Tasks (graded teamwork): please,

1. prove that
   1.1 $\log_a(n)$ is $\Theta(\log_b(n))$ (bases are asymptotically irrelevant)
   1.2 if $c_1 n^{c_2}$ is $\Theta(c_3 n^{c_4})$ then $c_2 = c_4$ (constant exponents are relevant)
2. verify the increasing ordering of these ACs (*trick:* log-log ;-)
3. what are the ACs of our summation laws on Page 35?

Hint (logarithm laws): recall that

$$\begin{aligned}\log_a(xy) &= \log_a x + \log_a y \\ \log_a x^y &= y \log_a x \\ x^{\log_a y} &= y^{\log_a x}\end{aligned} \qquad \log_a x = \frac{\log_b x}{\log_b a}$$

AnD | Hofmann-Roth-Zoller | ASYMPTOTICS

1.) prove that a) $\log_a(n)$ is $\Theta(\log_b(n))$

b) if $c_1 n^{c_2}$ is $\Theta(c_3 n^{c_4}) \Rightarrow c_2 = c_4$

a) by definition:

f is $\Theta(g) \Leftrightarrow f$ is $O(g) \wedge f$ is $\Omega(g)$

using the definitions of O and Ω:

$\exists(c \in \mathbb{R}_{>0})\,\exists(n_0 \in \mathbb{R}_{>0})\,\forall(n \in \mathbb{R}_{\geq 0})(n_0 \leq n \Rightarrow f(n) \leq c \cdot g(n))$

$\wedge$ 〃 ——————— $\Rightarrow f(n) \geq c \cdot g(n))$

$=$

If both conditions apply, then .... $f(n) = c \cdot g(n)$

Using the functions

$$\log_a(n) = c \cdot \log_b(n)$$

with: $\log_a x = \frac{\log_b x}{\log_b a} \Leftrightarrow \log_b a = \frac{\log_b x}{\log_a x}$

$$c = \frac{\log_a(n)}{\log_b(n)} = \frac{1}{\log_b a}$$

$\Rightarrow$ for two bases a and b, c is fixed, thus a) is true

b) if $c_1 n^{c_2}$ is $\Theta(c_3 n^{c_4})$ then $c_1 n^{c_2} = a \cdot c_3 \cdot n^{c_4}$

↑ any number $\neq f(n)$

$\Rightarrow \frac{c_1}{c_3} \cdot \frac{n^{c_2}}{n^{c_4}} = a$

$\Rightarrow \underbrace{n^{c_2 - c_4}} = \underbrace{a \cdot \frac{c_3}{c_1}}_{\text{a number}}$

has to be a fixed number (by definition). Can only be true if $c_2 - c_4 = 0 \Rightarrow c_2 = c_4$ q.e.d.

2) Verify increasing ordering of ACs (asymptotic complexity)

$$\Theta(1) < \Theta(\log(n)) < \Theta(\sqrt{n}) < \Theta(n) < \Theta(n \cdot \log n) < \Theta(n^2) < \Theta(n^3) < \Theta(c^n) < \Theta(n!)$$

a) $\Theta(1)$ and $\Theta(\log_b(n))$

For $n=b$ $\log_b(n) = 1$

After that, $\Theta(1)$ stays constant, while $\log_b(n)$ grows.

$\Rightarrow \Theta(1) < \Theta(\log_b(n))$

b) $\Theta(\log_b(n))$ and $\Theta(\sqrt{n})$

@ n=1: $\sqrt{n} = 1$

$\log_b(n) = 0$

Steigung für $n \geq 1$: $(\sqrt{n})' = \frac{1}{\sqrt{n}}$ $\quad (\ln(x))' = \frac{1}{x}$ $\quad \left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$

$$(\log_b(n))' = \left(\frac{\ln n}{\ln b}\right)' = \frac{\frac{1}{n} \cdot \ln b - \ln x \cdot 0}{(\ln b)^2} = \frac{1}{n \cdot \ln b}$$

$$\Rightarrow \frac{1}{\sqrt{n}} > \frac{1}{n \cdot \ln b}$$

$\Rightarrow \sqrt{n}$ steigt schneller als $\log_b(n)$ $(\forall b \in \mathbb{N})$

c) $\Theta(\sqrt{n})$ and $\Theta(n)$

@ n=1 : $\sqrt{n} = 1$

$n = 1$

Steigung nach $n=1$ $\quad \left.\begin{array}{l} n' = 1 \\ (\sqrt{n})' = \frac{1}{\sqrt{n}} \end{array}\right\} \; 1 > \frac{1}{\sqrt{n}},\ n > 1$

d) n and $n \cdot \log_b n$

$n = n \cdot \log_b n$ when $n = b$

Steigung danach $n' = 1$

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d) $(n \cdot \log_b n)' = 1 \cdot \log_b n + \cancel{n} \frac{1}{\cancel{n} \cdot \ln b}$ $(f \cdot g)' = f' \cdot g + f \cdot g'$

$$= \underbrace{\underbrace{\log_b n}_{>1 \text{ for } n>b} + \frac{1}{\ln b}}_{>1}$$

thus AC of $n \cdot \log_b n$ > AC of n

e) $\Theta(n \cdot \log n)$ and $\Theta(n^2)$

@ $n=0$ $n \log n = 0$

$n^2 = 0$

Steigung danach : $\log_b n + \frac{1}{\ln b}$

$(n^2)' = 2n$

since $n > \log_b n \Rightarrow 2n > \log_b n + \frac{1}{\ln b}$

f) $\Theta(n^2)$ and $\Theta(n^3)$: obvious

for $n=0$ both are 0

Steigung $(n^2)' = 2n$

$(n^3)' = 3n^2$

$3n^2 > 2n$

g) $\Theta(n^3)$ and $\Theta(c^n)$, for $\underline{c>1}$!

$c^n = n^3$ Schnittpunkte berechnen

$\log c^n = \log n^3$ (entspricht y-Achse logarithmisieren)

$\underbrace{n \cdot \log c}_{} = \underbrace{3 \cdot \log n}_{}$ -> Nullpunkte erfüllen $\frac{3}{\log c} = \frac{n}{\log n}$

linear AC > logarithmic AC (in lin-log Koord. system)

q.e.d.

h) $\Theta(n^n)$ and $\Theta(c^n)$

$$n^n = c^n$$

$$\underbrace{n \cdot \log n}_{\text{sub-quadratic AC}} = \underbrace{n \cdot \log c}_{\text{linear AC}}$$

logarithmische - Y-Achse

sub-quadratic AC > linear AC

3) Asymptotic complexity of

a) $\frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}$

n^2 "is much stronger" than n, thus for $n \to \infty$: $\frac{n^2}{2} + \frac{n}{2} \to \underline{\underline{n^2}}$

b) $\frac{x^{n+1} - 1}{x - 1}$ for $x \to \infty$: $\frac{x^{n+1} \cancel{-1}}{x \cancel{-1}} \to \frac{x^{n+1}}{x} = \underline{\underline{x^n}}$

Task 2 – Graphical representation: (zoomed version)