

The Correlation Coefficient and Regression

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Introduction

- ▶ Lately we've been working with *regression models*, a general statistical approach that uses one or more explanatory variables to model a numeric outcome
- ▶ *Simple linear regression* describes the special case involving a single numeric explanatory variable
 - ▶ You may recall the *correlation coefficient* is used in this same scenario (numeric explanatory variable, numeric response variable)
 - ▶ This lecture will cover the relationship between these two methods

Correlation (review)

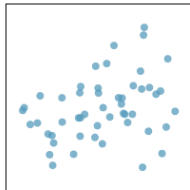
The correlation coefficient measures the strength of linear association between two numeric variables, X and Y :

$$r_{xy} = \frac{1}{n-1} \sum_i \left(\frac{x_i - \bar{x}}{s_x} \right) \left(\frac{y_i - \bar{y}}{s_y} \right)$$

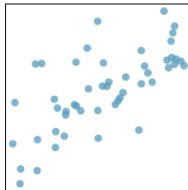
- ▶ It is useful to understand this calculation as the *average product of Z-scores* across the two variables
 - ▶ For example, if cases tend to be either above average in both X and Y or below average in both X and Y , the correlation coefficient will be positive

Correlation (review)

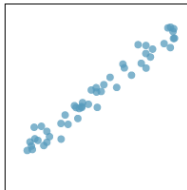
Below are some examples of different correlation coefficients:



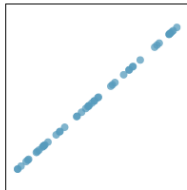
$R = 0.33$



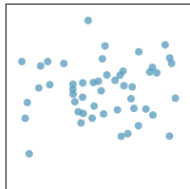
$R = 0.69$



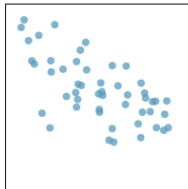
$R = 0.98$



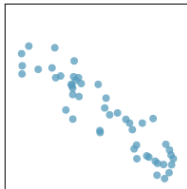
$R = 1.00$



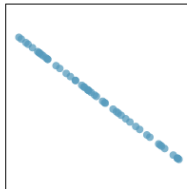
$R = 0.08$



$R = -0.64$



$R = -0.92$



$R = -1.00$

Correlation (review)

As a reminder, you should always graph the data before blindly interpreting a correlation coefficient:

From Cook & Swayne's *Interactive and Dynamic Graphics for Data Analysis*:

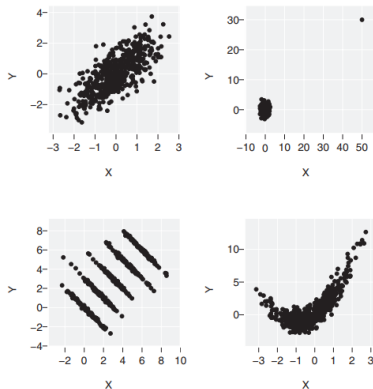


Fig. 6.1. Studying dependence between X and Y. All four pairs of variables have correlation approximately equal to 0.7, but they all have very different patterns. Only the top left plot shows two variables matching a dependence modeled by correlation.

Example - Pearson's Height Data

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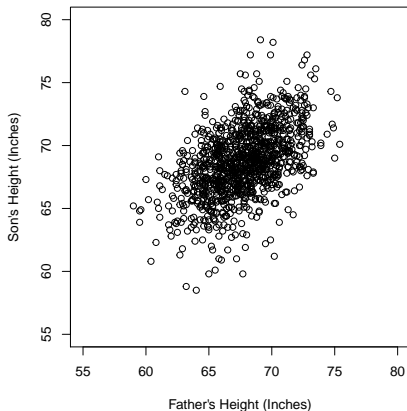
Example - Pearson's Height Data

- ▶ Francis Galton and Karl Pearson, two pioneers of modern statistics, lived in Victorian England at a time when the scientific community was fascinated by the idea of quantifying heritable traits
- ▶ Wondering if height is heritable, they measured the heights of 1,078 fathers and their (fully grown) sons:

Father	Son
65	59.8
63.3	63.2
65	63.3
65.8	62.8
...	...

Example - Pearson's Height Data

Does height appear heritable? To what degree?



Hypothesis Testing and Correlation

- ▶ While tall fathers tend to have tall sons, there are plenty of exceptions, so do these data provide convincing evidence of an association?
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 - ▶ What hypothesis might we want to test?

```
pearson <- read.delim("https://remiller1450.github.io/data/Pearson.txt", sep = "\t")
cor.test(x = pearson$Father, y = pearson$Son)
```

```
##
## Pearson's product-moment correlation
##
## data: pearson$Father and pearson$Son
## t = 18.997, df = 1076, p-value < 2.2e-16
## alternative hypothesis: true correlation is not equal to 0
## 95 percent confidence interval:
##  0.4550726 0.5445746
## sample estimates:
##          cor
## 0.5011627
```

- ▶ There is overwhelming statistical evidence of a non-zero correlation between father and son heights

Hypothesis Testing and Correlation (a second example)

- ▶ Do larger colleges tend to have a higher proportion of female students? What do you make of these results?

```
colleges19 <- read.csv("https://remiller1450.github.io/data/Colleges2019.csv")
cor.test(colleges19$PercentFemale, colleges19$Enrollment)
```

```
##
## Pearson's product-moment correlation
##
## data: colleges19$PercentFemale and colleges19$Enrollment
## t = -2.1114, df = 1606, p-value = 0.03489
## alternative hypothesis: true correlation is not equal to 0
## 95 percent confidence interval:
## -0.101237512 -0.003740064
## sample estimates:
## cor
## -0.05261417
```

Using Correlation to make Predictions

- ▶ Suppose we want to use Pearson and Galton's height data to make predictions

```
summary(pearson)
```

##	Father	Son
##	Min. :59.00	Min. :58.50
##	1st Qu.:65.80	1st Qu.:66.90
##	Median :67.80	Median :68.60
##	Mean :67.69	Mean :68.68
##	3rd Qu.:69.60	3rd Qu.:70.50
##	Max. :75.40	Max. :78.40

- ▶ What height would predict for a future son of a father who is 67.69 inches tall?

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- ▶ What height would predict for a future son of a father who is 67.69 inches tall?
 - ▶ This father is exactly average height, so the logical prediction is that the son is also average height, or 68.68 inches

Using Correlation to make Predictions

- ▶ How would you predict the son's height if the father were 65.0 inches, or 1 standard deviation below the average?
 - ▶ You'd be wise to predict a below average height for the son, but by how much?

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- ▶ Part of the answer is *standardization*
 - ▶ 65.0 inches is 1 standard deviation below the average for father's height
- ▶ But we know that father's height and son's height aren't perfectly correlated, so we shouldn't expect the son to be *exactly* 1 standard deviation below average

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- ▶ Part of the answer is *standardization*
 - ▶ 65.0 inches is 1 standard deviation below the average for father's height
- ▶ But we know that father's height and son's height aren't perfectly correlated, so we shouldn't expect the son to be *exactly* 1 standard deviation below average
 - ▶ Since the correlation is 0.50 between father/son heights, the "best" prediction we can make is that the son will be $0.5 * 1$ standard deviations between below average height

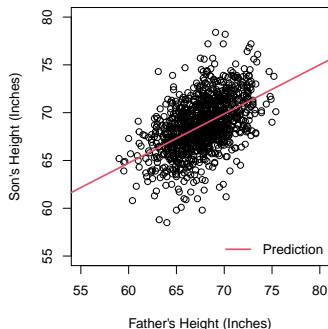
Using Correlation to make Predictions

The logic on the previous slide can be proceduralized:

1. Standardize the explanatory variable (In the previous example, $z_x = -1$)
2. Use the correlation coefficient to make a prediction (ie: $z_y = z_x * r_{xy} = -1 * .5$)
3. Un-standardize the prediction to get an answer in the original units (ie: predicted son's height = $\bar{y} + z_y * s_y$)

Using Correlation to make Predictions

The approach can be used to make predictions for *any* father's height:



This line is the simple linear regression model!

How Regression got its Name

- ▶ The correlation coefficient is always less than 1 (in absolute value)
 - ▶ So being 1 standard deviation above/below average in the explanatory variable *a/ways* leads to the prediction that a case is less than 1 standard deviation above/below average in the response variable
 - ▶ Galton described this phenomenon as: “regression to mediocrity”

Article Link: “Is the ‘Madden’ cover curse still a thing? A look back at 20 years of NFL stars offers a verdict”

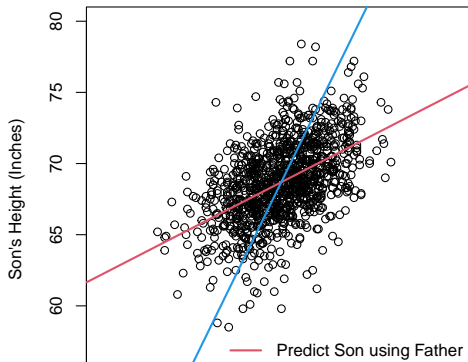
- ▶ Madden is an iconic video game whose cover features a different NFL player each year, usually a player who performed exceptionally well in the previous season
- ▶ Frequently, the player featured on the Madden cover suffers from a decline in play or sustains an injury in their next season (see the article)
- ▶ Is the “Madden Curse” real? What might be a more statistically sound explanation?

Regression to Mediocrity

- ▶ Each player featured on the Madden cover was selected because they had exceptional season
- ▶ Performance in the subsequent season is correlated with that of the prior season, but the correlation is nowhere near 1
- ▶ The best prediction is for these players to regress
- ▶ The NFL is such that seasons near the league's statistical averages are not generally regarded as “good”
 - ▶ In 2017, the 16th rated passer was Tyrod Taylor, with 2799 yds, 14 tds, 4 ints
 - ▶ The 16th rusher was Lamar Miller with 888 yds, 3 tds

Regression is Asymmetric

- ▶ Correlation is a **symmetric** statistical method: $r_{x,y} = r_{y,x}$
- ▶ Regression is an **asymmetric** statistical method: the choice of explanatory and response variables matters



The Coefficient of Variation (R^2)

- ▶ Correlation shares another connection with regression, the **coefficient of variation**, more commonly known as R^2
 - ▶ R^2 describes the *proportion of total variability explained by the explanatory variable*

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- ▶ Correlation shares another connection with regression, the **coefficient of variation**, more commonly known as R^2
 - ▶ R^2 describes the *proportion of total variability explained by the explanatory variable*
 - ▶ We can express proportion this using sums of squares (as we defined in ANOVA):

$$R^2 = \frac{SST - SSE}{SST}$$

- ▶ Recall SST is the *sum of squares total* (the maximum possible amount of modeling error), and SSE is the *sum of squared error* (how much error remains in our model)

Example in R (directions)

- 1) Fit a simple linear regression model that uses father's height to predict son's height
- 2) Use the `anova()` function to view an ANOVA table summarizing this model
- 3) Calculate the model's R^2 based upon the information in the ANOVA table, then compare your answer with the R^2 given by the `summary()` function
- 4) Take the square-root of the model's R^2 , then compare this value with the correlation coefficient found using `cor()`

Example in R(solution)

```
## Fit the simple linear regression
pearson <- read.delim("https://remiller1450.github.io/data/Pearson.txt", sep = "\t")
model <- lm(Son ~ Father, data = pearson)

## Get ANOVA table and extract SSE/SST
anova(model)

## Analysis of Variance Table
##
## Response: Son
##           Df Sum Sq Mean Sq F value    Pr(>F)
## Father      1 2145.4  2145.35   360.9 < 2.2e-16 ***
## Residuals 1076  6396.3     5.94
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

SSE <- 6396.3
SST <- SSE + 2145.4

## Calculate R^2
(SST - SSE)/SST

## [1] 0.2511678
```

Example in R(solution)

Notice the `summary()` function will also calculate our model's R^2 value

```
##
## Call:
## lm(formula = Son ~ Father, data = pearson)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -8.8910 -1.5361 -0.0092  1.6359  8.9894
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  33.89280     1.83289   18.49  <2e-16 ***
## Father        0.51401     0.02706    19.00  <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 2.438 on 1076 degrees of freedom
## Multiple R-squared:  0.2512, Adjusted R-squared:  0.2505
## F-statistic: 360.9 on 1 and 1076 DF,  p-value: < 2.2e-16
```

Example in R(solution)

```
## Square-root  $R^2$   
R2 <- (SST - SSE)/SST  
sqrt(R2)
```

```
## [1] 0.5011664
```

```
## Compare with cor()  
cor(x = pearson$Father, y = pearson$Son)
```

```
## [1] 0.5011627
```

- ▶ Much like the correlation coefficient, R^2 can be thought of as summarizing the strength of linear association between explanatory and response variables

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- ▶ Because one-way ANOVA is also a regression model, R^2 can be used to summarize the association between a categorical *explanatory variable* and a *numeric response variable*
 - ▶ This is useful because the correlation coefficient can only be calculated when both variables are numeric

Conclusion

We've now covered analysis approaches for every possible comparison of two variables:

- ▶ Two numeric variables - scatterplot, simple linear regression, correlation coefficient, R^2

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We've also covered methods for univariate analyses:

- ▶ One numeric variable - boxplot/histogram, t -test
- ▶ One categorical variable - bar charts or pie charts, exact binomial test (binary), Chi-square goodness of fit test (nominal)