LECTURE 3. DIFFERENTIALLY CLOSED FIELDS

We apply the results of the previous section to define the theory DCF_0 of *differentially closed fields of characteristic zero*. We work in the language of differentials rings

$$\mathcal{L}_\partial = \{0, 1, +, \times, -, \partial\} = \mathcal{L}_{\text{rings}} \cup \{\partial\}$$

where ∂ is a unary function symbol.

3.1. **Existential closedness (Robinson).** We are interested in th following class of $\mathcal{L}_\partial$-structures:

(A1) R is an integral domain

(A2) ∂ is a derivation, that is, for all x, y

$$\partial(x + y) = \partial(x) + \partial(y) \text{ and } \partial(xy) = x\partial(y) + y\partial(x).$$

(A3) R expands the field $\mathbb{Q}$ of rational numbers.

In the relative situation (A3) is replaced by: R is a differential k-algebra for some given differential field k. This axiom becomes expressible after adding constants to $\mathcal{L}_\partial$ accounting for the element of k.

Exercise 3.1. *Show that (A1) and (A2) can be expressed as $\forall$-sentences and that (A3) can be expressed as a $\forall\exists$-sentence.*

Lemma 3.2. *Let R be a differential ring. Every $\mathcal{L}_\partial$-term with variables $x_1, \ldots, x_n$ and parameters in R is of the form*

$$t^R(x_1, \ldots, x_n) = P(x_1, \ldots, x_n)$$

for some $P \in R\{X_1, \ldots, X_n\}$.

Proof. By induction on the complexity of the term. □

Definition 3.3. An integral (differential) ring R (or an integral (differential) k-algebra) is *existentially closed* if

> for every *embedding* of integral (differential) rings $i : R \to S$ and every existential sentence $\psi = \exists x \phi(x)$ with parameters in R, if S satisfies ψ then so does R.

Note that in the previous definition, we allow x to denote a tuple $x_1, \ldots, x_n$ of variables. The lemma implies that one can find a finite collection $P_{i,j}, Q_i \in R\{X_1, \ldots, X_n\}$ such that

$$R \models \phi(x) \leftrightarrow \bigvee_i \Big(\bigwedge_j P_{i,j}(x) = 0, Q_j(x) \neq 0 \Big).$$

As a consequence, a differential ring is existentially closed if and only if:

> *every system of equations and inequations which admits a solution in a differential ring extension of R already admits a solution in R.*

Definition 3.4. A *differentially closed field* is an existentially closed integral differential ring.

Proposition 3.5 (amalgamation)**.** *Let k be a differential field and R, S be integral differential k-algebras. There exists an integral differential k-algebra T in which R and S k-embed.*

Proof. It is enough to prove the statement when R is a finitely generated k-algebra and S is a differential field. The second part follows from the fact that a derivation extends from an integral domain to its field of fractions. The first part is by compactness. We write $S = K$ and

$$R = k\{X_1, \ldots, X_n\}/I$$

Consider the embedding $k\{X_1, \ldots, X_n\} \to K\{X_1, \ldots, X_n\}$ and the ideal $J = I \cdot K\{X_1, \ldots, X_n\}$. Then J is a differential ideal but not necessarily prime. The following claim shows that J is reduced.

Claim. *Let K/k be an extension of fields in characteristic zero. If R is an integral k-algebra then $R \otimes_k K$ is reduced.*

Proof. It is enough to prove the statement when K/k is finitely generated and R is finitely generated. So we can write $R = k[x_1, \ldots, x_n]/I$ and $K = k(x_1, \ldots, x_r)(x_{r+1})$ where $x_1, \ldots, x_r$ a transcendance basis and x_{r+1} algebraic over $L = k(x_1, \ldots, x_r)$. First, note that

$$R \otimes_k L = R \otimes_k \operatorname{Frac}(k[x_1, \ldots, x_r]) \subset \operatorname{Frac}(R \otimes_k k[x_1, \ldots, x_r])$$

Since $R \otimes_k k[x_1, \ldots, x_r] = R[x_1, \ldots, x_r]$ is an integral domain, we get that $R_L := R \otimes_k L$ is an integral domain. Denoting by μ the minimal polynomial of x_{r+1} over L, we get

$$R \otimes_k L = R_L \otimes_L K = R_L \otimes L[X]/(\mu) = R_L[X]/(\mu)$$

Since μ is separable (its roots are distinct) then this embeds in a product of fields. A product of fields is reduced so that $R \otimes_k L$ is reduced. □

By Ritt-Raudenbush theory, we can write $J = P_1 \cap \ldots \cap P_n$. We claim that

Claim. *$P_i \cap k\{X_1, \ldots, X_n\} = I$ for some i.*

Indeed, otherwise we can find $a_i \in P_i \cap k\{X_1, \ldots, X_n\} \setminus I$ for all i. Hence, we have $a = a_1 a_2 \cdots a_n \in I$. This contradicts that I is prime. It follows that the induced morphism

$$k\{X_1, \ldots, X_n\}/I \to K\{X_1, \ldots, X_n\}/J$$

is injective, as required. □

Corollary 3.6. *Every integral differential ring R embeds into a differentially closed field.*

Note that a differential ring is existentially closed if and only if it is existentially closed in all its finitely generated extensions.

Proof. We may assume that R is a differential field k. List all the quantifier-free formulas $(\psi_s(x), s \in)$ which are realized in some extension R_s of k. We want to show that the collection of sentences expressing that

$$\{R \text{ is an integral differential ring extending } k\} \cup \{R \models \exists x \psi_s(x) \mid @s \in S\}$$

is consistent. It is enough to see that it is finitely consistent and this follows from amalgamation.

Hence, we have constructed an integral differential ring k_1 which we may assume to be a field such that every qf-formula with parameters from k which has a solution in some extension of k has a solution in k_1. Iterating the process, we get a chain:

$$k_1 \subset k_2 \subset \cdots \subset k_n \subset \cdots$$

and the limit $k_\infty = \cup k_n$ is differentially closed. □

3.2. **Elimination of quantifiers (Blum).**

Proposition 3.7. *let R be a differentially closed field. Then R satisfies*

(A4) *R is an* algebraically closed field,
(A5) *for every nonconstant $P, Q \in R\{X\}$ with* $\mathrm{ord}(Q) < \mathrm{ord}(P)$*, the differential system*

$$P(x) = 0 \text{ and } Q(x) \neq 0$$

admits a solution in R.

Proof. Assume that R is an existentially closed differential integral domain. Then R is existentially closed integral domain. This follows from:

(*separability*) given a field extension L/k, any derivation on k extends to a derivation on L in characteristic zero

It follows that R is an algebraically closed field that we denote k. Now it is enough to consider (A4) for pairs $P, Q \in k\{X\}$ such that P is irreducible and $\mathrm{ord}(P) > \mathrm{ord}(Q)$. Denote by $I(P)$ the prime differential ideal associated to P by Ritt-Raudenbush theory. An easy computation shows that $Q \notin I(P)$. It follows that the image a of X in the integral differential domain $S = k\{X\}/I(P)$ is a solution of (3.7) in S. By existential closedness, such a solution already exist in R. □

Exercise 3.8. *Show that (A4) and (A5) can be expressed as an infinite set of $\forall\exists$-sentences.*

Definition 3.9. The theory DCF_0 is the first-order theory in the language $\mathcal{L}_\partial$ expressing the axioms (A1)-(A5).

Theorem 3.10 (Elimination of quantifiers)**.** *The theory* DCF_0 *eliminates the quantifiers in the language $\mathcal{L}_\partial$.*

Recall that a theory T in a language $\mathcal{L}$ eliminates quantifiers (or has QE) if every $\mathcal{L}$-formula is equivalent to a quantifier-free formula. Let T be a theory in a language $\mathcal{L}$. T has QE if and only if

whenever $M, N \models T$ extend a common finitely generated substructure A, $\overline{a} \in A^n$, $m \in M$ and $\phi(x, \overline{y})$ a quantifier-free $\mathcal{L}$-formula (without parameters) such that

$$M \models \phi(m, \overline{a}) \Rightarrow N \models \exists x \phi(x, \overline{a})$$

Furthermore, up to replacing N by an elementary overstructure, we may assume that N is ω-saturated[1] in order to check $(*)$.

Proof. Consider $K, L \models \mathrm{DCF}_0$ containing a common finitely generated differential subring A and assume that L is ω-saturated. A is an integral subring and since we are in characteristic zero:

[1]This means that every countable set of $\mathcal{S} = \{\phi_i(x, \overline{l}) \mid i \in \mathbb{N}\}$ which is *finitely* satisfiable in N is satisfiable in N.

the *derivation on A extends uniquely to the algebraic closure k of the fraction field of A*

As K, L are algebraically closed differential fields (A4), the embeddings $i_K : A \to K$ (resp. $i_L : A \to L$) extends uniquely to embeddings

$$\overline{i_K} : k \to K \text{ (resp. } \overline{i_L} : k \to L).$$

Consider $m \in K$, $\overline{s} \in k$ and $\phi(x, \overline{y})$ quantifier-free such that $K \models \phi(m, \overline{s})$. Note that if $n \in L$ satisfies that there exists an isomorphism

$$k\langle m \rangle \simeq k\langle n \rangle$$

sending m to n then m and n satisfy the same quantifier free-formulas. So to show that $L \models \exists \phi(x, \overline{s})$, it is enough to find a differential subfield of L isomorphic to $k\langle m \rangle$. They are two cases:

- Case 1. $m \in K$ satisfies a nontrivial differential equation over k that is

$$I(m/k) = \{f \in k\{X\} \mid f(m) = 0\}$$

is a nonzero ideal of $k\{X\}$.

By the first theorem of Ritt (Lecture 2), $I(m/k) = I(f)$ where f is a minimal nonzero polynomial in $I(m/k)$ with respect to $\ll$ which is irreducible. Axioms (A3) of DCF_0 implies that the countable set of formulas

$$\{f(x) = 0 \wedge g(x) \neq 0 \mid g(x) \in k\{x\} \text{ with } \mathrm{ord}(g) < \mathrm{ord}(f)\}$$

is finitely satisfiable in L and hence by ω-saturation satisfiable in L. By hypothesis, $I(n/k)$ is a prime ideal containing f and no differential polynomial of lower order. Since f is irreducible, we have

$$I(n/k) = I(f) = I(m/k) \Rightarrow k\langle m \rangle \simeq k\langle n \rangle$$

as required.

- Case 2. $m \in K$ satisfies no nontrivial differential equation over k.

In that case, by ω-saturation of L, the countable set of formulas

$$\{f(x) \neq 0 \mid f(x) \in k\{X\}\}$$

is finitely satisfiable and hence satisfiable in L say by $n \in L$. By construction, we have

$$k\langle m \rangle \simeq k\langle X \rangle \simeq k\langle n \rangle.$$

as required. This completes the proof of the theorem. □

Corollary 3.11. *The theory* DCF_0 *is complete. In particular, any two differentially closed fields satisfy the same first order theory.*

Proof. Every differentially closed field contains $\mathbb{Q}$ equipped with the trivial derivation as a substucture. A theory with QE whose models share a common substructure is complete (exercise). □

3.3. **Kolchin topology.** Consider $K \models \mathrm{DCF}_0$ arbitrary.

Definition 3.12. A *Kolchin-closed subset* of K^n is a subset of the form

$$V(f_1, \ldots, f_s) = \{\overline{x} \in K^n \mid f_1(x) = \ldots = f_s(x) = 0\}$$

where $f_1, \ldots, f_s \in K\{X_1, \ldots, X_n\}$. When $f_1, \ldots, f_s \in k\{X_1, \ldots, X_n\}$, we say that V is a Kolchin-closed set defined over k.

Note that any function f belonging to the radical differential ideal generated by $f_1, \ldots, f_s$ vanishes on $V(f_1, \ldots, f_s)$. Indeed, writing

$$f^n = a_1 f_1 + \cdots + a_s f_s + b_1 f_1' + \cdots b_n f_n' + c_1 f_1'' + \cdots$$

it is enough to see that all the derivatives of the f_i vanish on V. But

$$f(x) = 0 \Rightarrow \partial(f(x)) = 0$$

Exercise 3.13. *A Kolchin-closed set is defined over k if and only if it is a k-definable set, that is, the set of realization of a formula with parameters in k.*

Lemma 3.14. *The Kolchin-closed subset of K^n are the closed subset of a quasi-compact topology on K^n.*

Proof. Note that any function belongs to the radical differential ideal $f_1, \ldots, f_s$ and $g_1, \ldots, g_r$ generate the same radical differential ideal □

Corollary 3.15 (Differential Nullstellensatz)**.** *We have an inclusion reversing one-to-one correspondence*

$$\begin{array}{rcl} \{\text{ Kolchin-closed subset of } K^n\} & \leftrightarrows & \{\text{ radical differential ideals of } K\{\overline{X}\}\} \\ \Sigma & \to & I(\Sigma) = \{f \in K\{\overline{X}\} \mid f(x) = 0 \text{ for all } x \in \Sigma\} \\ V(I) = \{\overline{x} \in K^n \mid f(\overline{x}) = 0 \text{ for all } f \in I\} & \leftarrow & I \end{array}$$

Furthermore, the Kolchin-topology of K^n is a noetherian topology and irreducible Kolchin-closed subsets correspond to prime ideals.

Proof. Clearly, $I(\Sigma)$ is an ideal. It is radical and differential since for every $\overline{x} \in K^n$,

$$f^n(\overline{x}) = 0 \Rightarrow f(\overline{x}) = 0 \text{ and } f(\overline{x}) = 0 \Rightarrow \partial(f)(\overline{x}) = 0$$

as the evaluation is a morphism of differential rings. Conversely, $V(I)$ is a Kolchin-closed set since by the second theorem of Ritt (Lecture 2), $I = \{f_1, \ldots, f_n\}$ is finitely generated. Furthermore, we have

$$V(I(\Sigma)) = \Sigma \text{ and } I(V(\Sigma)) = I.$$

Indeed, the first equality follows from the second one as by definition any Kolchin-closed set Σ can be written $V(\{f_1, \ldots, f_p\})$ so that assuming the second equality, we get

$$V(I(\Sigma)) = V(I(V(\{f_1, \ldots f_p\}))) = V(\{f_1, \ldots, f_p\}) = \Sigma.$$

It is therefore enough to prove the second equality. To that end, note that $I \subset I(V(\Sigma))$ and consider $f \in K\{\overline{X}\} \setminus I$. Write

$$I = \bigcap_{j=1}^{n} I_j$$

where the I_j are prime differential ideals so that $f \notin I_j$ for some j. It follows that

$$L = \mathrm{Frac}(K\{\overline{X}\}/I_j) \subset \mathcal{U} \models \mathrm{DCF}_0$$

is a differential field. By construction, The image of $\overline{x}$ of $\overline{X}$ in $\mathcal{U}$ satisfies

$$\overline{x} \in \Sigma \wedge f(\overline{x}) \neq 0$$

so that $\mathcal{U} \models \exists \overline{x}(\overline{x} \in \Sigma) \wedge f(\overline{x}) \neq 0$ which is a sentence with parameters from K. It follows from Theorem 3.10 that modulo DCF_0, this formula is equivalent to a quantifier-free formula which is satisfied in $\mathcal{U}$ iff it is satisfied in K. It follows that

$$K \models \exists \overline{x}(\overline{x} \in \Sigma) \wedge f(\overline{x}) \neq 0$$

and hence that $f \notin I(V(\Sigma))$ as required. The second part of the statement is left as an exercise using the second theorem of Ritt. $\square$

Corollary 3.16. *The theory* DCF_0 *is noetherian theory, that is, for every model* $K \models \mathrm{DCF}_0$, K^n *is equipped with a noetherian topology such that the definable sets with* n *variables are Boolean combination of closed subsets for this topology.*

3.4. **References.** The concept of differentially closed field was introduced by Abraham Robinson [Rob59] as a differential analogue of the concept of algebraically closed field. The presentation of the theory DCF_0 based on the schemes of axioms (A1) to (A3) and the fact that differentially closed fields enjoy the elimination of quantifiers in the language of differential rings is due to Lenore Blum in her PhD thesis [Blu69]. See also [MMP96] for another presentation of these results.

References

[Blu69] Lenore Carol Blum. *Generalized algebraic theories: A model theoreticapproach.* ProQuest LLC, Ann Arbor, MI, 1969. Thesis (Ph.D.)–Massachusetts Institute of Technology.

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[Rob59] Abraham Robinson. On the concept of a differentially closed field. *Bull. Res. Council Israel Sect. F*, 8F:113–128 (1959), 1959.