# LECTURE 1. WHAT IS THIS COURSE ABOUT?

This course is an introduction to the techniques used in model theory and differential algebra aimed to study

- *transcendence problems* about holomorphic functions satisfying algebraic differential equations and,
- *integrability problems* about algebraic differential equations.

1.1. **Prerequisites.** We will assume some basic knowledge on the following subjects:

- differential Galois theory for linear differential equations: our standard reference will be the first chapter (p3-p36) of the book *Differential Galois theory* of Marius van der Put and Michael Singer.
- a basic course in model theory and $\omega$-stable structures: our standard reference will be the first two chapters (p1-p43) of the book *Model Theory and Algebraic Geometry* edited by Elisabeth Bouscaren.
- some basic knowledge on algebraic geometry over the complex numbers: the first chapter of Robin Hartshorne's book *Algebraic geometry* will be our standard reference.

1.2. **Basic computational problems.** Given holomorphic functions $f_1, \ldots, f_n$ defined on some complex domain $U$, each satisfying an algebraic differential equation

$$(E) : P(z, y, y', \ldots, y^{(k)}) = 0$$

where $P \in k[X_0, \ldots, X_n]$ where $k = \mathbb{C}, \mathbb{C}(z)$ or $\mathbb{C}(z)^{alg}$.

(Q1) *How to compute a system of generators of the ideal*

$$I(f_1, \ldots, f_n) = \{Q \in \mathbb{C}(z)[X_1, \ldots, X_n] \mid Q(z, f_1(z), \ldots, f_n(z)) = 0 \text{ for all (non singular) values } z \in U\}?$$

This ideal $I(f_1, \ldots, f_n)$ is the called the ideal of algebraic relations among $f_1, \ldots, f_n$. Equally interesting questions also arise by replacing the ideal $I(f_1, \ldots, f_n)$ by its differential analogue, that is, by considering the algebraic relations among the $f_i$ and their successive derivatives.

A simpler problem is simply to count the number of such algebraic relations:

(Q2) *How to compute*

$$\#\begin{cases} \textbf{independent} \text{ alg. relations} \\ \text{satisfied by } f_1, \ldots, f_n \end{cases} = n - \mathrm{td}(f_1, \ldots, f_n/\mathbb{C}(z))?$$

Here, $\mathrm{td}(f_1, \ldots, f_n/\mathbb{C}(z))$ denotes the transcendence degree of the field $\mathbb{C}(z, f_1(z), \ldots, f_n(z))$ generated by $f_1, \ldots, f_n$ over $\mathbb{C}(z)$ in the field of meromorphic functions on $U$.

Finally, we have the integrability problem. Consider a class $\mathcal{C}$ of mermorphic functions of interest. For example, one can take $\mathcal{C}$ to be the class of algebraic functions or

- the class of elementary functions obtained by adding the complex exponential and the logarithm to our class of functions of interest,
- the class of Liouvillian functions obtained by closing $\mathcal{C}$ under "taking a primitive",

- the class of PV-functions obtained by closing the class $\mathcal{C}$ under "solving a linear differential equation.

(Q3) *How to determine if an algebraic differential equation can be solved within the class $\mathcal{C}$ of functions of interest?*

1.3. **The linear case.** In the case where the differential equations are all *linear*, these questions can be handled using *differential Galois theory.* This approach is based on the construction

$$\mathrm{PV} : (\text{linear differential equations}) \longrightarrow \text{ differential ring over } \mathbb{C}(z)$$

which to a linear differential equation $(L)$ associates its PV-ring $R$. Recall that if $(L)$ is given in a matrix form as

$$(L) : Y' = A \cdot Y \text{ where } A \in \mathrm{Mat}_n(\mathbb{C}(z))$$

its PV-ring $R$ is characterized as follows:

PV1 the ring $R$ is a simple differential ring and hence is an integral domain whose ring of fractions has $\mathbb{C}$ as a field of constants.

PV2 $(L)$ admits a fundamental system of solutions $P \in \mathrm{GL}_n(R)$ in $R$.

PV3 $R$ is generated (as a ring extension over $\mathbb{C}(z)$) by the entries of the matrix $P$ and the inverse of its determinant.

A fundamental result is that these three properties characterize a unique differential ring extension of $k$ up to isomorphism and one sets

$$\mathrm{Gal}(L) = \mathrm{Aut}_\delta(R/k).$$

which measures the failure of the PV-extension to be well-defined up to a unique automorphism. This group can then be equipped with the structure of an algebraic group over the constants providing a Galois correspondence:

$$(\text{closed algebraic subgroups of } G) \leftrightsquigarrow (\text{ differential subfields of } K/k)$$

where $K = \mathrm{Frac}(R)$.

**Fact 1.** *Assume that $f_1, \dots, f_n$ are holomorphic functions solutions of a common linear differential system $(L)$ defined over $\overline{\mathbb{Q}}(z)$ with Galois group $G \subset \mathrm{GL}_N(\overline{\mathbb{Q}})$*

(Q1) *There is a decidable procedure returning the equations of $G$ in $GL_N(\overline{\mathbb{Q}})$ and the ideal $I(f_1, \dots, f_n)$ of algebraic relations.*

(Q2) *Setting $H = \mathrm{Stab}_G(f_1, \dots, f_n)$ then the number of independent algebraic relations between $f_1, \dots, f_n$ is given by*

$$r = n - \mathrm{td}(f_1, \dots, f_n/\mathbb{C}(z)) = n - \dim(G/H).$$

(Q3) *Given a linear differential equation $(L)$, it is integrable by algebraic functions if and only if* $\mathrm{Gal}(L)$ *is finite. It is integrable by Liouvillian functions if and only if* $\mathrm{Gal}(L)$ *is soluble by finite.*

For order one linear equations, we have:

**Exercise 1** (first-order linear differential equations)**.** *Consider a linear differential equation of order one*

$$(L) : y' = f(z) \cdot y \text{ where } f(z) \in \mathbb{C}(z)^{alg}$$

*and set $k = \mathbb{C}(z, f(z))$.*

(a) *Show that either* $\mathrm{Gal}(L/k) \simeq \mathbb{C}^*$ *or* $\mathrm{Gal}(L/k) \simeq \mathbb{Z}/n\mathbb{Z}$ *and that these two cases depend on whether* $(L)$ *admits a no nonzero algebraic solution or not.*

(b) *Show that if* $(L)$ *admits a algebraic solution then* $(L)$ *admits an algebraic solution of the form*

$$y'(t) = \sqrt[n]{\frac{P(z, f(z))}{Q(z, f(z))}}$$

*where* $P, Q$ *are polynomials.*

As an application, we prove

**Theorem 1.1** (Functional Lindemann-Weierstrass Theorem). *Let* $f_1(z), \ldots, f_n(z) \in \mathbb{C}(z)^{alg}$ *be non constant algebraic functions defined on some complex domain* $U$.

*if* $f_1(z), \ldots, f_n(z)$ *are* $\mathbb{Q}$*-lin ind. modulo* $\mathbb{C}$ *then their exponentials* $e^{f_1(z)}, \ldots, e^{f_n(z)}$ *are alg. ind. over* $\mathbb{C}(z)$.

The proof will be in two steps. The first step will be to compute the Galois group of a linear differential equation of the form

$$y' = f'(z) \cdot y$$

when $f'(z) \in \mathbb{C}(z)^{alg}$ is the derivative of an algebraic function $f(z)$. The second step is to use the Galois correspondence to obtain the conclusion of the theorem.

**Claim** (Step 1). *Let* $h(x) \in \mathbb{C}(x)^{alg}$ *be an algebraic function and*

$$(L) : y' = h'(x) \cdot y$$

*Then* $\mathrm{Gal}(L) = G_m(\mathbb{C})$ *iff* $h(x) \notin \mathbb{C}$.

*Proof.* One direction is obvious. Assume $\mathrm{Gal}(L) \neq \mathbb{G}_m(\mathbb{C})$ and and that $h$ is not a constant algebraic function for the sake a contradiction. The previous exercise implies that

$$(L) : y' = h' \cdot y$$

admits a nonzero algebraic solution $\phi \in \mathbb{C}(z)^{alg}$. To obtain an analytic realization, note that $h$ as an algebraic function satisfies an (irreducible) algebraic equation of the form

$$y^n + a_{n-1}(z) \cdot y^{n-1} + \ldots + a_0(z) = 0$$

and we can realize $h$ as an analytic function $h(z) \in \mathrm{Hol}(U)$ defined on a simply connected domain $U$ avoiding the poles of the $a_i(z)$. Furthermore, up to restricting $U$ even more, we may assume that

- $h : U \to \mathbb{C}$ is a biholomorphism onto its image.
- the algebraic solution $\phi : U \to \mathbb{C}$ on $U$ is also an holomorphic function on $U$.

A direct computation shows that

$$\psi = \phi \circ h^{-1} : z \mapsto \phi(h^{-1}(z))$$

is a solution of the differential equation $y' = y$. Indeed, the chain rule gives:

$$\frac{d\psi}{dz} = \frac{d\phi}{dx}(h^{-1}(z)) \cdot \frac{dh^{-1}(z)}{dz} = h'(h^{-1}(z)) \cdot \phi(h^{-1}(z)) \cdot \frac{1}{h'(h^{-1}(z)}) = \psi.$$

To conclude, we will use the classical fact from analytic geometry/o-minimal geometry.

**Fact 2** (Analytic geometry)**.** *The class of algebraic functions is stable under composition and compositional inverse (whenever this operation make sense).*

It follows from this fact that $\psi$ is an algebraic solution of $y' = y$ and hence must be equal to zero and hence so is $\phi$ which is our contradiction. □

*Proof of the theorem.* Let $f_1(z), \ldots, f_n(z) \in \mathbb{C}(z)^{alg}$ be nonconstant, set $g_i(z) = e^{f_i(z)}$ and assume that $(*)$

$$\mathrm{td}(g_1(z), \ldots, g_n(z)/\mathbb{C}(z)) < n$$

Note that the $g_i(z)$ are nonzero solutions of

$$(L_i) : y' = f_i'(z) \cdot y \text{ for } i = 1, \ldots, n$$

and generate the PV-extensions associated to the $(L_i)$ which by the previous claim all have Galois group $\mathbb{C}^*$. Now consider the composite

$$L = L_1 \cdots L_n = \mathbb{C}(z)^{alg}(g_1(z), \ldots, g_n(z))/\mathbb{C}(z)^{alg}$$

the composite of all the PV-extensions associated to the $(L_i)$. Since the composite of PV-extensions is a PV-extension, we can consider

$$G = \mathrm{Gal}(L/\mathbb{C}(z)^{alg}).$$

On the one hand, the action of $G$ on $L$ preserves each the $L_i$ and define an algebraic group embedding:

$$i : \mathrm{Gal}(L/k) \to \mathrm{Gal}(L_1/k) \times \ldots \times \mathrm{Gal}(L_n/k) = \mathbb{G}_m^n(\mathbb{C})$$

which is given by the formula

$$\sigma \mapsto (\sigma(f_1) \cdot f_1^{-1}, \ldots, \sigma(f_n) \cdot f_n^{-1})$$

and on the other hand

$$\dim(\mathrm{Gal}(L/k)) = \mathrm{td}(L/k) < n$$

by assumption. It follows that the image of $i$ the is a proper algebraic subgroup of $\mathbb{G}_m^n(\mathbb{C})$. The structure of the subgroups of the multiplicative torus then implies that any automorphism $\sigma$ of $G$ satisfies an equation of the form

$$\prod y_i^{e_i} = 1 \text{ where } y_i = \sigma(f_i) \cdot f_i^{-1}$$

Setting $g = \prod g_i^{e_i} \in L$ and reordering the factors, we obtain that $\sigma(g) = g$ for all $\sigma \in G$ and therefore by applying the Galois correspondence that $g \in \mathbb{C}(z)^{alg}$. Note that $g$ is an algebraic solution of the linear differential equation

$$y' = (e_1 \cdot f_1 + \ldots e_n \cdot f_n)' \cdot y$$

and hence by the applying Step 1 again, we conclude that $e_1 \cdot f_1 + \ldots + e_n \cdot f_n$ is a constant as required. □

1.4. **Decidability and Poincaré's problem.** The objective of this course is to study the algebraic nonlinear case that is:

$$\text{(linear differential equations)} \rightsquigarrow \text{(algebraic differential equations)}.$$

The first observation is that even in the order one case, there is no hope for decidability: indeed, we still don't know the answer to the following question.

**Problem 1** (Poincaré's problem, 1891)**.** *Can one decide whether an order one algebraic equation*

$$y' = f(z,y) \text{ where } f(X,Y) \in \mathbb{C}(X,Y) \tag{1}$$

*admits some algebraic solution or whether all its solutions are algebraic?*

A geometric formulation of this question is: *determine whether a given two -dimensional system of the form*

$$\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases} \quad \text{where } f(X,Y), g(X,Y) \in k[X,Y] \tag{2}$$

*admits an invariant curve, that is, an algebraic curve tangent to the vector field at every point.* Partial answers to Poincaré's problem are known under assumptions on the singularities of (2). In general, it is one of the prominent wide open problems in differential algebra.

In this course, we will not focus on this specific problem but on several general trends which appear in the functional transcendence of the solutions of algebraic differential equations.

1.5. **Definable sets.** Study the functor

$$\mathrm{Sol} : \text{(alg. differential equation)} \to \mathrm{Def}(\mathrm{DCF}_0) \tag{3}$$

which associates to an algebraic differential equation a definable set in the theory DCF of differentially closed fields. It plays a similar role as the role played by PV-ring construction in the linear case described above. It is described the following theorem:

**Theorem A** (Blum-Poizat)**.** *In characteristic zero, the theory of differentially closed field is a* complete noetherian theory *and eliminates* quantifiers and imaginaries *in the language of differential rings.*

The completeness of the theory $T = \mathrm{DCF}_0$ expresses that the properties of the right-hand side of (3) do not depend on the chosen differential closed fields. The fact that this theory is noetherian means that it is equipped with a *definable noetherian topology* such that

every definable set is a boolean combination of

Other examples of such first order theory include the theory CCM which describe the *analytic compact complex manifolds*. This theory shares many of the properties of $\mathrm{DCF}_0$. Note also that a nonexample is the theory $\mathrm{ACFA}_0$ which is both not complete and not noetherian. In our case, this topology is called the *Kolchin topology* and the closed sets in this topology are precisely the image of the functor Sol.

The elimination of quantifiers can be seen as an effective form of this property and expresses that every definable set can be decomposed as closed subsets for this topology. The eliminations of imaginaries expresses that one can take quotient in the category $\mathrm{Def}(\mathrm{DCF}_0)$: if $D$ is a definable set and $R$ is a definable equivalence relation on $D$, then $R/D$ can be identified with an object of $\mathrm{Def}(\mathrm{DCF}_0)$. It is of central importance in the model-theoretic of algebraic differential equations.

1.6. **Orthogonality and automatic algebraic independence.** Let $D_1, D_2$ be two definable sets say defined over the differential field $k = \mathbb{C}(z)^{alg}$ of functions. In the rest of the course, it will be more convenient to works with types rather than definable sets but for clarity we will first stick with definable sets first.

**Definition 1.2.** The definable sets $D_1$ and $D_2$ are *fully orthogonal* if whenever $f_1, \ldots, f_n \in D_1$ and $g_1, \ldots, g_m \in D_2$, we have

$$f_1, \ldots, f_n \underset{k}{\overset{\mathrm{DCF}}{\perp}} g_1, \ldots g_m.$$

This model-theoretic notation reads as $f_1, \ldots, f_n$ are independent of $g_1, \ldots, g_m$ over $k$. Since $k$ is algebraically closed, it expresses the additivity of the transcendence degree

$$\mathrm{td}(\phi, \psi/k) = \mathrm{td}(\phi/k) + \mathrm{td}(\psi/k)$$

whenever $\phi$ and $\psi$ are respectively algebraic combinations of elements of $D_1$ and $D_2$ and their derivatives.

**Example 1.3** (Umemura, Nagloo-Pillay)**.** The solution set of the Painlevé equation is fully orthogonal to every linear differential equation and to every order one differential equation.

The theorem of Nagloo-Pillay applies in fact for a broad classes of algebraic differential equations.

**Definition 1.4.** A definable set $D$ is strongly minimal if every definable subset of $D$ (with parameters) is either finite or cofinite in $D$.

Equivalently, the Kolchin-closed set $D$ defined by the vanishing of

$$y^{(n)} = F(y, \ldots, y^{(n-1)}) \text{ where } n \geq 1, f \in k[Y_0, \ldots, Y_n] \tag{4}$$

is strongly minimal if and only if

- *topology.* $D$ equipped with the Kolchin topology is a curve, that is, its Krull dimension is one and $D$ is irreducible.
- *transcendence.* whenever $y \in D$ and $k$ is a differential field of definition of $D$, we have $\mathrm{td}(y/k) = 0$ or $n$.

**Theorem B** (Nagloo-Pillay)**.** *Assume that the solution set of (4) is strongly minimal and that $n \geq 2$. Then the solution set of (4) is fully orthogonal to the solution set of any linear differential equation and of any algebraic differential equation of order $< n$.*

Note that the case $n = 1$ of order one equations is excluded in the previous theorem. In that case, strong minimality of (4) is automatic and hence, for obvious reasons, this statement has no direct analogue. Nevertheless, one can give a complete description of automatic independence statements for order one algebraic differential equations as well.

1.7. **Ax-Schanuel Theorems.** The second type of "qualitative" transcendence statements we will describe are presented in *Ax-Schanuel form.* The prototype example is Schanuel's conjecture which (conjecturally) describes the exponential function in number theory:

whenever $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ are $\mathbb{Q}$-linearly independent, we have

$$\operatorname{td}(\alpha_1, \ldots, \alpha_1, \exp(\alpha_1), \ldots, \exp(\alpha_n)/\overline{\mathbb{Q}}) \geq n. \tag{5}$$

Note that when $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{Q}}$, then (5) implies that

$$\operatorname{td}(\exp(\alpha_1), \ldots, \exp(\alpha_n)/\overline{\mathbb{Q}})) = n \tag{6}$$

This consequence is in fact known; this is the *Lindemann-Weierstrass Theorem* whose functional version is Theorem 1.1. The characteristic feature of Schanuel's conjecture is that the hypothesis $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{Q}}$ at the expense of introducing the $\alpha_i$ in the left-hand side of (5). Moreover, one can not hope for an equality and any value between $n$ and $2n$ is possible.

**Theorem C** (Ax-Schanuel)**.** *Let $f_1(z), \ldots, f_n(z)$ be holomorphic functions. Then*

$$\operatorname{td}(f_1(z), \ldots, f_n(z), e^{f_1(z)}, \ldots, e^{f_n(z)}/\mathbb{C}) \geq n + 1$$

*provided $f_1(z), \ldots, f_n(z)$ are $\mathbb{Q}$-linearly independent modulo $\mathbb{C}$.*

This theorem is a striking applications of the methods of differential algebra to functional transcendence. There have been many successful attempts to describe similarly the functional transcendence of many important transcendental functions and we will focus on the techniques used in the case of the exponential function in this course.

1.8. **Application to computational questions.** As an illustration of these techniques, we will (partially) treat some of the following problems about (Q1), (Q2) and (Q3) of the following form.

**Problem 2** (Rosenlicht)**.** *The error function given by*

$$\operatorname{erf}(z) = \int e^{-z^2} dz$$

*can not be expressed as an elementary function in the sense of Liouville.*

**Problem 3** (Duan-Nagloo)**.** *Consider distinct solutions $(x_i, y_i)$ of the prey-predator system*

$$\begin{cases} x' = ax - bxy \\ y' = -cy + dxy \end{cases} \quad \text{where } a, b, c, d \text{ are nonzero complex numbers and } a \neq c.$$

*Then we have:* $\operatorname{td}(x_1, \ldots, x_n, y_1, \ldots, y_n/\mathbb{C}(z)) = 2n$.

**Problem 4** (J.-Kirby)**.** *Any distinct solutions of Kepler's equation*

$$w + \sin(w) = z$$

*are algebraically independent over $\mathbb{C}(z)$.*

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