

# Central Limit Theorem and Confidence Intervals

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# Outline

1. The Normal distribution
2. Central Limit Theorem
3. Confidence intervals using the Central Limit theorem

We've previously used *bootstrapping* to estimate the *standard error* of a *point estimate*, which allowed us to form *confidence intervals*:

$$\text{Point Estimate} \pm c * SE$$

- ▶ For bell-shaped sampling distributions,  $c = 2$  produces 95% confidence intervals (this is the 2-SE method)
  - ▶ The 2-SE method for bell-shaped distributions is justified by the 99-95-68 percent rule
  - ▶ Thus, we could use a *different multiplier* to achieve a *different confidence level*

# The Normal distribution

The **Normal curve**, or Normal probability function, is a mathematical function that yields a bell-shaped distribution:

$$f(X) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(X-\mu)^2}{2\sigma^2}}$$

- ▶  $\mu$  is a constant that defines the *center* of the bell-curve
- ▶  $\sigma$  is a constant that defines the *standard deviation* of the bell-curve (how peaked or flat it is)

# The Normal distribution

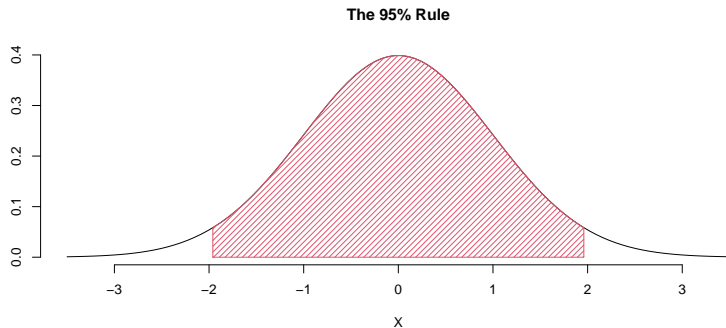
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- ▶  $\mu$  is a constant that defines the *center* of the bell-curve
- ▶  $\sigma$  is a constant that defines the *standard deviation* of the bell-curve (how peaked or flat it is)
- ▶ There infinitely many different Normal curves, one for each combination of  $\mu$  and  $\sigma$ 
  - ▶ We'll reference them using the shorthand:  $N(\mu, \sigma)$

# The Normal distribution

When data follow a Normal distribution, the *area under the curve* describes the likelihood you see a value within a particular range:



The Normal probability function doesn't have a closed-form integral, so we must rely upon software to find these areas

The Theoretical Distributions section of StatKey allows us to work with various Normal curves:

- 1) Consider a *standard Normal distribution*, or  $N(0,1)$ , what values define the middle 90% of this distribution?
- 2) Consider a  $N(10,5)$  distribution, what proportion of this distribution is larger than 16?

## Practice (solution)

- 1) The values of  $-1.645$  and  $+1.645$  define the middle 90% of the curve. This suggests we could use 1.645 as a multiplier on the SE to form a 90% CI estimate (if the sampling distribution is approximately Normal).
- 2) The area to the right of 16 on the  $N(10,5)$  curve is 0.115. This suggests there's a 11.5% chance of observing a value 16 or larger if the data follows this distribution.



# Confidence intervals (using the Normal distribution)

If the distribution of a sample estimate is approximately Normal, we can create a  $P\%$  confidence interval estimate for the population parameter via:

$$\text{Point Estimate} \pm c * SE$$

where  $c$  is a value taken from the  $N(0,1)$  distribution that defines the middle  $P\%$  of the distribution.

- ▶ So far, we've used *bootstrapping* to find the  $SE$  (a necessary component of this formula)
  - ▶ We've also used *bootstrapping* to determine Normality (of the sampling distribution)

# Central Limit Theorem

The **Central Limit Theorem** (CLT) is a theoretical result that establishes a Normal distribution (with known  $SE$ !) for a variety of different sample estimates (provided a sufficient sample size):

$$\text{Estimate} \sim N(\text{Population Parameter}, SE)$$

- ▶ The sample size needed for CLT to hold depends on the parameter we're estimating
  - ▶ For example,  $n = 30$  is considered sufficient when estimating  $\mu$  (a population's mean)
- ▶ CLT also provides a mathematical formula for the  $SE$ !
  - ▶ This formula will depend upon the parameter we're estimating

# Central Limit Theorem (one proportion)

When estimating a *single proportion*, CLT suggests:

$$\hat{p} \sim N\left(p, \sqrt{\frac{p(1-p)}{n}}\right)$$

- ▶ This implies that  $SE = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ 
  - ▶ We can then choose  $c$  from the  $N(0,1)$  curve and calculate a  $P\%$  CI estimate of  $p$
- ▶ The sample size condition for this result is:  $n\hat{p} \geq 10$  and  $n(1 - \hat{p}) \geq 10$

A 2021 study looked at the true-positive rate (sensitivity) of the Abbott Diagnostics rapid test for Covid-19. Of the 84 cases with symptomatic Covid-19 that took the test, 38 had a “positive” result. Our goal is to estimate  $p$ , the overall sensitivity of this test in the target population (ie: all symptomatic Covid cases).

- 1) Verify that the conditions are met to use the CLT Normal approximation to construct a confidence interval estimate
- 2) Find the values of  $\hat{p}$ , the  $SE$ , and  $c$  necessary to construct a 99% CI estimate of  $p$
- 3) Calculate and interpret the 99% CI

## Practice (solution)

- 1) First,  $\hat{p} = 38/84 = 0.452$ . Then,  $n\hat{p} = 84 * 0.452 = 38$  and  $n(1 - \hat{p}) = 84 * (1 - 0.452) = 46$ . Because both are larger than 10, the CLT Normal approximation is reasonable.
- 2)  $\hat{p} = 38/84 = 0.452$ ,  $SE = \sqrt{\frac{0.452(1-0.452)}{84}} = 0.054$ , and  $c = 2.576$  (this defines the middle 99% of a  $N(0,1)$  curve)
- 3) The 99% CI is  $0.452 \pm 2.576 * 0.054 = (0.313, 0.591)$ . Our sample suggests, with 99% confidence, that the true sensitivity of the Abbott rapid test is somewhere between 31.3% and 59.1%

# Central Limit Theorem (two proportions)

When estimating a *difference of two proportions*, CLT suggests:

$$\hat{p}_1 - \hat{p}_2 \sim N\left(p_1 - p_2, \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}\right)$$

- ▶ Using the sample proportions:  $\hat{p}_1$  and  $\hat{p}_2$ , as well as their denominators:  $n_1$  and  $n_2$  this result can be used to find the *SE* necessary to construct a confidence interval estimate of  $p_1 - p_2$
- ▶ The sample size condition to use this result is  $n_1\hat{p}_1 \geq 10$ ,  $n_1(1 - \hat{p}_1) \geq 10$ ,  $n_2\hat{p}_2 \geq 10$ , and  $n_2(1 - \hat{p}_2) \geq 10$

The previously mentioned study also examined a test produced by Siemens. Of the 72 cases with symptomatic Covid-19 that took the Siemens test, 39 had a “positive” result. Recall that 38 of 84 symptomatic cases tested positive on the Abbott test. Suppose our goal is estimate  $p_1 - p_2$ , the difference in sensitivity of these two tests (at the population level)

- 1) Let  $\hat{p}_1 = 38/84 = 0.45$  be the sample proportion for the Abbott test, and  $\hat{p}_2 = 39/72 = 0.54$  be the sample proportion for the Siemens test. Find the SE for the difference in proportions,  $\hat{p}_1 - \hat{p}_2$ .
- 2) Using the CLT Normal approximation, find and interpret a 95% CI estimate for  $p_1 - p_2$

## Practice (solution)

$$1) SE = \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}} = \sqrt{\frac{0.45(1-0.45)}{84} + \frac{0.54(1-0.54)}{72}} = 0.08$$

- 2) Since  $c = 1.96$  (for 95% confidence),  $\hat{p}_1 - \hat{p}_2 = 0.54 - 0.45 = -0.09$ , and  $SE = 0.08$ , we calculate:  
 $-0.09 \pm 1.96 * 0.08 = (-0.247, 0.067)$ .

- ▶ This interval represent a plausible range of differences in the sensitivity of these tests at the population level (estimated with 95% confidence). Since zero is included in this interval, it's plausible that the tests are no different.



# Central Limit Theorem (one mean)

When estimating a *single mean*, CLT suggests:

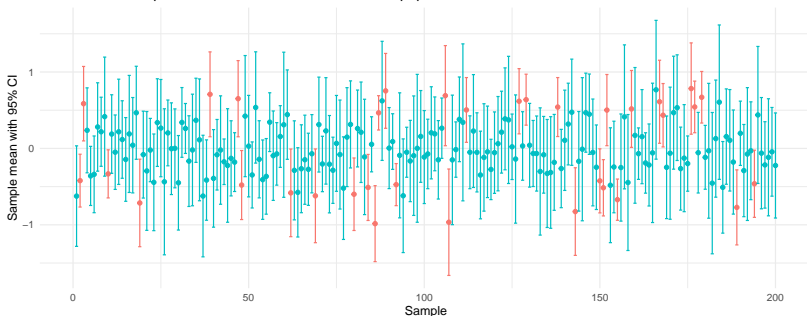
$$\bar{x} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$$

- ▶  $\sigma$  is the standard deviation of the population (something that's almost always unknown)
  - ▶ This result is not directly applicable in most real-world scenarios (see next few slides)

# William Gosset and the t-distribution

- ▶ The previous results involves a *second unknown parameter*,  $\sigma$  (the population's standard deviation)
  - ▶ It seems natural to simply replace  $\sigma$  with an estimate from the sample,  $s$ , but this is what happens:

200 different samples of  $n = 8$  from a Standard Normal population



# William Gosset and the t-distribution

- ▶ Clearly this procedure for constructing 95% CIs is *invalid*, too many of these intervals did not contain  $\mu$
- ▶ William Gosset, an employee at Guinness Brewing, became aware of this issue in the 1890s
  - ▶ His work evaluating the yields of different barley strains often involved statistical analyses on small, Normally distributed samples

# William Gosset and the t-distribution

- ▶ Clearly this procedure for constructing 95% CIs is *invalid*, too many of these intervals did not contain  $\mu$
- ▶ William Gosset, an employee at Guinness Brewing, became aware of this issue in the 1890s
  - ▶ His work evaluating the yields of different barley strains often involved statistical analyses on small, Normally distributed samples
- ▶ In 1906, Gosset took a leave of absence from Guinness to study under Karl Pearson (developer of the correlation coefficient)
  - ▶ Gosset discovered the issue was due to using  $s$  (sample standard deviation) interchangeably with  $\sigma$  (population standard deviation)

# William Gosset and the t-distribution

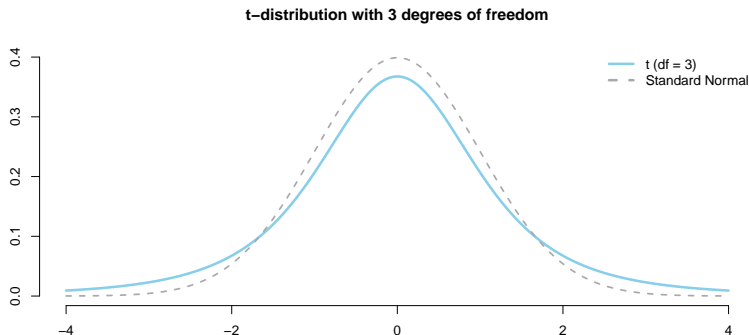
- ▶ Treating  $s$  as if it were a perfect estimate of  $\sigma$  results in a systematic underestimation of the total amount of variability involved in estimating  $\mu$ 
  - ▶ To account for the additional variability introduced by estimating  $\sigma$  using  $s$ , Gosset proposed a modified distribution that's slightly more spread out than the Standard Normal curve

# William Gosset and the t-distribution

- ▶ Treating  $s$  as if it were a perfect estimate of  $\sigma$  results in a systematic underestimation of the total amount of variability involved in estimating  $\mu$ 
  - ▶ To account for the additional variability introduced by estimating  $\sigma$  using  $s$ , Gosset proposed a modified distribution that's slightly more spread out than the Standard Normal curve
- ▶ Typically the inventor of a new method gets to name it after themselves
  - ▶ However, Gosset was forced to publish his new distribution under the pseudonym “student” because Guinness didn't want it's competitors knowing they employed statisticians!
  - ▶ Student's  $t$ -distribution is now among the most widely used statistical results of all time

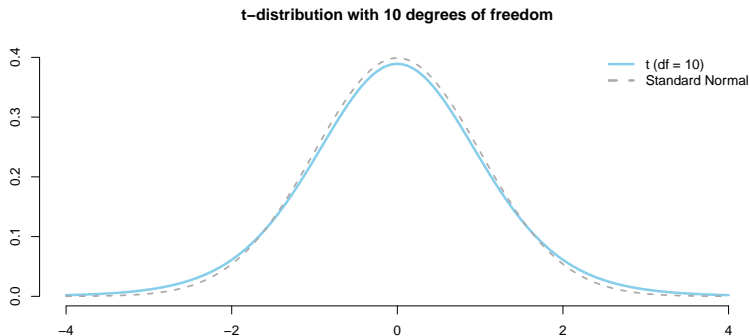
# The $t$ -distribution

The  $t$ -distribution accounts the additional uncertainty in small samples using a parameter known as *degrees of freedom*, or  $df$ :



When estimating a single mean,  $df = n - 1$

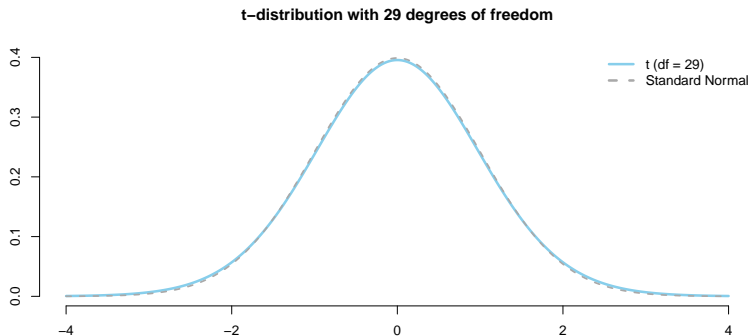
# The $t$ -distribution



To achieve the same level of confidence, one must go further into the tails of the  $t$ -distribution (as it's more spread out)



# The $t$ -distribution



As  $df$  increases, the  $t$ -distribution becomes more similar to the Normal curve (nearly indistinguishable past  $n = 30$ )

While waiting at an airport, a traveler notices 6 flights to similar a similar part of the country were delayed 6, 10, 13, 23, 45, 55 minutes. The mean delay in this sample was 25.33, with a sample standard deviation of  $s = 20.2$ . Assuming these data are a representative sample, answer the following:

- 1) How many degrees of freedom are involved when using the  $t$ -distribution to form a CI estimate? What is the value of  $c$  that should be used for 95% confidence?
- 2) What is the 95% CI estimate for the average delay of flights to the part of the country this traveler is heading?

## Practice (solution)

- 1) Because  $n = 6$ , we'd use  $df = n - 1 = 5$ . For  $df = 5$ ,  $c = 2.571$  defines the middle 95% of the distribution.
- 2) Point Estimate  $\pm$  *MOE*, Point estimate  $= \bar{x} = 25.33$ , Margin of error  $= c * SE = 2.571 * \frac{20.2}{\sqrt{6}}$ 
  - ▶ All together, 95% CI:  $25.33 \pm 2.571 * \frac{20.2}{\sqrt{6}} = (4.1, 46.5)$
  - ▶ We are 95% confident the *average* delay is somewhere between 4.1 minutes and 46.5 minutes

Note: had we erroneously used a Normal model (instead of the  $t$ -distribution), we'd get an interval that is much narrower (9.2, 41.5), but this interval wouldn't have the proper confidence level (ie: it wouldn't really be a 95% CI because it would miss too often )

# When to use the $t$ -distribution

- ▶ The  $t$ -distribution was designed for small, Normally distributed samples
  - ▶ However, it can also be reliably used on large samples, regardless of their shape

|             | Data are approximately Normal | Data are non-Normal or skewed       |
|-------------|-------------------------------|-------------------------------------|
| $n \geq 30$ | Use $t$ -distribution         | Use $t$ -distribution               |
| $n < 30$    | Use $t$ -distribution         | <i>do not</i> use $t$ -distribution |

Note: for small, non-Normal samples, robust methods (such as bootstrapping) should be used

# Central Limit Theorem (two means)

For a *difference of two means*, CLT states:

$$\bar{x}_1 - \bar{x}_2 \sim N\left(\mu_1 - \mu_2, \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}\right)$$

- ▶ Similar to applications estimating a single mean, the  $t$ -distribution should be used when  $s_1$  and  $s_2$  are used as estimates of  $\sigma_1$  and  $\sigma_2$ 
  - ▶ Degrees of freedom is complicated, we'll use the smaller of  $n_1 - 1$  and  $n_2 - 1$  as a conservative approach

# Practice

To explore whether artificial light at night contributes to weight gain (in  $g$ ), researchers randomly assigned 18 young mice to live in lab environments with either complete darkness or an artificial nightlight during evening hours:

Summary Statistics

| Statistics         | Light | Dark  | Overall |
|--------------------|-------|-------|---------|
| Sample Size        | 10    | 8     | 18      |
| Mean               | 6.732 | 4.114 | 5.568   |
| Standard Deviation | 2.966 | 1.557 | 2.729   |
| Minimum            | 1.71  | 2.27  | 1.71    |
| $Q_1$              | 4.99  | 2.68  | 4.00    |
| Median             | 6.19  | 4.11  | 5.16    |
| $Q_3$              | 9.17  | 5.28  | 6.94    |
| Maximum            | 11.67 | 6.52  | 11.67   |

- 1) Compare the means and medians of each group as a crude assessment of whether its reasonable to assume these data came from a Normally distributed population
- 2) Find a 95% CI estimate for the difference in mean weight gain experienced in each group (Light - Dark)

## Practice (solution)

- 1) Because the means and medians are reasonably close, we do not have a sufficient reason to doubt Normality
- 2) First, we should use  $df = 7$  because  $n_2 - 1$  is smaller than  $n_1 - 1$ . Thus,  $c = 2.365$  is necessary for 95% confidence. Next,  $SE = \sqrt{2.966^2/10 + 1.557^2/8} = 1.09$ , therefore the 95% CI estimate is  $(6.732 - 4.114) \pm 2.365 * 1.09 = (0.04, 5.20)$ . With 95% confidence we can conclude that light-exposed mice exhibit a larger weight gain, with the average difference being between +0.04g and +5.20g relative to mice without exposure.

# Central Limit Theorem (summary)

The table below summarizes the standard errors suggested by CLT in a variety of common scenarios:

| Estimate                | Standard Error   | CLT Conditions  |
|-------------------------|--|---|
| $\hat{p}$               | $\sqrt{\frac{p(1-p)}{n}}$                                | $np \geq 10$ and $n(1-p) \geq 10$                               |
| $\bar{x}$               | $\frac{\sigma}{\sqrt{n}}$                                | normal population or $n \geq 30$                                |
| $\hat{p}_1 - \hat{p}_2$ | $\sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}$ | $n_i p_i \geq 10$ and $n_i(1-p_i) \geq 10$ for $i \in \{1, 2\}$ |
| $\bar{x}_1 - \bar{x}_2$ | $\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$ | normal populations or $n_1 \geq 30$ and $n_2 \geq 30$           |
| $r$                     | $\sqrt{\frac{1-\rho^2}{n-2}}$                            | normal populations or $n > 30$                                  |



# Factors impacting CI width (summary)

If all other factors are held constant, the table below summarizes the impact of certain changes on the width of confidence intervals:

| Change  | Impact on CI width            |
|---|-------------------------------|
| Increasing $n$  | decreases width (narrower CI) |
| Increasing confidence level                               | increases width (wider CI)    |
| Increasing $SE$   | increases width (wider CI)    |
| Increasing number of bootstrap samples (if bootstrapping) | no impact on width            |
| Using $t$ rather than Normal                              | increases width (wider CI)    |