① Prove that little o and little omega (ω) are not reflexive.

Def. Reflexivity: $\forall(a \in A)(aRa \Leftrightarrow R$ is reflexive on the set $A)$

Little o and little ω are not reflexive if there is a case (for each relation) where $f(n)$ is not $o(f(n))$, $\omega(f(n))$ respectively.

Little o: Def.: $\exists(c \in \mathbb{R})\exists(n_0 \in \mathbb{R}_{\geq 0})\forall(n \in \mathbb{R}_{\geq 0})(n_0 \leq n \Rightarrow f(n) < c \cdot g(n))$

Reflexivity of little o would mean:

(1) $\exists(c \in \mathbb{R})\exists(n_0 \in \mathbb{R}_{\geq 0})\forall(n \in \mathbb{R}_{\geq 0})(n_0 \leq n \Rightarrow f(n) < c \cdot f(n))$

Counterexample: for $c = 1$ and $n_0 \geq 0$ it is not the case that $f(n) < c \cdot f(n)$. So (1) is false.

Little ω: Def. $\exists(c \in \mathbb{R}_{\geq 0})\exists(n_0 \in \mathbb{R}_{\geq 0})\forall(n \in \mathbb{R}_{\geq 0})(n_0 \leq n \Rightarrow f(n) > c \cdot g(n))$

Reflexivity of little ω would mean:

(2) $\exists(c \in \mathbb{R}_{\geq 0})\exists(n_0 \in \mathbb{R}_{\geq 0})\forall(n \in \mathbb{R}_{\geq 0})(n_0 \leq n \Rightarrow f(n) > c \cdot g(n))$

Counterexample: for $c = 1$ and $n_0 \geq 0$ it is not the case that $f(n) > c \cdot f(n)$. So (2) is false.

② Prove that Big O, Big Ω and Θ are reflexive and transitive.

Def Big O: $\exists(c \in \mathbb{R}_{\geq 0})\exists(n_0 \in \mathbb{R}_{\geq 0})\forall(n \in \mathbb{R}_{\geq 0})(n_0 \leq n \Rightarrow f(n) \leq c \cdot g(n))$

Def Big Ω: " (" $\geq$ ")

Def Θ: $f(n)$ is $\Theta(g(n)) \Leftrightarrow [f(n)$ is $O(g(n)) \wedge f(n)$ is $\Omega(g(n))]$

Reflexivity

Reflexivity of Big O: Def: $\forall(n \in \mathbb{R}_{\geq 0})(f(n)$ is $O(f(n)) \Leftrightarrow O$ is reflexive$)$

Reflexivity of Big Ω: Def: $\forall(n \in \mathbb{R}_{\geq 0})(f(n)$ is $\Omega(f(n)) \Leftrightarrow \Omega$ is reflexive$)$

Proof for Big O: If $c = 1$ and $n_0 = 0$, $f(n) \leq c \cdot f(n)$. So Big O is reflexive

Proof for Big Ω: " , $f(n) \geq c \cdot f(n)$. So Big Ω is reflexive

Proof for Θ: 1) If two Relations are reflexive, so is their union.

Proof: $(A \times A \subseteq R_1 \wedge A \times A \subseteq R_2) \Rightarrow A \times A \subseteq R_1 \cap R_2$

Θ is the union of Big O and Big Ω. Because of (1), Θ is reflexive.

② Transitivity of Big O, Big Ω and Θ.

<u>Transitivity of Big O</u>: <u>Def.</u>

$\exists(c \in \mathbb{R}_{\geq 0}) \exists(n_0 \in \mathbb{R}_{\geq 0}) \forall(n \in \mathbb{R}_{\geq 0})\left(n_0 \leq n \Rightarrow \left[\left(f(n) \leq c_1 \cdot g(n) \wedge g(n) < c_2 \cdot h(n)\right) \Rightarrow \left(f(n) \leq c_3 \cdot h(n)\right)\right]\right)$

<u>Transitivity of Big Ω</u>: <u>Def.</u>

$\exists($ " $)($ $\left[\left(f(n) \geq c_1 \cdot g(n) \wedge g(n) \geq c_2 \cdot h(n)\right) \Rightarrow \left(f(n) \geq c_3 \cdot h(n)\right)\right]$

1) <u>Transitivity of Θ Def.</u> $\left[f(n) \text{ is } \Theta(g(n)) \wedge g(n) \text{ is } \Theta(h(n))\right] \Rightarrow f(n) \text{ is } \Theta(h(n))$

<u>Proof for Big O:</u>

If: 1) $f(n) \leq c_1 g(n)$ $\wedge$
2) $g(n) \leq c_2 \cdot h(n)$

Then: $f(n) \leq c_1 \cdot (c_2 \cdot h(n))$
$f(n) \leq c_1 \cdot c_2 \cdot h(n)$

So Big O is transitive.

<u>Proof for Big Ω:</u>

If: 1) $f(n) \geq c_1 \cdot g(n)$ $\wedge$
2) $g(n) \geq c_2 \cdot h(n)$

Then: $f(n) \geq c_1 \cdot c_2 \cdot h(n)$ So Big ~~[illegible]~~ Ω is transitive.

<u>Proof for Θ:</u>

If: 1) $f(n) = c_1 g(n)$ $\wedge$
2) $g(n) = c_2 h(n)$

Then: $f(n) = c_1 \cdot c_2 \cdot h(n)$ So ~~Big~~ Θ is transitive.

4 Prove that Θ is a symmetric (and thus, see proofs in 1 and 2 an equivalence) relation.

<u>Def. symmetry:</u> $\left[\forall a \forall b\,(a \in A, b \in A)(aRb \rightarrow bRa)\right] \Leftrightarrow R$ is symmetric

<u>Def Θ:</u> $\exists(c \in \mathbb{R}_{\geq 0})\exists(n_0 \in \mathbb{R}_{\geq 0})\forall(n \in \mathbb{R}_{\geq 0})\left(n_0 \leq n \Rightarrow f(n) = c \cdot g(n)\right)$

<u>Proof:</u> $\left(f(n) = c \cdot g(n)\right) \rightarrow \left(g(n) = \frac{1}{c} \cdot f(n)\right)$

$f(n)$ is $\Theta(g(n)) \rightarrow g(n)$ is $\Theta(f(n))$