LECTURE 3. DIFFERENTIALLY CLOSED FIELDS

We now apply the previous results to describe the model-theoretic properties of the theory DCF_0 of *differentially closed fields of characteristic zero.* This theory will be written in the language of differentials rings

$$\mathcal{L}_\partial = \{0, 1, +, \times, -, \partial\} = \mathcal{L}_{rings} \cup \{\partial\}$$

which is the language of rings expanded with a unary function symbol ∂. A $\mathcal{L}_\partial$-structure (K, ∂) is a model of *the theory* DCF_0 if it satisfies the following (schemes of) axioms:

(A1) $K \models \mathrm{ACF}_0$ is an algebraically closed field of characteristic zero,

(A2) ∂ is additive and satisfies the Leibniz rule:

$$\partial(x + y) = \partial(x) + \partial(y) \text{ and } \partial(xy) = x\partial(y) + y\partial(x)$$

for all $x, y \in K$.

(A3) for every nonconstant differential polynomial $f, g \in K\{X\}$ with $\mathrm{ord}(g) < \mathrm{ord}(f)$, there exists $x \in K$ such that

$$f(x) = 0 \wedge g(x) \neq 0.$$

Lemma 3.1. *The theory* DCF_0 *is consistent. Futhermore, any differential field k is contained in a model of* DCF_0.

Proof. Let k be a differential field and let $f, g \in k\{X\}$ with $\mathrm{ord}(g) < \mathrm{ord}(f)$. Denote by f_1 an irreducible factor of f so that $\mathrm{ord}(f_1) = \mathrm{ord}(f)$ and consider $I(f_1)$ the prime differential ideal given by Theorem **??**. By construction of this ideal, $g \notin I(f_1)$ as g has lower order than f_1. It follows that the differential field

$$l = \mathrm{Frac}(k\{X\}/I(f_1))$$

extends k and contains an element a — the image of X — such that $f(a) = 0 \wedge g(a) \neq 0$.

- Iterating the process we produce a differential field extension $k_1 \mid k$ such that every system of equation and differential equations as above with coefficients in k has a solution in k_1.
- Iterating this new process, we obtain a chain of differential field extension

$$k \subset k_1 \subset k_2 \subset \ldots \subset k_n \subset \ldots$$

such that every system of equation and differential equations as above with coefficients in k_i has a solution in k_{i+1}.

Clearly, the limit $k = \cup_{i\in\mathbb{N}} k_i$ is a differentially closed field containing k. □

3.1. Elimination of quantifiers.

Theorem 3.2 (Elimination of quantifiers). *The theory* DCF_0 *admits the elimination of quantifiers in the language* $\mathcal{L}_\partial$ *of differential rings.*

Recall the following criterion for quantifier-elimination: Let T be a theory in a language $\mathcal{L}$. The theory T has QE in the language $\mathcal{L}$ if and only if

$(*)$ whenever $M, N \models T$ extend a common finitely generated substructure A, $\overline{a} \in A^n$, $m \in M$ and $\phi(x, \overline{y})$ a quantifier-free $\mathcal{L}$-formula (without parameters) such that

$$M \models \phi(m, \overline{a}) \Rightarrow N \models \exists x \phi(x, \overline{a})$$

Furthermore, up to replacing N by an elementary overstructure, we may assume that N is ω-saturated[1] in order to check $(*)$.

[1]This means that every countable set of $\mathcal{S} = \{\phi_i(x, \bar{l}) \mid i \in \mathbb{N}\}$ which is *finitely* satisfiable in N is satisfiable in N.

Proof. Consider $K, L \models \mathrm{DCF}_0$ containing a common finitely generated $\mathcal{L}_\partial$-substructure A and assume that L is ω-saturated. By definition of the language, A is a finitely generated substructure means that A is a finitely generated differential subring of K and L respectively and in particular is an integral domain. Denote by k the algebraic closure of the fraction field of A. Since *the derivation on A extends uniquely to k* and K, L are algebraically closed differential fields (by axioms (A1) and (A2)), the inclusions $i_K : A \to K$ (resp. $i_L : A \to L$) extends uniquely to inclusions

$$\overline{i_K} : k \to K \text{ (resp. } \overline{i_L} : k \to L)$$

Consider $m \in K$, $\overline{s} \in k$ and $\phi(x, \overline{y})$ quantifier-free such that $K \models \phi(m, \overline{s})$. To show that $L \models \exists \phi(x, \overline{s})$, a direct inspection of quantifier-free formulas[2] shows that it is enough to find $n \in L$ such that

$$k\langle m \rangle \simeq k\langle n \rangle$$

as differential fields over k sending m to n. If $m \in k$ then there is nothing to do. Otherwise, we distinguish according to the position of m with respect to k.

- Case 1. $m \in K$ satisfies a nontrivial differential equation over k that is

$$I(m/k) = \{f \in k\{X\} \mid f(m) = 0\}$$

  is a nonzero ideal of $k\{X\}$.

By Theorem **??**, $I(m/k) = I(f)$ where f is a minimal nonzero polynomial in $I(m/k)$ with respect to $\ll$ and in particular is irreducible. The axiom (A3) of DCF_0 implies that the countable set of formulas

$$\{f(x) = 0 \wedge g(x) \neq 0 \mid g(x) \in k\{x\} \text{ with } \mathrm{ord}(g) < \mathrm{ord}(f)\}$$

is finitely satisfiable in L and hence by ω-saturation satisfiable in L. By hypothesis, $I(n/k)$ is a prime ideal containing f and no differential polynomial of lower order. Since f is irreducible, we have

$$I(n/k) = I(f) = I(m/k) \Rightarrow k\langle m \rangle \simeq k\langle n \rangle$$

as required.

- Case 2. $m \in K$ satisfies no nontrivial differential equation over k.

In that case, by ω-saturation of L, the countable set of formulas

$$\{f(x) \neq 0 \mid f(x) \in k\{X\}\}$$

is finitely satisfiable and hence satisfiable in L say by $n \in L$. By construction, we have

$$k\langle m \rangle \simeq k\langle X \rangle \simeq k\langle n \rangle.$$

as required. This completes the proof of the theorem. □

Corollary 3.3. *The theory* DCF_0 *is complete.*

Proof. Every differentially closed field contains $\mathbb{Q}$ equipped with the trivial derivation as a substucture. A theory with QE whose models share a common substructure is complete (exercise). □

3.2. **Geometric consequences.** *Fix for the rest of the section* $K \models \mathrm{DCF}_0$.

Definition 3.4. A *Kolchin-closed subset* Σ of K^n is a set of the form

$$\Sigma = \{\overline{x} \in K^n \mid f_1(x) = \ldots = f_n(x) = 0\}$$

where $f_1, \ldots, f_n \in K\{X_1, \ldots, X_n\}$ are differential polynomial of n variables. A Kolchin-closed set is called irreducible if it can not be written as the union

$$\Sigma = \Sigma_1 \cup \Sigma_2 \text{ with } \Sigma_1 \not\subset \Sigma_2 \text{ and } \Sigma_2 \not\subset \Sigma_1.$$

[2]The quantifier-free formulas with parameters from k are the boolean combination of formulas of the form $P(x, \overline{s}) = 0$ where $P \in k\{x\}$ is a differential polynomial

Corollary 3.5 (Differential Nullstenlensatz). *We have an inclusion reversing one-to-one correspondence*

$$\begin{array}{rcl} \{ \text{ Kolchin-closed subset of} K^n\} & \leftrightarrows & \{ \text{ radical differential ideals of } K\{\overline{X}\}\} \\ \Sigma & \to & I(\Sigma) = \{f \in K\{\overline{X}\} \mid f(x) = 0 \text{ for all } x \in \Sigma\} \\ V(I) = \{\overline{x} \in K^n \mid f(\overline{x}) = 0 \text{ for all } f \in I\} & \leftarrow & I \end{array}$$

Furthermore, the Kolchin-topology of K^n is a noetherian topology and irreducible Kolchin-closed subsets correspond to prime ideals.

Proof. Clearly, $I(\Sigma)$ is an ideal. It is radical and differential since for every $\overline{x} \in K^n$,

$$f^n(\overline{x}) = 0 \Rightarrow f(\overline{x}) = 0 \text{ and } f(\overline{x}) = 0 \Rightarrow \partial(f)(\overline{x}) = 0$$

as the evaluation is a morphism of differential rings. Convesely, $V(I)$ is a Kolchin-closed set since by Theorem **??**, $I = \{f_1, \ldots, f_n\}$ is finitely generated. Furthermore, we have

$$V(I(\Sigma)) = \Sigma \text{ and } I(V(\Sigma)) = I.$$

Indeed, the first equality is trivial (and does not use the fact that $K \models \mathrm{DCF}_0$). To prove the second one, note that $I \subset I(V(\Sigma))$ and consider $f \in K\{\overline{X}\} \setminus I$. Write

$$I = \bigcap_{j=1}^{n} I_j$$

where the I_j are prime differential ideals so that $f \notin I_j$ for some j. It follows that

$$L = \mathrm{Frac}(K\{\overline{X}\}/I_j) \subset \mathcal{U} \models \mathrm{DCF}_0$$

is a differential field. By construction, The image of $\overline{x}$ of $\overline{X}$ in $\mathcal{U}$ satisfies

$$\overline{x} \in \Sigma \wedge f(\overline{x}) \neq 0$$

so that $\mathcal{U} \models \exists \overline{x}(\overline{x} \in \Sigma) \wedge f(\overline{x}) \neq 0$ which is a sentence with parameters from K. It follows from Theorem 3.2 that modulo DCF_0, this formula is equivalent to a quantifier-free formula which is satisfied in $\mathcal{U}$ iff it is satisfied in K. It follows that

$$K \models \exists \overline{x}(\overline{x} \in \Sigma) \wedge f(\overline{x}) \neq 0$$

and hence that $f \notin I(V(\Sigma))$ as required. The second part of the statement is left as an exercise using Theorem **??**. □

Corollary 3.6 (Description of types). *Let $k \subset K$ be a differential subfield. The function*

$$I : \begin{cases} S_n(k) & \to \mathrm{Spec}_\partial K\{X_1, \ldots X_n\} \\ p & \to I = \{f \in k\{X_1, \ldots, X_n \mid \text{“}f(x) = 0'' \in p\} \end{cases}$$

is a bijection where $S_n(k)$ denotes the model-theoretic space of types and $\mathrm{Spec}_\partial K\{X_1, \ldots X_n\}$ *is the set of differential prime ideals of $k\{X_1, \ldots, X_n\}$.*

Proof. We first need to show that I is well defined and that I is a prime ideal. Take $a \models p$. By enlarging K if necessary, we can find a realization $a = a_1, \ldots, a_n$ of p in a model of DCF. By construction of a, we have that

$$I = I(a) = \{f \in k\{X_1, \ldots, X_n\} \mid f(a) = 0\}$$

and the fact that I is a prime ideal follows easily from this presentation. It remains to show that I is injective and surjective. The second part is automatic. The first part follows directly from quantifier elimination: since every formula is equivalent to boolean combination of formulas of the form $f(x) = 0$, a type $p \in S_n(k)$ is determined by the function

$$f(x) \mapsto \chi_p : \begin{cases} 0 \text{ if “}f(x) = 0'' \in p \\ 1 \text{ otherwise} \end{cases}$$

which is the characteristic function of the subset I in $k\{X_1, \ldots, X_n\}$. Surjectivity follows from the differential Nullstellensatz as any partial type $\pi(x)$ (a consistent set of formulas) can be extended to a complete type. □

Theorem 3.7. *The theory* DCF_0 *is ω-stable (in the sense of model theory).*

Proof. (Counting types) A theory T in a countable language if for any set of parameters A, we have that

$$| S_1(A) |=| A |$$

$\square$

3.3. **Elimination of imaginaries.** Recall that a complete theory T in a language of $\mathcal{L}$ admits the *elimination of imaginaries* if for every definable equivalence relation E on some definable set $D \subset M^n$ of some model $M \models T$, there exists a definable function $f : D \to M^s$ such that

$$xEy \Leftrightarrow f(x) = f(y).$$

One can then identify D/E with the definable set $f(D)$ and therefore under this condition take quotient without leaving the category of definable sets. To prove the elimination of imaginaries, we will use a rather indirect path. *Fix once for all $K \models \mathrm{DCF}_0$ an ω saturated and ω-homogeneous model.*

Definition 3.8. Let $\phi(x, a)$ be a formula (in the language of differential rings). A *differential field of definition* for $\phi(x, a)$ is a differential subfield $k \subset K$ such that there exists a formula $\psi(x, b)$ with parameters $b = b_1, \dots, b_n$ from k such that

$$\psi(x, b) \leftrightarrow \phi(x, a).$$

Similarly, if I is a differential ideal of $K\{X_1, \dots, X_n\}$, *a differential field of definition for I* is a differential subfield of K which contains a system of generators for I.

Lemma 3.9. *Let $\phi(x, a)$ be a formula and let k be a differential subfield of K. k is a field of definition of* $\phi(x, a)$ if and only if *for every $\sigma \in \mathrm{Aut}_\partial(K)$,*

$$\sigma \textit{ fixes } k \textit{ pointwise } \Rightarrow \sigma(D) = D \textit{ setwise}$$

where $D = \phi(K, a)$ is the definable set defined by $\phi(x, a)$.

Proof. We only need to prove the converse. We first claim that the second part of the statement implies that for any type $p \in S(k)$ and any two realizations $a, b \models M$ We first claim that ω $\square$

Proposition 3.10. *The following properties are equivalent:*

(i) $T = \mathrm{DCF}_0$ *admits the elimination of imaginaries,*
(ii) *every formula admits a smallest (finitely generated) differential field of definition,*
(iii) *every radical differential ideal of $K\{X_1, \dots, X_n\}$ admits a smallest (finitely generated) differential field of definition.*

Proof. (i) $\Rightarrow$ (ii). Let $\phi(x, a)$ be a formula with $a = a_1, \dots, a_n$. Consider the definable equivalence relation $E(y, z)$ on K^n defined by

$$E(y, z) \text{ iff } K \models \forall x(\phi(x, y) \leftrightarrow \phi(x, z))$$

and denote by $f_E : K^n \to K^m$ the function witnessing elimination of imaginaries. We first claim that the differential field k generated by $\alpha = f_E(a)$ is the smallest differential field k of definition of $\phi(x, a)$. Indeed, by construction

$$\sigma \text{ fixes } k \text{ pointwise iff } \sigma(\alpha) = \alpha \text{ iff } K \models \phi(x, a) \leftrightarrow \phi(x, \sigma(a)) \text{ iff } \sigma(D) = D$$

and we conclude by the previous lemma that k is the smallest differential field of definition of $\phi(x, a)$.

(ii) $\Rightarrow$ (i) By Ritt-Raudenbush Theorem, every radical differential ideal I can be written as

$$I = \{f_1, \dots, f_s\}$$

where $f_1, \dots, f_s$ is a finite set of differential polynomials. Consider the formula

$$\phi(x, a) := \text{“}f_1(x) = 0 \wedge \ldots \wedge f_s(x) = 0''$$

where $a \in k$ is the tuple consisting of all the coefficients of the f_i. Clearly, by the differential Nullstenlensatz, a differential field of definition for $\phi(x, a)$ is a differential field of definition of I.

(iii) $\Rightarrow$ (i) Let $E(y, z)$ be a definable equivalence relation on some definable set D defined over k. For $a \in D$, denote by

$$[a]_E = \{x \in D \mid xEa\}$$

and by $\overline{[a]_E}$ its Kolchin-closure. We claim that

$$\overline{[a]_E} = \overline{[b]_E} \text{ iff } aEb.$$

Indeed, clearly $aEb \Rightarrow [a]_E = [b]_E \Rightarrow \overline{[a]_E} = \overline{[b]_E}$. Conversely, if $\overline{[a]_E} = \overline{[b]_E}$ then $[a]_E$ contains a dense open Kolchin-subset U_a of $\overline{[a]_E}$ and so does $[b]_E$. Since any two dense Kolchin subset intersect, we have $U_a \cap U_b \neq \emptyset$ which implies (by transitivity) that aEb as required.

Now fix $a \in D$ and set α for a generator of the differential field of definition of $I(\overline{[a]_E})$. Since $[a]_E$ is $k\langle a\rangle$-definable so is $\overline{[a]_E}$ and therefore

$$\alpha \in k\langle a\rangle$$

so that there exits a k-definable function $f_a : D \to K^m$ such that the preimages of α is the equivalence class of a. Since this is true for any $a \in D$, a compactness argument finishes the proof (exercise). □

Theorem 3.11 (Field of definition of an ideal)**.** *Every ideal I of $K[X_1, \dots, X_n]$ admits a smallest field of definition.*

Proof. Denote by M a basis of monomials of $K[\overline{X}]/I$ as a K-vector space. Each monomial u of $K[\overline{X}]$ can be uniquely written as

$$u = \sum_{m \in M} a_{u,m} m + f_u$$

where $f_u \in I, a_{u,m} \in K$.

Claim. *The field*

$$k = \mathbb{Q}[a_{u,m} \mid u \text{ monomial of } K[\overline{X}], m \in M]$$

is the smallest field of definition of I.

- Step 1. *We show that k is a field of definition of I.*

For $f \in I$, we can write

$$f = \sum_{u \text{ mon. of } K[\overline{X}]} b_u u = \sum_{u \text{ mon.} K[\overline{X}]} b_u \cdot \Big(u - \sum_{m \in M} a_{u,m} m\Big) + \sum_{m \in M} \Big(\sum_{u \text{ mon. of } K[\overline{X}]} b_u a_{u,m} \Big) \cdot m$$

Since by definition the left term lies in I and M is a K-basis of $K[\overline{X}]/I$, we conclude that all the coefficients of the right term must be zero and hence that

$$f = \sum_{u \text{ mon.} K[\overline{X}]} b_u \cdot \Big(u - \sum_{m \in M} a_{u,m} m\Big).$$

It follows that I is generated by the $u - \sum_{m \in M} a_{u,m} m \in k[\overline{X}]$ where u ranges over all monomials of $K[\overline{X}]$ so that k is indeed a field of definition for I.

- Step 2. *Consider l another field of definition of I. We show that $k \subset l$.*

Note that every automorphism of K extends to an automorphism of $K[X_1, \dots, X_n]$ by setting:

$$\sigma(\sum_{m \in mon.k[X]} f_m \cdot m) = \sum_{m \in mon.k[X]} \sigma(f_m) \cdot m$$

Since l is a field of definition of I, for every $\sigma \in \mathrm{Aut}(K/l)$, we have $\sigma(I) = I$. It follows that for every monomial u, we have

$$u = \sigma(u) = \sum_{m \in M} \sigma(a_{u,m}) \cdot m + \sigma(f_u)$$

By uniqueness of the decomposition, it follows that $\sigma(a_{u,m}) = a_{u,m}$ for every $\sigma \in \mathrm{Aut}(K/l)$ and every u, m. We have therefore shown that k is a subset of l. □

Theorem 3.12 (Elimination of imaginaries)**.** *The theory* DCF_0 *eliminates imaginaries in the language $\mathcal{L}_\partial$ of differential rings.*

Proof. It is enough to show that every radical differential ideal I admits a smallest differential field of definition using Proposition 3.10. By the Ritt-Raudenbush Theorem, we can find a finite set of differential polynomial such that

$$I = \{f_1, \dots, f_n\}$$

Consider N large enough so that $f_1, \dots, f_n \in K[X, X', \dots, X^{(N)}]$ and set J for the ideal they generate. By Theorem 3.11, J has a smallest field of definition $k \subset K$. It is now easy to see that the differential field $\tilde{k}$ generated by k is the smallest differential field of definition of I. □

Example 3.13. Let $K \models \mathrm{DCF}_0$ and denote by $\mathcal{C}$ the field of constants of K.

- The imaginary K^*/C^* is eliminated by the function
$$\partial log : y \mapsto \partial(y)/y$$
- Consider the action of the affine group $Aff_2(C)$ on the affine line K. The imaginary $K/Aff_2(C)$ is eliminated by the affine distorsion:
$$y \mapsto \partial^2(y)/\partial(y)$$
- Consider the action of $PSL_2(C)$ on the projective line $\mathbb{P}^1(K)$. The imaginary $\mathbb{P}^1(K)/PSL_2(C)$ is eliminated by the Schwarzian derivative
$$y \mapsto (y''/y')' - 1/2(y''/y')^2$$