First-order separation over countable ordinals

FoSSaCS '22, Munich

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Finite words

First-order logic (FO)

Goal: to better understand first-order logic on countable ordinals. **Warm-up:** finite words.

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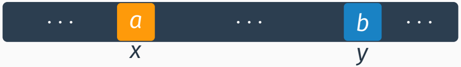
iff $u \quad =$



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Let $u \in \Sigma^*$ where $A = \{a, b, c, \ldots\}$.

$$u \models \exists x. \exists y. x < y \wedge a(x) \wedge b(y)$$

$$\text{iff } u = \cdots \underset{x}{a} \cdots \underset{y}{b} \cdots$$

$$\text{i.e. } u \in \Sigma^* a \Sigma^* b \Sigma^*.$$

FO-definability

FO-DEFINABILITY:

Input: Morphism $f : \Sigma^* \to M$

Question: Is f **FO-definable**?

$$f(u) := \begin{cases} m_1 & \text{if } u \models \varphi_1 \\ m_2 & \text{if } u \models \varphi_2 \\ \vdots & \vdots \\ m_n & \text{if } u \models \varphi_n \end{cases}$$

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Theorem [Schützenberger '65 & McNaughton-Papert '71]:
A morphism $f : \Sigma^* \to M$ is FO-definable IFF $\operatorname{im} f$ is **aperiodic**.

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every group in $\operatorname{im} f$ is trivial

Corollary: FO-DEFINABILITY is decidable.

Example!

$$
\begin{aligned}
f : \{a, b\}^* &\to M \\
u &\mapsto \begin{cases} 1 & \text{if } u = \varepsilon \\ a & \text{if } u \in a(aa)^*, \\ aa & \text{if } u \in (aa)^+, \\ 0 & \text{if } u \text{ contains a `b'} \end{cases}
\end{aligned}
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$\cdot$	1	a	aa	0
1	1	a	aa	0
a	a	aa	a	0
aa	aa	a	aa	0
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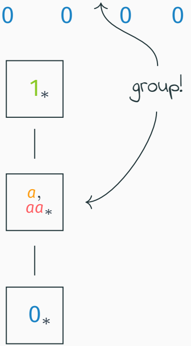

f is not FO-definable...

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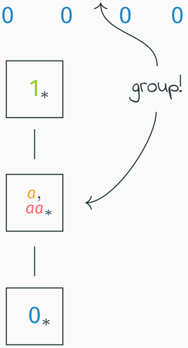
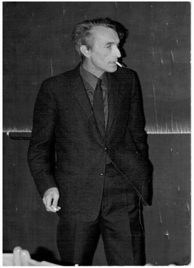


f is not FO-definable...
but still carries "FO-describable information"

Qualitative vs. quantitative

$f : \Sigma^* \to M$



$\operatorname{im} f$ aperiodic → f is FO-definable

$\operatorname{im} f$ not aperiodic → f is not FO-definable

Qualitative vs. quantitative

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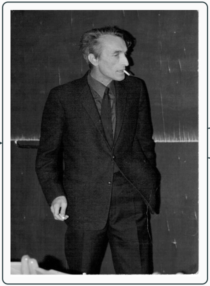

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Can we make a quantitative version of Schützenberger-McNaughton-Papert?

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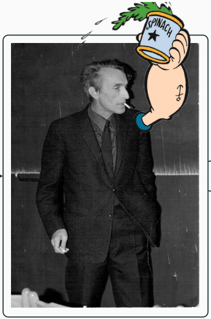

im f aperiodic → f is FO-definable

im f not aperiodic → f is not FO-definable → extract all FO-definable information from f

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Henckell's theorem: motivation & statement

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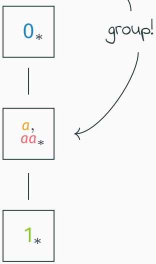

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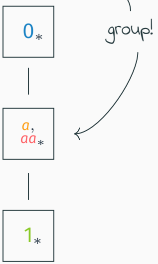

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$\exists\ \langle M\rangle^{*,\mathrm{grp}}$ computable submonoid of $\mathcal{P}(M)$ s.t.:

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Idea behind $\langle M\rangle^{*,\mathrm{grp}}$: "saturate" your monoid with groups.

Definition: $\langle M\rangle^{*,\mathrm{grp}}$ is the smallest submonoid $\mathcal{N}$ of $\mathcal{P}(M)$ containing all singletons and such that:

$$\begin{array}{rl} \text{IF} & \mathcal{G} \subseteq \mathcal{N} \text{ is a group,} \\ \text{THEN} & \bigcup \mathcal{G} \in \mathcal{N}. \end{array}$$

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Words over countable ordinals

Beyond finite words

ω**-words**: FO cannot capture *group-like phenomena*
[Perrin '84] (qualitative)
[Place & Zeitoun '16] (quantitative).

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Words indexed by countable ordinals:

Example: bca, $cabc(ab)^\omega$, $(ab^\omega c)^\omega$, etc.

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The word has a last position, and it is an 'a'.

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$a^\omega c b^\omega c a$	$(ab)^\omega b$	a^ω
yes	no	no

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FO cannot capture group-like phenomena over countable ordinals:

[Bedon '01] (qualitative)

[Colcombet, van Gool & M., '22] (quantitative).

Languages over countable ordinals: example

Word	a^ω	$(a^\omega a)^\omega$	$(a^\omega)^\omega a^{53}$	$a^{\omega\cdot\alpha+k}$
Longest finite suffix (LFS)	0	0	53	k

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group!

Henckell's theorem over countable ordinals

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The statement of the theorem is easy to generalise.
The proof isn't.

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$$f : u \mapsto \begin{cases} 1 & \text{if } u \text{ is the empty word} \\ a & \text{if } u \text{ is finite \& odd} \\ aa & \text{if } u \text{ is finite \& even} \\ a^\omega & \text{if } u \text{ is infinite \& even LFS} \\ a^\omega a & \text{if } u \text{ is infinite \& odd LFS} \end{cases}$$

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$$f : u \mapsto \begin{cases} 1 & \text{if } u \text{ is the empty word} \\ a & \text{if } u \text{ is finite \& odd} \\ aa & \text{if } u \text{ is finite \& even} \\ a^\omega & \text{if } u \text{ is infinite \& even LFS} \\ a^\omega a & \text{if } u \text{ is infinite \& odd LFS} \end{cases}$$

$$\langle M \rangle^{\text{ord,grp}} = \{\{1\}, \{a\}, \{aa\}, \{a^\omega\}, \{a^\omega a\}, \{a, aa\}, \{a^\omega, a^\omega a\}\}.$$

Countable ordinal words

Goal: Extract as many FO-definable information from $f : \Sigma^{\text{ord}} \to M$ as possible.

Main tool: closure $\langle M \rangle^{\text{ord,grp}}$ of M under product, ω-iteration and "groupisation" of the singletons of M.

Theorem [Colcombet, van Gool & M. '22]:

- for every set $X \in \langle M \rangle^{\text{ord,grp}}$, "elements of X cannot be distinguished by FO"
- for every $f : \Sigma^{\text{ord}} \to M$, there exists an FO-approximant $g : \Sigma^{\text{ord}} \to \langle M \rangle^{\text{ord,grp}}$.

The statement of the theorem is easy to generalise.
The proof isn't.

Succinct proof sketch

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  Start from a (partial) FO-approximant on $L \subseteq \Sigma^{\mathrm{ord}}$ and build a (partial) FO-approximant on $L' \supseteq L$.
  - True on finite words [Henckell '88, Place & Zeitoun '16]
  - True on ω-words [Place & Zeitoun '16]
  - True on a^{ord}
  - Inductive case: Detect some patterns in your word using first-order logic.

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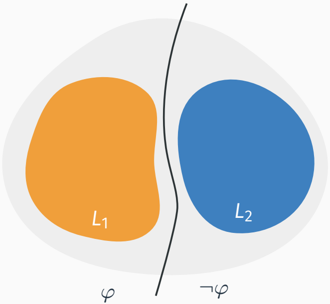

L_1 and L_2 are **FO-separable** whenever there exists $\varphi \in$ FO such that

$$u \models \varphi \text{ for all } u \in L_1 \qquad v \not\models \varphi \text{ for all } v \in L_2$$

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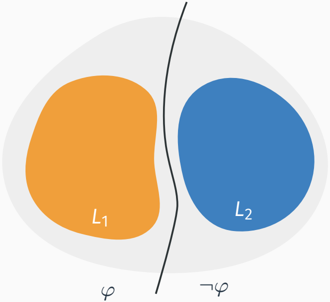

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$u \vDash \varphi$ for all $u \in L_1$ $\qquad$ $v \nvDash \varphi$ for all $v \in L_2$

FO-SEPARABILITY:
Input: L_1, L_2 regular languages
Question: Are L_1 and L_2 FO-separable?

Decidable!

Open questions & ongoing work

Domain (count. linear order)	Characterisation of FO: non-trivial ... are forbidden	Qualitative	Quantitative
Finite	groups	[Schützenberger '65, McNaughton & Papert '71]	[Henckell '88]
ω	groups	[Perrin '84]	[Place & Zeitoun '16]
Ordinals	groups	[Bedon '01]	[Colcombet, van Gool & M. '22]

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- **finite words**: for some *varieties*, the *saturation algorithm* works (ex: aperiodic), for some it doesn't (ex: $\mathcal{J}$-trivial). Can we characterise varieties for which it works? [van Gool & Steinberg '19]
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Scattered	groups, gaps	[Bès & Carton '11]	ongoing work
Countable	groups, gaps, shuffles	[Colcombet & Sreejith '15]	ongoing work