

VE475

Introduction to Cryptography

Homework 6

Manuel — UM-JI (Summer 2022)

Non-programming exercises:

- Write in a neat and legible handwriting, or use \LaTeX
- Clearly explain the reasoning process
- Write in a complete style (subject, verb and object)

Programming exercises:

- Write a README file for each program
- Upload an archive with all the programs onto Canvas

Ex. 1 — Application of the the DLP

Bob wants to prove his identity to Alice. Alice knows that Bob can compute $\log_{\alpha}\beta$ in $\mathbb{Z}/p\mathbb{Z}$, where α is a generator of the group $\mathbb{Z}/p\mathbb{Z}$, and p is a known prime. Unfortunately Bob is not willing to share the result with her, so he offers to apply the following strategy.

- (i) Bob generates a random integer r and sends $\gamma = \alpha^r \bmod p$ to Alice;
- (ii) Upon receiving γ Alice randomly requests r or $x + r \bmod (p - 1)$;
- (iii) Bob replies accordingly;

We now want to study Bob's idea.

1. In the previous protocol,
 - a) Why are r and $x + r$ considered modulo $(p - 1)$?
 - b) Prove that neither Bob nor Alice can cheat, while Bob can successfully prove his identity.
2. How many times should this be repeated for a
 - a) 128 bits security level?
 - b) 256 bits security level?
3. What type of protocol is this?

Ex. 2 — Pohlig-Hellman

Search and explain in details how the Pohlig-Hellman algorithm computes the discrete logarithm of an element in a multiplicative group whose order can be completely factorized into small primes. As an example calculate $\log_3 3344$ in $G = U(\mathbb{Z}/24389\mathbb{Z})$, knowing that 3 is generator of G .

Ex. 3 — Elgamal

1. Prove that the polynomial $X^3 + 2X^2 + 1$ is irreducible over $\mathbb{F}_3[x]$, and conclude that it defines the field \mathbb{F}_{3^3} , which has 27 elements.
2. Explain how to define a simple map from the set of the letters of the alphabet into \mathbb{F}_{3^3} .
3. What is the order of the subgroup generated by X ?
4. If we set the secret key to be 11, determine the public key.
5. Encrypt the message "goodmorning", and then decrypt the ciphertext.

Ex. 4 — Simple questions

1. Let n be the product two large primes, p and q . We define $h(x) \equiv x^2 \bmod n$. Is h (i) pre-image

resistant, (ii) second pre-image resistant, and (iii) collision resistant?

2. Supposed a message m is divided into blocks of 160 bits: $m = m_1 \| m_2 \| \dots \| m_l$. Which properties of a hash function does the function $h(m) = m_1 \oplus m_2 \oplus \dots \oplus m_l$ verify?

Ex. 5 — Merkle-Damgård construction

The Merkle-Damgård construction provided in the slides is only valid when $t \leq 2$, therefore we now use the same notations as in the slides to provide an alternative construction for $t = 1$.

Let g be a compression function from $\{0, 1\}^{m+1} \rightarrow \{0, 1\}^m$, and f be the function defined by $f(0) = 0$ and $f(1) = 01$. The map from x to y is defined by $y = 11 \| f(x_1) \| f(x_2) \| \dots \| f(x_{|x|})$, where x_i represents the i -th bit of x . Assuming $|y| = k$, compute

$$\begin{cases} z_1 = g(0^m \| y_1) \\ z_{i+1} = g(z_i \| y_{i+1}), & 1 \leq i \leq k-1, \end{cases}$$

and define $h(x)$ as z_k .

1. Check that
 - a) The map s from x to y is injective.
 - b) There is no strings $x \neq x'$ and z such that $s(x) = z \| s(x')$.
2. Explain why the two previous conditions are of a major importance.
3. Following a similar strategy as in the case $t \geq 2$, prove that h is a collision resistant hash function.

Ex. 6 — Programming

Implement the Pollard-rho factorization algorithm.