LECTURE 5. THE SEIDENBERG EMBEDDING THEOREM

Notation 5.1. If $y(z)$ in a holomorphic function of the complex variable z on some complex domain U, we denote by y the same function seen as an element of the differential field $\mathrm{Frac}(\mathrm{Hol}(U))$ equipped with the derivation

$$g(z) \mapsto g'(z) = \lim_{h \mapsto 0} \frac{g(z+h) - g(z)}{h}.$$

This differential field called the *differential field of meromorphic functions on U* and denoted $\mathcal{M}(U)$.

5.1. **Analytic solutions of algebraic equations.** Consider an *irreducible* algebraic equation of

$$P(z; y) = a_n(z) \cdot y^n + a_{n-1}(z) \cdot y + \ldots + a_1(z) \cdot y = 0 \in \mathrm{Hol}(U)[y] \text{ and } a_n \neq 0$$

whose coefficients $a_1, \ldots, a_n$ are holomorphic functions on some complex domain U.

Denote by M the analytic curve of $U \times \mathbb{C}$ defined by the previous equation and consider the (first) projection

$$\pi : M \subset U \times \mathbb{C} \to U.$$

The complex analytic solutions of $P(z; y) = 0$ are obtained by taking local analytic inverse of π. We make the following observations:

- we need to discard the set S of poles of the equations: the $z \in U$ such that $a_i(z) = 0$ for all i. They are the points $z \in U$ over which π has an infinite fiber. Outside of (the possibly infinite set) S in U, every point z has a finite preimage.
- Assume that P is separable (e.g. irreducible), we also discard the ramification points, that is, the points $z \in U$ such that this preimage has cardinal smaller than n. They can we describe as

$$R = \{z \in U \mid \exists y \in \mathbb{C}, P(z; y) = P'(z; y) = 0\}$$

Using the holomorphic function $\Delta(z) = \mathrm{Res}(P, P')(z)$ given by the *discriminant of the equation*, then this can be written as $\Delta(z) = 0$.

Lemma 5.2 (analytic inversion theorem). *Assume that P is separable and fix $(z_0, y_0) \in U \times \mathbb{C}$ such that*

$$P(z_0, y_0) = 0 \text{ and } \Delta(z_0) \neq 0.$$

There exists an analytic function $y(z)$ defined in a neighborhood of z_0 and satisfying $y_0 = y(z_0)$ around t_0, satisfying $y(t_0) = y_0$ and $P(z; y(z)) = 0$ for all z.

Proof. Since the zero set of an analytic function is discrete, for every compact $K \subset U$, we have that $(S \cup R) \cap K$ is finite. it follows that every point z in $U \setminus (S \cup R)$ admits a neighborhood which meets $(S \cup R)$ trivially. We can then apply the analytic inversion theorem. $\square$

5.2. **Analytic solutions of differential equations.** Let U be a complex domain. Consider an irreducible differential polynomial of order n and degree d

$$P(z;y,y',\dots,y^{(n)})=0 \text{ with } P\in \mathrm{Hol}(U)[X_0,\dots,X_n]$$

whose coefficients are holomorphic functions on U. In contrast with the convention from differential algebra, we have emphasized the dependence in t in the notation. We write

$$P(z;X_0,\dots,X_n)=\sum_{i=0}^{d} a_i(z;X_0,\dots,X_{n-1})\cdot X_n^i \tag{1}$$

so that $a_d(z,X_0,\dots,X_{n-1})$ can be identified with the defining polynomial of *the initial of* P. Instead of a curve M, consider

$$M:=\{(z,x_0,\dots,x_n)\in U\times\mathbb{C}^{n+1}\mid P(z;x_0,\dots,x_n)=0\}\subset U\times\mathbb{C}^{n+1}.$$

It is an analytic subvariety of dimension $n+1$ in $U\times\mathbb{C}^{n+1}$ equipped with the projection $\pi: M\to U\times\mathbb{C}^n$ Instead of the set of poles, set

$$S:=\{(z,x_0,\dots,x_{n-1})\in U\times\mathbb{C}^n\mid i_P(z,x_0,\dots,x_{n_1})\neq 0\}$$

and denote by V its complement in $U\times\mathbb{C}^n$ and

$$\pi: M\cap p^{-1}(V)\to V.$$

We make the same observation as previously: all fibers of π are finite since we discarded S and we need to get rid of the ramification points.

Lemma 5.3. *With the notation above,*

(i) *π ramifies — that is the differential $d\pi$ of π is not surjective — precisely at the points $(z;x_0,\dots,x_n)\in M\cap p^{-1}(V)$ where*

$$s_P(z;x_0,\dots,x_n):=\frac{\partial P}{\partial X_n}(z;x_0,\dots,x_n)=0.$$

(iii) *The image of the ramification locus (which is closed by (i)) under π is given by the zero set of*

$$\Delta(P)=\mathrm{Res}(P,s_P)\in \mathrm{Hol}(U)[X_0,\dots,X_{n-1}].$$

Theorem 5.4 (Cauchy-Kovalevskaya)**.** *Consider an irreducible diff. polynomial of order n and degree d*

$$P(t;y,y',\dots,y^{(n)})=0 \textit{ with } P\in\mathrm{Hol}(U)[X_0,\dots,X_n]$$

and $(t_0,c_0,\dots,c_n)\in U\times\mathbb{C}^{n+1}$ satisfying

$$P(t_0,c_0\dots,c_n)=0, i_f(t_0,c_0,\dots,c_n)\neq 0 \textit{ and } s_f(t_0,c_0,\dots,c_n)\neq 0.$$

Then:

- (Existence) *there exists a complex disk $\mathbb{D}$ centered at t_0 and $y\in\mathrm{Hol}(\mathbb{D})$ such that*
$$y(t_0)=c_0,\dots,y^{(n)}(t_0)=c_n \textit{ and } P(t;y(t),\dots,y^{(n)}(t))=0 \textit{ for all } t\in\mathbb{D}.$$
- (Uniqueness) *if $(\mathbb{D}_i,y_i)$ for $i=1,2$ are two solutions of the previous initial value problem, then*
$$y_{1|\mathbb{D}_1\cap\mathbb{D}_2}=y_{2|\mathbb{D}_1\cap\mathbb{D}_2}.$$

5.3. **Seidenberg embedding theorem.** The following theorem makes the connection between analytic geometry and differential algebra. Because of the finiteness assumption in the theorem, it is often used in combination with the compactness theorem from model theory.

Theorem 5.5 (Seidenberg's embedding theorem)**.** *Let k be a countable differential field and l/k a finitely generated extension of differential fields. Assume that we are given a (differential embedding)*

$$i_k : k \to \mathcal{M}(U)$$

for some complex domain U. Then there exists a complex domain $V \subset U$ and an embedding $i_l : l \to \mathcal{M}(V)$ such that the following diagram commutes

$$\begin{tikzcd} k \arrow[r, "i_k"] \arrow[d, "\subset"] & \mathcal{M}(U) \arrow[d, "\subset"] \\ l \arrow[r, "i_l"'] & \mathcal{M}(V) \end{tikzcd}$$

Definition 5.6. Let $\mathcal{K} \subset \mathcal{M}(U)$ be a countable subfield. We say that a point $x \in U$ is $\mathcal{K}$ -*generic* if it is not the pole nor the zero of any function from $\mathcal{K}$.

Note that by the theorem of isolated zeroes for holomorphic functions, every function from $\mathcal{K}$ has (at most) countably many zeroes and poles in U. Since $\mathcal{K}$ is countable and that a countable union of countable sets is countable, it follows that $\mathcal{K}$-generic points do exist.

Lemma 5.7. *Let $\mathcal{K} \subset \mathcal{M}(U)$ be a countable subfield and let $x \in U$ be a $\mathcal{K}$-generic point. The evaluation*

$$\mathrm{ev}_x : \mathcal{K} \to \mathbb{C}$$

identifies $\mathcal{K}$ with a countable subfield $\mathcal{K}(x)$ of $\mathbb{C}$.

Proposition 5.8. *Let $\mathcal{K} \subset \mathcal{M}(U)$ be a countable subfield and let $x \in U$ be a $\mathcal{K}$-generic point. Let f be an holomorphic function in a neighborhood of x and assume that*

$$f(x), f'(x), \ldots, f^{(n)}(x)$$

are algebraically independent over $\mathcal{K}(x)$ in $\mathbb{C}$. Then f satisfies no algebraic differential equation with order $\leq n$ and parameters from $\mathcal{K}$.

Proof. Otherwise, the function f satisfies a nontrivial algebraic differential equation

$$P(y, y', \ldots, y^{(n)}) = 0$$

where $P \in \mathcal{M}(U)[X_0, \ldots, X_n]$ can be written as

$$P = \sum a_{i_0,\ldots,i_n}(z) \cdot X_0^{i_0} \cdots X_n^{i_n}$$

Since x is $\mathcal{K}$-generic, the coefficients of P can be evaluated at x and the previous polynomial relation implies that

$$\sum a_{i_0,\ldots,i_n}(x) \cdot f(x)^{i_0} \cdots (f(x)^{(n)})^{i_n} = 0$$

which contradicts our assumption that $f(x), \ldots, f(x)^{(n)}$ are algebraically independent over $\mathcal{K}(x)$. $\square$

Corollary 5.9. *Let $\mathcal{K}$ be a countable differential subfield of $\mathcal{M}(U)$. Then $\mathcal{M}(U)$ has infinite differential transcendence degree over $\mathcal{K}$.*

Proof of Seidenberg theorem. Let $\mathcal{K}$ be a countable differential field and $L/\mathcal{K}$ a finitely generated extension of differential fields. Assume that we are given a (differential embedding)

$$i_{\mathcal{K}} : \mathcal{K} \to \mathcal{M}(U)$$

for some complex domain U. It suffices to see that for any differential field extension $L = \mathcal{K}\langle\xi\rangle/\mathcal{K}$ generated by a single element can be embedded inside a field of meromorphic functions.

- Assume that ξ is differentially transcendental over $\mathcal{K}$. We use the previous corollary to find a function $g \in \mathcal{M}(U)$ which is differentially transcendental over $\mathcal{K}$ and extend $i_{\mathcal{K}}$ to
$$i_L : L \simeq \mathcal{K}(g) \subset \mathcal{M}(U).$$
- Assume that ξ is differentially algebraic over $\mathcal{K}$ and denote by $P \in \mathcal{K}\{X\}$ the irreducible differential polynomial generating the prime differential ideal $I(\xi/\mathcal{K})$.

The description of prime differential ideals of $\mathcal{K}\{X\}$ shows that it is enough to find $g \in \mathcal{M}(V)$ which satisfies

$$(E) : P(z, y, y', \ldots, y^{(n)}).$$

and no other differential equation of minimal order. To do so consider, $z_0 \in U$ which is $\mathcal{K}$-generic and choose $c_0 \ldots, c_{n-1} \in \mathbb{C}$ algebraically independent over $\mathcal{K}(z_0)$. By algebraic independence,

$$i_f(z_0; c_0, \ldots, c_{n-1}) \neq 0$$

and we can choose c_n such that $P(z_0, c_0, \ldots, c_{n-1}, c_n) = 0$. Note that

$$s_f(z_0; c_0, \ldots, c_n) \neq 0$$

since $\Delta(P)(z_0; c_0, \ldots, c_{n-1}) \neq 0$ again by algebraic independence. Applying Cauchy-Kovalevskaya Theorem and the previous corollary finishes the proof. $\square$

References

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[Sei58] A. Seidenberg. Abstract differential algebra and the analytic case. *Proceedings of the American Mathematical Society*, 9(1):159–164, 1958.