

Chapter 1

Gravitoelectromagnetism: A Brief Review

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Abstract. The main theoretical aspects of gravitoelectromagnetism (“GEM”) are presented. Two basic approaches to this subject are described and the role of the gravitational Larmor theorem is emphasized. Some of the consequences of GEM are briefly mentioned.

1.1 Introduction

The analogy between gravitation and electromagnetism has a long history. The similarity between Newton’s law of gravitation and Coulomb’s law of electricity naturally led to a gravitoelectric description of Newtonian gravitation. Moreover, on the basis of advances in electrodynamics in the second half of the nineteenth century, Holzmüller [1] and Tisserand [2] postulated that the gravitational force exerted by the Sun on the planets of the solar system had an additional “magnetic” component. This extra force led to the precession of the planetary orbits; therefore, it could be adjusted in order to account for the excess perihelion precession of Mercury. Decades later, however, Einstein’s general relativity provided a beautiful explanation of the excess motion of Mercury’s perihelion in terms of a relativistic gravitoelectric correction to the Newtonian gravitational potential of the Sun [3]. Furthermore, general relativity, which is a field theory of gravitation, contains a gravitomagnetic field due to mass current [4]. Indeed, to bring together Newtonian gravitation and Lorentz invariance in a consistent field-theoretic framework, the introduction of a gravitomagnetic field is unavoidable.

According to general relativity, the proper rotation of the Sun produces a gravitomagnetic field and the influence of this field on planetary orbits was first considered by de Sitter [5] and later in a more general form by Lense and Thirring [4]. The gravitomagnetic contribution to the excess motion of Mercury's perihelion turns out to be much smaller and in the opposite sense compared to the main gravitoelectric motion; in fact, it turns out that the Lense-Thirring precession of planetary orbits is too small to be measurable at present. On the other hand, evidence for the gravitomagnetic field of the Earth has been offered by Ciufolini by studying the motion of laser-ranged satellites LAGEOS and LAGEOS II [6]. The precise measurement of this field via superconducting gyroscopes in a drag-free satellite in polar orbit about the Earth is one of the aims of NASA's GP-B [7].

Within the framework of general relativity, gravitoelectromagnetism ("GEM") has been discussed by a number of authors [3, 8]; a more extensive list of references is provided in [9]. The purpose of this review is to present the two principal approaches to GEM and briefly describe some of their consequences.

1.2 Linear Perturbation Approach to GEM

We are interested in the general linear solution of the gravitational field equations [3]. It is assumed that a global background inertial frame with coordinates $x^\mu = (ct, \mathbf{x})$ and Minkowski metric $\eta_{\mu\nu}$ is perturbed due to the presence of gravitating sources such that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x)$. It proves useful to define the trace-reversed amplitude $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$, where $h = \eta^{\mu\nu}h_{\mu\nu}$ is the trace of $h_{\mu\nu}$. To linear order in the perturbation, Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} \quad (1.1)$$

take the form

$$\square \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu} \quad (1.2)$$

after imposing the transverse gauge condition $\bar{h}^{\mu\nu}_{,\nu} = 0$. The general solution of (1.2) is a superposition of a particular solution together with the general solution of the wave equation; however, we are only interested in the special retarded solution of (1.2) given by

$$\bar{h}_{\mu\nu} = \frac{4G}{c^4} \int \frac{T_{\mu\nu}(ct - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x'. \quad (1.3)$$

Let us define the matter density ρ and matter current $\mathbf{j} = \rho\mathbf{v}$ via $T^{00} = \rho c^2$ and $T^{0i} = c j^i$, respectively. Moreover, it is useful to define the GEM potentials Φ and \mathbf{A} in terms of ρ and \mathbf{j} as $\bar{h}_{00} = 4\Phi/c^2$ and $\bar{h}_{0i} = -2A_i/c^2$, respectively. Assuming that the source consists of a finite distribution of slowly moving matter with $|\mathbf{v}| \ll c$, $T_{ij} \sim \rho v_i v_j + p \delta_{ij}$, where p is the pressure, and (1.3) imply that $\bar{h}_{ij} = O(c^{-4})$.

All terms of $O(c^{-4})$ will be neglected in this analysis. Under these conditions, the spacetime metric has the GEM form

$$ds^2 = -c^2 \left(1 - 2\frac{\Phi}{c^2}\right) dt^2 - \frac{4}{c}(\mathbf{A} \cdot d\mathbf{x})dt + \left(1 + 2\frac{\Phi}{c^2}\right) \delta_{ij} dx^i dx^j. \quad (1.4)$$

In the Newtonian limit, Φ reduces to the Newtonian gravitational potential, while $\mathbf{A} = O(c^{-1})$. If the source distribution is confined around the origin of spatial coordinates, then far from the source

$$\Phi \sim \frac{GM}{r}, \quad \mathbf{A} \sim \frac{G}{c} \frac{\mathbf{J} \times \mathbf{x}}{r^3}, \quad (1.5)$$

where $r = |\mathbf{x}|$ and M and \mathbf{J} are the mass and angular momentum of the source, respectively. Moreover, the transverse gauge condition reduces to

$$\frac{1}{c} \frac{\partial \Phi}{\partial t} + \nabla \cdot \left(\frac{1}{2} \mathbf{A}\right) = 0. \quad (1.6)$$

We define the GEM fields via

$$\mathbf{E} = -\nabla \Phi - \frac{1}{c} \frac{\partial}{\partial t} \left(\frac{1}{2} \mathbf{A}\right), \quad \mathbf{B} = \nabla \times \mathbf{A}, \quad (1.7)$$

in direct analogy with electromagnetism. It follows from these definitions that the GEM fields have dimensions of acceleration and

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial}{\partial t} \left(\frac{1}{2} \mathbf{B}\right), \quad \nabla \cdot \left(\frac{1}{2} \mathbf{B}\right) = 0. \quad (1.8)$$

Furthermore, (1.2) implies that

$$\nabla \cdot \mathbf{E} = 4\pi G\rho, \quad \nabla \times \left(\frac{1}{2} \mathbf{B}\right) = \frac{1}{c} \frac{\partial}{\partial t} \mathbf{E} + \frac{4\pi G}{c} \mathbf{j}. \quad (1.9)$$

The GEM field equations (1.8) and (1.9) contain the continuity equation $\nabla \cdot \mathbf{j} + \partial\rho/\partial t = 0$, as expected.

For a complete GEM theory, we need an analogue of the Lorentz force law. The Lagrangian for the motion of a test particle of mass m , $L = -mc ds/dt$, can be written to linear order in Φ and \mathbf{A} as

$$L = -mc^2 \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} + m\gamma \left(1 + \frac{v^2}{c^2}\right) \Phi - \frac{2m}{c} \gamma \mathbf{v} \cdot \mathbf{A}, \quad (1.10)$$

where γ is the Lorentz factor. The equation of motion, $d\mathbf{p}/dt = \mathbf{F}$, where $\mathbf{p} = \gamma m \mathbf{v}$ is the kinetic momentum, takes a simple familiar form if $\partial\mathbf{A}/\partial t = 0$ and \mathbf{F} is expressed to lowest order in \mathbf{v}/c , Φ and \mathbf{A} ; then,

$$\mathbf{F} = -m\mathbf{E} - 2m\frac{\mathbf{v}}{c} \times \mathbf{B}. \quad (1.11)$$

The canonical momentum of the particle is given in this case by $\mathbf{p} + (-2m/c)\mathbf{A}$.

Let us now discuss the gauge freedom of the GEM potentials. Under a coordinate transformation $x^\mu \rightarrow x'^\mu = x^\mu - \epsilon^\mu$, $h_{\mu\nu} \rightarrow h'_{\mu\nu} = h_{\mu\nu} + \epsilon_{\mu,\nu} + \epsilon_{\nu,\mu}$ to linear order in ϵ_μ . Therefore,

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} + \epsilon_{\mu,\nu} + \epsilon_{\nu,\mu} - \eta_{\mu\nu}\epsilon^\alpha{}_{,\alpha}. \quad (1.12)$$

Under this transformation, the Riemann tensor remains invariant, but the connection changes. We must restrict ϵ_μ in such a way that those elements of the connection defining GEM fields also remain invariant. The new metric perturbation satisfies the transverse gauge condition as well, provided $\square\epsilon_\mu = 0$. Therefore, let $\epsilon_0 = O(c^{-3})$ and $\epsilon_i = O(c^{-4})$; then, (1.12) implies that

$$\Phi' = \Phi - \frac{1}{c} \frac{\partial}{\partial t} \Psi, \quad \frac{1}{2} \mathbf{A}' = \frac{1}{2} \mathbf{A} + \nabla \Psi, \quad (1.13)$$

where $\Psi = c^2 \epsilon^0/4$ and $\square\Psi = 0$. Under the GEM gauge transformation (1.13), the GEM fields are invariant in close analogy with electrodynamics.

It is important to discuss the stress-energy tensor for GEM in the context of our approximation scheme. For the sake of simplicity we set $c = G = 1$ in what follows, unless specified otherwise. The Landau-Lifshitz pseudotensor $t_{\mu\nu}$ can be employed to determine the local stress-energy content of the GEM fields. The general gauge-dependent result is somewhat complicated and is given in [10]; however, for a stationary configuration (i.e. $\partial\Phi/\partial t = 0$ and $\partial\mathbf{A}/\partial t = 0$), one finds

$$4\pi G t_{00} = -\frac{7}{2} E^2 + \sum_{i,j} A_{(i,j)} A_{(i,j)}, \quad (1.14)$$

$$4\pi G t_{0i} = 2(\mathbf{E} \times \mathbf{B})_i, \quad (1.15)$$

$$4\pi G t_{ij} = \left(E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right) + \left(B_i B_j + \frac{1}{2} \delta_{ij} B^2 \right). \quad (1.16)$$

There is some similarity between these and the corresponding relations in classical electrodynamics. In particular, the GEM Poynting vector is given by

$$\mathcal{S} = -\frac{c}{2\pi G} \mathbf{E} \times \mathbf{B}. \quad (1.17)$$

For instance, gravitational energy circulates around a stationary source of mass m and angular momentum $\mathbf{J} = J\hat{\mathbf{z}}$ with a flow velocity

$$\mathbf{v}_g = k \frac{J}{Mr} \sin\theta \hat{\phi} \quad (1.18)$$

in the same sense as the rotation of the mass. Here we employ spherical polar coordinates and $k = 4/7$. The flow given by (1.18) is divergence-free and the corresponding circulation is independent of the radial distance r and is given by $2\pi k(J/M) \sin^2\theta$.

1.3 Gravitational Larmor Theorem

There is a certain degree of arbitrariness in the definitions of the GEM potentials and fields that leads to extraneous numerical factors as compared to electrodynamics. To develop GEM in a way that would provide the closest possible connection with the standard formulas of electrodynamics, we have adopted a convention that, among other things, provides a clear statement of the gravitational Larmor theorem. We assume that a test particle of inertial mass m has gravitoelectric charge $q_E = -m$ and gravitomagnetic charge $q_B = -2m$. For a source that is a rotating body of mass M , the corresponding charges are positive, $Q_E = M$ and $Q_B = 2M$, respectively, in order to preserve the attractive nature of gravity. Moreover, the ratio of the gravitomagnetic charge to the gravitoelectric charge is always 2, since linearized gravity is a spin-2 field. This is consistent with the fact that for a spin-1 field such as in Maxwell's theory, the corresponding ratio is always unity.

The Larmor theorem originally established a basic local equivalence between magnetism and rotation [11]. In fact, the electromagnetic force on a test particle of mass m and charge q in the linear approximation is the same as the inertial force experienced by the free particle with respect to an accelerated system of reference with translational acceleration $\mathbf{a}_L = -q_E \mathbf{E}/m$ and frequency of rotation $\boldsymbol{\Omega}_L = q_B \mathbf{B}/(2mc)$. In electromagnetism $q_E = q_B = q$ and for all particles with the same charge-to-mass ratio q/m , the electromagnetic field can be replaced by the same accelerated system. This circumstance takes on a universal character in the case of gravity since the gravitational charge-to-mass ratio is the same for all particles according to the principle of equivalence of gravitational and inertial masses. The universality of the gravitational interaction thus leads to a geometric theory of gravitation, i.e. general relativity. An analogous approach to electrodynamics is impossible due to the fact that q/m for different particles can, for instance, be positive, negative or zero.

To develop a gravitational analogue of Larmor's theorem, we recall from the previous section that in the linear approximation of general relativity, the exterior gravity of a rotating source can be described in terms of GEM fields. In a sufficiently small neighborhood of the exterior region, the GEM fields may be considered locally uniform. These fields may then be locally replaced by an accelerated system in Minkowski spacetime. To this end, let us imagine an accelerated observer following a worldline $x_0^\mu(\tau)$. Here τ is the observer's proper time and $u^\mu = dx_0^\mu/d\tau$ and $A^\mu = du^\mu/d\tau$ are its velocity and acceleration vectors, respectively. Let $\lambda_{(\alpha)}^\mu$ be the orthonormal tetrad frame of the observer such that $\lambda_{(0)}^\mu = u^\mu$ and

$$\frac{d\lambda_{(\alpha)}^\mu}{d\tau} = \phi_{\alpha}^{\beta} \lambda_{(\beta)}^\mu, \quad (1.19)$$

where $\phi_{\alpha\beta}(\tau)$ is the antisymmetric acceleration tensor of the observer. In analogy with the Faraday tensor, $\phi_{\alpha\beta}$ consists of an "electric" part $\phi_{0i} = a_i$ and a "magnetic"

part $\phi_{ij} = \epsilon_{ijk}\Omega^k$. Here \mathbf{a} and $\boldsymbol{\Omega}$ are spacetime scalars that represent respectively the translational acceleration, $a_i = a_\mu \lambda^\mu_{(i)}$, and the rotational frequency of the local spatial frame with respect to the local nonrotating (i.e. Fermi-Walker transported) frame. Consider now a geodesic system of coordinates X^μ established along the worldline of the fiducial observer. At any event τ along the worldline, the straight spacelike geodesic lines orthogonal to the worldline span a hyperplane that is Euclidean space. Let x^μ be the coordinates of a point on this hyperplane; then,

$$x^\mu = x_0^\mu + X^i \lambda^\mu_{(i)}(\tau), \quad \tau = X^0. \quad (1.20)$$

The Minkowski metric $\eta_{\mu\nu}dx^\mu dx^\nu$ with respect to the new coordinates takes the form $g_{\mu\nu}dX^\mu dX^\nu$, where

$$g_{00} = -(1 + \mathbf{a} \cdot \mathbf{X})^2 + (\boldsymbol{\Omega} \times \mathbf{X})^2, \quad (1.21)$$

$$g_{0i} = (\boldsymbol{\Omega} \times \mathbf{X})_i, \quad g_{ij} = \delta_{ij}. \quad (1.22)$$

These geodesic coordinates are admissible if $g_{00} < 0$; a detailed discussion of the nature of the boundary of the admissible region is given in [12].

A comparison of the metric given by (1.21) and (1.22) with (1.4) reveals that they are Larmor equivalent at the linear order once

$$\Phi = -\mathbf{a}_L \cdot \mathbf{X}, \quad \mathbf{A} = -\frac{1}{2}\boldsymbol{\Omega}_L \times \mathbf{X}, \quad (1.23)$$

and we neglect spatial curvature. The corresponding GEM fields to lowest order are $\mathbf{E} = -\nabla\Phi = \mathbf{a}_L$ and $\mathbf{B} = \nabla \times \mathbf{A} = -\boldsymbol{\Omega}_L$, as expected from the traditional Larmor theorem with $q_E = -m$ and $q_B = -2m$.

The gravitational Larmor theorem [13] is essentially Einstein's principle of equivalence formulated within the GEM framework. Einstein's heuristic principle of equivalence traditionally refers to the Einstein "elevator" and its translational acceleration in connection with the gravitoelectric field of the source. However, it follows from the gravitational Larmor theorem that a rotation of the elevator is generally necessary as well in order to take due account of the gravitomagnetic field of the source.

In classical electrodynamics, a charged spinning test particle has a magnetic dipole moment $\boldsymbol{\mu} = q\mathbf{S}/(2mc)$, where m , q and \mathbf{S} are respectively the mass, charge and the spin of the particle. In an external magnetic field \mathbf{B} , the test dipole has an interaction energy $-\boldsymbol{\mu} \cdot \mathbf{B}$ and precesses due to a torque $\boldsymbol{\mu} \times \mathbf{B}$. In a similar way, a test gyroscope of spin \mathbf{S} with $q \rightarrow q_B = -2m$ has a gravitomagnetic dipole moment $\boldsymbol{\mu}_g = -\mathbf{S}/c$ and precesses in the exterior field of a rotating source of mass M and spin \mathbf{J} with the frequency [14]

$$\boldsymbol{\Omega}_P = \frac{GJ}{c^2 r^3} [3(\hat{\mathbf{J}} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \hat{\mathbf{J}}], \quad (1.24)$$

where $\mathbf{\Omega}_P = \mathbf{B}/c$ and the gravitomagnetic field \mathbf{B} is given by the curl of the vector potential in (1.5) that corresponds to a source with gravitomagnetic dipole moment \mathbf{J}/c . The related interaction energy is $\mathbf{S} \cdot \mathbf{\Omega}_P$. A major aim of the GP-B is to measure (1.24) for gyros in a polar orbit about the Earth.

It follows from (1.24) that $2\pi/\Omega_P$ is a characteristic timescale for the gravitomagnetic field. More generally, gravitomagnetic effects reveal an interesting temporal structure around a rotating mass; this can be further illustrated by the phenomena associated with the gravitomagnetic clock effect [15] and the gravitomagnetic time delay [16].

A more exact long-term post-Schwarzschild analysis of the orbital motion of an ideal test gyroscope in the field of a rotating source reveals that besides the gravitoelectric geodetic (i.e. de Sitter-Fokker) precession of the gyro axis there is a complex gravitomagnetic component involving precessional as well as nutational motions—the latter is known as relativistic nutation [17]. The net gravitomagnetic spin motion reduces in the post-Newtonian approximation to equation (1.24).

1.4 Spacetime Curvature Approach to GEM

The main elements that underlie the gravitational Larmor theorem apply equally well but in a different context to an alternative treatment of GEM based on spacetime curvature. Unlike the previous treatment, the new approach is not limited to perturbations of flat spacetime and can be employed in an arbitrary curved spacetime.

Consider a congruence of test observers following geodesics in a gravitational field. Choosing a reference observer in this congruence, we set up a Fermi coordinate system along its path. This amounts to constructing an inertial system of coordinates in the immediate neighborhood of the reference observer [18]. Let $\lambda^\mu_{(\alpha)}(\tau)$ be the orthonormal tetrad of the reference observer. Here $\lambda^\mu_{(0)}$ is the vector tangent to the worldline of the observer and τ is the proper time along its path. The spatial frame $\lambda^\mu_{(i)}$, $i = 1, 2, 3$, consists of unit vectors along ideal gyro directions that are parallel transported along the worldline. The Fermi frame is a geodesic reference system that is based on the nonrotating orthonormal tetrad $\lambda^\mu_{(\alpha)}$. The metric of spacetime in Fermi coordinates $X^\mu = (T, \mathbf{X})$ is then given by

$$g_{00} = -1 - R_{0i0j}X^iX^j + \dots, \quad (1.25)$$

$$g_{0i} = -\frac{2}{3}R_{0jik}X^jX^k + \dots, \quad (1.26)$$

$$g_{ij} = \delta_{ij} - \frac{1}{3}R_{ikjl}X^kX^l + \dots, \quad (1.27)$$

where $R_{\alpha\beta\gamma\delta}(T)$ is the projection of the Riemann curvature tensor on the orthonormal tetrad of the reference observer

$$R_{\alpha\beta\gamma\delta} = R_{\mu\nu\rho\sigma}\lambda^\mu_{(\alpha)}\lambda^\nu_{(\beta)}\lambda^\rho_{(\gamma)}\lambda^\sigma_{(\delta)}. \quad (1.28)$$

Along the reference geodesic $T = \tau$, $\mathbf{X} = 0$ and $g_{\mu\nu} = \eta_{\mu\nu}$ by construction. The Fermi coordinates are admissible within a cylindrical spacetime region of radius $\sim \mathcal{R}$ around the worldline of the reference observer. Here \mathcal{R} is the radius of curvature of spacetime.

The metric in Fermi coordinates—within the limited region of admissibility—has the form of a perturbation about Minkowski spacetime; therefore, using the previous GEM approach and comparing (1.25) and (1.26) with (1.4), we define the new GEM potentials as

$$\Phi(T, \mathbf{X}) = -\frac{1}{2}R_{0i0j}(T)X^iX^j + \dots, \quad (1.29)$$

$$A_i(T, \mathbf{X}) = \frac{1}{3}R_{0jik}(T)X^jX^k + \dots, \quad (1.30)$$

where the spatial curvature has been ignored. The GEM fields are defined in terms of the potentials as before and are given to lowest order as

$$E_i(T, \mathbf{X}) = R_{0i0j}(T)X^j + \dots, \quad (1.31)$$

$$B_i(T, \mathbf{X}) = -\frac{1}{2}\epsilon_{ijk}R^{jk}_{0l}(T)X^l + \dots \quad (1.32)$$

It is interesting to note that in this approach the gravitoelectric field is directly connected with the “electric” components of the curvature tensor R_{0i0j} and the gravitomagnetic field is directly connected with the “magnetic” components of the curvature tensor R_{0ijk} [19]. It is possible to combine (1.31) and (1.32) in the GEM Faraday tensor

$$F_{\alpha\beta} = -R_{\alpha\beta 0i}X^i \quad (1.33)$$

to linear order in \mathbf{X} , where $F_{0i} = -E_i$ and $F_{ij} = \epsilon_{ijk}B^k$. Then, Maxwell’s equations $F_{[\alpha\beta,\gamma]} = 0$ and $F^{\alpha\beta}_{,\beta} = 4\pi J^\alpha$ are satisfied in this case to lowest order in $|\mathbf{X}|/\mathcal{R}$ with

$$4\pi J_\alpha(T, \mathbf{0}) = -R_{0\alpha} = -8\pi G \left(T_{0\alpha} - \frac{1}{2}\eta_{0\alpha}T^\beta_\beta \right) \quad (1.34)$$

along the reference trajectory in Fermi coordinates. Here we have used the gravitational field equations as well as the symmetries of the Riemann curvature tensor.

The new approach to GEM naturally contains the analogue of the Lorentz force law. The motion of free test particles in the congruence relative to the reference particle at the spatial origin of Fermi coordinates can be expressed as

$$\begin{aligned} \frac{d^2 X^i}{dT^2} + R_{0i0j}X^j + 2R_{ikj0}V^kX^j + \left(2R_{0kj0}V^iV^k \right. \\ \left. + \frac{2}{3}R_{ikjl}V^kV^l + \frac{2}{3}R_{0kjl}V^iV^kV^l \right) X^j = 0, \end{aligned} \quad (1.35)$$

valid to linear order in the separation \mathbf{X} . This geodesic deviation equation is a generalized Jacobi equation [20] in which the rate of geodesic separation (i.e. the

relative velocity of the test particle) $\mathbf{V} = d\mathbf{X}/dT$ is in general arbitrary ($|\mathbf{V}| < 1$ at $\mathbf{X} = \mathbf{0}$). To linear order in velocity, one can show that (1.35) takes the Lorentz form

$$m \frac{d^2 \mathbf{X}}{dT^2} = q_E \mathbf{E} + q_B \mathbf{V} \times \mathbf{B}, \quad (1.36)$$

where $q_E = -m$ and $q_B = -2m$ as before.

The stress-energy tensor in the new approach can be constructed essentially from the Faraday tensor (1.33) as in Maxwell's theory, i.e.

$$GT^{\alpha\beta} = \frac{1}{4\pi} \left(F^\alpha_\gamma F^{\beta\gamma} - \frac{1}{4} g^{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta} \right), \quad (1.37)$$

where an extra factor of G has been introduced due to dimensional considerations. From equations (1.33) and (1.37), we find

$$\mathcal{T}^{\alpha\beta} = \frac{1}{4\pi G} \left(R^\alpha_{\gamma 0i} R^{\beta\gamma}_{0j} - \frac{1}{4} \eta^{\alpha\beta} R_{\gamma\delta 0i} R^{\gamma\delta}_{0j} \right) X^i X^j. \quad (1.38)$$

This tensor as well as the Faraday tensor (1.33) vanishes along the worldline of the reference observer in the Fermi system; indeed, this is an immediate consequence of the inertial character of this system along the reference trajectory and a realization of Einstein's principle of equivalence. Thus this treatment depends on our choice of a reference observer and the corresponding Fermi coordinate system.

To obtain a coordinate-independent measure of the stress-energy content of the gravitational field, we invoke the notion that the physical measurement of such a quantity requires an averaging process [21]. Starting from an event $(T, \mathbf{0})$ on the reference worldline, we average the tensor given in (1.38) over a small sphere of radius ϵL , where $0 < \epsilon \ll 1$ and L is an invariant length scale that is characteristic of the source of the gravitational field under consideration. For instance, L could be GM/c^2 or, in the absence of such a scale, the Planck length. The quadratic nature of (1.38) in the spatial coordinates implies that the averaging involves

$$\langle X^i X^j \rangle = k(\epsilon L)^2 \delta_{ij}, \quad (1.39)$$

where $k = 1/5$ or $1/3$ depending on whether the averaging involves the volume or the surface of the sphere, respectively. In either case, the constant k can be absorbed in the definition of L . Thus

$$\langle \mathcal{T}_{\alpha\beta} \rangle = \frac{k\epsilon^2 L^2}{4\pi G} \bar{T}_{\mu\nu\rho\sigma} \lambda_{(\alpha)}^\mu \lambda_{(\beta)}^\nu \lambda_{(0)}^\rho \lambda_{(0)}^\sigma, \quad (1.40)$$

where $\bar{T}_{\mu\nu\rho\sigma}(x)$ is the Bel tensor given by

$$\bar{T}_{\mu\nu\rho\sigma}(x) = \frac{1}{2} (R_{\mu\xi\rho\zeta} R_\nu^\xi{}_\rho{}^\zeta + R_{\mu\xi\sigma\zeta} R_\nu^\xi{}_\rho{}^\zeta) - \frac{1}{4} g_{\mu\nu} R_{\alpha\beta\rho\gamma} R^{\alpha\beta}{}_\sigma{}^\gamma. \quad (1.41)$$

This tensor bears a certain similarity with the Maxwell stress-energy tensor and on this basis was first defined by Bel [22] for Einstein spaces in 1958. The Bel superenergy tensor is symmetric and trace-free in its first pair of indices and only symmetric in the second pair of indices. In a Ricci-flat spacetime, the Riemann tensor reduces to the Weyl conformal tensor $C_{\mu\nu\rho\sigma}$ and the Bel tensor reduces to the completely symmetric and trace-free Bel-Robinson tensor $T_{\mu\nu\rho\sigma}$ given by

$$T_{\mu\nu\rho\sigma} = \frac{1}{2}(C_{\mu\xi\rho\zeta}C_{\nu}{}^{\xi}{}_{\sigma}{}^{\zeta} + C_{\mu\xi\sigma\zeta}C_{\nu}{}^{\xi}{}_{\rho}{}^{\zeta}) - \frac{1}{16}g_{\mu\nu}g_{\rho\sigma}C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}. \quad (1.42)$$

An invariant average GEM stress-energy tensor of the gravitational field can thus be defined up to a constant positive multiplicative factor by [23]

$$\bar{T}_{(\alpha)(\beta)} = \frac{L^2}{G}\bar{T}_{\mu\nu\rho\sigma}\lambda_{(\alpha)}^{\mu}\lambda_{(\beta)}^{\nu}\lambda_{(0)}^{\rho}\lambda_{(0)}^{\sigma}, \quad (1.43)$$

where $\bar{T}_{(\alpha)(\beta)}$ is symmetric and traceless. In the Ricci-flat case, $\bar{T}_{(\alpha)(\beta)} \rightarrow T_{(\alpha)(\beta)}^*$, where $T_{(\alpha)(\beta)}^*$ is the *gravitational stress-energy tensor*. This designation refers to the fact that in a Ricci-flat spacetime, the spatial components of the curvature tensor in (1.27), which were essentially ignored in our GEM analysis, are indeed basically given by its electric components; therefore, $T_{(\alpha)(\beta)}^*$ involves all of the components of the spacetime curvature. The stress-energy tensors $\bar{T}_{\mu\nu}$ and $T_{\mu\nu}^*$ have properties reminiscent of Maxwell's electrodynamics and have been discussed in detail in [23]. For instance, in the exterior of a rotating mass the gravitational Poynting flux based on the new approach has a flow velocity given by (1.18) with $k = 3$.

A significant generalization of the concept of superenergy tensors has been developed in [24].

The gravitomagnetic contribution to the spacetime curvature due to a rotating source involves subtle cumulative effects [25] that can be measured in principle via relativistic gravity gradiometry [26].

1.5 Spin-Rotation-Gravity Coupling

An issue of fundamental interest is whether intrinsic spin is affected by a gravitomagnetic field in basically the same way as the classical spin of an ideal gyroscope. This question is related to the inertia of intrinsic spin. The description of physical states in the quantum theory is based upon the irreducible unitary representations of the inhomogeneous Lorentz group, which are characterized by means of mass and spin. The inertial properties of mass in moving frames of reference are already well known: for instance, via Coriolis, centrifugal and other mechanical effects, as well as their quantum mechanical counterparts. The inertial properties of intrinsic spin involve the phenomena associated with the spin-rotation-gravity coupling.

The coupling of intrinsic spin with rotation reveals the rotational inertia of intrinsic spin. That is, as a particle moves, its intrinsic spin keeps its aspect with

respect to an inertial frame; therefore, the spin appears to rotate with respect to a rotating observer. To this motion of spin corresponds, according to quantum mechanics, a Hamiltonian $H = -\gamma \mathbf{\Omega} \cdot \mathbf{S}$. The general formula for the transformation of energy turns out to be $\mathcal{E}' = \gamma(\mathcal{E} - \hbar \mathbf{\Omega} \mathcal{M})$, where $\mathbf{\Omega}$ is the frequency of rotation of the observer and $\hbar \mathcal{M}$ is the component of the total angular momentum along the axis of rotation; that is, $\mathcal{M} = 0, \pm 1, \pm 2, \dots$ for a scalar or a vector particle, while $\mathcal{M} \mp \frac{1}{2} = 0, \pm 1, \pm 2, \dots$ for a Dirac particle. This formula relates the energy of a quantum system measured by a rotating observer \mathcal{E}' to measurements performed in a global inertial frame and can be written in the JWKB approximation as $\mathcal{E}' = \gamma(\mathcal{E} - \mathbf{\Omega} \cdot \mathbf{J})$, where $\mathbf{J} = \mathbf{r} \times \mathbf{p} + \mathbf{S}$ is the total angular momentum. Thus, $\mathcal{E}' = \gamma(\mathcal{E} - \mathbf{v} \cdot \mathbf{p}) - \gamma \mathbf{\Omega} \cdot \mathbf{S}$, so that in the absence of intrinsic spin we recover the classical expression for the energy of a particle as measured in the rotating frame with $\mathbf{v} = \mathbf{\Omega} \times \mathbf{r}$. The spin-rotation coupling therefore involves an energy shift given by the Hamiltonian $H = -\gamma \mathbf{\Omega} \cdot \mathbf{S}$ [27].

Observational evidence for such an energy shift in the case of fermions has been provided in certain high-precision experiments by way of a small frequency offset due to the coupling between the nuclear spin of mercury and the rotation of the Earth [28, 29]; moreover, a direct approach using neutron or atom interferometry has been proposed in [30]. For photons, helicity-rotation coupling has been confirmed to rather high accuracy using rotating GPS receivers [31]; moreover, experimental evidence exists for such a coupling in the microwave and optical regimes in terms of the frequency shift of polarized radiation [30]. The modifications of Doppler effect and aberration due to the coupling of photon spin with the rotation of the source and/or receiver have been the subject of recent studies [32, 33].

Let us now turn to the coupling of spin with gravitomagnetic fields; the spin-gravity coupling is naturally related to the spin-rotation coupling by way of Einstein's principle of equivalence. That is, starting from the spin-rotation Hamiltonian, the transformation $\mathbf{\Omega} \rightarrow -\mathbf{\Omega}_P$ leads, according to the gravitational Larmor theorem, to the spin-gravity Hamiltonian.

It follows from these ideas that in Earth-based experiments, to every Hamiltonian we must add the spin-rotation-gravity interaction Hamiltonian $\delta H \cong -\mathbf{\Omega}_{\oplus} \cdot \mathbf{S} + \mathbf{\Omega}_P \cdot \mathbf{S}$, where $\mathbf{\Omega}_{\oplus}$ and $\mathbf{\Omega}_P$ refer to the rotation frequency of the Earth and the corresponding gravitomagnetic precession frequency, respectively. Thus in the approximation under consideration here a particle with intrinsic spin behaves essentially like an ideal gyroscope. The energy difference corresponding to a spin-1/2 particle polarized vertically up and down relative to the surface of the Earth can be estimated from $\hbar \mathbf{\Omega}_{\oplus} \cong 10^{-19} \text{eV}$ and $\hbar \mathbf{\Omega}_P \cong 10^{-29} \text{eV}$. The measurement of the latter term is beyond present capabilities by several orders of magnitude. In this connection, however, we note that near Jupiter $\hbar \mathbf{\Omega}_P \cong 10^{-27} \text{eV}$, and therefore it is likely that with further improvements in magnetometer design, the spin-gravitomagnetic coupling could become measurable in a satellite in orbit near the surface of Jupiter in the foreseeable future [34]. It is important to recognize that such a relativistic

quantum gravitational effect, like all other gravitational effects, is subject to the whole mass-energy content of the universe. It follows that our treatment has been based on certain cosmological assumptions regarding the distribution of angular momentum in the universe; specifically, we have assumed that on the largest scales there is no preferred sense of rotation. Moreover, in δH the spin-gravity coupling term has a gradient. Therefore, there exists a gravitomagnetic Stern-Gerlach force $-\nabla(\boldsymbol{\Omega}_P \cdot \mathbf{S})$ on a spinning particle that is independent of its mass and hence violates the universality of the gravitational acceleration. The weight of a body thus depends on its spin, but the effect is too small to be directly measurable in the foreseeable future. It is interesting to note that the Stern-Gerlach force has an exact analogue in the classical Mathisson-Papapetrou spin-curvature force. The results of this section are in agreement with the consequences of Dirac-type wave equations in the gravitational field of a rotating mass [35].

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