Exercise 1

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1 Mass point hanging from the ceiling

see source code

2 Analytic solution and results analysis

2.1 Analytic solution

$$y(t) = c_1 e^{at} cos(\beta t) + c_2 e^{at} cos(\beta t) - L - \frac{mg}{k}$$

with $\alpha = -\frac{\gamma}{2m}$ and $\beta = \frac{\sqrt{4km-\gamma^2}}{2m}$
Inserting the initial conditions $y(0) = -L$ and $y'(0) = 0$ we find

$$c_1 = \frac{mg}{k}$$

$$c_2 = -\frac{\alpha}{\beta} c_1$$

2.2 Error convergence analysis

The absolute displacement error is shown in table 1 and the error convergence rate in table 2.

Timestep	Absolute displacement error			
	Euler	*Simplectic Euler*	*Midpoint*	*Backwards Euler*
0.003	3.74E-04	3.74E-04	2.32E-02	4.53E-04
0.0015	9.32E-05	9.32E-05	1.18E-02	1.18E-04
0.00075	2.33E-05	2.33E-05	5.93E-03	3.80E-05
0.000375	5.81E-06	5.81E-06	2.97E-03	3.51E-06
0.0001875	1.45E-06	1.45E-06	1.49E-03	3.10E-06
0.00009375	3.63E-07	3.63E-07	7.45E-04	4.62E-07
0.00004688	9.07E-08	9.07E-08	3.72E-04	5.13E-07
0.00002344	2.27E-08	2.27E-08	1.86E-04	5.22E-08
0.00001172	5.67E-09	5.67E-09	8.75E-04	9.68E-04
0.00000586	1.42E-09	1.42E-09	4.37E-04	4.84E-04

Table 1: Absolute displacement error

Timestep (@ i+1)	Error convergence rate (e_i / e_1+1)			
	Euler	*Simplectic Euler*	*Midpoint*	*Backwards Euler*
0.0015	4.0126	4.0126	1.9708	3.8443
0.00075	4.0064	4.0064	1.9835	3.1002
0.000375	4.0032	4.0032	1.9987	10.8293
0.0001875	4.0016	4.0016	1.9923	1.132
0.00009375	4.0008	4.0008	2	6.7126
0.00004688	4.0004	4.0004	2.0025	0.9011
0.00002344	4.0002	4.0002	1.9971	9.8169
0.00001172	4.0001	4.0001	0.2127	0.0001
0.00000586	4	4	2.0014	2.0011

Table 2: Error convergence

All the Eulerian methods converge with 2nd order accuracy, while the midpoint method converges with 1st order accuracy. However, the backwards Euler cannot maintain its 2nd order convergence to the very smallest time step. This can be verified in figure 1. Using damping reduces the error converges for all methods.

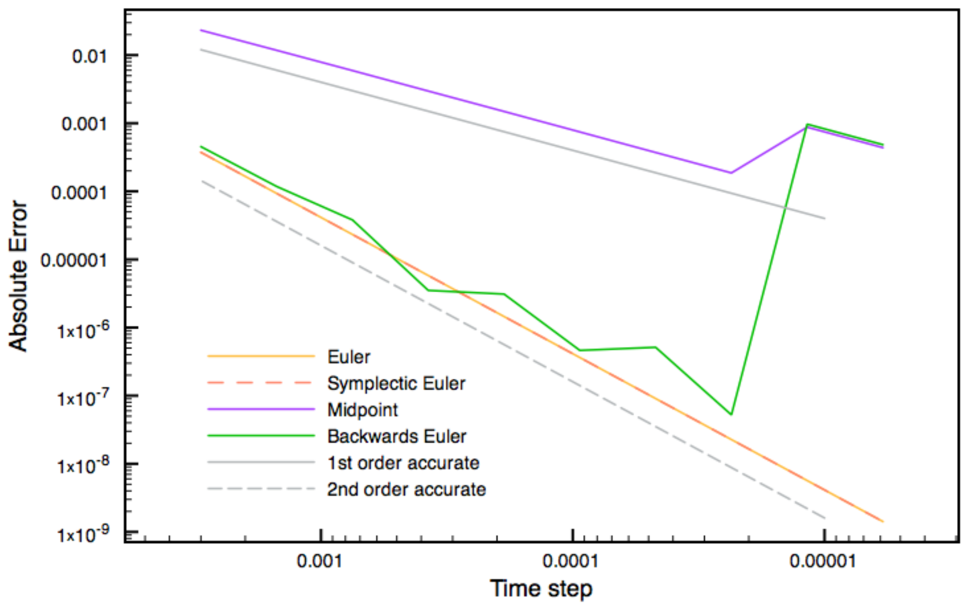

Figure 1: Error convergence

2.3 Stability analysis

Running the simulation with damping=0 gives the following results for the stability_measurement testcase:

Max amplitude table:

step	euler	symplectic	midpoint	backwards	analytic
0.003	2.52082	2.19621	2.30542	2	2.1962
0.006	39.1643	2.19624	4.00152	2	2.1962
0.012	1.92E+09	2.19638	16146.8	2	2.1962
0.024	3.16E+39	2.19691	1.17E+20	2	2.1962
0.048	8.86E+148	2.19915	2.76E+86	2	2.1962
0.096	1.#INF	2.20992	2.87E+306	2	2.1962
0.192	1.#INF	2.44846	1.#INF	2	2.1962
0.384	1.#INF	1.#INF	1.#INF	2	2.1962
0.768	1.#INF	1.#INF	4.12E+307	2	2.1962
1.536	1.#INF	1.#INF	1.80E+307	2	2.1962

The Symplectic Euler integrator shows a good stability behavior, while the other integrators are quite unstable. That is due to the fact that with no damping we have periodic system. And approximating a periodic function with tangents is not stable. The Symplectic Euler integrator is tailor made for Hamiltonian system. I.e. an ODE that can be formulated in the following way:

$$\frac{\partial p}{\partial t} = -\frac{\partial H}{\partial q}$$

$$\frac{\partial q}{\partial t} = \frac{\partial H}{\partial p}$$

Where the Hamiltonian H(p, q, t) gives the total energy of the system, p is the momentum and q the position.

The Symplectic Euler integrator exactly conserves momentum and nearly conserves energy. Running the test again with damping =1.9 gives the following results:

Max amplitude table:

step	euler	symplectic_	midpoint	backwards_	analytic
0.003	2.09811	2.0981	2.0981	2	2.0981
0.006	2.09811	2.0981	2.0981	2	2.0981
0.012	2.09811	2.0981	2.0981	2	2.0981
0.024	2.09813	2.0981	2.0981	2	2.0981
0.048	2.09823	2.0981	2.0981	2	2.0981
0.096	2.10632	1.#INF	2.0981	2	2.0981

We have now a much better stability for the other methods. That is caused by the fact, that increasing the damping makes the system less periodic. (If the damping ration $\frac{\gamma}{2\sqrt{mk}}$ is > 1 we have an exponential decay.)

3 Triangle colliding with the ground

see source code