Statistical Inference for One-sample quantitative data

Ryan Miller

Outline

- Video #1
  - Confidence intervals and the t-distribution
- Video #2
  - The one-sample t-test
- Video #3
  - Model assumptions for using the t-distributions
- Video #4
  - Simulation-based alternatives to statistical inference on quantitative data

Introduction

So far, we've covered two major types of *statistical inference*:

1) **Confidence interval estimation** - using sample data to derive a range of plausible estimates for some characteristic of a population
2) **Hypothesis testing** - using sample data to evaluate a conjecture about a population

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1) **Confidence interval estimation** - using sample data to derive a range of plausible estimates for some characteristic of a population
2) **Hypothesis testing** - using sample data to evaluate a conjecture about a population

- We've now seen numerous examples of these approaches for *one-sample categorical data*, or scenarios that can be summarized by a *single proportion*
- This week, we'll see how the approach differ for *one-sample quantitative data*, or scenarios that can be summarized by a *single mean*

Interval Estimation

- Recall that we've used Normal probability models as the basis for constructing $P\%$ confidence intervals:

$$\text{Point Estimate} \pm z^* SE$$

- Until now, the only population characteristic we've considered estimating is the *population proportion*, or p:

$$\hat{p} \pm z^* \sqrt{\frac{\hat{p} * (1 - \hat{p})}{n}}$$

- Recall that the SE formula was based upon the results of the *Central Limit theorem*

CLT for Means

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- For means, $SE(\bar{x}) = \sqrt{\frac{\sigma^2}{n}} = \frac{\sigma}{\sqrt{n}}$
  - This formula is more challenging because we don't know σ (the standard deviation of cases in the population)

Interval Estimation for Quantitative Data

- A simple solution is to estimate σ (the standard deviation of cases in the population) using the sample data
  - The standard deviation of the cases in the sample is denoted by s, but is it really valid to use s in place of σ when estimating the population mean?

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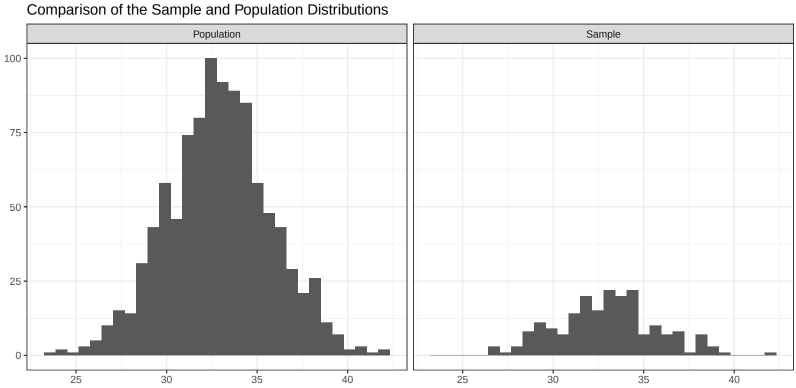

William Gosset

- William Gosset was an English chemist working for Guinness Brewing in the 1890s
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- In 1906, Gosset took a leave of absence to go work with Karl Pearson (creator of the correlation coefficient) on the problem

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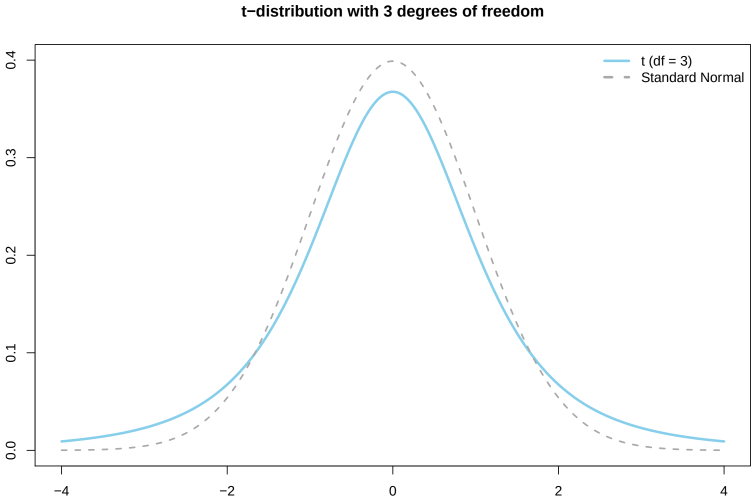
- Simply "plugging in" s into the CLT result introduces a new source of variability (due to the imperfect estimation of σ)
- Usually the person who discovers an important results gets to name it
  - However, Gosset had to publish his work under the name "Student" because Guinness didn't want competitors knowing it employed statisticians!
  - Gosset's result, called Student's t-distribution, is among the most widely-used statistical results of all time

The t-distribution



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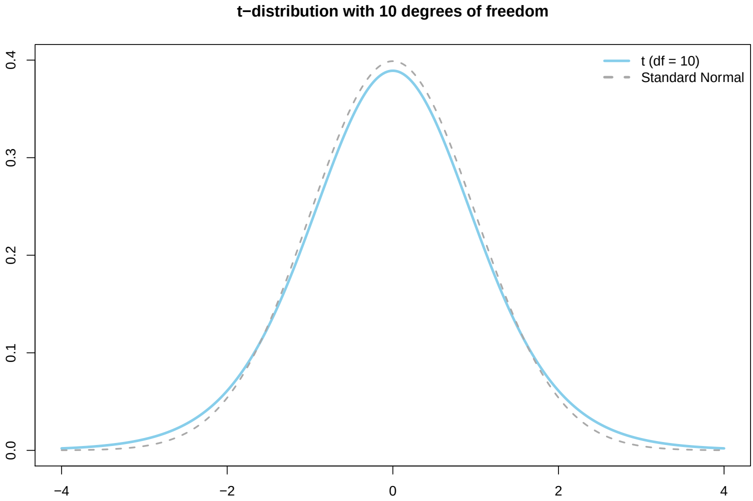

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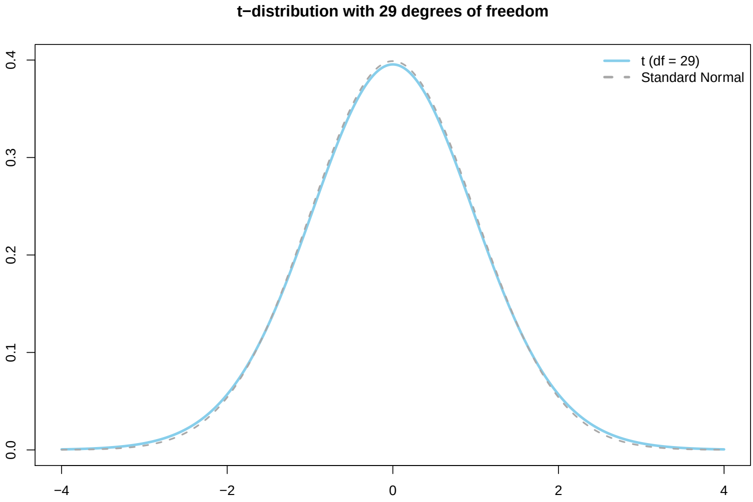

How to use the t-distribution

When estimating a single mean, we use the t-distribution to construct a $P\%$ confidence interval via:

$$\bar{x} \pm t^*_{n-1} \frac{s}{\sqrt{n}}$$

- t^*_{n-1} is a percentile from the t-distribution with $n-1$ degrees of freedom defining the middle $P\%$ of the distribution
- $\frac{s}{\sqrt{n}}$ is the *standard error* (SE) of the sample mean, $\bar{x}$

Example

- While waiting at an airport, a passenger notices 6 flights to similar a similar part of the country were delayed 6, 10, 13, 23, 45, 55 minutes
  - The mean delay of this sample was 25.33
  - The standard deviation of delays in the sample was 20.2
- Assuming these data are representative, use them to come up with a 95% confidence interval estimate for the average flight delay from this airport to your destination

Example (solution)

- 95% CI for a population mean: Point Estimate $\pm$ MOE
  - Point estimate $= \bar{x} = 25.33$
  - Margin of error $= t^*_{df=5} * SE = 2.571 * \frac{20.2}{\sqrt{6}}$

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  - We are 95% confident the average delay is somewhere between 4.1 minutes and 46.5 minutes
- Note: if we'd erroneously used a Normal model, we'd get an interval that is much narrower (9.2, 41.5), but this interval wouldn't have the confidence level we are advertising (ie: it wouldn't really be a 95% CI because it would miss too often)

Closing Remarks

- We've now introduced the t-distribution, a necessary modification to the Normal model in scenarios involving a quantitative data
  - In these situations, the standard error required estimating an extra parameter, thus the t-distribution modifies the Normal model to account for this added uncertainty
- In this class, you should generally expect to use the t-distribution for means and the Normal distribution for proportions (aside a few exceptions where assumptions aren't met)
  - We will cover exceptions in a later video

Introduction

Previously, we introduced the Z-test:

1) State the null hypothesis (a conjecture about the population that would be useful to disprove)
2) Use the null hypothesis (and corresponding null model) to find a Z-value describing the *sample estimate*
3) Locate the Z-value on the Standard Normal curve to find the p-value
4) Use the p-value to make a decision regarding the null hypothesis

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This approach needs to be adjusted for scenarios involving means (since there is added uncertainty induced by estimating an extra parameter), the resulting procedure is known as the T-test

The T-test

- Procedurally, only difference between the T-test and Z-test is the probability distribution used to calculate the p-value
  - When analyzing one-sample *categorical* data, the Z-test compares $z = \frac{\hat{p}-p}{SE}$ to the Standard Normal distribution
  - When analyzing one-sample *quantitative* data, the T-test compares $t = \frac{\bar{x}-\mu}{SE}$ to a t-distribution with $df = n - 1$

Example

- According to thr Australian government, the mean birthweight of all babies born in Australia is 7.86 lbs
- A hospital in Missouri reports the average birthweight of 112 born there last year was 7.68, with a sample standard deviation of 1.31
- Assuming this Missouri hospital is a representative sample of all babies born in the US, do these data support the hypothesis that birthweight of US babies is different from that of Australian babies?

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3) We next must locate $t = -1.45$ on a t-distribution with $df = n - 1 = 111$ using StatKey
4) The two-sided p-value is 0.15, so we conclude insufficient evidence to believe the mean birthweight of babies in the US differs from that of Australia

Comparison vs. The Z-test

- As previously mentioned, the only difference between the T-test and Z-test is null distribution
  - We use the T-test to account for the small amount of additional uncertainty introduced when using the sample standard deviation, s, to estimate σ, the standard deviation of the population

Comparison vs. The Z-test

- As previously mentioned, the only difference between the T-test and Z-test is null distribution
  - We use the T-test to account for the small amount of additional uncertainty introduced when using the sample standard deviation, s, to estimate σ, the standard deviation of the population
- Thus, we know the p-value of a T-test will always be higher than the corresponding Z-test
  - Here, the p-value is 0.15 comparing our T-value of -1.45 to the $t_{df=111}$ distribution
  - If we compared -1.45 to the Standard Normal curve, we'd get a p-value of 0.148

Closing Remarks

- The T-test is a modified version of the Z-test that makes the Normal results of CLT suitable for statistical inference on quantitative data
  - You should expect to use the Z-test for hypothesis testing on proportions (categorical data)
  - You should expect to use the T-test for hypothesis testing on means (quantitative data)

Conditions for the Normal model (one proportion)

When performing statistical inference on a *proportion*, we've used the following Normal model:

$$\hat{p} \sim N(p, \sqrt{\frac{p(1-p)}{n}})$$

- This model works well when $n * p \geq 10$ and $n * (1 - p) \geq 10$

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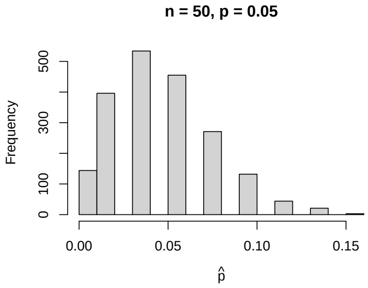
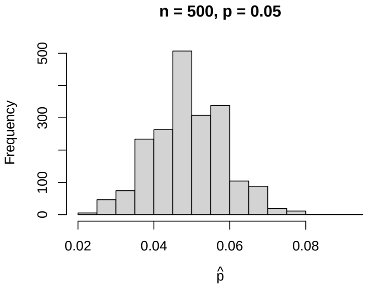
- In hypothesis testing, we use this model to determine what might have been observed in our sample if H_0 were true
  - For this reason, we use value specified in H_0 in place of the unknown population parameter, p
- In confidence interval estimation, we use this model to determine the variability of possible sample proportions
  - For this reason, we used our best estimate of p, which is the sample proportion $\hat{p}$

Examples of Violations (proportion)

The conditions $n * p \geq 10$ and $n * (1 - p) \geq 10$ can be violated in two ways:

1) p is too close to a boundary value (a proportion of 0 or 1) relative to the sample size

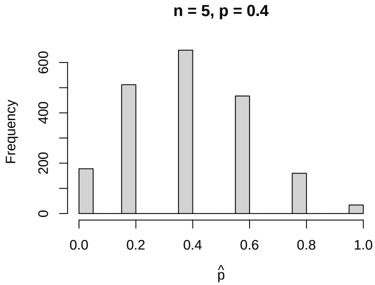


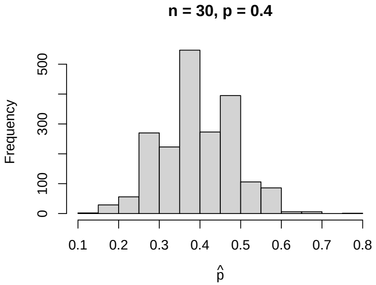


Examples of Violations (proportion, part 2)

The conditions $n * p \geq 10$ and $n * (1 - p) \geq 10$ can be violated in two ways:

2) p isn't near a boundary, but n is too small





Conditions for the t-distribution (one mean)

When performing statistical inference on a *mean*, we've used the t-distribution:

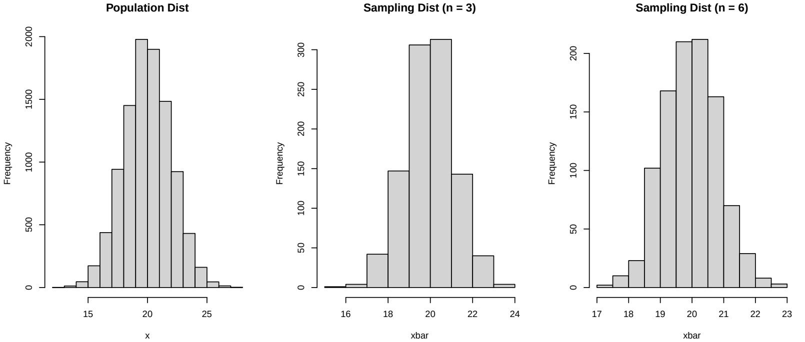
$$\frac{\bar{x} - \mu}{s/\sqrt{n}} \sim t_{n-1}$$

This model works well in two situations:

1) the population we sampled from is Normally distributed (regardless of sample size)
2) the sample size is large ($n \geq 30$)

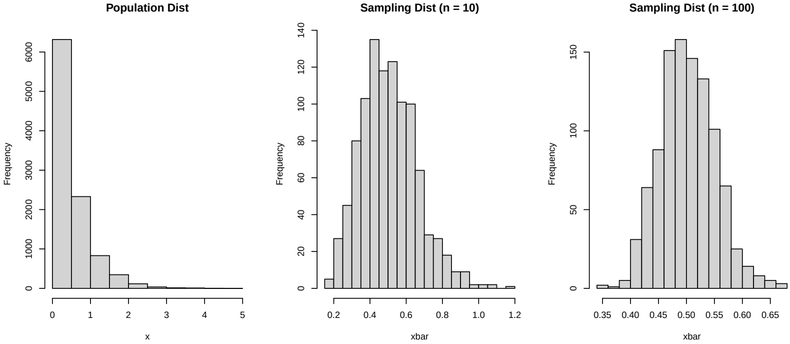
Examples of Violations (mean)

An illustration of the first situation (Normal population, any sample size):



Examples of Violations (mean, part 2)

An illustration of the second situation (Skewed population, large samples):



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- Each of the examples used in the lecture are *hypothetical* in the sense that we'd never be able to see thousands of replications of any real-world study
  - That said, they illustrate the importance of checking the conditions that are recommended for the models we've using

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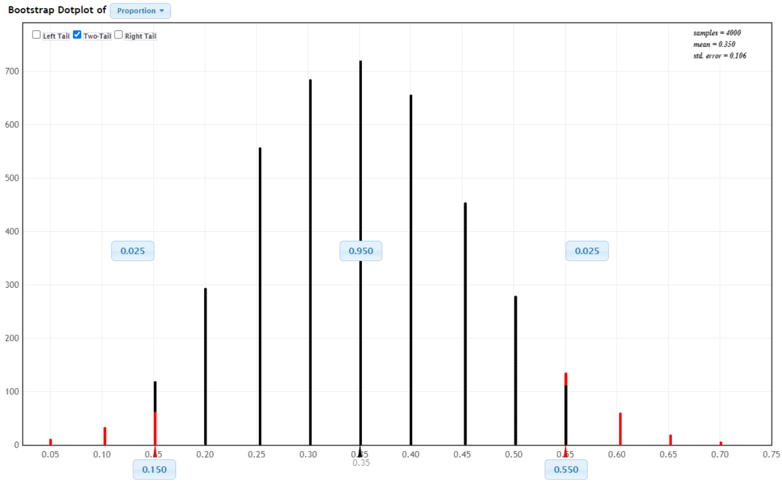
- Each of the examples used in the lecture are *hypothetical* in the sense that we'd never be able to see thousands of replications of any real-world study
  - That said, they illustrate the importance of checking the conditions that are recommended for the models we've using
- But can we still do inference when these conditions aren't met?
  - The answer is yes, but we'll need to estimate the sampling/null distribution in another way (simulation)

Simulation for One Proportion (CI)

- Consider a large calculus class at a University
  - In a survey of 20 students from this class, only 7 report getting an A or B on a midterm exam
- Can these data be used to estimate the proportion of the *entire* class who received an A or B?
  - Notice $n * \hat{p} = 20 * \frac{7}{20} = 7$, which does not meet the conditions for using a Normal model

Simulation for One Proportion (CI) - solution

Using simulation via StatKey, we estimate with 95% confidence that between 15% and 55% of the class got an A or B

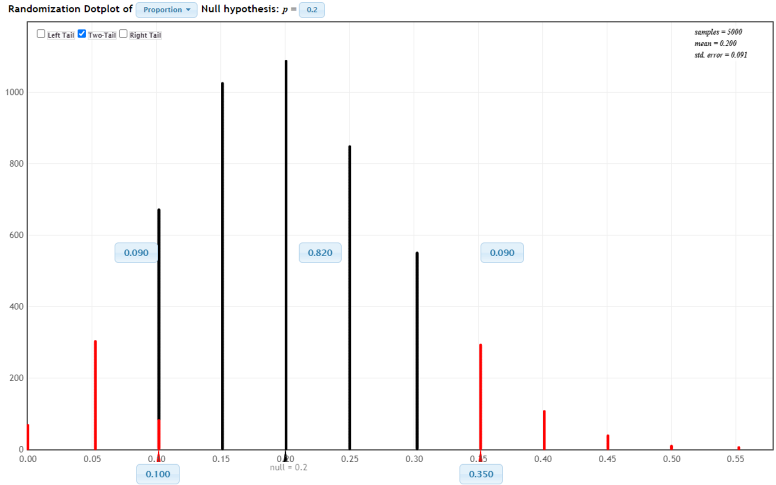


Simulation for One Proportion (Testing)

- Does this sample (where only 7 of 20 reported getting an A or B) provide convincing evidence that more than 20% of the class got an A or B?
  - $H_0 : p = 0.2$
  - Notice $n * p = 20 * 0.2 = 4$, which does not meet the conditions for using a Normal model

Simulation for One Proportion (Testing) - solution

Using simulation via StatKey, the two-sided p-value of this test is approximately 0.18

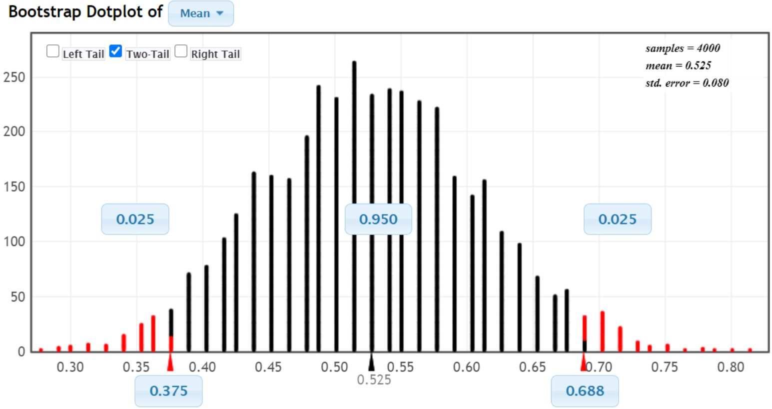


Simulation for One Mean (CI)

- The EPA recommends homeowners take action when radon levels above 0.4 pCi/L are consistently present
  - Suppose the basement of a home is tested on 8 randomly selected dates, and resulting in the following measurements $\{2, .7, .3, .9, .5, .3, .7, .6\}$
- Can these data be used to estimate the true radon levels of this home?
  - Notice the sample size is small and we aren't sure if the population being sampled is Normally distributed

Simulation for One Mean (CI)

Using simulation via StatKey, the 95% *bootstrapped* confidence interval is (0.375, 0.688)



Simulation for One Mean (Testing)

- The EPA recommends homeowners *requires* action if random levels are above 4 pCi/L
  - Suppose the basement of a home is tested on 8 randomly selected dates, and resulting in the following measurements $\{2, .7, .3, .9, .5, .3, .7, .6\}$
  - Do these 8 measurements provide sufficient evidence that the EPA *does not* need to intervene? (ie: evidence that $\mu < 4$)

Simulation for One Mean (Testing)

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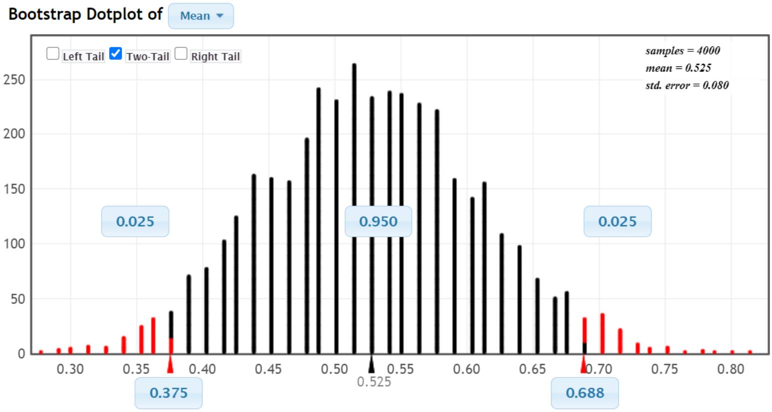
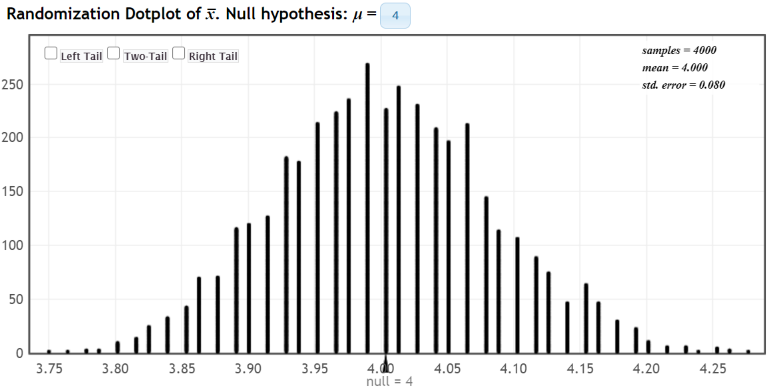


- Are these samples *convincing evidence* that the basement's radon levels are below 4 pCi/L?
  - Again, recognize the sample size is small and that we aren't

Simulation for One Mean (Testing)

Using simulation via StatKey, a *randomization test* provides a p-value of essentially zero (recall $\bar{x} = 0.525$)



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- This lecture reviewed the conditions necessary for responsibly using probability models inspired by the Central Limit theorem for statistical inference
- It also introduced simulation-based alternatives that can be used when these conditions are not met
  - In this class, I am less concerned with you being able to execute these simulation-based approaches, and more concerned with your ability to identify situations when they are warranted (ie: violated conditions)
- Recognize that *p*-values and confidence intervals obtained via simulation are interpreted identically to those obtained using more traditional methods
  - That is, a confidence interval always describes a range of plausible values for a population parameter
  - A *p*-value always measures how compatible the sample data are with a null hypothesis