Two-sample Categorical Data

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Introduction

- So far in this course all of our applications of statistical inference have focused on *one-sample* (one group) settings
  - We've used Normal models (the Z-test) for inference involving one proportion
  - We've used the t-distribution (the T-test) for inference involving one mean
- This week we will learn how to extend these ideas to *two-sample* settings, or situations involving the comparison of two proportions or two means

Surgical Site Infections

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- At the time, many experts believed these infections were due to "bad air"
  - Hospitals had policies that required their wards open their windows at midday to air out
- It was customary for surgeons to move quickly from patient to patient with out any sort of special precautions
  - In fact, many took pride the accumulated stains on their surgical gowns as a measure of experience

Louis Pasteur and Joseph Lister

- In 1862, Louis Pasteur discovered that food spoilage was caused by the growth and proliferation of harmful micro-organisms
- Pasteur identified three methods for eliminating these micro-organisms: heat, filtration, and chemical disinfectants
  - The method of heating became known as pasteurization (named for Pasteur) and is widely applied to milk, beer, and many other food products

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  - The method of heating became known as pasteurization (named for Pasteur) and is widely applied to milk, beer, and many other food products
- Joseph Lister, a Professor of Surgery at the Glasgow Royal Infirmary, became aware of Pasteur's work and theorized that it might explain the infections that frequently occurred after surgery
  - How would you recommend Lister evaluate his theory?

Lister's Experiment

- Lister proposed a new "sterile" protocol where surgeons were required to wash their hands, wear clean gloves, and disinfect their instruments with a carbolic acid solution
  - He randomly assigned 75 patients undergoing surgery to receive either his new "sterile" protocol or a control group
  - He then tracked how many patients in each group survived until their discharge from the hospital

	Died	Survived
Control	16	19
Sterile	6	34

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2) Confounding variables? No, we'd expect random assignment to have balanced the two groups
3) Random chance? . . . This is where hypothesis testing is useful

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  - In words, what should the null model be for Lister's experiment?
- The null model is that the Lister's proposed sterilization procedure makes no difference
  - That is, equal proportions of the "Sterile" and "Control" groups are expected to die prior to discharge

$$H_0 : p_1 - p_2 = 0$$

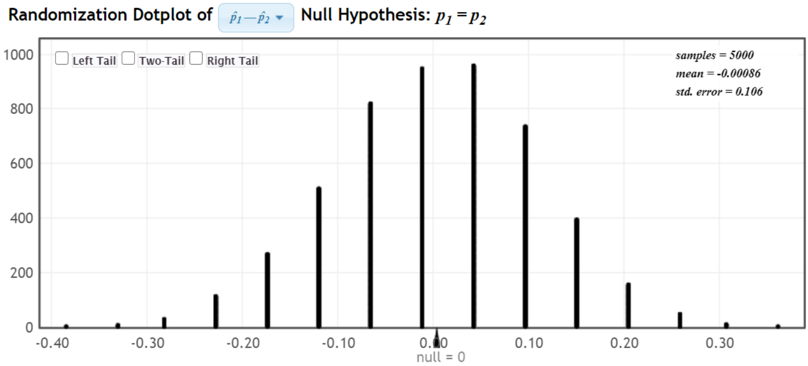
- Here, p_1 denotes the proportion of deaths among the "Control" group, and p_2 is the proportion of deaths among the "Sterile" group

Simulating the Null Distribution

- If the sterilization protocol made no difference, any deaths observed in this study data occurred at random (ie: the assigned group made no difference)
    - Thus, under the null model, we can assume the *overall death rate* (estimated by 22/75, or 29%) applies equally to both groups
- We can use StatKey to simulate possible outcomes that could occur under this null model using sets of $n_1 = 35$ and $n_2 = 40$ "weighted coin-flips", where each flip represents a 29% chance of death

Simulating the Null Distribution

If both groups had the same death rate (29%), we could expect to have observed the following differences in proportions:



The study saw a difference of $\hat{p}_1 - \hat{p}_2 = 16/35 - 6/40 = 0.31$, only 2 of 5000 simulated outcomes were this extreme!

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  - Central Limit theorem (combined with some probability theory and linear algebra) suggests the following:

$$\hat{p}_1 - \hat{p}_2 \sim N\left(p_1 - p_2, \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}\right)$$

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- One challenge in applying this theoretical result is that our null hypothesis only specifies that $p_1 = p_2$ (which can be satisfied by many different values)
  - The most common solution is to used the *pooled* (overall) proportion in place of both p_1 and p_2
  - In our example, this would be applying the overall death rate of 29% to both groups

The Two-sample Z-test (categorical data)

1) State the null hypothesis (ie: $H_0 : p_1 = p_2$ for two-sample categorical data)
2) Calculate a Z-value using the sample data and an appropriate Normal model (ie: $Z = \frac{(\hat{p}_1 - \hat{p}_2) - 0}{\sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}}$)
3) Compare the Z-value to the Standard Normal curve to find the p-value
4) Use the p-value to make a decision (remember to consider context!)

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$Z = \frac{0.31-0}{\sqrt{0.29*(1-0.29)/35+0.29*(1-0.29)/40}} = 2.94$
3) Using StatKey, a Z-value of 2.94 corresponds to a two-sided p-value of 0.0032
4) We conclude there is overwhelming statistical evidence that the new sterilization procedure leads to lower death rates

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  - This is a fundamentally different question from "is the observed difference large enough to change how we should act?"
- Confidence intervals are an important tool for answering the later question
  - The Normal model we've already presented makes for the easy construction of these interval estimates:

$$\text{Point Estimate} \pm \text{Margin of Error}$$

$$\hat{p}_1 - \hat{p}_2 \pm z^* \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$

Confidence Intervals for a Difference in Proportions

- One subtle difference when comparing the Normal model for confidence intervals with the one we used for hypothesis testing is the standard error
  - For hypothesis testing, the standard error used a *pooled proportion* in order to be consistent with the null hypothesis (which says $p_1 = p_2$)
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- Thus, for Lister's experiment we can 95% confident that between 11.1% and 50.9% more of a sterile surgery group will survive

$$\hat{p}_1 - \hat{p}_2 \pm z^* \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$

$$0.46 - 0.15 \pm 1.96\sqrt{\frac{0.46(1-0.46)}{35} + \frac{0.15(1-0.15}{40}} = (0.111, 0.509)$$

A Second Example

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- Among all smokers, 0.44% are expected to develop lung cancer in a 10-year period
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- The difference in proportions is only 0.0039 (0.39%), so is it really worthwhile to get people to quit smoking?
  - I'd argue "yes", as the cancer risk is nearly *10 times higher*!

Odds Ratios

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- In our smoking example, the odds of a smoker developing lung cancer are $\frac{0.00438}{1-0.00438} = 0.00440$
  - Similarly, the odds of a non-smoker developing lung cancer are $\frac{0.00045}{1-0.00045} = 0.00045$

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  - Similarly, the odds of a non-smoker developing lung cancer are $\frac{0.00045}{1-0.00045} = 0.00045$
- Thus, the *odds ratio* is $\frac{0.00440}{0.00045} = 9.8$
  - We say that the odds of a smoker developing lung cancer are 9.8 times those of a non-smoker developing lung cancer

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  - Instead, I only expect that you're able to properly interpret a CI estimate of an odds ratio
- For example, the 95% CI for the odds ratio in Lister's experiment is (1.4, 17.2)
  - This means the odds of dying in the control group are estimated (with 95% confidence) to be between 1.4 times and 17.2 times higher than the odds of dying in the sterile surgery group (quite an improvement!)

Conclusion

- This presentation introduced statistical methods for two-sample categorical data
  - Similar to one-sample categorical data, these methods are built upon a Normal model:

$$\hat{p}_1 - \hat{p}_2 \sim N\left(p_1 - p_2, \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}\right)$$

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- When using this model for hypothesis testing, we use a *pooled proportion* to calculate the standard error
- When using this model for confidence interval estimation, we use the *sample proportions* to calculate the standard error