The Algebras for Automatic Relations*

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Abstract

We introduce "synchronous algebras", an algebraic structure tailored to recognize automatic relations (*a.k.a.* synchronous relations, or regular relations). They are the equivalent of monoids for regular languages, however they conceptually differ in two points: first, they are typed and second, they are equipped with a dependency relation expressing constraints between elements of different types.

We first show that the three pillars of algebraic language theory hold for synchronous algebras: (a) any relation admits a *syntactic* synchronous algebra recognizing it, and moreover, the relation is synchronous if, and only if, its minimal algebra is finite; (b) classes of synchronous relations with desirable closure properties (called "pseudovarieties") correspond to pseudovarieties of synchronous algebras; and (c) pseudovarieties of synchronous algebras are exactly the classes of synchronous algebras defined by a generalization of profinite equalities called "profinite dependencies".

Building on these results, we show how algebraic characterizations of pseudovarieties of regular languages can be lifted to the pseudovarieties of synchronous relations that they induce. A typical (and running) example of such a pseudovariety is the class of "group relations", defined as the relations recognized by finite-state synchronous permutation automata.

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❧ This pdf contains internal links: clicking on a notion leads to its *definition*.

1 Introduction

1.1 Background

The landscape of rationality for k-ary relations of finite words ($k \geq 2$) is far more complex than for languages,[1] as depicted in Figure 1. Perhaps the most natural class is the one of *rational relations*, *a.k.a.* *rational transductions*, defined as relations accepted by non-deterministic two-tape[2] automata that can move independently its two heads from left to right—see [14, §2.1] for a formal definition.

Our paper focuses on synchronous relations, *a.k.a.* automatic relations or regular relations, defined as the rational relations that can be recognized by synchronous automata, a subclass of the machines described above obtained by keeping a single head that moves synchronously from left to right, reading one pair of letters after the other; we add padding symbols $\_$ at the end of the shorter word—see Figure 2. This class is effectively closed under Boolean operations—see *e.g.* [8, Lemma XI.1.3, p. 627]. Moreover, synchronous relations play a central role in the definitions of automatic structures—introduced by Hodgson [25, 24, 26] and rediscovered by Khoussainov & Nerode [27], see [8, §XI, pp. 627–762]. They also have

[1] Recall that languages can be seen as unary relations of finite words.
[2] An input (u, v) is described by writing u on the first tape and v and the second tape.

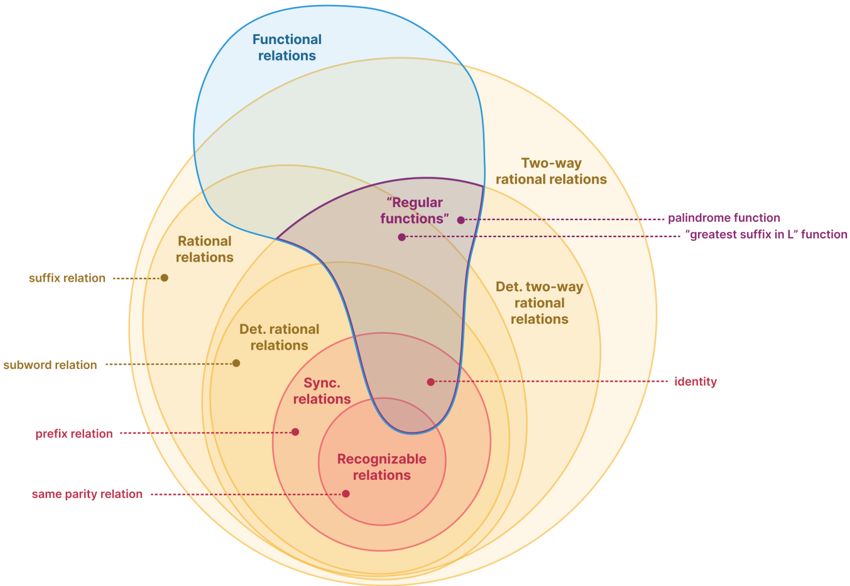

Figure 1 The landscape of rationality for binary relations.

been studied in the context of graph databases [5, Definition 3.1, p.7 & Theorem 6.3, p. 13], see [18, §8, p. 17] for more context & results on *extended* conjunctive regular path queries.

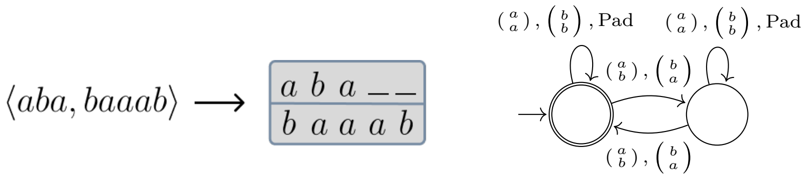

Figure 2 Encoding a pair of words of $\Sigma^* \times \Sigma^*$ into an element of $(\Sigma^2_{\_})^*$ (left) and a deterministic complete synchronous automaton (right) over $\Sigma = \{a, b\}$ accepting the binary relation of pairs (u, v) such that the number of a's in $u_1 \ldots u_k$ and in $v_1 \ldots v_k$ are the same mod 2, where $k = \min(|u|, |v|)$. Pad denotes the set of transitions $\{\binom{a}{\_}, \binom{b}{\_}, \binom{\_}{a}, \binom{\_}{b}\}$.

▶ Remark 1.1. All our results are described for binary relations, but can be extended to k-ary synchronous relations, see Section 7.2.

Synchronous relations stand at the frontier between expressiveness and undecidability: for instance, Carton, Choffrut and Grigorieff showed that it is decidable whether an automatic relation is *recognizable* [14, Proposition 3.9, p. 265], meaning that it can be written as a finite union of Cartesian products of regular languages.[3][4] Moreover, inclusion (and subsequent problems: universality, emptiness, equivalence…) is decidable by reduction to classical automata, contrary to the equivalence problem over rational relations which is undecidable [6, Theorem 8.4, p. 81]. However, some seemingly easy problems are undecidable: Köcher

[3] For instance, the relation "having the same length modulo 2" is recognizable, since it can be written as $(aa)^* \times (aa)^* \cup a(aa)^* \times a(aa)^*$.

[4] The problem was latter shown to be NL-complete and PSPACE-complete depending on whether the input automaton is deterministic or not in [4, Theorem 1, p. 3].

showed that it is undecidable if the (infinite) graph defined by a synchronous relation is 2-colourable—[29, Proposition 6.5, p. 43], and Barceló, Figueira and Morvan showed that undecidability also holds for regular 2-colourability [3, Theorem 4.4, p. 8]. On the other hand, one can decide if said graph contains an infinite clique, see *e.g.* [28, Corollary 5.5, p. 32].

1.2 Contributions

In this paper, we develop an algebraic theory of synchronous relations. The algebraic approach usually provides more than decidability: it attaches canonical algebras to languages/relations (*e.g.* monoids for languages of finite words), and often simple ways to characterize complex properties (*e.g.* first-order definability, see *e.g.* [12, Theorem 2.6, p. 40]). Our synchronous algebras differ from monoids in two points:

- they are typed—a quite common feature in algebraic language theory, shared *e.g.* by ω-semigroups [30, §4.1, p. 91];
- they are equipped with a dependency relation, which expresses constraints between elements of different types—to our knowledge, this feature is entirely novel.[5]

Moreover, these algebras arise from a monad, but to our knowledge none of the meta-theorems developing algebraic language theories over monads apply to it, see Appendix A for more details.

The Three Pillars. We show that synchronous algebras satisfy the following three properties:

- existence of syntactic algebras (Theorem 3.11): each relation $\mathcal{R}$ admits a unique canonical & minimal algebra $\mathbf{A}_{\mathcal{R}}$, which is finite *iff* the relation is synchronous;
- correspondence between classes of finite algebras and classes of synchronous relations (Theorem 4.7)—we assume suitable closure properties; these classes are called "pseudo-varieties";
- pseudovarieties of finite algebras are exactly the classes that can be described by profinite dependencies, a generalization of profinite equalities that takes into account the dependency relation (Theorem 5.12).

While the proof structures follow the classic proofs, see *e.g.* [32], the dependency relation has to be taken into account quite carefully, leading for instance to a surprising definition of residuals, see Definition 4.5.

The Lifting Theorems. One of our motivations to introduce these algebras is the following: given a class $\mathcal{V}$ of languages, we want to characterize the class of relations it induces, called $\mathcal{V}$-relations—see Section 2 for a formal definition. For instance, the relation of Figure 2 is a $\mathcal{V}$-relation where $\mathcal{V}$ is the class of all group languages—these relations can be alternatively described as those recognized by a deterministic complete synchronous automaton whose transitions functions are permutations of states. We show that assuming that $\mathcal{V}$ is a $*$-pseudovariety of regular languages,[6] then the algebraic characterization of $\mathcal{V}$ can be easily lifted to characterize $\mathcal{V}$-relations.

▶ **Theorem 4.2** (*Lifting theorem: Elementary Formulation*)**.** *Given a relation $\mathcal{R}$ and a $*$-pseudovariety of regular languages $\mathcal{V}$ corresponding to a pseudovariety of monoids $\mathbb{V}$, the following are equivalent:*

[5] Note that algebras equipped with binary relations have been studied before, *e.g.* Pin's ordered ω-semigroups—see [31, §2.4, p. 7]—but the constraints (here the orderings) are always defined between elements of the *same type*.

[6] Corresponding to pseudovarieties of finite monoids.

1. *$\mathcal{R}$ is a $\mathcal{V}$-relation,*
2. *$\mathcal{R}$ is recognized by a finite synchronous algebra $\mathbf{A}$ whose underlying monoids are all in $\mathbb{V}$,*
3. *all underlying monoids of $\mathbf{A}_{\mathcal{R}}$ are in $\mathbb{V}$.*

Motivated by the fact that many classes of regular languages are not $*$-pseudovarieties but only $+$-pseudovarieties,[7] we introduce positive synchronous algebras, which are to synchronous algebras as semigroups are to monoids. Surprisingly, we show that the simple characterization above fails if naively generalized to $+$-pseudovarieties, see Example 5.6. Motivated by numerous examples, we then show how algebraic characterizations of $\mathcal{V}$ can be lifted to $\mathcal{V}$-relations using the profinite topology.

▶ **Theorem 6.8** (*Lifting Theorem for Semigroups*)**.** *Given a relation $\mathcal{R} \subseteq (\Sigma^* \times \Sigma^*) \smallsetminus \{(\varepsilon, \varepsilon)\}$ and a $+$-pseudovariety of regular languages $\mathcal{V}$ corresponding to a pseudovariety of semigroups $\mathbb{V}$, letting $\mathcal{E}_{\mathbb{V}}$ denote the set of* all *profinite equalities satisfied by all semigroups of $\mathbb{V}$, then the following are equivalent:*

1. *$\mathcal{R}$ is a $\mathcal{V}$-relation,*
2. *$\mathcal{R}$ is recognized by a finite positive synchronous algebra $\mathbf{A}$ satisfying all profinite dependencies ${\mathcal{E}_{\mathbb{V}}}^{\mathsf{sync}}$ induced by $\mathcal{E}_{\mathbb{V}}$,*
3. *the syntactic positive synchronous algebra of $\mathcal{R}$ satisfies all profinite dependencies of ${\mathcal{E}_{\mathbb{V}}}^{\mathsf{sync}}$.*

In short, induced profinite dependencies are obtained from profinite equalities by guessing the type of variables in a consistent way.

Organization. After giving preliminary results in Section 2, we introduce the synchronous algebras in Section 3 and show the existence of syntactic algebras. We then proceed to prove the lifting theorem for $*$-pseudovarieties in Section 4, and after introducing $*$-pseudovarieties of synchronous relations, we provide a more algebraic reformulation of the lifting theorem (Theorem 4.9). We then proceed to study positive synchronous algebras in Section 5 before explaining the lifting theorem for $+$-pseudovarieties in Section 6. We conclude the paper with a short discussion in Section 7.

1.3 Related Work

The algebraic framework has been extended far beyond languages of finite words: let us cite amongst other for weighted languages, Reutenauer's "algèbre associative syntactique" [35, Théorème I.2.1, p. 451] and their associated Eilenberg theorem [35, Théorème III.1.1, p. 469]; for languages of ω-words, Wilke's algebras and ω-semigroups, see [30, §II, pp. 75–131 & §VI, pp. 265–306]; more generally, for languages over countable linear orderings, see Carton, Colcombet & Puppis' nifty "⊛-monoids" and "⊛-algebras" [15, §3, p. 7]. A systemic approach has been recently developed using monads, see Appendix A. Non-linear structures are also suited to such an approach, see *e.g.* Bojańczyk & Walukiewicz's forest algebras [10, §1.3, p. 4] [12, §5, p. 159], or Engelfriet's hyperedge replacement algebras for graph languages [16, §2.3, p. 100] [11, §6.2, p. 194]. For relations over words (*a.k.a.* transductions), recognizable relations are exactly the ones recognized by monoid morphisms $\Sigma^* \times \Sigma^* \to M$ where M is finite. This can be trivially generalized to show that a relation $\mathcal{R}$ is a finite union of Cartesian products of languages in $\mathcal{V}$ if, and only if, it is recognized by a monoid from $\mathbb{V}$, the pseudovariety of monoids corresponding to $\mathcal{V}$, see Appendix B. For

[7] Corresponding to pseudovarieties of finite semigroups.

bigger classes, the algebraic approach is prone to have less desirable properties than our synchronous algebras—indeed, the existence of syntactic algebras, together with the fact that they are computable implies decidability of the equivalence problem. Yet, in 2023, Bojańczyk & Nguyễn managed to develop an algebraic structure called "transducer semigroups" for "regular functions" [9, Theorem 3.2, p. 6], an orthogonal class of relations to ours—see Figure 1.

The counterpart of $\mathcal{V}$-relations for rational relations—that we call here $\mathcal{V}$-rational relations—was studied by Filiot, Gauwin & Lhote [20]: they show that if $\mathcal{V}$ has decidable membership, then "$\mathcal{V}$-rational transductions" also have decidable membership [20, Theorem 4.10, p. 26]. "Rational transductions" correspond in Figure 1 to the intersection of functional relations with rational relations: hence this class is both orthogonal to synchronous relations and to "regular functions". A different problem—focussing more on the semantics of the transduction—, called "$\mathcal{V}$-continuity" was studied by Cadilhac, Carton & Paperman [13, Theorem 1.3, p. 3], although it has to be noted that their results only concern (1) *functional* rational relations and (2) a finite number of pseudovarieties.

2 Preliminaries

2.1 Automata & Relations

We assume familiarity with basic algebraic language theory over finite words, see [9, §1, 2, 4, pp. 3–66 & pp. 107–156] for a succinct and monad-driven approach, or [32, §I–XIV, pp. 3–247] for a more detailed presentation of the domain. More precise pointers are given in Appendix C.

A *relation* is a subset of $\Sigma^* \times \Sigma^*$—or, starting from Section 5, a subset of $\Sigma^* \times \Sigma^* \smallsetminus \{(\varepsilon, \varepsilon)\}$—where Σ is an alphabet, namely a non-empty finite set. We define its *complement* $\neg\mathcal{R}$ as the relation $\{(u, v) \in \Sigma^* \times \Sigma^* \mid (u, v) \notin \mathcal{R}\}$. Letting $\Sigma^2_{\_}$ as $\Sigma \times \Sigma \cup \Sigma \times \{\_\} \cup \{\_\} \times \Sigma$, a *synchronous automaton* is a finite-state machine with initial states, final states, and non-deterministic transitions labelled by elements of $\Sigma^2_{\_}$. We denote by $\mathsf{WellFormed}_\Sigma$ the set of *well-formed* words over $\Sigma^2_{\_}$ where the padding symbols are placed consistently, namely: if some padding symbol occurs on tape/component, then the following symbols of this tape/component must all be padding symbols. Note that elements of $\mathsf{WellFormed}_\Sigma$ are in natural bijection with $\Sigma^* \times \Sigma^*$—see Figure 2. The relation recognized by a synchronous automaton is the set of pairs $(u, v) \in \Sigma^* \times \Sigma^*$ such that their corresponding element in $\mathsf{WellFormed}_\Sigma$ is the label of an accepting run of the automaton. We say that a relation is *synchronous* if it is recognized by such a machine.

▶ Remark 2.1. Crucially, in the semantics of synchronous automata we *never* try to feed them inputs where the padding symbols are not consistent: for instance, while

$$\binom{aab}{b\_a}, \text{ or } \binom{aba\_}{a\_\_b}$$

are sequences of $(\Sigma^2_{\_})^*$, the behaviour of a synchronous automaton on such sequences are completely disregarded to define the relation they recognize.

▶ **Fact 2.2.** *Given a synchronous automaton, its semantic as a synchronous automaton can be written as the intersection of its semantic as a classical automaton over $\Sigma^2_{\_}$ with $\mathsf{WellFormed}_\Sigma$. In particular relation $\mathcal{R}$ is synchronous if, and only if, it is a regular language when seen as a subset of $(\Sigma^2_{\_})^*$.*

However, note that building a deterministic complete automaton over $\Sigma^2_{\_}$ from a deterministic complete synchronous automaton over Σ, does not preserve many properties of the form "the automaton belongs in some well-behaved class $\mathcal{V}$ of automaton", as we will prove in Example 2.4.

2.2 Induced Relations

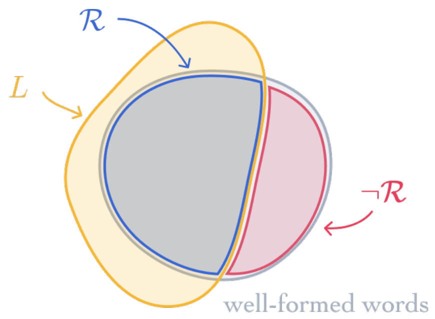

Figure 3 Drawing in $(\Sigma^2_{-})^*$ of a $\mathcal{V}$-relation $\mathcal{R}$ and $\neg\mathcal{R}$, where $\mathcal{R}$ is defined as $L \cap \mathsf{WellFormed}_\Sigma$ with $L \in \mathcal{V}$.

▶ **Question 2.3.** *Given a class $\mathcal{V}$ of languages, can we characterize relations $\mathcal{R}$ that can be written as the intersection of a language from $\mathcal{V}$ and $\mathsf{WellFormed}_\Sigma$?*

Relations as above are called *$\mathcal{V}$-relations*. For instance, if $\mathcal{V}$ is the class of all regular languages, then by Fact 2.2, $\mathcal{V}$-relations are exactly the regular relations, *a.k.a.* synchronous relations! However, because of Remark 2.1, the minimal automaton for a relation, seen as a language over Σ^2_{-}, can be significantly more complex than a deterministic complete synchronous automaton recognizing it, see Figure 4.

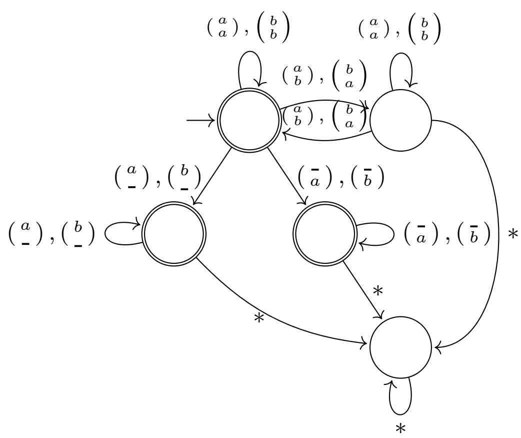

Figure 4 Minimal (deterministic complete) "classical" automaton for the binary relation of pairs (u, v) such that the number of a's in $u_1 \dots u_k$ and in $v_1 \dots v_k$ are the same mod 2, where $k = \min(|u|, |v|)$, seen as a language over Σ^2_{-}. Each label $*$ is defined so that the automaton is deterministic and complete.

Note that if $\mathcal{R}$ belongs to $\mathcal{V}$ (when $\mathcal{R}$ is seen as a language over Σ^2_{-}), then $\mathcal{R}$ is a $\mathcal{V}$-relation. The converse implication is not true in general.

▶ **Example 2.4** (Group relations). If $\mathcal{V}$ is the class of group languages, namely languages recognizing by a permutation automata[8] or equivalently by a finite group, then we call $\mathcal{V}$-relations *group relations*. They can be characterized as relations recognized by permutation synchronous automaton. For instance, the relation of Figure 2 is a group relation as witnessed by the permutation synchronous automaton of Figure 2. Note however that it is not a group language, when seen as a language over Σ_{-}^{2}, since its minimal automaton over Σ_{-}^{2} is not a permutation automaton, see Figure 4 on Page 6.

▶ **Fact 2.5.** *If $\mathcal{V}$ is a class of languages closed under intersection which contains* $\mathsf{WellFormed}_\Sigma$*, then a relation $\mathcal{R}$ is a $\mathcal{V}$-relation if, and only if, it belongs to $\mathcal{V}$ when seen as a language over Σ_{-}^{2}.*

Classes of languages $\mathcal{V}$ satisfying the previous assumption (*e.g.* first-order definable languages, piecewise-testable languages, etc.) are easy to capture when it comes to $\mathcal{V}$-relations since this class reduces to $\mathcal{V}$-languages. So, in the remaining of the paper, we will focus on classes $\mathcal{V}$ which do not satisfy the assumptions of Fact 2.5, such as group languages, or nilpotent semigroups.

▶ **Fact 2.6.** *Given a relation $\mathcal{R}$ and a class $\mathcal{V}$ of languages, the following are equivalent:*
1. *$\mathcal{R}$ is a $\mathcal{V}$-relation;*
2. *$\mathcal{R}$ and $\neg\mathcal{R}$ are $\mathcal{V}$-separable[9] as languages over Σ_{-}^{2}.*

Proof. By definition, see Figure 3, on page 6. ◀

And so, if the $\mathcal{V}$-separability problem is decidable, then the class of $\mathcal{V}$-relations is decidable. However, there are pseudovarieties $\mathcal{V}$ with decidable membership but undecidable separability problem [36, Corollary 1.6, p. 478].[10] Moreover, some of these classes do not contain $\mathsf{WellFormed}_\Sigma$ [36, Corollary 1.7, p. 478]. But beyond this, even when separation algorithm exist, they can be conceptually much harder than their membership counterpart: for instance, deciding membership for group languages is trivial—it boils down to checking if a monoid is a group—, yet the decidability of the separation problem for group languages follows from Ash's infamous type II theorem [2, Theorem 2.1, p. 129], see [23, Theorem 1.1, p. 3] for a presentation of the result in terms of pointlike sets, see also [33, §III, Theorem 8, p. 5] for an elegant automata-theoretic reformulation.

▶ **Example 2.7** (Nilpotent relations). Say that a finite semigroup S is *nilpotent* if it contains a unique idempotent element e, which is a zero—*i.e.* $ex = e = xe$ for all $x \in S$. Equationally, nilpotent semigroups, whose class is denoted by $\mathbf{Nil}$, are defined by the profinite equalities $x^\omega y = x^\omega$ and $yx^\omega = x^\omega$. It is well known [32, Proposition XI.4.14, p. 196 & Proposition XIV.1.18, p. 238] that $\mathbf{Nil}$ corresponds to the $+$-pseudovariety $(co)\mathcal{F}in$ of regular languages which are either finite or cofinite. It is routine to check that $(co)\mathcal{F}in$-relations consists of relations $\mathcal{R}$ which are finite, or such that $\neg\mathcal{R}$ is finite. On the other hand, note that if $\neg\mathcal{R}$ is finite, then $\mathcal{R}$ is not cofinite as a subset of $(\Sigma_{-}^{2})^{+}$, but only as a subset of $\mathsf{WellFormed}_\Sigma$. In particular, in this case, the syntactic semigroup of $\mathcal{R}$, seen as a subset of $(\Sigma_{-}^{2})^{+}$ will not be nilpotent.

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[8] A permutation automata is a finite-state deterministic complete automaton whose transition functions are all permutation of states.
[9] Meaning that there is a language in $\mathcal{V}$ which contains $\mathcal{R}$ and does not intersect $\neg\mathcal{R}$.
[10] The paper cited only claims undecidability of pointlikes, but it was noted in [22, §1, pp. 1–2] that undecidability of the 2-pointlikes also holds, which is a problem equivalent to separability by [1, Proposition 3.4, p. 6].

3 Synchronous Algebras

In this section, we introduce and study the "elementary" properties of synchronous algebras.

3.1 Types & dependent Sets

Motivation. The axiomatization of a semigroup reflects the algebraic structure of finite words: these objects can be concatenated, in a associative way—reflecting the linearity of words. Now observe that elements of $\mathsf{WellFormed}_\Sigma$ are still linear, but not all words can be concatenated together: for instance, $\binom{a}{b}$ cannot follow $\binom{a}{\_}$. Formally, given two words $u, v \in \mathsf{WellFormed}_\Sigma$, to decide if $uv \in \mathsf{WellFormed}_\Sigma$ it is necessary and sufficient to know if the last pair of u and first pair of v consists of a pair of proper letters (denoted by L/L), a pair of a proper letter and a blank/padding symbol (L/B) or a pair of a blank/padding symbol and a proper letter (B/L). This information is called the *letter-type* of an element of $\Sigma^2_{\_}$.

We then define the *type* of a word of $(\Sigma^2_{\_})^+$ as the pair (α, β), usually written $\alpha \to \beta$, of the letter-types of its first and last letters. It is then routine to check that all the possible types of well-formed words are

$$\mathcal{T} \mathrel{\hat{=}} \left\{ \text{L}/\text{L} \to \text{L}/\text{L},\ \text{L}/\text{L} \to \text{L}/\text{B},\ \text{L}/\text{B} \to \text{L}/\text{B},\ \text{L}/\text{L} \to \text{B}/\text{L},\ \text{B}/\text{L} \to \text{B}/\text{L} \right\}.$$

For the sake of readability, we will write α instead of $\alpha \to \alpha$ for $\alpha \in \{\text{L}/\text{L}, \text{L}/\text{B}, \text{B}/\text{L}\}$.

One non-trivial point lies in the following innocuous question: what is the type of the empty word? Any type of $\mathcal{T}$ sounds as an acceptable answer. But then it would be natural to say that the concatenation of $\binom{aaa}{aaa}$ of type L/L with the empty word of type $\text{L}/\text{L} \to \text{L}/\text{B}$ should be $\binom{aaa}{aaa}$ of type $\text{L}/\text{L} \to \text{L}/\text{B}$. Automata-wise, this would represent a sequence of transitions $\binom{a}{a}, \binom{a}{a}, \binom{a}{a}$ together with the promise that the next transition would have a padding symbol on its second tape. But then, one has to formalize the idea that the two elements $\binom{aaa}{aaa}$ of type L/L and $\text{L}/\text{L} \to \text{L}/\text{B}$ represent the same underlying pair of words of $\Sigma^* \times \Sigma^*$: this idea will be captured by what we call a dependency relation.[11]

A *$\mathcal{T}$-typed set* (or *typed set* for short) consists of a tuple $\mathbf{X} = (X_\tau)_{\tau \in \mathcal{T}}$, where each X_τ is a set. Instead of $x \in X_\tau$, we will often write $x_\tau \in \mathbf{X}$. A *map between typed sets* $\mathbf{X}$ and $\mathbf{Y}$ is a collection of functions $X_\tau \to Y_\tau$ for each type τ.

▶ **Definition 3.1.** *A dependency relation over a typed set $\mathbf{X}$ consists of a reflexive and symmetric relation $\asymp$ over $\biguplus \mathbf{X} \mathrel{\hat{=}} \bigcup_{\tau \in \mathcal{T}} X_\tau \times \{\tau\}$, such that:*

- *for all $x_\sigma, y_\tau \in \mathbf{X}$, if $x_\sigma \asymp y_\tau$ and $\sigma = \tau$ then $x_\sigma = y_\tau$, and*
- *the relation is locally transitive: for all $x_\sigma, x'_\sigma, y_\tau, y'_\tau \in \mathbf{A}$, if $x'_\sigma \asymp x_\sigma$, $x_\sigma \asymp y_\tau$ and $y_\tau \asymp y'_\tau$, then $x'_\sigma \asymp y'_\tau$.*[12]

Crucially, we do not ask for this relation to be transitive.[13] *A dependent set is a $\mathcal{T}$-typed set together with a dependency relation over it. A closed subset of a dependent set $\langle \mathbf{X}, \asymp \rangle$ is a subset $C \subseteq \mathbf{X}$ such that for all $x, x' \in \mathbf{X}$, if $x \asymp x'$ then $x \in C \iff x' \in C$.*[14]

[11] Another solution would be to disallow the empty word: this solution is explored in Section 5, but interestingly, while it is possible then to define algebras without a dependency relation, very few classes of relations can be described by the properties satisfied by such algebras. Hence, we will also equip these algebras with a dependency relation, which will yet again play a crucial role in Sections 5 and 6.

[12] In particular, it implies that $\asymp$ is transitive when restricted to elements of the same type.

[13] Many natural examples we will see of dependency relations will be equivalence relations, but this non-transitivity is actually an important feature, motivated amongst other by the syntactic congruence and Corollary 3.14.

[14] In other words, C is a union of equivalence classes of the transitive closure of $\asymp$.

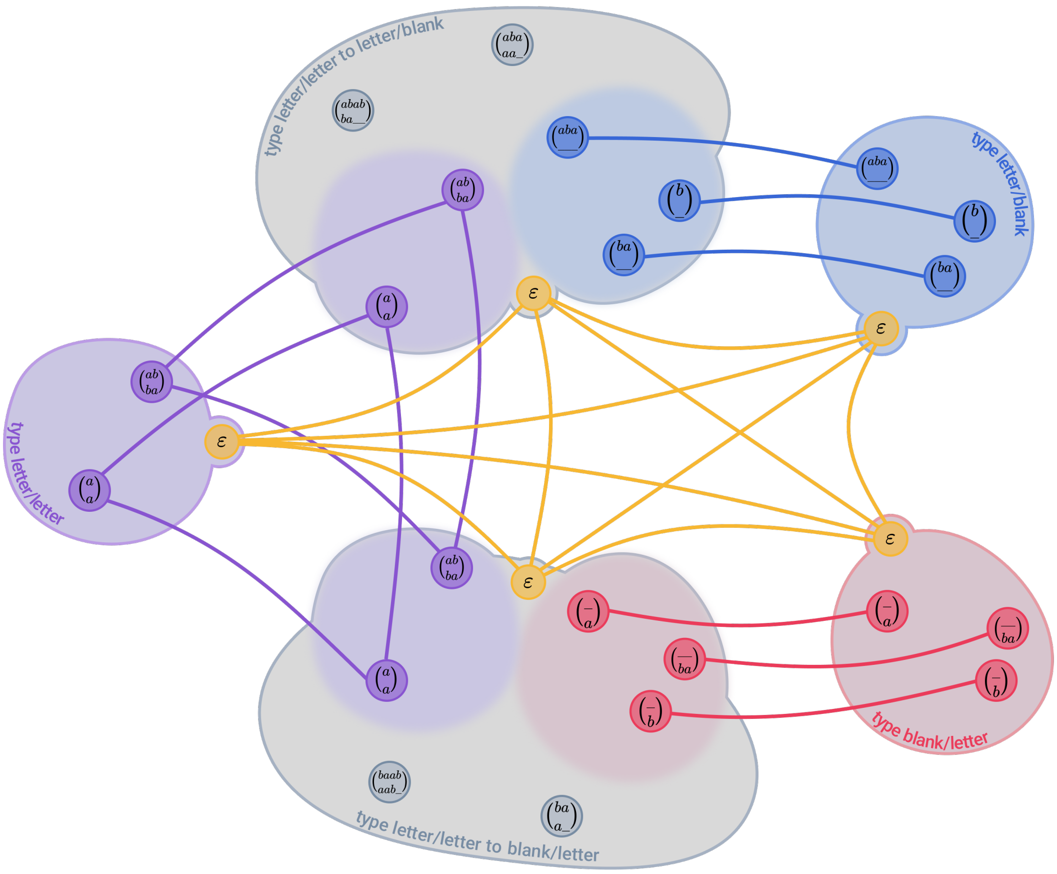

Figure 5 Representation of the dependent set $\mathbf{S}_2\Sigma$ of synchronous words. Coloured edges represent the dependency relation, and self-loops are not drawn.

▶ **Example 3.2.** Given a finite alphabet Σ, let $\mathbf{S}_2\Sigma$ be[15] the dependent set of *synchronous words* defined by:

- $(\mathbf{S}_2\Sigma)_{\text{L/L}} \mathrel{\hat{=}} (\Sigma \times \Sigma)^*$,
- $(\mathbf{S}_2\Sigma)_{\text{L/L}\to\text{L/B}} \mathrel{\hat{=}} (\Sigma \times \Sigma)^*(\Sigma \times \_)^*$,
- $(\mathbf{S}_2\Sigma)_{\text{L/B}} \mathrel{\hat{=}} (\Sigma \times \_)^*$,
- $(\mathbf{S}_2\Sigma)_{\text{L/L}\to\text{B/L}} \mathrel{\hat{=}} (\Sigma \times \Sigma)^*(\_ \times \Sigma)^*$,
- $(\mathbf{S}_2\Sigma)_{\text{B/L}} \mathrel{\hat{=}} (\_ \times \Sigma)^*$,
- $\asymp$ identifies u_σ with u_τ whenever u appears both in $(\mathbf{S}_2\Sigma)_\sigma$ and $(\mathbf{S}_2\Sigma)_\tau$. For instance:

$$\binom{abb}{bab}_{\text{L/L}} \asymp \binom{abb}{bab}_{\text{L/L}\to\text{L/B}} \quad \text{and} \quad \varepsilon_{\text{L/B}} \asymp \varepsilon_{\text{B/L}}.$$

Note that the dependency relation of $\mathbf{S}_2\Sigma$ is an equivalence relation, as depicted in Figure 5.

▶ **Fact 3.3.** *Any relation $\mathcal{R} \subseteq \Sigma^* \times \Sigma^*$ induces a unique closed subset of $\mathbf{S}_2\Sigma$, denoted by $\underline{\mathcal{R}}$.*

In fact this map is also surjective, and this establishes a bijection between relations and closed subsets of $\mathbf{S}_2\Sigma$.

3.2 Synchronous Algebras

One key property of types is that some of them can be concatenated to produce other types. We say that two types $\sigma, \tau \in \mathcal{T}$ are *compatible* when there exists non-empty words $u, v \in \mathsf{WellFormed}_\Sigma$ of type σ and τ, respectively, such that uv is well-formed. Said otherwise, $\alpha \to \beta$ is compatible with $\beta' \to \gamma$ if either $\beta = \beta'$ or $\beta = \text{L/L}$—indeed, for this last case note that *e.g.* the concatenation of $\binom{aaa}{aaa}$ of type L/L with $\binom{\_\_}{aa}$ of type B/L is well-formed. Lastly, if $\alpha \to \beta$ is compatible with $\beta' \to \gamma$, we define their product as $(\alpha \to \beta)\cdot(\beta' \to \gamma) \mathrel{\hat{=}} \alpha \to \gamma$. Note that this partial operation is associative, in the following sense: for $\rho, \sigma, \tau \in \mathcal{T}$, $(\rho\cdot\sigma)\cdot\tau$ is well-defined if and only if $\rho\cdot(\sigma\cdot\tau)$ is well-defined, in which case both types are equal. Note that this implies that the notion of compatibility of types can be unambiguously lifted to finite lists of types.

▶ **Definition 3.4.** *A* synchronous algebra $\langle \mathbf{A}, \cdot, \asymp \rangle$ *consists of a dependent set $\langle \mathbf{A}, \asymp \rangle$ together with a partial binary operation $\cdot$ on $\mathbf{A}$, called* product *such that:*

- *for $x_\sigma, y_\tau \in \mathbf{A}$, $x_\sigma \cdot y_\tau$ is defined iff σ and τ are compatible,*
- associativity: *for all $x_\rho, y_\sigma, z_\tau \in \mathbf{A}$, if ρ, σ, τ are compatible:*

$$(x_\rho \cdot y_\sigma) \cdot z_\tau = x_\rho \cdot (y_\sigma \cdot z_\tau),$$

- compatibility: *for all $x_\sigma, x'_{\sigma'}, y_\tau \in \mathbf{A}$, if $x_\sigma \asymp x'_{\sigma'}$ and both σ, τ and σ', τ are compatible, then $x_\sigma \cdot y_\tau \asymp x'_{\sigma'} \cdot y_\tau$,*
  dually if τ, σ and τ, σ' are compatible, then $y_\tau \cdot x_\sigma \asymp y_\tau \cdot x'_{\sigma'}$,
- units: *for each type τ there is an element $1_\tau \in \mathbf{A}$ such that for any $x_\sigma \in \mathbf{A}$, then $1_\tau \cdot x_\sigma \asymp x_\sigma$ if τ and σ are compatible, and $x_\sigma \cdot 1_\tau \asymp x_\sigma$ if σ and τ are compatible, and moreover, $1_{\text{L/L}\to\beta} = 1_{\text{L/L}} \cdot 1_\beta$ for $\beta \in \{\text{L/B}, \text{B/L}\}$.*

Note in particular that for any type $\tau \in \{\text{L/L}, \text{L/B}, \text{B/L}\}$, then $1_\tau \cdot x_\tau \asymp x_\tau$ but since $1_\tau \cdot x_\tau$ has type τ and $\asymp$ is a dependency relation, then $1_\tau \cdot x_\tau = x_\tau$. This implies in particular that restricting $\langle \mathbf{A}, \cdot \rangle$ to a type L/L, L/B or B/L yields a monoid. These are called the three

[15] The index refers to the arity of the relations we are considering: here we focus on binary relations, but all constructions can be generalized to higher arities.

underlying monoids of $\mathbf{A}$. The canonical example of synchronous algebras is synchronous words $\mathbf{S}_2\Sigma$ under concatenation. Its underlying monoids are $(\Sigma \times \Sigma)^*$, $(\Sigma \times \{\_\})^*$ and $(\{\_\} \times \Sigma)^*$.

▶ **Fact 3.5.** *Any closed subset of* $\mathbf{A}$ *either contains all units, or none of them.*

Note that the product induces a monoid left (resp. right) action of the underlying monoid $\mathbf{A}_{\mathrm{L/L}}$ (resp. $\mathbf{A}_{\mathrm{L/B}}$) on set $\mathbf{A}_{\mathrm{L/L} \to \mathrm{L/B}}$. Moreover, $x_{\mathrm{L/L}} \mapsto x_{\mathrm{L/L}} \cdot 1_{\mathrm{L/L} \to \mathrm{L/B}}$ identifies any element of type $\mathrm{L/L}$ with an element of type $\mathrm{L/L} \to \mathrm{L/B}$. Over $\mathbf{S}_2\Sigma$, these identifications are injective, but it need not be the case in general.

▶ Remark 3.6. There exists a monad over the category of dependent sets whose Eilenberg-Moore algebras exactly correspond to synchronous algebras, see Appendix A.

Morphisms of synchronous algebras are defined naturally as maps that preserve the type, the product and the dependency relation.

Free algebras $\mathbf{S}_2\Sigma$ is free in the sense that for any synchronous algebra $\mathbf{A}$, there is a natural bijection between synchronous algebra morphisms $\mathbf{S}_2\Sigma \to \mathbf{A}$ and maps of typed sets $\Sigma^2_{\_} \to \mathbf{A}$. Said otherwise, synchronous algebra morphisms are uniquely defined by their value on $\Sigma^2_{\_}$.

3.3 Recognizability

Given a synchronous algebra $\mathbf{A}$, a morphism $\varphi \colon \mathbf{S}_2\Sigma \to \mathbf{A}$ and a closed subset $\mathrm{Acc} \subseteq \mathbf{A}$ called "accepting set", we say that $\langle \varphi, \mathbf{A}, \mathrm{Acc} \rangle$ *recognizes* a relation $\mathcal{R}$ when $\underline{\mathcal{R}} = \varphi^{-1}[\mathrm{Acc}]$. We extend the notion of recognizability to $\langle \varphi, \mathbf{A} \rangle$ or to simply $\mathbf{A}$ by existential quantification over the missing elements in the tuple $\langle \varphi, \mathbf{A}, \mathrm{Acc} \rangle$.

Synchronous algebra induced by a monoid A monoid morphism $\varphi \colon (\Sigma^2_{\_})^* \to M$ naturally *induces* a synchronous algebra morphism $\tilde{\varphi} \colon \mathbf{S}_2\Sigma \to \mathbf{A}_M$, where:

- $\mathbf{A}_M$ has for every type τ a copy of M,
- $\asymp$ is $\{(x_\sigma, y_\tau) \mid x = y \text{ in } M\}$,
- for all $x_\sigma, y_\tau \in \mathbf{A}_M$ with compatible type, $x_\sigma \cdot y_\tau \mathrel{\hat{=}} (x \cdot y)_{\sigma \cdot \tau}$,
- $\tilde{\varphi}\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \mathrel{\hat{=}} \left(\varphi\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\right)_{\mathrm{L/L}}$, $\tilde{\varphi}\left(\begin{smallmatrix} a \\ \_ \end{smallmatrix}\right) \mathrel{\hat{=}} \left(\varphi\left(\begin{smallmatrix} a \\ \_ \end{smallmatrix}\right)\right)_{\mathrm{L/B}}$, and $\tilde{\varphi}\left(\begin{smallmatrix} \_ \\ a \end{smallmatrix}\right) \mathrel{\hat{=}} \left(\varphi\left(\begin{smallmatrix} \_ \\ a \end{smallmatrix}\right)\right)_{\mathrm{B/L}}$.

Put simply, the algebra simply duplicates M as many times as needed and identifies two elements together when then originated from the same element of M.

▶ **Fact 3.7.** *If* φ *recognizes* $\mathcal{R}$ *for some relation* $\mathcal{R} \subseteq \Sigma^* \times \Sigma^*$ *(seen as a subset of* $(\Sigma^2_{\_})^*$*), then* $\tilde{\varphi}$ *recognizes* $\mathcal{R}$*.*

Consolidation of a synchronous algebra Given a synchronous algebra morphism $\varphi \colon \mathbf{S}_2\Sigma \to \mathbf{A}$, define its *consolidation*[16] as the semigroup morphism $\varphi^0 \colon (\Sigma^2_{\_})^* \to \mathbf{A}^0$, where:

- $\mathbf{A}^0$ is the semigroup obtained from $\biguplus \mathbf{A}$ by first merging units, by adding a zero (denoted by 0), and extending $\cdot$ to be a total function by letting all missing products equal 0,
- φ^0 sends a word $u \in (\Sigma^2_{\_})^*$ to
  - 0 if u is not well-formed,
  - $\varphi(u_{\mathrm{L/L}})$ if $u \in (\Sigma \times \Sigma)^*$,

[16] Named by analogy with Tilson's construction [40, §3, p. 102].

- $\varphi(u_{\mathrm{L/B}})$ if $u \in (\Sigma \times \_)^+$,
- $\varphi(u_{\mathrm{B/L}})$ if $u \in (\_ \times \Sigma)^+$,
- $\varphi(u_{\mathrm{L/L}\to\mathrm{L/B}})$ if $u \in (\Sigma \times \Sigma)^+(\Sigma \times \_)^+$, and
- $\varphi(u_{\mathrm{L/L}\to\mathrm{B/L}})$ if $u \in (\Sigma \times \Sigma)^+(\_ \times \Sigma)^+$.

Note that this operation disregards the dependency relation of $\mathbf{A}$.

▶ **Fact 3.8.** *If φ recognizes some relation $\underline{\mathcal{R}}$, then φ^0 recognizes $\mathcal{R}$, when seen as a subset of $(\Sigma^2_\_)^*$.*

The following result follows from Facts 2.2, 3.7, and 3.8.

▶ **Proposition 3.9.** *A relation is synchronous if and only if it is recognized by a finite synchronous algebra.*

Let us continue with a slightly less trivial example of algebra.

▶ **Example 3.10** (Group relations: Example 2.4, cont'd.)**.** Fix $p, q \in \mathbb{N}$. Let $\mathbf{Z}_{p,q}$ denote the algebra whose underlying monoids are:

- the trivial monoid $(0, +)$ for type $\mathrm{L/L}$,
- the cyclic monoid $(\mathbb{Z}/p\mathbb{Z}, +)$ for type $\mathrm{L/B}$,
- the cyclic monoid $(\mathbb{Z}/q\mathbb{Z}, +)$ for type $\mathrm{B/L}$.

Moreover, the sets $Z_{\mathrm{L/L}\to\mathrm{L/B}}$ and $Z_{\mathrm{L/L}\to\mathrm{L/B}}$ are defined as $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/q\mathbb{Z}$, respectively, with the natural operations:

- $x_{\mathrm{L/L}} \cdot y_{\mathrm{L/L}\to\beta} \mathrel{\hat{=}} y_{\mathrm{L/L}\to\beta}$,
- $y_{\mathrm{L/L}\to\beta} \cdot z_{\beta\to\beta} \mathrel{\hat{=}} (y+z)_{\mathrm{L/L}\to\beta}$,
- $x_{\mathrm{L/L}} \cdot z_{\beta\to\beta} \mathrel{\hat{=}} z_{\mathrm{L/L}\to\beta}$.

for all $\beta \in \{\mathrm{L/B}, \mathrm{B/L}\}$ and all $x_{\mathrm{L/L}}, y_{\mathrm{L/L}\to\beta}, z_{\beta\to\beta} \in \mathbf{Z}_{p,q}$. The dependency relation only identifies x_σ with $1_\tau \cdot x_\sigma$ and $x_\sigma \cdot 1_\tau$ when the types are compatible, and nothing else.

Let $\varphi\colon \mathbf{S}_2\Sigma \to \mathbf{Z}_{p,q}$ be the synchronous algebra morphism defined by

$$\varphi\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \mathrel{\hat{=}} \bar{0}_{\mathrm{L/L}}, \quad \varphi\left(\begin{smallmatrix} a \\ \_ \end{smallmatrix}\right) \mathrel{\hat{=}} \bar{1}_{\mathrm{L/B}} \quad \text{and} \quad \varphi\left(\begin{smallmatrix} \_ \\ a \end{smallmatrix}\right) \mathrel{\hat{=}} \bar{1}_{\mathrm{B/L}}.$$

This morphism recognizes any relation of the form

$$\begin{aligned} \mathcal{R}^{I,J} \mathrel{\hat{=}} \big\{(u,v) \;\big|\; & |u| > |v| \text{ and } (|u| - |v| \bmod p) \in I, \text{ or} \\ & |u| < |v| \text{ and } (|v| - |u| \bmod q) \in J. \quad \big\}, \end{aligned}$$

where $I \subseteq \mathbb{Z}/p\mathbb{Z}$ and $J \subseteq \mathbb{Z}/q\mathbb{Z}$ are such that $\bar{0} \notin I$ and $\bar{0} \notin J$. This last condition is necessary because the accepting set has to be a closed subset of $\mathbf{Z}_{p,q}$: if $\bar{0}$ was in I, then we would need $\bar{0} \in J$, but also to add $\bar{0}_{\mathrm{L/L}}$ to the accepting set: this would recognize

$$\begin{aligned} \big\{(u,v) \;\big|\; & |u| > |v| \text{ and } (|u| - |v| \bmod p) \in I, \text{ or} \\ & |u| < |v| \text{ and } (|v| - |u| \bmod q) \in J, \text{ or} \\ & \qquad\qquad\qquad\qquad |u| = |v| \quad \big\}. \end{aligned}$$

Note also that all relations $\mathcal{R}^{I,J}$ with $\bar{0} \notin I$ and $\bar{0} \notin J$ are group relations: letting G be the group $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$, $\mathcal{R}$ can be written as $\mathsf{WellFormed}_\Sigma \cap \psi^{-1}[I \times \{0\} \cup \{0\} \times J]$ where $\psi\colon (\Sigma^2_\_)^* \to G$ is the monoid morphism defined by $\psi\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \mathrel{\hat{=}} (\bar{0}, \bar{0})$, $\psi\left(\begin{smallmatrix} a \\ \_ \end{smallmatrix}\right) \mathrel{\hat{=}} (\bar{1}, \bar{0})$ and $\psi\left(\begin{smallmatrix} \_ \\ a \end{smallmatrix}\right) \mathrel{\hat{=}} (\bar{0}, \bar{1})$.

3.4 Syntactic Morphisms & Algebras

▶ **Theorem 3.11** (*Syntactic morphism theorem*). *For each relation $\mathcal{R}$, there exists a surjective synchronous algebra morphism*

$$\eta_{\mathcal{R}}\colon \mathbf{S}_2\Sigma \twoheadrightarrow \mathbf{A}_{\mathcal{R}}$$

which recognizes $\mathcal{R}$ and is such that for any other surjective synchronous algebra morphism $\varphi\colon \mathbf{S}_2\Sigma \twoheadrightarrow \mathbf{B}$ recognizing $\mathcal{R}$, there exists a synchronous algebra morphism $\psi\colon \mathbf{B} \twoheadrightarrow \mathbf{A}_{\mathcal{R}}$ such that the diagram

$$\begin{tikzcd} \mathbf{S}_2\Sigma \arrow[r, two heads, "\eta_{\mathcal{R}}"] \arrow[dr, two heads, "\varphi"'] & \mathbf{A}_{\mathcal{R}} \\ & \mathbf{B}, \arrow[u, two heads, "\psi"'] \end{tikzcd}$$

commutes. Objects $\eta_{\mathcal{R}}$ and $\mathbf{A}_{\mathcal{R}}$ are called the syntactic synchronous algebra morphism *and* syntactic synchronous algebra *of $\mathcal{R}$, respectively. Moreover, these objects are unique up to isomorphisms of the algebra.*

▶ **Corollary 3.12** (of Proposition 3.9 and Theorem 3.11). *A relation is synchronous if and only if its syntactic synchronous algebra is finite.*

The proof Theorem 3.11—see Appendix D.1—relies, as in the case of monoids, on notion of congruence.

Given a synchronous algebra $\langle \mathbf{A}, \asymp, \cdot \rangle$, a *congruence* is any reflexive, symmetric and locally transitive relation $\Bumpeq$ over $\mathbf{A}$ that is coarser than $\asymp$.

The *quotient structure* $\mathbf{A}/\Bumpeq$ of $\mathbf{A}$ by a congruence $\Bumpeq$ is defined as follows:

- its underlying typed set consists of the equivalence classes of $\mathbf{A}$ under the equivalence relation $\{(x_\sigma, y_\sigma) \mid x_\sigma \Bumpeq y_\sigma\}$, such a class being abusively denoted by $[x]^{\Bumpeq}$,
- its product is the product induced by $\mathbf{A}$, in the sense that $[x] \cdot [y] \triangleq [xy]$, and
- its dependency relation is the relation induced by $\Bumpeq$, *i.e.* $[x] \asymp [y]$ whenever $x \Bumpeq y$,
- its units are defined as the equivalence classes of the units of $\mathbf{A}$.

Moreover, $x \mapsto [x]$ defines a surjective morphism of synchronous algebras from $\mathbf{A}$ to $\mathbf{A}/\Bumpeq$.

Given a synchronous algebra $\langle \mathbf{A}, \asymp, \cdot \rangle$ and a closed subset $C \subseteq \mathbf{A}$, we define a congruence $\Bumpeq_C$, called *syntactic congruence* of C over $\mathbf{A}$ by letting $a_\sigma \Bumpeq_C b_\tau$ when for all $x, y \in \mathbf{A}$, if both $xa_\sigma y$ and $xb_\tau y$ are defined, then $xa_\sigma y \in C$ *iff* $xb_\tau y \in C$.[17] It is routine to check that the syntactic congruence is indeed a congruence. Note however that while the relation is locally transitive, it is not transitive in general, which explains Definition 3.1.

In fact more precisely, the construction done on the free synchronous algebra $\mathbf{S}_2\Sigma$ can be trivially generalized—which justifies the terminology "syntactic congruence".

▶ **Proposition 3.13.** *Let $\varphi\colon \mathbf{S}_2\Sigma \to \mathbf{A}$ be a synchronous algebra morphism, that recognizes $\mathcal{R}$, say $\underline{\mathcal{R}} = \varphi^{-1}[Acc]$, then*

$$\begin{array}{rccc} \varphi/\Bumpeq_{Acc}\colon & \mathbf{S}_2\Sigma & \twoheadrightarrow & \mathbf{A}/\Bumpeq_{Acc} \\ & u & \mapsto & [\varphi(u)]^{\Bumpeq_{Acc}} \end{array}$$

is the syntactic morphism of $\mathcal{R}$.

[17] Starting from now, we will sometimes omit the type of *some* elements. When this happens, we do not make any assumption on the hidden types.

▶ **Corollary 3.14.** *In the syntactic synchronous algebra $\mathbf{A}_{\mathcal{R}}$, the syntactic congruence $\asymp_{\mathrm{Acc}}$ and the dependency relation $\asymp$ coincide.*

Proof. By Proposition 3.13 applied to the syntactic morphism, $\mathbf{A}_{\mathcal{R}}/{\asymp_{\mathrm{Acc}}}$ is $\mathbf{A}_{\mathcal{R}}$ itself. Since the dependency relation of $\mathbf{A}_{\mathcal{R}}/{\asymp_{\mathrm{Acc}}}$ is, by definition, $\asymp_{\mathrm{Acc}}$ this shows that $\asymp_{\mathrm{Acc}}$ and $\asymp$ are the same relation. ◀

3.5 Operations on Synchronous Algebras

Boolean operations Given two synchronous algebras $\mathbf{A}$ and $\mathbf{B}$, define their *Cartesian product* $\mathbf{A} \times \mathbf{B}$ by taking, for each type τ, the Cartesian product $A_\tau \times B_\tau$. Units, product and dependency are defined naturally. Then $\neg\mathcal{R}$ is recognized by $\mathbf{A}$, and $\mathcal{R} \cup \mathcal{S}$ and $\mathcal{R} \cap \mathcal{S}$ are recognized by $\mathbf{A} \times \mathbf{B}$.

Composition One particularity of relations, compared with languages, is that they can be composed. We introduce in Appendix D.2 an operation on synchronous algebras that simulates this operation.

4 The Lifting Theorem & Pseudovarieties

4.1 Elementary Formulation

▶ **Example 4.1** (Group relations: Example 3.10 cont'd)**.** We want to decide when the relation

$$\begin{aligned}\mathcal{R}_{I,J} \mathrel{\hat=} \big\{(u,v) \;\big|\; & u > v \text{ and } (|u| - |v| \bmod p) \in I, \text{ or} \\ & u < v \text{ and } (|v| - |u| \bmod q) \in J. \quad \big\}\end{aligned}$$

from Example 3.10 is a group relation. By definition this happens if and only if there exists a finite group G, together with a monoid morphism $\varphi\colon (\Sigma^2_{\_})^* \to G$ and a subset $\mathrm{Acc} \subseteq G$ *s.t.* $\forall u \in \mathsf{WellFormed}_\Sigma$, $u \in \mathcal{R}^{I,J}$ *iff* $\varphi(u) \in \mathrm{Acc}$. We claim:

$$\mathcal{R}_{I,J} \text{ is a group relation} \quad \textit{iff} \quad \big(\bar 0 \notin I \text{ and } \bar 0 \notin J\big). \tag{✻}$$

We prove the implication from left to right: let n be the order of G so that $x^n = 1$ for all $x \in G$. In particular, we have: $\varphi\big((\begin{smallmatrix} a \\ \_ \end{smallmatrix})^{pqn}\big) = 1 = \varphi\big((\begin{smallmatrix} a \\ a \end{smallmatrix})^{pqn}\big)$. Since $\varphi\big((\begin{smallmatrix} a \\ a \end{smallmatrix})^{pqn}\big) \notin \mathcal{R}_{I,J}$, it follows that $(\begin{smallmatrix} a \\ \_ \end{smallmatrix})^{pqn} \notin \mathcal{R}_{I,J}$ *i.e.* $\bar 0 \notin I$. Also, $\bar 0 \notin J$ by symmetry.

Reciprocally, assume that $\bar 0 \notin I$ and $\bar 0 \notin J$. Let $G \mathrel{\hat=} \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$, and $\varphi\colon (\Sigma^2_{\_})^* \to G$ be defined by, for $a, b \in \Sigma$:

$$\varphi\,(\begin{smallmatrix} a \\ b \end{smallmatrix}) \mathrel{\hat=} (\bar 0, \bar 0), \quad \varphi\,(\begin{smallmatrix} a \\ \_ \end{smallmatrix}) \mathrel{\hat=} (\bar 1, \bar 0) \quad \text{and} \quad \varphi\,(\begin{smallmatrix} \_ \\ a \end{smallmatrix}) \mathrel{\hat=} (\bar 0, \bar 1).$$

Then let $\mathrm{Acc} \mathrel{\hat=} I \times \{\bar 0\} \cup \{\bar 0\} \times J$. It follows that $\varphi^{-1}[\mathrm{Acc}] \cap \mathsf{WellFormed}_\Sigma = \mathcal{R}^{I,J}$, which concludes the proof.

Even more generally, we can decide if a relation $\mathcal{R}$ is a group relation by simply looking at the syntactic synchronous algebra of $\mathcal{R}$.

▶ **Theorem 4.2** (*Lifting theorem: Elementary Formulation*)**.** *Given a relation $\mathcal{R}$ and a $*$-pseudovariety of regular languages $\mathcal{V}$ corresponding to a pseudovariety of monoids $\mathbb{V}$, the following are equivalent:*

1. *$\mathcal{R}$ is a $\mathcal{V}$-relation,*
2. *$\mathcal{R}$ is recognized by a finite synchronous algebra $\mathbf{A}$ whose underlying monoids are all in $\mathbb{V}$,*

3. *all underlying monoids of $\mathbf{A}_{\mathcal{R}}$ are in $\mathbb{V}$.*
   See the proof in Appendix E.1.

▶ Remark 4.3. In light of Theorem 4.2, one can wonder whether the notion of synchronous algebra is necessary to characterize $\mathcal{V}$-relation, or if it is enough to look at the languages corresponding to the underlying monoids. Unsurprisingly, synchronous algebras are indeed necessary, as there are relations $\mathcal{R}$ such that:

$$\begin{aligned}&\underline{\mathcal{R}} \cap (\Sigma \times \Sigma)^* \in \mathcal{V}_{\Sigma\times\Sigma},\\ &\underline{\mathcal{R}} \cap (\Sigma \times \_)^* \in \mathcal{V}_{\Sigma\times\_}, \text{ and}\\ &\underline{\mathcal{R}} \cap (\_ \times \Sigma)^* \in \mathcal{V}_{\_\times\Sigma}.\end{aligned} \tag{✵}$$

but $\mathcal{R}$ is *not* a $\mathcal{V}$-relation. Indeed, this happens for instance if $\mathcal{V}$ is the $*$-pseudovariety of all regular languages and

$$\mathcal{R} \mathrel{\hat{=}} \{(u, v) \mid |u| > |v| > 0 \text{ and } |u| - |v| \text{ is prime}\}.$$

Notice that there is a subtle but crucially important difference between (✵) and the second item of the Lifting Theorem: while the underlying monoids of a synchronous algebra $\mathbf{A}$ recognizing $\mathcal{R}$ only accept words of the form $(\Sigma \times \Sigma)^*$, $(\Sigma \times \_)^*$ or $(\_ \times \Sigma)^*$, elements of $(\Sigma\times\Sigma)^+(\Sigma\times\_)^+$ or $(\Sigma\times\Sigma)^+(\_\times\Sigma)^+$ influence the underlying monoids of $\mathbf{A}$ via the axioms of synchronous algebras.

From Theorem 4.2 we obtain a decidability (meta-)result for $\mathcal{V}$-relations.

▶ **Corollary 4.4.** *The class of $\mathcal{V}$-relations has decidable membership iff $\mathcal{V}$ has decidable membership.*

4.2 Pseudovarieties of Synchronous Relations

We introduce the notion of pseudovariety of synchronous algebras and $*$-pseudovariety of synchronous relations. We show an Eilenberg correspondence between these two notions. We then reformulate the Lifting Theorem to show that any Eilenberg correspondence between monoids and regular languages lifts to an Eilenberg correspondence between synchronous algebras and synchronous relations.

Say that a synchronous algebra $\mathbf{A}$ is a *quotient* of $\mathbf{B}$ when there exists a surjective synchronous algebra morphism from $\mathbf{B}$ to $\mathbf{A}$. A *subalgebra* of $\mathbf{B}$ is any closed subset of $\mathbf{B}$ closed under product and containing the units. We then say that synchronous algebra $\mathbf{A}$ *divides* $\mathbf{B}$ when $\mathbf{A}$ is a quotient of a subalgebra of $\mathbf{B}$.

Observe that $\mathbf{S}_2\Sigma$ admits the following property: elements of type $\mathrm{L}/\mathrm{L} \to \mathrm{L}/\mathrm{B}$ and $\mathrm{L}/\mathrm{L} \to \mathrm{B}/\mathrm{L}$ are generated by the underlying monoids. Since syntactic synchronous algebras are homomorphic images of $\mathbf{S}_2\Sigma$, they also satisfy this property. In general, we say that a synchronous algebra $\mathbf{A}$ is *locally generated* if every element of type $\mathrm{L}/\mathrm{L} \to \mathrm{L}/\mathrm{B}$ (resp. $\mathrm{L}/\mathrm{L} \to \mathrm{B}/\mathrm{L}$) can be written as the product of an element of type L/L with an element of type L/B (resp. B/L)

A *pseudovariety of synchronous algebras* is any class $\mathbb{V}$ of locally generated finite synchronous algebras closed under

- *finite product:* if $\mathbf{A}, \mathbf{B} \in \mathbb{V}$ then $\mathbf{A} \times \mathbf{B} \in \mathbb{V}$,
- *division:* if $\mathbf{A}$ divides $\mathbf{B}$ and $\mathbf{B} \in \mathbb{V}$, then $\mathbf{A} \in \mathbb{V}$.

Because of Theorem 3.11, a synchronous relation is recognized by a finite synchronous algebra of a pseudovariety $\mathbb{V}$ *iff* its syntactic synchronous algebra belongs to $\mathbb{V}$.

A *$*$-pseudovariety of synchronous relations* is a function $\mathcal{V}\colon \Sigma \mapsto \mathcal{V}_\Sigma$ such that for any finite alphabet Σ, $\mathcal{V}_\Sigma$ is a set of synchronous relations over Σ such that $\mathcal{V}$ is closed under

- *Boolean combinations:* if $\mathcal{R}, \mathcal{S} \in \mathcal{V}_\Sigma$, then $\neg\mathcal{R}$, $\mathcal{R} \cup \mathcal{S}$ and $\mathcal{R} \cap \mathcal{S}$ belong to $\mathcal{V}_\Sigma$ too,
- *Syntactic derivatives:* if $\mathcal{R} \in \mathcal{V}_\Sigma$, then any relation recognized by the syntactic synchronous algebra morphism of $\mathcal{R}$ also belongs to $\mathcal{V}_\Sigma$.
- *Inverse morphisms:* if $\varphi\colon \mathbf{S}_2\Gamma \to \mathbf{S}_2\Sigma$ is a synchronous algebra morphism and $\mathcal{R} \in \mathcal{V}_\Sigma$ then $\varphi^{-1}[\mathcal{R}] \in \mathcal{V}_\Gamma$.

To recover a more traditional definition (of the form "closure under Boolean operations, residuals[18] and inverse morphisms"), we need to properly define what are the residuals of a relation. It turns out that the answer is quite surprising and less trivial than what one would expect.

▶ **Definition 4.5** (Residuals). *Let $\mathbf{A}$ be a synchronous algebra, $x_\sigma \in \mathbf{A}$, and $C \subseteq \mathbf{A}$ be a closed subset. The* left residual *and* right residual *of C by x_σ are defined by*

$$x_\sigma^{-1}C \triangleq \{y_\tau \in \mathbf{A} \mid \exists y'_{\tau'} \asymp_C y_\tau,\ x_\sigma y'_{\tau'} \in C\}, \text{ and}$$
$$Cx_\sigma^{-1} \triangleq \{y_\tau \in \mathbf{A} \mid \exists y'_{\tau'} \asymp_C y_\tau,\ y'_{\tau'} x_\sigma \in C\},$$

respectively. We refer indiscriminately to both these notions as residuals*. We extend these notions to sets, by letting $X^{-1}C \triangleq \bigcup_{x\in X} x^{-1}C$ and $CX^{-1} \triangleq \bigcup_{x \in X} Cx^{-1}$.*

For the sake of readability, we will sometimes drop the type of elements when dealing with residuals. It is routine to check that residuals are always closed subsets (since $\asymp_C$ is coarser than the dependency relation), or that $(x^{-1}C)y^{-1} = x^{-1}(Cy^{-1})$. As an example, consider the algebra $\mathbf{S}_2 a$ and the relation $\mathcal{R} = (aa)^* \times a(aa)^*$. Then

$$\underline{\mathcal{R}}\left(\begin{smallmatrix} a \\ a \end{smallmatrix}\right)^{-1} = \underline{a(aa)^* \times (aa)^*}.$$

Interestingly, the naive definition of this residual, namely

$$\{y_\tau \in \mathbf{S}_2 a \mid y_\tau \cdot \left(\begin{smallmatrix} a \\ a \end{smallmatrix}\right) \in \mathcal{R}\}$$

would simply yield the empty set! Indeed, for $y_\tau \cdot \left(\begin{smallmatrix} a \\ a \end{smallmatrix}\right)$ to be well-defined, one needs τ to be ${}^{\mathrm{L}}/_{\mathrm{L}}$, *i.e.* y encodes a pair of two words (u, v) of the same length. But then $(ua, va) \notin \mathcal{R}$.

▶ **Lemma 4.6.** *A class $\mathcal{V}\colon \Sigma \mapsto \mathcal{V}_\Sigma$ is a $*$-pseudovariety of synchronous relations if, and only if, it is closed under Boolean combinations, residuals and inverse morphisms.*

See the proof in Appendix E.2.

Let $\mathbb{V} \to \mathcal{V}$ denote the map (called *correspondence*) that takes a pseudovariety of synchronous algebras and maps it to

$$\mathcal{V}\colon \Sigma \mapsto \{\mathcal{R} \subseteq \Sigma^* \times \Sigma^* \mid \mathbf{A}_\mathcal{R} \in \mathbb{V}\}.$$

Dually, let $\mathcal{V} \to \mathbb{V}$ denote the *correspondence* that takes a $*$-pseudovariety of synchronous relations $\mathcal{V}$ and maps it to the pseudovariety of synchronous algebras generated by all $\mathbf{A}_\mathcal{R}$ for some $\mathcal{R} \in \mathcal{V}_\Sigma$. Here, the *variety generated* by a class C is the smallest pseudovariety containing all algebras of C, or equivalently,[19] the class of all synchronous algebras that divide a finite product of algebras of C.

[18] Also called "quotient" *e.g.* in [32, §III.1.3, p. 39], or "polynomial derivative" in [11, §4, p. 19].
[19] The proof is straightforward, see *e.g.* [32, Proposition XI.1.1, p. 190] for a proof in the context of semigroups.

▶ **Theorem 4.7** (*An Eilenberg theorem for synchronous relations*)**.** *The correspondences $\mathbb{V} \to \mathcal{V}$ and $\mathcal{V} \to \mathbb{V}$ define mutually inverse bijective correspondences between pseudovarieties of synchronous algebras and $*$-pseudovarieties of synchronous relations.*

See the proof in Appendix E.3.

As consequence of Theorem 4.7, if $\mathcal{V}$ is a $*$-pseudovariety of synchronous relations and $\mathbb{V}$ is a pseudovariety of synchronous algebras, we write $\mathcal{V} \leftrightarrow \mathbb{V}$ to mean that either $\mathcal{V} \to \mathbb{V}$ or, equivalently, $\mathbb{V} \to \mathcal{V}$. This relation is called an *Eilenberg-Schützenberger correspondence*.

▶ **Proposition 4.8.** *If $\mathbb{V}$ is a pseudovariety of monoids, then*

$$\mathbb{V}^{sync} \triangleq \{\mathbf{A} \text{ locally generated finite synchronous algebra } \text{s.t. all underlying monoids of } \mathbf{A} \text{ are in } \mathbb{V}\}$$

is a pseudovariety of synchronous algebras. Similarly, if $\mathcal{V}$ is an $*$-pseudovariety of regular languages, then

$$\mathcal{V}^{sync} \colon \Sigma \mapsto \{\mathcal{R} \subseteq \Sigma^* \times \Sigma^* \mid \exists L \in \mathcal{V}_{\Sigma^2}, \mathcal{R} = L \cap \mathsf{WellFormed}_{\Sigma}\}$$

is a $*$-pseudovariety of synchronous relations.

Proof. The first point is straightforward. The second one follows from it and Theorems 4.2 and 4.7. ◀

Finally, Theorem 4.2 can be elegantly rephrased by saying that correspondences between pseudovarieties of monoids and $*$-pseudovarieties of regular languages lift to correspondences between pseudovarieties of synchronous algebras and $*$-pseudovarieties of synchronous relations.

▶ **Theorem 4.9** (*Lifting Theorem: Pseudovariety Formulation*)**.** *Given an Eilenberg correspondence $\mathcal{V} \leftrightarrow \mathbb{V}$ between pseudovarieties of monoids and $*$-pseudovarieties of regular languages, then there is an Eilenberg correspondence $\mathcal{V}^{sync} \leftrightarrow \mathbb{V}^{sync}$ between the pseudovariety of synchronous algebras $\mathbb{V}^{sync}$ and the $*$-pseudovariety of synchronous relations $\mathcal{V}^{sync}$.*

5 Positive Synchronous Algebras

5.1 Axiomatization

The only drawback from Sections 3 and 4 is that it deals with $*$-pseudovarieties of regular languages, but many interesting classes do not satisfy this property but only are $+$-pseudovarieties of regular languages.[20] This includes *e.g.* the class $(co)\mathcal{F}in$. or the class of *locally trivial languages*

$$\mathcal{LI}_{\Sigma} \triangleq \{F\Sigma^*G \cup H \mid F, G, H \subseteq_{\mathsf{finite}} \Sigma^+\}.$$

In this section, we introduce positive synchronous algebras, which are essentially synchronous algebras without the restriction to have units.

[20] These $+$-pseudovarieties correspond to pseudovarieties of semigroups. Crucially, note that pseudovarieties of semigroups and of finite monoids do not only differ by the presence of units: pseudovarieties of semigroups have slightly weaker closure properties.

▶ **Definition 5.1.** *A positive synchronous algebra $\langle \mathbf{A}, \cdot, \asymp \rangle$ consists of a dependent set $\langle \mathbf{A}, \asymp \rangle$ together with a partial binary operation $\cdot$ on $\mathbf{A}$, called product such that:*

- *for $x_\sigma, y_\tau \in \mathbf{A}$, $x_\sigma \cdot y_\tau$ is defined iff σ and τ are compatible,*
- associativity: *for all $x_\rho, y_\sigma, z_\tau \in \mathbf{A}$, if ρ, σ, τ are compatible, then*

$$(x_\rho \cdot y_\sigma) \cdot z_\tau = x_\rho \cdot (y_\sigma \cdot z_\tau).$$

- compatibility: *for all $x_\sigma, x'_{\sigma'}, y_\tau \in \mathbf{A}$, if $x_\sigma \asymp x'_{\sigma'}$ and both σ, τ and σ', τ are compatible, then $x_\sigma \cdot y_\tau \asymp x'_{\sigma'} \cdot y_\tau$;*
  dually if τ, σ and τ, σ' are compatible, then $y_\tau \cdot x_\sigma \asymp y_\tau \cdot x'_{\sigma'}$.

In other words, a positive synchronous algebra is like a synchronous algebra, potentially without units. Note that the restriction of a positive synchronous algebra $\mathbf{A}$ to its elements of type L/L, L/B or B/L yields three semigroups, called the *underlying semigroups* of $\mathbf{A}$. Moreover, any semigroup S *induces* a positive synchronous algebra $\mathbf{A}_S$, obtained by taking one copy of the semigroup for each type, and making all elements issued from the same element of the semigroup dependent. In fact, any semigroup morphism $\varphi\colon (\Sigma^2_{\_})^+ \to S$ *induces* a morphism of positive synchronous algebras $\tilde{\varphi}\colon \mathbf{S}_2^+\Sigma \to \mathbf{A}_S$. *Morphisms of positive synchronous algebras* are defined as maps of typed sets which preserve the type. Note that by definition, in any synchronous algebra, there exists elements which are dependent but not equal, because of the units. This is not true for positive algebras.

Free algebras. Let $\mathbf{S}_2^+\Sigma$ be the dependent set defined by:

- $\mathbf{S}_2^+\Sigma_{\text{L}/\text{L}} \triangleq (\Sigma \times \Sigma)^+$,
- $\mathbf{S}_2^+\Sigma_{\text{L}/\text{L}\to\text{L}/\text{B}} \triangleq (\Sigma \times \Sigma)^+(\Sigma \times \_)^+$,
- $\mathbf{S}_2^+\Sigma_{\text{L}/\text{B}} \triangleq (\Sigma \times \_)^+$,
- $\mathbf{S}_2^+\Sigma_{\text{L}/\text{L}\to\text{B}/\text{L}} \triangleq (\Sigma \times \Sigma)^+(\_ \times \Sigma)^+$,
- $\mathbf{S}_2^+\Sigma_{\text{B}/\text{L}} \triangleq (\_ \times \Sigma)^+$,
- $\asymp$ is the equality relation.

Together with natural concatenation, this defines a positive synchronous algebra, which is moreover free, in the sense that positive synchronous algebra morphisms from $\mathbf{S}_2^+\Sigma \to \mathbf{A}$ are in bijection with maps of typed sets $\Sigma^2_{\_} \to \mathbf{A}$. Every relation $\mathcal{R} \subseteq \Sigma^* \times \Sigma^* \smallsetminus \{(\varepsilon, \varepsilon)\}$ induces a unique closed subset, denoted by $\underline{\mathcal{R}}$, of $\mathbf{S}_2^+\Sigma$.

5.2 Properties of Positive Algebras

The whole setting developed in Section 3 can be easily adapted to deal with positive algebras.

▶ **Proposition 5.2** (Syntactic morphism theorem). *For each relation $\mathcal{R} \subseteq \Sigma^* \times \Sigma^* \smallsetminus \{(\varepsilon, \varepsilon)\}$, there exists a surjective synchronous positive morphism*

$$\eta_{\mathcal{R}}\colon \mathbf{S}_2\Sigma \twoheadrightarrow \mathbf{A}_{\mathcal{R}}$$

which recognizes $\mathcal{R}$ and is such that for any other surjective synchronous positive morphism $\varphi\colon \mathbf{S}_2\Sigma \twoheadrightarrow \mathbf{B}$ recognizing $\mathcal{R}$, there exists a synchronous positive morphism $\psi\colon \mathbf{B} \twoheadrightarrow \mathbf{A}_{\mathcal{R}}$ such that the diagram

$$\begin{tikzcd}
\mathbf{S}_2\Sigma \arrow[r, two heads, "\eta_{\mathcal{R}}"] \arrow[dr, two heads, "\varphi"'] & \mathbf{A}_{\mathcal{R}} \\
& \mathbf{B}, \arrow[u, two heads, "\psi"']
\end{tikzcd}$$

commutes. Objects $\eta_{\mathcal{R}}$ and $\mathbf{A}_{\mathcal{R}}$ are called the syntactic positive synchronous algebra morphism and syntactic positive synchronous algebra of $\mathcal{R}$, respectively. Moreover, these objects are unique up to isomorphisms of the algebra.

Congruences, quotient structures and syntactic congruences are defined in the same way as for synchronous algebras, modulo units, see Section 3.4.

▶ **Proposition 5.3.** *For any positive synchronous algebra morphism $\varphi\colon \mathbf{S}_2^+\Sigma \to \mathbf{A}$ recognizing a relation $\mathcal{R} \subseteq (\Sigma^* \times \Sigma^*) \smallsetminus \{(\varepsilon, \varepsilon)\}$, say $\mathcal{R} = \varphi^{-1}[Acc]$, then*

$$\begin{array}{rccc} \varphi/{\precapprox_{Acc}}\colon & \mathbf{S}_2^+\Sigma & \twoheadrightarrow & \mathbf{A}/{\precapprox_{Acc}} \\ & u & \mapsto & [\varphi(u)]^{\precapprox_{Acc}} \end{array}$$

is the syntactic morphism of $\mathcal{R}$.

▶ **Corollary 5.4.** *In the syntactic positive synchronous algebra $\mathbf{A}_{\mathcal{R}}$, the syntactic congruence $\precapprox_{Acc}$ and the dependency relation $\asymp$ coincide.*

Consolidation of the syntactic positive synchronous algebra. Lastly, note that one key difference between the two settings is that any word $u \in (\Sigma^2_{-})^+$ can be unambiguously identified as a unique word of $(\Sigma^* \times \Sigma^*) \smallsetminus \{(\varepsilon, \varepsilon)\}$. This leads to some nice relationship between the consolidation of the syntactic positive synchronous algebra of $\mathcal{R}$ and its syntactic semigroup (when seen as a language over Σ^2_{-}). In short, the consolidation[21] of the syntactic positive synchronous algebra of $\mathcal{R}$ is the smallest semigroup recognizing $\mathcal{R}$, which can be $\mathcal{T}$-typed. The consolidation of a positive algebra is defined similarly to the consolidation of an algebra. Consider the consolidation $\mathbf{A}^0_{\mathcal{R}}$ of the syntactic positive algebra.

The unique morphism $\mathbf{A}_{\mathcal{R}} \to \mathbf{1}$ induces a semigroup morphism $\pi_{\mathsf{type}}\colon \mathbf{A}^0_{\mathcal{R}} \to \mathbf{1}^0$ where $\mathbf{1}$ is the trivial positive synchronous algebra which has one element of each type on which $\asymp$ is the full relation. Elements of $\mathbf{1}^0$ can be thought as types or zero, and π_{type} assigns to every element of $\mathbf{A}_{\mathcal{R}}$ to its type, and maps $0 \in \mathbf{A}^0_{\mathcal{R}}$ to $0 \in \mathbf{1}^0$. Similarly, there is a map assigning to each word of $(\Sigma^2_{-})^+$ its type or 0. This semigroup morphism is denoted by $\mathrm{type}\colon (\Sigma^2_{-})^+ \to \mathbf{1}^0$. Note in fact that $\mathbf{1}^0$ is the syntactic semigroup of $\mathsf{WellFormed}_\Sigma$.

On the other hand $\eta_{\mathcal{R}}\colon \mathbf{S}_2^+\Sigma \twoheadrightarrow \mathbf{A}_{\mathcal{R}}$ induces a semigroup morphism $\eta^0_{\mathcal{R}}\colon (\mathbf{S}_2^+\Sigma)^0 \twoheadrightarrow \mathbf{A}^0_{\mathcal{R}}$. But then note that there is a canonical surjective morphism $\sigma\colon (\Sigma^2_{-})^+ \to (\mathbf{S}_2^+\Sigma)^0$: it sends a well-formed word to its corresponding element, and a non-well-formed word to 0. Then, by construction $\eta^0_{\mathcal{R}} \circ \sigma\colon (\Sigma^2_{-})^+ \twoheadrightarrow \mathbf{A}^0_{\mathcal{R}}$ recognizes $\mathcal{R}$, seen as a subset of $(\Sigma^2_{-})^+$: we denote this morphism by $\eta^{\mathsf{sgp}}_{\mathcal{R}}$. So, letting

$$\zeta_{\mathcal{R}}\colon (\Sigma^2_{-})^+ \twoheadrightarrow S_{\mathcal{R}}$$

denote the syntactic semigroup morphism of $\mathcal{R}$, by definition, there exists a surjective semigroup morphism $\pi_{\mathsf{val}}\colon \mathbf{A}^0_{\mathcal{R}} \to S_{\mathcal{R}}$, such that $\pi_{\mathsf{val}} \circ \eta^{\mathsf{sgp}}_{\mathcal{R}} = \zeta_{\mathcal{R}}$.

▶ **Proposition 5.5.** *The following diagram commutes:*

$$\begin{tikzcd} & (\Sigma^2_{-})^+ \arrow[dl, "\mathrm{type}"'] \arrow[d, two heads, "\eta^{\mathsf{sgp}}_{\mathcal{R}}"] \arrow[dr, "\zeta_{\mathcal{R}}"] & \\ \mathbf{1}^0 & \mathbf{A}^0_{\mathcal{R}} \arrow[l, two heads, "\pi_{\mathsf{type}}"] \arrow[r, two heads, "\pi_{\mathsf{val}}"'] & S_{\mathcal{R}}. \end{tikzcd}$$

[21] Defined analogously to the consolidation of a synchronous algebra, except that the units are not merged—it would be hard to merge what does not exist.

In other words, every element of $\mathbf{A}^0_{\mathcal{R}}$ can be assigned both a type-ish (namely an element of $\mathbf{1}^0$) and a value (in the syntactic semigroup $S_{\mathcal{R}}$ of $\mathcal{R}$) consistently with the types and values of $(\Sigma^2_{\text{-}})^+$. In fact, we can show that it is the *canonical* object satisfying this property, see Appendix F.1.

5.3 Pseudovarieties & Profinite Dependencies

While many of the elementary properties of synchronous algebras—and monoids and semigroups—remain true on positive synchronous algebras, *e.g.* the existence of syntactic algebras, or the Eilenberg correspondence between varieties of relations and varieties of algebras, the Lifting Theorem fails. For instance, there are relations whose underlying semigroups of their syntactic positive synchronous algebra are all groups, but these relations are not group relations!

▶ **Example 5.6** (Group relations: Example 4.1, cont'd.). Consider the relation $\mathcal{R} \mathrel{\hat{=}} \{(u,v) \mid u > v \text{ and } |u| \equiv |v| \bmod p\}$. Then the underlying semigroups of the syntactic positive synchronous algebra of $\mathcal{R}$ of type ${}^{\text{L}}/_{\text{L}}$, ${}^{\text{L}}/_{\text{B}}$, ${}^{\text{B}}/_{\text{L}}$ are groups—namely $\mathbb{Z}/1\mathbf{Z}$, $\mathbb{Z}/p\mathbf{Z}$ and $\mathbb{Z}/1\mathbf{Z}$, respectively. Yet, $\mathcal{R}$ is not a group relation.

So, in order to still manage to characterize $\mathcal{V}$-relations via their syntactic positive synchronous algebras, we will introduce profinite dependencies—which are the equivalent of "profinite equalities" adapted to our setting—and show a Reiterman-style result—namely that pseudovarieties of positive synchronous algebras are exactly the class of positive synchronous algebras characterized by profinite dependencies. Our results rely on an adaptation of the theory of profinite words to our settings: we follow [32, §X—XIII], and most proof can be caried to our setting with ease.

Pseudovarieties of positive synchronous algebras are defined as classes of locally generated finite positive synchronous algebras closed under Cartesian product and division. A *+-pseudovariety of synchronous relations* is a class $\mathcal{V}\colon \Sigma \mapsto \mathcal{V}_\Sigma$ *s.t.* for any finite alphabet Σ, $\mathcal{V}_\Sigma$ is a set of synchronous relations $\mathcal{R} \subseteq (\Sigma^* \times \Sigma^*) \smallsetminus \{(\varepsilon,\varepsilon)\}$ closed under Boolean combinations, syntactic derivatives and inverse morphisms of free positive synchronous algebras.

▶ Remark 5.7. Note that the last condition is the main difference with $*$-pseudovariety of synchronous relations, since morphisms of positive synchronous algebras

$$\varphi\colon \mathbf{S}_2^+\Gamma \to \mathbf{S}_2^+\Sigma$$

have to be non-erasing, *i.e.* $\varphi(u) \neq \varepsilon$, contrary to morphisms of synchronous algebras from $\mathbf{S}_2\Gamma \to \mathbf{S}_2\Sigma$. This implies that the closure properties imposed to be a $+$-pseudovariety are much weaker than those imposed to be $*$-pseudovariety.

▶ **Lemma 5.8.** *A class $\mathcal{V}\colon \Sigma \mapsto \mathcal{V}_\Sigma$ is a $+$-pseudovariety of synchronous relations if, and only if, it is closed under Boolean combinations, residuals and inverse morphisms of free positive synchronous algebras.*

▶ **Lemma 5.9** (An Eilenberg theorem for positive synchronous relations). *The correspondences $\mathbb{V} \to \mathcal{V}$ and $\mathcal{V} \to \mathbb{V}$ define mutually inverse bijective correspondences between pseudovarieties of positive synchronous algebras and $+$-pseudovarieties of synchronous relations.*

We then adapt a classical result from algebraic language theory, known as Reiterman's theorem [34, Theorem 3.1, p. 4] [32, Theorem XI.3.13, p. 195], that shows that pseudovarieties of semigroups are exactly the classes of finite semigroups defined by "profinite equalities"—*e.g.* commutative semigroups are exactly the semigroups satisfying $xy = yx$, and aperiodic semigroups are those satisfying $x^{\omega+1} = x^\omega$ where $-^\omega$ denotes the idempotent power.

Profinite words. Given a typed set X with only elements of type ${}^{\mathrm{L}}/_{\mathrm{L}}, {}^{\mathrm{L}}/_{\mathrm{B}}, {}^{\mathrm{B}}/_{\mathrm{L}}$,[22] consider the space[23] $\mathbf{S}_2^+ X$ of all words over X equipped with the ultrametric

$$\begin{aligned} \mathbf{d}(u,v) &\mathrel{\hat{=}} 2^{-\mathbf{r}(u,v)} \text{ where} \\ \mathbf{r}(u,v) &\mathrel{\hat{=}} \min \big\{ n \in \mathbb{N} \mid \varphi(u) \neq \varphi(v) \text{ for some positive} \\ &\qquad\qquad \text{synchronous algebra morphism } \varphi \\ &\qquad\qquad \text{whose codomain has at most } n \text{ elements} \big\}. \end{aligned}$$

We then consider *Cauchy sequences* of words over X, namely sequences $(u_n)_{n\in\mathbb{N}} \in (\mathbf{S}_2^+ X)^{\mathbb{N}}$ such that for every $\varepsilon > 0$, there exists $n_\varepsilon > 0$ such that for all $p, q \geq n_\varepsilon$, we have $\mathbf{d}(u_p, u_q) \leq \varepsilon$. Say that two Cauchy sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are equivalent when for every $\varepsilon > 0$, then eventually always $\mathbf{d}(u_n, v_n) \leq \varepsilon$.

Notice that if two words u and v have different type, then $\mathbf{r}(u,v) = |\mathcal{T}|$, *i.e.* it is equal to the size of the trivial positive algebra. In particular, in a Cauchy sequence, all words must eventually all have the same type, and this is how we define the type of a Cauchy sequence.

▶ **Definition 5.10.** Profinite words *over X are defined as the space of Cauchy sequences, up to equivalence, together with the metric induced by $\mathbf{d}$. They form a compact topological space, denoted by $\widehat{\mathbf{S}_2^+} X$. Moreover, when equipped with piecewise concatenation, $\widehat{\mathbf{S}_2^+} X$ is a positive synchronous algebra where the dependency relation coincides with equality.*

Note that, by definition, any morphism of positive synchronous algebras $\varphi\colon \mathbf{S}_2^+ X \to \mathbf{A}$, where $\mathbf{A}$ is finite, can be extended to $\widehat{\varphi}\colon \widehat{\mathbf{S}_2^+} X \to \mathbf{A}$ by letting $\widehat{\varphi}([(u_n)_{n\in\mathbb{N}}])$ be the stationary value of $(\varphi(u_n))_{n\in\mathbb{N}}$. For instance, if x has type $\tau \in \{{}^{\mathrm{L}}/_{\mathrm{L}}, {}^{\mathrm{L}}/_{\mathrm{B}}, {}^{\mathrm{B}}/_{\mathrm{L}}\}$, then $((x_\tau)^{n!})_{n\in\mathbb{N}}$ is a Cauchy sequence, whose equivalence class is denoted by x_τ^ω, and $\widehat{\varphi}(x_\tau^\omega)$ is the unique *idempotent power* of $\varphi(x_\tau)$ in the underlying semigroup of type τ.[24] On the other hand, note that it does not make sense to speak about the idempotent power of an element of type ${}^{\mathrm{L}}/_{\mathrm{L}} \to {}^{\mathrm{L}}/_{\mathrm{B}}$ or ${}^{\mathrm{L}}/_{\mathrm{L}} \to {}^{\mathrm{B}}/_{\mathrm{L}}$.

▶ **Definition 5.11.** *Let X be a finite typed set. A* profinite dependency *over variables X, consists of a pair $(u_\sigma, v_\tau) \in \widehat{\mathbf{S}_2^+} X$ denoted by $u \asymp v$. A positive synchronous algebra $\mathbf{A}$* satisfies *this profinite dependency when for all morphisms[25] $\varphi\colon \mathbf{S}_2^+ X \to \mathbf{A}$, $\widehat{\varphi}(u_\sigma) \asymp^{\mathbf{A}} \widehat{\varphi}(v_\tau)$.*

More generally, if $\mathcal{E}$ is a (potentially infinite) set of profinite dependencies and $\mathbb{V}$ is a class of positive synchronous algebras, we say that $\mathbb{V}$ *satisfies* $\mathcal{E}$ when all $\mathbf{A} \in \mathbb{V}$ satisfy all $(u_\sigma \asymp v_\tau) \in \mathcal{E}$. We say that $\mathbb{V}$ is *defined* by $\mathcal{E}$ when $\mathbb{V}$ is exactly the set $[\![\mathcal{E}]\!]$ of all positive algebras that satisfy $\mathcal{E}$.

▶ **Theorem 5.12** (*A Reiterman theorem for positive synchronous algebras*)**.** *A class of locally generated finite positive synchronous algebras is a pseudovariety if, and only if, it is defined by a set of profinite dependencies.[26] Moreover, w.l.o.g., all variables have type ${}^{\mathrm{L}}/_{\mathrm{L}}$, ${}^{\mathrm{L}}/_{\mathrm{B}}$ or ${}^{\mathrm{B}}/_{\mathrm{L}}$.*

The proof can be adapted from [32, §XI.3.3, pp. 194–196], see Appendix F.2. Note also that a similar statement holds for (non-positive) synchronous algebras.

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[22] This assumption is there because we are only interested in locally generated algebras.
[23] We only gave a definition of this when X had the form $\Sigma_{\perp}^2$ but given a more general definition is quite easy, see Appendix A.
[24] In other words, it is the unique element e of the form x_τ^n with $n \in \mathbb{N}_{>0}$ *s.t.* $e = e \cdot e$. See [32, §II.6, p. 32] for more details.
[25] Which are in bijection with maps of typed sets from X to $\mathbf{A}$, and so it can be thought of a valuation of variables from X with values in $\mathbf{A}$.
[26] Note however that this set can be infinite.

6 The Lifting Theorem for Positive Algebras

6.1 Some Examples

In most of this section, we fix an Eilenberg-Schützenberger correspondence $\mathcal{V} \leftrightarrow \mathbb{V}$ between a $+$-pseudovariety of regular languages and a pseudovariety of semigroups.

In Example 5.6, we showed that not all relations whose syntactic positive synchronous algebra whose underlying semigroups are all groups are necessarily group relations. In fact, it is quite easy to find a necessary and sufficient condition on the syntactic positive algebra to characterize group relations.

▶ **Example 6.1** (Group relations: Example 5.6, cont'd). Let $\mathcal{R} \subseteq (\Sigma^* \times \Sigma^*) \setminus \{(\varepsilon, \varepsilon)\}$. Then $\mathcal{R}$ is a group relation if and only if it is recognized by a finite positive synchronous algebra $\mathbf{A}$ such that:

- every underlying semigroup of $\mathbf{A}$ is a group, and moreover
- for each $1_\sigma, x_\tau \in \mathbf{A}$, $1_\sigma \cdot x_\tau \asymp x_\tau$ if σ, τ are compatible and $x_\tau \cdot 1_\sigma \asymp x_\tau$ if τ, σ are compatible, where 1_σ is the unit of type σ.

The left-to-right implication follows from the construction of the positive synchronous algebra induced by a semigroup. The right-to-left implication follows from the fact that $\mathbf{A}$ must in fact be a synchronous algebra and Theorem 4.2.

Notice how the second item of this condition is an axiom of (non-positive) synchronous algebras.

Interestingly, some $\mathcal{V}$-relations can still be characterized by solely looking at the underlying semigroups of their syntactic positive synchronous algebra.

▶ **Example 6.2** (Commutative relations). Let $\mathcal{R} \subseteq (\Sigma^* \times \Sigma^*) \setminus \{(\varepsilon, \varepsilon)\}$. Define *commutative relations* as the $\mathcal{V}$-relations where $\mathcal{V}$ is the class of commutative languages, corresponding to the pseudovariety of commutative semigroups. Then $\mathcal{R}$ is a commutative relation if and only if it is recognized by a finite positive synchronous algebra $\mathbf{A}$ such that every underlying semigroup of $\mathbf{A}$ is a commutative semigroup. The left-to-right implication is, as always, trivial, by considering the positive synchronous algebra induced by a semigroup. For the converse, let $S_{\mathrm{L/L}}, S_{\mathrm{L/B}}$ and $S_{\mathrm{B/L}}$. Consider the *unit adjunction* operation $-^1$, which consists of taking a semigroup and *always* adding a unit to it, making it into a monoid. Then

$$M \mathrel{\hat{=}} S^1_{\mathrm{L/L}} \times S^1_{\mathrm{L/B}} \times S^1_{\mathrm{B/L}}$$

is a commutative semigroup, and recognizes $\mathcal{R}$ as a subset of $(\Sigma^2_-)^+$, using the same construction as Theorem 4.2. Note that proving this must *crucially* rely on the fact that the unit adjunction adds a new unit to every semigroup (even if the original semigroup already has a unit).

More generally, the construction above can be generalized with ease to any pseudovariety of semigroups closed under unit adjunction. For such classes, $\mathcal{V}$-relations are exactly those recognized by positive synchronous algebras whose underlying semigroups belong to $\mathbb{V}$. Both Examples 6.2 and 5.6 have the particularity to deal with pseudovarieties that particularly interact nicely with units, and hence are not too different from the setting of Theorem 4.2.

A natural question is to then study classes which are intrinsically allergic to units: can a nice characterization of $\mathcal{V}$-relations be found in this case?

▶ **Example 6.3** (Nilpotent relations: Example 2.7, cont'd). A relation $\mathcal{R} \subseteq (\Sigma^* \times \Sigma^*) \setminus \{(\varepsilon, \varepsilon)\}$ is a *(co)$\mathcal{F}in$-relation* if, and only if, it is recognizing by a finite positive synchronous algebra $\mathbf{A}$ such that:

- every underlying semigroup of $\mathbf{A}$ is nilpotent, and moreover
- for each $0_\sigma, x_\tau \in \mathbf{A}$, $0_\sigma \cdot x_\tau \asymp 0_\sigma$ if σ, τ are compatible and $x_\tau \cdot 0_\sigma \asymp 0_\sigma$ if τ, σ are compatible, where 0_σ denotes either the zero of the underlying semigroup of type σ, if $\sigma \in \{\text{L/L}, \text{L/B}, \text{B/L}\}$, or $0_{\text{L/L}} \cdot 0_\beta$ for $\beta \in \{\text{L/B}, \text{B/L}\}$ if $\sigma = \text{L/L} \to \beta$.

To prove the converse implication, one can notice that any accepting set must either contain all zeroes of $\mathbf{A}$, or none of them. In the former case, one can then show that the preimage of Acc is finite, and in the latter set, its complement is finite. The conclusion follows from Example 2.7.

It turns out that the characterizations of Examples 6.1–6.3 can be described in a unified way using profinite dependencies: the idea is to look at some profinite equalities defining $\mathbb{V}$ (for instance $\{x^\omega y = y,\, yx^\omega = y\}$ for groups), and transform them into profinite dependencies by (a) non-deterministically guessing the type of the variables so that both sides of the equation are well-typed, and (b) changing the equality symbol into a dependency symbol. For instance, $x^\omega y = y$ gives rise, among others, to the profinite dependencies $x^\omega_{\text{L/L}} y_{\text{L/L}} \asymp y_{\text{L/L}}$, $x^\omega_{\text{L/L}} y_{\text{L/B}} \asymp y_{\text{L/B}}$ and $x^\omega_{\text{L/L}} y_{\text{L/L}\to\text{L/B}} \asymp y_{\text{L/L}\to\text{L/B}}$. It can then be checked that the characterization of Examples 6.1–6.3 exactly corresponds to the profinite dependencies induced by the profinite equalities $\{x^\omega y = y,\, yx^\omega = y\}$ (for groups), $\{xy = yx\}$ (for commutative semigroups), and $\{x^\omega y = x^\omega,\, yx^\omega = x^\omega\}$ (for nilpotent semigroups), respectively.

6.2 Induced Profinite Dependencies & the Lifting Theorem

We now proceed to give a formal definition of the set of profinite dependencies induced by a profinite equality. Given a set X, and a function $\mathbf{t}\colon X \to \mathcal{T}$ called *typing*, we denote by $X^{\mathbf{t}}$ the typed set $\langle \mathbf{t}^{-1}[\tau] \rangle_{\tau\in\mathcal{T}}$. We say that τ is *consistent* with a word $u \in X^+$ if the list of types $\langle \mathbf{t}(u_i) \rangle_{i\in\text{dom}(u)}$ is compatible. We say that τ is *consistent* with a profinite word $u = (u_n)_{n\in\mathbb{N}}$, if there exists some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\mathbf{t}$ is consistent with word $u_n \in X^+$. Clearly, if $\mathbf{t}$ is consistent with word $u \in X^+$, then the word $u^{\mathbf{t}}$ obtained by replacing every letter x by the associated element in $X^{\mathbf{t}}$ is an element of $\mathbf{S}_2^+ X^{\mathbf{t}}$.

▶ **Fact 6.4.** *If $\mathbf{t}$ is consistent with two words $u, v \in X^+$, then the smallest semigroup separating u and v is upper-bounded, up to an additive constant, by the size of the smallest positive synchronous algebra separating $u^{\mathbf{t}}$ and $v^{\mathbf{t}}$.*[27]

It follows that if $\mathbf{t}$ is consistent with a profinite word u, say $(u_n)_{n\in\mathbb{N}} \in \widehat{X^+}$, then $u^{\mathbf{t}} \triangleq (u_n^{\mathbf{t}})_{n\in\mathbb{N}}$ is Cauchy and hence belongs in $\widehat{\mathbf{S}_2^+ X^{\mathbf{t}}}$.

▶ **Definition 6.5.** *Given a profinite equality $u = v$ over variables X, the profinite dependencies induced by $u = v$ is defined as the set of profinite dependencies*

$$(u = v)^{\textsf{sync}} \triangleq \left\{ u^{\mathbf{t}} \asymp v^{\mathbf{t}} \;\middle|\; \mathbf{t} \text{ is consistent with both } u \text{ and } v \right\}.$$

We extend the notation additively to sets of equations.

Then Example 6.1 can be reformulated as follows: a relation is a group relation if, and only if, it is recognized by a positive synchronous algebra which satisfies all profinite dependencies of $\{x^\omega y = y,\, yx^\omega = y\}^{\textsf{sync}}$.

Remark that the typing $\mathbf{t}\colon X \to \mathcal{T}$ that is always equal to σ, for $\sigma \in \{\text{L/L}, \text{L/B}, \text{B/L}\}$, is always consistent with any profinite word. Also recall that $x_\sigma \asymp y_\sigma$ always implies $x_\sigma = y_\sigma$.

[27] This result can be proven by considering the consolidation.

▶ **Fact 6.6.** *For any set $\mathcal{E}$ of profinite equalities defining $\mathbb{V}$, if a finite positive synchronous algebra $\mathbf{A}$ satisfies all profinite dependencies induced by $\mathcal{E}$, then all underlying semigroups of $\mathbf{A}$ belong to $\mathbb{V}$.*

Note that given a set $\mathcal{E}$ of profinite equalities defining $\mathbb{V}$, the class of all finite locally generated positive synchronous algebras $[\![\mathcal{E}^{\mathsf{sync}}]\!]$—namely the algebras satisfying all profinite dependencies induced by $\mathcal{E}$—is a pseudovariety of positive synchronous algebras as a consequence of Theorem 5.12. In light of the examples presented above, it is quite natural to conjecture that this is the class corresponding to $\mathcal{V}$-relations. We show that, somewhat surprisingly, the choice of $\mathcal{E}$ matters.

▶ **Example 6.7** (Locally trivial relations)**.** The pseudovariety of semigroups corresponding to $\mathcal{LI}$ can be equivalently defined by the profinite equality $x^\omega y x^\omega = x^\omega$, or by $x^\omega y z^\omega = x^\omega z^\omega$, see [32, Proposition XI.4.17, p. 197]. A relation is a $\mathcal{LI}$-relation *iff* it is recognized by a finite positive synchronous algebra satisfying all profinite dependencies induced by $x^\omega y z^\omega = x^\omega z^\omega$. On the other hand, there are relations whose syntactic positive synchronous algebra satisfies all profinite dependencies induced by $x^\omega y x^\omega = x^\omega$, but which are not $\mathcal{LI}$-relations. To intuitively grasp why

$$[\![(x^\omega y x^\omega = x^\omega)^{\mathsf{sync}}]\!] \subsetneq [\![(x^\omega y z^\omega = x^\omega z^\omega)^{\mathsf{sync}}]\!]$$

one can notice that there are much fewer typings consistent with $x^\omega y x^\omega$ than with $x^\omega y z^\omega$.

Examples 6.1–6.3 and 6.7 lead to the following result.

▶ **Theorem 6.8** (*Lifting Theorem for Semigroups*)**.** *Given a relation $\mathcal{R} \subseteq (\Sigma^* \times \Sigma^*) \smallsetminus \{(\varepsilon, \varepsilon)\}$ and a $+$-pseudovariety of regular languages $\mathcal{V}$ corresponding to a pseudovariety of semigroups $\mathbb{V}$, letting $\mathcal{E}_{\mathbb{V}}$ denote the set of* all *profinite equalities satisfied by all semigroups of $\mathbb{V}$, then the following are equivalent:*

1. *$\mathcal{R}$ is a $\mathcal{V}$-relation,*
2. *$\mathcal{R}$ is recognized by a finite positive synchronous algebra $\mathbf{A}$ satisfying all profinite dependencies ${\mathcal{E}_{\mathbb{V}}}^{\mathsf{sync}}$ induced by $\mathcal{E}_{\mathbb{V}}$,*
3. *the syntactic positive synchronous algebra of $\mathcal{R}$ satisfies all profinite dependencies of ${\mathcal{E}_{\mathbb{V}}}^{\mathsf{sync}}$.*

See the proof in Appendix G.

Proof sketch. Implications (1) ⇒ (2) ⇒ (3) are relatively straightforward and similar to the proof of (1) ⇒ (2) ⇒ (3) of Theorem 4.2. The interesting implication is (3) ⇒ (1): we need to show that if the class of all $\mathcal{V}$-relations satisfies some profinite dependency $u \asymp v$, then $\mathcal{V}$ satisfies $u^{\mathbf{f}} = v^{\mathbf{f}}$ where $u^{\mathbf{f}}$ and $v^{\mathbf{f}}$ are obtained from u and v by forgetting the type of variables. Directly proving this is particularly hard because of the definition of residuals. Our proof uses a generalization of profinite equations, namely explicit profinite equivalence introduced by Gehrke, Grigorieff & Pin [21, §8—9]—see Appendix G.1. Our main technical lemma shows that if $\mathcal{V}$-relations satisfy an explicit profinite equivalence $u \leftrightarrow v$, then $\mathcal{V}$ satisfy $u^{\mathbf{f}} \leftrightarrow v^{\mathbf{f}}$. We conclude using an adaptation of Reiterman theorem to streams (a generalisation of pseudovarieties, see Proposition G.3). ◀

Note however Theorem 6.8 does not imply decidability: first, there are uncountably many profinite words, and second, $\mathcal{E}_{\mathbb{V}}$ is infinite.

7 Discussion

7.1 The Lifting Theorem

Observe that the implication (3) ⇒ (1) of Theorem 6.8 is non-constructive: our proof does not provide a way to construct a language $L \in \mathcal{V}_\Sigma$ such that $\mathcal{R} = L \cap \mathsf{WellFormed}_\Sigma$ from $\mathbf{A}_\mathcal{R}$. Moreover, the set of profinite dependencies used to characterize $\mathcal{V}$-relations is infinite.

▶ **Question 7.1.** *Is is true that if $\mathbb{V}$ can be defined by finitely many profinite equalities, then the pseudovariety of positive synchronous algebras defined by $\mathcal{E}_{\mathbb{V}}{}^{\mathsf{sync}}$ can be defined by finitely many profinite dependencies?*

7.2 Beyond Synchronous Relations: Path algebras & the Figueira-Libkin Synchronization Problem

The construction of synchronous algebras can be generalized to any type system defined by a finite graph giving rise to the notion of "path algebras"[28], and that the syntactic morphism theorem, the Eilenberg correspondence theorem and the Reiterman-style characterization of pseudovarieties hold. Similarly, the lifting theorem for monoids can be proven provided that the graph defining the type system is acyclic. In particular, this extends the results of this paper to synchronous relations of arbitrary arity, or to co-synchronous (*a.k.a.* right synchronous) relations.

In [19, §3], Figueira and Libkin introduced a "systematic way of defining classes of relations on words", which generalized the classes of recognizable relations, synchronous relations, co-synchronous relations or rational relations. Any regular language L over $[\![1,k]\!]$ gives rise to a class of relations, called "L-controlled". They showed that, given $L \subseteq \{1,2\}^*$, it is decidable if L-controlled relations correspond exactly to recognizable relations (resp. synchronous relations, resp. rational relations) [17, §, Theorem 1, p. 6]. Descotte, Figueira & Puppis then showed that the problem of given $K, L \subseteq \{1,2\}^*$, it was decidable if all K-controlled relations are also L-controlled relations [17, Corollary 18, p. 12]. The case of k-ary relations for $k > 2$ was left wide open [17, §9, p. 13]: no simple characterization (even without putting any decidability constraint) is known.

For any regular language $L \subseteq [\![1,k]\!]^*$, we claim that we can define a type system $\mathcal{T}_L$ such that finite $\mathcal{T}_L$-path algebras exactly recognize L-controlled relations.

▶ **Question 7.2.** *Is it true that for any regular languages $K, L \subseteq [\![1,k]\!]^*$, K-controlled relations are included in L-controlled relations if, and only if, there is an adjunction from the category of $\mathcal{T}_K$-path algebras to the category of $\mathcal{T}_L$-path algebras?*

[28] In short, they are the adaptation of the free category generated by a graph to dependent sets. See also Appendix A.

A Monads Everywhere!

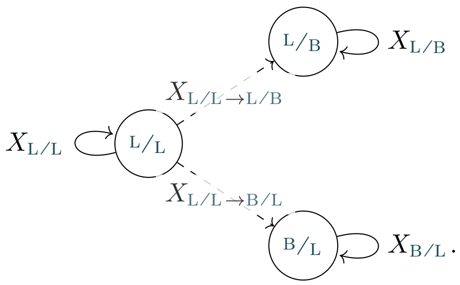
We denote by $\mathsf{Set}^{\mathcal{S}}$ and $\mathsf{Pos}^{\mathcal{S}}$ the category of $\mathcal{S}$-typed sets and $\mathcal{S}$-partially ordered sets—note that in this model, each type is equipped with its own order and that elements of different types cannot be compared. Similarly, let $\mathsf{Dep}^{\mathcal{S}}$ be the category of $\mathcal{S}$-dependent sets.

A.1 The Synchronous Monads

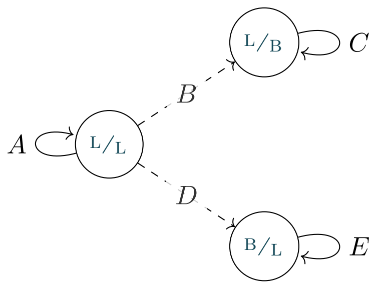
We claim that synchronous algebras correspond to Eilenberg-Moore algebras of some monad over the category $\mathsf{Dep}^{\mathcal{T}}$. For the sake of readability, we represent the underlying typed set of a $\mathcal{T}$-dependent set

$$\mathbf{X} = \langle X_{\mathrm{L/L}}, X_{\mathrm{L/L}\to\mathrm{L/B}}, X_{\mathrm{L/B}}, X_{\mathrm{L/L}\to\mathrm{B/L}}, X_{\mathrm{B/L}} \rangle$$

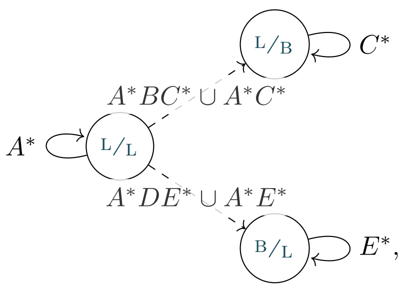
as follows:



We define the *synchronous monad* $\mathbb{S}_2$ over $\mathsf{Dep}^{\mathcal{T}}$ as the functor which maps

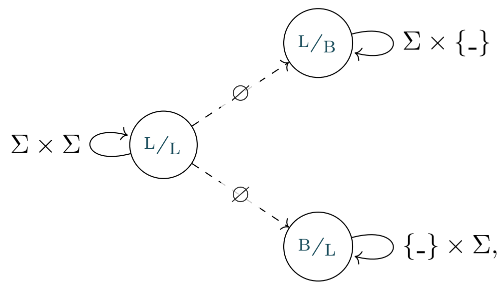


equipped with a dependency relation $\asymp$ to the dependent set



and two words are dependent if their domain are isomorphic and their letters are pairwise dependent. The unit and free multiplication are defined naturally.

Note in particular that all five empty words are mutually dependent, and that synchronous words $\mathbf{S}_2\Sigma$ correspond to applying $\mathbb{S}_2$ to



equipped with equality. Moreover, synchronous algebras exactly correspond to $\mathbb{S}_2$-algebras.

Similarly, positive synchronous algebras correspond to the algebras for the monad of positive synchronous words $\mathbb{S}_2^+$ over the category $\mathsf{Dep}^{\mathcal{T}}$ defined analogously to $\mathbb{S}_2$, but where each $-^*$ is replaced by a $-^+$.

A.2 Where Monadic Meta-Theorems Go to Die

A systemic approach to algebraic language theory was proposed by Bojańczyk using the formalism of monads [11], for monads over finitely typed sets $\mathsf{Set}^{\mathcal{S}}$. Urbat, Adámek, Chen & Milius then extended these results to capture monads over varieties of typed (ordered) algebras [41]. Lastly, Blumensath extended those results to monads over the category of typed posets $\mathsf{Pos}^{\mathcal{S}}$ when the set $\mathcal{S}$ of types is infinite [7].

Observe that the category of dependent sets is not captured by any of the results above since the dependency relation can compare elements of different types, contrary to typed posets & co.

B Characterizing Induced Classes of Recognizable Relations

In this section, we prove a simple characterization of recognizable relations whose languages are "simple".

▶ **Proposition B.1.** *Let $\mathbb{V}$ be a pseudovariety of finite monoids and $\mathcal{V}$ be the corresponding pseudovariety of regular languages. Given a relation $\mathcal{R} \subseteq \Sigma^* \times \Sigma^*$, the following are equivalent:*

1. *$\mathcal{R}$ is recognized by a monoid in $\mathbb{V}$, i.e. there is a monoid morphism $\varphi\colon \Sigma^* \times \Sigma^* \to M$ for some monoid $M \in \mathbb{V}$ such that $\mathcal{R} = \varphi^{-1}[Acc]$, and*
2. *There exists $n \in \mathbb{N}$ and $K_1, \ldots, K_n, L_1, \ldots, L_n$ in $\mathcal{V}$ such that $\mathcal{R} = \bigcup_{i=1}^{n} K_i \times L_i$,*

in which case we say that $\mathcal{R}$ is $\mathcal{V}$-recognizable.

The statement of the result when $\mathbb{V}$ is the pseudovariety of all regular languages is folklore, and the proof of the proposition is easy.

Proof. ❧ (1) ⇒ (2). By definition:

$$\mathcal{R} = \bigcup_{z \in \mathrm{Acc}} \varphi^{-1}[z].$$

Observe then that $\varphi(u, v) = \varphi((u, \varepsilon) \cdot (\varepsilon, v)) = \varphi((u, \varepsilon)) \cdot \varphi((\varepsilon, v))$ for all $u, v \in \Sigma^* \times \Sigma^*$, and hence:

$$\mathcal{R} = \bigcup_{\substack{x,y \in M \\ s.t.\ x \cdot y \in \mathrm{Acc}}} \underbrace{\{u \in \Sigma^* \mid \varphi(u, \varepsilon) = x\}}_{\triangleq K_x} \times \underbrace{\{v \in \Sigma^* \mid \varphi(\varepsilon, v) = y\}}_{\triangleq L_y}.$$

Since M is finite, the union is finite, and moreover, each K_x and L_y is recognized by M, and hence belong to $\mathcal{V}$.

❧ (2) ⇒ (1). If $\mathcal{R} = \bigcup_{i=1}^{n} K_i \times L_i$ where all language belong to $\mathcal{V}$, then let $M_i, N_i \in \mathbb{V}$ be their syntactic monoid, ζ_i, η_i be their syntactic morphism, and $\mathrm{Acc}_i, \mathrm{Bcc}_i$ be their accepting sets. Consider the monoid morphism

$$\begin{array}{rcl} \Sigma^* \times \Sigma^* & \to & \prod_i (M_i \times N_i) \\ (u, v) & \mapsto & \big(\zeta_i(u), \eta_i(v)\big)_i. \end{array}$$

Then $\mathcal{R}$ is the preimage by this morphism of

$$\bigcup_{i=1}^{n} (\ldots \times (M_{i-1} \times N_{i-1}) \times (\mathrm{Acc}_i \times \mathrm{Bcc}_i) \times (M_{i+1} \times N_{i+1}) \times \ldots).$$

The conclusion follows from the fact that $\mathbb{V}$ is closed under (finite) products. ◀

C Pointers to Notions of Algebraic Language Theory

- *idempotent* elements: see [32, §II.1.2, p. 14];
- semigroup division and *monoid division*: see [32, §II.3.3, p. 21];
- *pseudovarieties of semigroups* and *pseudovarieties of monoids*: see [32, §XI.1, p. 189] under the name “variety”;
- *$*$-pseudovarieties of regular languages* see [32, §XIII.3, p. 226];
- *$+$-pseudovarieties of regular languages*: see [32, §XIII.4, “Eilenberg’s $+$-varieties”, p. 229];
- *Eilenberg correspondence*: see [32, Theorem XIII.4.10, p. 228];
- profinite topology and the $\widehat{-}$ notation: see [32, §X.2, p. 178];
- *profinite equality* and the $=$ notation: see [32, §XI.3, p. 193] under the name “profinite identity for semigroups/monoids”;
- *profinite implication* and the $\to$ notation: see [32, §XIII.1, p. 223];
- *satisfiability* of profinite equality: see [32, §XI.3, p. 193];
- equations *defining* a pseudovariety: see [32, §XI.3.3, p. 194].

D Details on Synchronous Algebras

D.1 Proof of the Syntactic Morphism Theorem

▶ **Theorem 3.11** (*Syntactic morphism theorem*)**.** *For each relation $\mathcal{R}$, there exists a surjective synchronous algebra morphism*

$$\eta_{\mathcal{R}} \colon \mathbf{S}_2\Sigma \twoheadrightarrow \mathbf{A}_{\mathcal{R}}$$

which recognizes $\mathcal{R}$ and is such that for any other surjective synchronous algebra morphism $\varphi \colon \mathbf{S}_2\Sigma \twoheadrightarrow \mathbf{B}$ recognizing $\mathcal{R}$, there exists a synchronous algebra morphism $\psi \colon \mathbf{B} \twoheadrightarrow \mathbf{A}_{\mathcal{R}}$ such that the diagram

$$\begin{tikzcd} \mathbf{S}_2\Sigma \arrow[r, two heads, "\eta_{\mathcal{R}}"] \arrow[dr, two heads, "\varphi"'] & \mathbf{A}_{\mathcal{R}} \\ & \mathbf{B}, \arrow[u, two heads, "\psi"'] \end{tikzcd}$$

commutes. Objects $\eta_{\mathcal{R}}$ and $\mathbf{A}_{\mathcal{R}}$ are called the syntactic synchronous algebra morphism and syntactic synchronous algebra of $\mathcal{R}$, respectively. Moreover, these objects are unique up to isomorphisms of the algebra.

Proof of Theorem 3.11. ❧ *Uniqueness.* Consider two potential syntactic morphism, say $\eta_1 \colon \mathbf{S}_2\Sigma \twoheadrightarrow \mathbf{A}_1$ and $\eta_2 \colon \mathbf{S}_2\Sigma \twoheadrightarrow \mathbf{A}_2$. Then by the universal property of η_1 (resp. η_2), there exists $\psi_1 \colon \mathbf{A}_2 \twoheadrightarrow \mathbf{A}_1$ and $\psi_2 \colon \mathbf{A}_1 \twoheadrightarrow \mathbf{A}_2$ such that $\eta_1 = \psi_1 \circ \eta_2$ and $\eta_2 = \psi_2 \circ \eta_1$. Overall, it implies that the following digram commutes

$$\begin{tikzcd}
& \mathbf{A}_1 \\
\mathbf{S}_2\Sigma \arrow[ur, two heads, "\eta_1"] \arrow[r, two heads, "\eta_2"] \arrow[dr, two heads, "\eta_1"'] & \mathbf{A}_2 \arrow[u, two heads, "\psi_1"'] \\
& \mathbf{A}_1, \arrow[u, two heads, "\psi_2"']
\end{tikzcd}$$

and so $\psi_1 \circ \psi_2 \circ \eta_1 = \eta_1$. Since η_1 is surjective, and hence right-cancellative, $\psi_1 \circ \psi_2 = \mathrm{id}_{\mathbf{A}_1}$. Symmetrically, $\psi_2 \circ \psi_1 = \mathrm{id}_{\mathbf{A}_2}$, showing that ψ_1 and ψ_2 are mutually inverse isomorphisms of synchronous algebras.

❧ *Existence.* We will instead prove a stronger property. Consider the identity morphism $\mathrm{id} \colon \mathbf{S}_2\Sigma \twoheadrightarrow \mathbf{S}_2\Sigma$, which recognizes $\mathcal{R}$ since $\underline{\mathcal{R}} = \mathrm{id}^{-1}[\underline{\mathcal{R}}]$ and $\underline{\mathcal{R}}$ is closed. Consider the syntactic congruence $\mathrel{\underline{\asymp}}_{\mathcal{R}}$. We claim that

$$\begin{array}{rccc}
\mathrm{id}/{\mathrel{\underline{\asymp}}_{\mathcal{R}}} \colon & \mathbf{S}_2\Sigma & \twoheadrightarrow & \mathbf{S}_2\Sigma/{\mathrel{\underline{\asymp}}_{\mathcal{R}}} \\
& u & \mapsto & [u]^{\mathrel{\underline{\asymp}}_{\mathcal{R}}}
\end{array}$$

is *a*, and hence *the*, syntactic synchronous algebra morphism of $\mathcal{R}$. First, it clearly recognizes $\mathcal{R}$ since for all pairs of words $u, v \in \mathbf{S}_2\Sigma$, if $u \mathrel{\underline{\asymp}}_{\mathcal{R}} v$ then $u \in \mathcal{R}$ *iff* $v \in \mathcal{R}$. Then, given another surjective morphism $\varphi \colon \mathbf{S}_2\Sigma \twoheadrightarrow$ which recognizes $\mathcal{R}$, say $\underline{\mathcal{R}} = \varphi^{-1}[\mathrm{Acc}]$, then for each $a_\sigma \in \mathbf{A}$, there exists $u_\sigma \in \mathbf{S}_2\Sigma$ such that $\varphi(u_\sigma) = a_\sigma$. This defines a map $\psi \colon \mathbf{A} \to \mathbf{S}_2\Sigma/{\mathrel{\underline{\asymp}}_{\mathcal{R}}}$ which sends a_σ to $[u_\sigma]^{\mathrel{\underline{\asymp}}_{\mathcal{R}}}$.

We claim that $\mathrm{id}/{\mathrel{\underline{\asymp}}_{\mathcal{R}}} = \psi \circ \varphi$: indeed, for $u_\sigma \in \mathbf{S}_2\Sigma$, $\psi(\varphi(u))$ equals $[v_\sigma]^{\mathrel{\underline{\asymp}}_{\mathcal{R}}}$ for some $v_\sigma \in \mathbf{S}_2\Sigma$ such that $\varphi(v_\sigma) = \varphi(u_\sigma)$. This in turns implies that $u_\sigma \mathrel{\underline{\asymp}}_{\mathcal{R}} v_\sigma$ since for all $x, y \in \mathbf{S}_2\Sigma$, if xuy and xvy are well-defined, then so are $\varphi(x)\varphi(u)\varphi(v)$ and $\varphi(x)\varphi(v)\varphi(y)$, and both elements are equal, so one belongs to Acc *iff* the other does, and it follows that $xuy \in \underline{\mathcal{R}}$ *iff* $xvy \in \underline{\mathcal{R}}$. Hence $\psi(\varphi(u)) = [u_\sigma]^{\mathrel{\underline{\asymp}}_{\mathcal{R}}}$.

From $\mathrm{id}/{\mathrel{\underline{\asymp}}_{\mathcal{R}}} = \psi \circ \varphi$ it follows that the map ψ preserves the product[29] and is surjective. Lastly, we claim that it preserves the dependency relation. Indeed, by surjectivity of φ, it boils down to proving that for all $u_\sigma, v_\tau \in \mathbf{S}_2\Sigma$, if $\varphi(u) \asymp \varphi(v)$ then $\psi(\varphi(u)) \asymp \psi(\varphi(v))$, namely $[u_\sigma]^{\mathrel{\underline{\asymp}}_{\mathcal{R}}} \asymp [v_\sigma]^{\mathrel{\underline{\asymp}}_{\mathcal{R}}}$. By definition of the dependency relation $\asymp$ over the quotient structure $\mathbf{S}_2\Sigma/{\mathrel{\underline{\asymp}}_{\mathcal{R}}}$, the latter property rewrites as $u_\sigma \mathrel{\underline{\asymp}}_{\mathcal{R}} v_\tau$. So pick x, y such that both xuy and xvy are well-defined. Then $\varphi(x)\varphi(u)\varphi(y)$ and $\varphi(u)\varphi(v)\varphi(y)$ have the same type as xuy and xvy, respectively, so they are well-defined, but since $\varphi(u) \asymp \varphi(v)$, then $\varphi(x)\varphi(u)\varphi(y) \asymp \varphi(x)\varphi(u)\varphi(y)$ and since Acc is a closed subset, one of these elements belongs it it *iff* the other ones does too, from which it follows that $xuv \in \mathcal{R}$ *iff* $xvy \in \mathcal{R}$ *i.e.* $u_\sigma \mathrel{\underline{\asymp}}_{\mathcal{R}} v_\tau$. Overall, this proves that ψ is a surjective synchronous algebra morphism, and concludes our proof. ◀

[29] See *e.g.* [11, Lemma 3.2, p. 10].

D.2 Composition of Algebras

While the formal definition of the composition is quite different, it is somehow conceptually similar to Schützenberger's product—introduced in [38, §2], see also [39, §1] for a modern presentation—in the sense that is relies on a powerset construction to simulate existential quantification.

The main idea is that, given $f\colon \mathbf{S}_2\Sigma \to \mathbf{A}$ and $g\colon \mathbf{S}_2\Sigma \to \mathbf{B}$, when we will define $h\colon \mathbf{S}_2\Sigma \to \mathbf{A} \circ \mathbf{B}$, which will recognize any composition[30] of a relation recognized by f and by g, we must encode in $h\left(\begin{smallmatrix} u \\ w \end{smallmatrix}\right)$ the set of possible values $\left(f\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right), g\left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right)\right)$, when v ranges over finite words.

▶ **Definition D.1** (Composition). *Given two synchronous algebras* $\mathbf{A}$ *and* $\mathbf{B}$*, define their composition* $\mathbf{A} \circ \mathbf{B}$ *as the algebra* $\mathbf{C}$ *where:*

- $C_{a/b \to c/d} = \mathcal{P}\left(\bigcup_{e,f \in \{l,b\}} A_{a/e \to c/f} \times B_{e/b \to f/d}\right)$ *for all* $a/b \to c/d \in \mathcal{T}$,
- *the product of* $X_\tau \in \mathbf{C}$ *with* $X'_{\tau'} \in \mathbf{C}$ *is defined as*

$$\Big\{\big((x \cdot x')_{\rho \cdot \rho'}, (y \cdot y')_{\sigma \cdot \sigma'}\big) \;\Big|\; (x_\rho, y_\sigma) \in X_\tau,\ (x'_{\rho'}, y'_{\sigma'}) \in X'_{\tau'} \text{ and both } (\rho, \rho') \text{ and } (\sigma, \sigma') \text{ are compatible}\Big\}_{\tau \cdot \tau'}.$$

▶ **Fact D.2.** *If two relations* $\mathcal{R}$ *and* $\mathcal{S}$ *are recognized by* $\mathbf{A}$ *and* $\mathbf{B}$*, respectively, then* $\mathcal{R} \circ \mathcal{S}$ *is recognized by* $\mathbf{A} \circ \mathbf{B}$*.*

Proof sketch. Say that $\varphi\colon \mathbf{S}_2\Sigma \to \mathbf{A}$ recognizes $\mathcal{R}$ and $\psi\colon \mathbf{S}_2\Sigma \to \mathbf{B}$ recognizes $\mathcal{S}$. Then we define $\chi\colon \mathbf{S}_2\Sigma \to \mathbf{A} \circ \mathbf{B}$ as follows: for each pair $(u, w) \in \Sigma^* \times \Sigma^*$, $\chi(u, w)$ is defined as the set $\{\langle \varphi(u, v), \varphi(v, w)\rangle \mid v \in \Sigma^*\}$. Then $\mathcal{R} \circ \mathcal{S}$ is the preimage by χ of the set of all subsets containing a pair $\langle a, b\rangle \in \mathbf{A} \times \mathbf{B}$ *s.t.* $a \in \varphi[\underline{\mathcal{R}}]$ and $b \in \psi[\underline{\mathcal{S}}]$. ◀

E Details on Pseudovarieties

E.1 Proof of the Lifting Theorem

▶ **Theorem 4.2** (*Lifting theorem: Elementary Formulation*). *Given a relation* $\mathcal{R}$ *and a* $*$-*pseudovariety of regular languages* $\mathcal{V}$ *corresponding to a pseudovariety of monoids* $\mathbb{V}$*, the following are equivalent:*

1. $\mathcal{R}$ *is a* $\mathcal{V}$*-relation,*
2. $\mathcal{R}$ *is recognized by a finite synchronous algebra* $\mathbf{A}$ *whose underlying monoids are all in* $\mathbb{V}$*,*
3. *all underlying monoids of* $\mathbf{A}_{\mathcal{R}}$ *are in* $\mathbb{V}$*.*

Proof. ❧ *(1)* ⇒ *(2).* Since $\mathcal{R}$ is a $\mathcal{V}$-relation, there exists $\mathcal{L} \in \mathcal{V}_{\Sigma^2_-}$ such that $\underline{\mathcal{R}} = \mathcal{L} \cap \mathsf{WellFormed}_\Sigma$. Hence, there exists a morphism of monoids $\varphi\colon (\Sigma^2_-)^* \to M$ such that $M \in \mathbb{V}$ and $\mathcal{L} = \varphi^{-1}[\mathrm{Acc}]$ for some $\mathrm{Acc} \subseteq M$. It follows that $\underline{\mathcal{R}} = \mathcal{L} \cap \mathsf{WellFormed}_\Sigma\Sigma$ rewrites as "for all $u \in \mathsf{WellFormed}_\Sigma$, $\varphi(u) \in \mathrm{Acc}$ *iff* $u \in \underline{\mathcal{R}}$". Letting $\mathbf{A}_M$ be the synchronous algebra induced by the monoid M, define $\psi\colon \mathbf{S}_2\Sigma \to \mathbf{A}_M$ by $\psi(u_\sigma) \mathrel{\hat{=}} (\varphi(u))_\sigma$ for $u_\sigma \in \mathbf{S}_2\Sigma$. Let $\mathrm{Acc}' \mathrel{\hat{=}} \{x_\sigma \mid x \in \mathrm{Acc} \wedge \sigma \in \mathcal{T}\}$. We claim that $\psi^{-1}[\mathrm{Acc}'] = \underline{\mathcal{R}}$. Indeed, for $u_\sigma \in \mathbf{S}_2\Sigma$, $u_\sigma \in \underline{\mathcal{R}}$ *iff* $u \in \mathcal{L}$, *i.e.* $\varphi(u) \in \mathrm{Acc}$, that is $\psi(u_\sigma) = (\varphi(u))_\sigma \in \mathrm{Acc}'$. Notice then that all underlying monoids of $\mathbf{A}_M$ are M, and hence they belong to $\mathbb{V}$.

[30] We chose the natural, and hence non-standard, definition, namely $\mathcal{R} \circ \mathcal{S}$ is the set of pairs (u, w) *s.t.* $(u, v) \in \mathcal{R}$ and $(v, w) \in \mathcal{S}$ for some v.

❧ *(2) ⇒ (3).* By Theorem 3.11, the syntactic synchronous algebra of $\mathcal{R}$ divides any algebra $\mathbf{B}$ recognizing $\mathcal{R}$. In particular, its underlying monoids divide the underlying monoids of $\mathbf{B}$. The conclusion follows since $\mathbb{V}$ is closed under division.

❧ *(3) ⇒ (1).* Denote by $M_{\mathrm{L/L}}, M_{\mathrm{L/B}}$ and $M_{\mathrm{B/L}}$ the underlying monoids of $\mathbf{A}_{\mathcal{R}}$. Let $\mathrm{Acc} \subseteq \mathbf{A}_{\mathcal{R}}$ be the accepting set such that $\underline{\mathcal{R}} = \eta_{\mathcal{R}}^{-1}[\mathrm{Acc}]$. Define $M \triangleq M_{\mathrm{L/L}} \times M_{\mathrm{L/B}} \times M_{\mathrm{B/L}}$, and

$$\begin{array}{rccc} \varphi\colon & (\Sigma^2_{-})^* & \to & M \\ & \binom{a}{b} & \mapsto & \langle \eta_{\mathcal{R}}\binom{a}{b}_{\mathrm{L/L}}, 1_{\mathrm{L/B}}, 1_{\mathrm{B/L}} \rangle \\ & \binom{a}{-} & \mapsto & \langle 1_{\mathrm{L/L}}, \eta_{\mathcal{R}}\binom{a}{-}_{\mathrm{L/B}}, 1_{\mathrm{B/L}} \rangle \\ & \binom{-}{a} & \mapsto & \langle 1_{\mathrm{L/L}}, 1_{\mathrm{L/B}}, \eta_{\mathcal{R}}\binom{-}{a}_{\mathrm{B/L}} \rangle \end{array}$$

and finally, let

$$\begin{aligned} \mathrm{Acc}' \triangleq\ & \big\{ \langle x_{\mathrm{L/L}}, 1_{\mathrm{L/B}}, 1_{\mathrm{B/L}} \rangle \mid x_{\mathrm{L/L}} \in \mathrm{Acc} \big\} \\ & \cup \big\{ \langle 1_{\mathrm{L/L}}, y_{\mathrm{L/B}}, 1_{\mathrm{B/L}} \rangle \mid y_{\mathrm{L/B}} \in \mathrm{Acc} \big\} \\ & \cup \big\{ \langle 1_{\mathrm{L/L}}, 1_{\mathrm{L/B}}, z_{\mathrm{B/L}} \rangle \mid z_{\mathrm{B/L}} \in \mathrm{Acc} \big\} \\ & \cup \big\{ \langle x_{\mathrm{L/L}}, y_{\mathrm{L/B}}, 1_{\mathrm{B/L}} \rangle \mid x_{\mathrm{L/L}} \cdot y_{\mathrm{L/B}} \in \mathrm{Acc} \big\} \\ & \cup \big\{ \langle x_{\mathrm{L/L}}, 1_{\mathrm{L/B}}, z_{\mathrm{B/L}} \rangle \mid x_{\mathrm{L/L}} \cdot z_{\mathrm{B/L}} \in \mathrm{Acc} \big\}. \end{aligned}$$

We first claim that

$$\begin{aligned} &\text{For every } u_{\mathrm{L/L}\to\mathrm{L/B}} \in \mathbf{S}_2\Sigma, \\ &\qquad \varphi(u) \text{ is of the form } \langle a, b, 1 \rangle \text{ and moreover,} \\ &\qquad \eta_{\mathcal{R}}(u_{\mathrm{L/L}\to\mathrm{L/B}}) = a \cdot b, \end{aligned} \tag{♪}$$

which can trivially be proven by induction on u. Analogous results hold for words of different type. We then prove that for each $u_\sigma \in \mathbf{S}_2\Sigma$,

$$\eta_{\mathcal{R}}(u_\sigma) \in \mathrm{Acc} \quad \textit{iff} \quad \varphi(u) \in \mathrm{Acc}'. \tag{§}$$

The direct implication is straightforward, using Equation (♪). The converse implication is more tricky: assume *e.g.* that $\sigma = {}^{\mathrm{L}}/_{\mathrm{L}} \to {}^{\mathrm{L}}/_{\mathrm{B}}$, say $t_\sigma = u_{\mathrm{L/L}} v_{\mathrm{L/B}}$. If $\varphi(t) \in \mathrm{Acc}'$, using Equation (♪) then it implies either that

1. $\eta_{\mathcal{R}}(u_{\mathrm{L/L}}) \in \mathrm{Acc}$ and $\eta_{\mathcal{R}}(v_{\mathrm{L/B}}) = 1$, or
2. $\eta_{\mathcal{R}}(u_{\mathrm{L/L}}) = 1_{\mathrm{L/L}}$ and $\eta_{\mathcal{R}}(v_{\mathrm{L/B}}) \in \mathrm{Acc}$, or
3. $\eta_{\mathcal{R}}(u_{\mathrm{L/L}}) = 1_{\mathrm{L/L}}$ and $\eta_{\mathcal{R}}(v_{\mathrm{L/B}}) = 1_{\mathrm{L/L}}$, and $1_{\mathrm{B/L}} \in \mathrm{Acc}$, or
4. $\eta_{\mathcal{R}}(u_{\mathrm{L/L}}) \cdot \eta_{\mathcal{R}}(v_{\mathrm{L/B}}) \in \mathrm{Acc}$, or even
5. $\eta_{\mathcal{R}}(v_{\mathrm{L/B}}) = 1_{\mathrm{L/B}}$ and $\eta_{\mathcal{R}}(u_{\mathrm{L/L}}) \cdot 1_{\mathrm{B/L}} \in \mathrm{Acc}$.

Clearly, (4) implies the desired conclusion, namely that $\eta_{\mathcal{R}}(t_\sigma) = \eta_{\mathcal{R}}(u_{\mathrm{L/L}})\eta_{\mathcal{R}}(v_{\mathrm{L/B}}) \in \mathrm{Acc}$. In all other cases, we will make heavy use of the dependency relation. If (1) holds, then $\eta_{\mathcal{R}}(t_\sigma) = \eta_{\mathcal{R}}(u_{\mathrm{L/L}}) \cdot 1_{\mathrm{L/B}} \asymp \eta_{\mathcal{R}}(u_{\mathrm{L/L}})$. Since $\eta_{\mathcal{R}}(u_{\mathrm{L/L}}) \in \mathrm{Acc}$ and Acc is closed, it follows that $\eta_{\mathcal{R}}(t_\sigma) \in \mathrm{Acc}$. Case (2) is handled similarly. For case (3), we have that $\eta_{\mathcal{R}}(t_\sigma) = 1_{\mathrm{L/L}\to\mathrm{L/B}}$. From $1_{\mathrm{B/L}} \in \mathrm{Acc}$, Fact 3.5 yields $1_{\mathrm{L/L}} \cdot 1_{\mathrm{L/B}} = 1_{\mathrm{L/L}\to\mathrm{L/B}} \in \mathrm{Acc}$, since Acc is closed. Lastly, in case (5), $\eta_{\mathcal{R}}(u_{\mathrm{L/L}}) \asymp \eta_{\mathcal{R}}(u_{\mathrm{L/L}}) \cdot 1_{\mathrm{B/L}} \in \mathrm{Acc}$ so $\eta_{\mathcal{R}}(u_{\mathrm{L/L}}) \in \mathrm{Acc}$ and hence $\eta_{\mathcal{R}}(t_{\mathrm{L/L}\to\mathrm{L/B}}) = \eta_{\mathcal{R}}(u_{\mathrm{L/L}}) \cdot 1_{\mathrm{L/B}} \asymp \eta_{\mathcal{R}}(u_{\mathrm{L/L}}) \in \mathrm{Acc}$ which yields $\eta_{\mathcal{R}}(t_{\mathrm{L/L}\to\mathrm{L/B}}) \in \mathrm{Acc}$. This concludes the proof of (§), from which we deduce that $\underline{\mathcal{R}} = \varphi^{-1}[\mathrm{Acc}'] \cap \mathsf{WellFormed}_\Sigma$. ◀

E.2 Proof of Lemma 4.6

▶ **Proposition E.1.** *Let $\varphi\colon \mathbf{A} \twoheadrightarrow \mathbf{B}$ be a surjective morphism, and Acc be a closed subset of* $\mathbf{B}$. *Let $a, a' \in \mathbf{A}$. Then*

$$a \asymp_{\varphi^{-1}[Acc]} a' \quad \textit{iff} \quad \varphi(a) \asymp_{Acc} \varphi(a').$$

Proof. ❧ *Direct implication.* Pick any $b_\ell, b_r \in \mathbf{B}$ such that both $b_\ell\varphi(a)b_r$ and $b_\ell\varphi(a')b_r$ are well-defined. By surjectivity of φ, there exists $a_\ell, a_r \in \mathbf{A}$ such that $\varphi(a_\ell) = b_\ell$ and $\varphi(a_r) = b_r$. Then both $a_\ell a a_r$ and $a_\ell a' a_r$ are well-defined since they have the same type as $b_\ell\varphi(a)b_r$ and $b_\ell\varphi(a')b_r$, respectively. From $a \asymp_{\varphi^{-1}[\mathrm{Acc}]} a'$, it follows that $a_\ell a a_r$ belongs to $\varphi^{-1}[\mathrm{Acc}]$ *iff* $a_\ell a' a_r$ does. And hence

$$b_\ell\varphi(a)b_r \in \mathrm{Acc} \quad \textit{iff} \quad b_\ell\varphi(a')b_r \in \mathrm{Acc}.$$

❧ *Converse implication.* Dually, pick any $a_\ell, a_r \in \mathbf{A}$ such that both $a_\ell a a_r$ and $a_\ell a' a_r$ are well-defined. Then $\varphi(a_\ell)\varphi(a)\varphi(a_r)$ and $\varphi(a_\ell)\varphi(a')\varphi(a_r)$ are also well-defined since they have the same type as their preimage, and $\varphi(a) \asymp_{\mathrm{Acc}} \varphi(a')$ implies that the element $\varphi(a_\ell)\varphi(a)\varphi(a_r)$ belongs to Acc *iff* $\varphi(a_\ell)\varphi(a')\varphi(a_r)$ does. It follows that $a_\ell a a_r \in \varphi^{-1}[\mathrm{Acc}]$ *iff* $a_\ell a' a_r \in \varphi^{-1}[\mathrm{Acc}]$. ◀

▶ **Proposition E.2** (Inverse images of surjective morphisms preserve residuals)**.** *Let $\varphi\colon \mathbf{A} \twoheadrightarrow \mathbf{B}$ be a surjective morphism, and $Acc \subseteq \mathbf{B}$ be a closed subset. Let $u \in \mathbf{A}$. Then*

$$u^{-1}\varphi^{-1}[Acc] = \varphi^{-1}[\varphi(u)^{-1}Acc].$$

Proof. ❧ *Left-to-right inclusion.* Let $a \in u^{-1}\varphi^{-1}[\mathrm{Acc}]$. Then there exists $a' \in \mathbf{A}$ such that $a \asymp_{\varphi^{-1}[\mathrm{Acc}]} a'$ and $ua' \in \varphi^{-1}[\mathrm{Acc}]$. By Proposition E.1 $a \asymp_{\varphi^{-1}[\mathrm{Acc}]} a'$ implies $\varphi(a) \asymp_{\mathrm{Acc}} \varphi(a')$, and $ua' \in \varphi^{-1}[\mathrm{Acc}]$ yields $\varphi(u)\varphi(a') \in \mathrm{Acc}$. Overall, this shows that $a \in \varphi^{-1}[\varphi(u)^{-1}\mathrm{Acc}]$.

❧ *Right-to-left inclusion.* Let $a \in \varphi^{-1}[\varphi(u)^{-1}\mathrm{Acc}]$. Then $\varphi(a) \in \varphi(u)^{-1}\mathrm{Acc}$, so there exists $b' \in \mathbf{B}$ such that $\varphi(a) \asymp_{\mathrm{Acc}} b'$ and $\varphi(u)b' \in \mathrm{Acc}$. By surjectivity of φ and Proposition E.1, there exists $a' \in \mathbf{A}$ such that $\varphi(a') = b'$ and $a \asymp_{\varphi^{-1}[\mathrm{Acc}]} a'$. ◀

▶ **Lemma 4.6.** *A class $\mathcal{V}\colon \Sigma \mapsto \mathcal{V}_\Sigma$ is a $*$-pseudovariety of synchronous relations if, and only if, it is closed under Boolean combinations, residuals and inverse morphisms.*

Proof of Lemma 4.6. ❧ *Direct implication.* By Proposition E.2, the residual of any relation recognized by some morphism φ is also recognized by φ. Hence, being closed under syntactic derivatives implies being closed under residuals.

❧ *Converse implication.* Consider some relation $\mathcal{R}$. We will show that any relation recognized by $\eta_{\mathcal{R}}$ can be expressed as a Boolean combination of residuals of $\mathcal{R}$.[31] Let Acc be the closed subset of $\mathbf{A}_{\mathcal{R}}$ such that $\underline{\mathcal{R}} = \eta_{\mathcal{R}}{}^{-1}[\mathrm{Acc}]$. Pick $x \in \mathbf{A}_{\mathcal{R}}$. Let $\Lambda \mathrel{\hat{=}} \{s, t \in \mathbf{A}_{\mathcal{R}} \mid \exists x' \in \mathbf{A}_{\mathcal{R}},\ x' \asymp x \text{ and } sx't \in \mathrm{Acc}\}$. We claim that

$$[x]^{\asymp_{\mathbf{A}_{\mathcal{R}}}} = \left(\bigcap_{(s,t)\in\Lambda} s^{-1}\mathrm{Acc}\, t^{-1}\right) \smallsetminus \left(\bigcup_{(s,t)\notin\Lambda} s^{-1}\mathrm{Acc}\, t^{-1}\right). \tag{❧}$$

To prove the inclusion from left-to-right, first notice that $x \in s^{-1}\mathrm{Acc}\, t^{-1}$ for all $(s,t) \in \Lambda$. Then, assume by contradiction that there exists $(s,t) \notin \Lambda$ *s.t.* $x \in s^{-1}\mathrm{Acc}\, t^{-1}$. Then there

[31] This result can be put in perspective with [32, Lemma XIII.4.11, p. 229] which proves a similar result in the context of monoids.

would exist $x' \preccurlyeq_{\mathrm{Acc}} x$ such that such that $sx't \in \mathrm{Acc}$. But since $\eta_{\mathcal{R}}$ is the syntactic synchronous algebra of $\mathcal{R}$, $\preccurlyeq_{\mathrm{Acc}}$ is precisely the relation $\asymp$ by Corollary 3.14. Contradiction. Hence, x belongs to the right-hand side (RHS). But then, this latter set is a Boolean combination of residuals of a closed subset, so it is also closed, and hence $[x]^{\asymp_{\mathbf{A}_{\mathcal{R}}}}$ is included in the RHS.

Dually, any element y of the RHS satisfies that for all $s, t \in \mathbf{A}_{\mathcal{R}}$, $x \in s^{-1}\mathrm{Acc}\, t^{-1}$ *iff* $y \in s^{-1}\mathrm{Acc}\, t^{-1}$. We claim that $x \preccurlyeq_{\mathrm{Acc}} y$. Pick $s, t \in \mathbf{B}$ and assume that both sxt and syt are well-defined. If $sxt \in \mathrm{Acc}$ then $x \in s^{-1}\mathrm{Acc}\, t^{-1}$ so $y \in s^{-1}\mathrm{Acc}\, t^{-1}$ and hence, there exists $y' \asymp_{\mathbf{A}_{\mathcal{R}}} y$ *s.t.* $sy't \in \mathrm{Acc}$. But syt is also well-defined and $y \asymp_{\mathbf{A}_{\mathcal{R}}} y'$ so $syt \in \mathrm{Acc}$. By symmetry, we have shown that $sxt \in \mathrm{Acc}$ *iff* $syt \in \mathrm{Acc}$, and hence $x \preccurlyeq_{\mathrm{Acc}} y$. Using again the fact that $\mathbf{A}_{\mathcal{R}}$ is the syntactic algebra of $\mathcal{R}$, it follows that $x \asymp_{\mathbf{A}_{\mathcal{R}}} y$. This concludes the proof of (❧). By taking the union, it follows that any closed subset of $\mathbf{A}_{\mathcal{R}}$ is a Boolean combination of residuals of Acc. Applying Proposition E.2 then yields that any relation recognized by φ is a Boolean combination of residuals of $\mathcal{R}$. Hence, any class closed under Boolean combinations and residuals is also closed under syntactic derivatives. ◀

E.3 Proof of Lemma 4.7

▶ **Theorem 4.7** (*An Eilenberg theorem for synchronous relations*). *The correspondences $\mathbb{V} \to \mathcal{V}$ and $\mathcal{V} \to \mathbb{V}$ define mutually inverse bijective correspondences between pseudovarieties of synchronous algebras and $*$-pseudovarieties of synchronous relations.*

Proof. We very roughly follow the proof schema of [32, §XIII.4, pp. 226–229]—which is a proof of Eilenberg's theorem in the context of monoids.

❧ *The correspondence $\mathbb{V} \to \mathcal{V}$ produces varieties.* First we have to show that if $\mathbb{V}$ is a pseudovariety of synchronous algebras and $\mathbb{V} \to \mathcal{V}$, then $\mathcal{V}$ is a $*$-pseudovarieties of synchronous relations. Since $\mathbb{V}$ is closed under finite products, $\mathcal{V}$ is closed under Boolean operations.

Syntactic derivatives: Then let $\mathcal{R} \in \mathcal{V}_\Sigma$, and let $\mathcal{S}$ be any other relation recognized by $\mathbf{A}_{\mathcal{R}}$. This implies that $\mathbf{A}_{\mathcal{S}}$ divides $\mathbf{A}_{\mathcal{R}}$, and so $\mathbf{A}_{\mathcal{S}} \in \mathbb{V}$, from which we have $\mathbf{A}_{\mathcal{S}} \in \mathcal{V}_\Sigma$.

Inverse morphisms: Lastly, if $\mathcal{R} \in \mathcal{V}_\Sigma$, say $\underline{\mathcal{R}} = \eta_{\mathcal{R}}^{-1}[\mathrm{Acc}]$, if $\psi \colon \mathbf{S}_2\Gamma \to \mathbf{S}_2\Sigma$ is a synchronous algebra morphism, then $\psi^{-1}[\mathcal{R}] = (\eta_{\mathcal{R}} \circ \psi)^{-1}[\mathrm{Acc}]$, so $\psi^{-1}[\mathcal{R}]$ is recognized by $\mathbf{A}_{\mathcal{R}}$, that is $\mathbf{A}_{\psi^{-1}[\mathcal{R}]}$ divides $\mathbf{A}_{\mathcal{R}}$. Since $\mathbf{A}_{\mathcal{R}} \in \mathbb{V}$ and $\mathbb{V}$ is closed by division, it follows that $\mathbf{A}_{\psi^{-1}[\mathcal{R}]} \in \mathbb{V}$ and hence $\psi^{-1}[\mathcal{R}] \in \mathcal{V}_\Gamma$. This concludes the proof that $\mathcal{V}$ is a $*$-pseudovariety of synchronous relations.

❧ *Inverse bijections: part 1.* Assume that $\mathbb{V} \to \mathcal{V}$ and $\mathcal{V} \to \mathbb{W}$. Then

$$\mathcal{V} \colon \Sigma \mapsto \{\mathcal{R} \subseteq \Sigma^* \times \Sigma^* \mid \mathbf{A}_{\mathcal{R}} \in \mathbb{V}\},$$

and so $\mathbb{W}$ is the pseudovariety generated by all syntactic synchronous algebras that belong to $\mathbb{V}$. It follows that $\mathbb{W} \subseteq \mathbb{V}$. To prove that $\mathbb{V} \subseteq \mathbb{W}$, let $\mathbf{A} \in \mathbb{V}$. Let $\Sigma_{\mathbf{A}}$ be an alphabet big enough so that there are injections from $\mathbf{A}_{\mathrm{L/L}}$ to $\Sigma_{\mathbf{A}} \times \Sigma_{\mathbf{A}}$, and from $\mathbf{A}_{\mathrm{L/B}}$ and $\mathbf{A}_{\mathrm{B/L}}$ to $\Sigma_{\mathbf{A}} \times \_$ and $\_ \times \Sigma_{\mathbf{A}}$, respectively. Since $\mathbf{A}$ is locally generated, this allows us to define a surjective synchronous algebra morphism $\varphi \colon \mathbf{S}_2\Sigma_{\mathbf{A}} \twoheadrightarrow \mathbf{A}$. We then claim that $\mathbf{A}$ divides $\mathbf{B} \mathrel{\hat{=}} \prod_{x_\tau \in \mathbf{A}} \mathbf{B}_{x_\tau}$ where $\mathbf{B}_{x_\tau}$ is the syntactic synchronous algebra of $\varphi^{-1}[x_\tau]$. Indeed, let $\psi_{x_\tau} \colon \mathbf{S}_2\Sigma_{\mathbf{A}} \twoheadrightarrow \mathbf{B}_{x_\tau}$ be the syntactic algebra of $\varphi^{-1}[x_\tau]$, say $\varphi^{-1}[x_\tau] = \psi_{x_\tau}^{-1}[\mathrm{Acc}_{x_\tau}]$. Then consider

$$\begin{array}{rccc} \Psi \colon & \mathbf{S}_2\Sigma_{\mathbf{A}} & \to & \mathbf{B} \\ & u_\sigma & \mapsto & \langle \psi_{x_\tau}(u_\sigma) \rangle_{x_\tau \in \mathbf{A}}, \end{array}$$

and let $\mathbf{B}_0$ be its image. Observe that for each $u_\sigma \in \mathbf{S}_2\Sigma_{\mathbf{A}}$, $\psi_{x_\tau}(u_\sigma) \in \mathrm{Acc}_{x_\tau}$ *iff* $u_\sigma \in \varphi^{-1}[x_\tau]$ *i.e.* $\varphi(u_\sigma) = x_\tau$—note by the way that it implies $\sigma = \tau$. This implies that for any

$(\langle y_{x_\tau}\rangle_{x_\tau \in \mathbf{A}})_\sigma \in \mathbf{B}_0$, there exists a unique x_τ *s.t.* $y_{x_\tau} \in \mathrm{Acc}_{x_\tau}$. This defined a map $\chi\colon \mathbf{B}_0 \to \mathbf{A}$. Since moreover it makes the following diagram commute

$$\begin{tikzcd} \mathbf{S}_2\Sigma_{\mathbf{A}} \arrow[r, two heads, "\Psi"] \arrow[dr, two heads, "\varphi"'] & \mathbf{B}_0 \arrow[d, "\chi"] \\ & \mathbf{A} \end{tikzcd}$$

it follows that χ is in fact a surjective synchronous algebra morphism.[32] Hence, $\mathbf{A}$ is a quotient of $\mathbf{B}_0$, which is a subalgebra of $\mathbf{B}$, which in turns in a product of algebras from $\mathbb{W}$, and so $\mathbf{A} \in \mathbb{W}$. It concludes the proof that $\mathbb{V} = \mathbb{W}$.

❧ *Inverse bijections: part 2.* Assume now that $\mathcal{V} \to \mathbb{V}$ and $\mathbb{V} \to \mathcal{W}$. Then for each Σ, for each $\mathcal{R} \in \mathcal{V}_\Sigma$, $\mathbf{A}_{\mathcal{R}} \in \mathbb{V}$ so $\mathcal{R} \in \mathcal{W}_\Sigma$, and hence $\mathcal{V} \subseteq \mathcal{W}$.

We then want to show the converse inclusion, namely $\mathcal{W} \subseteq \mathcal{V}$. Let $\mathcal{R} \in \mathcal{W}_\Sigma$ for some Σ, *i.e.* $\mathbf{A}_{\mathcal{R}} \in \mathbb{V}$. Hence there exists Γ and relations $\mathcal{S}_1 \in \mathcal{V}_{\Gamma_1}, \dots \mathcal{S}_k \in \mathcal{V}_{\Gamma_k}$ such that $\mathbf{A}_{\mathcal{R}}$ divides $\mathbf{B} \mathrel{\hat=} \mathbf{A}_{\mathcal{S}_1} \times \cdots \times \mathbf{A}_{\mathcal{S}_k}$, *i.e.* there is a subalgebra $\mathbf{C} \subseteq \mathbf{B}$ which is a quotient of $\mathbf{B}$. Then $\mathbf{C}$ also recognizes $\mathcal{R}$, say $\mathcal{R} = \varphi^{-1}[\mathrm{Acc}]$ for some morphism $\varphi\colon \mathbf{S}_2\Sigma \twoheadrightarrow \mathbf{C}$ and $\mathrm{Acc} \subseteq \mathbf{C}$. Let $\iota\colon \mathbf{C} \to \mathbf{B}$ be the canonical embedding, $\pi_i\colon \mathbf{B} \twoheadrightarrow \mathbf{A}_{\mathcal{S}_i}$ be the canonical projection, and $\varphi_i \mathrel{\hat=} \pi_i \circ \iota \circ \varphi\colon \mathbf{S}_2\Sigma \to \mathbf{A}_{\mathcal{S}_i}$ for $i \in [\![1,k]\!]$. Then notice that since $\eta_{\mathcal{S}_i}\colon \mathbf{S}_2\Gamma_i \twoheadrightarrow \mathbf{A}_{\mathcal{S}_i}$ is surjective, then there exists $\psi_i\colon \mathbf{S}_2\Sigma \to \mathbf{S}_2\Gamma_i$ such that $\eta_{\mathcal{S}_i} \circ \psi_i = \varphi_i$. Indeed, it suffices to send $\binom{a}{b}$ (resp. $\binom{a}{\_}$), resp. $\binom{\_}{a}$) on any element $u_{\mathrm{L/L}} \in \mathbf{S}_2\Gamma_i$ (resp. $u_{\mathrm{L/B}}$, resp. $u_{\mathrm{B/L}}$) such that $\eta_{\mathcal{S}_i}(u_{\mathrm{L/L}}) = \varphi\binom{a}{b}$ (resp. $\eta_{\mathcal{S}_i}(u_{\mathrm{L/B}}) = \varphi\binom{a}{\_}$, resp. $\eta_{\mathcal{S}_i}(u_{\mathrm{B/L}}) = \varphi\binom{\_}{a}$). Overall, the following diagram commutes

$$\begin{tikzcd} \mathbf{S}_2\Sigma \arrow[rr, "\psi_i"] \arrow[d, "\varphi"'] \arrow[drr, "\varphi_i"] & & \mathbf{S}_2\Gamma_i \arrow[d, "\eta_{\mathcal{S}_i}"] \\ \mathbf{C} \arrow[r, "\iota"'] & \mathbf{B} \arrow[r, "\pi_i"'] & \mathbf{A}_{\mathcal{S}_i}. \end{tikzcd}$$

Our goal is to show that $\mathcal{R} \in \mathcal{V}_\Sigma$. Observe that:

$$\mathcal{R} = \varphi^{-1}[\mathrm{Acc}] = \cup_{x \in \mathrm{Acc}} \varphi^{-1}[x]$$

but then $\mathrm{Acc} \subseteq \mathbf{B}$, so x is a tuple $\langle x_1, \dots, x_n\rangle$ (all elements having the same type), and by definition:

$$\varphi^{-1}[x] = \bigcap_{i=1}^{n} \varphi^{-1}[\iota^{-1}[\pi_i^{-1}[x_i]]] = \bigcap_{i=1}^{n} \varphi_i^{-1}[x_i].$$

But then $\varphi_i^{-1}[x_i] = \psi_i^{-1}[{\eta_{\mathcal{S}_i}}^{-1}[x_i]]$. Since $\mathcal{V}$ is closed under syntactic derivatives and $\mathcal{S}_i \in \mathcal{V}_{\Gamma_i}$ we have ${\eta_{\mathcal{S}_i}}^{-1}[x_i] \in \mathcal{V}_{\Gamma_i}$, and then since $\mathcal{V}$ is closed under Inverse morphisms and $\psi_i\colon \mathbf{S}_2\Sigma \to \mathbf{S}_2\Gamma_i$ is a morphism between free algebras, $\psi_i^{-1}[{\eta_{\mathcal{S}_i}}^{-1}[x_i]] \in \mathcal{V}_\Sigma$. Thus $\mathcal{R}$ is a Boolean combination of elements of $\mathcal{V}_\Sigma$, and hence it also belongs to $\mathcal{V}_\Sigma$. This concludes the proof of $\mathcal{W} \subseteq \mathcal{V}$. ◀

[32] See *e.g.* [11, Lemma 3.2, p. 10] for a proof in a similar (but different) context.

F Details on Positive Synchronous Algebras

F.1 Canonicity of the Consolidation of the Syntactic Positive Synchronous Algebra Morphism

We formalize the idea that $\mathbf{A}^0_{\mathscr{R}}$ is the *canonical* and *minimal* (in an algebraic sense, as usual) semigroup recognizing $\mathscr{R}$ that can be typed.

▶ **Proposition F.1.** *For any semigroup T together with surjective semigroup morphisms*

$$\varphi\colon (\Sigma^2_{-})^+ \twoheadrightarrow T, \qquad \rho_{type}\colon T \twoheadrightarrow \mathbf{1}^0 \qquad \textit{and} \qquad \rho_{val}\colon T \twoheadrightarrow S_{\mathscr{R}}$$

such that $\rho_{type} \circ \varphi = \mathrm{type}$ and $\rho_{val} \circ \varphi = \zeta_{\mathscr{R}}$, there exists a semigroup morphism $\psi\colon T \twoheadrightarrow \mathbf{A}^0_{\mathscr{R}}$ such that $\eta^{\mathsf{sgp}}_{\mathscr{R}} = \psi \circ \varphi$. In other words, the following diagram commutes:

$$\begin{tikzcd}
& (\Sigma^2_{-})^+ \arrow[ddl, "\mathrm{type}"'] \arrow[d, two heads, "\varphi"] \arrow[ddr, "\zeta_{\mathscr{R}}"] \arrow[ddd, two heads, "\eta^{\mathsf{sgp}}_{\mathscr{R}}"', near start] & \\
& T \arrow[dl, two heads, "\rho_{type}"'] \arrow[dr, two heads, "\rho_{val}"] \arrow[dd, dashed, two heads, "\psi", near end] & \\
\mathbf{1}^0 & & S_{\mathscr{R}}. \\
& \mathbf{A}^0_{\mathscr{R}} \arrow[ul, two heads, "\pi_{type}"] \arrow[ur, two heads, "\pi_{val}"'] &
\end{tikzcd}$$

The previous proposition can be nicely reformulated in categorical terms: denote by $(\Sigma^2_{-})^+ /\!\!\!\nearrow \mathsf{Sgp}$ the full subcategory of the coslice category $(\Sigma^2_{-})^+/\mathsf{Sgp}$ induced by surjective morphisms, *i.e.* it is the category whose objects are surjective semigroup morphisms $\varphi\colon (\Sigma^2_{-})^+ \twoheadrightarrow T$, and whose morphisms from $\varphi\colon (\Sigma^2_{-})^+ \twoheadrightarrow T$ to $\psi\colon (\Sigma^2_{-})^+ \twoheadrightarrow U$ are simply the semigroup morphisms $\chi\colon T \to U$ such that $\psi = \chi \circ \varphi$. Then the syntactic semigroup morphism of well-formed words (namely type) and of $\mathscr{R}$, seen as objects of $(\Sigma^2_{-})^+ /\!\!\!\nearrow \mathsf{Sgp}$, admit a Cartesian product, and this product is the composition $\eta^{\mathsf{sgp}}_{\mathscr{R}}\colon (\Sigma^2_{-})^+ \twoheadrightarrow \mathbf{A}^0_{\mathscr{R}}$ of the of the consolidation of the syntactic synchronous morphism of $\mathscr{R}$ together with the canonical surjection $\sigma\colon (\Sigma^2_{-})^+ \twoheadrightarrow (\mathbf{S}_2\Sigma)^0$.

▶ Remark F.2. Finally, note that this property implies that if $\mathbf{1}^0 \in \mathbb{V}$, then $\mathbf{A}^0_{\mathscr{R}} \in \mathbb{V}$ *iff* $S_{\mathscr{R}} \in \mathbb{V}$. Indeed, if $\mathbf{A}^0_{\mathscr{R}} \in \mathbb{V}$, then, as a quotient, $S_{\mathscr{R}} \in \mathbb{V}$. Conversely, if $S_{\mathscr{R}} \in \mathbb{V}$ then $S_{\mathscr{R}} \times \mathbf{1}^0 \in \mathbb{V}$ since $\mathbf{1}^0 \in \mathbb{V}$, and since $S_{\mathscr{R}} \times \mathbf{1}^0$ comes naturally equipped with projections onto $S_{\mathscr{R}}$ and $\mathbf{1}^0$, by Proposition F.1, $\mathbf{A}^0_{\mathscr{R}}$ is a quotient of $S_{\mathscr{R}} \times \mathbf{1}^0$ and hence belongs to $\mathbb{V}$. This statement is the algebraic counterpart of Fact 2.5.

F.2 Profinite Topology & Pseudovarieties

For any pseudovariety of positive synchronous algebras $\mathbb{V}$, let

$$\begin{aligned}
\mathbf{d}_{\mathbb{V}}(u,v) &\mathrel{\widehat{=}} 2^{-\mathbf{r}_{\mathbb{V}}(u,v)} \text{ where} \\
\mathbf{r}_{\mathbb{V}}(u,v) &\mathrel{\widehat{=}} \min\big\{n \in \mathbb{N} \mid \varphi(u) \not\asymp \varphi(v) \text{ for some morphism} \\
&\qquad\qquad \varphi \text{ whose codomain has at most} \\
&\qquad\qquad n \text{ elements and is in } \mathbb{V}.\big\}.
\end{aligned}$$

Then define the congruence $\asymp_{\mathbb{V}}$ over $\widehat{\mathbf{S}^+_2\Sigma}$ by $u \asymp_{\mathbb{V}} v$ *iff* $\mathbf{d}_{\mathbb{V}}(u,v) = 0$. Then $\widehat{\mathbf{F}}_{\mathbb{V}}\Sigma \mathrel{\widehat{=}} \widehat{\mathbf{S}^+_2\Sigma}/{\asymp_{\mathbb{V}}}$ is a positive synchronous algebra, called the *free pro-$\mathbb{V}$ positive synchronous algebra*.

Note however that contrary to the free pro-$\mathbb{V}$ semigroup—see [32, §XI.2, p. 190]—it is not a compact metric space, for the simple reason that one can have $\mathbf{d}_{\mathbb{V}}(u,v) = 0$ for some $u \neq v \in \widehat{\mathbf{S}_2^+\Sigma}/{\preceq_{\mathbb{V}}}$ of different type. However, the free pro-$\mathbb{V}$ positive synchronous algebra is a compact ultrapseudometric space, and all restrictions of $\widehat{\mathbf{S}_2^+\Sigma}/{\preceq_{\mathbb{V}}}$ are metric (*a.k.a.* T_0). Most properties of metric space can easily be generalized to pseudometric spaces, see *e.g.* [37, Theorem 17.33, p. 470] for a characterization of compactness.

Sending a profinite word to its equivalence class under $\preceq_{\mathbb{V}}$ defines a canonical surjective morphism

$$\pi_{\mathbb{V}} \colon \widehat{\mathbf{S}_2^+\Sigma} \twoheadrightarrow \widehat{\mathbf{F}}_{\mathbb{V}}\Sigma$$

which happens to be 1-Lipschitz and hence uniformly continuous.

▶ **Proposition F.3.** *Every morphism $\varphi \colon \mathbf{S}_2^+\Sigma \to \mathbf{A}$ where $\mathbf{A} \in \mathbb{V}$ extends to a unique uniformly continuous morphism $\widehat{\mathbf{F}}_{\mathbb{V}}\Sigma \to \mathbf{A}$.*

Proof. Similar to [32, Proposition XI.2.6, p. 192]. ◀

▶ **Proposition F.4.** *A finite Σ^2_--generated algebra belongs to $\mathbb{V}$ if and only if it is a continuous quotient of $\widehat{\mathbf{F}}_{\mathbb{V}}\Sigma$, is the sense that it is the image of $\widehat{\mathbf{F}}_{\mathbb{V}}\Sigma$ by a continuous morphism.*

Proof. Similar to [32, Proposition XI.2.7, p. 192]. ◀

▶ **Proposition F.5.** *Let $u, v \in \widehat{\mathbf{S}_2^+X}$. Then all algebras of $\mathbb{V}$ satisfy $u \asymp v$ iff $\pi_{\mathbb{V}}(u) \asymp \pi_{\mathbb{V}}(v)$.*

Proof. Similar to [32, Proposition XI.3.11, p. 195]. ◀

▶ **Theorem 5.12** (*A Reiterman theorem for positive synchronous algebras*)**.** *A class of locally generated finite positive synchronous algebras is a pseudovariety if, and only if, it is defined by a set of profinite dependencies.*[33] *Moreover, w.l.o.g., all variables have type ${}^{L}\!/\!{}_{L}$, ${}^{L}\!/\!{}_{B}$ or ${}^{B}\!/\!{}_{L}$.*

Proof of Theorem 5.12. We follow the proof of [32, Theorem XI.3.13, p. 195].

❧ *Converse implication.* Let $\mathscr{E}$ be a set of profinite dependencies. Let us show that $[\![\mathscr{E}]\!]$ is a pseudovariety of positive synchronous algebras. Let $\mathbf{A}_1, \mathbf{A}_2 \in [\![\mathscr{E}]\!]$. For any $(u_\sigma \asymp v_\tau) \in \mathscr{E}$, let X be its set of variables. Then for every morphism $\varphi \colon \mathbf{S}_2^+X \to \mathbf{A}_1 \times \mathbf{A}_2$, let π_1, π_2 be the canonical projection from $\mathbf{A}_1 \times \mathbf{A}_2$ to $\mathbf{A}_1$ and $\mathbf{A}_2$, respectively. Then $\pi_i \circ \varphi$ is a morphism $\mathbf{S}_2^+X \to \mathbf{A}_i$ and hence $\widehat{\pi_i \circ \varphi}(u_\sigma) \asymp^{\mathbf{A}_i} \widehat{\pi_i \circ \varphi}(v_\tau)$. From $\widehat{\pi_i \circ \varphi} = \pi_i \circ \widehat{\varphi}$ it follows that

$$\begin{aligned}\widehat{\varphi}(u_\sigma) &= \langle \widehat{\pi_1 \circ \varphi}(u_\sigma), \widehat{\pi_2 \circ \varphi}(u_\sigma) \rangle \\ &\asymp^{\mathbf{A}_1 \times \mathbf{A}_2} \langle \widehat{\pi_1 \circ \varphi}(v_\tau), \widehat{\pi_2 \circ \varphi}(v_\tau) \rangle = \widehat{\varphi}(v_\tau).\end{aligned}$$

Hence $\mathbf{A}_1 \times \mathbf{A}_2 \in [\![\mathscr{E}]\!]$. Observe then that $[\![\mathscr{E}]\!]$ is obviously closed under subalgebras. For quotients, let $\mathbf{A} \in [\![\mathscr{E}]\!]$, and consider a quotient $\psi \colon \mathbf{A} \twoheadrightarrow \mathbf{A}/{\preceq}$. Then for every profinite dependency $u \asymp v$ over X, for every $\varphi \colon \mathbf{S}_2^+X \to \mathbf{A}/{\preceq}$, then there exists $\chi \colon \mathbf{S}_2^+X \to \mathbf{A}$ *s.t.* $\varphi = \psi \circ \chi$.[34] And hence, from $\widehat{\psi \circ \chi} = \psi \circ \widehat{\chi}$, it follows that

$$\widehat{\varphi}(u_\sigma) = \psi(\widehat{\chi}(u_\sigma)) \asymp \psi(\widehat{\chi}(v_\tau)) = \widehat{\varphi}(v_\tau).$$

[33] Note however that this set can be infinite.
[34] Map $x \in X$ to any element $a \in \mathbf{A}$ *s.t.* $\varphi(x) = [a]^{\preceq}$. Such an element must exist by surjectivity of ψ.

❧ *Direct implication.* Let $\mathcal{E}$ be the set of profinite dependencies satisfied by all algebras of $\mathbb{V}$. Let $\mathbb{W} \mathrel{\hat{=}} [\![\mathcal{E}]\!]$. By the previous implication, it is a pseudovariety of positive synchronous algebras, and by construction $\mathbb{V} \subseteq \mathbb{W}$. We want to show the converse inclusion, so let $\mathbf{A} \in \mathbb{W}$. Since $\mathbf{A}$ is finite and locally generated, there exists a finite alphabet Σ and a surjective morphism $\varphi\colon \mathbf{S}_2^+\Sigma \twoheadrightarrow \mathbf{A}$. It induces a uniformly continuous morphism $\widehat{\varphi}\colon \widehat{\mathbf{S}_2^+\Sigma} \twoheadrightarrow \mathbf{A}$. Pick $u, v \in \widehat{\mathbf{S}_2^+\Sigma}$. Observe that if $\pi_{\mathbb{V}}(u) \asymp \pi_{\mathbb{V}}(v)$ then by Proposition F.5, $u \asymp v$ is satisfied by all algebras in $\mathbb{V}$ and hence belongs to $\mathcal{E}$. Hence, $\mathbf{A}$ satisfies $u \asymp v$, and thus $\widehat{\varphi}(u) \asymp \widehat{\varphi}(v)$. By second isomorphism theorem, it follows that there exists a morphism $\gamma\colon \widehat{\mathbf{F}}_{\mathbb{V}}\Sigma \twoheadrightarrow \mathbf{A}$ such that $\widehat{\varphi} = \gamma \circ \pi_{\mathbb{V}}$.

We then claim that γ is continuous. Indeed, $\mathbf{A}$ is discrete, and for each x_σ, $\gamma^{-1}[x_\sigma] = \pi_{\mathbb{V}}[\widehat{\varphi}^{-1}[x_\sigma]]$. By continuity of $\widehat{\varphi}$, $\widehat{\varphi}^{-1}[x_\sigma]$ is closed (in fact clopen), and hence compact as a closed subset of the compact space $\widehat{\mathbf{S}_2^+\Sigma}$. Then $\pi_{\mathbb{V}}[\widehat{\varphi}^{-1}[x_\sigma]]$ is compact, as the image of a compact set by a continuous map, and hence in particular it is closed.

Therefore, γ is a continuous morphism, proving that $\mathbf{A}$ is a continuous quotient of $\widehat{\mathbf{F}}_{\mathbb{V}}\Sigma$. Hence, by Proposition F.4, $\mathbf{A} \in \mathbb{V}$which concludes the proof that $\mathbb{V} = \mathbb{W}$. ◀

G Details on the Lifting Theorem for Positive Algebras

G.1 Boolean Algebas & Streams of Relations

A *profinite equivalence* over variables X, consists of a pair $(u_\sigma, v_\tau) \in \widehat{\mathbf{S}_2^+X}$ denoted by $u \leftrightarrow v$. A relation $\mathcal{R}$ *satisfies* this profinite equivalence when for all positive synchronous algebra $\mathbf{A}$ recognizing $\mathcal{R}$, for all morphism $\varphi\colon \mathbf{S}_2^+X \to \mathbf{A}$, $\widehat{\varphi}(u_\sigma) \in \varphi[\mathcal{R}] \Leftrightarrow \widehat{\varphi}(v_\tau) \in \varphi[\mathcal{R}]$.

An *explicit profinite equivalence* ove the alphabet Σ consists of a pair $(u_\sigma, v_\tau) \in \widehat{\mathbf{S}_2^+\Sigma}$ also denoted by $u \leftrightarrow v$. A relation $\mathcal{R}$ over alphabet Σ *satisfies* it when $u \in \bar{\mathcal{R}}$ *iff* $v \in \bar{\mathcal{R}}$.

▶ **Proposition G.1.** *Fix a finite alphabet Σ. A set of synchronous relations is a Boolean algebra if, and only if, it can be defined by a set of explicit profinite equivalences over Σ.*

Proof. The proof can be easily generalized from [32, Theorem XII.2.6, p. 211]. One of the key argument is that if $\mathcal{R}$ is synchronous, then $\bar{\mathcal{R}}$ is clopen in $\widehat{\mathbf{S}_2^+\Sigma}$. ◀

A *+-stream of relations* consists of a map $\mathcal{V}\colon \Sigma \mapsto \mathcal{V}_\Sigma$ where each $\mathcal{V}_\Sigma$ is a set of relations over Σ, such that:

- $\mathcal{V}_\Sigma$ is closed under Boolean operator,
- $\mathcal{V}$ is closed under inverse morphisms of the form $\mathbf{S}_2^+\Gamma \to \mathbf{S}_2^+\Sigma$.

▶ **Fact G.2.** *Let $\mathcal{V}$ be a +-stream of relations, and Σ be a finite alphabet. Then the following set coincides:*

- *$\mathcal{E}_{\mathcal{V}}^{\Sigma}$: the intersection of the set of all profinite equivalences satisfied by all relations of $\mathcal{V}$, and of $\widehat{\mathbf{S}_2^+\Sigma} \times \widehat{\mathbf{S}_2^+\Sigma}$;*
- *$\mathcal{F}_{\mathcal{V}}^{\Sigma}$: the set of all explicit profinite equivalences satisfied by all relations of $\mathcal{V}_\Sigma$.*

Proof. By construction, $\mathcal{E}_{\mathcal{V}}^{\Sigma} \subseteq \mathcal{F}_{\mathcal{V}}^{\Sigma}$. Conversely, let $(u \leftrightarrow v) \in \mathcal{F}_{\mathcal{V}}^{\Sigma}$. To show that $(u \leftrightarrow v) \in \mathcal{E}_{\mathcal{V}}^{\Sigma}$, we will show that all relations $\mathcal{R} \in \mathcal{V}_\Sigma$ satisfy $u \leftrightarrow v$, seen as a (non-explicit) profinite dependency. So, let $\mathcal{R} \in \mathcal{V}_\Sigma$ and $\varphi\colon \mathbf{S}_2^+\Sigma \to \mathbf{S}_2^+\Sigma$ be a morphism. Since $\mathcal{V}$ is closed under preimages, $\varphi^{-1}[\mathcal{R}] \in \mathcal{V}_\Sigma$. By continuity arguments, $\widehat{\varphi}^{-1}[\bar{\mathcal{R}}] = \varphi^{-1}[\mathcal{R}]$. Hence, $\widehat{\varphi}(u) \in \bar{\mathcal{R}}$ *iff* $u \in \widehat{\varphi}^{-1}[\bar{\mathcal{R}}] = \varphi^{-1}[\mathcal{R}]$. Since $\varphi^{-1}[\mathcal{R}] \in \mathcal{V}_\Sigma$ satisfies the explicit profinite equivalence $u \asymp v$, it follows that $u \in \varphi^{-1}[\mathcal{R}] \Leftrightarrow v \in \varphi^{-1}[\mathcal{R}]$, and hence:

$$\widehat{\varphi}(u) \in \bar{\mathcal{R}} \Leftrightarrow \widehat{\varphi}(v) \in \bar{\mathcal{R}}.$$

This concludes the proof of $\mathcal{E}_{\mathcal{V}}^{\Sigma} \supseteq \mathcal{F}_{\mathcal{V}}^{\Sigma}$. ◀

We can then show an equivalent version of Reiterman's theorem for streams. This idea was introduced in the context of languages by Gehrke, Grigorieff and Pin, see [21, §8—9] or [32, Theorem XIII.1.2, p. 224].

▶ **Proposition G.3** (A Reiterman theorem for streams of relations). *Let $\mathcal{V}$ be a class of relations. $\mathcal{V}$ is a $+$-stream of relations if, and only if, it is described by a set of profinite equivalences.*

Proof. The right-to-left implication is easy and similar to the right-to-left implication of Theorem 5.12.

So, let $\mathcal{V}$ be a $+$-stream of relations. Let $\mathcal{E}$ be the set of all profinite dependencies satisfied by $\mathcal{V}$. Clearly, $\mathcal{V} \subseteq [\![\mathcal{E}]\!]$. Conversely, let $\mathcal{R} \in [\![\mathcal{E}]\!]$. Let Σ be the alphabet over which $\mathcal{R}$ is written. By Fact G.2, $\mathcal{R}$ satisfies all explicit profinite equivalences satisfied by all relations of $\mathcal{V}_\Sigma$. It follows from Proposition G.1 that $\mathcal{R} \in \mathcal{V}_\Sigma$. ◀

▶ **Fact G.4.** *For any $+$-pseudovariety of regular languages $\mathcal{V}$,*[35] *$\mathcal{V}$-relations form a $+$-stream of relations.*

Proof. Closure under Boolean operations follows from the closure under Boolean operations of $\mathcal{V}$. Similarly, if $\varphi \colon \mathbf{S}_2^+\Gamma \to \mathbf{S}_2^+\Sigma$ is a morphism and $\mathcal{R} \in \mathcal{V}_\Sigma$, then let $L \in \mathcal{V}_\Sigma$ be such that $\mathcal{R} = L \cap \mathsf{WellFormed}_\Sigma$. Define $\psi \colon (\Gamma^2_-)^+ \to (\Sigma^2_-)^+$ by $\psi(\vec{a}) \mathrel{\hat{=}} \varphi(\vec{a})$. Then

$$\varphi^{-1}[\mathcal{R}] = \psi^{-1}[L] \cap \mathsf{WellFormed}_\Gamma.$$

Since $\mathcal{V}$ is closed by inverse morphisms, $\psi^{-1}[L] \in \mathcal{V}_\Gamma$ and it follows that $\varphi^{-1}[\mathcal{R}]$ is a $\mathcal{V}$-relation. ◀

G.2 Proof of the Lifting Theorem for Positive Semigroups

▶ **Theorem 6.8** (*Lifting Theorem for Semigroups*). *Given a relation $\mathcal{R} \subseteq (\Sigma^* \times \Sigma^*) \smallsetminus \{(\varepsilon, \varepsilon)\}$ and a $+$-pseudovariety of regular languages $\mathcal{V}$ corresponding to a pseudovariety of semigroups $\mathbb{V}$, letting $\mathcal{E}_\mathbb{V}$ denote the set of* all *profinite equalities satisfied by all semigroups of $\mathbb{V}$, then the following are equivalent:*

1. *$\mathcal{R}$ is a $\mathcal{V}$-relation,*
2. *$\mathcal{R}$ is recognized by a finite positive synchronous algebra $\mathbf{A}$ satisfying all profinite dependencies ${\mathcal{E}_\mathbb{V}}^{\mathsf{sync}}$ induced by $\mathcal{E}_\mathbb{V}$,*
3. *the syntactic positive synchronous algebra of $\mathcal{R}$ satisfies all profinite dependencies of ${\mathcal{E}_\mathbb{V}}^{\mathsf{sync}}$.*

▶ **Fact G.5.** *Let $\mathcal{V}$ be a $+$-pseudovariety of regular languages. Let $u, v \in \widehat{X^+}$. The pseudovariety $\mathcal{V}$ satisfies $u \to v$ iff it satisfies $u = v$.*

Proof. First, $\mathcal{V}$ satisfies $u = v$ *iff* it satisfies[36]

$$u \leftrightarrow v, \quad xu \leftrightarrow xv, \quad uy \leftrightarrow vy, \quad \text{and} \quad xuy \leftrightarrow xvy,$$

where x, y are new letters not occurring in u or v. Then, we will show that $\mathcal{V}$ satisfies $xuy \leftrightarrow xvy$, the other cases being similar. Let Σ be an alphabet and $L \in \mathcal{V}_\Sigma$. Let X be the set of variables of u and v, and $\bar{X} \mathrel{\hat{=}} X \cup \{x, y\}$. Let S be the syntactic semigroup of

[35] Of course the assumption can be weakened to $\mathcal{V}$ being a stream.
[36] This is because we are dealing with $+$-pseudovarieties and hence variables are interpreted as non-emtpy words.

L, and let $\varphi\colon \bar{X}^+ \to S$ be a semigroup morphism. We want to show that $\widehat{\varphi}(xuy) \in \varphi[L]$ *iff* $\widehat{\varphi}(xvy) \in \varphi[L]$. Let $\mathrm{Acc} \mathrel{\hat{=}} \widehat{\varphi}(x)^{-1}\varphi[L]\widehat{\varphi}(y)^{-1}$. Denote by φ_X the restriction of φ to X^+. Since $\mathcal{V}$ is a $+$-pseudovariety, $\varphi_X^{-1}[\mathrm{Acc}] \in \mathcal{V}_\Sigma$, and hence it satisfies $u = v$. It follows that $\widehat{\varphi}(u) = \widehat{\varphi_X}(u) \in \mathrm{Acc}$ *iff* $\widehat{\varphi}(v) = \widehat{\varphi_X}(v) \in \mathrm{Acc}$, and so, by definition of Acc, $\widehat{\varphi}(xuy) \in \varphi[L]$ *iff* $\widehat{\varphi}(xvy) \in \varphi[L]$. Hence, L satisfies $xuy \leftrightarrow xvy$. ◀

Proof of Theorem 6.8. ❧ (1) ⇒ (2). If $\mathcal{R}$ is a $\mathcal{V}$-relation, then there exists a semigroup $S \in \mathbb{V}$ together with a semigroup morphism $\varphi\colon (\Sigma^2_-)^+ \to S$ such that $\mathcal{R} = \varphi^{-1}[\mathrm{Acc}] \cap \mathsf{WellFormed}_\Sigma$ for some $\mathrm{Acc} \subseteq S$. Consider the morphism it induces, namely

$$\tilde{\varphi}\colon \mathbf{S}_2^+\Sigma \to \mathbf{A}_S.$$

Then $\tilde{\varphi}^{-1}[\mathrm{Acc}'] = \underline{\mathcal{R}}$ where Acc' is the closed subset $\{x_\sigma \mid x \in \mathrm{Acc} \wedge \sigma \in \mathcal{T}\}$ and so φ recognizes $\mathcal{R}$.

Moreover, we claim that $\mathbf{A}_S$ satisfies all profinite dependencies of $\mathcal{E}_{\mathbb{V}}{}^{\mathsf{sync}}$. Pick any $(u = v) \in \mathcal{E}_{\mathbb{V}}$, and let $\mathbf{t}$ be a typing consistent with both profinite words u and v. Fix a morphism $\psi\colon \mathbf{S}_2^+ X^{\mathbf{t}} \to \mathbf{A}_S$, where X denotes the set of variables of the profinite equality $u = v$. We must show that $\widehat{\psi}(u^{\mathbf{t}}) \asymp \widehat{\psi}(v^{\mathbf{t}})$.

Consider the "untyped" version of ψ, $\psi_{\mathsf{u}}\colon X^+ \to S$. Then $\widehat{\psi_{\mathsf{u}}}(u) = \widehat{\psi_{\mathsf{u}}}(v)$. It is then routine to check that $\widehat{\psi}(u^{\mathbf{t}}) = (\widehat{\psi_{\mathsf{u}}}(u))_\sigma$ where σ is the type of $u^{\mathbf{t}}$, and similarly, $\widehat{\psi}(v^{\mathbf{t}}) = \widehat{\psi_{\mathsf{u}}}(v)_\tau$ where τ is the type of $v^{\mathbf{t}}$. Hence, from $\widehat{\psi_{\mathsf{u}}}(u) = \widehat{\psi_{\mathsf{u}}}(v)$ in S, it follows that $\widehat{\psi}(u^{\mathbf{t}}) \asymp \widehat{\psi}(v^{\mathbf{t}})$ in $\mathbf{A}_S$, which concludes the proof that $\mathbf{A}_S$ satisfies the profinite dependency $u^{\mathbf{t}} \asymp v^{\mathbf{t}}$.

❧ (2) ⇔ (3). Indeed, $[\![\mathcal{E}_{\mathbb{V}}{}^{\mathsf{sync}}]\!]$ is a pseudovariety of positive synchronous algebras.

❧ (3) ⇒ (1). This implication is quite surprising because its proof is not constructive—one could except the proof to explicitly build a semigroup S accepting a language L such that $\mathcal{R} = L \cap \mathsf{WellFormed}_\Sigma$ from $\mathbf{A}_{\mathcal{R}}$. Since $\mathcal{V}$-relations form a $+$-stream of relations by Fact G.4, they can be defined by a set of profinite equivalences by Proposition G.3. Let $\mathcal{F}$ be the set of all profinite equivalences satisfied by $\mathcal{V}$-relations.

▶ **Fact G.6.** *Let $u, v \in \widehat{\mathbf{S}_2^+\Sigma}$. Define $u^{\mathbf{f}}$ as the profinite word of $\widehat{(\Sigma^2_-)^+}$ obtained from u by forgetting the types of variables. If $u^{\mathbf{f}} = v^{\mathbf{f}} \notin \mathcal{E}_{\mathbb{V}}$ then $(u \leftrightarrow v) \notin \mathcal{F}$.*

Assume that $u^{\mathbf{f}} = v^{\mathbf{f}} \notin \mathcal{E}_{\mathbb{V}}$. Then by Fact G.5, $\mathcal{V}$ does not satisfy $u^{\mathbf{f}} \leftrightarrow v^{\mathbf{f}}$. Let Σ and $L \in \mathcal{V}_\Sigma$ be such that L does not satisfy $u^{\mathbf{f}} \leftrightarrow v^{\mathbf{f}}$.

To show $(u \leftrightarrow v) \notin \mathcal{F}$, it suffices then to build a $\mathcal{V}$-relation $\mathcal{R}_L$ whose syntactic positive synchronous algebra does not satisfy $u \leftrightarrow v$. Let Γ be any alphabet big enough so that $|\Gamma| \geq |\Sigma|$, and let μ be a collection of three surjective maps $\Gamma \times \Gamma \twoheadrightarrow \Sigma$, $\Gamma \times \{\_\} \twoheadrightarrow \Sigma$ and $\{\_\} \times \Gamma \twoheadrightarrow \Sigma$. This naturally extends to a morphism $\mu\colon (\Gamma^2_-)^+ \to \Sigma^+$, and define $\mathcal{R}$ as $\mu^{-1}[L] \cap \mathsf{WellFormed}_\Gamma$. Since $\mathcal{V}$ is a $+$-pseudovariety of regular languages, $\mu^{-1}[L] \in \mathcal{V}_\Gamma$ and hence $\mathcal{R}$ is a $\mathcal{V}$-relation. By construction, for any $w_\sigma \in \mathbf{S}_2^+\Gamma$:

$$w_\sigma \in \mathcal{R} \textit{ iff } \mu(w_\sigma) \in L. \tag{1}$$

By continuity of $\widehat{\mu}$, and since $\bar{L}$ is clopen in $\widehat{\Sigma^+}$ and $\bar{\mathcal{R}}$ is clopen in $\widehat{\mathbf{S}_2^+\Gamma}$—the proof of this claim can be easily adapted from [32, Proposition X.3.16, p. 186]—, it follows that for all $w_\sigma \in \widehat{\mathbf{S}_2^+\Gamma}$:

$$w_\sigma \in \bar{\mathcal{R}} \textit{ iff } \widehat{\mu}(w_\sigma) \in \bar{L}. \tag{2}$$

By construction, L does not satisfy $u^{\mathbf{f}} \leftrightarrow v^{\mathbf{f}}$, so there exists a semigroup morphism $\varphi\colon (X^{\mathbf{f}})^+ \to \Sigma^+$—where X is the set of variables of u and v—such that $\widehat{\varphi}(u^{\mathbf{f}}) \in \bar{L}$ but $\widehat{\varphi}(v^{\mathbf{f}}) \notin \bar{L}$, or symmetrically. Pick any map $\psi\colon \mathbf{S}_2^+ X \to \mathbf{S}_2^+\Gamma$ such that

$$\widehat{\mu \circ \psi}(u) = \widehat{\varphi}(u^{\mathbf{f}}) \text{ and } \widehat{\mu \circ \psi}(v) = \widehat{\varphi}(v^{\mathbf{f}}).$$

Such a map must exist by construction of Γ and μ. Indeed, for $x \in X$, consider the word $\varphi(x^{\mathbf{f}}) \in \Sigma^+$. Recall—see Theorem 5.12—that *w.l.o.g.* all variables of X have type L/L or L/B or B/L. Assume *e.g.* that x has type L/B (the other cases are similar), then since $\mu\colon \Gamma \times \{\text{\_}\} \twoheadrightarrow \Sigma$ is surjective, there exists $u \in (\Gamma \times \{\text{\_}\})^+$, or equivalently $u_{\mathrm{L}/\mathrm{B}} \in \mathbf{S}_2^+\Gamma$, such that $\mu(u) = \varphi(x^{\mathbf{f}})$. Letting $\psi(x_{\mathrm{L}/\mathrm{B}}) \mathrel{\hat{=}} u_{\mathrm{L}/\mathrm{B}}$ defines a morphism $\psi\colon \mathbf{S}_2^+X \to \mathbf{S}_2^+\Gamma$ which satisfies the desired property.

From $\widehat{\varphi}(u^{\mathbf{f}}) \in \bar{L}$ and $\widehat{\varphi}(v^{\mathbf{f}}) \notin \bar{L}$, it follows by (❦) that $\widehat{\psi}(u) \in \bar{\mathcal{R}}$ but $\widehat{\psi}(v) \notin \bar{\mathcal{R}}$. Hence, the $\mathcal{V}$-relation $\mathcal{R}$ does not satisfy $u \leftrightarrow v$. Therefore, $(u \leftrightarrow v) \notin \mathcal{F}$, which concludes the proof of Fact G.6.

By (1) $\Rightarrow$ (3), $\mathcal{E}_{\mathbb{V}}{}^{\mathsf{sync}} \subseteq \mathcal{F}$, in the sense that for each $u \leftrightarrow v$, then all four equations

$$u \leftrightarrow v, \quad xu \leftrightarrow xv, \quad uy \leftrightarrow vy, \quad \text{and} \quad xuy \leftrightarrow xvy$$

belong to $\mathcal{F}$. By the contraposition of Fact G.6, if $(u \leftrightarrow v) \in \mathcal{F}$, then $u^{\mathbf{f}} = v^{\mathbf{f}} \in \mathcal{E}_{\mathbb{V}}$, which implies $(u \asymp v) \in \mathcal{E}_{\mathbb{V}}{}^{\mathsf{sync}}$ and by (1) $\Rightarrow$ (3), $(u \asymp v) \in \mathcal{F}$.

Let $\mathcal{F}_0$ be the subset of profinite dependencies of $\mathcal{F}$. By the paragraph above, a relation satisfies $\mathcal{F}$ *iff* it satisfies $\mathcal{F}_0$, and so, by definition of $\mathcal{F}$:

a relation is a $\mathcal{V}$-relation *iff* it satisfies $\mathcal{F}_0$. (☙)

We claim that $\mathcal{E}_{\mathbb{V}}{}^{\mathsf{sync}} \supseteq \mathcal{F}_0$. If $(u \asymp v) \in \mathcal{F}_0$ then $(u^{\mathbf{f}} = v^{\mathbf{f}}) \in \mathcal{E}_{\mathbb{V}}$ by Fact G.6, and so $(u \asymp v) \in \mathcal{E}_{\mathbb{V}}{}^{\mathsf{sync}}$. Hence, if $\mathcal{R}$ satisfies all profinite dependencies of $\mathcal{E}_{\mathbb{V}}{}^{\mathsf{sync}}$, then it satisfies all profinite dependencies of $\mathcal{F}_0$ and hence by (☙), it is a $\mathcal{V}$-relation. This concludes the proof of (3) $\Rightarrow$ (1). ◀

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