Computer Vision:
Maths review

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Why we talk about maths?

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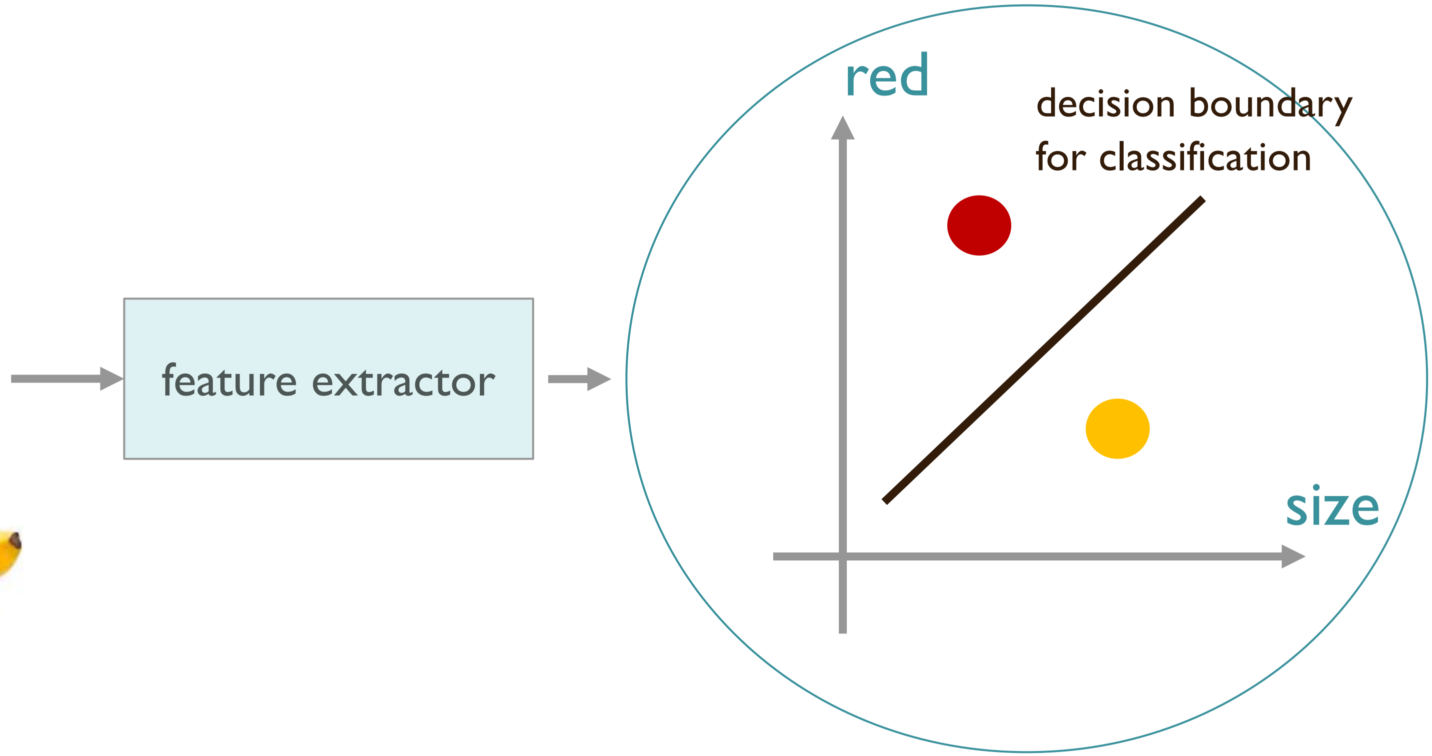


image space

feature space

vector space

Outline

Vector spaces

Derivatives

Vector spaces

DEFINITION 2.1 (VECTOR SPACE) A *vector space* over a field of scalars $\mathbb{C}$ (or $\mathbb{R}$) is a set of vectors, V, together with operations of vector addition and scalar multiplication. For any x, y, z in V and α, β in $\mathbb{C}$ (or $\mathbb{R}$), these operations must satisfy the following properties:

(i) *Commutativity:* $x + y = y + x$.

(ii) *Associativity:* $(x + y) + z = x + (y + z)$ and $(\alpha\beta)x = \alpha(\beta x)$.

(iii) *Distributivity:* $\alpha(x + y) = \alpha x + \alpha y$ and $(\alpha + \beta)x = \alpha x + \beta x$.

Furthermore, the following hold:

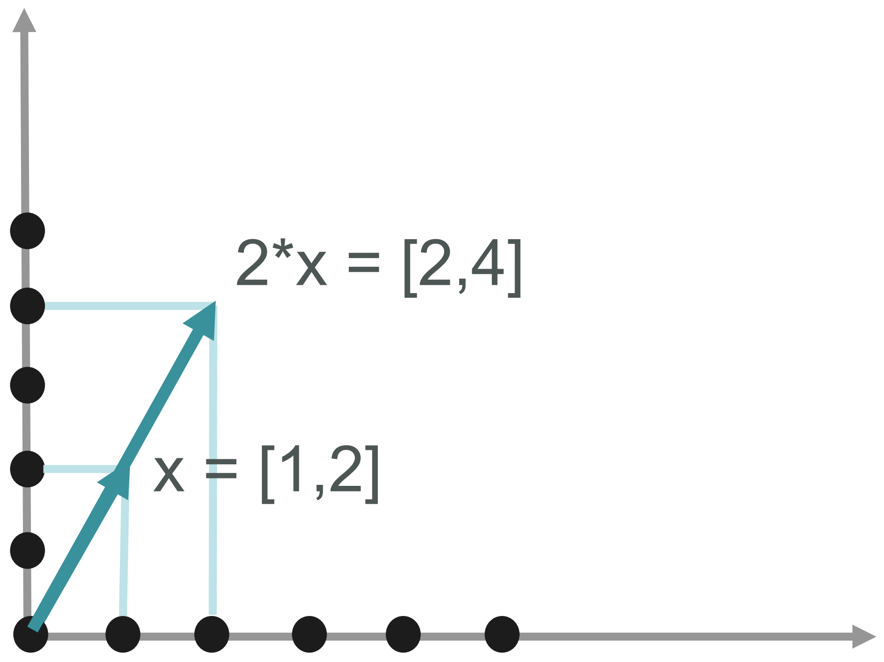
(iv) *Additive identity:* There exists an element $\mathbf{0}$ in V such that $x + \mathbf{0} = \mathbf{0} + x = x$ for every x in V.

(v) *Additive inverse:* For each x in V, there exists a unique element $-x$ in V such that $x + (-x) = (-x) + x = \mathbf{0}$.

(vi) *Multiplicative identity:* For every x in V, $1 \cdot x = x$.

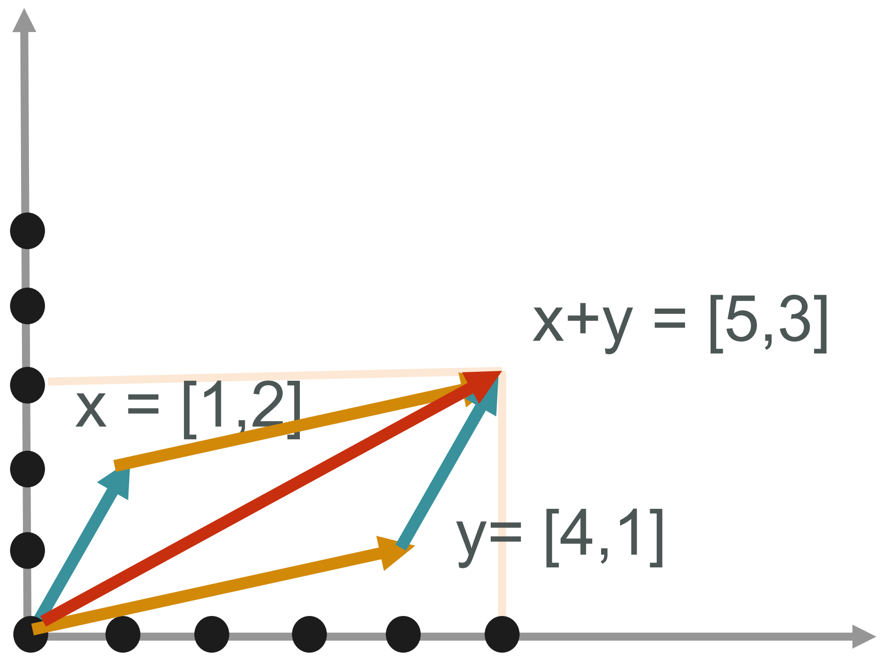
Vector spaces

Scale (vector, scalar -> vector)



Vector spaces

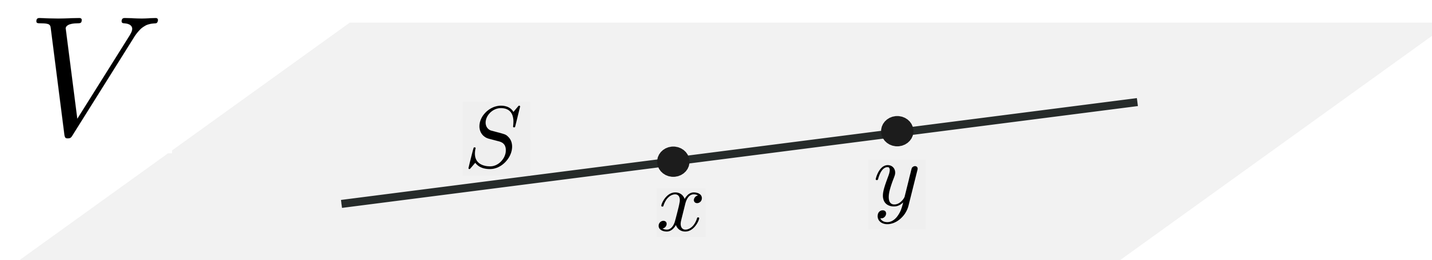
Add (vector, vector -> vector)



Vector spaces

Definition 2.2 (Subspace) A nonempty subset S of a vector space V is a *subspace* when it is closed under the operations of vector addition and scalar multiplication:

(i) For all x and y in S, $x + y$ is in S.
(ii) For all x in S and α in $\mathbb{C}$ (or $\mathbb{R}$), αx is in S.



Vector spaces

DEFINITION 2.7 (INNER PRODUCT) An *inner product* on a vector space V over $\mathbb{C}$ (or $\mathbb{R}$) is a complex-valued (or real-valued) function $\langle \cdot, \cdot \rangle$ defined on $V \times V$, with the following properties for any $x, y, z \in V$ and $\alpha \in \mathbb{C}$ (or $\mathbb{R}$):

(i) *Distributivity:* $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$.

(ii) *Linearity in the first argument:* $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$.

(iii) *Hermitian symmetry:* $\langle x, y \rangle^* = \langle y, x \rangle$.

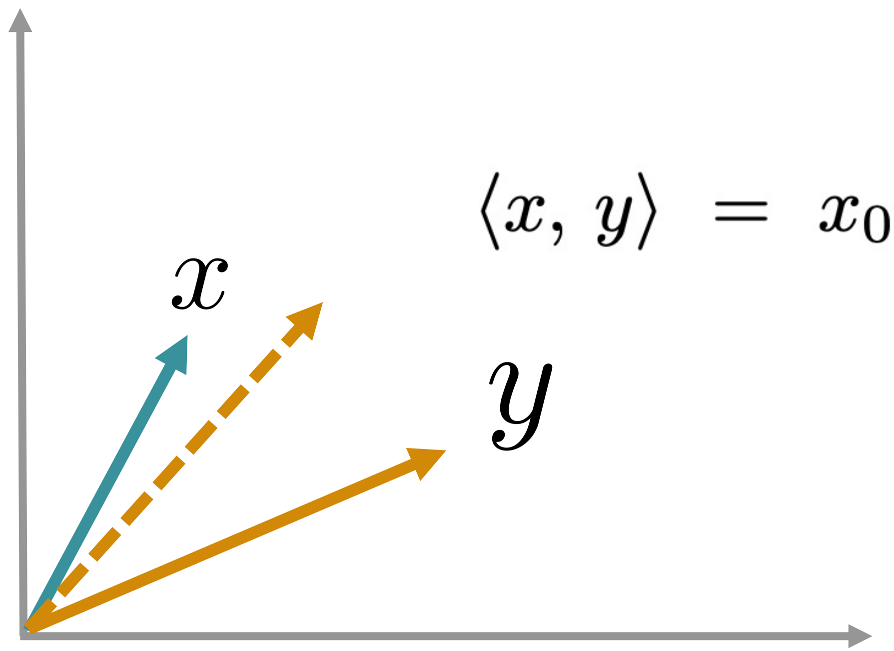
(iv) *Positive definiteness:* $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ if and only if $x = \mathbf{0}$.

"Similarity" $$\langle x, y \rangle = \sum_{n=0}^{N-1} x_n y_n^* = y^* x$$

Vector spaces

Inner product (vector, vector -> scalar)

$$\langle x,\ y\rangle \ = \ x_0 y_0 + x_1 y_1$$



Vector spaces

Definition 2.8 (Orthogonality)

(i) Vectors x and y are said to be *orthogonal* when $\langle x, y \rangle = 0$, written as $x \perp y$.

(ii) A set of vectors S is called *orthogonal* when $x \perp y$ for every x and y in S such that $x \neq y$.

(iii) A set of vectors S is called *orthonormal* when it is orthogonal and $\langle x, x \rangle = 1$ for every x in S.

(iv) A vector x is said to be *orthogonal* to a set of vectors S when $x \perp s$ for all $s \in S$, written as $x \perp S$.

(v) Two sets S_0 and S_1 are said to be *orthogonal* when every vector s_0 in S_0 is orthogonal to the set S_1, written as $S_0 \perp S_1$.

(vi) Given a subspace S of a vector space V, the *orthogonal complement* of S, denoted $S^\perp$, is the set $\{x \in V \mid x \perp S\}$.

Vector spaces

DEFINITION 2.9 (NORM) A *norm* on a vector space V over $\mathbb{C}$ (or $\mathbb{R}$) is a real-valued function $\|\cdot\|$ defined on V, with the following properties for any $x, y \in V$ and $\alpha \in \mathbb{C}$ (or $\mathbb{R}$):

(i) *Positive definiteness:* $\|x\| \geq 0$, and $\|x\| = 0$ if and only if $x = \mathbf{0}$.
(ii) *Positive scalability:* $\|\alpha x\| = |\alpha| \, \|x\|$.
(iii) *Triangle inequality:* $\|x + y\| \leq \|x\| + \|y\|$, with equality if and only if $y = \alpha x$.

"length"

standard norm

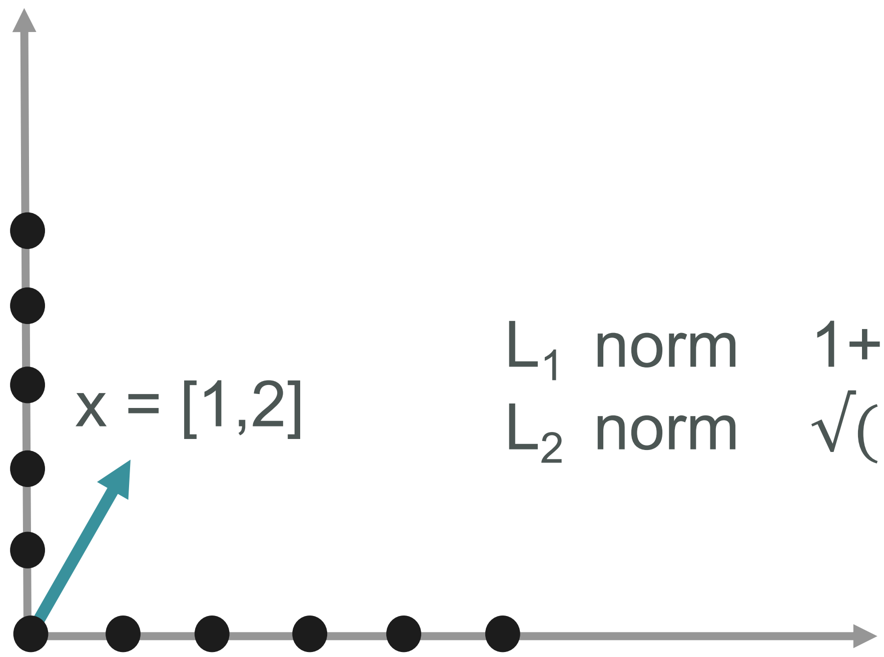
$$\|x\| = \sqrt{\langle x, x\rangle} = \left(\sum_{n=0}^{N-1} |x_n|^2\right)^{1/2}$$

p-norm

$$\|x\|_p = \left(\sum_{n=0}^{N-1} |x_n|^p\right)^{1/p}$$

Vector spaces

Norm (vector -> scalar)



L_1 norm $1+2 = 3$

L_2 norm $\sqrt{(1^2 + 2^2)}$

Vector spaces

Definition 2.10 (Metric, or distance) In a normed vector space, the *metric*, or *distance*, between vectors x and y is the norm of their difference:

$$d(x, y) = \|x - y\|.$$

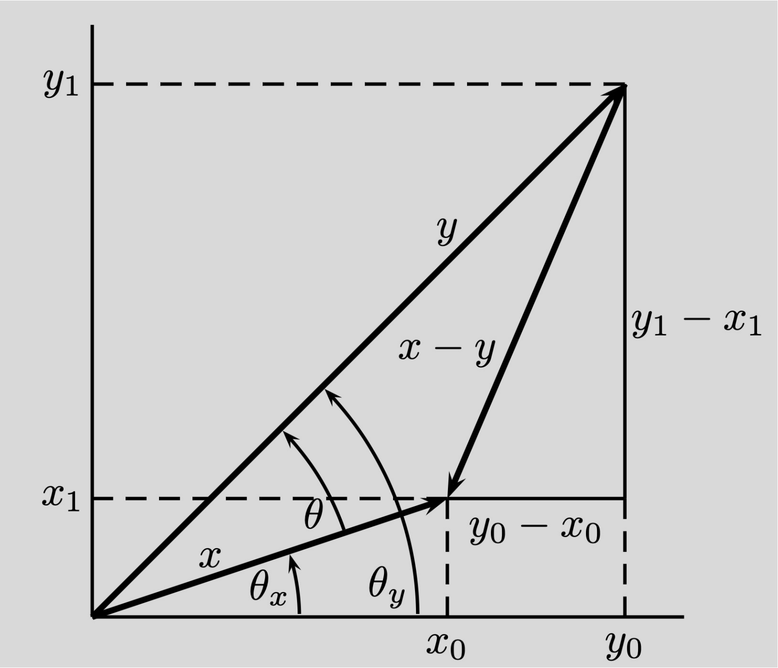
"length of difference"

Vector spaces: 2D example

Inner product

$$x = \begin{bmatrix} x_0 & x_1 \end{bmatrix}^\top \quad y = \begin{bmatrix} y_0 & y_1 \end{bmatrix}^\top$$

$$\begin{aligned}
\langle x,\, y\rangle &= x_0 y_0 + x_1 y_1 \\
&= (\|x\| \cos\theta_x)(\|y\| \cos\theta_y) + (\|x\| \sin\theta_x)(\|y\| \sin\theta_y) \\
&= \|x\|\,\|y\|(\cos\theta_x \cos\theta_y + \sin\theta_x \sin\theta_y) \\
&= \|x\|\,\|y\| \cos(\theta_x - \theta_y).
\end{aligned}$$



Norm

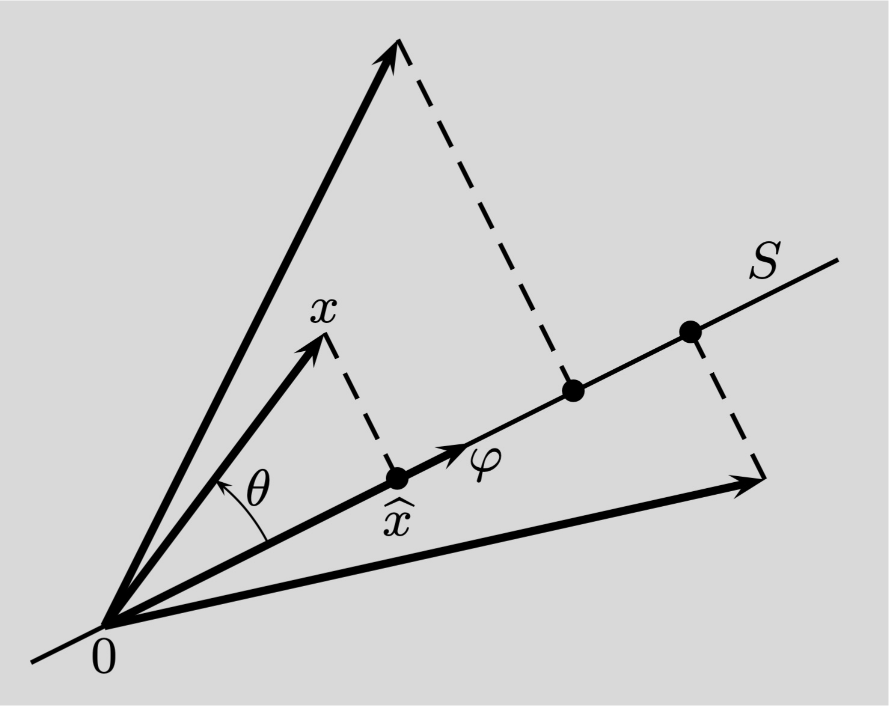
$$\|x\| = \sqrt{\langle x,\, x\rangle} = \sqrt{x_0^2 + x_1^2}.$$

Distance

$$d(x, y) = \|x - y\| = \sqrt{\langle x - y,\, x - y\rangle}$$

Vector spaces

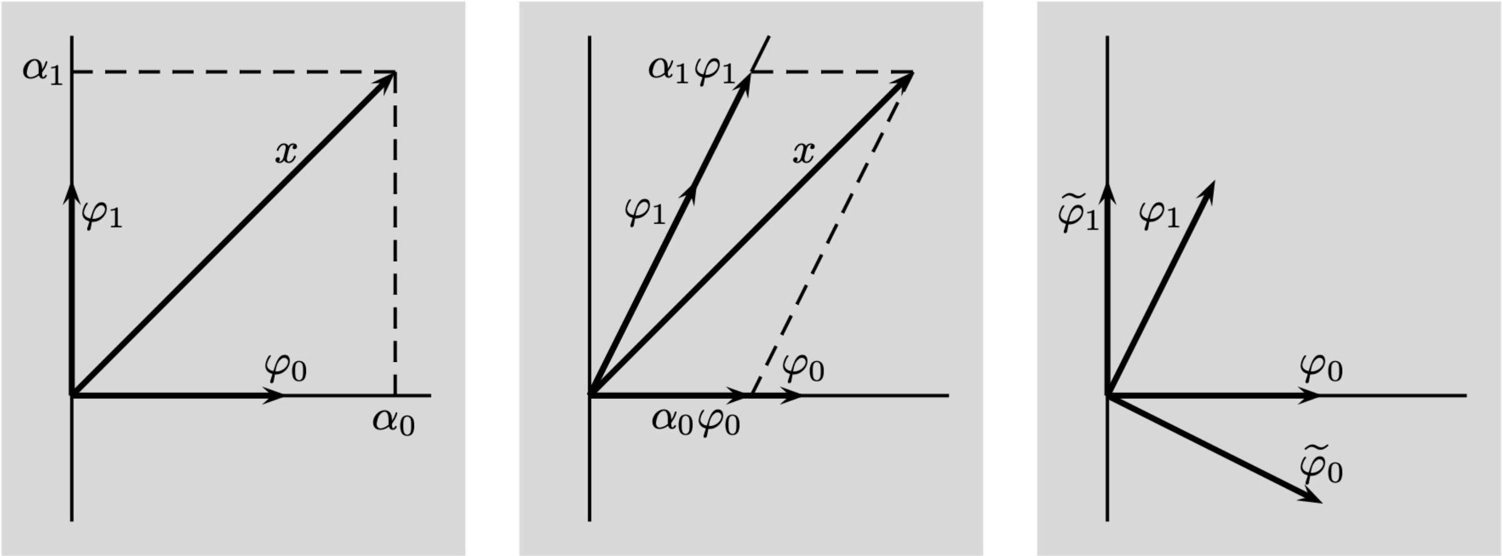
Projection



$$\widehat{x} \overset{(a)}{=} (\|x\| \cos\theta)\frac{\varphi}{\|\varphi\|} = (\|x\|\,\|\varphi\|\cos\theta)\frac{\varphi}{\|\varphi\|^2} \overset{(b)}{=} \frac{1}{\|\varphi\|^2}\langle x,\, \varphi\rangle\varphi$$

Vector spaces

Projection



(a) Expansion with an orthonormal basis.

(b) Expansion with a nonorthogonal basis.

(c) Basis $\{\varphi_0, \varphi_1\}$ and its dual $\{\widetilde{\varphi}_0, \widetilde{\varphi}_1\}$.

Vector spaces

Projection

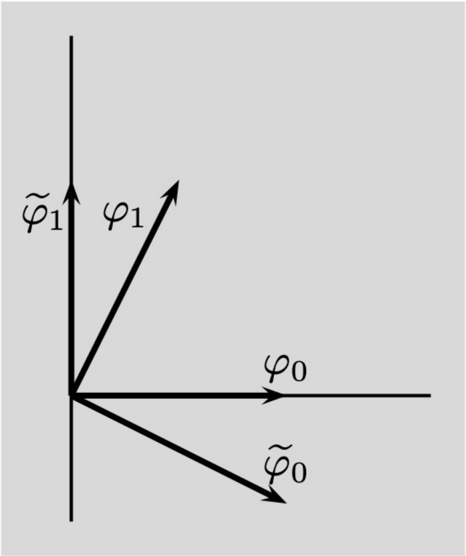
analysis

$$\alpha = \begin{bmatrix}\alpha_0 \\ \alpha_1\end{bmatrix} = \begin{bmatrix}\langle x, \widetilde{\varphi}_0\rangle \\ \langle x, \widetilde{\varphi}_1\rangle\end{bmatrix} = \underbrace{\begin{bmatrix}\widetilde{\varphi}_{00} & \widetilde{\varphi}_{01} \\ \widetilde{\varphi}_{10} & \widetilde{\varphi}_{11}\end{bmatrix}}_{\widetilde{\Phi}^\top} \begin{bmatrix}x_0 \\ x_1\end{bmatrix} = \widetilde{\Phi}^\top x.$$

synthesis

$$x = \begin{bmatrix}x_0 \\ x_1\end{bmatrix} = \alpha_0 \begin{bmatrix}\varphi_{00} \\ \varphi_{01}\end{bmatrix} + \alpha_1 \begin{bmatrix}\varphi_{10} \\ \varphi_{11}\end{bmatrix} = \underbrace{\begin{bmatrix}\varphi_{00} & \varphi_{10} \\ \varphi_{01} & \varphi_{11}\end{bmatrix}}_{\Phi} \begin{bmatrix}\alpha_0 \\ \alpha_1\end{bmatrix} = \begin{bmatrix}\varphi_0 & \varphi_1\end{bmatrix} \begin{bmatrix}\alpha_0 \\ \alpha_1\end{bmatrix}$$

$$= \Phi\alpha = \Phi\widetilde{\Phi}^\top x,$$



(c) Basis $\{\varphi_0, \varphi_1\}$ and its dual $\{\widetilde{\varphi}_0, \widetilde{\varphi}_1\}$.

Vector spaces

DEFINITION 2.34 (BASIS) The set of vectors $\Phi = \{\varphi_k\}_{k\in\mathcal{K}} \subset V$, where $\mathcal{K}$ is finite or countably infinite, is called a *basis* for a normed vector space V when

(i) it is *complete* in V, meaning that, for any $x \in V$, there is a sequence $\alpha \in \mathbb{C}^{\mathcal{K}}$ such that

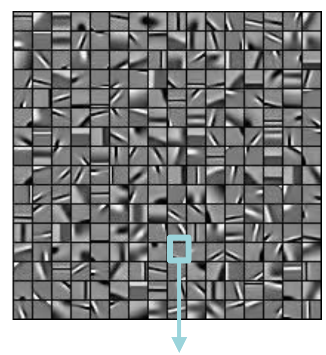
$$x = \sum_{k\in\mathcal{K}} \alpha_k \varphi_k; \tag{2.87}$$

and

(ii) for any $x \in V$, the sequence α that satisfies (2.87) is unique.

x

synthesis



$\varphi_k \quad \alpha_k$

Vector spaces

THEOREM 2.39 (ORTHONORMAL BASIS EXPANSIONS) Let $\Phi = \{\varphi_k\}_{k\in\mathcal{K}}$ be an orthonormal basis for a Hilbert space H. The unique expansion with respect to Φ of any x in H has expansion coefficients

analysis

$$\alpha_k = \langle x, \varphi_k \rangle \quad \text{for } k \in \mathcal{K}, \quad \text{or}, \tag{2.93a}$$

$$\alpha = \Phi^* x. \tag{2.93b}$$

Synthesis with these coefficients yields

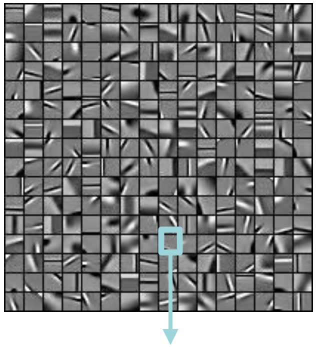
synthesis

$$x = \sum_{k\in\mathcal{K}} \langle x, \varphi_k \rangle \varphi_k \tag{2.94a}$$

$$= \Phi\alpha = \Phi\Phi^* x. \tag{2.94b}$$

analysis

synthesis



x

$\varphi_k \quad \alpha_k$

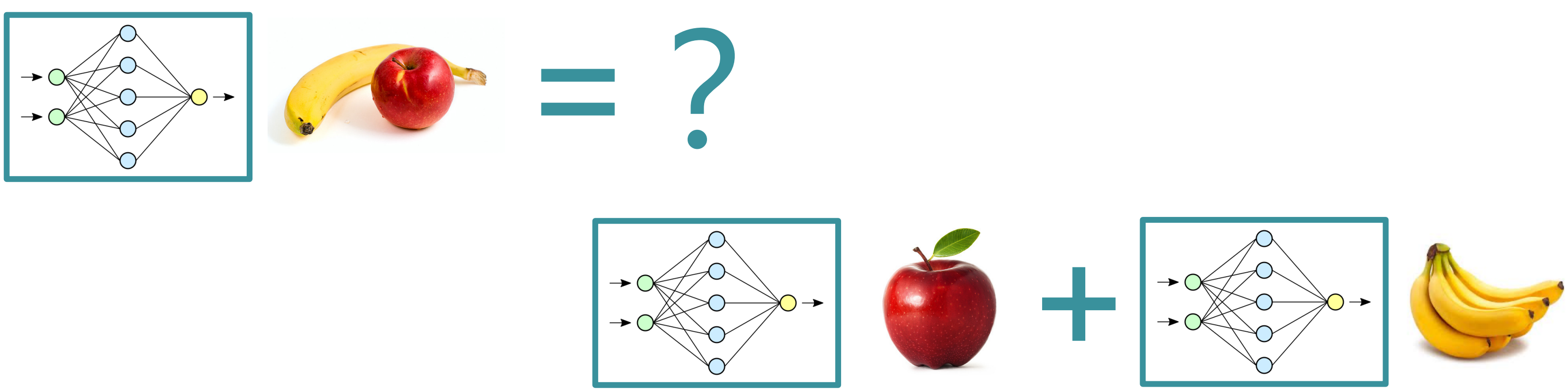
Vector spaces

Definition 2.17 (Linear operator) A function $A : H_0 \to H_1$ is called a *linear operator* from H_0 to H_1 when, for all x, y in H_0 and α in $\mathbb{C}$ (or $\mathbb{R}$), the following hold:

(i) *Additivity:* $A(x+y) = Ax + Ay$.

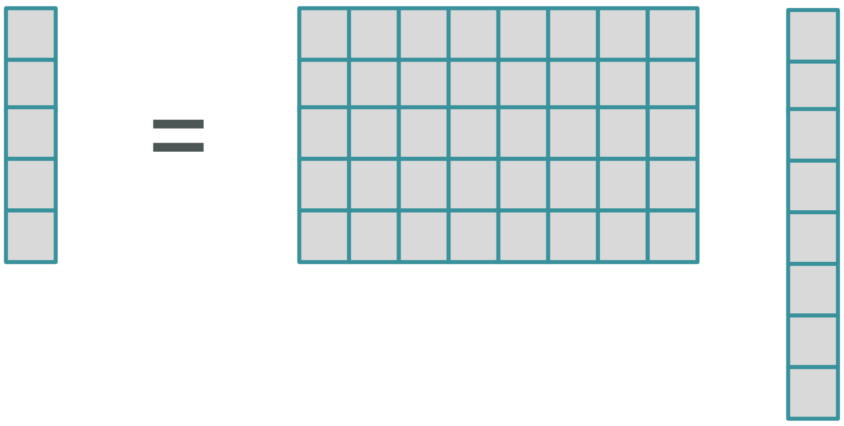
(ii) *Scalability:* $A(\alpha x) = \alpha(Ax)$.

When the domain H_0 and the codomain H_1 are the same, A is also called a linear operator on H_0.



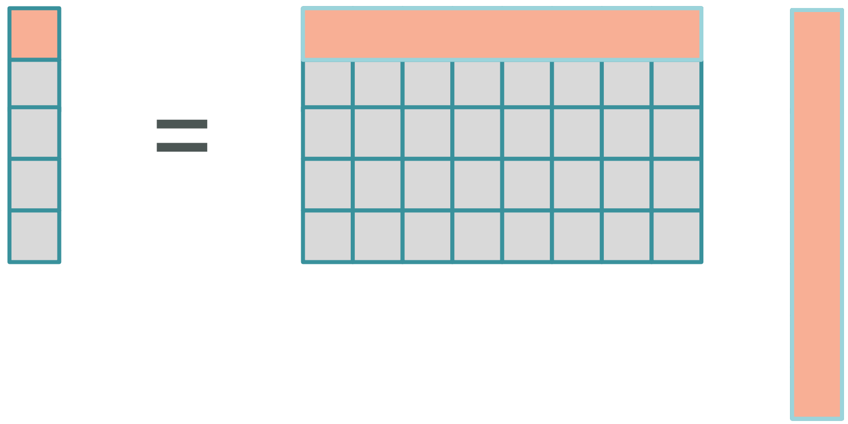
Vector spaces

Matrix-vector product



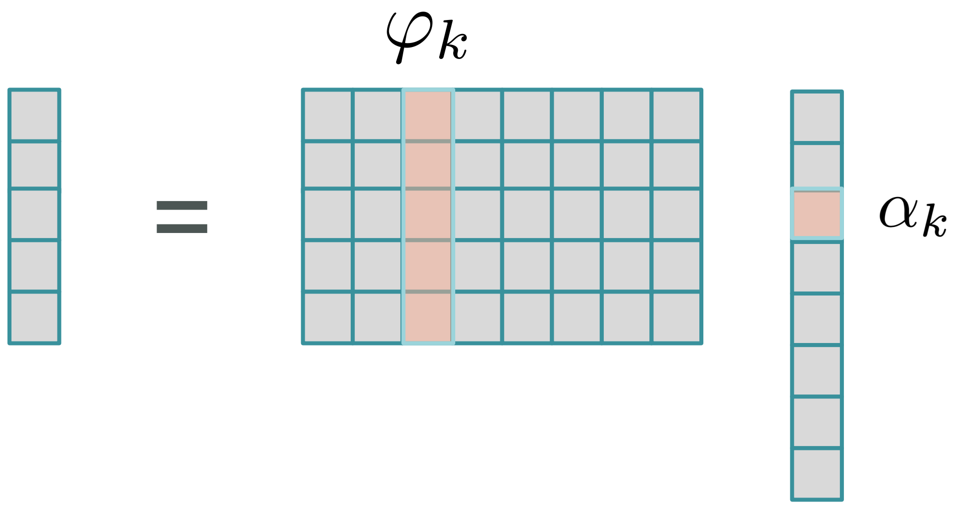
Vector spaces

Matrix-vector product



Vector spaces

Matrix-vector product



$$x = \sum_{i \in \mathcal{K}} \alpha_i \varphi_i$$

Vector spaces

Subspaces about a matrix

- Column space

  $$\mathcal{R}(A) \;=\; \mathrm{span}(\{a_0,\, a_1,\, \ldots,\, a_{N-1}\}) \;=\; \{y \in \mathbb{R}^M \mid y = Ax \text{ for some } x \in \mathbb{R}^N\}$$

- Row space

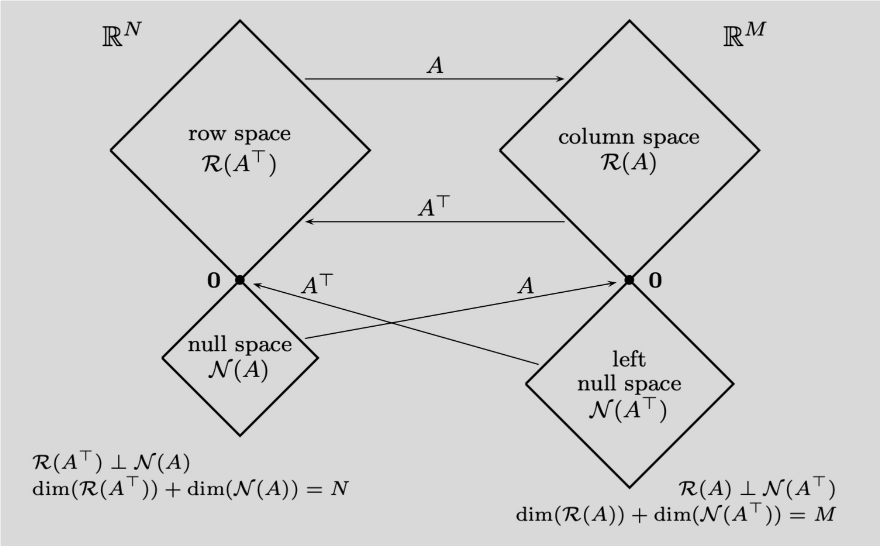
  $$\mathcal{R}(A^\top) \;=\; \mathrm{span}(\{b_0^\top,\, b_1^\top,\, \ldots,\, b_{M-1}^\top\}) \;=\; \{x \in \mathbb{R}^N \mid x = A^\top y \text{ for some } y \in \mathbb{R}^M\}$$

- Null space

  $$\mathcal{N}(A) \;=\; \{x \in \mathbb{R}^N \mid Ax = \mathbf{0}\}$$

- Left null space

  $$\mathcal{N}(A^\top) \;=\; \{y \in \mathbb{R}^M \mid A^\top y = \mathbf{0}\}$$



Vector spaces

Subspaces about a matrix

Space	**Symbol**	**Definition**	**Dimension**
Column space (range)	$\mathcal{R}(A)$	$\{y \in \mathbb{C}^M \mid y = Ax \text{ for some } x \in \mathbb{C}^N\}$	$\operatorname{rank} A$
Left null space	$\mathcal{N}(A^*)$	$\{y \in \mathbb{C}^M \mid A^*y = 0\}$	$M - \operatorname{rank} A$
	$\dim(\mathcal{R}(A)) + \dim(\mathcal{N}(A^*)) = M$		
Row space	$\mathcal{R}(A^*)$	$\{x \in \mathbb{C}^N \mid x = A^*y \text{ for some } y \in \mathbb{C}^M\}$	$\operatorname{rank} A$
Null space (kernel)	$\mathcal{N}(A)$	$\{x \in \mathbb{C}^N \mid Ax = 0\}$	$N - \operatorname{rank} A$
	$\dim(\mathcal{R}(A^*)) + \dim(\mathcal{N}(A)) = N$		

Vector spaces

DEFINITION 2.18 (OPERATOR NORM AND BOUNDED LINEAR OPERATOR) The *operator norm* of A, denoted by $\|A\|$, is defined as

$$\|A\| = \sup_{\|x\|=1} \|Ax\|. \tag{2.45}$$

A linear operator is called *bounded* when its operator norm is finite.

"strength of operation"

Vector spaces

Matrix norm

- Frobenius norm $$\|A\|_{\mathrm{F}} = \sqrt{\sum_{i=0}^{M-1}\sum_{j=0}^{N-1}|A_{i,j}|^2} = \sqrt{\mathrm{tr}(AA^*)}$$
- Operator norm $$\|A\|_{p,q} = \sup_{\|x\|_p=1} \|Ax\|_q$$
- Some examples

$$
\begin{aligned}
\|A\|_1 &= \|A\|_{1,1} = \max_{0\le j\le N-1} \textstyle\sum_{i=0}^{M-1}|A_{i,j}|,\\
\|A\|_{1,2} &= \max_{0\le j\le N-1} \left(\textstyle\sum_{i=0}^{M-1}|A_{i,j}|^2\right)^{1/2},\\
\|A\|_{1,\infty} &= \max_{0\le i\le M-1,\, 0\le j\le N-1} |A_{i,j}|,\\
\|A\|_2 &= \|A\|_{2,2} = \sigma_{\max}(A) = \sqrt{\lambda_{\max}(A^*A)},\\
\|A\|_{2,\infty} &= \max_{0\le i\le M-1} \left(\textstyle\sum_{j=0}^{N-1}|A_{i,j}|^2\right)^{1/2},\\
\|A\|_\infty &= \|A\|_{\infty,\infty} = \max_{0\le i\le M-1} \textstyle\sum_{j=0}^{N-1}|A_{i,j}|.
\end{aligned}
$$

Vector spaces

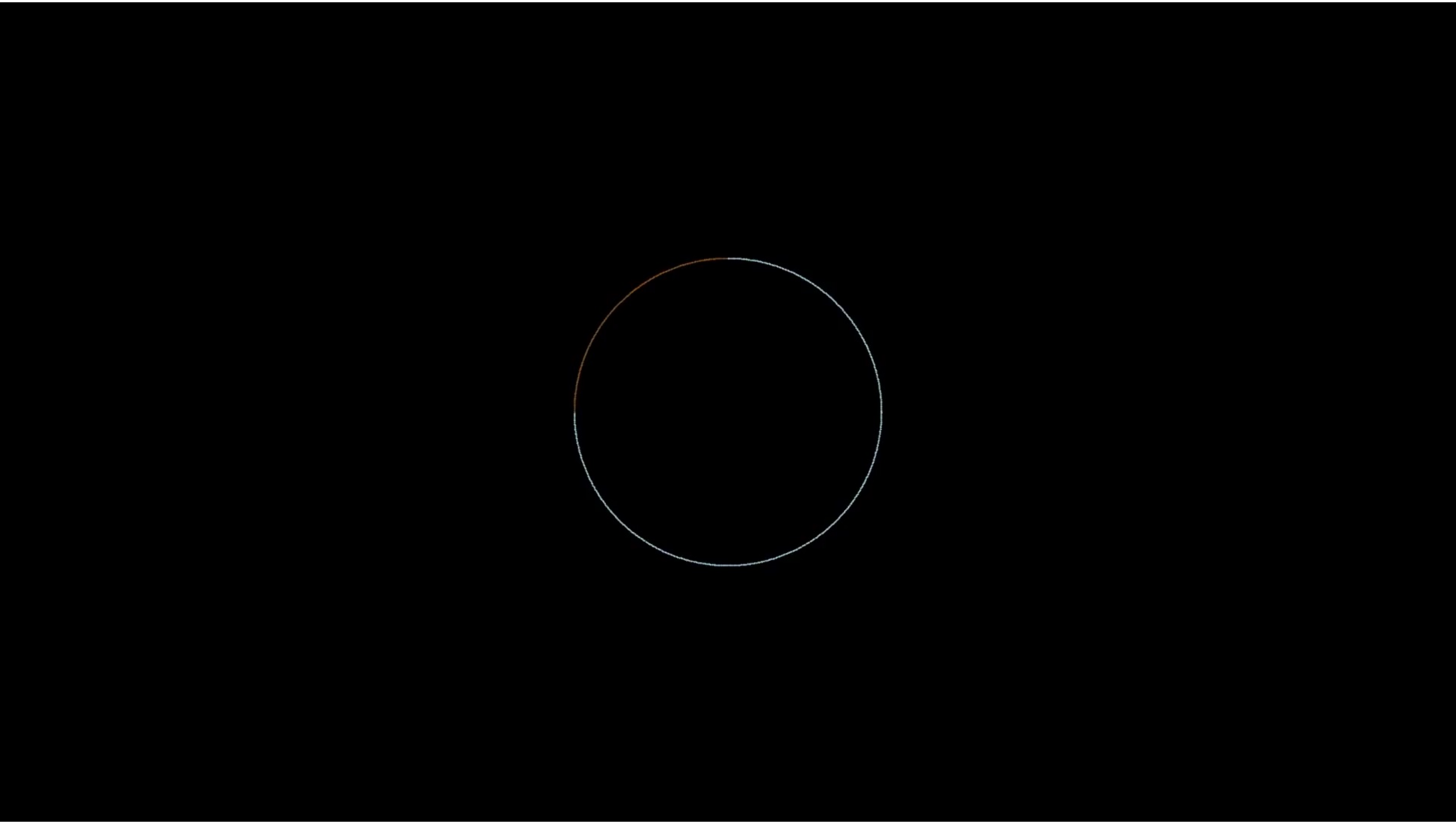
Eigen-decomposition

$$Av = \lambda v$$

$$A = V \Lambda V^{-1}$$

$$Ax = A \left(\sum_{k=0}^{N-1} \alpha_k v_k \right) \overset{(a)}{=} \sum_{k=0}^{N-1} \alpha_k (A v_k) \overset{(b)}{=} \sum_{k=0}^{N-1} (\alpha_k \lambda_k) v_k$$

Vector spaces



https://www.youtube.com/watch?v=fNk_zzaMoSs&list=PLZHQObOWTQDPD3MizzM2xVFitgF8hE_ab

Vector spaces

Singular value decomposition

$$A = U\Sigma V^*$$

$$AA^* = (U\Sigma V^*)(V\Sigma^* U^*) = U\Sigma^2 U^*$$
$$A^*A = (V\Sigma^* U^*)(U\Sigma V^*) = V\Sigma^2 V^*$$

$$\sigma^2(A) = \lambda(AA^*) = \lambda(A^*A)$$

Vector spaces: Implementation

Sum across axes

Let **A** be a matrix of shape (N, 2):

```
A = np.random.randn(N, 2)
```

$$\boldsymbol{A} = \begin{bmatrix} x_1 & y_1 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix}$$

Vector spaces: Implementation

Broadcasting

You Probably Saw Matrix Addition

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

What is this? FYI: e is a scalar

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + e =$$

https://web.eecs.umich.edu/~justincj/slides/eecs442/WI2021/442_WI2021_lecture04.pdf

Vector spaces: Implementation

Broadcasting

Given: Matrix **P** of shape (N, 2) vector **v** of shape (2, 1)
Want: Difference matrix **D** of shape (N, 2)

$$\boldsymbol{P} = \begin{bmatrix} x_1 & y_1 \\ \vdots & \vdots \\ x_N & y_N \end{bmatrix} \quad \boldsymbol{v} = \begin{bmatrix} a \\ b \end{bmatrix} \quad \boldsymbol{D} = \begin{bmatrix} x_1 - a & y_1 - b \\ \vdots & \vdots \\ x_N - a & y_N - b \end{bmatrix}$$

Vector spaces: Implementation

Broadcasting

```
x = np.ones(10, 20)
y = np.ones(20)
z = x + y
print(z.shape)
(10,20)
```

```
x = np.ones(10, 20)
y = np.ones(10)
z = x + y
print(z.shape)
ERROR
```

```
x = np.ones(10, 20)
y = np.ones(10, 1)
z = x + y
print(z.shape)
(10,20)
```

```
x = np.ones(1, 20)
y = np.ones(10, 1)
z = x + y
print(z.shape)
(10,20)
```

Vector spaces: Implementation

Broadcasting

The same broadcasting rules apply to tensors with any number of dimensions!

```
x = np.ones(30)
y = np.ones(20, 1)
z = np.ones(10, 1, 1)
w = x + y + z
print(w.shape)

(10, 20, 30)
```

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Vector spaces: Implementation

Vectorization example

- Suppose I have two sets of (D-dimensional) vectors $\{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$ and $\{\mathbf{y}_1, \ldots, \mathbf{y}_M\}$ and I want to compute all pairwise distances $d_{i,j} = ||x_i - y_j||$
- Identity: $\|\boldsymbol{x} - \boldsymbol{y}\|^2 = \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 - 2\boldsymbol{x}^T\boldsymbol{y}$
- Or: $\|\boldsymbol{x} - \boldsymbol{y}\| = (\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 - 2\boldsymbol{x}^T\boldsymbol{y})^{1/2}$

Vector spaces: Implementation

Vectorization example

$$\boldsymbol{X} = \begin{bmatrix} - & \boldsymbol{x}_1 & - \\ & \vdots & \\ - & \boldsymbol{x}_N & - \end{bmatrix} \quad \boldsymbol{Y} = \begin{bmatrix} - & \boldsymbol{y}_1 & - \\ & \vdots & \\ - & \boldsymbol{y}_M & - \end{bmatrix} \quad \boldsymbol{Y}^T = \begin{bmatrix} | & & | \\ \boldsymbol{y}_1 & \cdots & \boldsymbol{y}_M \\ | & & | \end{bmatrix}$$

N×D M×D

Compute a Nx1 vector of norms (can also do Mx1)

$$\boldsymbol{\Sigma}(\boldsymbol{X}^2, \boldsymbol{1}) = \begin{bmatrix} \|\boldsymbol{x}_1\|^2 \\ \vdots \\ \|\boldsymbol{x}_N\|^2 \end{bmatrix}$$

Compute a NxM matrix of dot products

$$\left(\boldsymbol{X}\boldsymbol{Y}^T\right)_{ij} = \boldsymbol{x}_i^T \boldsymbol{y}_j$$

Vector spaces: Implementation

Vectorization example

$$\mathbf{D} = \left(\Sigma(\boldsymbol{X}^{\mathbf{2}}, 1) + \Sigma(\boldsymbol{Y}^{\mathbf{2}}, 1)^{\boldsymbol{T}} - 2\boldsymbol{X}\boldsymbol{Y}^{\boldsymbol{T}}\right)^{1/2}$$

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Vector spaces: Implementation

Vectorization example

$$\mathbf{D} = \left(\Sigma(\boldsymbol{X}^2, 1) + \Sigma(\boldsymbol{Y}^2, 1)^T - 2\boldsymbol{X}\boldsymbol{Y}^T\right)^{1/2}$$

$$\mathbf{D}_{ij} = \|\boldsymbol{x}_i\|^2 + \|\boldsymbol{y}_j\|^2 + 2\boldsymbol{x}^T\boldsymbol{y}$$

Numpy code:

```
XNorm = np.sum(X**2,axis=1,keepdims=True)
YNorm = np.sum(Y**2,axis=1,keepdims=True)
D = (XNorm+YNorm.T-2*np.dot(X,Y.T))**0.5
```

N×1

N×1

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Vector spaces: Implementation

Vectorization example

$$\mathbf{D} = \left(\Sigma(\boldsymbol{X}^2, 1) + \Sigma(\boldsymbol{Y}^2, 1)^{T} - 2\boldsymbol{X}\boldsymbol{Y}^{T}\right)^{1/2}$$

$$\mathbf{D}_{ij} = \|\boldsymbol{x}_i\|^2 + \|\boldsymbol{y}_j\|^2 + 2\boldsymbol{x}^T\boldsymbol{y}$$

Numpy code:

```
XNorm = np.sum(X**2,axis=1,keepdims=True)
YNorm = np.sum(Y**2,axis=1,keepdims=True)   M×1
D = (XNorm+YNorm.T-2*np.dot(X,Y.T))**0.5
       N×1      M×1
```

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Vector spaces: Implementation

Vectorization example

$$\mathbf{D} = \left(\Sigma(\boldsymbol{X}^2, 1) + \Sigma(\boldsymbol{Y}^2, 1)^{\boldsymbol{T}} - 2\boldsymbol{X}\boldsymbol{Y}^{\boldsymbol{T}}\right)^{1/2}$$

$$\mathbf{D}_{ij} = \|\boldsymbol{x}_i\|^2 + \left\|\boldsymbol{y}_j\right\|^2 + 2\boldsymbol{x}^{\boldsymbol{T}}\boldsymbol{y}$$

Numpy code:

```
XNorm = np.sum(X**2,axis=1,keepdims=True)
YNorm = np.sum(Y**2,axis=1,keepdims=True)
D = (XNorm+YNorm.T-2*np.dot(X,Y.T))**0.5
```

N×M

N×1 1×M N×M

Vector spaces: Implementation

Vectorization example

Computing pairwise distances between 300 and 400 128-dimensional vectors

1. for x in X, for y in Y, using native python: 9s
2. for x in X, for y in Y, using numpy to compute distance: 0.8s
3. vectorized: 0.0045s (~2000x faster than 1, 175x faster than 2)

Expressing things in primitives that are optimized is usually faster

Even more important with special hardware like GPUs or TPUs!

https://web.eecs.umich.edu/~justincj/slides/eecs442/WI2021/442_WI2021_lecture04.pdf

Outline

Vector spaces

Derivatives

Derivatives

Rate at which a function f(x) changes at a point as well as the **direction** that increases the function

Given quadratic function f(x)

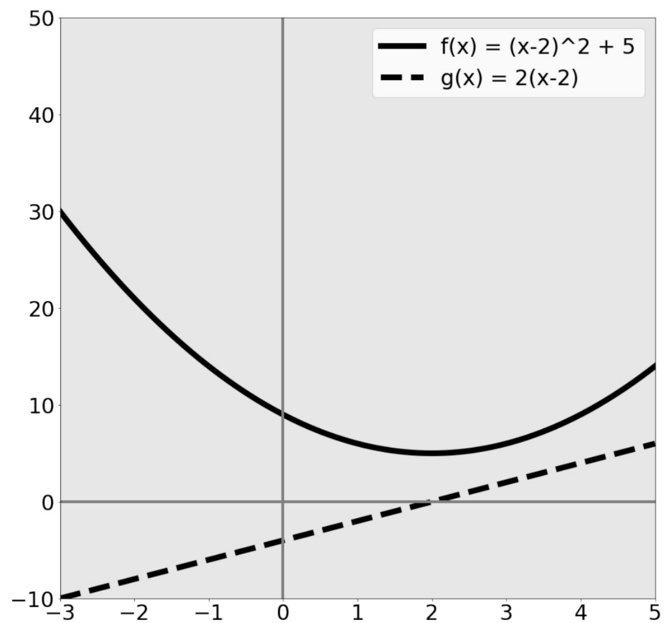
$$f(x, y) = (x - 2)^2 + 5$$

$f(x)$ is function

$$g(x) = f'(x)$$

aka

$$g(x) = \frac{d}{dx} f(x)$$



Derivatives

Rate at which a function f(x) changes at a point as well as the **direction** that increases the function

Given quadratic function f(x)

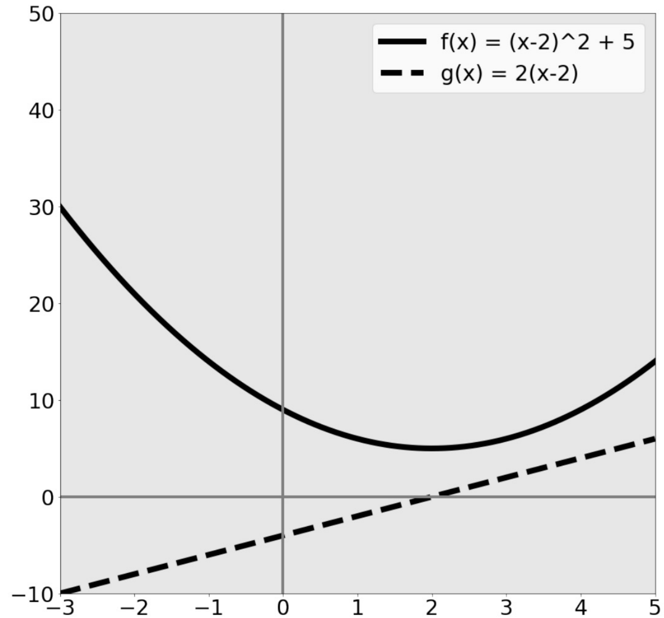
$$f(x, y) = (x-2)^2 + 5$$

What's special about x=2?

$f(x)$ minim. at 2
$g(x) = 0$ at 2

a = minimum of f → $g(a) = 0$

Reverse is ***not true***



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Derivatives

Rate at which a function f(x) changes at a point as well as the **direction** that increases the function

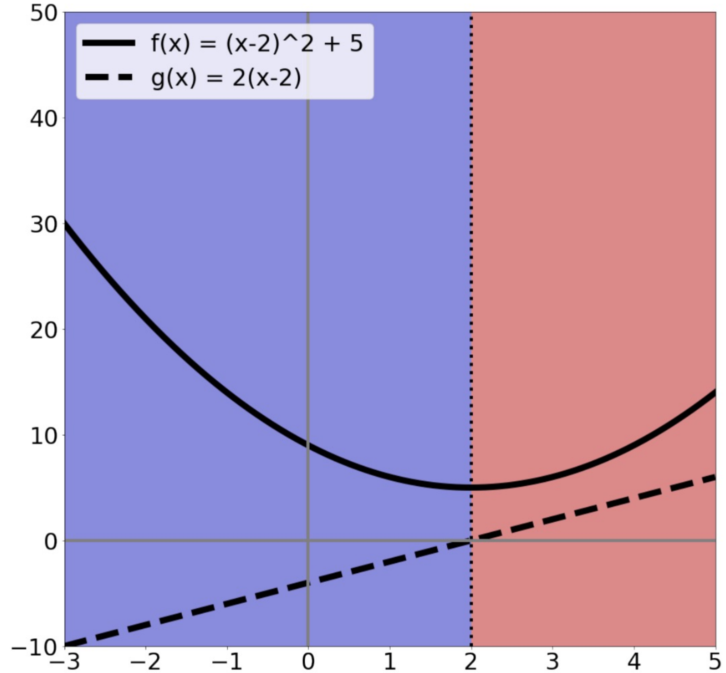
Rates of change

$$f(x,y) = (x-2)^2 + 5$$

Suppose I want to increase f(x) by changing x:

Blue area: move left
Red area: move right

Derivative tells you direction of ascent and rate



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Derivatives

Basics

- Given a scalar-valued function $f: \mathbb{R} \to \mathbb{R}$, the **derivative** $f'(x) \in \mathbb{R}$ of f at the point $x \in \mathbb{R}$ is the rate at which the function changes at that point

Derivatives

Partial derivatives

- Pretend other variables are constant, take a derivative. That's it.
- Make our function a function of two variables

$$f(x) = (x-2)^2 + 5$$

$$\frac{\partial}{\partial x} f(x) = 2(x-2) * 1 = 2(x-2)$$

$$f_2(x,y) = (x-2)^2 + 5 + \boxed{(y+1)^2}$$

Pretend it's constant → derivative = 0

$$\frac{\partial}{\partial x} f_2(x) = 2(x-2)$$

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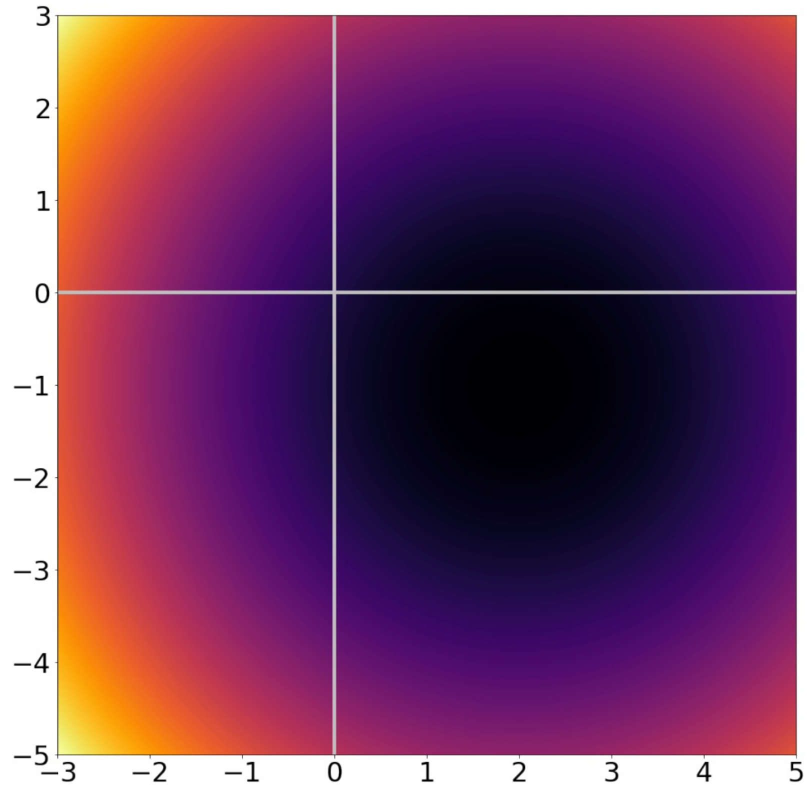
Derivatives

Partial derivatives

Zooming Out

$$f_2(x, y) = (x-2)^2 + 5 + (y+1)^2$$

Dark = f(x,y) low
Bright = f(x,y) high



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Derivatives

Partial derivatives

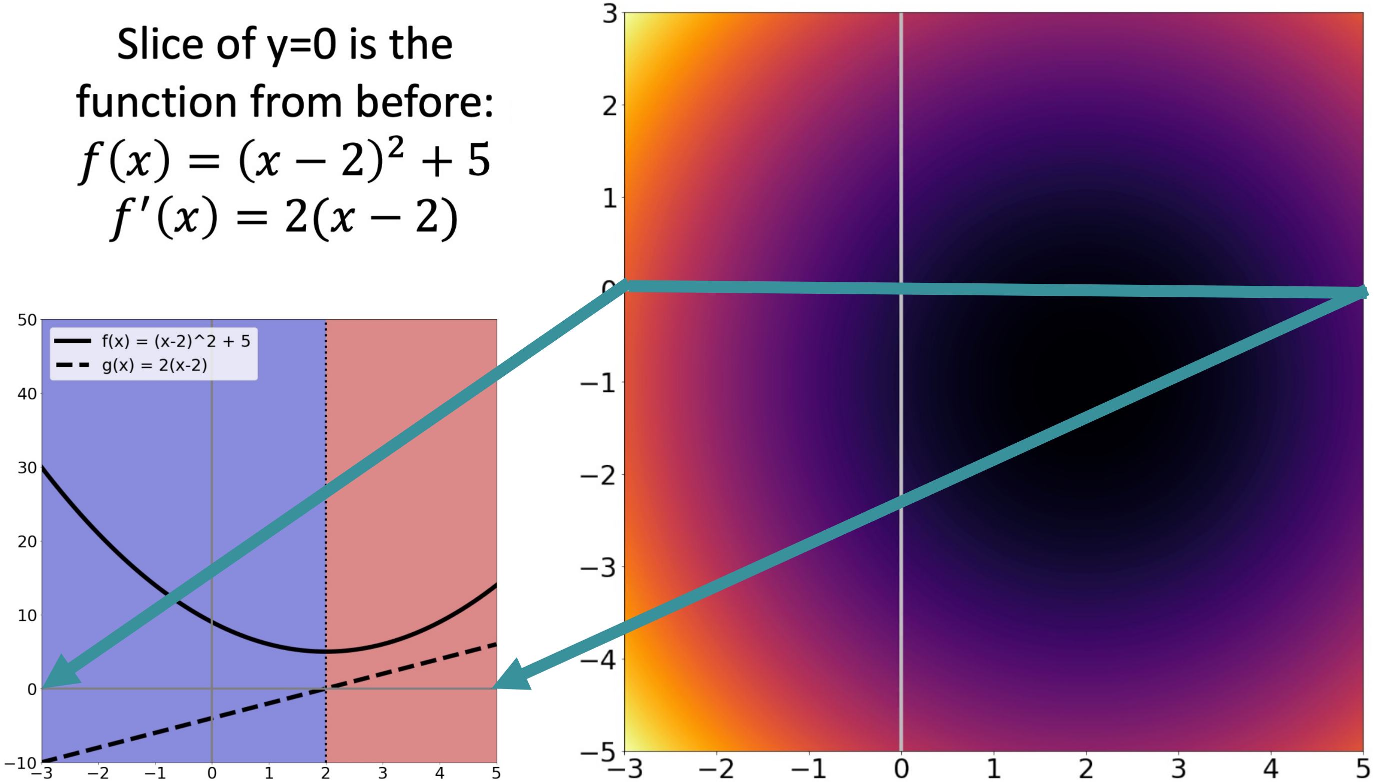
Taking a slice of

$$f_2(x, y) = (x-2)^2 + 5 + (y+1)^2$$

Slice of y=0 is the function from before:

$$f(x) = (x-2)^2 + 5$$

$$f'(x) = 2(x-2)$$



Derivatives

Partial derivatives

Zooming Out

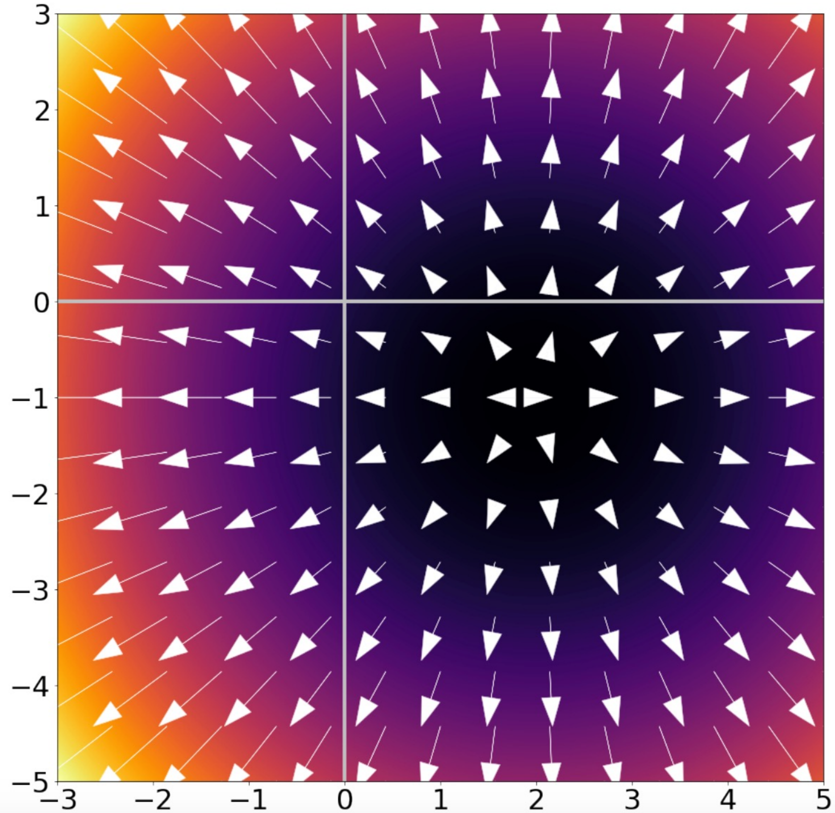
$$f_2(x, y) = (x-2)^2 + 5 + (y+1)^2$$

Gradient/Jacobian: Making a vector of

$$\nabla_f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right]$$

gives rate and direction of change.

Arrows point OUT of minimum / basin.



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Next lecture: Image filtering

The first assignment is due on this Friday

extremely simple! 10 mins work,
But 4 points !

Thank you very much!

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