AnD	Hofmann - Roth - Zoller	Nonlinear recurrences

Given $f(n) = n \cdot \left(f\left(\frac{n}{2}\right)\right)^2$ ⓪ nonlinear recursive definition of $f(n)$

1.) change of variable $g(m) = f(2^m)$ ①

2.) change of range $h(m) = \log_2(g(m))$ ②

using ⓪ with $n = 2^m$:

$$f(2^m) = 2^m \cdot \left[f\left(\frac{2^m}{2}\right)\right]^2$$

$$f(2^m) = 2^m \cdot \left[f(2^{m-1})\right]^2 \quad / \log_2 \ldots$$

$$\log_2\left[f(2^m)\right] = \log_2\left[(2^m) \cdot \left(f(2^{m-1})\right)^2\right]$$

$$= \log_2 2^m + \log_2 \left(f(2^{m-1})\right)^2$$

$$\log_2\left[f(2^m)\right] = m + 2 \cdot \log_2\left(f(2^{m-1})\right) \quad ③$$

① & ② : $h(m) = \log_2(f(2^m))$ ④

Replace ④ in ③:

$$h(m) = m + 2 \cdot h(m-1)$$

⑤ $\underline{h(m) - 2\,h(m-1) = m}$ inhomogeneous linear recurrence

General definition of inhomogeneous linear recurrence:

$$\underbrace{a_0 f(m) + a_1 f(m-1) + \ldots + a_c f(m-c)}_{\underbrace{h(m)}_{a_0=1} \underbrace{-2}_{a_1=-2} h(m-1)} = \underbrace{\sum_{i=1}^{k} d_i^m \cdot p_i(m)}_{=m}$$

$a_0 = 1$, $a_1 = -2$, $c = 1$

thus $k = 1$ $d_1 = 1$

$p_1(m) = m^{c_1} \Rightarrow c_1 = 1$

SF's characteristic polynomial is: (=0 !!!)

$$\left(a_0 x^c + a_1 x^{c-1} + \dots + a_c\right) \cdot \prod_{i=1}^{k} (x - d_i)^{c_i+1} = 0$$

$$(1 \cdot x - 2)^1 \cdot (x-1)^2 = 0$$

$$\underbrace{(x-2)}_{i=1} \underbrace{(x-1)^2}_{i=2} = 0$$

$r_1 = 2 \qquad r_2 = 1$

$M_1 = 1 \qquad M_2 = 2$

Solution schema:

$$h(m) = \sum_{i=1}^{2} \sum_{j=1}^{M_i} b_{ij} \cdot m^{j-1} \cdot r_i^m$$

$$= \sum_{i=1}^{1} \sum_{j=1}^{M_1=1} b_{11} \cdot m^0 \cdot r_1^m + \sum_{i=2}^{2} \sum_{j=1}^{M_2=2} b_{2j} \cdot m^{j-1} \cdot r_2^m$$

$$= b_{11} \cdot r_1^m + b_{21} \cdot 1 \cdot r_2^m + b_{22} \cdot m \cdot r_2^m$$

$$= b_{11} \cdot r_1^m + b_{21} \cdot r_2^m + b_{22} \cdot m \cdot r_2^m$$

$$= b_{11} \cdot 2^m + b_{21} \cdot 1^m + b_{22} \cdot m \cdot 1^m$$

$$\boxed{\begin{array}{l} h(m) = b_{11} \cdot 2^m + b_{21} + b_{22} \cdot m \\ h(m) = a \cdot 2^m + b + c \cdot m \end{array}}$$ (6) (new variable names w/out indexes)

(6) into (5) $h(m) - 2h(m-1) = m$

$$(a \cdot 2^m + b + c \cdot m) - 2\left(a \cdot 2^{m-1} + b + c(m-1)\right) = m$$

$$\cancel{a2^m} + b + cm - \cancel{2a2^{m-1}} - 2b - 2cm + 2c = m$$

$$-b - c \cdot m + 2c = m$$

$$2c - b = m(1+c) \quad \cancel{(7)}$$

$m = 0 \Rightarrow 2c = b$ (7)

$m = 1 \Rightarrow 2c - b = 1 + c \Rightarrow c = 1 + b$ (8)

AnD	Hofmann-Roth-Zoller	Nonlinear recurrences (2)

(8) into (7) $\rightarrow$ $2c = b$

$$2(1+b) = b \Rightarrow \underline{\underline{b = -2}} = b_{21}$$

$$\Rightarrow \underline{\underline{c = -1}} = b_{22}$$

Thus (6): $h(m) = a \cdot 2^m - 2 - m$

using (2): $h(m) = \log_2(g(m))$

$$a \cdot 2^m - 2 - m = \log_2(g(m)) \quad / 2^{\cdots}$$

$$2^{a \cdot 2^m - 2 - m} = g(m)$$

$$2^{a \cdot 2^m - (2+m)} =$$

$$2^{a \cdot 2^m} \cdot \frac{1}{2^{2+m}} =$$

$$2^{a \cdot 2^m} \cdot \frac{1}{2^2 \cdot 2^m} = g(m)$$

using $2^m = n$ and $g(m) = f(2^m)$

$$g(m) = f(2^m) = \frac{2^{a \cdot 2^m}}{4 \cdot 2^m}$$

$$f(n) = \frac{2^{a \cdot n}}{4 \cdot n}$$

$$(6) \rightarrow \boxed{f(n) = \frac{2^{b_{11} \cdot n}}{4 \cdot n}}$$