AnD	Hofmann - Roth - Zoller	Nonlinear recurrences

Given $f(n) = n \cdot \left(f\left(\frac{n}{2}\right)\right)^2$ ⓪ nonlinear recursive definition of $f(n)$

1.) change of variable $g(m) = f(2^m)$ ①

2.) change of range $h(m) = \log_2(g(m))$ ②

using ⓪ with $n = 2^m$:

$$f(2^m) = 2^m \cdot \left[f\left(\frac{2^m}{2}\right)\right]^2$$

$$f(2^m) = 2^m \cdot \left[f\left(2^{m-1}\right)\right]^2 \quad / \log_2 \ldots$$

$$\log_2\left[f(2^m)\right] = \log_2\left[(2^m) \cdot \left(f\left(2^{m-1}\right)\right)^2\right]$$

$$= \log_2 2^m + \log_2\left(f\left(2^{m-1}\right)\right)^2$$

$$\log_2\left[f(2^m)\right] = m + 2 \cdot \log_2\left(f\left(2^{m-1}\right)\right) \quad ③$$

① & ② : $h(m) = \log_2\left(f(2^m)\right)$ ④

Replace ④ in ③:

$$h(m) = m + 2 \cdot h(m-1)$$

⑤ $\underline{h(m) - 2\,h(m-1) = m}$ inhomogeneous linear recurrence

General definition of inhomogeneous linear recurrence:

$$\underbrace{a_0 f(m) + a_1 f(m-1) + \ldots + a_c f(m-c)}_{\underbrace{h(m)}_{a_0=1} \underbrace{-2}_{a_1=-2} h(m-1)} = \underbrace{\sum_{i=1}^{k} d_i^m \cdot p_i(m)}_{=m}$$

$a_0 = 1$ $a_1 = -2$, $c = 1$

thus $k = 1$ $d_1 = 1$

$p_1(m) = m^{c_1} \Rightarrow c_1 = 1$

AnD	Hofmann-Roth-Zoller	Nonlinear recurrences (2)

(8) into (7) $\to$ $2c = b$

$$2(1+b) = b \Rightarrow \underline{\underline{b = -2}} = b_{21}$$

$$\Rightarrow \underline{\underline{c = -1}} = b_{22}$$

Thus (6): $h(m) = a \cdot 2^m - 2 - m$

using (2): $h(m) = \log_2(g(m))$

$$a \cdot 2^m - 2 - m = \log_2(g(m)) \quad / 2^{\cdots}$$

$$2^{a \cdot 2^m - 2 - m} = g(m)$$

$$2^{a \cdot 2^m - (2+m)} =$$

$$2^{a \cdot 2^m} \cdot \frac{1}{2^{2+m}} =$$

$$2^{a \cdot 2^m} \cdot \frac{1}{2^2 \cdot 2^m} = g(m)$$

using $2^m = n$ and $g(m) = f(2^m)$

$$g(m) = f(2^m) = \frac{2^{a \cdot 2^m}}{4 \cdot 2^m}$$

$$f(n) = \frac{2^{a \cdot n}}{4 \cdot n}$$

$$(6) \to \boxed{f(n) = \frac{2^{b_{11} \cdot n}}{4 \cdot n}}$$