# Normal Approximations

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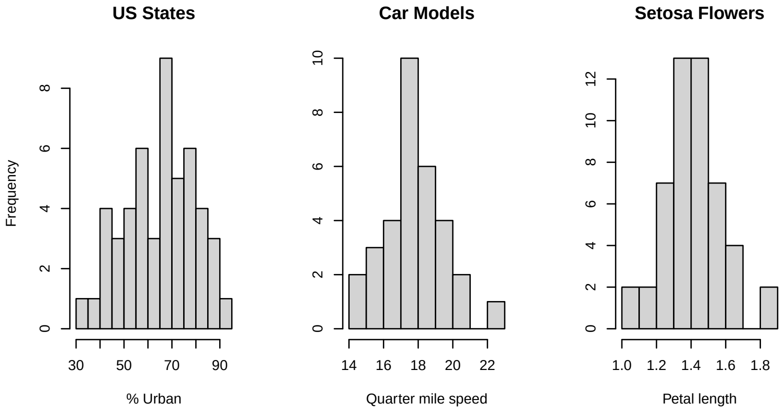
# Different but Similar Data?



Practically speaking, these data are very different, but do you see any similarities (ignoring the units)?

# The Normal Distribution

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  - To make this work, we needed proper values of $\mu$ and $\sigma$, the parameters dictating the normal curve's *center* and *spread*

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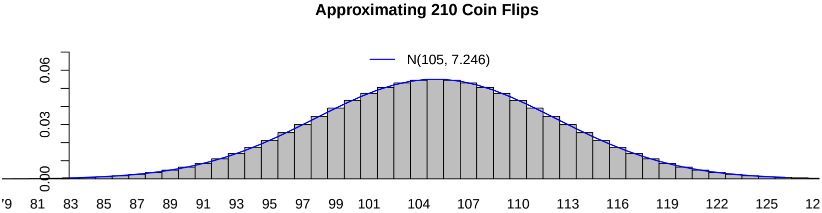
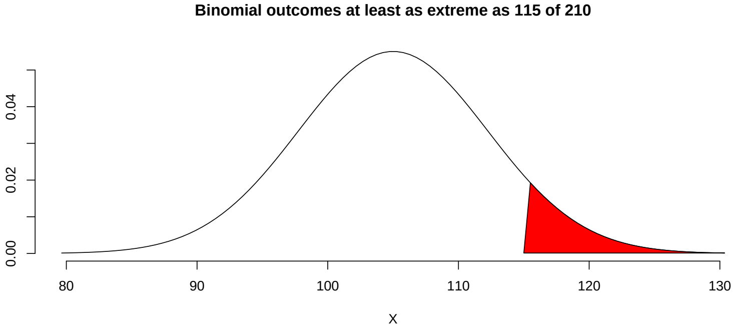
- Last time, we introduced the idea of approximating another distribution with a normal curve
  - To make this work, we needed proper values of $\mu$ and $\sigma$, the parameters dictating the normal curve's *center* and *spread*
- We used the random variable's *expected value* and *standard deviation* (square root of variance)
  - For a *binomial random variable*, $E(X) = n * p$ and $StdDev(X) = \sqrt{n * p * (1 - p)}$



# Probability and the Normal Curve

- The normal distribution is useful because the area under it can be used to find probabilities
  - Shaded in red is the area representing $P(X \geq 115)$
  - This area *approximates* the one-sided $p$-value for 115 successes in 210 trials under the null model that $p = 0.5$



# Probability and the Normal Curve

- Recall we could have calculated this *p*-value *exactly* using the *binomial distribution* by summing many different discrete probabilities, $P(X = 115) + P(X = 116) + \ldots + P(X = 210)$
  - But would this summation approach work if $X$ were continuous?

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# Probability and the Normal Curve

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  - But would this summation approach work if $X$ were continuous?
- For continuous random variables, it *only* makes sense calculate probabilities using areas
- A continuous variable can theoretically be measured with infinite precision
  - Thus, the probability of observing an average height of precisely 70.25 inches in a sample is zero (because there are infinitely many possibilities!)
  - However, the probability of observing an average height in the interval $70.25 \pm \epsilon$ is calculable

# Functions in R

- `pnorm()` is the primary R function for *calculating probabilities* using the normal distribution, the main arguments are:
  - `q` - the boundary value defining your area (ie: $x = 115$)
  - `mean` - the normal distribution's mean (ie: $\mu = 105$)
  - `sd` - the distributions standard deviation (ie: $\sigma = 7.246$)
  - `lower.tail` - whether to take the area to the left of the boundary value (TRUE) or to the right of the boundary value (FALSE)

```r
pnorm(115, mean = 105, sd = 7.246, lower.tail = TRUE)
```

```
## [1] 0.9162177
```

```r
pnorm(115, mean = 105, sd = 7.246, lower.tail = FALSE)
```

```
## [1] 0.08378228
```

```r
pnorm(115, mean = 115, sd = 7.246,lower.tail = FALSE)
```

```
## [1] 0.5
```

# Example

- The National Health and Nutrition Examination Survey (NHANES) is a national study designed assess US population health and nutrition
- The NHANES sample included 2649 adult women
  - The average height was 63.5 inches
  - The standard deviation of these heights was 2.75 inches

**Question**: Suppose you have a friend who is very shallow, and is only interested in dating adult women who are between 5'3 (63 in) and 5'6 (66 in). First, describe this scenario using a random variable, $X$. Then, use a normal approximation of the NHANES data to estimate the probability that a randomly selected adult female is between 5'3 and 5'6.

# Example (solution)

- Let the random variable $X$ denote the height of a randomly chosen adult female
  - We want to determine $P(63 \leq X \leq 66)$
- `R` can provide us with the areas to the left of 63 and 66 inches, then area in between these values can be found via subtraction:

```
p66 <- pnorm(66, mean = 63.5, sd = 2.75, lower.tail = TRUE)
p63 <- pnorm(63, mean = 63.5, sd = 2.75, lower.tail = TRUE)
p66 - p63
```

```
## [1] 0.3904862
```

- So, we estimate there's a 39.0% chance that a randomly selected women meets our friend's height criteria
  - In case you're curious, in the actual NHANES sample, 38.8% of women were in this range

# Standardization

- While we could calculate probabilities using any normal curve, doing so (without software like `R`) is non-trivial because the normal curve does not have a closed-form integral

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

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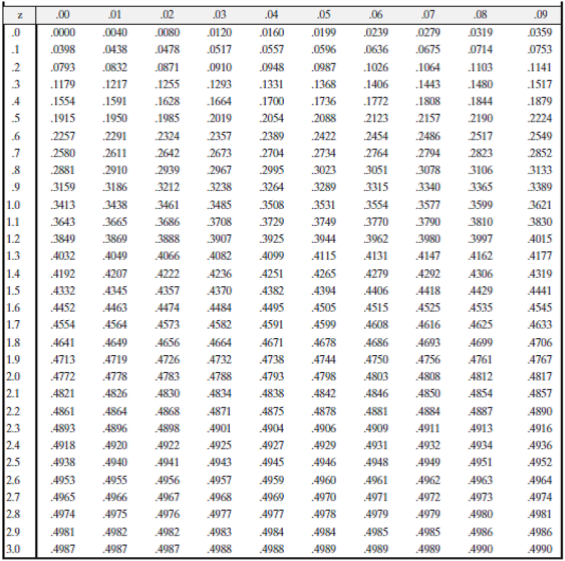
- For mostly historical reasons, this has led statisticians to **standardize** their data in order to make use of the $N(0, 1)$ curve
  - Until recently, statistics textbooks devoted many pages to tables detailing various areas under $N(0, 1)$ curve
  - Modern statistical software has made using such tables obsolete

# Normal Table



| z | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| .0 | .0000 | .0040 | .0080 | .0120 | .0160 | .0199 | .0239 | .0279 | .0319 | .0359 |
| .1 | .0398 | .0438 | .0478 | .0517 | .0557 | .0596 | .0636 | .0675 | .0714 | .0753 |
| .2 | .0793 | .0832 | .0871 | .0910 | .0948 | .0987 | .1026 | .1064 | .1103 | .1141 |
| .3 | .1179 | .1217 | .1255 | .1293 | .1331 | .1368 | .1406 | .1443 | .1480 | .1517 |
| .4 | .1554 | .1591 | .1628 | .1664 | .1700 | .1736 | .1772 | .1808 | .1844 | .1879 |
| .5 | .1915 | .1950 | .1985 | .2019 | .2054 | .2088 | .2123 | .2157 | .2190 | .2224 |
| .6 | .2257 | .2291 | .2324 | .2357 | .2389 | .2422 | .2454 | .2486 | .2517 | .2549 |
| .7 | .2580 | .2611 | .2642 | .2673 | .2704 | .2734 | .2764 | .2794 | .2823 | .2852 |
| .8 | .2881 | .2910 | .2939 | .2967 | .2995 | .3023 | .3051 | .3078 | .3106 | .3133 |
| .9 | .3159 | .3186 | .3212 | .3238 | .3264 | .3289 | .3315 | .3340 | .3365 | .3389 |
| 1.0 | .3413 | .3438 | .3461 | .3485 | .3508 | .3531 | .3554 | .3577 | .3599 | .3621 |
| 1.1 | .3643 | .3665 | .3686 | .3708 | .3729 | .3749 | .3770 | .3790 | .3810 | .3830 |
| 1.2 | .3849 | .3869 | .3888 | .3907 | .3925 | .3944 | .3962 | .3980 | .3997 | .4015 |
| 1.3 | .4032 | .4049 | .4066 | .4082 | .4099 | .4115 | .4131 | .4147 | .4162 | .4177 |
| 1.4 | .4192 | .4207 | .4222 | .4236 | .4251 | .4265 | .4279 | .4292 | .4306 | .4319 |
| 1.5 | .4332 | .4345 | .4357 | .4370 | .4382 | .4394 | .4406 | .4418 | .4429 | .4441 |
| 1.6 | .4452 | .4463 | .4474 | .4484 | .4495 | .4505 | .4515 | .4525 | .4535 | .4545 |
| 1.7 | .4554 | .4564 | .4573 | .4582 | .4591 | .4599 | .4608 | .4616 | .4625 | .4633 |
| 1.8 | .4641 | .4649 | .4656 | .4664 | .4671 | .4678 | .4686 | .4693 | .4699 | .4706 |
| 1.9 | .4713 | .4719 | .4726 | .4732 | .4738 | .4744 | .4750 | .4756 | .4761 | .4767 |
| 2.0 | .4772 | .4778 | .4783 | .4788 | .4793 | .4798 | .4803 | .4808 | .4812 | .4817 |
| 2.1 | .4821 | .4826 | .4830 | .4834 | .4838 | .4842 | .4846 | .4850 | .4854 | .4857 |
| 2.2 | .4861 | .4864 | .4868 | .4871 | .4875 | .4878 | .4881 | .4884 | .4887 | .4890 |
| 2.3 | .4893 | .4896 | .4898 | .4901 | .4904 | .4906 | .4909 | .4911 | .4913 | .4916 |
| 2.4 | .4918 | .4920 | .4922 | .4925 | .4927 | .4929 | .4931 | .4932 | .4934 | .4936 |
| 2.5 | .4938 | .4940 | .4941 | .4943 | .4945 | .4946 | .4948 | .4949 | .4951 | .4952 |
| 2.6 | .4953 | .4955 | .4956 | .4957 | .4959 | .4960 | .4961 | .4962 | .4963 | .4964 |
| 2.7 | .4965 | .4966 | .4967 | .4968 | .4969 | .4970 | .4971 | .4972 | .4973 | .4974 |
| 2.8 | .4974 | .4975 | .4976 | .4977 | .4977 | .4978 | .4979 | .4979 | .4980 | .4981 |
| 2.9 | .4981 | .4982 | .4982 | .4983 | .4984 | .4984 | .4985 | .4985 | .4986 | .4986 |
| 3.0 | .4987 | .4987 | .4987 | .4988 | .4988 | .4989 | .4989 | .4989 | .4990 | .4990 |

*Source:* Abridged from Table 1 of A. Hald, *Statistical Tables and Formulas* (New York: John Wiley & Sons, Inc.), 1952. Reproduced by permission of the publisher.

# Z-Scores

Suppose a random variable follows a normal distribution (ie: $X \sim N(\mu, \sigma)$), then:

$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1)$$

- These transformed, or *standardized*, values are called $Z$-scores
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  - A $Z$-score can be interpreted as how many standard deviations an observed data-point is above or below its expected value
- For example, suppose $X$ is a random variable from a $N(\mu = 3.13, \sigma = 2.23)$ distribution and we observe $x = 5.19$
  - The corresponding $Z$-score is $z = (5.19 - 3.13)/2.23 = 0.923$
  - So this data-point is slightly less than one standard deviation above average (meaning it is quite typical)

# Z-scores (interpretation)

- $Z$-scores can be useful in helping non-experts understand variables with obtuse units
  - For example, if a doctor tells me that my blood urea nitrogen is 8 nmol/L above average I don't know if I should worry
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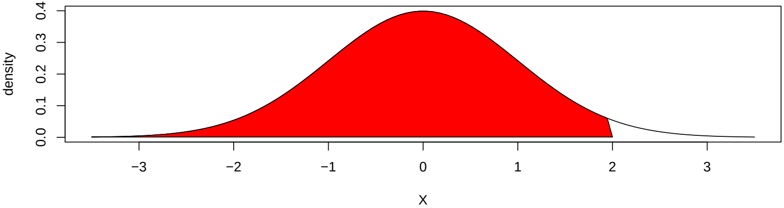
  - For example, if a doctor tells me that my blood urea nitrogen is 8 nmol/L above average I don't know if I should worry
  - But if they tell me it's 4 standard deviations above average I know I should be concerned
- $Z$-scores can also be mapped to *percentiles* of the normal distribution
  - A $Z$-score of 2 indicates a value larger than 97.7% of the data-points (we'll see how to find this in `R` next)



# Functions in R

- `qnorm()` allows to *find percentiles* of a normal distribution its main arguments are:
  - `p` - the percentile of interest
  - `mean` - the normal distribution's mean (ie: $\mu = 0$)
  - `sd` - the distributions standard deviation (ie: $\sigma = 1$)

```r
qnorm(.5, mean = 0, sd = 1)
```

```
## [1] 0
```

```r
qnorm(.75, mean = 0, sd = 1)
```

```
## [1] 0.6744898
```

```r
qnorm(.99, mean = 0, sd = 1)
```

```
## [1] 2.326348
```

# Example (revisited)

Recall the NHANES sample included 2649 adult women, who had an average height was 63.5 inches with a standard deviation 2.75 inches

1) For our very shallow friend who was interested in dating adult women who are between 5'3 (63 in) and 5'6 (66 in), use $Z$-scores and the *standard normal distribution* to express and calculate the probability of a randomly chosen women falling within this range.
2) Suppose a different (but equally shallow) friend wants to maximize his chances of having a son who plays college basketball, thus he is only interested in dating women who are above the 95% percentile for height. Use the NHANES sample and a normal approximation to determine the $95^{th}$ percentile.

# Example (solution #1)

- Let $X$ be a random variable describing a randomly chosen female's height.
  - The $Z$-score for our friend's lower threshold is $z_L = \frac{63-63.5}{2.75} = -0.18$
  - The upper threshold is $z_U = \frac{66-63.5}{2.75} = 0.91$
- Thus, $P(63 \leq X \leq 66) = P(-0.18 \leq Z \leq 0.91)$, which we can find using `R` using `pnorm()`:

```r
pzu <- pnorm(.91, mean = 0, sd = 1, lower.tail = TRUE)
pzl <- pnorm(-.18, mean = 0, sd = 1, lower.tail = TRUE)
pzu - pzl
```

```
## [1] 0.3900125
```

# Example (solution #2)

We could can find the $Z$-score corresponding to the 95% percentile using R using `qnorm()`, then work backwards (un-standardize) to find $x$

```r
qnorm(.95, mean = 0, sd = 1)
```

```
## [1] 1.644854
```

$$1.645 = \frac{x - 63.5}{2.75} \implies x = 1.645 * 2.75 + 63.5 = 68.02$$

Alternatively, we could specify the mean/sd of our data in `qnorm()`

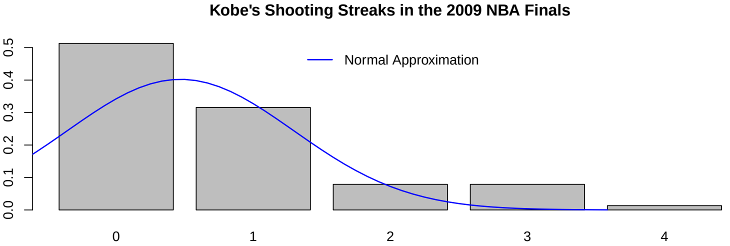
```r
qnorm(.95, mean = 63.5, sd = 2.75)
```

```
## [1] 68.02335
```

Either way, we see the $95^{th}$ percentile is roughly 5‘8

# Approximation Accuracy

- In the two examples we've seen so far, approximating via the normal distribution has been remarkably accurate
- Unfortunately, the normal curve cannot be applied to all situations
  - The graph below shows a normal approximation of Kobe's shooting streaks, would you feel comfortable using this curve to estimate the probability of 3+ shot shooting streak?



# Conclusion

- It turns out, there is something special going on in our two examples (binomial random variable with \$n = 210, and adult female heights) that led to the normal approximation being accurate

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- Next time, we'll explore a landmark theoretical result that provides tremendous insight regarding scenarios where a normal distribution will provide an accurate approximation
    - In fact, statisticians have *proven* the normal distribution to be apply to certain scenarios

# Conclusion

- It turns out, there is something special going on in our two examples (binomial random variable with \$n = 210, and adult female heights) that led to the normal approximation being accurate
- Next time, we'll explore a landmark theoretical result that provides tremendous insight regarding scenarios where a normal distribution will provide an accurate approximation
  - In fact, statisticians have *proven* the normal distribution to be apply to certain scenarios
- From this presentation, you should take away three things:
  - Be able to use the normal distribution to calculate probabilities
  - Understand $Z$-scores, standardization, and the standard normal distribution
  - Be able to find values representing certain percentiles of a normally distributed random variable