HOMEWORK: MODEL THEORY OF DIFFERENTIAL FIELDS

DUE ON FRIDAY MARCH 8TH

Convention/notation. All differential fields are of characteristic zero. If k is a differential field, $k\langle \alpha_1, \dots, \alpha_n\rangle$ always denotes the differential field generated by $\alpha_1, \dots, \alpha_n$ and k.

1. Part 1. Kolchin's primitive element theorem

The goal of this exercise to prove:

Theorem A (Kolchin, 1942). *Let k be a nonconstant differential field and let $K = k\langle \alpha_1, \dots, \alpha_n\rangle$ be a finitely generated differential field extension of k such that each α_i is a solution of a nonzero differential polynomial $A_i \in k\{X\}$. Then there exists $\gamma \in K$ such that*

$$K = k\langle \alpha_1, \dots, \alpha_n\rangle = k\langle \gamma\rangle.$$

An element γ satisfying the conclusion of the theorem is called a *primitive element* of the differential field extension K/k.

1.1. Primitive elements and automorphisms.
Let k be a differential field and fix $\mathcal{U} \models \mathrm{DCF}_0$ an ω-saturated and ω-homogeneous extension of k.

(1a) Show that to prove the theorem, we may assume that k is a finitely generated differential field.

(1b) Let $\alpha_1, \dots, \alpha_n \in \mathcal{U}$ and set $K = k\langle \alpha_1, \dots, \alpha_n\rangle$. Under the assumption of (1a), show that an element $\gamma \in K$ is a primitive element of K/k if and only if for every $\sigma \in \mathrm{Aut}_\partial(\mathcal{U}/k)$,

$$\sigma(\gamma) = \gamma \Rightarrow \sigma(\alpha_i) = \alpha_i \text{ for } i = 1, \dots, n.$$

1.2. Non-vanishing of non-zero differential polynomials.
Let k be a nonconstant differential field. The goal of this question is to prove that *if $A(X_1, \dots, X_n) \in k\{X_1, \dots, X_n\}$ is a nonzero differential polynomial then there exist $a_1, \dots, a_n \in k$ such that*

$$A(a_1, \dots, a_n) \neq 0.$$

(2a) Show that it is enough to prove the previous statement for $n = 1$.

(2b) Consider $g \in k$ with $\partial(g) \neq 0$. Replacing the derivation ∂ by the derivation

$$D = \frac{\partial}{\partial(g)} \in \mathrm{Der}(k)$$

show that we may assume that k contains an element t such that $\partial(t) = 1$ in order to prove the previous statement.

(2c) Consider a nonzero $A(X) \in k\{X\}$ and fix $t \in k$ an element with $\partial(t) = 1$. Show that we can find a polynomial $\chi = \chi(t) \in \mathbb{Q}[t]$ such that

$$A(\chi) = P(\chi, \chi', \dots, \chi^{(n)}) \neq 0.$$

1.3. Existence of primitive elements.
Let $A_1(X), \dots, A_n(X) \in k\{X\}$ be nonzero differential polynomials. Consider the differential radical ideal I of $k\{X_1 \dots, X_n, Y_1, \dots, Y_n, T_1, \dots, T_n\}$ given by

$$I = \{A_1(X_1), \dots, A_n(X_n), A_1(Y_1), \dots, A_n(Y_n), T_1 \cdot (X_1 - Y_1) + \dots + T_n \cdot (X_n - Y_n)\}$$

and the decomposition

$$I = I_1 \cap \dots \cap I_n$$

into prime differential ideals given by Ritt second theorem.

(3a) Show that for every i, we can find $x_1, \dots, x_n, y_1, \dots, y_n, t_1, \dots, t_n \in \mathcal{U}$ such that

$$I_i = I(x_1, \dots, x_n, y_1, \dots, y_n, t_1, \dots, t_n/k)$$

(3b) With the previous notation, show that if $x_i \neq y_i$ for some i then there exist a nontrivial differential polynomial $L_i \in k\{T_1, \ldots, T_n\}$ such that

$$L_i(t_1, \ldots, t_n) = 0$$

(3c) Conclude that there exists $\tau_1, \ldots, \tau_n \in k$ such that the function

$$x_1, \ldots, x_n \mapsto \tau_1 x_1 + \ldots + \tau_n x_n$$

is injective when evaluated in the cartesian product of the solution sets in $\mathcal{U}$ of $A_1(X), \ldots, A_n(X)$.

1.4. Conclusion.

(4) Prove Theorem A.

2. Part 2. Liouville's theorem on integration in finite terms

The goal of this second exercise is to prove:

Theorem B (Liouville, 1833)**.** *Let $f(t)$ be an algebraic function. Assume that the primitive $\int f(t)dt$ is an elementary function. Then*

$$\int f(t)dt = R_0(t, f(t)) + \sum_{i=1}^{n} \ln(R_i(t, f(t)))$$

where the c_i are complex numbers and the $R_i(X, Y)$ are rational functions of two variables.

Fix k a differential field with an algebraically closed field of constants.

2.1. **Elementary differential fields extensions.** We say that an extension of differential fields K/k is *elementary* if K and k have the same field of constants and there exists a tower of differential fields

$$k = k_0 \subset k_1 \subset \cdots \subset k_n = K$$

such that each k_{i+1}/k_i is either an algebraic extension, generated by a logarithm of an element of k_i or by the exponential of an element in k_i.

(1) Let K/k be an elementary differential field extension with $\mathrm{td}(K/k) = n$. Show that there exists n K-linearly independent one-forms $\omega_1, \ldots, \omega_n \in \Omega^1(K/k)$ of the form

$$\omega_i = dy_i/y_i - dx_i \text{ with } x_i \in K, y_i \in K^*$$

such that $\mathcal{L}_\partial(\omega_i) = 0$ for $i = 1, \ldots, n$.

We say that a meromorphic function $f(t) \in \mathcal{M}(U)$ is an elementary function if its restriction to some nonempty subset of U is contained in an elementary differential field extension of $\mathbb{C}(t)$.

2.2. **Reduction to the algebraic case.** Let $f(t)$ be an algebraic function and assume that $\int f(t)dt$ is an elementary function.

(2a) Consider $K/\mathbb{C}(t)^{alg}$ an elementary differential field extension containing $\int f(t)dt$. Show that there exists constants $c_1, \ldots, c_n \in \mathbb{C}$, $x_1, \ldots, x_n \in K$ and $y_1, \ldots, y_n \in K^*$ such that

$$df = \sum_{i=1}^{n} c_i \cdot \Big(\frac{dy_i}{y_i} - dx_i\Big) \in \Omega^1(K/\mathbb{C}(t)^{alg}).$$

(2b) Conclude that

$$f'(t) = S_0(t) + \sum_{i=1}^{m} \gamma_i \cdot \frac{S_i'(t)}{S_i(t)}$$

where the γ_i are complex numbers and $S_0, \ldots, S_m \in \mathbb{C}(t)^{alg}$.

2.3. **Reduction to the rational case.** Denote by L the normal extension of $\mathbb{C}(t, f(t))$ generated by $S_0(t), \ldots, S_m(t)$ and set $G = \mathrm{Gal}(L/\mathbb{C}(t, f(t))$.

(3) Using Galois theory, prove Theorem B.