5A ModIA

# Programming Practical with FEniCS

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# 1 Context

The goal of this project is to simulate the temperature distribution within a mountain hut over both space and time. This simulation depends on the initial uniform temperature $u_0$ and a firepower policy.

# 2 Equation

The thermal transfer model consists to solve the following equations:

$$\frac{\partial u(x,t)}{\partial t} - div(\mu \nabla u(x,t)) = 0$$

in $\Omega \times [0, T]$.

We simplified the model of the hut by a square of 1x1 meter and we chose T = 1 000 000s.

$$\Omega = \{(x, y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$$

The mesh of the hut is generated with $30 \times 30$ points. The finite element space $V$ is constructed using Lagrange elements of degree $k = 2$.

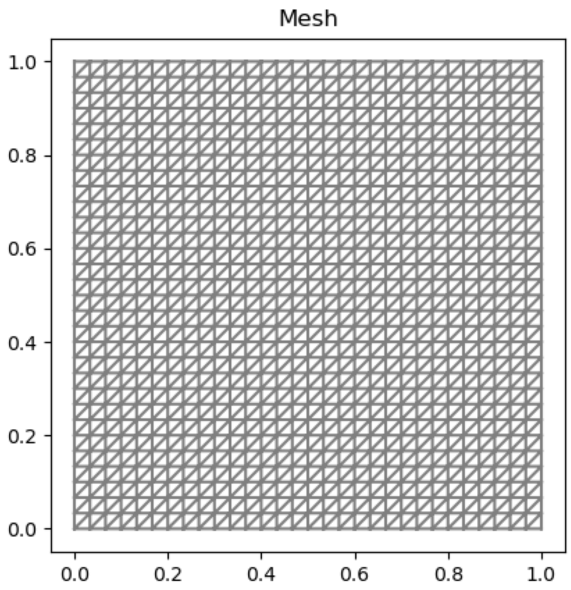


Figure 1: Hut Mesh with P2-Lagrange interpolation

The initial condition is $u(x, 0) = u_0 = 273K \quad \forall x \in \Omega$

The domain boundary $\partial\Omega$ is decomposed as follows:

$$\partial\Omega = \Gamma_{fire} \cup \Gamma_{dir}$$

where $\Gamma_{fire}$ represents the fireplace boundary (situated on the right of the hut), and $\Gamma_{dir}$ denotes a part of the boundary where the temperature is known (situated on the left of the hut).

On $\Gamma_{fire}$, we have:

$$-\mu\nabla u(x,t)\cdot n(x) = \sigma\left(u^4 - {u_{fire}}^4\right)(x,t)$$

where $\sigma$ is the Stefan-Boltzmann's constant, $\sigma \approx 5.67 \times 10^{-8}$ W. m$^{-2}$.K$^{-4}$, $u_{fire} =$ 1773 K (it is a little high for the value of a chimney fire but we have a dirichlet condition $u_{dirichlet} = 273$ K, it is as if we let the door opened...)

## 2.1 Stationnary equation

The stationnary equation is defined as followed:

$$-div(\mu\nabla u(x,t)) = 0$$

in $\Omega$.

With respect to Green's formula, the weak formulation is given by:

$$\forall v \in V_t, \int_\Omega \mu\nabla u\cdot\nabla v\,dx - \int_{\Gamma_{fire}} \sigma u^4 v\,ds = \int_{\Gamma_{fire}} \sigma u^4_{fire} v\,ds$$

with $V_t = \{$ v $\in H^1(\Omega)$ $\mid$ v = u$_{dirichlet}$ on $\Gamma_{dirichlet}\}$

The PDE is non linear, we compute its solution $u_h$ with the Newton-Raphson algorithm:

Given $u^{(k)}$ we solve (with a convergence criteria):

$$\frac{\partial a(u^{(k)},v)}{\partial u}\cdot\delta u = -a(u(k),v) + l(v) \quad \forall v \in V_t$$

i.e.

$$\int_\Omega \mu\nabla\delta u\nabla v\,dx - \int_{\Gamma_{fire}} 4\sigma u^{(k)3}\delta uv\,ds = \int_\Omega \mu\nabla u^{(k)}\cdot\nabla v\,dx - \int_{\Gamma_{fire}} \left(\sigma u^{(k)4} - \sigma u^{(k)4}_{fire}\right) ds$$

We obtain the solution and its convergence curve associated (figure 2 and figure 3).

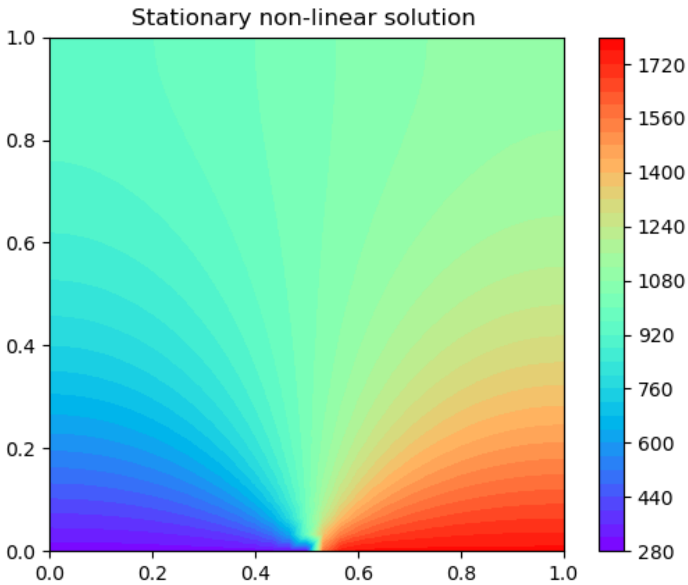


Figure 2: Stationnary solution

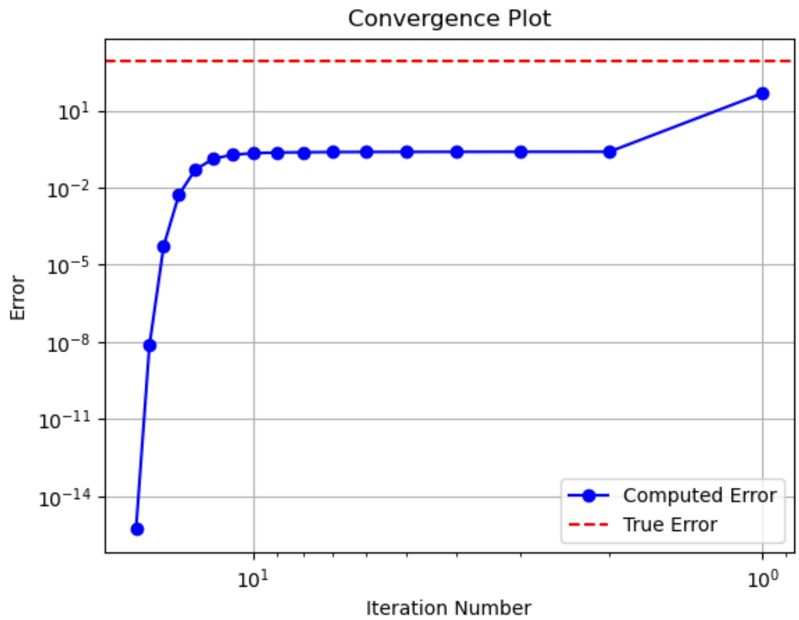


Figure 3: Convergence curve of the stationnary solution

## 2.2 Non-stationnary equation

We now add the time-dependant term. When t tends to $+\infty$ we should find a plot similar to the stationnary solution. We solved the time dependant non linear equation with a $\theta$ scheme, with $\theta = 1$ (implicit Euler) as defined in the course.

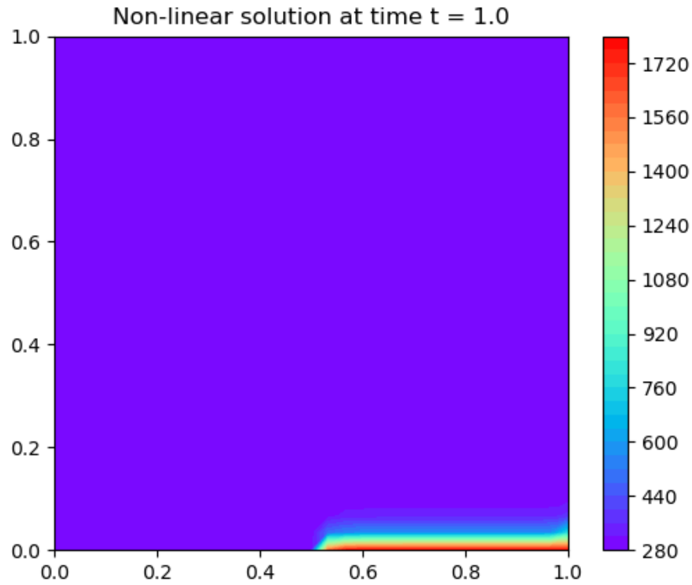


Figure 4: Non-stationnary solution at time t = 1

At t = 1 (figure 4), the whole hut is at 273 K and we heat up the room with a chimney fire (located in the right boundary). We let the door opened on the left, this is not optimal (dirichlet condition).

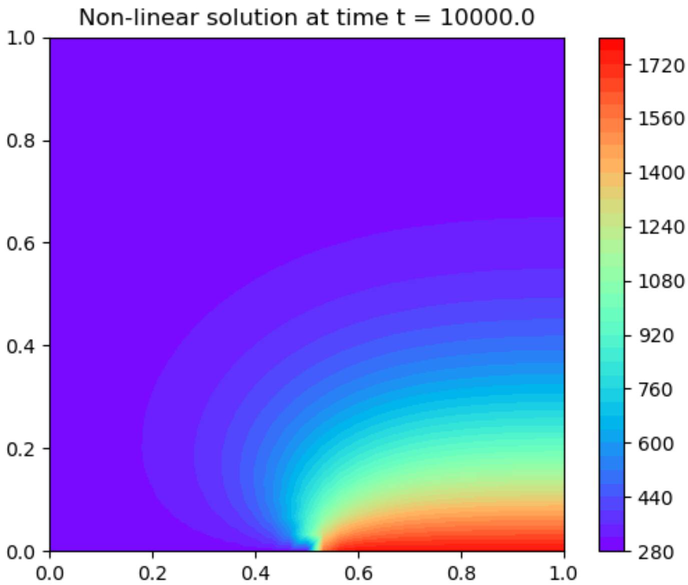


Figure 5: Non-stationnary solution at time t = 10000

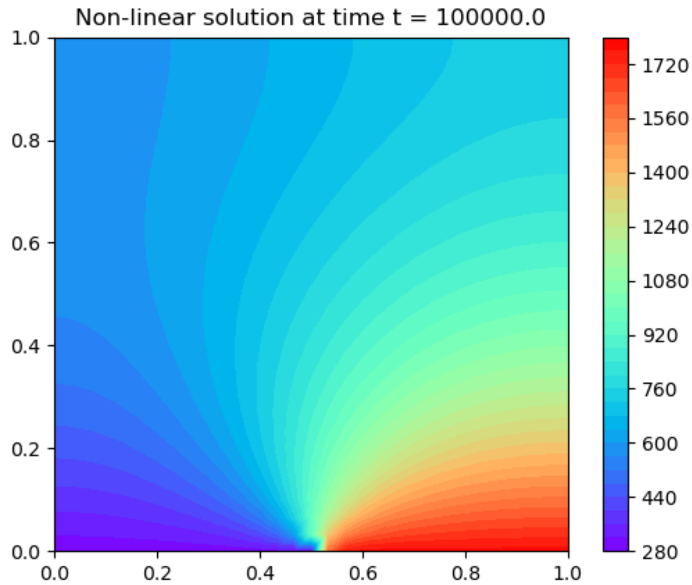


Figure 6: Non-stationnary solution at time t = 100000

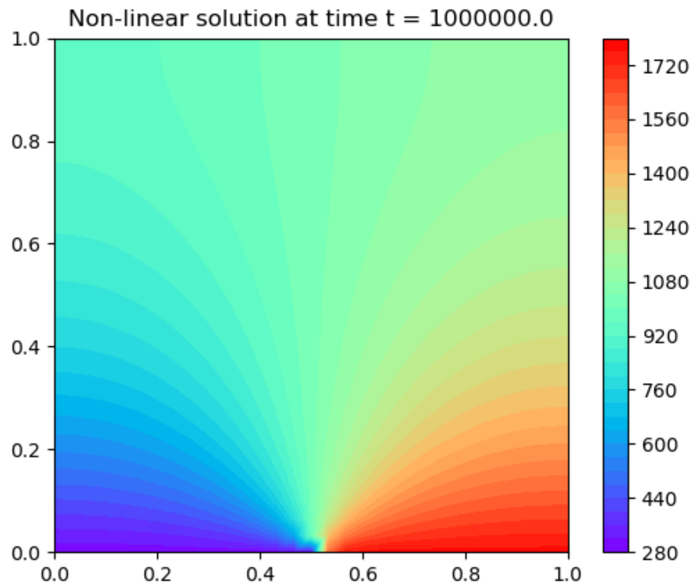


Figure 7: Non-stationnary solution at time t = 1000000

As expected, when t tends to $+\infty$ t we get the same result as the stationnary solution which translates the fact that our room was well heated up as the hut has an ambiant temperature of 1080 K.

The convergence curve (figure 8) associated (log(error) in function of time step) is linear. This is coherent because we used an implicit Euler scheme that typically have first-order convergence, i.e. that the error decreases at a rate proportional to the time step.

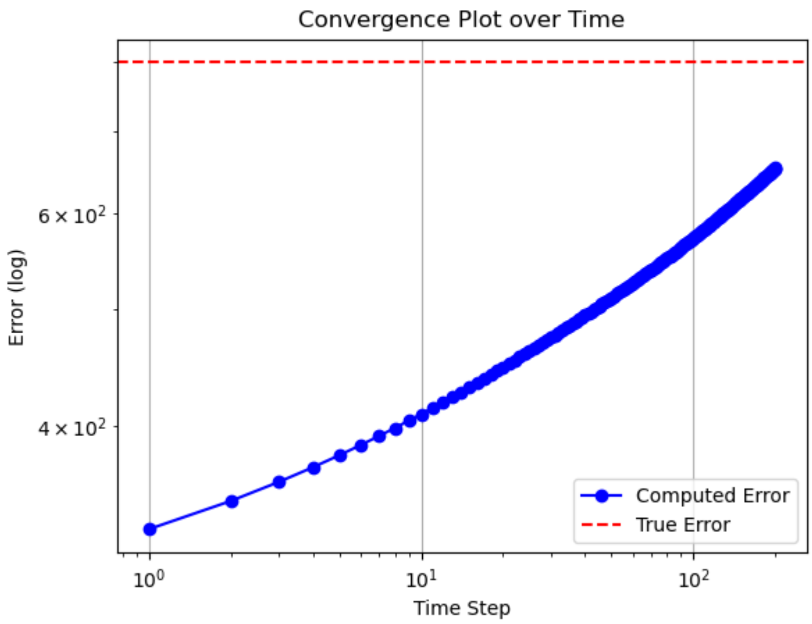


Figure 8: Convergence curve of the non-stationnary solution

# 3 Limitations

Several elements in the original problem were simplified as it was difficult for us to model and implement it. First, we didn't succeed in representing the whole hut (with the roof etc.) so we modelled it instead with a square of 1x1 meter. Then we supposed that the convective transfer and the diffusive transfer are neglected. Realistically, that means that the walls are perfectly isolated (homogeneous Neumann boundary condition) and that the convective contribution to heat transfer, which would arise from the movement of fluid, is effectively ignored. Lastly, we supposed that the source term $f(x, t) = 0$ i.e. we supposed that there is no other heat source inside the hut.