Linear Regression (part 1)

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Linear Regression

Using ANOVA, we modeled a quantitative outcome variable Y using a single categorical variable:

$$y_i = \mu_k + \epsilon_i$$

Generally speaking, **linear regression** models a quantitative outcome using a *linear combination* of variables:

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots \beta_p x_{ip} + \epsilon_i$$

ANOVA as Linear Regression

To fit our ANOVA model into the linear regression framework, we need to use *dummy variables* and *reference coding*:

Y	group
8.5	C
11.6	B
9.0	C
9.1	B
8.0	A
9.7	A

Y	X1	X2
8.5	0	1
11.6	1	0
9.0	0	1
9.1	1	0
8.0	0	0
9.7	0	0

- In this example, $y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \epsilon_i$
  - For those in group A, $X_1 = 0$ and $X_2 = 0$, suggesting $\beta_0 = \mu_A$
  - For those in group B, $X_1 = 1$ and $X_2 = 0$, suggesting $\beta_1 = \mu_B - \mu_A$
  - For those in group C, $X_1 = 0$ and $X_2 = 1$, suggesting $\beta_2 = \mu_C - \mu_A$

Model Estimation

- In regression, we assume the specified model *is true* at the population level
  - The specified model is something we decide, the data are then used to *estimate* the model's *parameters*
  - We understand the model through these estimates
- The using population model: $y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \epsilon_i$ results in the estimated model:

$$\hat{y}_1 = b_0 + b_1 x_{i1} + b_2 x_{i2}$$

- The *parameter estimates* are found using least squares, they are: $b_0 = 9.77$, $b_1 = 0.25$, $b_2 = 0.03$
  - In our example, $\bar{y}_A = 9.77$, $\bar{y}_B = 10.02$, $\bar{y}_C = 9.80$; this is not a coincidence

Interpretting Model Parameter Estimates

For the drug use and tailgating example, the *estimated* ANOVA model is:

$$\hat{Y} = b_0 + b_1 X_{MDMA} + b_2 X_{NODRUG} + b_3 X_{THC}$$

The *parameter estimates* are $b_0 = 36.8$, $b_1 = -9.2$, $b_2 = 10.5$, $b_3 = 5.8$, from this we know:

1. `ALC` is the reference group
2. The mean following distance in the `ALC` group is 36.8 ft (interpretation of b_0)
3. Predicted following distances in the `MDMA` group are 9.2 ft *less* than the reference group of `ALC` users (interpretation of b_1)
4. Predicted following distances in the `THC` group are 5.8 ft *more* than the reference group of `ALC` users (interpretation of b_3)

What About Uncertainty?

- Like any estimate, the regression estimates, $b_0, b_1, \ldots, b_p$, won't *exactly* match the population parameters, $\beta_0, \beta_1, \ldots, \beta_p$
- We won't go too far into the details, but most standard software will provide confidence interval estimates for the population parameters using the t-distribution

What About Inference?

- We can also use the regression estimates, $b_0, b_1, \ldots, b_p$, to perform hypothesis testing
- How might you describe the hypothesis $H_0 : \beta_1 = 0$? (Hint: think about reference coding and what β_1 is in terms of group means)
- $\beta_1 = 0$ implies that the mean of group 1 is no different from the mean of the reference group, essentially a two-sample t-test!

Example #1

With your group, load the College Data into Minitab and fit a linear regression using `REGION` to predict `AVGFACSAL` (Hint: you'll need to change the variable `REGION` to text by right clicking the column and selecting "format column"). Answer the following questions:

1. By default, which region is chosen as the reference category?
2. Which region has the highest average faculty salaries?
3. Are the population mean salaries in Region 3 (Great Lakes) significantly different from those in Region 4 (Midwest) (Hint: change the reference category using "Coding" in the regression menu)

Example #1 - Solution

1. Region 1 (Northeast) is the default reference category
2. Region 1 (Northeast) has the highest average salaries
3. No, the estimated difference of -2572 has a p-value of 0.66, there is not sufficient evidence for us to believe that colleges in these regions have different average faculty salaries

Simple Linear Regression

- The inference methods we used apply to quantitative predictor variables too
  - However, our interpretations must change
- *Simple linear regression* uses a single quantitative predictor and the population level model:

$$y_i = \beta_0 + \beta_1 x_{i1} + \epsilon_i$$

- The Tips Data documents the tips received by a server in a suburban national chain restaurant:

Example #2

With your group, load the Tips Data into Minitab and fit a regression model that uses `TotBill` to predict `Tip`, then answer the following:

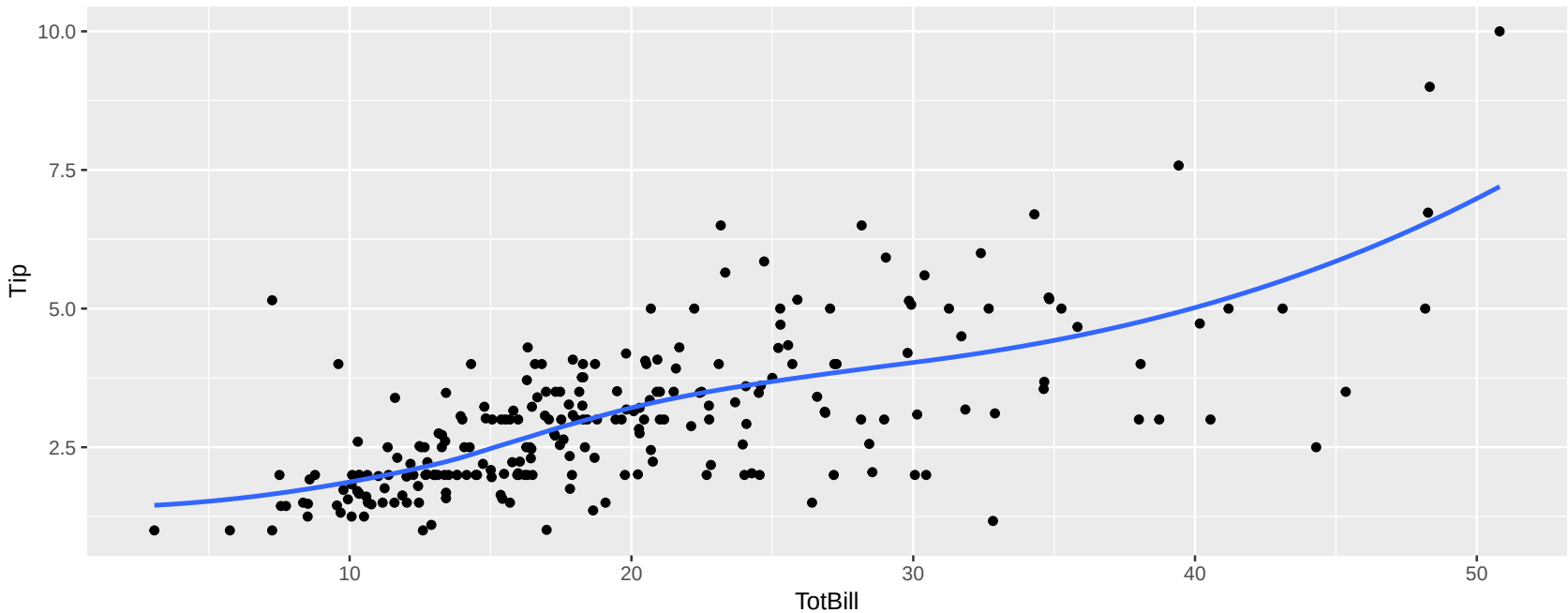
1. Interpret the estimate and 95% confidence interval for the slope coefficient of `TotBill`
2. Is it plausible that tips are unrelated to the total bill amount?

Example #2 - Solution

1. The estimate is 0.105, indicating that each \$1 increase in the total bill on average leads to a 10.5 cent increase in the tip. The 95% confidence interval suggests we are confident the actual population level effect for this server is between 9 cents and 12 cents
2. No, the 95% confidence interval doesn't contain zero (similarly, the hypothesis test has a very small p-value)

Non-linear Effects

- Sometimes, the relationship between a predictor and the outcome won't be linear
- The plot below adds a *loess* smoothing line (essentially the average calculated within a moving window) applied the Tips Data



Non-linear Effects

- The loess line shows some *curvature*, but it is drastic enough to suggest that a population model with a **quadratic effect** should be used?
  - We can compare these two models using ANOVA!
- Null model: $y_i = \beta_0 + \beta_1 x_{i1} + \epsilon_i$
- Alternative model: $y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i1}^2 + \epsilon_i$
  - All we need is the *sum of squares* for each of these models to perform the test

Non-linear Effects - Example

For the Tips Data, use Minitab fit the following models:

1. A model which predicts `Tip` using a linear effect for `TotBill`
2. A model which predicts `Tip` using a quadratic effect for `TotBill` (INSERT MINITAB DIRECTIONS)

Record the sum of squares for each model and test for superiority of the quadratic model using ANOVA

Non-linear Effects - Solution

SOLUTION COMING SOON

R-Squared

- When we first discussed regression in Chapter 2, we learned about the coefficient of variation or R^2
- We can express R^2 using sums of squares:

$$R^2 = SSE/SST$$

- Note: In calculating R^2, SST refers to the null model that predicts each observation as the mean $\bar{y}$. This is in contrast to some situations (like the last example) where we consider another model to be the "null model"

Loose-ends

- Linear regression models assume ϵ's (population level deviations from the model) are normally distributed with a mean of zero
  - We can check this assumption using the residuals
- ANOVA can only be used to compare *nested models*
  - $y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i1}^2 + \epsilon_i$ and $y_i = \beta_0 + \beta_1 x_{i1} + \epsilon_i$ are nested because forcing $\beta_2 = 0$ makes them identical

Conclusion

These notes cover Ch 9 of the textbook. Right now, you should...

1. Know the relationship between one-way ANOVA and linear regression
2. Understand how to perform on statistical inference on the parameters of linear regression model
3. Know how to compare two nested models using ANOVA

I encourage you to read Ch 9 of the book and its examples.