Intermediate research project defense

Extensional equalities in combinatory logic and λ-calculus

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Rémi Germe, supervised by Samuel Mimram

Contents

Introduction “with hands”

Field	Computer science	Mathematics
Studied objects	algorithms, programs	functions
Natural equality	do programs compute the same operations in the same order ? Are they rigorously defined in the exact same way ?	$\forall x, f(x) = g(x) \Rightarrow f = g$
Equality requirements	• results of programs • temporal complexity • spatial complexity • …	• results of functions
Equality type	**intensional (syntactic)**	**extensional**

GOAL : MOVE FROM INTENSIONAL EQUALITIES TO EXTENSIONAL EQUALITIES

1 - Lambda calculus

Definition

$V = \{x, y, z, \ldots\}$: set of variables

λ-terms

- **variable** : $x \in V$
- **application** : for all λ-terms M and N, MN is a λ-term
- **abstraction** : for all λ-term M and variable x, $\lambda x.M$ is a λ-term

Examples :

- $\lambda x.x$ is the identity, yet intensionally $\lambda x.x \not\equiv \lambda y.y$
- $\lambda xy.x \equiv \lambda x.(\lambda y.x)$ is the first projection

Goal : extensional equality

An equivalence relation such that $Mx = Nx \Rightarrow M = N$.

α-reduction

Up to a change of bound variables, terms are equivalent.

Example : $\lambda x.x \underset{\alpha}{=} \lambda y.y$

β-reduction, η-reduction

The evaluation (computation) of a term corresponds to the β-reduction : $(\lambda x.M)U \rhd_\beta [U/x]M$ where $[U/x]M$ denotes M where all free occurrences of x in M have been replaced by U.
Moreover, the η-reduction allows us to simplify terms : $\lambda x.Mx \rhd_\eta M$ if $x \notin \mathrm{FV}(M)$.

Examples :

- $(\lambda x.xw)(\lambda y.y) \rhd_\beta (\lambda y.y)w \rhd_\beta w$
- $(\lambda xy.xy)(\lambda z.wz) \rhd_\beta \lambda y.(\lambda z.wz)y \rhd_\beta \lambda y.wy \rhd_\eta w$
- so $w \underset{\beta\eta}{=} (\lambda x.xw)(\lambda y.y) \underset{\beta\eta}{=} (\lambda xy.xy)(\lambda z.wz)$

Extensionality

For extensionality :

$$(\zeta)\ \frac{Mx = Nx}{M = N} \text{ if } x \notin \mathrm{FV}(MN)$$

Extensionality of $\alpha\beta\eta$

The $\underset{\beta\eta}{=}$ relation is extensional (α is always implied).

In fact, it is equivalent to have (ζ) or $(\xi + \eta)$:

$$(\xi)\ \frac{M = N}{\lambda x.M = \lambda x.N}$$

$$(\eta)\ \lambda x.Mx = M \text{ if } x \notin \mathrm{FV}(M)$$

2 - Combinatory logic

Combinators

- **S**, **K** and **I** are combinators
- for all combinators U and V, UV is a combinator

No variables required, still expressive

Reducing **S**, **K** and **I** (*weak reduction*) :

$$
\begin{array}{lll}
\mathbf{I}U & \triangleright_w U & \text{identity} \\
\mathbf{K}UV & \triangleright_w U & \text{first projection} \\
\mathbf{S}UVZ & \triangleright_w UZ(VZ) & \text{distribution of a variable}
\end{array}
$$

Abstractions for CL

Example : $\lambda xy.yx$

$$\begin{aligned}\mathbf{S}(\mathbf{K}(\mathbf{SI}))(\mathbf{S}(\mathbf{KK})\mathbf{I})xy &\rhd_w \mathbf{K}(\mathbf{SI})x(\mathbf{S}(\mathbf{KK})\mathbf{I}x)y \\ &\rhd_w \cdots \\ &\rhd_w yx\end{aligned}$$

Procedure for abstractions

Adding variables :

- $[x].x \equiv \mathbf{I}$
- $[x].U \equiv \mathbf{K}U$ if $x \notin \mathrm{FV}(U)$
- (η) $[x].Ux \equiv U$ if $x \notin \mathrm{FV}(U)$: cas supplémentaire utilisé dans le livre, omis ici
- $[x].UV \equiv \mathbf{S}([x].U)([x].V)$

$$[x, y].yx \equiv \ldots \equiv \mathbf{S}(\mathbf{K}(\mathbf{SI}))(\mathbf{S}(\mathbf{KK})\mathbf{I})$$

Problem for extensionality

Problem : $(\xi)\ \dfrac{M = N}{[x].M = [x].N}$ does not stand !

$\mathbf{I}x \underset{w}{=} x$ but

- $[x].x \equiv \mathbf{I}$
- $[x].\mathbf{I}x \equiv \mathbf{S}(\mathbf{K}\mathbf{I})\mathbf{I}$
- for now, $\mathbf{I} \underset{w}{\neq} \mathbf{S}(\mathbf{K}\mathbf{I})\mathbf{I}$

Obtain extensionality

Several options :

- add (ζ) in the definition of the extensional equality
- add both (ξ) and (η)
- add a finite number of axioms to have (ξ) and (η)

Extensionality axioms

Process for $\mathbf{I}$:

- we have $\mathbf{I}U = U$
- we want $[x].\mathbf{I}U = [x].U$

$$\begin{aligned}
[x].\mathbf{I}U &\equiv \mathbf{S}(\mathbf{KI})([x].U) \\
&\underset{w}{=} ([u].\mathbf{S}(\mathbf{KI})u)([x].U) \\
&\underset{\mathrm{ax}}{=} ([u].u)([x].U) \\
&\underset{w}{=} [x].U
\end{aligned}$$

Here we found an axiom : $[u].\mathbf{S}(\mathbf{KI})u \underset{\mathrm{ax}}{=} [u].u$ (there are no variables left in this equality).

Same processes for $\mathbf{K}$ and $\mathbf{S}$, with $\mathbf{K}UV = U$ and $\mathbf{S}UVZ = UZ(VZ)$.

Extensionality axioms (ii)

Process for (η), $x \notin \mathrm{FV}(U)$:

- we want $[x].Ux = U$
  - $[x].Ux \equiv \mathbf{S}(\mathbf{K}U)\mathbf{I}$
  - we'd like to add $\mathbf{S}(\mathbf{K}U)\mathbf{I} \underset{\text{ax}}{=} U$ as an axiom BUT we'd need to add it for all U

Solution : "close" the term and ask for $[U, x].Ux = [U].U$

It is then possible to use the previous obtained axiom to simulate an (η)-reduction.

3 - Linear combinatory logic

B, C, I

Linear λ-calculus / combinatory logic

Linear λ-calculus : bound variables are used once and only once.

The corresponding linear combinatory logic is defined using $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{I}$ such that :

- $\mathbf{B}UVZ \rhd_w U(VZ)$ the variable is distributed on the right
- $\mathbf{C}UVZ \rhd_w UZV$ the variable is distributed on the left
- $\mathbf{I}U \rhd_w U$ the identity

Some processes to get extensionality axioms (some details).

For second trimester : formal proofs of results in Agda.