Linearization procedures in the semi-minimal analysis of algebraic differential equations

Rémi Jaoui

University of Notre Dame

March 26, 2021

Introduction

The questions I am going to talk about come from two different origins:

- geometric stability theory specialized to the theory $\mathbf{DCF}_0$ of differentially closed fields of characteristic 0.
- the transcendence properties of the solutions of algebraic differential equations going back to Painlevé.

Two examples

- The first Painlevé equation $(E) : y'' = 6y^2 + t$ (Nishioka,Umemura, Nagloo-Pillay)

Painlevé, Umemura, Nishioka	Geometric stability theory
the analytic solutions of (E) are "new" transcendental functions.	the set $\mathcal{S}(E)$ of solutions of (E) in a diff.closed field is strongly minimal.
any tuple of distinct solutions of (E) are, together with their first derivatives, algebraically independent.	the set $\mathcal{S}(E)$ of solutions of (E) in a diff.closed field is a pure infinite set.

- An order one equation $f(y, y't) = 0$ "without movable singularities" (Painlevé, Buium).

Painlevé, Umemura, Nishioka	Geometric stability theory
the analytic solutions of (E) can be expressed using classical meromorphic functions such as exponentials, logarithms and elliptic functions.	the set $\mathcal{S}(E)$ of solutions of (E) in a diff.closed field is internal to the constants.

Table of contents

1. How to describe a definable set of $\mathbf{DCF_0}$?
2. Linearization along an algebraic solution (in dimension 2)
3. Linearization along a generic solution (in dimension 2)

Autonomous differential equations

I will consider the case of complex autonomous algebraic differential equations:

$$F(y, y', \dots, y^{(n)}) = 0 \text{ where } F \in \mathbb{C}[X_0, \dots, X_n]$$

and focus on the behavior of their generic solutions.

- Such an equation will be presented geometrically (forgetting about the embedding in a jet space) as a pair (X, v) where X is an (irreducible) algebraic variety and v is a vector field on X. The vector field v induces a derivation δ_v on $\mathbb{C}(X)$ defined by

$$\delta_v(f) = df(v).$$

- I will write

$$\phi : (X, v) \to (Y, w)$$

to mean that ϕ is a morphism of complex algebraic varieties such that $d\phi(v) = w$.

We obtain a category $\mathcal{C}$ called the category of complex D-varieties by Buium.

Autonomous differential equations

I will consider the case of complex autonomous algebraic differential equations:

$$F(y, y', \ldots, y^{(n)}) = 0 \text{ where } F \in \mathbb{C}[X_0, \ldots, X_n]$$

and focus on the behavior of their generic solutions.

- Such an equation will be presented geometrically (forgetting about the embedding in a jet space) as a pair (X, v) where X is an (irreducible) algebraic variety and v is a vector field on X. The vector field v induces a derivation δ_v on $\mathbb{C}(X)$ defined by

$$\delta_v(f) = df(v).$$

- I will write

$$\phi : (X, v) \to (Y, w)$$

  to mean that ϕ is a morphism of complex algebraic varieties such that $d\phi(v) = w$.

We obtain a category $\mathcal{C}$ called the category of complex D-varieties by Buium.

(1) This category has (fiber) products: if (X, v) and (Y, w) are objects of $\mathcal{C}$.

$$(X, v) \times (Y, w) = (X \times Y, v \times w).$$

(2) A closed invariant subvariety of (X, v) is a closed algebraic subvariety Z of (X, v) which is tangent to the vector field v.

Solution sets in a differentially closed field

We fix once for all a saturated model $(\mathcal{U}, \delta_{\mathcal{U}})$ of the theory $\mathbf{DCF_0}$. We consider the "functor"

$$(E) : F(y, y', \ldots, y^{(n)}) = 0 \to \mathcal{S}(E) = \begin{cases} \text{set of sol. of } (E) \\ \text{in } (\mathcal{U}, \delta_U) \end{cases}$$

- Starting from the category $\mathcal{C}$, we obtain a solution functor:

$$\mathcal{S} : \mathcal{C} \to Def(\mathbf{DCF_0}/\mathbb{C})$$

defined by

$$\mathcal{S}(X, v) = (X, v)^{\delta} = \{\overline{x} \in X(\mathcal{U}) \mid \delta_{\mathcal{U}}(\overline{x}) = v(\overline{x})\}.$$

- Since X is irreducible, the conditions

$$x \in (X, v)^{\delta} \text{ and } x \notin Z(\mathcal{U}) \text{ for every proper invariant algebraic subvariety } Z \text{ of } X$$

isolate a complete type $p \in S(\mathbb{C})$ which will be called the generic type of (X, v).

Analytic geometry	Geometric stability theory
the solutions of (X, v) are the analytic curves $t \mapsto x(t)$ on $X(\mathbb{C})^{an}$ tangent to v.	the solutions of (X, v) are elements of an abstract differential field.
the solutions $t \mapsto x(t)$ which are Zariski-dense in X.	the realizations of the generic type of (X, v).

The semi-minimal analysis in $\mathbf{DCF_0}$

Let (X, v) be an autonomous differential equation of positive dimension.

- The set of solutions $(X, v)^\delta$ is a definable set of finite Morley rank and more precisely:

$$1 \leq MR((X, v)^\delta) \leq dim(X).$$

- When X is a curve, then $(X, v)^\delta$ is always a strongly minimal set and one can use the powerful structure theory for strongly minimal sets in $\mathbf{DCF_0}$ to describe the structure of autonomous differential equations of dimension one (Hrushovski-Itaï).
- I will be interested in the case where $dim(X) = 2, 3, ...$ In that case, $(X, v)^\delta$ is only a definable set with "small" finite Morley rank but not always strongly minimal.

The semi-minimal analysis in $\mathbf{DCF_0}$

Let (X, v) be an autonomous differential equation of positive dimension.

- The set of solutions $(X, v)^\delta$ is a definable set of finite Morley rank and more precisely:

$$1 \leq MR((X, v)^\delta) \leq dim(X).$$

- When X is a curve, then $(X, v)^\delta$ is always a strongly minimal set and one can use the powerful structure theory for strongly minimal sets in $\mathbf{DCF_0}$ to describe the structure of autonomous differential equations of dimension one (Hrushovski-Itaï).
- I will be interested in the case where $dim(X) = 2, 3, \ldots.$ In that case, $(X, v)^\delta$ is only a definable set with "small" finite Morley rank but not always strongly minimal.

A well-known principle: one can recover information about a definable set D of finite Morley rank in a stable theory as follows:

- first obtain some information about the strongly minimal sets of the stable theory T (Zilber's trichotomy).
- study how D interacts these with strongly minimal sets (orthogonality, internality, analyzability).

I will refer to this process as the semi-minimal analysis of the definable set D (or of a type living on D).

Question: How to describe this process from the point of view of analytic geometry?

Non locally modular strongly minimal sets

The most obvious strongly minimal set in the theory $\mathbf{DCF_0}$ is the field of constants defined by

$$\mathcal{C} = \{x \in \mathcal{U} \mid \delta(x) = 0\}.$$

Theorem (Hrushovski)

Let X be a non locally modular strongly minimal set in $\mathbf{DCF_0}$. Then X is almost internal to the constants: there exists a definable finite to one map (with parameters)

$$\phi : X \to W \subset \mathcal{C}^n$$

onto a definable subset W of $\mathcal{C}^n$.

Non locally modular strongly minimal sets

The most obvious strongly minimal set in the theory $\mathbf{DCF_0}$ is the field of constants defined by

$$\mathcal{C} = \{x \in \mathcal{U} \mid \delta(x) = 0\}.$$

Theorem (Hrushovski)

Let X be a non locally modular strongly minimal set in $\mathbf{DCF_0}$. Then X is almost internal to the constants: there exists a definable finite to one map (with parameters)

$$\phi : X \to W \subset \mathcal{C}^n$$

onto a definable subset W of $\mathcal{C}^n$.

Examples:

(i) $y' = y$ and $y' = 1/t$ define non locally modular strongly minimal sets.

(ii) More generally, any Ricatti equation (e.g. over $\mathbb{C}(t)$)

$$y' = a(t)y^2 + b(t)y + c(t) \text{ with } a(t), b(t), c(t) \in \mathbb{C}(t).$$

(iii) any elliptic equation $(y')^2 = y^3 + ay + b$ where $a, b \in \mathbb{C}$.

Observation: the solutions of (i), (ii) and (iii) are "classical" meromorphic functions: the exponential function, the complex logarithm for (i), the solutions of (ii) can be written using a solution of an order 2 linear equation, the solutions of (iii) are elliptic functions.

Analyzability and orthogonality to the constants

Let (X, v) be an autonomous differential equation. The two following definitions describe two opposite situations for the interaction between $(X, v)^\delta$ and the field of constants:

Definition

We say that the generic type of (X, v) is analyzable in the constants if there exists an iterated fibration:

$$(X, v) \dashrightarrow (X_2, v_2) \dashrightarrow \cdots \dashrightarrow (X_r, v_r)$$

where each $\phi_i : (X_i, v_i) \to (X_{i+1}, v_{i+1})$ is a dominant rational morphism between autonomous differential equations and

the generic fibre of $\phi_i^\delta : (X_i, v_i)^\delta \to (X_{i+1}, v_{i+1})^\delta$ is almost internal to the constants.

Definition

We say that the generic type of (X, v) is orthogonal to the constants if for every (Y, w) such that the generic type of (Y, w) is analyzable in the constants (as above), there are no **proper** closed invariant subvariety

$$Z \subset (X, v) \times (Y, w) = (X \times Y, v \times w)$$

projecting generically on both factors.

Classical meromorphic functions

Let $K \subset (\mathcal{M}(U), \frac{d}{dt})$ be a differential subfield of the field of meromorphic functions on a connected open set $U \subset \mathbb{C}$.

Definition (Umemura)

We say that K is a differential field of classical meromorphic functions (or of meromorphic functions of the class $\mathcal{C}_0$) if there exists a tower of differential fields

$$K_0 = \mathbb{C}(t) \subset K_1 \subset \ldots \subset K_n = K$$

such that each step of this tower is obtained from the previous one using one of the following operations:

(A1) Solving an *algebraic equation*: $K_{i+1} = K_i(\xi)$ is generated by a solution ξ of:

$$\xi^r + b_{r-1}\xi^{r-1} + \ldots b_1\xi + b_0 = 0 \text{ with } b_0, \ldots b_{r-1} \in K_i.$$

(C1) Solving a *linear differential equation* (of arbitrary order): $K_{i+1} = K_i\langle\xi\rangle$ is generated by a solution ξ of:

$$y^{(r)} + b_{r-1}y^{(r-1)} + \ldots + b_1 y = 0 \text{ où } b_1, \ldots b_{r-1} \in K_i.$$

(C2) *Composing* meromorphic functions of K_i *with an abelian function* : given a lattice Λ of $\mathbb{C}^r$ such that $\mathbb{C}^r/\Lambda$ is an abelian variety, $K_{i+1} = K_i\langle\xi\rangle$ is the differential field generated by a meromorphic function ξ of the form

$$\xi = \theta \circ \pi \circ (b_1, \ldots, b_r) : U \to \mathbb{C}^r \to \mathbb{C}^r/\Lambda \to \mathbb{C}$$

where θ is a meromorphic function on $\mathbb{C}^r/\Lambda$, $\pi : \mathbb{C}^r \to \mathbb{C}^r/\Lambda$ the canonical projection and $b_1, \ldots, b_r \in K_i$.

Analyzability in the constants and solvability in the class $\mathcal{C}_0$

Let (X, v) be an autonomous differential equation. We will require that (X, v) does not admit non-trivial rational integral:

$$\text{for every rational function } f \in \mathbb{C}(X),\ \delta_v(f) = 0 \Rightarrow f \in \mathbb{C}.$$

Proposition

The generic type of (X, v) is analyzable in the constants if and only if the "generic" analytic solutions of (X, v) belong to the class $\mathcal{C}_0$:

if $t \mapsto x(t)$ is a Zariski-dense analytic solution of (X, v) and $f \in \mathbb{C}(X)$, then the meromorphic function $t \mapsto f(x(t))$ is in the class $\mathcal{C}_0$.

$\Leftarrow$ is the hard part. It uses the theory of the binding group and the full extend of Kolchin's Galois theory (for non linear algebraic groups).

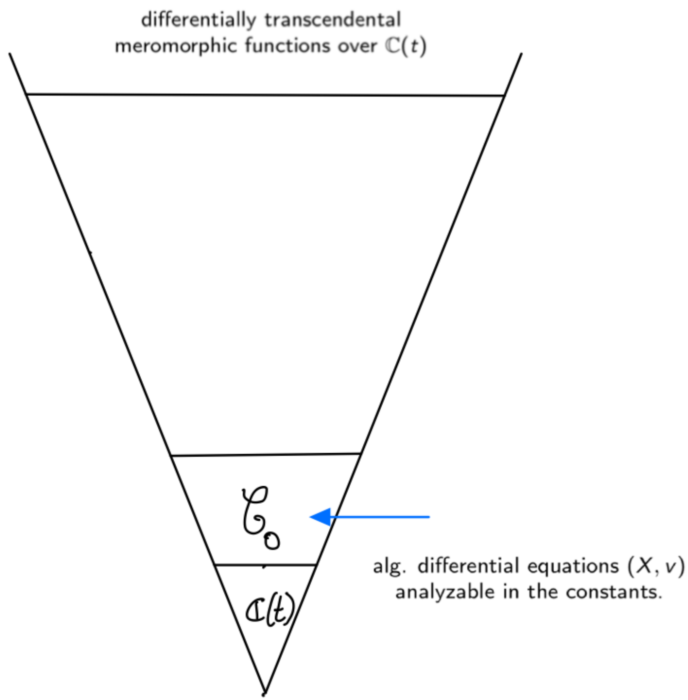
Corollary

The generic type of (X, v) is orthogonal to the constants if and only if the "generic" analytic solutions of (X, v) are alg. ind. over $\mathbb{C}$ from every meromorphic function in the class $\mathcal{C}_0$:

if $t \mapsto x(t)$ is a Zariski-dense analytic solution of (X, v) and $f \in \mathbb{C}(X) \setminus \mathbb{C}$, then the meromorphic function $t \mapsto f(x(t))$ is alg ind. from every $\phi(t) \in \mathcal{C}_0$.

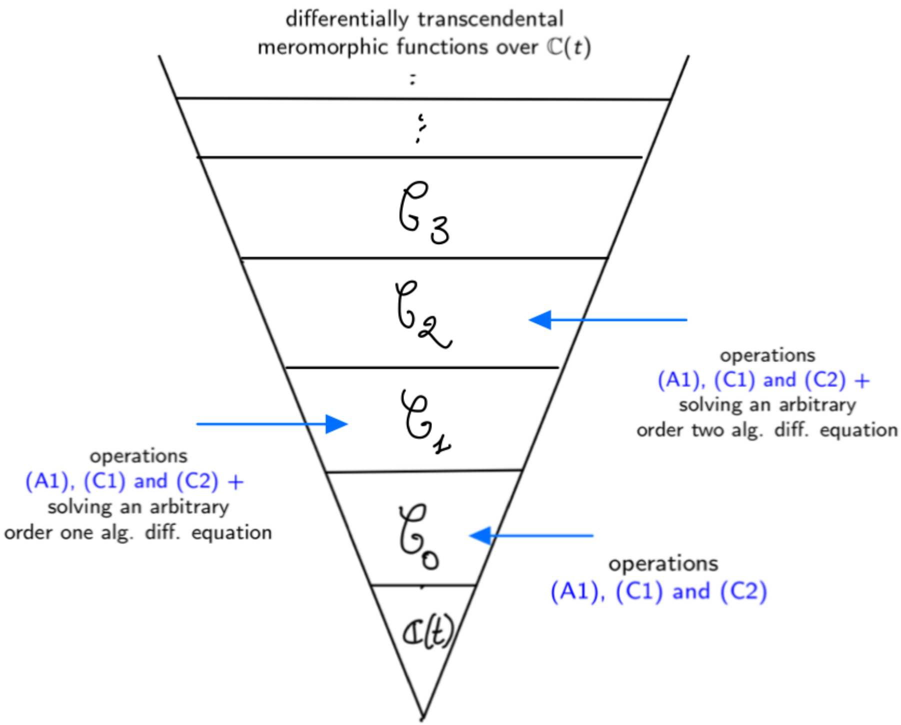
Painlevé's hierarchy of meromorphic functions

We will represent the differential field $(\mathcal{M}(U), \frac{d}{dt})$ of meromorphic functions of a connected open set $U \subset \mathbb{C}$ as follows:



Painlevé's hierarchy of meromorphic functions

We will represent the differential field $(\mathcal{M}(U), \frac{d}{dt})$ of meromorphic functions of a connected open set $U \subset \mathbb{C}$ as follows:



The classes $\mathcal{C}_k$ for $k \geq 1$

Definition (Umemura)

Let $k \geq 1$. We say that K is a differential field of meromorphic functions in the class $\mathcal{C}_k$ if there exists a tower of differential fields

$$K_0 = \mathbb{C}(t) \subset K_1 \subset \ldots \subset K_n = K$$

such that each step of this tower using one of the operations (A1),(C1),(C2) and

(P_k) Solving an algebraic differential equation of order $r \leq k$: $K_{i+1} = K_i\langle \xi \rangle$ is generated by a solution ξ of:

$$P(y, y', \ldots, y^{(r)}) = 0 \text{ where } P \in K_i[X_0, \ldots, X_n].$$

Proposition

Let (X, v) be an autonomous algebraic differential equation of dim. $n > 1$ without non-trivial rational integral. TFAE:

(i) *the generic type of (X, v) is minimal.*

(ii) *the analytic solutions of (X, v) are new meromorphic functions (in the sense of Painlevé):*

if $t \mapsto x(t)$ is a Zariski-dense analytic solution of (X, v) and $f \in \mathbb{C}(X) \setminus \mathbb{C}$, then

$$\text{the meromorphic function } t \mapsto f(x(t)) \in \mathcal{C}_n$$

is algebraically independent (over $\mathbb{C}$) of all meromorphic functions in the class $\mathcal{C}_{n-1}$.

Summary

Let (X, v) be an autonomous differential equation of dimension $n \geq 2$ without non-trivial rational integral. Denote by q the generic type of (X, v) and by $t \mapsto x(t)$ a Zariski-dense analytic solution of (X, v).

Analytic formulation	Geometric stability theory
$\forall f \in \mathbb{C}(X) \setminus \mathbb{C}$, $t \mapsto f(x(t))$ is alg. ind. from all mer. functions of the class $\mathcal{C}_0$.	q is orthogonal to the constants.
$\forall f \in \mathbb{C}(X) \setminus \mathbb{C}$, $t \mapsto f(x(t))$ is alg. ind. from all mer. functions of the class $\mathcal{C}_{n-1}$.	q is minimal.

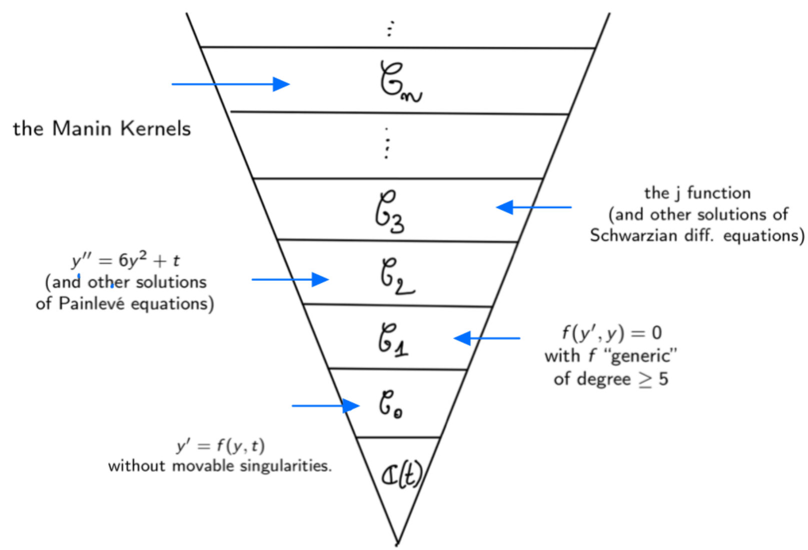


Table of contents

1. How to describe a definable set of $\mathbf{DCF}_0$?
2. Linearization along an algebraic solution (in dimension 2)
3. Linearization along a generic solution (in dimension 2)

Linearization procedures

By a linearization procedure, I mean an argument of the following form:

$$\text{autonomous differential equation } (X, v) \rightarrow \text{a linear differential equation}$$

such that one can recover information about the generic solutions of (X, v) using information about this linear differential equation and its differential Galois group (or its binding group).

Linearization procedures

By a linearization procedure, I mean an argument of the following form:

$$\text{autonomous differential equation } (X, v) \to \text{a linear differential equation}$$

such that one can recover information about the generic solutions of (X, v) using information about this linear differential equation and its differential Galois group (or its binding group).

An example (in the analytic setting): Assume that we are able to compute **one particular** analytic solution

$$t \mapsto x(t) \text{ of the differential equation } (X, v).$$

- If $y(t)$ is another analytic solution sufficiently close to $x(t)$ (for small times t) then we can write (in a chart)

$$x(t) = y(t) + \epsilon(t)$$

So that

$$\epsilon'(t) = v(x(t)) - v(y(t)) = dv_{x(t)}(\epsilon(t)) + \text{higher order terms (in } \epsilon).$$

- If we abandon the higher order terms, we obtain a linear differential equation

$$\epsilon'(t) = dv_{x(t)}(\epsilon(t))$$

called the linearization of (X, v) along the solution $x(t)$.

Goal: study a generic solution of (X, v) using properties of a particular solution and of the linearization along this solution.

Linear differential equations

Let (X, v) be an autonomous differential equation.

Definition

A linear differential equation over (X, v) of order r consists of the following data:

- a vector bundle $\pi : E \to X$ over X of rank r.
- a partial connection ∇_v on the vector bundle E along the vector field v.

Such a partial connection can be identified with a vector field v_E on the total space E such that

$$\pi : (E, v_E) \to (X, v) \text{ is a morphism of } \mathcal{C}$$

and the vector field v_E is linear on the fibres.

Linear differential equations

Let (X, v) be an autonomous differential equation.

Definition

A linear differential equation over (X, v) of order r consists of the following data:

- a vector bundle $\pi : E \to X$ over X of rank r.
- a partial connection ∇_v on the vector bundle E along the vector field v.

Such a partial connection can be identified with a vector field v_E on the total space E such that

$$\pi : (E, v_E) \to (X, v) \text{ is a morphism of } \mathcal{C}$$

and the vector field v_E is linear on the fibres.

Local picture: If $U \subset \mathbb{C}^n$ (or more generally an étale cover of such an open set) and $E = U \times \mathbb{C}^r$ is the trivial bundle. Then

- v has the form $v = f_1 \frac{\partial}{\partial x_1} + \ldots + \ldots f_n \frac{\partial}{\partial x_n}$ where $f_i \in \mathcal{O}_X(U)$.
- In coordinates $(x_1, \ldots, x_n, \epsilon_1, \ldots, \epsilon_r)$, v_E has the form

$$v_E = f_1 \frac{\partial}{\partial x_1} + \ldots + \ldots f_n \frac{\partial}{\partial x_n} + (\sum a_{i,1}\epsilon_i) \frac{\partial}{\partial \epsilon_1} + \ldots + (\sum a_{i,r}\epsilon_i) \frac{\partial}{\partial \epsilon_r}$$

  where $a_{i,j} \in \mathcal{O}_X(U)$ is a function on U.
- The generic fibre of π is a linear differential equation

$$Y' = AY \text{ where } A = (a_{i,j}) \in M_n(\mathbb{C}(X))$$

Linearization along a closed invariant subvariety

Let X be a smooth algebraic variety and Z a smooth closed subvariety of X. The normal bundle of Z in X denoted $N_{X/Z}$ is defined by the exact sequence of vector bundles on Z:

$$0 \to T_Z \to T_{X|Z} \to N_{X/Z} \to 0.$$

Construction

Assume that v is a vector field on X such that Z is invariant under v (i.e. tangent to the vector field v). Then the vector field v induces:

- *a vector field v_Z obtained by restricting the vector field v on X to Z.*
- *a partial connection ∇_v on $N_{X/Z}$ along the vector field v_Z.*

The linear equation $(N_{X/Z}, \nabla_v)$ will be call the (normal) linearization of X along the invariant subvariety Z.

Linearization along a closed invariant subvariety

Let X be a smooth algebraic variety and Z a smooth closed subvariety of X. The normal bundle of Z in X denoted $N_{X/Z}$ is defined by the exact sequence of vector bundles on Z:

$$0 \to T_Z \to T_{X|Z} \to N_{X/Z} \to 0.$$

Construction

Assume that v is a vector field on X such that Z is invariant under v (i.e. tangent to the vector field v). Then the vector field v induces:

- *a vector field v_Z obtained by restricting the vector field v on X to Z.*
- *a partial connection ∇_v on $N_{X/Z}$ along the vector field v_Z.*

The linear equation $(N_{X/Z}, \nabla_v)$ will be call the (normal) linearization of X along the invariant subvariety Z.

- As described before, after choosing some coordinates, we can identify the generic fibre of $\pi : (N_{X/Z}, \nabla_v) \to (Z, v_Z)$ of the form

$$Y' = A.Y \text{ where } A \in M_n(\mathbb{C}(Z)).$$

- The group $GL_r(\mathbb{C}(Z))$ acts on such differential equation by gauge transformation:

$$A \mapsto BAB^{-1} + B.B^{\delta}.$$

  We denote by $[N_{X/Z}]$ the equivalence class of the generic fibre modulo gauge transformation.

Linearization in dimension two

For any differential field (K, δ), we can identify the equivalence classes of linear differential equation of **order one** with the cokernel of

$$dlog : \begin{cases} (K^*, \times) \to (K, +) \\ x \mapsto \delta(x)/x. \end{cases}$$

Theorem (with L. Jimenez and A. Pillay)

Let (X, v) be a smoth autonomous differential equation with an invariant algebraic curve C. Assume that

(i) *the differential equation (C, v_C) is orthogonal to the constants.*

(ii) *For all $n \geq 1$, $n.[(N_{X/Z}, \nabla_v)] \notin \mathbb{C} + dlog(\mathbb{C}(C)^*)$*

Then the generic type of (X, v) is orthogonal to the constants.

- The condition (ii) expresses that up to gauge transformation over $\mathbb{C}(C)^{alg}$, the linear differential equation $(N_{X/Z}, \nabla_v)$ does not descend to the constants.

Linearization in dimension two

For any differential field (K, δ), we can identify the equivalence classes of linear differential equation of **order one** with the cokernel of

$$dlog : \begin{cases} (K^*, \times) \to (K, +) \\ x \mapsto \delta(x)/x. \end{cases}$$

Theorem (with L. Jimenez and A. Pillay)

Let (X, v) be a smoth autonomous differential equation with an invariant algebraic curve C. Assume that

(i) *the differential equation (C, v_C) is orthogonal to the constants.*

(ii) *For all $n \geq 1$, $n.[(N_{X/Z}, \nabla_v)] \notin \mathbb{C} + dlog(\mathbb{C}(C)^*)$*

Then the generic type of (X, v) is orthogonal to the constants.

- The condition (ii) expresses that up to gauge transformation over $\mathbb{C}(C)^{alg}$, the linear differential equation $(N_{X/Z}, \nabla_v)$ does not descend to the constants.
- When C is a rational curve, the conditions (i) and (ii) can be ensured by computing residues of rational functions on $\mathbb{P}^1$. For example

$$(E_1) \begin{cases} y' = yx + g_2(x)y^2 + \ldots + g_m(x)y^m \\ x' = x^3(x-1) + f_1(x)y + \ldots + f_n(x)y^m \end{cases} \quad \text{and } (E_2) : \begin{cases} y' = yx \\ x' = x^2(x-1) \end{cases}$$

admit $y = 0$ as an invariant curve and satisfy (i). (E_1) does satisfy (ii) but not (E_2).

Explanations

- Denote by q the generic type of (X, v).

  q is non-orthogonal to the constants if and only if for $n \gg 0$, $(X, v)^n$ admits a non-trivial (i.e. dominant) rational integral $f : (X, v)^n \to (\mathbb{A}^1, 0)$.

- Fix $n \geq 1$ and consider

$$C_n = C \times \ldots \times C \to X \times \ldots \times X.$$

  C_n is invariant under the vector field $v \times \ldots \times v$ and we can consider the linearization of $(X, v)^n$ along the closed invariant subvariety C_n. We obtain a linear differential equation

$$(E_n) : Y' = A_n Y$$

  where A_n is a matrix of size n with parameters in $\mathbb{C}(C_n)$. We denote by G_n the Galois group of this linear differential equation.

Explanations

- Denote by q the generic type of (X, v).

  q is non-orthogonal to the constants if and only if for $n \gg 0$, $(X, v)^n$ admits a non-trivial (i.e. dominant) rational integral $f : (X, v)^n \to (\mathbb{A}^1, 0)$.

- Fix $n \geq 1$ and consider

  $$C_n = C \times \ldots \times C \to X \times \ldots \times X.$$

  C_n is invariant under the vector field $v \times \ldots \times v$ and we can consider the linearization of $(X, v)^n$ along the closed invariant subvariety C_n. We obtain a linear differential equation

  $$(E_n) : Y' = A_n Y$$

  where A_n is a matrix of size n with parameters in $\mathbb{C}(C_n)$. We denote by G_n the Galois group of this linear differential equation.

Observation : If $f : (X, v)^n \to (\mathbb{A}^1, 0)$ is a non-trivial rational integral then f induces a non-trivial rational integral of the linearization (E_n) of $(X, v)^n$ along C_n. In particular an **obstruction** to the existence of non-trivial rational integral is

$$G_n \text{ acts transitively on the set of solutions of } (E_n) \text{ for all } n \geq 1.$$

Conclusion We want to ensure not only that the "first" Galois group $G_1 = \mathbb{G}_m$ is as large as possible but also that all the "higher dimensional" ones

$$G_2, G_3, \ldots G_n, \ldots$$

are sufficiently large.

Explanations II

When n varies, the Galois group G_n are Galois groups of linear differential equations defined over a varying differential field $\mathbb{C}(C_n)$.

- an idea of L. Jimenez PhD is to replace the Galois groups by Galois groupoids. In contrast with the Galois group, the Galois groupoid of (E_n) will always be definable over $\mathbb{C}$.
- In particular, replacing groups by groupoids, Jimenez was able to use the projections $\pi_i : X^{n+1} \to X^n$ to obtain "face maps"

$$\pi_{i*} : G_{n+1} \to G_n \text{ for } i = 1, \ldots, (n+1)$$

Explanations II

When n varies, the Galois group G_n are Galois groups of linear differential equations defined over a varying differential field $\mathbb{C}(C_n)$.

- an idea of L. Jimenez PhD is to replace the Galois groups by Galois groupoids. In contrast with the Galois group, the Galois groupoid of (E_n) will always be definable over $\mathbb{C}$.
- In particular, replacing groups by groupoids, Jimenez was able to use the projections $\pi_i : X^{n+1} \to X^n$ to obtain "face maps"

$$\pi_{i*} : G_{n+1} \to G_n \text{ for } i = 1, \ldots, (n+1)$$

Lemma

When $G_1 = \mathbb{G}_m$ and the curve (C, v_C) is orthogonal to the constants, the face maps are surjective. Moreover, the following are equivalent

(i) *The sequence $(G_n, n \in \mathbb{N})$ collapses: for $n \gg 0$, the face maps*

$$\pi_{i*} : G_{n+1} \to G_n$$

have finite kernels. In particular, the sequence $(dim(G_n))_{n\in\mathbb{N}}$ is eventually constants.

(ii) $G_2 \neq \mathbb{G}_m \times \mathbb{G}_m$

(iii) *There is $n \geq 1$ such that $n.[(N_{X/Z}, \nabla_v)] \in \mathbb{C} + dlog(\mathbb{C}(C)^*)$*

Comments

- **In progress**: generalizations to higher dimension (more precisely to higher codimensions). A particularly interesting case for applications:

  an invariant curve C in a autonomous diff. equation (X, v) of dimension 3 such that the generic fibre of the linearization of (X, v) along C has Galois group $SL_2(\mathbb{C})$.

- Using a perturbation technique, this method can also be applied to certain autonomous differential equations which do not admit any invariant curve:

Corollary

Let $\{(X(\alpha), v_\alpha), \alpha \in S(\mathbb{C}\}$ be a smooth family of autonomous differential equation indexed by an (irreducible) algebraic variety S. If some fixed $\alpha_0 \in S(\mathbb{C})$, $(X(\alpha_0), v_{\alpha_0})$ satisfies the assumptions of the theorem then the conclusion of the theorem holds for almost all $\alpha \in S(\mathbb{C})$

- But this method can not distinguish whether the generic solutions of (X, v) are new meromorphic functions of the class $\mathcal{C}_2$ or meromorphic functions of the class $\mathcal{C}_1$.

Why? Because (X, v) can be equal to its own linearization $(N_{X/C}, \partial_v)$! In that case, the analytic solutions of (X, v) are always in the class $\mathcal{C}_1$.

Table of contents

1. How to describe a definable set of $\mathbf{DCF}_0$?
2. Linearization along an algebraic solution (in dimension 2)
3. Linearization along a generic solution (in dimension 2)

Linearization at the generic point

Let (X, v) be an autonomous differential equation and $\pi : T_X \to X$ the tangent bundle of X.

Construction

The vector field v can be extended to a partial connection ∇_v on T_X along v so that

$$\pi : (T_X, \nabla_v) \to (X, v) \textit{ is a morphism of } \mathcal{C}.$$

It is characterized by the property that if $f \in \mathbb{C}(X)$ then

$$\delta_{\nabla_v}(df) = d(\delta_v(f)) \textit{ where } df \textit{ is identified with a function on } T_X.$$

The linear differential equation (T_X, ∇_v) will be called the linearization of (X, v) at the generic point.

Linearization at the generic point

Let (X, v) be an autonomous differential equation and $\pi : T_X \to X$ the tangent bundle of X.

Construction

The vector field v can be extended to a partial connection ∇_v on T_X along v so that

$$\pi : (T_X, \nabla_v) \to (X, v) \text{ is a morphism of } \mathcal{C}.$$

It is characterized by the property that if $f \in \mathbb{C}(X)$ then

$$\delta_{\nabla_v}(df) = d(\delta_v(f)) \text{ where } df \text{ is identified with a function on } T_X.$$

The linear differential equation (T_X, ∇_v) will be called the linearization of (X, v) at the generic point.

- I will describe the case where $dim(X) = 2$. In that case, the generic fibre of

$$\pi : (T_X, \nabla_v) \to (X, v)$$

  is a linear differential equation of order 2 with parameters in $\mathbb{C}(X)$.
- We will denote by $G(X, v)$ its Galois group which can be identified with an algebraic subgroup of $GL_2(\mathbb{C})$. The vector field v defines tautological section

$$v : (X, v) \to (T_X, \nabla_v) \text{ compatible with the diff. structure.}$$

- The Galois group $G(X, v)$ has to fix this section, so that $G(X, v)$ is always a subgroup of $Aff_2(\mathbb{C})$.

Linearization at the generic point (dimension two)

When this Galois group $G(X, v)$ is maximal (i.e. $G(X, v) = Aff_2(\mathbb{C})$), we obtain:

Theorem (J.)

Let (X, v) be an autonomous differential equation of dimension two. Assume that

(i) *the Galois group $G(X, v)$ of the generic fibre of $\pi : (T_X, \nabla_v) \to (X, v)$ is the affine group.*

(ii) *the generic type of (X, v) is orthogonal to the constants .*

Then the generic type of (X, v) is minimal: if $t \mapsto x(t)$ is a Zariski-dense analytic solution of (X, v) and $f \in \mathbb{C}(X) \setminus \mathbb{C}$ then

$t \mapsto f(x(t))$ is a new meromorphic function of the class $\mathcal{C}_2$ (i.e. algebraically independent over $\mathbb{C}$ of every function $\phi(t) \in \mathcal{C}_1$).

Linearization at the generic point (dimension two)

When this Galois group $G(X, v)$ is maximal (i.e. $G(X, v) = Aff_2(\mathbb{C})$), we obtain:

Theorem (J.)

Let (X, v) be an autonomous differential equation of dimension two. Assume that

(i) *the Galois group $G(X, v)$ of the generic fibre of $\pi : (T_X, \nabla_v) \to (X, v)$ is the affine group.*

(ii) *the generic type of (X, v) is orthogonal to the constants .*

Then the generic type of (X, v) is minimal: if $t \mapsto x(t)$ is a Zariski-dense analytic solution of (X, v) and $f \in \mathbb{C}(X) \setminus \mathbb{C}$ then

$t \mapsto f(x(t))$ is a new meromorphic function of the class $\mathcal{C}_2$ (i.e. algebraically independent over $\mathbb{C}$ of every function $\phi(t) \in \mathcal{C}_1$).

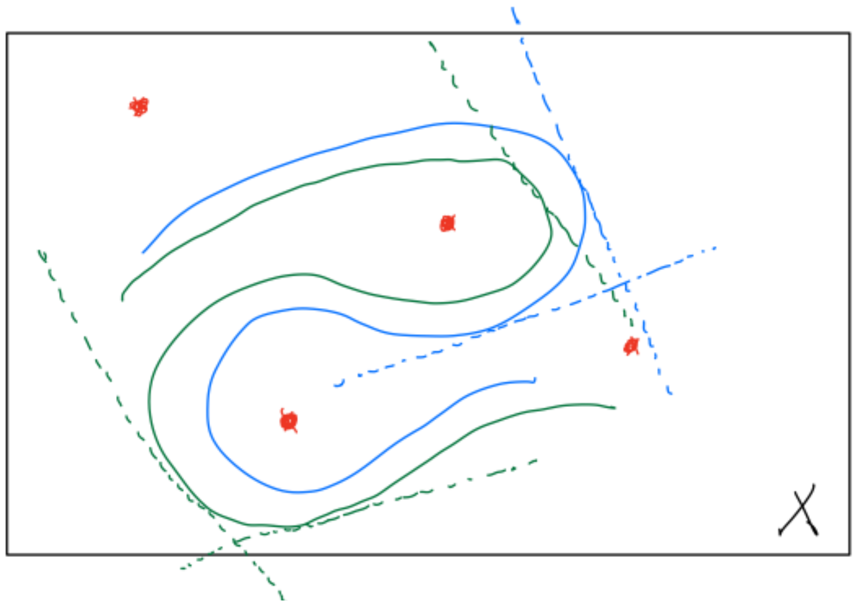
In practice,

- To apply this theorem in practice, I showed that if v admits an (isolated) complex zero which is not contained in any closed invariant algebraic curve then condition (i) holds.
- To obtain (ii), we can apply the linearization theorem in dimension two that I described before.

Explanations

The first tool both in the proof of this theorem and to compute the Galois group $G(X, v)$ in practice are the notions of invariant foliations and invariant webs (of foliations).

- The foliations are the rational sections $\sigma_{\mathcal{F}} : X \dashrightarrow T_X$ of the projectivization
$$\pi : \mathbb{P}(T_X) \to X$$
of the tangent bundle of X and the d-webs are the algebraic sections of degree d of π.
- Such a foliation $\mathcal{F}$ defines outside of its singular locus a partition of $X(\mathbb{C})^{an}$ in analytic Riemann surfaces immersed in $X(\mathbb{C})^{an}$ which are tangent at every point to the field of lines defined by $\mathcal{F}$.



Explanations

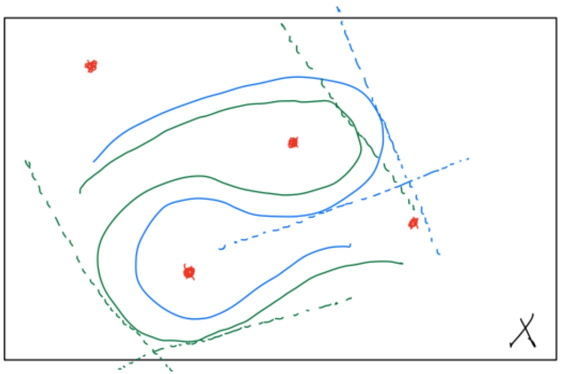
The first tool both in the proof of this theorem and to compute the Galois group $G(X, v)$ in practice are the notions of invariant foliations and invariant webs (of foliations).

- The foliations are the rational sections $\sigma_{\mathcal{F}} : X \dashrightarrow T_X$ of the projectivization

$$\pi : \mathbb{P}(T_X) \to X$$

  of the tangent bundle of X and the d-webs are the algebraic sections of degree d of π.

- Such a foliation $\mathcal{F}$ defines outside of its singular locus a partition of $X(\mathbb{C})^{an}$ in analytic Riemann surfaces immersed in $X(\mathbb{C})^{an}$ which are tangent at every point to the field of lines defined by $\mathcal{F}$.



- We say that a foliation is invariant under the vector field v if this partition is preserved by the solutions of the differential equation (X, v):

  if $t \mapsto x(t)$ and $t \mapsto y(t)$ are two analytic solutions of (X, v) and $x(0)$ and $y(0)$ are in the same $\mathcal{F}$-leaves then so are $x(t)$ and $y(t)$ for all times t where both solutions are defined.

- In particular, the foliation $\mathcal{F}(v)$ tangent to v is always invariant.

Explanations II

The Galois group $G(X, v)$ and its connected component G^0 encode some information about the structure of the foliations and webs on X invariant under v:

Proposition

Let (X, v) be an autonomous differential equation. Exactly one of the three following cases occur:

(1) $G(X, v) = G^0 = Aff_2(\mathbb{C})$ *if and only if apart from $\mathcal{F}(v)$, there is no other invariant foliation or any other invariant (irreducible) web.*

(2) $G^0 = \mathbb{G}_m(\mathbb{C})$ *if and only if there are exactly two irreducible webs invariant under v. One of them is $\mathcal{F}(v)$ and the other one $\mathcal{W}$ is:*

  (2a) *either a foliation,*

  (2b) *or a 2-web.*

(3) $G^0 = 0$ *if and only if v admits at least three invariant irreducible webs if and only if v admits infinitely many.*

Explanations II

The Galois group $G(X, v)$ and its connected component G^0 encode some information about the structure of the foliations and webs on X invariant under v:

Proposition

Let (X, v) be an autonomous differential equation. Exactly one of the three following cases occur:

(1) *$G(X, v) = G^0 = Aff_2(\mathbb{C})$ if and only if apart from $\mathcal{F}(v)$, there is no other invariant foliation or any other invariant (irreducible) web.*

(2) *$G^0 = \mathbb{G}_m(\mathbb{C})$ if and only if there are exactly two irreducible webs invariant under v. One of them is $\mathcal{F}(v)$ and the other one $\mathcal{W}$ is:*
   - (2a) *either a foliation,*
   - (2b) *or a 2-web.*

(3) *$G^0 = 0$ if and only if v admits at least three invariant irreducible webs if and only if v admits infinitely many.*

- The second ingredient of the proof: the notion of modularity (or one-basedness) in geometric stability theory. We use our hypothesis (ii) to show that if the generic type of (X, v) is not modular then we can produce an invariant foliation $\mathcal{F} \neq \mathcal{F}(v)$.
- Once we know the type is one-based, we use the good properties of the semi-minimal analysis of one-based types to produce an irreducible invariant web $W \neq \mathcal{F}(v)$ when the type is not minimal.

An application: planar algebraic vector fields

Theorem (J.)

Consider a differential equation of the form:

$$(E_v) : \begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases} \quad \textit{with } f(x,y), g(x,y) \in \mathbb{C}[x,y]$$

Assume that $f(x,y)$ and $g(x,y)$ are polynomials of degree $d \geq 3$ and that the coefficients of $f(x,y)$ and $g(x,y)$ satisfy no non-trivial algebraic relation with coefficients in $\mathbb{Q}$ (in particular, they are all no zero). Then the set of solutions of (E_v) is:

(A) *strongly minimal : the coordinates of every non-constant analytic solutions are new meromorphic functions of the class $\mathcal{C}_2$.*

(B) *disintegrated (or geometrically trivial): if $(x_1(t), y_1(t)), \ldots, (x_r(t), y_r(t))$ are r non-cconstant analytic solutions which are not algebraically independent then*

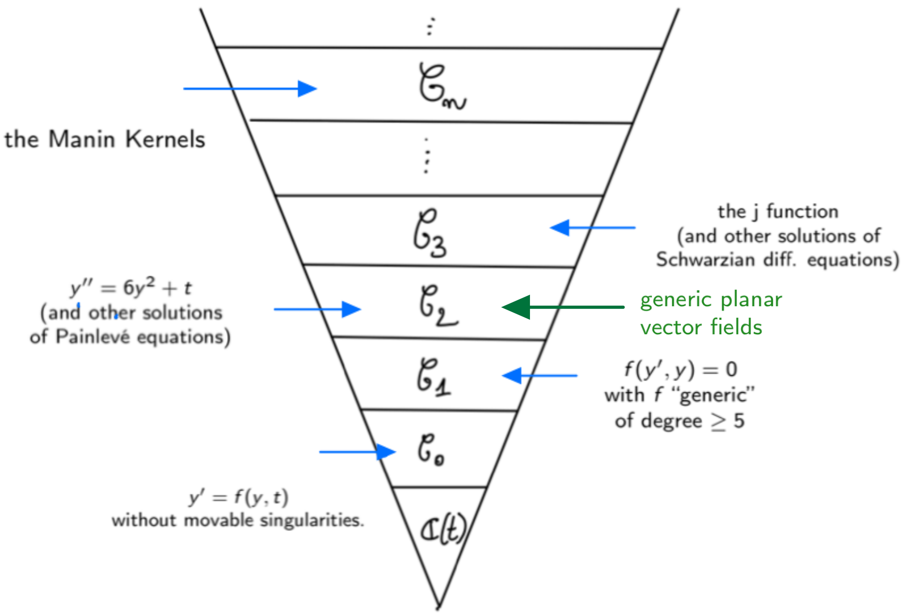
$$t \mapsto (x_1(t), y_1(t), x_2(t), \ldots, x_r(t), y_r(t)) \textit{is not Zariski-dense in } \mathbb{C}^{2n}$$

then there are $i \neq j$ such that:

$$t \mapsto (x_i(t), y_i(t), x_j(t), y_j(t)) \textit{ is not Zariski-dense in } \mathbb{C}^4.$$

Generic differential equations

(Poizat 1980): à propos des corps différentiels, on est souvent amené à faire des conjectures dont on est persuadé qu'elles ne peuvent être fausses que pour des équations très particulières et que pourtant on arrive à montrer que dans des cas encore plus particuliers.



- (Shelah 73') : A "sufficiently general" algebraic differential equation of order $n \geq 2$ is minimal.
- In other words, the generic solutions of such equations are new meromorphic functions of the class $\mathcal{C}_n$.

Thank you for your attention