Kungliga Tekniska Högskolan
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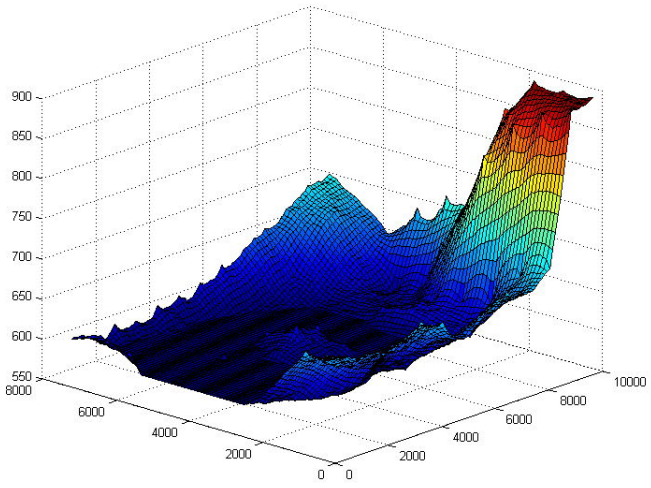
Parallel Computations for Large-Scale Problems

« Fractal Terrain Generation »

March 30^{th} - May 15^{th}



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Introduction

Stochastic algorithms for the generation of fractal terrains has a considerable interest in computer graphics, because of their ability to mimic the behaviour of natural terrain. The applications of fractal terrains is not limited to the generation of terrain, but includes the generation of clouds, textures, noise and image filters. In this project we provide and compare three different algorithms that generates fractal landscapes on distributed-memory parallel environments using MPI.

The first, *Diamond-Square*, is a simple and widely used recursive algorithm designed to compute high resolution height maps given a low resolution one. Then we introduce a little-known algorithm that we call *Linear Displacement*, that is an attempt to avoid the recursiveness of Diamond-Square. Finally we discuss an algorithm that uses the *Fast Fourier Transform* to do the transform of pink noise, that has an amplitude of $1/f^{\alpha}$.

1 Diamond-Square

1.1 Sequential algorithm

Fractal terrain generation can be achieved using the Diamond-Square algorithm introduced by Fournier, Fussel, Carpenter[3] and Miller[5]. This algorithm computes a high resolution terrain through multiple iterations, each containing a *square* and *diamond* pass.

The final resolution is as follows :

$$(width_K, height_K) = 2^K((width_0, height_0) - 1) + 1 \tag{1}$$

With L the terrain width or height and K the number of iterations. Therefore, each iteration increases the resolution according to $L_{k+1} = 2L_k - 1$.

Note that if we don't use an initial terrain, $2^K = L$.

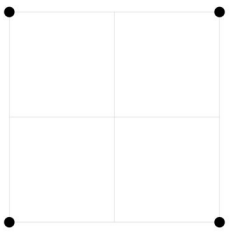

FIGURE 1 – Initial matrix

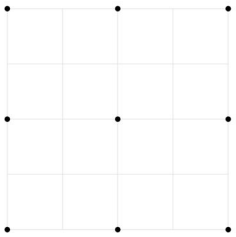

FIGURE 2 – Matrix after the first iteration

To reduce the computation time, we have optimized the memory management by removing the memory allocation and release required by the change of matrix dimensions. Hence, we use a single matrix which has the final terrain dimension. This matrix is actually a simple height map. However, to simplify the algorithm, each iteration only consider a *virtual* matrix which is a part of the final matrix. This matrix has the size expected at

the end of the iteration, and we use the following formula to switch from virtual to real matrix coordinates :

$$(i_K, j_K) = 2^{K-k}(i_k, j_k) \tag{2}$$

1.1.1 Randomness

In order to make our terrain less smooth and thereforemore more realistic, we add a random factor when computing the new points of our terrain.

The random factor r is generated according to $0 \leq r \leq scale_k$, with $scale_{k+1} = \frac{scale_k}{2}$.

1.1.2 Square pass

The square pass aims at computing the value located at the center of a *square*. This value is the average of those four points.

$$m(i,j) = \frac{m(i-1,j-1) + m(i-1,j+1) + m(i+1,j-1) + m(i+1,j+1) + r)}{4} \tag{3}$$

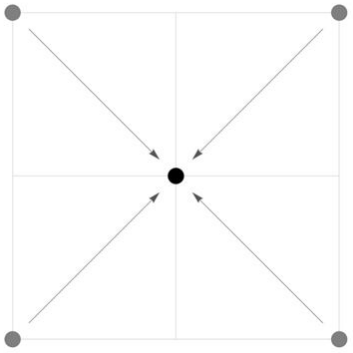

FIGURE 3 – Square pass

1.1.3 Diamond pass

In a similar way, the diamond pass computes the average of four other points as described in figure 4.

$$m(i,j) = \frac{m(i-1,j) + m(i+1,j) + m(i,j-1) + m(i,j+1) + r)}{4} \tag{4}$$

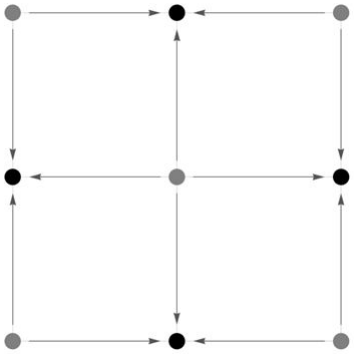

FIGURE 4 – Diamond pass

1.1.4 Water level

Optionally, every value in the height map below a certain thresold can be set to this threshold after the K iterations. This operation can be considered as pouring water on the terrain to give a more realistic rendering.

1.2 Parallelization

1.2.1 Processes topology

In order to make our program more user friendly, this one takes as input the number of processes N and computes the best processes topology PQ. This is achieved by computing the two highest factors such that $N = PQ$.

A process coordinates (p, q) in the processes topology is therefore

$$\begin{aligned} p &= rank \mod P \\ q &= \frac{rank}{P} \end{aligned} \tag{5}$$

While its rank is $rank = qP + p$

1.2.2 Scatter and gather data

Since P_0 is the only process importing and exporting the terrain, this algorithm obviously contain a *scatter* (one-to-all) and *gather* (all-to-one) phase, while data is exchanged between P_0 and the other processes. To reduce the amount of data exchanged and the memory consumption, each process (except P_0) knows only the part of the terrain which has been assigned to it.

The matrix dimensions of a process given its coordinates in the processes grid is given below, with N the number of processes ($\sqrt{N}$ is P if we distribute the data width, Q otherwise). This formula guarantees the best linear data distribution on each axis. For

simplification, we use $L = length = width = height$

$$L = \frac{L + \sqrt{N} - 1}{\sqrt{N}} + \mathbb{1}_{rank<(L+\sqrt{N}-1) \mod \sqrt{N}} \tag{6}$$

1.2.3 Parallel Diamond-Square

The Diamond-Square algorithm computed by each process is almost the same as previously described. However, we can observe that the diamond pass requires data from other processes. Since each point located at the border of the matrix must be computed based on the values computed by the square pass of two different processes, those *ghost cells* must be exchanged at the end of each iteration using a two-dimensions red-black communication.

The data exchanged only contain $\frac{L_k-1}{2}$ data points, i.e. the output of the previous square pass for the corresponding terrain side.

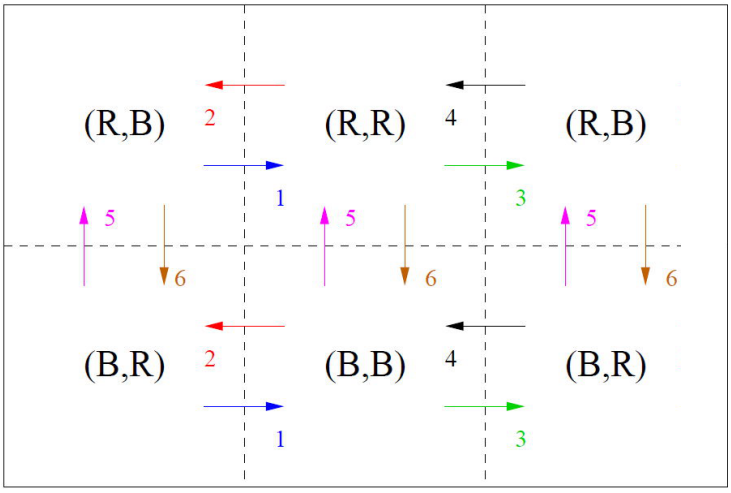

FIGURE 5 – Red-Black communication algorithm

Since the diamond pass computation also includes a random factor, this one is added to the ghost cells sent by a left-rigth or up-down communication so that each process work with the same data.

After this communication step, the diamond pass is finalized by computing the data points on the four sides of the height map.

1.2.4 Parallel algorithm

The parallelized algorithm is eventualy

```
initMPI(argc, argv, &N, &rank);

// Computes the process topology and return the process coordinates
getProcessTopology(N, &P, &Q, &p, &q);
```

```
// Read the initial matrix from an input file then send to each process the
    part it has to compute
data = importAndScatterData(rank, inputFile, &height, &width, &initWidth,
    &initHeight, iterations, P, Q);

// Diamond-Square algorithm, including data exchange
diamondSquare(data, initWidth, initHeight, iterations, p, q, P, Q);

// Pour water on the terrain so that every point below the water level is
    raised to this level
pourWater(data, height, width);

// Gather final data from every process then complete the data matrix for final
    result
gatherData(data, height, width, rank, P, Q, iterations);

if(rank == 0) {
    // Export the final terrain in the output file
    exportData(data, FINAL_TERRAIN_HEIGHT, FINAL_TERRAIN_WIDTH, outputFile);
}

// Release memory and finalizes MPI
clear(data, height);
```

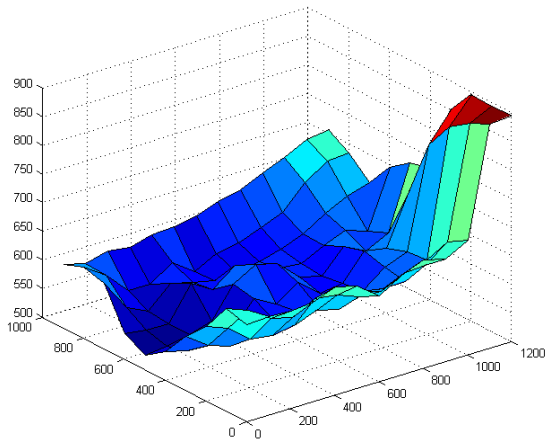

FIGURE 6 – Initial terrain (13x10)

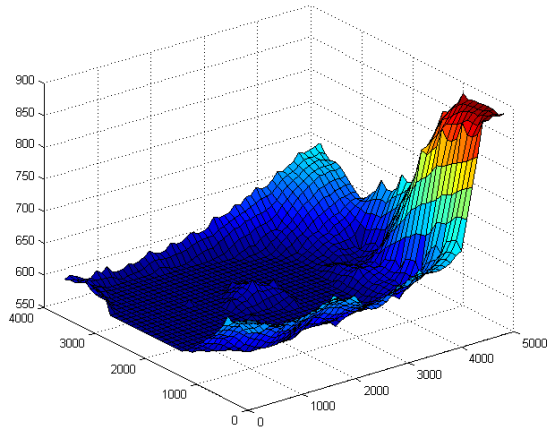

FIGURE 7 – Final terrain - 2 iterations (49x37)

1.3 Parallel efficiency

1.3.1 Complexity

The complexity of the Diamond-Square algorithm is obtained as follows, assuming a perfectly load balanced linear data distribution, $\sqrt{N} = P = Q$ Seeing that for the

Diamond-Square step we have a complexity of

$$\underbrace{\mathcal{O}\left((N-1)\left(\frac{L_0+N-1}{\sqrt{N}}\right)^2\right)}_{\text{scatter data}}+\mathcal{O}\left(\sum_{k=1}^{K}\text{square}+\text{diamond}+\text{exchange}+\text{update diamonds}\right)+ \\ +\underbrace{\mathcal{O}\left((N-1)\left(\frac{L_K}{\sqrt{N}}\right)^2\right)}_{\text{gather data}} \quad (7)$$

L_k is the data width or height at iteration k. Hence, L_K^2 is the final terrain dimension, with K the total number of iterations. The complexity for an iteration is

$$\underbrace{\underbrace{\mathcal{O}\left(\left(\frac{L_k}{2\sqrt{N}}\right)^2\right)}_{\text{square}}+\underbrace{\mathcal{O}\left(\frac{(L_k/\sqrt{N})^2}{2}\right)}_{\text{diamond}}+\underbrace{\mathcal{O}\left(\frac{4L_k}{2\sqrt{N}}\right)}_{\text{exchange}}+\underbrace{\mathcal{O}\left(\frac{4L_k}{2\sqrt{N}}\right)}_{\text{update diamonds}}}_{\text{Diamond-Square}} \quad (8)$$

We eventually get the total complexity

$$\begin{aligned}\mathcal{O}\left(L_0^2+\sum_{k=1}^{K}\left(\left(\frac{L_k}{\sqrt{N}}\right)^2+\left(\frac{L_k}{\sqrt{N}}\right)^2+\frac{L_k}{\sqrt{N}}+\frac{L_k}{\sqrt{N}}\right)+L_K^2\right) \\ =\mathcal{O}\left(L_0^2+\sum_{k=1}^{K}\frac{L_k^2}{N}\right) \\ =\mathcal{O}\left(\frac{L^2}{N}\right)\end{aligned} \quad (9)$$

This last step is obtained by solving the geometric series for $L_K \approx 2^K$ when starting with an empty terrain.

Eq. 6 gives us the data distribution. Based on this equation, the distribution is almost optimal :

$$\frac{L+\sqrt{N}-1}{\sqrt{N}}+\mathbb{1}_{rank<(L+\sqrt{N}-1) \mod \sqrt{N}} \underset{L\to+\infty}{\longrightarrow} \frac{L}{\sqrt{N}}$$

1.3.2 Benchmarks and speedup

According to fig. 8, we can observe that our algorithm scales quite poorly. If the computation of the terrain scales quite efficiently, the loss of performances is mostly caused by the important communication cost when gathering data. Indeed, fig. 9 shows a much more interesting speedup without data gathering.

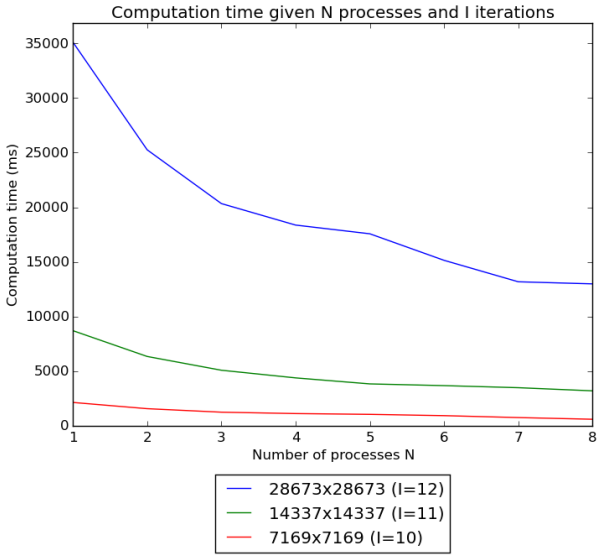

FIGURE 8 – Benchmarks

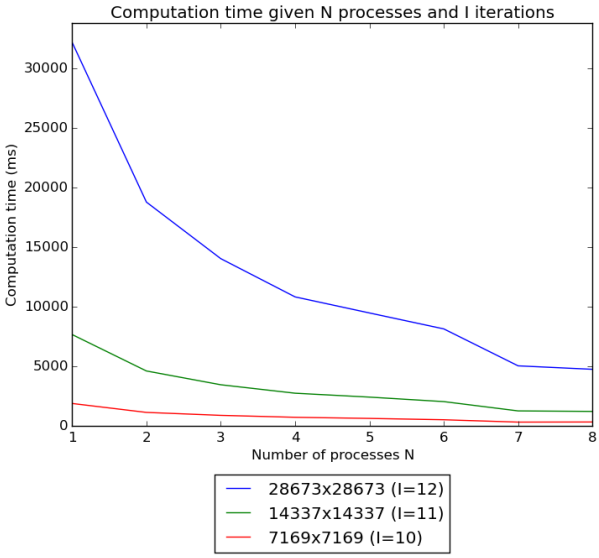

FIGURE 9 – Benchmarks - No data gathering

Because of the amound of data exchanged at the end of the algorithm, we obtain the speedups for an output resolution of 28673x28673 visible in table 1. We assume $T_s \approx T_1$ since our algorithm do not contain any communication or additional computation when $N = 1$.

	Gathering		Without Gathering	
Processes	T_p [s]	S_p	T_p [s]	S_p
2	25.256	1.39	18.773	1.71
4	18.370	1.91	10.820	2.97
8	12.990	2.70	4.746	6.78

TABLE 1 – Running time T_p and speedup S_p for different amount of processes. The running time for a single process is $T_s = 35.100$ s with gathering and $T_s = 32.186$ s without.

2 Linear Displacement

This algorithm is a simple but non-intuitive method to create fractal terrain. It is used as an example to model realistic planets[2], but can be easily be modified to generate rectangular heightmaps.

2.1 Algorithm

The approach of our algorithm can be summarized as following :

- pick a random line going through the heightmap
- increase or decrease (decide at random) by one the height of each point under the line
- repeat until you are satisfied with the result

These steps are shown in fig. 10, where we can see the algorithm at an increasing number of iterations. The algorithm is suprisingly short, yet requires a great deal of iterations to get good results. In our implementation the line is generated by choosing two random points in the grid and finding the equation of the line that passes through them. In cartesian coordinates, if (x_1, y_1) and (x_2, y_2) are two independent random points with uniform distribution, then the equation of the line will be

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1} \tag{10}$$

The fact that we lift (or lower) all the points *under* this line is irrelevant and does not qualitatively change the terrain. We could do it for the points *over* the line if we wanted to. The original algorithm[2], raises one side and lowers the other, but this requires to access all points in the array at each iteration. Our method allows us to access on average only half of the array points, without losing any quality.

Finally a note on the number of iterations : to have a decent quality (no presence of artifacts), we usually need a number of iterations proportional to the number of points

$$N_{iter} = qL_xL_y \tag{11}$$

where we define $0 < q < 1$ as the quality factor. To have acceptable results we empirically take $q \simeq 0.1$.

2.2 Parallelization

The parallelisation of this algorithm is quite simple, as the iterations do not depend on each other. Therefore we split the number of iterations along the processes and, once that is complete, perform a `MPI_reduce()` to superpose the iterations of the different processes on the root process.

2.3 Performance analysis

2.3.1 Complexity

For simplicity we assume $L = L_x = L_y$. At each iteration we have to access on average $L^2/2$ points. But, as discussed before, each process must perform N_{iter}/N iterations. Thus, by eq 11 we have a complexity of

$$\underbrace{\mathcal{O}(qL^2/N)\mathcal{O}(L^2/2)}_{\text{iterations}} + \underbrace{\mathcal{O}(L^2 \log N)}_{\text{reduce}} = \underbrace{\mathcal{O}(L^4/N)}_{\text{overall}} \tag{12}$$

since we can safely assume that $N \ll L^2$.

We remark that this algorithm is considerably slower as it does not scale as well as the other ones we present. However the constant hidden behind the $\mathcal{O}$ notation is considerably small — we simply increment the value of a memory location — making this algorithm an interesting study case nonetheless.

A tipical result is presented in fig. 11. We notice that the peaks are quite jagged. One feature of this algorithm is that it does not take any parameter to control the "smoothness" of the fractal.

2.3.2 Experimental results

Runs on the *Ferlin* computer at PDC confirm that this algorithm is significantly slower than the other two presented. For this reason the benchmarking has been done with significantly lower sizes.

By inspecting the performance results in fig. 12 and fig. 13 we see that the performance is very sensitive to the number of processes and the efficiency actually goes down with the size of the height map. This suggest that the major bottleneck is the communication step, where the processes communicate a very large amount of data.

The data collected is summarized in table 2, along with the average speedup.

$L =$	512	756	1024	
Processes	T_p [s]			$\bar{S}_p$
2	1.2160	5.2860	16.4160	1.98
4	0.6330	2.6900	29.8530	2.94
8	0.3430	2.2200	38.3460	4.20
16	0.2730	1.6430	18.1460	5.67
32	0.1430	0.9000	7.8560	10.9
64	0.1000	0.4360	3.6960	19.0

TABLE 2 – Running time T_p and average speedup $\bar{S}_p$ for different amount of processes and sizes.

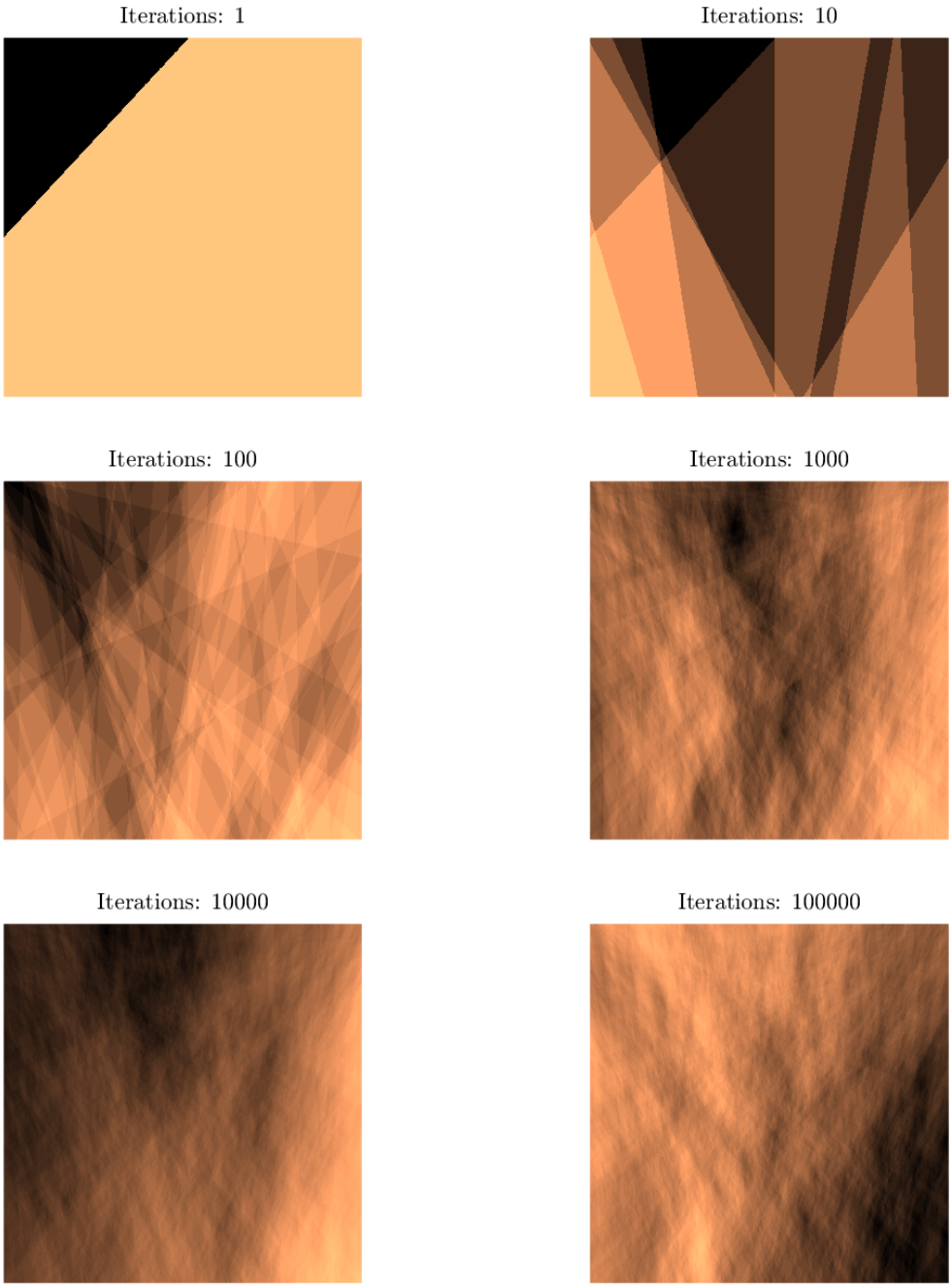

FIGURE 10 – Linear displacement algorithm with an increasing number of iterations.

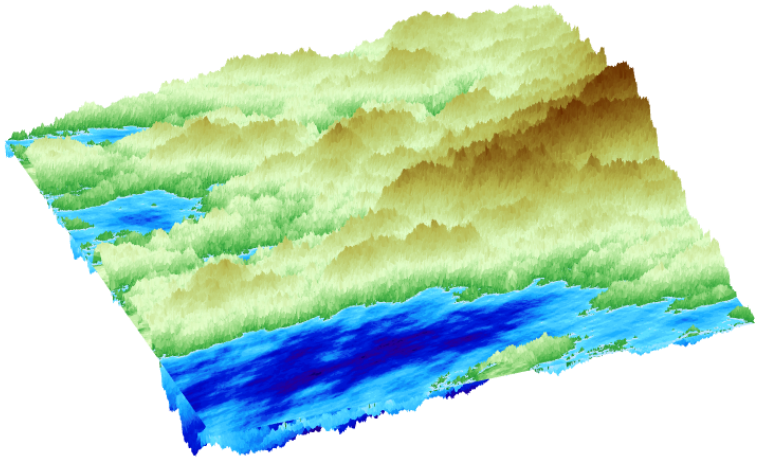

FIGURE 11 – Example of terrain generated with the lines displacement algorithm

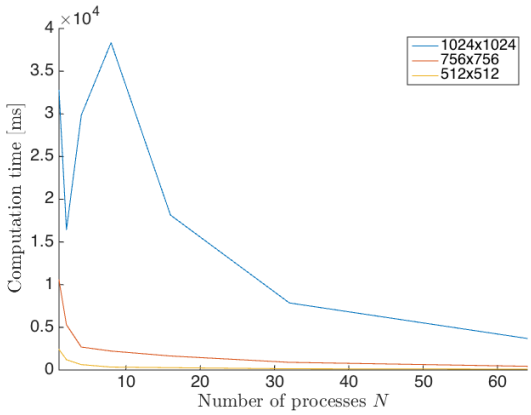

FIGURE 12 – Execution time for the linear displacement algorithm

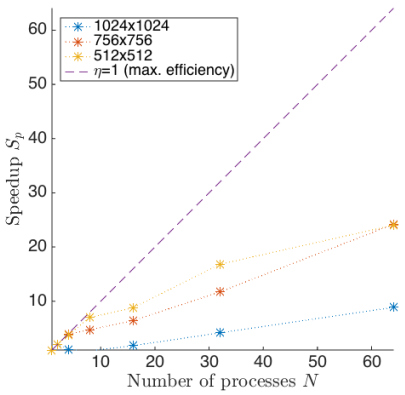

FIGURE 13 – Speedup for the linear displacement algorithm

3 Fast Fourier Transform

Fractal terrain generation with a Discrete Fourier Transform has been experimented previously[4]. The idea is to do the inverse transform of pink noise to generate a terrain. This can be done very quickly with a *Fast Fourier Transform* (which will be later referred as FFT). In order to do so, in our parallel implementation we use the *FFTW*[1] library.

3.1 Algorithm

In the first step each process initially generates it's share of complex coefficients with a pink noise damping. This means that in frequency space we have

$$P(x_i, y_j) = \frac{z}{(x_i^2 + y_j^2)^\alpha} \tag{13}$$

where $0 < \alpha < 2$ is a parameter that will determine the roughness of the terrain, and z is a random complex number with a normal distribution. Since C `stdlib.h` library generates uniformly distributed floating point numbers, this is done with a Box-Muller tranform. This is done by taking two independent random variables with a uniform distribution in the range $(0, 1)$, say u_1 and u_2. Then we can generate a complex number z with a normal distribution as following

$$z = \sqrt{-2 \ln u_1} \exp(2\pi i u_2) \tag{14}$$

Since the full height map will be a $L_x L_y$ matrix, each process with have a slice of $L_x L_y / N$ points, where N is the total number of processes used.

The second step of the algorithm is to perform a parallel FFT of the points we generated in frequency space. In theory we should do an inverse fourier transform, but we can do a direct transform, but this does not change the result qualitatively. It also saves some computations as it saves us from a rescaling factor. The transform is handled by the FFTW library, that has an MPI implementation. The advantage of this configuration is that the matrix is distributed along the processes, so that the algorithm can handle arrays that are too big to fit on a single process. The disadvantage of performing an FFT on a distributed memory system is that the algorithm requires to do a transpose, which involves a *complete exchange* : each process must communicate to all the others.

Finally, each process writes in order their results to `stdout` ; in a tipical usage the user will redirect this output to a file or to `/dev/null` for benchmarking.

One property of this algorithm is that, because of the periodicity of the Fourier transform, it can tiled. As can seen in fig. 14 the algorithm generates more "rolling hills", due to the smoothness and continuity of the sin or cos functions.

3.2 Performance analysis

3.2.1 Complexity

As before we will estimate the complexity with a square height map ($L = L_x = L_y$) with N processes. Since we do not know exactly the complexity of the FFTW routine,

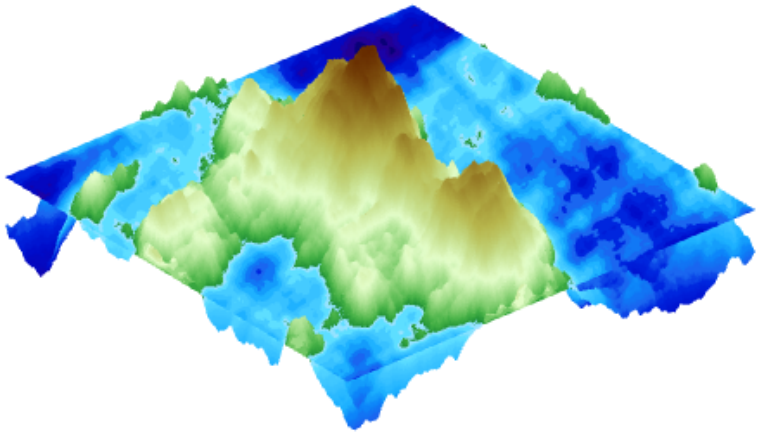

FIGURE 14 – Example of terrain generated with the FFT algorithm ($\alpha = 1.1$)

but we can estimate that at worst it runs in times comparable to a 2D Cooley-Tuckey algorithm. Therefore, ignoring writing times we have

$$\underbrace{\mathcal{O}(L^2/N)}_{\text{init. of coeffients}} + \underbrace{\mathcal{O}((L\log L)^2/N)}_{FFT} = \underbrace{\mathcal{O}(L^2\log^2 L/N)}_{\text{overall}} \tag{15}$$

3.2.2 Experimental results

The benchmarking from parallel runs on the *Ferlin* machine using a number of processes that are powers of 2, for which the FFT is most performant. The results are presented in fig. 15 and fig. 16, where we can observe the execution times and the speedup for different height map size and different number of processes. We remark that the execution times are quite slower compared to the Diamond-Square algorithm, but this algorithm performs impressively well when the scale becomes larger, since the efficiency η is close to 1 (see fig. 16). The data is presented in table 3.

$L =$	3584	7168	14336	
Processes	T_p [s]			$\bar{S}_p$
2	2.266	14.683	62.276	1.80
4	1.190	7.700	32.173	3.44
8	0.676	4.250	17.863	6.17
16	0.406	1.973	9.206	11.9
32	0.236	0.843	5.060	23.7
64	0.116	0.436	2.180	49.3

TABLE 3 – Running time T_p and average speedup $\bar{S}_p$ for different amount of processes and sizes.

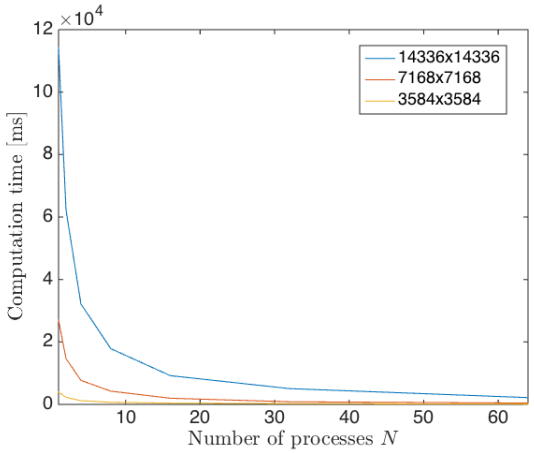

FIGURE 15 – Execution time for the FFT algorithm

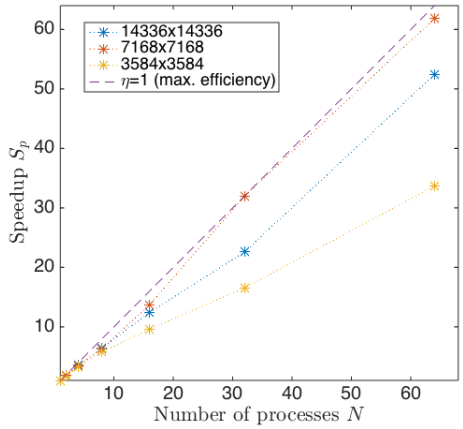

FIGURE 16 – Speedup for the FFT algorithm

4 Conclusion

In this work we provided the implementation and discussed three different stochastic algorithms for the generation of fractal landscapes in parallel environments.

The classic Diamond-Square algorithm has the lowest complexity and performed the fastest, although does not scale as nicely in distributed-memory parallel environments due to the fact that there is an increasing amount of communication required between processes. The somewhat unusual Linear Displacement algorithm was presented for academic interest, but performs quite poorly both at large scales and in large parallel systems. Finally, using the Fast Fourier Transform of pink noise we provided an algorithm that is slower than Diamond-Square but behaves more efficiently in large-scale environments, where the data has to be distributed along different memory locations and the number of processes is considerable.

More advanced terrain generations take in account that the fractal properties are not uniform everywhere. A more advanced technique uses *multifractal systems* where we combines different fractal algorithms. As a simple experiment we combined the results from the algorithms we implemented, as can be seen in fig. 17

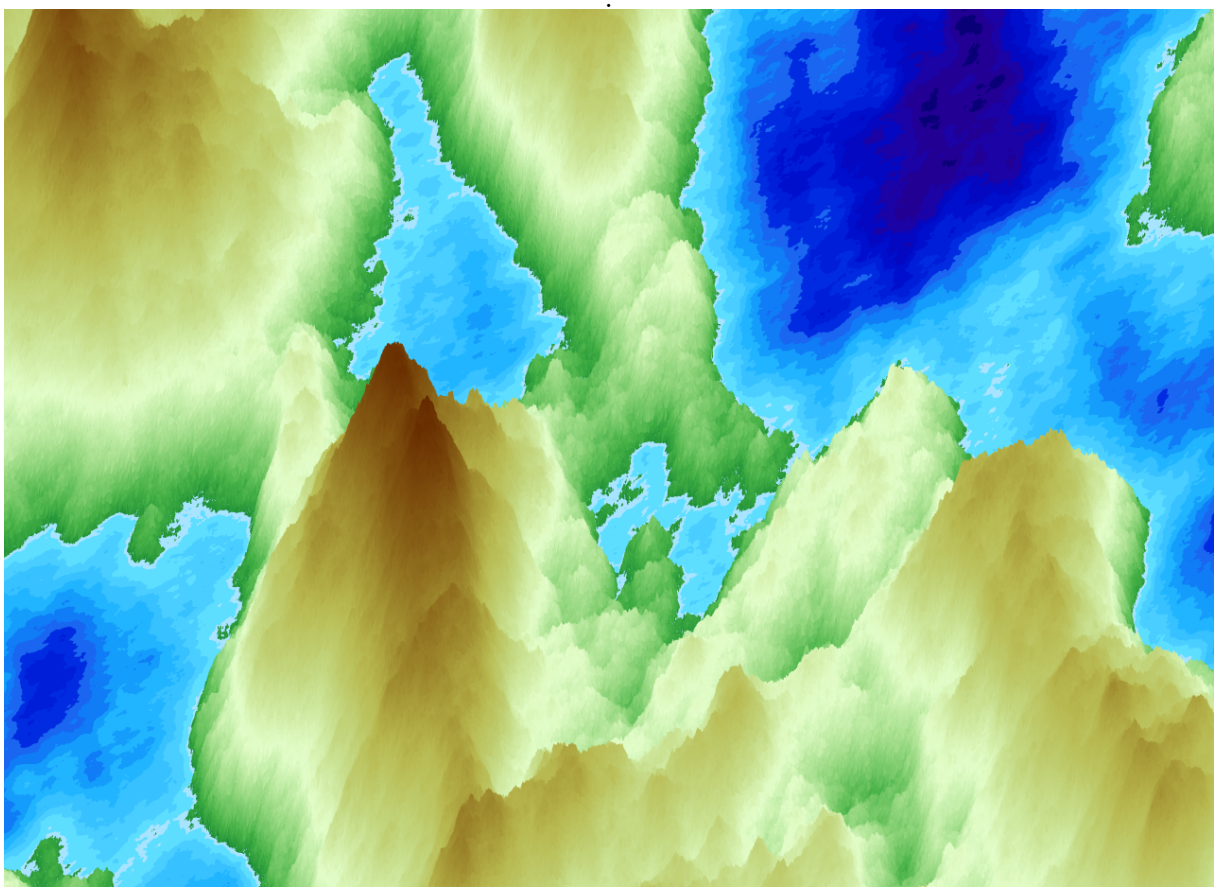

FIGURE 17 – Detail of a terrain generated using different fractal algorithms.

References

[1] FFTW library. `http://www.fftw.org`.

[2] Paul Bourke. Modelling fake planets. `http://paulbourke.net/fractals/noise/`, 2000.

[3] Alain Fournier, Don Fussell, and Loren Carpenter. Computer rendering of stochastic models. *Communications of the ACM*, 25(6) :371–384, 1982.

[4] Eng-Kiat Koh and DD Hearn. Fast generation and surface structuring methods for terrain and other natural phenomena. In *Computer Graphics Forum*, volume 11, pages 169–180. Wiley Online Library, 1992.

[5] Gavin SP Miller. The definition and rendering of terrain maps. In *ACM SIGGRAPH Computer Graphics*, volume 20, pages 39–48. ACM, 1986.