AnD	Hofmann - Roth - Zoller	Asymptotics	5.6.2018	1

1. Prove that o and ω are not reflexive by giving a counter example

Reflexive: $\forall (x \in X)\ (xRx)$ where R: „is in relation with"

o) Reflexive for o:

By definition: f is o(g) means

$$\exists (c \in \mathbb{R}_{>0}) \exists (n_0 \in \mathbb{R}_{\geq 0}) \underset{\text{for all}}{\forall} (n \in \mathbb{R}_{\geq 0}) (n_0 \leq n \Rightarrow f(n) < c \cdot g(n))$$

Reflexive would mean:

$$\exists (c \in \mathbb{R}_{>0}) \exists (n_0 \in \mathbb{R}_{\geq 0}) \forall (n \in \mathbb{R}_{\geq 0}) (n_0 \leq n \Rightarrow f(n) < c \cdot f(n))$$

Counter-example: if $f(n) = 0$ then $f(n) < c \cdot f(n)$ is NOT TRUE, because $c > 0$.

ω) Similar for ω:

$$\exists (c \in \mathbb{R}_{>0}) \exists (n_0 \in \mathbb{R}_{\geq 0}) \forall (n \in \mathbb{R}_{\geq 0}) (n_0 \leq n \Rightarrow f(n) > c \cdot f(n))$$

This is NOT TRUE, counter-example: $f(n) = 0 \not> c \cdot f(n)$ where $c > 0$.

2. Big Notions O, Ω, Θ relations are reflexive and transitive

reflexive:

Big O: $\exists (c \in \mathbb{R}_{>0}) \exists (n_0 \in \mathbb{R}_{\geq 0}) \forall (n \in \mathbb{R}_{\geq 0}) (n_0 \leq n \Rightarrow f(n) \leq c \cdot f(n))$

$\Rightarrow$ For all $c \geq 1$ this is true

Big Ω: $\exists (c \in \mathbb{R}_{>0}) \exists (n_0 \in \mathbb{R}_{\geq 0}) \forall (n \in \mathbb{R}_{\geq 0}) (n_0 \leq n \Rightarrow f(n) \geq c \cdot f(n))$

$\Rightarrow$ For all $0 < c < 1$ this is true

Transitivity: if $aRb \wedge bRc \Rightarrow aRc$

Big O: if $\exists (c_1 \in \mathbb{R}^+) \exists (n_1 \in \mathbb{R}^+_0) \forall (n \in \mathbb{R}^+_0) (n_1 \leq n \Rightarrow f(n) \leq c_1 \cdot g(n))$

and if $\exists (c_2 \cdots) \exists (n_2 \cdots) \forall (n \cdots) (n_2 \leq n \Rightarrow g(n) \leq c_2 \cdot h(n))$

then: $\exists (c_3 \quad) \exists (n_3 \quad) \forall (\quad) (n_3 \leq n \Rightarrow f(n) \leq c_3 \cdot h(n))$

$$f(n) \leq c_1 \cdot g(n) \leq c_1 \cdot c_2 \cdot h(n)$$

$$\Rightarrow f(n) \leq \underbrace{c_1 \cdot c_2}_{:= c_3} \cdot h(n) \qquad \text{q.e.d.}$$

Big Ω

if $\exists (c_1 \in \mathbb{R}^+) \exists (n_1 \in \mathbb{R}_0^+) \forall (n \in \mathbb{R}_0^+) (n_1 \leq n \Rightarrow f(n) \geq c_1 \cdot g(n))$

and if $\exists (c_2 \quad) \exists (n_2 \quad) \forall (\quad) (n_2 \leq n \Rightarrow g(n) \geq c_2 \cdot h(n))$

then $\exists (c_3 \quad) \exists (n_3 \quad) \forall (\quad) (n_3 \leq n \Rightarrow f(n) \geq c_3 h(n))$

$$f(n) \geq c_1 \cdot g(n) \geq c_1 \cdot c_2 \cdot h(n)$$

$$\Rightarrow f(n) \geq \underbrace{c_1 \cdot c_2}_{:= c_3} \cdot h(n) \qquad \text{q.e.d.}$$

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4) Prove that Θ defines a symmetric (i.e. equivalence) relation

Symmetry: if $aRb \Rightarrow bRa$ or $aRb \Leftrightarrow bRa$

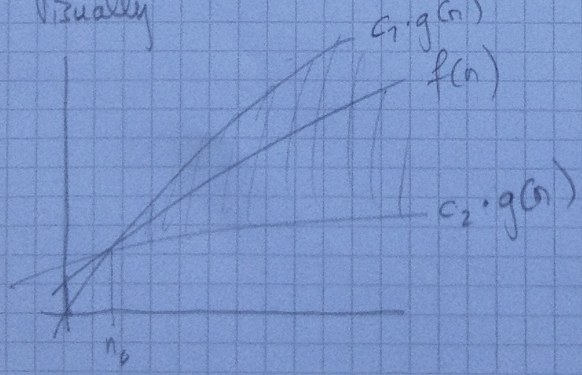
By definition: f is $\Theta(g)$ if f is $O(g) \wedge f$ is $\Omega(g)$

$\Rightarrow$ Symmetry thus means f is $\Theta(g) \Leftrightarrow g$ is $\Theta(f)$

$\Rightarrow$ which means: if $f(n) \leq c_1 \cdot g(n) \wedge f(n) \geq c_2 \cdot g(n)$

$\Rightarrow$ (then) $g(n) \leq c_3 \cdot f(n) \wedge g(n) \geq c_4 \cdot f(n)$

Visually



$\{c_1, c_2, c_3, c_4\} \in \mathbb{R}^+$, i.e. $c_i > 0$

$$f(n) \leq c_1 \cdot g(n) \Leftrightarrow g(n) \geq \frac{1}{c_1} \cdot f(n)$$

$$\Leftrightarrow \underline{g(n) \geq c_4 \cdot f(n)} \qquad \text{q.e.d.}$$

$$f(n) \geq c_2 \cdot g(n) \Leftrightarrow g(n) \leq \frac{1}{c_2} \cdot f(n) \Leftrightarrow \underline{g(n) \leq c_3 \cdot f(n)}$$

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1b) prove that o and ω are transitive

Transitive: if $aRb \wedge bRc \Rightarrow aRc$

if $\exists(c_1 \in \mathbb{R}_{>0})\exists(n_{01} \in \mathbb{R}_{\geq 0})\forall(n \in \mathbb{R}_{\geq 0})(n_{01} \leq n \Rightarrow f(n) < c_1 \cdot g(n))$ ①

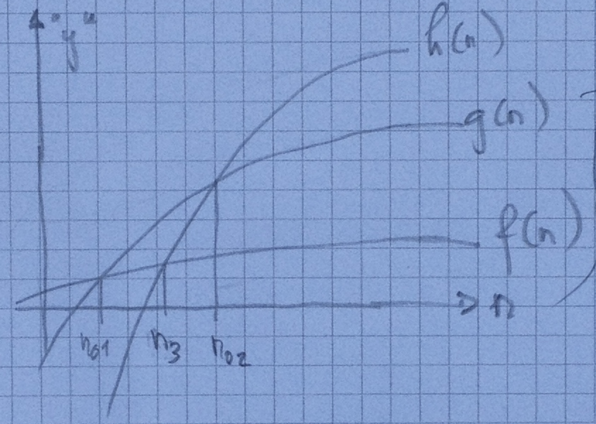
and

if $\exists(c_2 \in \mathbb{R}^+)\exists(n_{02} \in \mathbb{R}^+_0)\forall(n \in \mathbb{R}^+_0)(n_{02} \leq n \Rightarrow g(n) < c_2 \cdot h(n))$ ②

then:

$\exists(c_3 \in \mathbb{R}^+)\exists(n_3 \in \mathbb{R}^+_0)\forall(n \in \mathbb{R}^+_0)(n_3 \leq n \Rightarrow f(n) < c_3 \cdot h(n))$ ③

Visually



$g(n) < c_2 \cdot h(n)$

$f(n) < c_1 \cdot g(n)$

$\Rightarrow f(n) < c_1 \cdot g(n)$ ① where: $g(n) < c_2 \cdot h(n)$ ②

$c_1 \cdot g(n) < c_1 \cdot c_2 \cdot h(n)$

$\Rightarrow f(n) < c_1 \cdot g(n) < c_1 \cdot c_2 \cdot h(n)$

$\Rightarrow f(n) < \underbrace{c_1 \cdot c_2}_{=:c_3} \cdot h(n)$

$\Rightarrow f(n) < c_3 \cdot h(n)$ ③ is true

<u>Similar for ω</u>

if $\exists(c_1 \in \mathbb{R}^+)\exists(n_1 \in \mathbb{R}^+_0)\forall(n \in \mathbb{R}^+_0)(n_1 \leq n \Rightarrow f(n) > c_1 \cdot g(n))$

$\wedge$ if $\exists(c_2 \in \mathbb{R}^+)\exists(n_2 \in \mathbb{R}^+_0)\forall(n \in \mathbb{R}^+_0)(n_2 \leq n \Rightarrow g(n) > c_2 \cdot h(n))$

then: $\exists(c_3 \in \mathbb{R}^+)\exists(n_3 \in \mathbb{R}^+_0)\forall(n \in \mathbb{R}^+_0)(n_3 \leq n \Rightarrow f(n) > c_3 \cdot h(n))$

Similar $f(n) > c_1 \cdot g(n) > \underbrace{c_1 \cdot c_2}_{:=c_3} \cdot h(n) \Rightarrow \underline{f(n) > c_3 \cdot h(n)}$

q.e.d.

4) (continued):

$$g(n) \geq c_4 \cdot f(n) \quad \text{where} \quad c_4 = \frac{1}{c_1}$$

$$g(n) \leq c_3 \cdot f(n) \quad \text{where} \quad c_3 = \frac{1}{c_2}$$

If these two conditions are met, then Θ defines a symmetric relation.
These conditions are always met, since $c_1, c_2 > 0$.