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CS 4364 - Machine Learning
Assignment I
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**1. Probability**

(a) The number of possible orderings of picking the iPhones from the box for $K$:

$$\begin{cases} \text{K} = 0 \rightarrow \binom{20}{0} = 1 \\ \text{K} = 1 \rightarrow \binom{20}{1} = 20 \\ \text{K} = 2 \rightarrow \binom{20}{2} = 190 \end{cases} \tag{1}$$

Possible ordering of draws where at least one deficient iPhone is picked in the first four draws:

$$\begin{cases} \text{K} = 0 \rightarrow 1 \text{ - } \binom{16}{0} = 0 \\ \text{K} = 1 \rightarrow 20 \text{ - } \binom{16}{1} = 4 \\ \text{K} = 2 \rightarrow 190 \text{ - } \binom{16}{2} = 70 \end{cases} \tag{2}$$

Therefore, the individual probability that no deficient iPhone was picked in the first 4 draws:

$$\begin{cases} \text{K} = 0 \rightarrow \frac{1-0}{1} = 1 \\ \text{K} = 1 \rightarrow \frac{20-4}{20} = \frac{4}{5} \\ \text{K} = 2 \rightarrow \frac{190-70}{190} = \frac{12}{19} \end{cases} \tag{3}$$

Hence, because all possibilities were weighted equally, we can deduce that

$$\frac{1 * 95 + 4 * 19 + 12 * 5}{95} = \frac{231}{95} \simeq 2.43$$

We can therefore calculate the probability of each event:

$$\begin{cases} P(K=0 \mid \text{first 4 are not deficient}) = \boxed{\dfrac{1}{2.43} \simeq 0.41} \\ P(K=1 \mid \text{first 4 are not deficient}) = \boxed{\dfrac{0.8}{2.43} \simeq 0.33} \\ P(K=2 \mid \text{first 4 are not deficient}) = \boxed{\dfrac{\frac{12}{19}}{2.43} \simeq 0.26} \end{cases} \tag{4}$$

(b) i. $\frac{1}{3} + \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{3}$

$$= \boxed{\frac{5}{9}}$$

ii.

$$\begin{aligned} P(2) &\rightarrow \quad \tfrac{1}{9} \\ P(3) &\rightarrow \quad \tfrac{5}{9} \\ P(4) &\rightarrow \quad \tfrac{2}{9} \\ P(5) &\rightarrow \quad \tfrac{1}{9} \end{aligned}$$

$$E[N] = 2 \times \tfrac{1}{9} + 3 \times \tfrac{5}{9} + 4 \times \tfrac{2}{9} + 5 \times \tfrac{1}{9} = \boxed{\frac{30}{9} \simeq 3.34}$$

iii. $E[N|C] = \frac{4 \times \frac{2}{9} + 5 \times \frac{1}{9}}{\frac{1}{9} + \frac{2}{9}} = \frac{\frac{13}{9}}{\frac{3}{9}} = \boxed{\dfrac{13}{3} \simeq 4.34}$

**2. Linear Algebra**

(a) $det(A) = 3 \times (2 \times 1 - 2 \times 2) - 2 \times (2 \times 1 - 2 \times 0) + 0 \times (2 \times 2 - 2 \times 0)$
$= 3 \times (-2) - 2 \times 2 + 0 \times 4$
$= -6 - 4 + 0$
$\boxed{= \text{-10}}$

(b) To determine whether $A$ is positive semi-definite, we will compute its eigenvalues. If they are all non-negative, then $A$ is positive semi-definite, otherwise, it is not.

$$det(A-\lambda) = det\begin{pmatrix} 3-\lambda & 2 & 0 \\ 2 & 2-\lambda & 2 \\ 0 & 2 & 1-\lambda \end{pmatrix}$$

$$\begin{aligned} &= (3-\lambda)(2-\lambda)(1-\lambda) + 0 + 0 - 0 - 2\times 2\times(1-\lambda) - (3-\lambda)\times 2\times 2 \\ &= -\lambda^3 + 6\lambda^2 - 3\lambda - 10 \\ &= -(\lambda+1)(\lambda-2)(\lambda-5) = 0 \end{aligned}$$

$$\leftrightarrow \quad \boxed{\lambda_1 = -1, \quad \lambda_2 = 2, \quad \lambda_3 = 5}$$

$\lambda_1$ is negative, so A is not positive semi-definite.

(c) For any square matrix $A$, we can derive its polynomial as:

$$p(A) = a_0 I + a_1 A + a_2 A^2 + ... + a_n A^n$$

The characteristic polynomial of A gives:

$$p(\lambda) = det(A - \lambda I) = b_0 + b_1\lambda + b_2\lambda^2 + ... + b_n\lambda^n$$

We know by definition that $Ax = \lambda x$. Hence, by substitution, we get:

$$p(A) = a_0 I + a_1\lambda + a_2\lambda^2 + ... + a_n\lambda^n$$

Hence,

$$p(A) = p(\lambda)$$