

# HOMEWORK: MODEL THEORY OF DIFFERENTIAL FIELDS

DUE ON FRIDAY MARCH 8TH

**Convention/notation.** All differential fields are of characteristic zero. If  $k$  is a differential field,  $k\langle\alpha_1, \dots, \alpha_n\rangle$  always denotes the differential field generated by  $\alpha_1, \dots, \alpha_n$  and  $k$ .

## 1. PART 1. KOLCHIN'S PRIMITIVE ELEMENT THEOREM

The goal of this exercise is to prove:

**Theorem A** (Kolchin, 1942). *Let  $k$  be a nonconstant differential field and let  $K = k\langle\alpha_1, \dots, \alpha_n\rangle$  be a finitely generated differential field extension of  $k$  such that each  $\alpha_i$  is a solution of a nonzero differential polynomial  $A_i \in k\{X\}$ . Then there exists  $\gamma \in K$  such that*

$$K = k\langle\alpha_1, \dots, \alpha_n\rangle = k\langle\gamma\rangle.$$

An element  $\gamma$  satisfying the conclusion of the theorem is called a *primitive element* of the differential field extension  $K/k$ .

**1.1. Primitive elements and automorphisms.** Let  $k$  be a differential field and fix  $\mathcal{U} \models \text{DCF}_0$  an  $\omega$ -saturated and  $\omega$ -homogeneous extension of  $k$ .

- (1a) Show that to prove the theorem, we may assume that  $k$  is a finitely generated differential field.
- (1b) Let  $\alpha_1, \dots, \alpha_n \in \mathcal{U}$  and set  $K = k\langle\alpha_1, \dots, \alpha_n\rangle$ . Under the assumption of (1a), show that an element  $\gamma \in K$  is a primitive element of  $K/k$  if and only if for every  $\sigma \in \text{Aut}_\partial(\mathcal{U}/k)$ ,

$$\sigma(\gamma) = \gamma \Rightarrow \sigma(\alpha_i) = \alpha_i \text{ for } i = 1, \dots, n.$$

**1.2. Non-vanishing of non-zero differential polynomials.** Let  $k$  be a nonconstant differential field. The goal of this question is to prove that *if  $A(X_1, \dots, X_n) \in k\{X_1, \dots, X_n\}$  is a nonzero differential polynomial then there exist  $a_1, \dots, a_n \in k$  such that*

$$A(a_1, \dots, a_n) \neq 0.$$

- (2a) Show that it is enough to prove the previous statement for  $n = 1$ .
- (2b) Consider  $g \in k$  with  $\partial(g) \neq 0$ . Replacing the derivation  $\partial$  by the derivation

$$D = \frac{\partial}{\partial(g)} \in \text{Der}(k)$$

show that we may assume that  $k$  contains an element  $t$  such that  $\partial(t) = 1$  in order to prove the previous statement.

- (2c) Consider a nonzero  $A(X) \in k\{X\}$  and fix  $t \in k$  an element with  $\partial(t) = 1$ . Show that we can find a polynomial  $\chi = \chi(t) \in \mathbb{Q}[t]$  such that

$$A(\chi) = P(\chi, \chi', \dots, \chi^{(n)}) \neq 0.$$

**1.3. Existence of primitive elements.** Let  $A_1(X), \dots, A_n(X) \in k\{X\}$  be nonzero differential polynomials. Consider the differential radical ideal  $I$  of  $k\{X_1, \dots, X_n, Y_1, \dots, Y_n, T_1, \dots, T_n\}$  given by

$$I = \{A_1(X_1), \dots, A_n(X_n), A_1(Y_1), \dots, A_n(Y_n), T_1 \cdot (X_1 - Y_1) + \dots + T_n \cdot (X_n - Y_n)\}$$

and the decomposition

$$I = I_1 \cap \dots \cap I_n$$

into prime differential ideals given by Ritt second theorem.

- (3a) Show that for every  $i$ , we can find  $x_1, \dots, x_n, y_1, \dots, y_n, t_1, \dots, t_n \in \mathcal{U}$  such that

$$I_i = I(x_1, \dots, x_n, y_1, \dots, y_n, t_1, \dots, t_n/k)$$

- (3b) With the previous notation, show that if  $x_i \neq y_i$  for some  $i$  then there exist a nontrivial differential polynomial  $L_i \in k\{T_1, \dots, T_n\}$  such that

$$L_i(t_1, \dots, t_n) = 0$$

- (3c) Conclude that there exists  $\tau_1, \dots, \tau_n \in k$  such that the function

$$x_1, \dots, x_n \mapsto \tau_1 x_1 + \dots + \tau_n x_n$$

is injective when evaluated in the cartesian product of the solution sets in  $\mathcal{U}$  of  $A_1(X), \dots, A_n(X)$ .

#### 1.4. Conclusion.

- (4) Prove Theorem A.

### 2. PART 2. LIOUVILLE'S THEOREM ON INTEGRATION IN FINITE TERMS

The goal of this second exercise is to prove:

**Theorem B** (Liouville, 1833). *Let  $f(t)$  be an algebraic function. Assume that the primitive  $\int f(t)dt$  is an elementary function. Then*

$$\int f(t)dt = R_0(t, f(t)) + \sum_{i=1}^n c_i \cdot \ln(R_i(t, f(t)))$$

where the  $c_i$  are complex numbers and the  $R_i(X, Y)$  are rational functions of two variables.

Fix  $k$  a differential field with an algebraically closed field of constants.

**2.1. Elementary differential fields extensions.** We say that an extension of differential fields  $K/k$  is *elementary* if  $K$  and  $k$  have the same field of constants and there exists a tower of differential fields

$$k = k_0 \subset k_1 \subset \dots \subset k_n = K$$

such that each  $k_{i+1}/k_i$  is either an algebraic extension, generated by a logarithm of an element of  $k_i$  or by the exponential of an element in  $k_i$ .

- (1) Let  $K/k$  be an elementary differential field extension with  $\text{td}(K/k) = n$ . Show that there exists  $n$   $K$ -linearly independent one-forms  $\omega_1, \dots, \omega_n \in \Omega^1(K/k)$  of the form

$$\omega_i = dy_i/y_i - dx_i \text{ with } x_i \in K, y_i \in K^*$$

such that  $\mathcal{L}_\partial(\omega_i) = 0$  for  $i = 1, \dots, n$ .

We say that a meromorphic function  $f(t) \in \mathcal{M}(U)$  is an elementary function if its restriction to some nonempty subset of  $U$  is contained in an elementary differential field extension of  $\mathbb{C}(t)$ .

**2.2. Reduction to the algebraic case.** Let  $f(t)$  be an algebraic function and assume that  $\int f(t)dt$  is an elementary function.

- (2a) Consider  $K/\mathbb{C}(t)^{\text{alg}}$  an elementary differential field extension containing  $\int f(t)dt$ . Show that there exists constants  $c_1, \dots, c_n \in \mathbb{C}$ ,  $x_1, \dots, x_n \in K$  and  $y_1, \dots, y_n \in K^*$  such that

$$df = \sum_{i=1}^n c_i \cdot \left( \frac{dy_i}{y_i} - dx_i \right) \in \Omega^1(K/\mathbb{C}(t)^{\text{alg}}).$$

- (2b) Conclude that

$$f(t) = S'_0(t) + \sum_{i=1}^m \gamma_i \cdot \frac{S'_i(t)}{S_i(t)}$$

where the  $\gamma_i$  are complex numbers and  $S_0, \dots, S_m \in \mathbb{C}(t)^{\text{alg}}$ .

**2.3. Reduction to the rational case.** Denote by  $L$  the normal extension of  $\mathbb{C}(t, f(t))$  generated by  $S_0(t), \dots, S_m(t)$  and set  $G = \text{Gal}(L/\mathbb{C}(t, f(t)))$ .

- (3) Using Galois theory, prove Theorem B.