Central Limit Theorem and Confidence Intervals

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Outline

1. The Normal distribution
2. Central Limit Theorem
3. Confidence intervals using the Central Limit theorem

We've previously used *bootstrapping* to estimate the *standard error* of a *point estimate*, which allowed us to form *confidence intervals*:

$$\text{Point Estimate} \pm c * SE$$

- For bell-shaped sampling distributions, $c = 2$ produces 95% confidence intervals (this is the 2-SE method)
  - The 2-SE method for bell-shaped distributions is justified by the 99-95-68 percent rule
  - Thus, we could use a *different multiplier* to achieve a *different confidence level*

The Normal distribution

The **Normal curve**, or Normal probability function, is a mathematical function that yields a bell-shaped distribution:

$$f(X) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(X-\mu)^2}{2\sigma^2}}$$

- μ is a constant that defines the *center* of the bell-curve
- σ is a constant that defines the *standard deviation* of the bell-curve (how peaked or flat it is)

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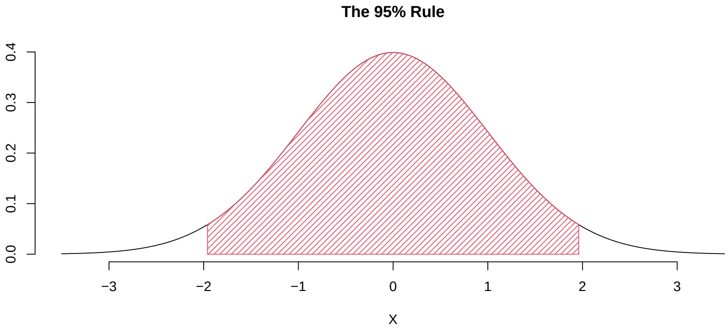
- μ is a constant that defines the *center* of the bell-curve
- σ is a constant that defines the *standard deviation* of the bell-curve (how peaked or flat it is)
- There infinitely many different Normal curves, one for each combination of μ and σ
    - We'll reference them using the shorthand: $N(\mu, \sigma)$

The Normal distribution

When data follow a Normal distribution, the *area under the curve* describes the likelihood you see a value within a particular range:



The Normal probability function doesn't have a closed-form integral, so we must rely upon software to find these areas

Practice

The Theoretical Distributions section of StatKey allows us to work with various Normal curves:

1) Consider a *standard Normal distribution*, or N(0,1), what values define the middle 90% of this distribution?
2) Consider a N(10,5) distribution, what proportion of this distribution is larger than 16?

Practice (solution)

1) The values of -1.645 and +1.645 define the middle 90% of the curve. This suggests we could use 1.645 as a multiplier of the SE to form a 90% CI estimate (if the sampling distribution is approximately Normal).
2) The area to the right of 16 on the N(10,5) curve is 0.115. This suggests there's a 11.5% chance of observing a value 16 or larger if the data follow this distribution.

Confidence intervals (using the Normal distribution)

If the distribution of a sample estimate is approximately Normal, we can create a P% confidence interval estimate for a population parameter via:

$$\text{Point Estimate} \pm c * SE$$

where c is a value taken from the $N(0,1)$ distribution that defines the middle P% of the distribution.

- So far, we've used *bootstrapping* to find the SE (a necessary component of this formula)
  - We've also used *bootstrapping* to assess Normality (of the sampling distribution)

Central Limit Theorem

The **Central Limit Theorem** (CLT) is a theoretical result that establishes a Normal distribution, with known SE, for a variety of different sample estimates (provided a sufficient sample size):

$$\text{Estimate} \sim N(\text{Population Parameter}, SE)$$

- The sample size needed for CLT to hold depends on the parameter we're estimating
  - For example, $n = 30$ is considered sufficient when estimating μ (a population's mean)
- CLT provides a mathematical formula for the SE!
  - This formula will depend upon the population parameter we're estimating

Central Limit Theorem (one proportion)

When estimating a *single proportion*, CLT suggests:

$$\hat{p} \sim N\left(p, \sqrt{\frac{p(1-p)}{n}}\right)$$

- This suggests $SE = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$
  - We can then choose c from the N(0,1) curve in order to find a P% CI estimate of p
- The sample size condition for this result is: $n\hat{p} \geq 10$ and $n(1-\hat{p}) \geq 10$

Practice

A 2021 study looked at the true-positive rate (sensitivity) of the Abbott Diagnostics rapid test for Covid-19. Of the 84 cases with symptomatic Covid-19 that took the test, 38 had a "positive" result. Our goal is to estimate p, the overall sensitivity of this test in the target population (ie: all symptomatic Covid cases).

1) Verify that the conditions are met to use the CLT Normal approximation to construct a confidence interval estimate
2) Find the values of $\hat{p}$, the SE, and c necessary to construct a 99% CI estimate of p
3) Calculate and interpret the 99% CI

Practice (solution)

1) First, $\hat{p} = 38/84 = 0.452$. Then, $n\hat{p} = 84 * 0.452 = 38$ and $n(1 - \hat{p}) = 84 * (1 - 0.452) = 46$. Because both are larger than 10, the CLT Normal approximation is reasonable.
2) $\hat{p} = 38/84 = 0.452$, $SE = \sqrt{\frac{0.452(1-0.452)}{84}} = 0.054$, and $c = 2.576$ (this defines the middle 99% of a N(0,1) curve)
3) The 99% CI is $0.452 \pm 2.576 * 0.054 = (0.313, 0.591)$. Our sample suggests, with 99% confidence, that the true sensitivity of the Abbott rapid test is somewhere between 31.3% and 59.1%

Central Limit Theorem (two proportions)

When estimating a *difference of two proportions*, CLT suggests:

$$\hat{p}_1 - \hat{p}_2 \sim N\left(p_1 - p_2, \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}\right)$$

- Using the sample proportions: $\hat{p}_1$ and $\hat{p}_2$, as well as their denominators: n_1 and n_2 this result can be used to find the *SE* necessary to construct a confidence interval estimate of $p_1 - p_2$
- The sample size condition to use this result is $n_1\hat{p}_1 \geq 10$, $n_1(1 - \hat{p}_1) \geq 10$, $n_2\hat{p}_2 \geq 10$, and $n_2(1 - \hat{p}_2) \geq 10$

Practice

The previously mentioned study also examined a test produced by Siemens. Of the 72 cases with symptomatic Covid-19 that took the Siemens test, 39 had a "positive" result. Recall that 38 of 84 symptomatic cases tested positive on the Abbott test. Suppose our goal is estimate $p_1 - p_2$, the difference in sensitivity of these two tests (at the population level)

1) Let $\hat{p}_1 = 38/84 = 0.45$ be the sample proportion for the Abbott test, and $\hat{p}_2 = 39/72 = 0.54$ be the sample proportion for the Siemens test. Find the SE for the difference in proportions, $\hat{p}_1 - \hat{p}_2$.
2) Using the CLT Normal approximation, find and interpret a 95% CI estimate for $p_1 - p_2$

Practice (solution)

1) $SE = \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}} = \sqrt{\frac{0.45(1-0.45)}{84} + \frac{0.54(1-0.54)}{72}} = 0.08$
2) Since $c = 1.96$ (for 95% confidence), $\hat{p}_1 - \hat{p}_2 = 0.45 - 0.54 = -0.09$, and $SE = 0.08$, we calculate: $-0.09 \pm 1.96 * 0.08 =$ (-0.247, 0.067).
   - This interval represent a plausible range of differences in the sensitivity of these tests at the population level (estimated with 95% confidence). Since zero is included in this interval, it's plausible that the tests are no different.

Central Limit Theorem (one mean)

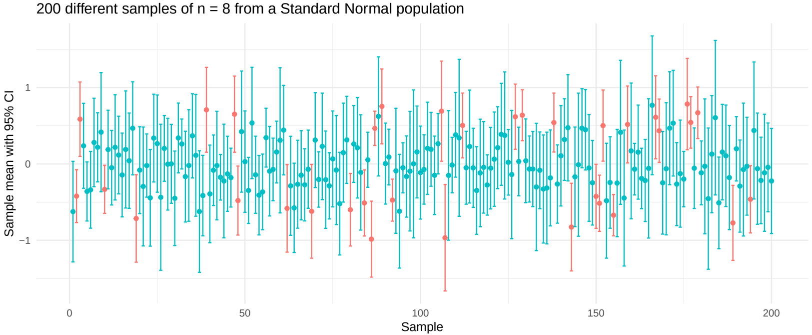
When estimating a *single mean*, CLT suggests:

$$\bar{x} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$$

- σ is the standard deviation of the population (something that's almost always unknown)
  - This result is not directly applicable in most real-world scenarios (see next few slides)

William Gosset and the t-distribution

- The previous results involves a *second unknown parameter*, σ (the population's standard deviation)
  - It seems natural to simply replace σ with an estimate from the sample, s, but this is what happens:



William Gosset and the t-distribution

- Clearly this procedure for constructing 95% CIs is *invalid*, too many of these intervals did not contain μ
- William Gosset, an employee at Guinness Brewing, became aware of this issue in the 1890s
  - His work evaluating the yields of different barley strains often involved statistical analyses on small, Normally distributed samples

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- William Gosset, an employee at Guinness Brewing, became aware of this issue in the 1890s
  - His work evaluating the yields of different barley strains often involved statistical analyses on small, Normally distributed samples
- In 1906, Gosset took a leave of absence from Guinness to study under Karl Pearson (developer of the correlation coefficient)
  - Gosset discovered the issue was due to using s (sample standard deviation) interchangeably with σ (population standard deviation)

William Gosset and the t-distribution

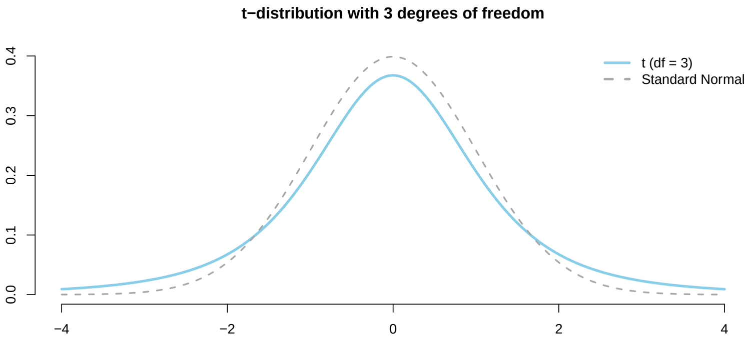
- Treating s as if it were a perfect estimate of σ results in a systematic underestimation of the total amount of variability involved in estimating μ
  - To account for the additional variability introduced by estimating σ using s, Gosset proposed a modified distribution that's slightly more spread out than the Standard Normal curve

William Gosset and the t-distribution

- Treating s as if it were a perfect estimate of σ results in a systematic underestimation of the total amount of variability involved in estimating μ
  - To account for the additional variability introduced by estimating σ using s, Gosset proposed a modified distribution that's slightly more spread out than the Standard Normal curve
- Typically the inventor of a new method gets to name it after themselves
  - However, Gosset was forced to publish his new distribution under the pseudonym "student" because Guinness didn't want it's competitors knowing they employed statisticians!
  - Student's t-distribution is now among the most widely used statistical results of all time

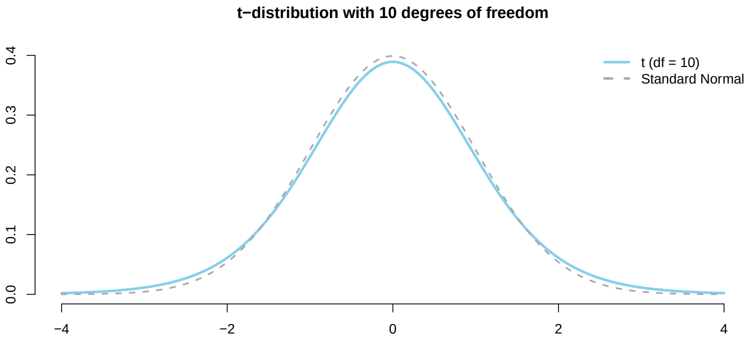
The t-distribution

The *t*-distribution accounts the additional uncertainty in small samples using a parameter known as *degrees of freedom*, or *df*:



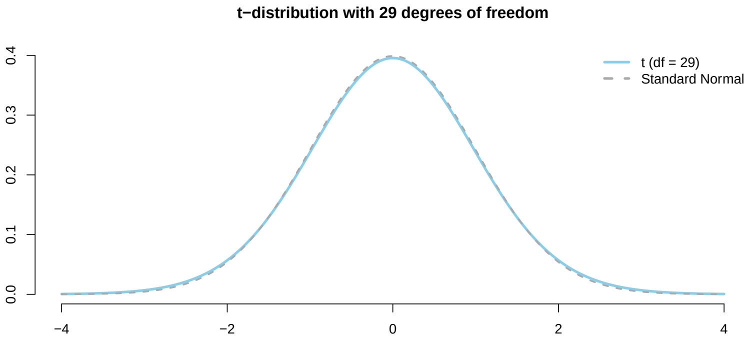
When estimating a single mean, $df = n - 1$

The t-distribution



To achieve the same level of confidence, one must go further into the tails of the *t*-distribution (as it's more spread out)

The t-distribution



As *df* increases, the *t*-distribution becomes more similar to the Normal curve (nearly indistinguishable past $n = 30$)

Practice

While waiting at an airport, a traveler notices 6 flights to similar a similar part of the country were delayed 6, 10, 13, 23, 45, 55 minutes. The mean delay in this sample was 25.33, with a sample standard deviation of $s = 20.2$. Assuming these data are a representative sample, answer the following:

1) How many degrees of freedom are involved when using the t-distribution to form a CI estimate? What is the value of c that should be used for 95% confidence?
2) What is the 95% CI estimate for the average delay of flights to the part of the country this traveler is heading?

Practice (solution)

1) Because $n = 6$, we'd use $df = n - 1 = 5$. For $df = 5$, $c = 2.571$ defines the middle 95% of the distribution.
2) Point Estimate $\pm$ *MOE*, Point estimate $= \bar{x} = 25.33$, Margin of error $= c * SE = 2.571 * \frac{20.2}{\sqrt{6}}$
   - All together, 95% CI: $25.33 \pm 2.571 * \frac{20.2}{\sqrt{6}} = (4.1, 46.5)$
   - We are 95% confident the *average* delay is somewhere between 4.1 minutes and 46.5 minutes

Note: had we erroneously used a Normal model (instead of the t-distribution), we'd get an interval that is much narrower (9.2, 41.5), but this interval wouldn't have the proper confidence level (ie: it wouldn't really be a 95% CI because it would miss too often)

When to use the t-distribution

- The t-distribution was designed for small, Normally distributed samples
  - However, it can also be reliably used on large samples, regardless of their shape

	Data are approximately Normal	Data are non-Normal or skewed
$n \geq 30$	Use t-distribution	Use t-distribution
$n < 30$	Use t-distribution	*do not* use t-distribution

Note: for small, non-Normal samples, robust methods (such as bootstrapping) should be used

Central Limit Theorem (two means)

For a *difference of two means*, CLT states:

$$\bar{x}_1 - \bar{x}_2 \sim N\left(\mu_1 - \mu_2, \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}\right)$$

- Similar to applications estimating a single mean, the t-distribution should be used when s_1 and s_2 are used as estimates of σ_1 and σ_2
  - Degrees of freedom is complicated, we'll use the smaller of $n_1 - 1$ and $n_2 - 1$ as a conservative approach

Practice

To explore whether artificial light at night contributes to weight gain (in g), researchers randomly assigned 18 young mice to live in lab environments with either complete darkness or an artificial nightlight during evening hours:

Summary Statistics

Statistics	Light	Dark	Overall
Sample Size	10	8	18
Mean	6.732	4.114	5.568
Standard Deviation	2.966	1.557	2.729
Minimum	1.71	2.27	1.71
Q_1	4.99	2.68	4.00
Median	6.19	4.11	5.16
Q_3	9.17	5.28	6.94
Maximum	11.67	6.52	11.67

1) Compare the means and medians of each group as a crude assessment of whether its reasonable to assume these data came from a Normally distributed population
2) Find a 95% CI estimate for the difference in mean weight gain experienced in each group (Light - Dark)

Practice (solution)

1) Because the means and medians are reasonably close, we do not have a sufficient reason to doubt Normality
2) First, we should use $df = 7$ because $n_2 - 1$ is smaller than $n_1 - 1$. Thus, $c = 2.365$ is necessary for 95% confidence. Next, $SE = \sqrt{2.966^2/10 + 1.557^2/8} = 1.09$, therefore the 95% CI estimate is $(6.732 - 4.114) \pm 2.365 * 1.09 = (0.04, 5.20)$. With 95% confidence we can conclude that light-exposed mice exhibit a larger weight gain, with the average difference being between $+0.04$g and $+5.20$g relative to mice without exposure.

Central Limit Theorem (summary)

The table below summarizes the standard errors suggested by CLT in a variety of common scenarios:

Estimate	Standard Error	CLT Conditions
$\hat{p}$	$\sqrt{\frac{p(1-p)}{n}}$	$np \geq 10$ and $n(1-p) \geq 10$
$\bar{x}$	$\frac{\sigma}{\sqrt{n}}$	normal population or $n \geq 30$
$\hat{p}_1 - \hat{p}_2$	$\sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}$	$n_i p_i \geq 10$ and $n_i(1-p_i) \geq 10$ for $i \in \{1,2\}$
$\bar{x}_1 - \bar{x}_2$	$\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$	normal populations or $n_1 \geq 30$ and $n_2 \geq 30$
r	$\sqrt{\frac{1-\rho^2}{n-2}}$	normal populations or $n > 30$

Factors impacting CI width (summary)

If all other factors are held constant, the table below summarizes the impact of certain changes on the width of confidence intervals:

Change	Impact on CI width
Increasing n	decreases width (narrower CI)
Increasing confidence level	increases width (wider CI)
Increasing SE	increases width (wider CI)
Increasing number of bootstrap samples (if bootstrapping)	no impact on width
Using t rather than Normal	increases width (wider CI)