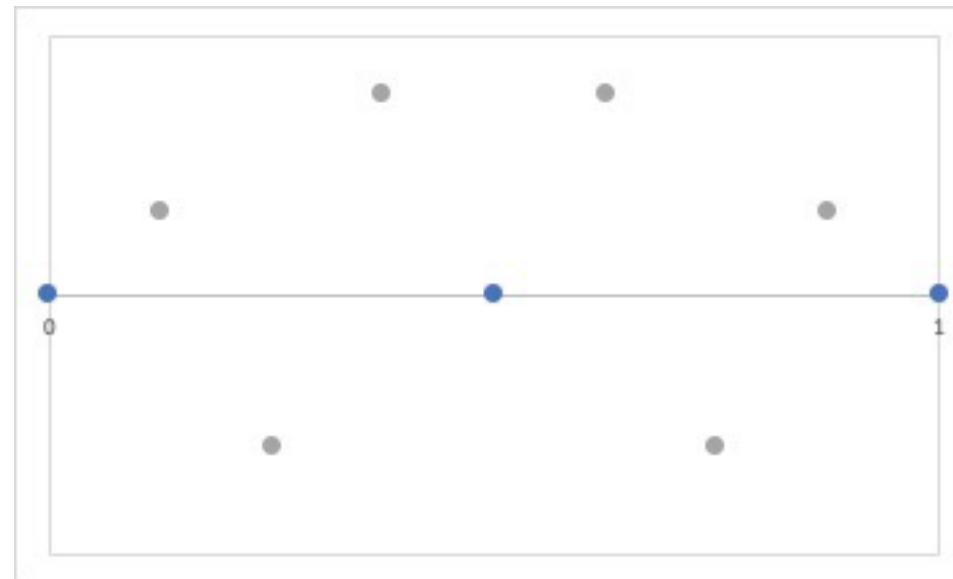


Large scale

# The Multi-scale approach

---

- Employ the local processing with simple arithmetic.
- Do this on different scales.



Large scale

# The Multi-scale approach

---

- Employ the local processing with simple arithmetic.
- Do this on different scales.



# The Multi-scale approach

---

- Employ the local processing with simple arithmetic.
- Do this on different scales.

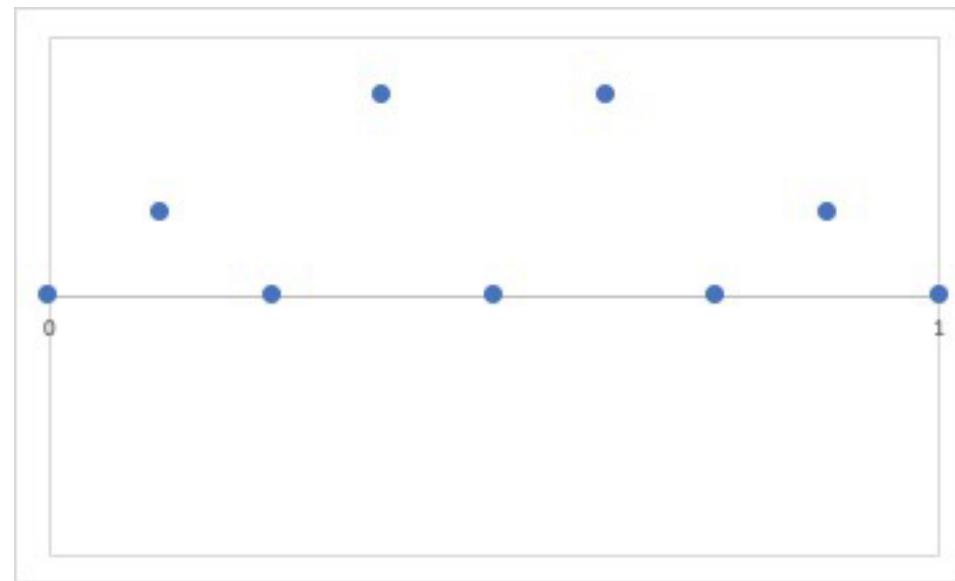


Intermediate scale

# The Multi-scale approach

---

- Employ the local processing with simple arithmetic.
- Do this on different scales.



Small scale

# The Multi-scale approach

---

- Employ the local processing with simple arithmetic.
- Do this on different scales.



Small scale

# How much do we save?

---

- Jacobi's method requires about  $N^2$  iterations and  $N^2 \cdot N = N^3$  operations to improve the accuracy by an order of magnitude
- The multiscale approach solves the problem in about  $\log_2(N)$  iterations (whistle blows) and only about  $N$  operations.
- Example: For  $N=1000$  we require about
  - Multiscale: 10 iterations and 1000 operations
  - Jacobi: 1 000 000 iterations and 1 000 000 000 operations

---

# Geometric Multigrid methods

# Elements of multigrid

---

Relaxation schemes and smoothing properties

# Model problem: 1D Poisson equation

- The 1D Poisson equation:

$$\begin{aligned}-u'' &= f \\ u(0) &= u(1) = 0\end{aligned}$$

- Grid points

$$h = \frac{1}{N}, \quad x_i = ih, \quad i = 0, \dots, N$$

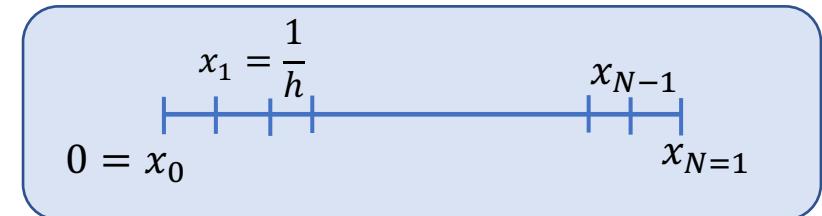
- Finite difference scheme

$$u_i'' \approx \frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} = f_i, \quad i = 1, \dots, N-1$$

$$u_0 = u_N = 0$$

- We obtain a linear system

$$A\mathbf{u} = \frac{1}{h^2} \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{N-1} \end{pmatrix}$$



# Eigenvectors and Eigenvalues

- The matrix is **spd** and **sparse** (not more than 3 non-zero entries per row and column).

- It has the eigenvalues ( $\lambda_k A = A \mathbf{w}_k$ )

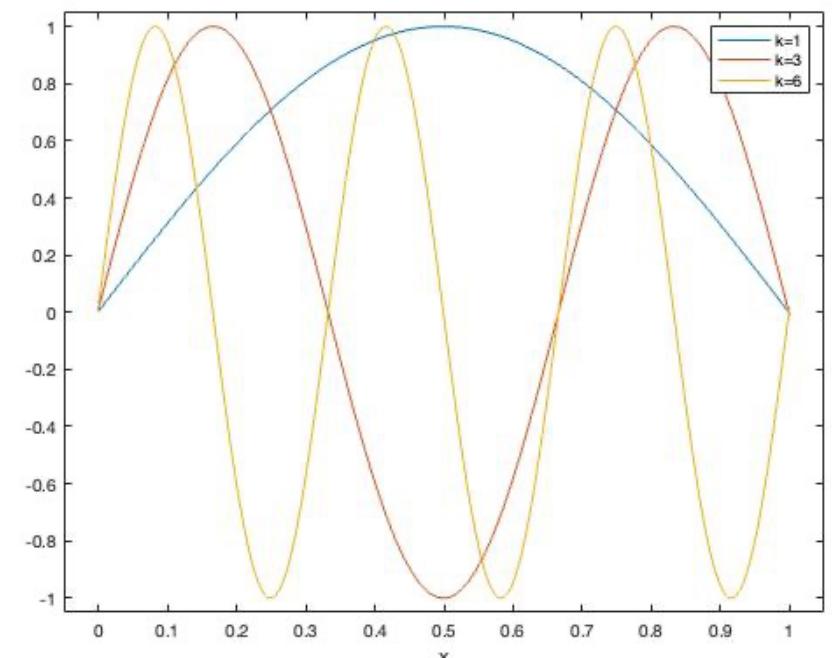
$$\lambda_k = 4 \sin^2 \left( \frac{k\pi}{2N} \right)$$

- And the eigenvectors

$$(\mathbf{w}_k)_j = \sin \left( \frac{jk\pi}{N} \right)$$

- This denotes the  $j$ -th component of the  $k$ -th eigenvector.
- The eigenvectors are Fourier modes.

$$A = \frac{1}{h^2} \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix}$$



# Definition

---

Modes in the upper half of the spectrum, i.e.

$$\frac{N}{2} \leq k < N - 1$$

are called **high-frequency** or **oscillatory modes**.

Modes in the lower half of the spectrum, i.e.

$$1 \leq k < \frac{N}{2}$$

are called **low-frequency** or **smooth modes**.

Note that the classification of smooth or oscillatory wave number depends on the total number  $N$  of grid points. A fixed wave number  $k$  might thus be smooth on one, but oscillatory on another grid.

## Application to our example: Jacobi method

- Remember the discretization at point  $i$  (this gives one matrix row):

$$-u_{i-1} + 2u_i - u_{i+1} = h^2 f_i$$

$$A = \frac{1}{h^2} \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$$

- Solving for  $u_i$ :

$$u_i = \frac{1}{2}(u_{i-1} + u_{i+1} + h^2 f_i)$$

- We can write this iteratively as

$$u_i^{(m+1)} = \frac{1}{2}(2u_{i-1}^{(m)} + u_{i+1}^{(m)} + h^2 f_i)$$

- This can be expressed in matrix form as

$$\begin{aligned} \mathbf{u}^{(m+1)} &= D^{-1}(L + U)\mathbf{u}^{(m)} + D^{-1}\mathbf{f} \\ &= \underbrace{(I - D^{-1}A)}_{:=R} \mathbf{u}^{(m)} + D^{-1}\mathbf{f} \end{aligned}$$

$$R = \frac{1}{2} \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & \ddots & & \\ & \ddots & \ddots & 1 & \\ & & & 1 & 0 \end{pmatrix}$$

# Weighted Jacobi method

---

To increase convergence properties, we next look at the weighted Jacobi method.

We compute

$$u_i^* = \frac{1}{2}(u_{i-1} + u_{i+1} + h^2 f_i)$$

This is now only an intermediate step, and we define the new weighted iterate by

$$u_j^{(m+1)} = (1 - \omega)u_j^{(m)} + \omega u_i^*, \quad i \leq j \leq N - 1$$

In matrix notation, we can express it by

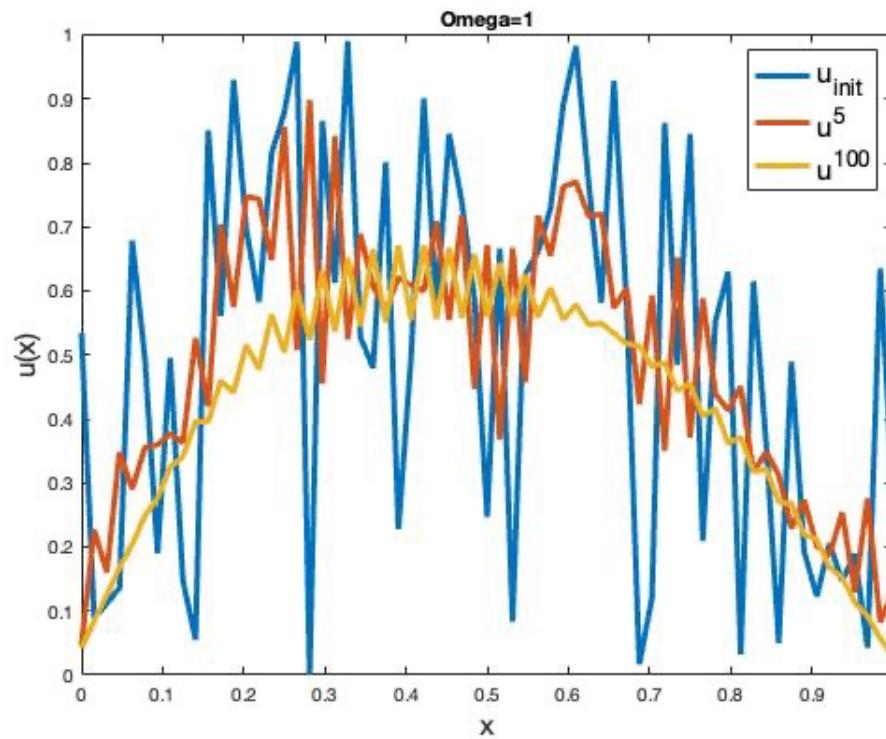
$$\begin{aligned} \mathbf{u}^{(m+1)} &= [(1 - \omega)\mathbf{I} + \omega S_J] \mathbf{u}^{(m)} + \omega D^{-1} \mathbf{f}^{(m)} \\ &= (1 - \omega D^{-1}) A \mathbf{u}^m + \omega D^{-1} \mathbf{f} \end{aligned}$$

Define

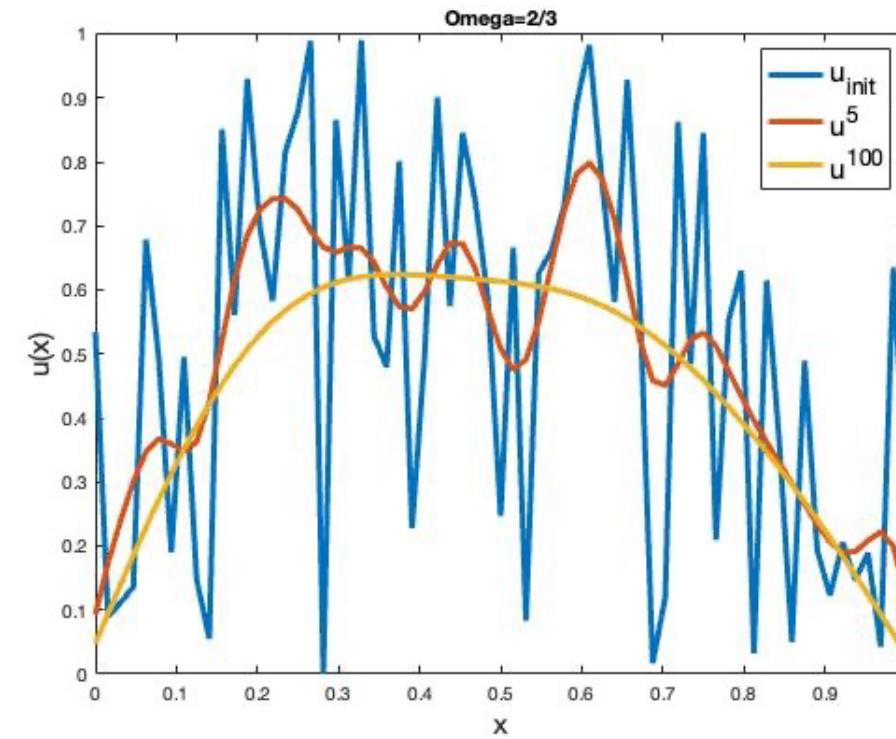
$$R_\omega := (I - \omega D^{-1} A)$$

# Smoothing with Jacobi and weighted Jacobi

- Let  $f = 0$  (thus solution  $u = 0$ ). Then random initial guess  $\rightarrow$  random error



$$\text{Jacobi } R = I - D^{-1}A$$

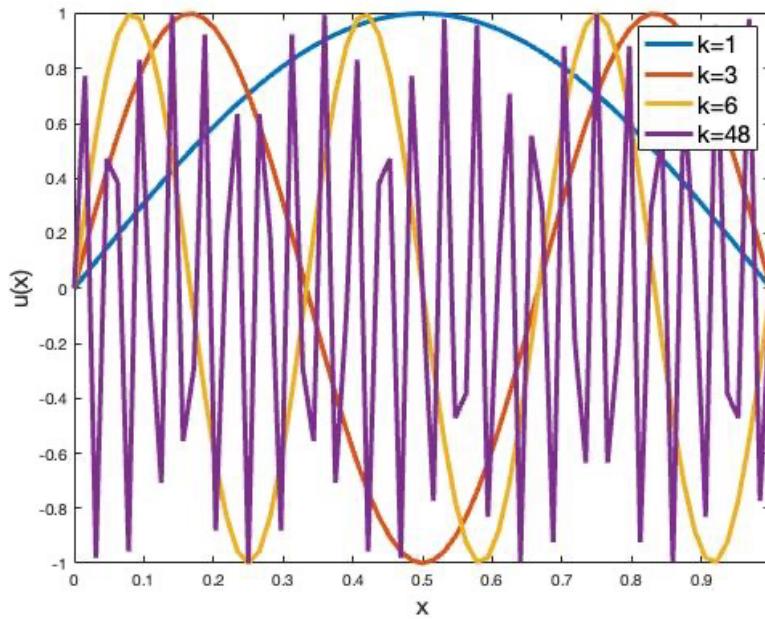


$$\text{Weighted Jacobi } R_\omega = I - \frac{2}{3}D^{-1}A$$

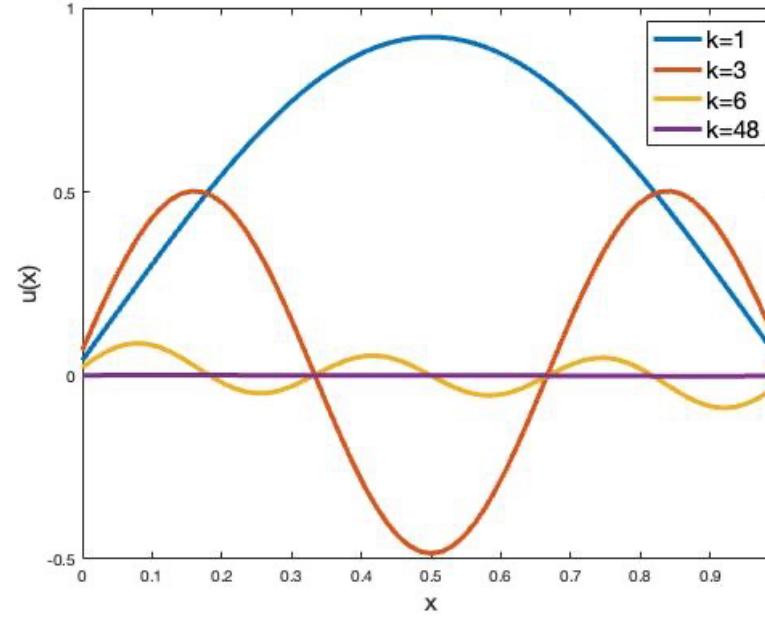
# Weighted Jacobi

- Let us use the four modes:  $k = 1, 3, 6, 48$ , as initial guess.
- Then smooth modes will dampen less quickly than higher ones.

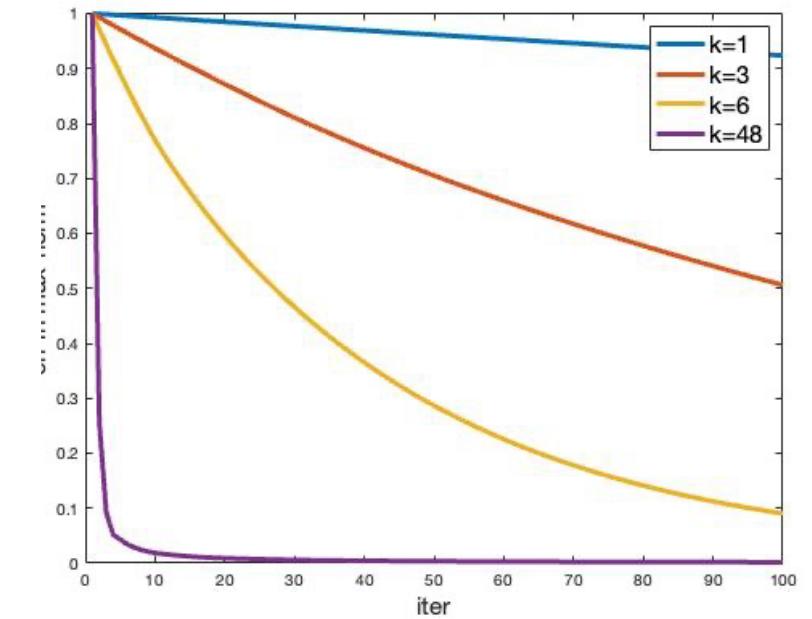
Initial error



Error after 100 iterations



Error decrease



WHY?

## A worked example: 1d Poisson

---

We now use the weighted Jacobi iteration matrix in the version

$$R_\omega = I - \omega D^{-1}A$$

For the 1D Poisson example, we obtain the iteration matrix

$$R_\omega = I - \frac{\omega}{2} \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{pmatrix}$$

It follows the relationship of the Eigenvalues

$$\lambda(R_\omega) = 1 - \frac{\omega}{2} \lambda(A).$$

# Eigenvalues and Eigenvectors

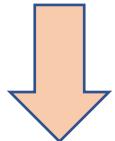
---

The Eigenvalues of  $A$  are given as

$$\lambda_k(A) = 4 \sin^2\left(\frac{k\pi}{2N}\right)$$

and the Eigenvectors as

$$(w_k)_j = \sin\left(\frac{jk\pi}{N}\right), \quad 1 \leq k \leq N-1, \quad 0 \leq j \leq N$$



It follows for  $R_\omega$ :

$$\lambda_k(R_\omega) = 1 - 2\omega \sin^2\left(\frac{k\pi}{2N}\right)$$

The Eigenvectors are the same as for  $A$ , thus

$$(w_k)_j = \sin\left(\frac{jk\pi}{N}\right), \quad 1 \leq k \leq N-1, \quad 0 \leq j \leq N$$

# Eigenvector expansion of error

We can represent the error  $\mathbf{e}^{(0)}$  using the Eigenvectors of  $A$ .

$$\mathbf{e}^{(0)} = \sum_{k=1}^n c_k \mathbf{w}_k$$

Where the coefficients  $c_k$  give the ‘amount’ of each mode in the error. We know furthermore that

$$\mathbf{e}^{(m)} = R_\omega^m \mathbf{e}^{(0)}.$$

Now using the eigenvector expansion for  $\mathbf{e}^{(0)}$ , we get

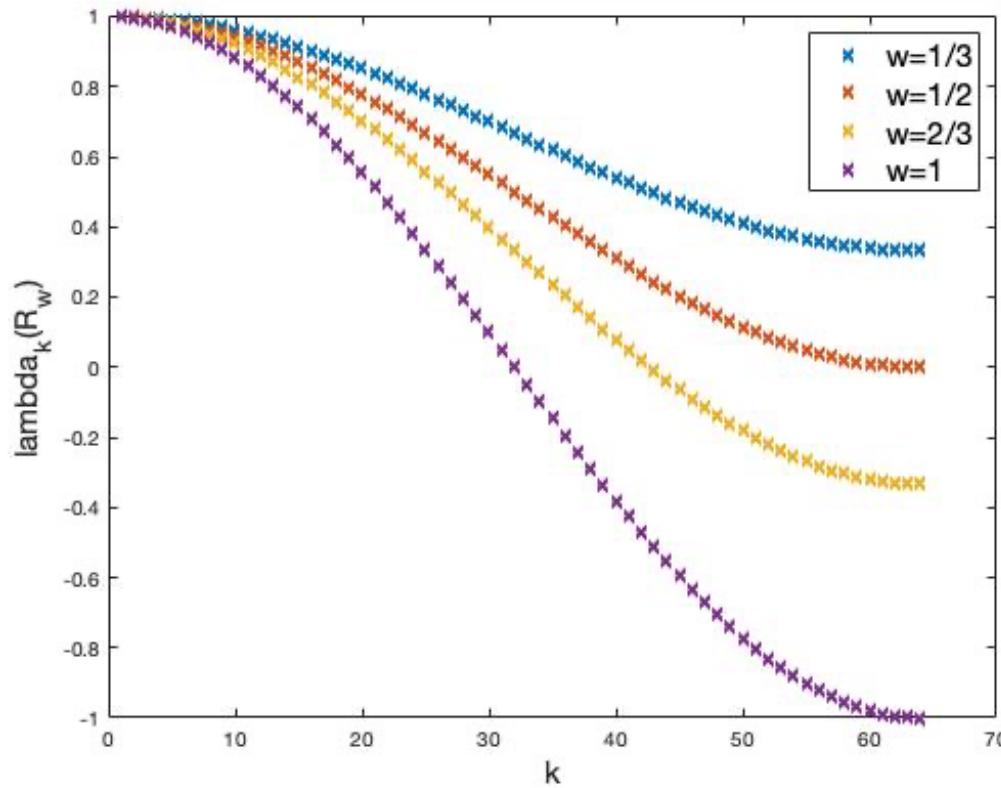
$$\mathbf{e}^{(m)} = R_\omega^m \mathbf{e}^{(0)} = \sum_{k=1}^n c_k R_\omega^m \mathbf{w}_k = \sum_{k=1}^n c_k \lambda_k^m(R_\omega) \mathbf{w}_k.$$

After  $m$  iterations, the  $k$ th mode of the initial error has been reduced by a factor of  $\lambda_k^m(R_\omega)$ .

- If  $|\lambda_k^m|$  is close to 1, then the reduction will be slow. If  $|\lambda_k^m|$  is close to 0, then it will be fast.
- What choice of  $\omega$  gives the best iterative scheme in our example?

# Choice of $\omega = 1$

---

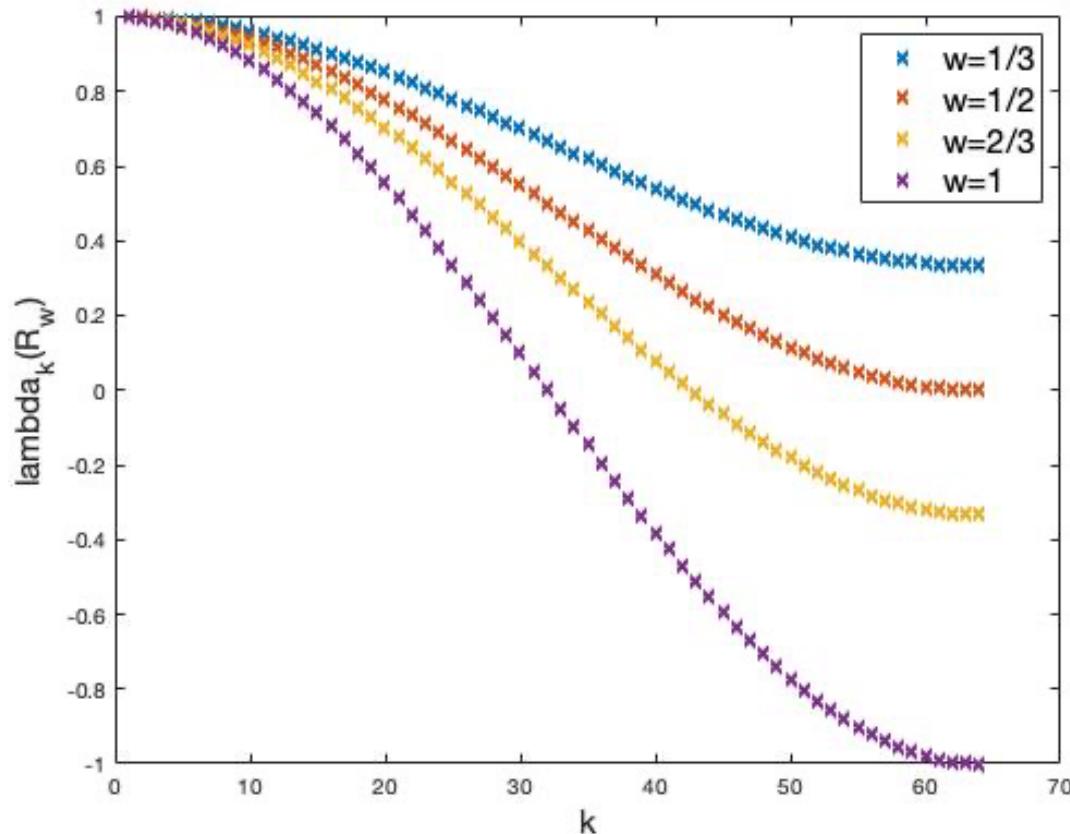


## Case $\omega = 1$

- We obtain the non-weighted Jacobi method.
- Here, we see that for small  $k$ , but also for larger  $k$  close to  $N-1$  in the oscillatory spectrum, the eigenvalues are close to 1.
- Convergence is thus also slow in these cases.
- This illustrates why the weighted Jacobi method is advantageous.

# Choice of $\omega$

- Recall: for  $0 < \omega \leq 1$ , we have  $|\lambda_k(R_\omega)| < 1$ .
- Find the value  $\omega$  that makes  $|\lambda_k(R_\omega)|$  as small as possible for all  $k$



For all values of  $\omega$ , the eigenvalues associated to the ‘smoothest’ modes are close to 1.

We have

$$\lambda_1 = 1 - 2\omega \sin^2\left(\frac{\pi h}{2}\right) \approx 1 - \frac{\omega \pi^2 h^2}{2}$$

Thus  $\lambda_1$  will always be close to one, no matter which  $\omega$ . It's getting even worse, the smaller  $h$ !

## Optimal choice of $\omega$

---

To find the optimal value of  $\omega$ , we search for the smallest interval  $[-\bar{\lambda}, \bar{\lambda}]$  with  $\lambda_k(R_\omega) \in [-\bar{\lambda}, \bar{\lambda}]$  for  $\frac{N}{2} \leq k \leq N - 1$ . This can be done for example by

$$-\lambda_{N/2}(R_\omega) = \lambda_{N-1}(R_\omega)$$

and we obtain

$$\omega = \frac{2}{3}$$

# The multigrid smoothing factor

---

- The **smoothing factor** of a relaxation method  $R$  is the maximum magnitude of the upper half of the spectrum

$$\max_{k \in [\frac{N}{2}, N]} |\lambda_k(R_\omega)|$$

- In the example above, we have

$$\max_{k \geq N/2} |\lambda_k(R_{2/3})| = \max_{k \geq \frac{N}{2}} \left| 1 - \frac{4}{3} \sin^2 \left( \frac{k\pi}{2N} \right) \right| \leq \frac{1}{3}$$

⇒ The oscillatory components are reduced at least **by a factor of 3** at each relaxation.

- We thus see furthermore, that the bound is independent of the mesh size  $h = \frac{1}{N}$ .

A common feature:

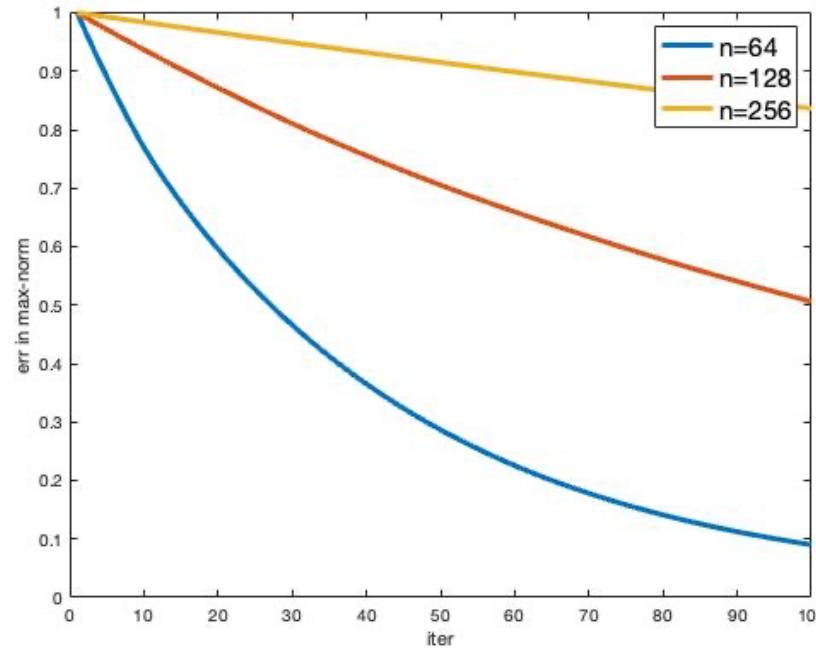
**Oscillatory** modes are quick to converge

**Smooth** modes are slow to converge

# Fixed wave number on different grids

---

- Let us still see what happens for the fixed wave number  $k=6$  on different grids.
- We use weighted Jacobi relaxation with  $\omega = \frac{2}{3}$  and show the error for the first 100 iterations



Observation: For a fixed wave number, the error is reduced better on a coarser than on a finer grid.

## Summary: Classification of modes

---

- Consider a fixed Fourier mode  $\sin(k\pi x)$  and its discrete representation  $\sin\left(\frac{jk\pi}{N}\right), j = 1, \dots, N - 1$ .

Then the classification of this mode depends on the fineness of the grid.

- If the grid is sufficiently **fine**, i.e.  $k < \frac{N}{2}$ , it is a **smooth mode**.
- If the grid is sufficiently **coarse**, i.e.  $\frac{N}{2} \leq k < N - 1$ , then it is an **oscillatory mode**.

The **damping** properties **depends** on the **smoothness** of the mode.

- If it is a **smooth mode**, then the weighted Jacobi method will **damp** it only **slowly**.
- If it is an **oscillatory mode**, then the weighted Jacobi method may **damp** it **quickly**.

# Relaxation schemes

---

## Gauss-Seidel method

- Calculate an entry  $u_k$ , with  $\mathbf{u} = (u_1, \dots, u_n)$ , of the new iteration and use it in the computation of  $u_i, i = k + 1, \dots, n$ .

$$\mathbf{u}^{(m+1)} = (D - L)^{-1}U \mathbf{u}^{(m)} + (D - L)^{-1}\mathbf{f}$$

Define  
 $R_{GS} := (D - L)^{-1}U$

## SOR method

$$\mathbf{u}^{(m+1)} = -(D - \omega L)^{-1}(-\omega U + (\omega - 1)D)\mathbf{u}^{(m)} + \omega(D - \omega L)^{-1}\mathbf{f}$$

$R_{GS\omega} := (D - \omega L)^{-1}(-\omega U + (\omega - 1)D)$

- If  $\omega = 1$ , then the Gauss-Seidel method is recovered.

# Gauss-Seidel method

---

- The eigenvalues of  $R_{GS} = (D - L)^{-1}U$  are given by

$$\lambda_k(R_{GS}) = \cos^2\left(\frac{k\pi}{N}\right)$$

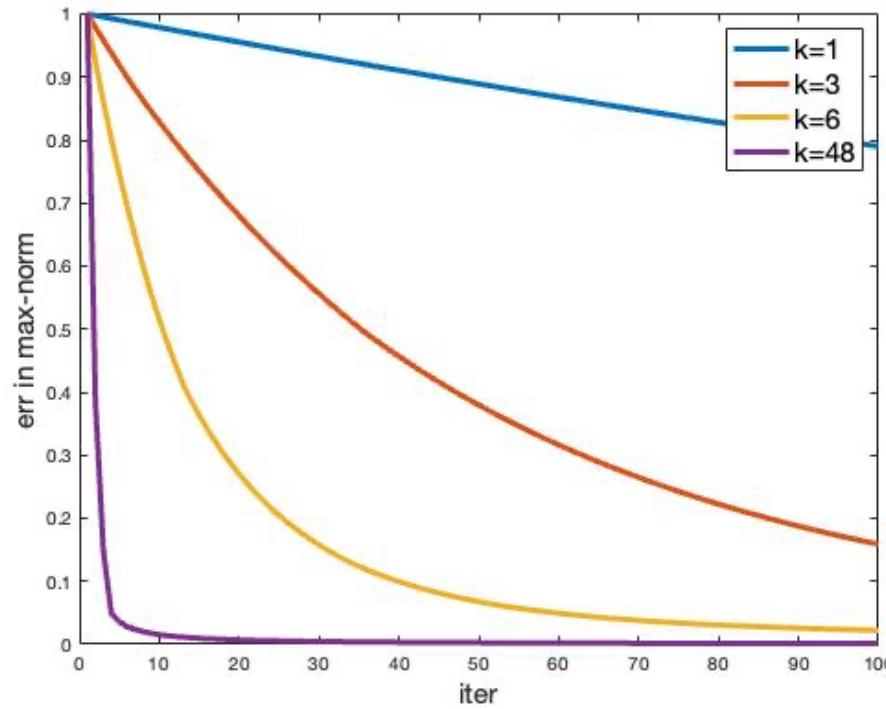
- The eigenvectors of  $R_{GS}$  are given by

$$w_{k,j}(R_{GS}) = \left[\cos\left(\frac{k\pi}{N}\right)\right]^2 \sin\left(\frac{jk\pi}{N}\right)$$

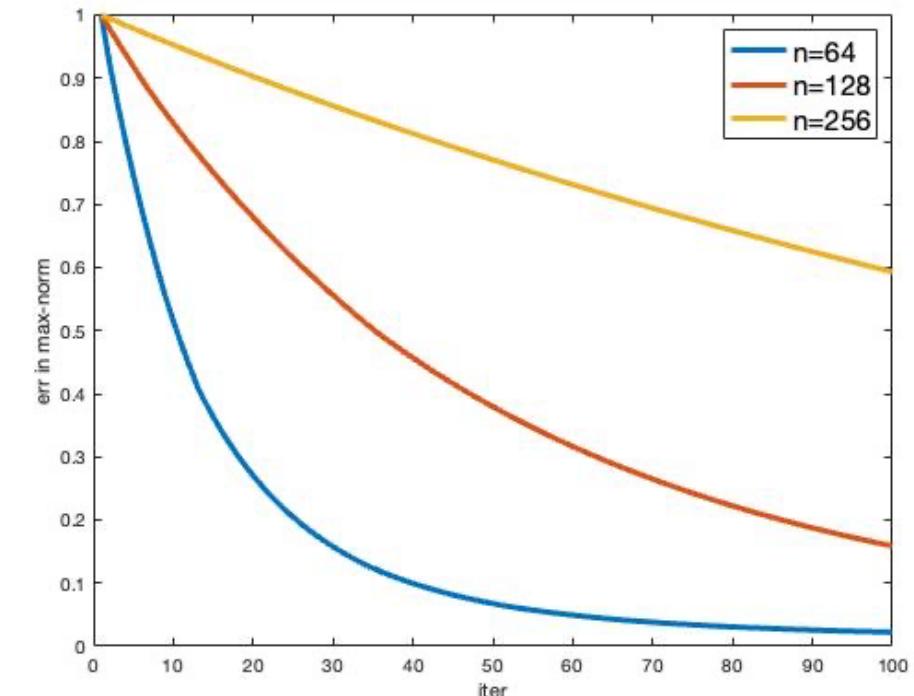
- These are **not** the **same** as for  $A$ . The convergence analysis has thus to be done carefully. We do not want to get into further details here.

# Gauss-Seidel method

- With the eigenvectors of  $A$  as initial guess, the damping behaviour of the Gauss-Seidel method is qualitatively the same.



Fixed number of nodes  $n=64$



Fixed wave number  $k=6$

## Summary: Relaxation schemes and damping

---

- Classical iterative schemes might **damp** highly **oscillating** discrete error modes **very quickly**.
- There is only a **slow damping** of the **smooth** discrete error modes.
- The **smoothness** of an error mode **depends** on the **grid**. A smooth error mode on a given grid is generally on a coarser grid less smooth.