Chapter 7
THERMAL NOISE

When we discussed the fundamental limits to the sensitivity of an interferometric measuring system, we were led to consider the inherent discreteness of light, and thus to touch on the quantum mechanical view of the world. An equally fundamental limit exists on the degree to which a test mass can remain at rest. This is the phenomenon usually called *thermal noise,* a generalization of Brownian motion. In a certain sense, this noise is also due to the discreteness of the world, but here it is the simple graininess of extended objects and their environment (due to the existence of atoms) that is responsible. Thus, as a thermodynamic phenomenon its magnitude depends on Boltzmann's constant k_B, instead of on Planck's constant $\hbar$. (Its proportionality to k_BT is what gives the effect the name "thermal noise".)

Thermal noise is a prototype of the kind of effect we referred to in the previous chapter as a *displacement noise,* which is characterized by a certain amplitude of motion of each test mass. This class of noise sources plays a crucial role in interferometer design, since (as we saw at the end of the previous chapter) it will set the limits to the degree to which an interferometer can be made compact by folding the optical path in its arms. The case of thermal noise is a good pedagogical example, but it isn't just that. As we will see, it will most likely determine the limiting sensitivity of gravitational wave interferometers in a frequency band centered on 100 Hz or so.

7.1 Brownian Motion

The classic form of thermal noise was discovered by the microscopist Robert Brown around 1828.[92] He observed a ceaseless jiggling motion of small grains of dust and pollen suspended in water. The motion was reminiscent of the activity of microorganisms, but was seen in grains made of any sort of material, whether organic or inorganic. (For some reason, Brown thought it important that the motion even was seen in dust made from a part of the Great Sphinx.) Brown's hypothesis was that the motion resulted from the action of a universal "vital force". A readable and illuminating account of Brownian motion and noise in general may be found in D. K. C. MacDonald's *Noise and Fluctuations: An Introduction.*[93]

The true source of Brownian motion was not understood until Einstein showed how it arose from fluctuations in the rate of impacts of individual water molecules on a grain. Recognizing that molecular impacts were also at the root of an explanation of the dissipation of a grain's kinetic energy as it moved through a fluid, Einstein showed that the mean-square displacement of a particle is

$$\overline{x^2_{therm}} = k_B T \frac{1}{3\pi a \eta} \tau, \tag{7.1}$$

where τ is the duration of the observation, a is the radius of a spherical grain, and η is the viscosity of the fluid in which the grain is suspended. This was the first of many links between a *fluctuation* phenomenon, the random displacement of the particle, and a mechanism for *dissipation,* the viscosity of the water.

It is easy to overlook how sensational the understanding of Brownian motion was at the time of its elucidation. It embodied the clearest reason yet to believe in the actual physical existence of atoms, which might previously have been thought to be only hypothetical entities. This is, among other reasons, because the linkage between molecular impacts and Brownian motion allowed one to determine the value of Avogadro's number, and thus the mass of an individual atom. (There is no property like weight to endow an object with physical significance.) The many ramifications of this discovery are discussed in a beautiful set of articles by A. Perrin, reprinted as a book called *Atoms*, first published in 1913.[94] Einstein's papers on Brownian motion are also available collected in a single volume.[95]

7.2 Brownian Motion of a Macroscopic Mass Suspended in a Dilute Gas

It is instructive to examine an example of Brownian motion in a case where the physics is more directly relevant to conditions in a gravitational wave interferometer. Let's model an interferometer test mass as a rectangular plate of mass m and cross-sectional area A. (See Figure 7.1.) We'll hang the plate as a pendulum from a long wire, with resonant frequency f_0.

The space around the plate is filled with a dilute gas of pressure $p = nk_BT$, where n is the number density of molecules in the gas. Consider the limit when p is small enough that the mean free path ℓ for gas molecules is large compared to any of the dimensions of the mass. Then we can neglect intermolecular collisions and concentrate only on collisions between the plate and individual molecules.

When the plate is at rest, it will be bombarded by gas molecules. On average, equal numbers arrive from each side. The rate of collisions with molecules arriving from one side is (see an introductory statistical mechanics text, such as Reif[96])

$$\mathcal{N} = \frac{1}{4} n\bar{v}A = nA\sqrt{\frac{k_BT}{2\pi\mu}}, \tag{7.2}$$

where μ is the mass of an individual molecule of the gas. Assume each of the molecules reflect elastically from the surface of the plate. The mean collision rate corresponds to a mean force from one side of just pA, or

$$F_+ = nk_BTA. \tag{7.3}$$

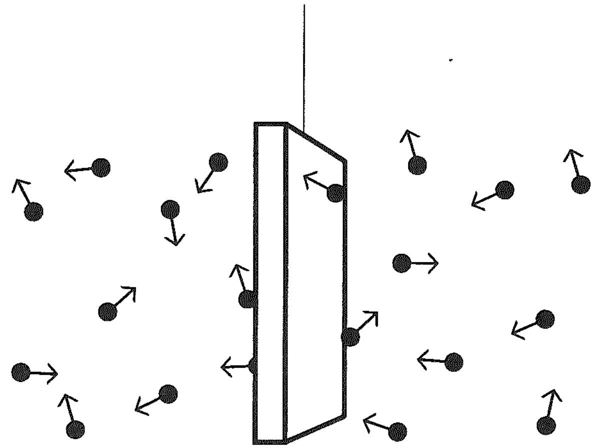

Figure 7.1 A schematic diagram of a macroscopic mass suspended in a dilute gas.

Pressure is the net force due to collisions from many individual molecules. When processes involve counting discrete things that arrive without correlation, we expect the fluctuations to obey Poisson statistics. In an integration time τ, we thus have a fluctuation in the total number $N\tau$ of molecules striking the plate from the left of

$$\frac{\sigma_{N\tau}}{N\tau} = \frac{1}{\sqrt{N\tau}}. \tag{7.4}$$

The fractional fluctuation in the force should be equal (perhaps up to factors of order unity) to this fractional fluctuation in the number of molecular collisions with our mass. Thus, we expect a force noise of order

$$\sigma_F^2 \sim (k_BT)^2 \frac{nA}{\bar{v}\tau}. \tag{7.5}$$

As we saw in our discussion of shot noise, we can translate this into a net force power spectrum of

$$F^2(f) \sim (k_BT)^2 \frac{nA}{\bar{v}}. \tag{7.6}$$

This is almost a useful result all by itself. But we will be rewarded if we examine the model further, to relate the fluctuating force to the frictional force the plate feels when it moves normal to its surface with velocity v_p. The plate feels a force in the direction opposite to its motion, because it receives more "backward momentum" from the molecules it sweeps up in the forward direction than it gets in "forward momentum" from the molecules trying to catch up with it from behind.

The frictional force is [97]

$$F_{fric} = -\frac{1}{4} n A \mu \bar{v} v_p \equiv -b v_p. \tag{7.7}$$

Careful comparison of this expression with Eq. 7.6 above reveals a suggestive relationship of much greater generality that this derivation would lead us to expect. With only a little algebra, we can recast Eq. 7.6 as

$$F^2(f) \sim k_B T b. \tag{7.8}$$

Here we see a basic pattern, that the force fluctuation power spectrum is proportional to the magnitude of the coefficient of dissipation.

7.3 The Fluctuation-Dissipation Theorem

We can imagine carrying out a full microphysical analysis of a number of dissipative systems, to explore how common is this relationship between fluctuation and dissipation. Fortunately, that is not necessary, because the relationship is established clearly by a theorem of great generality. It applies to any system that is linear and in thermodynamic equilibrium.

First, it is useful to make a couple of preliminary definitions. As long as a system is linear, then we can write its equation of motion in the frequency domain in terms of the amplitude of an external force $F_{ext}(f)$ necessary to cause the system to move with a sinusoidal velocity of amplitude $v(f)$. In other words, we can write the equation of motion in the form

$$F_{ext} = Zv. \tag{7.9}$$

Equivalently, we can write

$$v = Y F_{ext}. \tag{7.10}$$

The function $Z(f)$ is called the *impedance,* while $Y(f) \equiv Z^{-1}(f)$ is called the *admittance.*

These terms are useful for writing the most transparent form of the *Fluctuation-Dissipation Theorem,* as derived by H. B. Callen and his co-workers in 1951-2.[98] The theorem states that the power spectrum $F^2_{therm}(f)$ of the minimal fluctuating force on a system is given by

$$F^2_{therm}(f) = 4 k_B T \Re\left(Z(f)\right), \tag{7.11}$$

where $\Re(Z)$ indicates the real (i.e. dissipative) part of the impedance. In an alternative useful form, the power spectrum of the system's fluctuating motion is given directly as

$$x^2_{therm}(f) = \frac{k_B T}{\pi^2 f^2} \Re\left(Y(f)\right). \tag{7.12}$$

Let's make this more concrete by considering as an example our gas-damped pendulum. First, write the equation of motion of the system, neglecting the fluctuating force, as

$$F_{ext} = m\ddot{x} + b\dot{x} + kx. \tag{7.13}$$

(Here, $k = m(2\pi f_0)^2$ is the effective spring constant of the pendulum, not to be confused with Boltzmann's constant k_B.) What we want to do is to treat the mass as if it supplies an "inertia force" resisting acceleration, in a manner analogous to the way a spring resists displacement from its equilibrium length, or to the way a dashpot resists relative motion of its ends. This is appropriate since we will want to be able to apply the method in cases where we treat the entire system as if it were a black box.

We want to find the external force F_{ext} necessary to establish a given velocity v. So the next step is to re-express all of the forces in the frequency domain, and to treat each term as if it were proportional to velocity. That is, $x = v/i2\pi f$, while $\ddot{x} = i2\pi f v$. The impedance $Z \equiv F/v$ is then the sum of the proportionality coefficients, in general a function of frequency. For this particular system, we find

$$Z \equiv \frac{F_{ext}}{v} = b + i2\pi f m - i\frac{k}{2\pi f}. \tag{7.14}$$

In this case we have the simple expression $\Re(Z) = b$ and so

$$F^2_{therm}(f) = 4 k_B T b, \tag{7.15}$$

in line with our expectation, Eq. 7.8, from the kinetic theory analysis. Here, though, the theorem gives us the numerical constant directly.

If we want to find the power spectral density of the motion of the pendulum mass, we can use the definition of the impedance to write

$$v^2(f) = \left(\frac{1}{Z}\right)^2 F^2(f), \tag{7.16}$$

or, more familiarly,

$$x^2_{therm}(f) = \left(\frac{1}{i2\pi f Z}\right)^2 F^2(f). \tag{7.17}$$

A more direct way is to use the second form of Callen's fluctuation-dissipation theorem, Eq.7.12. We calculate the admittance, $Y(f) \equiv Z^{-1}(f)$, as

$$Y(f) = \frac{1}{b + i2\pi f m - ik/2\pi f}, \tag{7.18}$$

or, rationalizing the denominator,

$$Y(f) = \frac{b - i2\pi f m + ik/2\pi f}{b^2 + (2\pi f m - k/2\pi f)^2}. \tag{7.19}$$

Then we can pick out the real part of the admittance directly, and write

$$x^2_{therm}(f) = \frac{k_B T b}{\pi^2 f^2 \left(b^2 + (2\pi f m - k/2\pi f)^2\right)}. \tag{7.20}$$

7.4 Remarks on the Fluctuation-Dissipation Theorem

1. One immediate advantage of this formalism is that one does not need to make a detailed microscopic model of any dissipation phenomenon in order to predict the fluctuation associated with it. (Getting the factors of 2 right in the kinetic theory calculation of gas damping takes a lot of care.) This is especially useful, since it shows how to treat on an equal footing thermal noise from different sources of dissipation, including those from processes more complicated than gas damping. All that is needed is a macroscopic mechanical model specifying the impedance as a function of frequency. Furthermore, when several sources of dissipation act together, one simply includes their combined mechanical effects in one overall expression for the impedance.

2. Study of the papers of Callen *et al.* will reveal the essential unity of a wide variety of fluctuation phenomena. Perhaps the most well-known analog of Brownian motion is *Johnson noise* in a resistor R,

$$v^2(f) = 4k_B T R. \tag{7.21}$$

   This was discovered after the explanation of Brownian motion by Einstein. The fact that we attribute this as a distinct discovery to Johnson,[99] and give credit to Nyquist[100] for its explanation, is a relic of the long history during which we finally came to understand the unity of thermodynamic fluctuation phenomena.

3. The fluctuation-dissipation theorem shows us that the way to reduce the level of thermal noise is to reduce the amount of dissipation, since $x^2_{therm}(f)$ is proportional to b divided by a function of frequency. Yet, it seems to conflict with another of our cherished beliefs from thermodynamics, the *Equipartition Theorem*: every quadratic term in a system's Hamiltonian will have a mean energy of $k_B T/2$. A one-dimensional harmonic oscillator has two such terms: the kinetic energy $mv^2/2$ and the potential energy $kx^2/2$. With respect to the latter term, equipartition requires

$$\overline{x^2} = \frac{k_B T}{k}. \tag{7.22}$$

   This holds without regard to the strength of the dissipation.

   How can these two statements be consistent? Actually, they are not in conflict, because they refer to different aspects of the fluctuation power spectrum. The equipartition theorem refers to the mean square fluctuation, or in other

   words to the integral of the displacement power spectrum. The fluctuation-dissipation theorem describes the power spectral density at each frequency, $x^2_{therm}(f) \propto \Re(Y(f))$. Therein lies the hidden complication. Consider the expression for the admittance of the velocity-damped harmonic oscillator, Eq. 7.19 above. The real part of the numerator is just the damping coefficient b. But the denominator has a richer dependence on b. Far from the resonance, the denominator's magnitude depends almost entirely on either k alone (at low frequencies) or on m alone (at high frequencies). But the shape of the admittance near the resonance is dominated by b; a low value of b means a large resonance peak as the term $(2\pi f m - k/2\pi f)^2$ vanishes.

   So a small value of b means a small thermal noise driving term, and thus a small value of the thermal noise displacement spectral density at frequencies far from the resonant frequency. But in the vicinity of the resonant frequency, that is compensated by a large (resonant) response. The net effect is a displacement power spectrum $x^2(f)$ whose integral over all frequencies is independent of the amount of dissipation. This can be verified by direct integration of the power spectral density.[101]

   Thus we ought to sharpen our statement on the lesson of the fluctuation-dissipation theorem: the way to reduce the thermal noise displacement spectral density away from any resonances is to reduce the amount of dissipation. This gives us a design criterion for the interferometer test masses — they need to be not only nearly free, but also nearly dissipationless.

7.5 The Quality Factor, Q

The quality factor, or Q, of an oscillator is a dimensionless measure of how small the dissipation is at the resonant frequency. It was originally used as a description of electrical resonators by electrical engineers. The definition is[102]

$$Q \equiv \frac{f_0}{\Delta f}, \tag{7.23}$$

where f_0 is the resonant frequency, and Δf is the full width of the resonance peak in the frequency response of the system, measured at the level of half of the maximum power (that is, at $1/\sqrt{2} \approx 0.707$ of maximum amplitude.)

Consider the interpretation of this definition for the mechanical oscillator that we discussed in Chapter 4 as an example of a vibration filter. It has a transfer function

$$G(f) = \frac{k}{k - m(2\pi f)^2 + i2\pi f b}, \tag{7.24}$$

or, making its dimensionless character more obvious,

$$G(f) = \frac{f_0^2}{f_0^2 - f^2 + i f b/2\pi m}. \tag{7.25}$$