On uniform relative internality and orthogonality to the constants

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Introduction

This is a report on a joint work with Léo Jimenez and Anand Pillay which originated from the confluence of two different problems:

- Describe fibrations of definable sets in stable theories satisfying the canonical base property.

For example, when a fibration $\pi : Z \to X$ can be transformed (up to a generically finite to finite correspondence) into a trivial fibration $\pi_2 : Z_0 \times X \to X$?

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- Study the structure of the set of solutions of a differential equation (E) from an auxillary linear differential equation (E_0) obtained from (E) using a geometric linearization procedure.

Why is it useful for concrete computations? The structure of (E_0) is encoded by a linear algebraic group called the differential Galois Group of (E_0). Hrushovski showed that there exists an algorithm computing the Galois Group of every linear differential equation defined over $(\mathbb{C}(t), \frac{d}{dt})$.

Linearization of differential equation

- The base differential field is the field of complex numbers. Algebraic differential equations are presented geometrically as pairs (X, v) where X is a smooth complex algebraic variety endowed with a vector field v.
- In many geometric linearization procedures,

$$\text{differential equation } (X, v) \to \text{linear differential equation } (E_0)$$

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Definition

A linear differential equation over (Y, w) is a pair (E, v_E) where.

- $\pi : E \rightarrow Y$ is a vector bundle over Y.
- v_E is a vector field on E and $\pi : (E, v_E) \rightarrow (Y, w)$ is a morphism of D-varieties that is $d\pi(v_E) = w$.
- The vector field v_E is linear on the fibres of π.

The base (Y, w) is the differential equation (over the base differential field) satisfied by these additional parameters and the equation (E_0) appears as the generic fibre of π.

Linear differential equations

Let $\pi : (E, v_E) \to (Y, w)$ be a linear differential equation over (Y, w).

- After passing to the solutions, we obtain a fibration in the theory $\mathbf{DCF_0}$

$$\pi : (E, v_E)^\delta \to (Y, w)^\delta$$

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Taking into account these additional parameters, we distinguish two types of properties for a linear differential equation:

- What is the behavior of one generic fibre π? This is encoded by the differential Galois group of a linear differential equation.
- How do the fibres of π vary with the base $(Y, w)^\delta$? For example, does the generic fibre of π descend to the base differential field?

The second property is not encoded in the differential Galois group of the generic fibre. It is encoded by more sophisticated object — a definable groupoïd — defined by Jimenez (retractability and collapse).

Plan of the talk

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(1) Uniform relative internality

(2) Explicit computations in $\mathbf{DCF}_0$.

(3) Linearization along an invariant hypersurface.

(4) Some consequences for the CBP

Relative internality in a stable theory T

Let $q \in S(A)$, π an A-definable function whose domain contains q and $\mathcal{P}$ a collection of partial types over A.

Definition

We say that (q, π) is relatively internal to $\mathcal{P}$ if for some (any) realization a of q, $\mathrm{tp}(a/\pi(a), A)$ is internal to $\mathcal{P}$ that is

$$\text{there exists } e \underset{\pi(a),A}{\perp\!\!\!\!\perp} a \text{ such that } a \in dcl(A, \pi(a), e, \mathcal{P}).$$

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- Examples: $T = \mathbf{DCF}_0$ $\mathcal{P} = \{constant\ field\}$. If q the generic type of (the total space of) a linear differential equation $\pi : (E, v_E) \to (Y, w)$, then (q, π) is relatively internal to the constants

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- Two more examples: non homogenous linear differential equations and Ricatti equations: (q is the generic type and π is the projection on x)

$$\begin{cases} y' = a(x)y + b(x) \\ x' = f(x) \end{cases} \quad \text{and} \quad \begin{cases} y' = a(x)y^2 + b(x)y + c(x) \\ x' = f(x) \end{cases}$$

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- We will focus on the case where (q, π) is relatively minimal (the fibres are minimal). In that case, it is known that the binding group of the fibres is either abelian ($\mathbb{G}_m$ in the case of line bundle), the affine group Aff_2 (Affine equations) or PSL_2 (Ricatti equations).

Retractability and collapse

Let $q \in S(A)$, π an A-definable function whose domain contains q and and $\mathcal{P}$ a collection of partial types over A such that (q, π) is relatively $\mathcal{P}$-internal.

Definition (Retractibility)

We say that the fibration (q, π) is trivial if there exists a type $q_0 \in S(A)$ internal to $\mathcal{P}$ such that $q = \pi(q) \otimes q_0$: for some (any) realization a of q there exists a realization b of q_0 such that a and $(\pi(a), b)$ are interdefinable over A.

Definition (Collapse)

We say that the fibration (q, π) is *uniformly* relatively internal to $\mathcal{P}$ if for some (any) realization a of q

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- the second notion is the more robust: if $A \subset B$, then (q, π) is uniformly relatively $\mathcal{P}$-internal if and only if $(q_{|B}, \pi)$ is.

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- the second notion is the more robust: if $A \subset B$, then (q, π) is uniformly relatively $\mathcal{P}$-internal if and only if $(q_{|B}, \pi)$ is.
- For every notion, we define the almost version where *dcl* is replaced by *acl*.

Uniform relative internality over a model

Proposition

Assume that $\mathcal{P}$ consists of a single formula and that the parameter space A is a model of T. If (q, π) is relatively $\mathcal{P}$-internal TFAE:

- (q, π) *is uniformly relatively internal to* $\mathcal{P}$.
- (q, π) *is* $\mathcal{P}$*-definable: for some (any) realization* a *of* q

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This applies to the theory **CCM** over $\emptyset$: let E be a vector bundle of rank two over a compact complex manifold X. Denote by q the generic type of the projective space $\pi : \mathbb{P}(E) \to X$ then:

- the meromorphic (q, π) is uniformly relatively almost internal to $\mathbb{P}^1$.
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If X is a projective complex variety then every projective bundle is uniformly internal to $\mathbb{P}^1$. If X is a non algebraic K3-surface or a non algebraic torus of dimension two, non uniform projective bundles exist (Toma).

Uniform internality over arbitrary sets of parameters

Let $q \in S(A)$ and π an A-definable function whose domain contains q and $\mathcal{P}$ a collection of types over A.

Proposition (Jimenez)

Assume that (q, π) relatively internal to $\mathcal{P}$ and $\pi(q)$ is internal to $\mathcal{P}$. TFAE:

- *(q, π) is uniformly relatively almost internal to $\mathcal{P}$.*
- *q is almost internal to the constants.*

Moreover, Jimenez and Jin-Moosa independently construct examples where the fibration (q, π) is uniformly relatively internal and not almost trivial.

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Proposition

Assume that (q, π) relatively internal to $\mathcal{P}$, $\pi(q)$ is orthogonal to $\mathcal{P}$ and that (q, π) is relatively minimal. TFAE:

(a) *(q, π) is uniformly relatively almost internal to $\mathcal{P}$.*

(b) *The fibration (q, π) is almost trivial: there exists a type $q_0 \in S(A)$ internal to $\mathcal{P}$ such that q is almost $\pi(q) \otimes q_0$.*

(c) *q is not orthogonal to $\mathcal{P}$.*

Comments

- If $\pi(q)$ is internal to $\mathcal{P}$ and fundamental, Jimenez shows that — in the case of a trivial fibration — a decomposition $q = p \otimes q_0$ gives rise to a decomposition of the binding groups:

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  So to construct a fibration (q, π) uniformly relatively internal and not almost trivial, you start with an exact sequence of algebraic groups:

$$(*) : 0 \to H \to G \to G/H \to 0.$$

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  which does not almost split and then use inverse differential Galois theory to realize $(*)$ as an exact sequence of binding groups.
- If (q, π) is relatively minimal and $\pi(q)$ is orthogonal to the constants, there are only two possibilities:
  - either the fibration (q, π) is almost trivial.
  - or q is orthogonal to $\mathcal{P}$.

  Note that since (q, π) is relatively internal to $\mathcal{P}$, q always have a (non algebraic) forking extension non-orthogonal to $\mathcal{P}$.

Linear differential equation of relative dimension one

The base differential field is the field of complex numbers. We study the family of differential equations of the form:

$$(E) : \begin{cases} y' = g(x, x', x'', \dots, x^{(n)})y \\ f(x, x', x'', \dots, x^{(n)}) = 0 \end{cases}$$

where f, g are differential rational functions (with complex coefficients).

- Let q_E be the generic type of (E) and π_E the projection on x. The fibration (q_E, π_E) is relatively internal to the constants. The binding group of the fibres are always subgroups of the multiplicative group.

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- (Universality) If (q, π) is relatively internal to the constants, relatively minimal and the binding group of the fibre of (q, π) is $\mathbb{G}_m$ then (q, π) is interdefinable with (q_E, π_E) for some equation (E).

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- (Geometric presentation) The study of this family is equivalent to the study of linear differential equation $\pi : (L, v_L) \to (Y, w)$ supported by a line bundle.
- Assume that $\pi_E(q_E)$ is orthogonal to the constants. When is (q_E, π_E) uniformly internal to the constants?

Uniform relative internality with binding group $\mathbb{G}_m$

Let $g(x) = \frac{g_1(x,x',\ldots,x^{(n)})}{g_2(x,x',\ldots,x^{(n)})} \in \mathbb{C}\langle x\rangle$ be the quotient of two differential polynomials. Consider in a differentially closed field $(\mathcal{U}, \delta_{\mathcal{U}})$:

$$\mathcal{D}_{log} = \{(x,y) \in \mathcal{U}^2,\ g_2(x) \neq 0 \text{ and } y' = g(x)y\}.$$

It is a $\mathbb{C}$-definable set of rank $\omega + 1$, we denote by π the projection on the first coordinate.

Theorem

For every type $q \in S(\mathbb{C})$ living on $\mathcal{D}_{log}$ such that $\pi(q)$ is a type of finite rank orthogonal to the constants. TFAE:

(i) *The fibration (q, π) is uniformly internal to the constants.*

(ii) *Let x be any realization of $\pi(q)$. There exists $c \in \mathbb{C}$ such that the linear differential equation*

$$y' = (g(x) + c)y \text{ has a non-zero solution in } \mathbb{C}\langle x\rangle.$$

Here, $\mathbb{C}\langle x\rangle$ denotes the differential field generated by x over $\mathbb{C}$.

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- The almost version holds where (ii) is replaced by there exists $c \in \mathbb{C}$ such that the linear differential equation $y' = (g(x) + c)y$ has a non-zero solution in $\mathbb{C}\langle x\rangle^{alg}$. This condition can be rewritten as:

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- From a geometric perspective, the study of higher dimensional vector bundles looks also very interesting. In that case, we lose the equivalence beween uniform relative internality and triviality of the fibration.
- In the opposite situation where the type $\pi_E(q)$ is internal to the constants, Jin and Moosa obtain a similar classification of uniform relative internality (through analyzability in the constants) for pullbacks by the logarithmic derivative.

Examples in dimension two

For a differential equation of the form:

$$(E):\begin{cases} y' = g(x)y \\ x' = f(x) \end{cases} \quad \text{with } f, g \in \mathbb{C}(x).$$

- The generic type of (E) is orthogonal to the constants if and only if:
  - $\pi_E(q_E)$ is orthogonal to the constants
  - (q_E, π_E) is not uniformly relatively almost internal to the constants.

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- The generic type of (E) is orthogonal to the constants if and only if:
  - $\pi_E(q_E)$ is orthogonal to the constants
  - (q_E, π_E) is not uniformly relatively almost internal to the constants.
- In this family of differential equations, we reduce the question of orthogonality to the constants to computing poles and residues of rational functions $h(x) \in \mathbb{C}(x)$.

Corollary

The generic type of (E) is orthogonal to the constants if and only if:

- *(Rosenlicht) Either $1/f(x)$ admits at least a simple pole and a multiple pole or all the poles of $1/f(x)$ are simple and the quotient of the residues of $1/f(x)$ at two of these poles is not rational.*
- *For all $c \in \mathbb{C}$, $\frac{g(x)+c}{f(x)}$ has at least a multiple pole or a simple pole with a residue $\notin \mathbb{Q}$.*

Some explicit computations

Let's compare:

$$(E_1) : \begin{cases} y' = xy \\ x' = x(x-1)^2. \end{cases} \quad \text{and } (E_2) : \begin{cases} y' = xy \\ x' = x(x-1)^3. \end{cases}$$

For both of these equations, the results of Rosenlicht ensure that the base is orthogonal to the constants. We compute:

$$\frac{g_1(x)+c}{f_1(x)} = \frac{x+c}{x(x-1)^2} \text{ and } \frac{g_2(x)+c}{f_2(x)} = \frac{x+c}{x(x-1)^3}$$

The first one has simple poles with integer residues for $c = -1$, so the fibration (q_1, π_1) is uniformly internal and therefore trivial. The second one always admit 1 as a multiple pole so q_2 is orthogonal to the constants.

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The first one has simple poles with integer residues for $c = -1$, so the fibration (q_1, π_1) is uniformly internal and therefore trivial. The second one always admit 1 as a multiple pole so q_2 is orthogonal to the constants. Using a linearization

procedure, we now describe the effect of adding a small perturbation in y of the form:

$$(E)_{pert} : \begin{cases} y' = g(x)y + g_2(x)y^2 + g_3(x)y^3 + \ldots + g_n(x)y^n \\ x' = f(x) + f_1(x)y + f_2(x)y^2 + \ldots + f_n(x)y^n \end{cases}$$

The normal bundle of a closed invariant subvariety

Let X be a smooth complex algebraic variety endowed with a vector field v and consider Z a smooth closed invariant subvariety of (X, v).

- We denote by $\pi : N_{X/Z} \to Z$ the normal bundle of Z in X. It is a vector bundle of rank $\mathrm{codim}(Z)$ defined by the exact sequence:

$$0 \to T_Z \to (T_X)_{|Z} \to N_{X/Z} \to 0.$$

- If f is a function on an open set U of X vanishing on Z, the differential df also vanishes on T_Z and defines a function denoted $\overline{f}$ on the open set $\pi^{-1}(U \cap Z)$ of the normal bundle $N_{X/Z}$.

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Lemma

There exists a unique vector field v_{lin} on $N_{X/Z}$ such that

(i) $\pi : (N_{X/Z}, v_{lin}) \to (Z, v_Z)$ *is a linear differential equation over* (Z, v_Z).

(ii) *For every function f on an open set of X vanishing on Z,*

$$v_{lin}(\overline{f}) = \overline{v(f)}$$

Note that if f vanishes on Z then the derivative $v(f)$ of f with respect to v also vanishes on Z because Z is invariant.

Uniform internality of the normal bundle of an invariant hypersurface

When Z is a smooth invariant hypersurface of (X, v) — that is when Z has codimension one in X — then $N_{X/Z}$ is a line bundle on Z.

Theorem

Let X be a smooth complex algebraic variety, v be a vector field on X and let Z be a smooth irreducible invariant hypersurface of (X, v).
Denote by q the generic type of the normal bundle $(N_{X/Z}, v_{lin})$ and π the canonical projection. Assume that:

- *the type $\pi(q)$ (the generic type of (Z, v_Z)) is orthogonal to the constants.*
- *(q, π) is not uniformly relatively almost internal to the constants*

Then the generic type of (X, v) is orthogonal to the constants.

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- We know almost nothing when Z has codimension ≥ 2 (and the normal bundle is a vector bundle bundle of rank ≥ 2).
- In codimension one, the proof is based on the discrete valuation associated to an irreducible hypersurface.

Application to planar vector fields preserving a line

Consider a differential equation of the form:

$$(E) : \begin{cases} y' = g(x,y) \\ x' = f(x,y) \end{cases} \quad \text{with } f, g \in \mathbb{C}[x,y].$$

The line $y = 0$ is invariant if and only if y divides $g(x,y)$ in $\mathbb{C}[x,y]$. Therefore, the equation takes the form:

$$(E) : \begin{cases} y' = g_1(x)y + g_2(x)y^2 + \ldots + g_n(x)y^n \\ x' = f_0(x) + f_1(x)y + \ldots + f_n(x)y^n \end{cases} \quad \text{with } f_i, g_i \in \mathbb{C}[x].$$

The linearization along the the line $y = 0$ is given by:

$$(E_0) : \begin{cases} y' = g_1(x)y \\ x' = f_0(x) \end{cases} \quad \text{with } f_i, g_i \in \mathbb{C}[x].$$

To prove the theorem, we consider the *discrete* valuation defined by the hypersurface $y = 0$ and we view (E_0) as the leading term of (E) with respect to the valuation.

Orthogonality to the constants for vector fields preserving a line

- For the linear differential equations

$$(E_1) : \begin{cases} y' = xy \\ x' = x(x-1)^2. \end{cases} \quad \text{and } (E_2) : \begin{cases} y' = xy \\ x' = x(x-1)^3. \end{cases}$$

the generic type of (E_1) is non-orthogonal to the constants and the generic type of (E_2) orthogonal to the constants.

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- For a perturbated versions of (E_2)

$$(E_2)_p : \begin{cases} y' = xy + y^2 g_2(x) + \ldots + y^n g_n(x) \\ x' = x(x-1)^3 + y f_1(x) + \ldots + y^n f_n(x). \end{cases}$$

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  the generic type may be orthogonal to the constants or not. One has to study higher order obstructions.

Orthogonality to the constants on other complex algebraic varieties

Using a second linearization procedure in the neighborhood of an hyperbolic and non-resonant zero that I developped to study algebraic factors, we see that:

Corollary

Let v be a vector field on a complex algebraic surface X. Assume that:

- *v admits exactly one invariant complex algebraic curve C and the generic type of $(N_{X/C}, v_{lin})$ is orthogonal to the constants.*
- *v admits a zero $p \in X(\mathbb{C}) \setminus C(\mathbb{C})$ such that the eigenvalues $\lambda, \mu \in \mathbb{C}$ of the linear part of v around p are non-zero and satisfy $\lambda/\mu \notin \mathbb{R}_- \cup \mathbb{Q}_+$.*

Then the generic type of (X, v) is strongly minimal and disintegrated.

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It is also possible to show:

Corollary

Let $Z_0 \subset Z_1 \subset \ldots \subset Z_n$ be a flag of smooth irreducible complex algebraic varieties. There exists a rational vector field v on Z_n such that:

- *for every i, v is well-defined at the generic point of Z_i and Z_i is invariant.*
- *for every $i \geq 1$, the generic type of $(Z_i, v_{|Z_i})$ is orthogonal to the constants.*

Uniform versions of the CBP

Let T be a stable theory and let $\mathcal{P}$ be the collection of non-locally modular minimal types of T. Recall that a stable theory T satisfies:

- the CBP if $tp(a/b)$ is stationary and $b = Cb(tp(a/b))$ implies that $tp(b/a)$ is almost internal to $\mathcal{P}$.
- the UCBP if $tp(a/b)$ is stationary and $b = Cb(tp(a/b))$ implies that $tp(b/a)$ preserves internality to $\mathcal{P}$: for any e,

$$tp(a/e) \text{ is } \mathcal{P} - \text{internal} \Rightarrow stp(b/e) \text{ is } \mathcal{P} - \text{internal}$$

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- the $\overline{\text{UCBP}}$ if if $tp(a/b)$ is stationary and $b = Cb(tp(a/b))$ implies that

$$(tp(ab/\emptyset), \pi_1) \text{ is uniformly relatively almost } \mathcal{P}-\text{internal.}$$

Note that the CBP implies that $(tp(ab/\emptyset), \pi_1)$ is relatively almost $\mathcal{P}$-internal.

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Question

Chatzidakis showed that CBP $\Leftrightarrow$ UCBP. What about $\overline{UCBP}$?

Positive results

- Jimenez showed that in a stable theory T and any collection of partial types $\mathcal{P}$ over $\emptyset$:

Proposition

Suppose $tp(a/b)$ is stationary. If $(tp(ab/\emptyset), \pi_1)$ is uniformly relatively almost $\mathcal{P}$-internal then $tp(b/a)$ preserves internality to $\mathcal{P}$.
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- So $\overline{\text{UCBP}} \Rightarrow$ UCBP and using the results of Chatzidakis, $\overline{\text{UCBP}}$ holds at least locally when $tp(a/\emptyset)$ is internal to the collection of non locally modular minimal types.
- On the other hand, $\overline{\text{UCBP}}$ holds (for trivial reasons) locally when $tp(a/\emptyset)$ is one-based.

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- So $\overline{\text{UCBP}} \Rightarrow$ UCBP and using the results of Chatzidakis, $\overline{\text{UCBP}}$ holds at least locally when $tp(a/\emptyset)$ is internal to the collection of non locally modular minimal types.
- On the other hand, $\overline{\text{UCBP}}$ holds (for trivial reasons) locally when $tp(a/\emptyset)$ is one-based.
- We strongly suspect that $\overline{\text{UCBP}}$ does not hold globally in **CCM** and **$\mathbf{DCF_0}$** and that this is witnessed by types orthogonal to the constants and not one based such as the ones we studied before.

A (partial) counterexample

Consider the differential equation

$$(E) : \begin{cases} y' = xy + y^2/2 \\ x' = x^3(x-1). \end{cases}$$

It is a Ricatti equation (in the fibres) so that (q_E, π) is relatively internal to the constants. The line $y = 0$ is invariant and the linearization is

$$(E_0) : \begin{cases} y' = xy \\ x' = x^3(x-1). \end{cases}$$

so q_E is orthogonal to the constants and not one based

Proposition

The Kolchin tangent bundle of (E) preserves internality to the constants but is not uniformly relatively almost internal to the constants.

Thank you for your attention!