Benchmark 2024: Structural optimisation of a ten-bar truss

Sizing and shape optimisation, initial guidelines

A. Andrade-Campos, J. Dias-de-Oliveira

Updated Guidelines: 16/04/2024

Abstract Structural mechanics are an essential part of mechanical engineering. The impact of the quality and efficiency of structures on products is ever present, namely in terms of performance and costs. Nowadays, mechanical design must be performed efficiently, therefore, optimisation procedures must be applied when designing new mechanical parts. this leads to a particular field within engineering optimisation, concerning structural optimisation.

In this challenge, within the Non-Linear Optimisation in Engineering Course, the mass of a truss (bar/rod) structure is minimised according to two strategies. One concerns a sizing problem, where the aim is to select the ideal cross section area of each bar. The other concerns a shape optimisation approach, where the aim is instead to adjust the position of the different nodes of the structure. This must always comply with structural equilibrium and the material must work within elasticity. The problem was initially be presented with an example structure, made of 10 bars, and later expanded to a more complex structure, as a non-mandatory bonus test. This design problem is a constrained optimisation problem.

Keywords Mechanical design · Weight minimisation · Non-linear optimisation · Constrained optimisation · Benchmark

1 Introduction

Most engineering design problems are formulated as mathematical programming models. In the last decades, these nonlinear engineering problems have been investigated by different methods. To compare the performance of different optimisation algorithms, several structural engineering applications are often solved to validate or test the suitability of the optimisation algorithms. The 10-bar plane truss problem is a simple and well-known benchmark problem in structural optimisation. The base problem represents the design of a structure, posed as a sizing problem acting on the cross section area of the structural elements [1–3]. In the present case, the problem will also be approached as a shape optimisation challenge and later on expanded to a more challenging structure, with a higher number of variables. Students should develop their implementations so that they allow for different inputs and problem data.

The base structure is represented on Figure 1, showing 10 bars, reference dimensions L and two node load vectors $\mathbf{P}$. It also shows a possible numbering for the nodes (linkage points) and elements (structural connections). This structure make it of the articulated type, composed of linear elements (bars/rods) arranged and connected in such a way that they transmit only axial force, no bending moments.

Node coordinates are shown in Table 1. Table 2, in turn, shows the connectivity of the structure, *i.e.* the nodes to which each element is connected. Both are also illustrated in Figure 1.

Nodes 1 and 4 are pined, fixed in the original coordinates. Dimension $L = 915$ mm and each load norm $||\mathbf{P}|| = 1.0\times^5$ N. The bars have round sections, with a diameter of 40 mm. the fabrication process used for these bars doesn't allow for diameters lower than 5 mm. These are made of a 6061-T6 Aluminium alloy. For this material, the relevant properties in this study are a vol-

A. Andrade-Campos, J. Dias-de-Oliveira
Non-Linear Optimisation in Engineering
Year 2023/24
E-mail: gilac@ua.pt; jalex@ua.pt

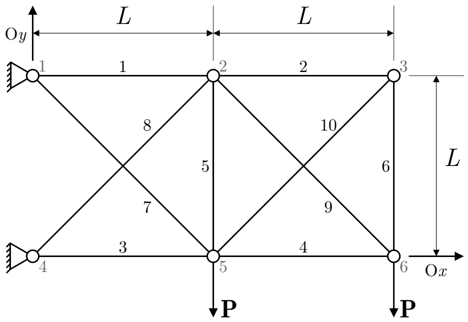


Figure 1. 10-bar plane truss structure [1].

Table 1. Coordinates of the nodes.

node	x	y
1	0	L
2	L	L
3	$2L$	L
4	0	0
5	L	0
6	$2L$	0

Table 2. Connectivity of the mesh (nodes n_1 and n_2 of each element.

element	n_1	n_2
1	1	2
2	2	3
3	4	5
4	5	6
5	2	5
6	3	6
7	1	5
8	4	2
9	2	6
10	5	3

ume mass of 2700 kg/m^3, Young modulus of 68.9 GPa and a tensile yield stress of 276 MPa [4].

2 Formulation of the optimisation problems

This problem is formulated as constrained nonlinear to minimise the mass objective function $m(\mathbf{x})$, subjected to the constraints $g_k(\mathbf{x})$, with $k = 1, 2, 3$. This is presented for two alternative approaches.

The first version of this benchmark is its usual form, as a sizing problem. In this case, the objective is to find the values of the continuous variables (10 for the 10 bar case)

$$x_i = A_i, \text{ with } i = 1, \cdots, n_{\text{el}}\,, \tag{1}$$

that define the area of each bar, A_i, to

$$\begin{aligned} \text{minimize} \quad & m(\mathbf{x}) = \sum_{i=1}^{n_{\text{el}}} m_i(\mathbf{x}) = \rho \sum_{i=1}^{n_{\text{el}}} l_i A_i \\ \text{subj. to} \quad & g_1(\mathbf{x}) = |\sigma_i(\mathbf{x})| - \sigma_{\text{adm}} \leq 0 \\ & g_2(\mathbf{x}) = A_{\text{min}} - A_i \leq 0 \\ & g_3(\mathbf{x}) = \mathbf{Ku} = \mathbf{f}\,, \end{aligned} \tag{2}$$

where ρ is the volume mass of the material, n_{el} is the number of elements, and, for element i, m_i is the mass, l_i is the length and σ_i is the axial stress. Constraints state that each bar should not go over the admissible (yield, in this case) stress, σ_{adm}, the area of each bar should not be under a minimum value A_{min}, and the structure should be in static equilibrium.

The second form of this benchmark is presented as a shape optimisation problem. In this case, the cross-section area is constant and equal for all elements, the continuous variables are

$$\mathbf{x}_j = \begin{Bmatrix} x_j \\ y_j \end{Bmatrix}, \text{ with } j = 2, 3, 5, \tag{3}$$

where x_j and y_j are the coordinates of node j. Coordinates of nodes 1, 4 and 6 are not variables and are fixed. An added domain constraint is used in this case, defining Ox coordinates as non-negative. The objective is still to

$$\begin{aligned} \text{minimize} \quad & m(\mathbf{x}) = \sum_{i=1}^{n_{\text{el}}} m_i(\mathbf{x}) = \rho A \sum_{i=1}^{n_{\text{el}}} l_i(\mathbf{x}) \\ \text{subj. to} \quad & g_1(\mathbf{x}) = |\sigma_i(\mathbf{x})| - \sigma_{\text{adm}} \leq 0 \\ & g_2(\mathbf{x}) = -l_i(\mathbf{x}) < 0 \\ & g_3(\mathbf{x}) = \mathbf{Ku} = \mathbf{f} \\ & g_4(\mathbf{x}) = -x_j \leq 0\,. \end{aligned} \tag{4}$$

3 Evaluation (simulation) of the structural problem

This section provides a quick summary on the topics of the Finite Element Method (FEM) [5] needed to solve this structural problem. Figure 2 shows the basic generic element for this summary.

3.1 Element stiffness

Whenever a structure consists of bars it is said to be an articulated truss structure. This designation is justified by the fact that the constituent elements of these structures are subject only to axial forces. This fact means that there is freedom of rotation in the unions between the bars, the nodes.

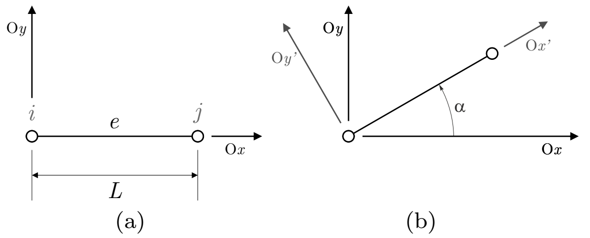


Figure 2. Generic rod element e from an articulated structure: (a) reference orientation and (b) Cartesian coordinate system in a two-dimensional space. .

The use of analytical methods for the calculation of these structures is limited and depends on the complexity of each structure. Thus, it becomes necessary to develop other alternative and systematic methods, which allow the analysis of the behaviour of these structures. One of these methods is based on matrix analysis of the overall equilibrium of the structure.

To better understand the method consider that a bar e (*vd.* Figure 2) of length l^e and cross-sectional area A^e is constructed using a material with Young modulus E^e. Using the FEM, this bar can also be called element e . This linear element has connectivity to nodes i and j.

Bar e only supports efforts in the direction of its axis, which for the case of figure 2 is represented by Ox. A single bar can only support loads in this direction, called nodal loads. The axial strain of bar e can be calculated as $\varepsilon = \frac{\Delta l^e}{l^e} = \frac{u_j^e - u_i^e}{l^e}$, where u_i^e and u_j^e are the displacements of nodes i and j, respectively. Knowing the axial strain ε and assuming that the material has a linear elastic behaviour, the axial stress in the bar can be determined using Hooke's law as $\sigma = E^e\varepsilon$.

Since the bar is only subjected to axial forces, the axial stress σ is constant in the section and is constant along of the bar axis. This allows the calculation of the axial force in the bar, N, multiplying the stress σ by its cross-sectional area, A^e, as

$$N = \sigma A^e = E^e A^e \frac{\Delta l}{l^e}. \tag{5}$$

This is the basis for, assuming that the bar is in static equilibrium, the matrix form that connects degrees of freedom, loads and stiffness, defined as

$$\begin{Bmatrix} f_i^e \\ f_j^e \end{Bmatrix} = \frac{EA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}. \tag{6}$$

The vector of the nodal loads f_i^e and f_j^e is called, from compact form, by $\mathbf{f}^e$ and the vector of nodal displacements u_i and u_j by $\mathbf{u}^e$. So, the matrix equation 6 can be written more compactly as

$$\mathbf{f}^e = \mathbf{k}^e \mathbf{u}^e, \tag{7}$$

where

$$\mathbf{k}^e = \frac{EA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{8}$$

Considering that the bar is homogeneous and has cross section constant, $\mathbf{k}^e$ is the proportionality constant between the loads and displacements of the modes of element e. Within the scope of the FEM, the $\mathbf{k}^e$ matrix is called the stiffness matrix of the bar element e.

3.2 Referential rotation

Element stiffness matrices are constructed in a reference frame, with the Ox axis aligned with the element. Placing the previous 1-D element in a 2-D space, as represented in Figure 2, its stiffness becomes

$$\mathbf{k}^e = \frac{EA}{l} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{9}$$

The element goes from 2 degrees-of-freedom, the axial displacement for each node, to 4 degrees of freedom, x and y displacements for each node. Note that, as stated before, this is still only able to provide a mechanical response along its axial (in this case horizontal) direction, as showed by the null lines and columns for degrees of freedom 2 (y for node i) and 4 (y for node j). In this sense, namely since not all elements in a plane truss will be in this position, to allow its correct influence on the global rigidity of the system the formulation of elements whose local referential is not coincident with the global one have to be rotated. These operations of rotation are defined as

$$\mathbf{K}_g^e = \boldsymbol{\Gamma}^{\mathrm{T}} \mathbf{k}^e \boldsymbol{\Gamma}, \tag{10}$$

where matrix $\boldsymbol{\Gamma}$ is the rotation matrix corresponding to the element e and $\mathbf{K}_g^e$ is the element stiffness matrix oriented according to the global frame of reference. To carry out the rotation of a vector, the operation is

$$\mathbf{v}_g = \boldsymbol{\Gamma}^{\mathrm{T}} \mathbf{v}. \tag{11}$$

Reverse rotations are performed by similar operations, but with the transposed rotation matrices relative to what is defined in equations 10 and 11. This operation is possible due to the rotation matrices of Cartesian reference frames are always orthogonal. $\mathbf{M}$ representing a generic orthogonal matrix.

In the case of a two-dimensional Cartesian space, such as the one illustrated in figure 2(b), the rotation operation is performed with a rotation matrix that can be expressed as

$$\boldsymbol{\Gamma} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & \sin\alpha \\ 0 & 0 & -\sin\alpha & \cos\alpha \end{bmatrix}. \tag{12}$$

The stiffness matrix that results from the previous operations can also be formulated in condensed form as

$$\mathbf{K}_g^e = \frac{EA}{l} \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}, \tag{13}$$

were c and s are, respectively, $\cos(\alpha)$ and $\sin(\alpha)$.

3.3 Global stiffness

To construct the global system equilibrium equations for a structure it is necessary to obtain the global stiffness matrix. In this process, the contributions of elements are summed to the corresponding positions (degrees of freedom) of the global matrix. So, if element

$$\mathbf{k}^e = \frac{EA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \mathrm{k}_{11}^e & \mathrm{k}_{12}^e \\ \mathrm{k}_{21}^e & \mathrm{k}_{22}^e \end{bmatrix}, \tag{14}$$

has connectivity with nodes i and j, its values will be added to the global matrix in the corresponding positions (i,i), (i,j), (j,i) and (j,j)s, as

$$\begin{Bmatrix} \vdots \\ f_i \\ \vdots \\ f_j \\ \vdots \end{Bmatrix} = \begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} \overset{\qquad i \qquad\quad j}{\begin{bmatrix} & \vdots & & \vdots & \\ \cdots & \mathrm{k}_{11}^e & \cdots & \mathrm{k}_{12}^e & \cdots \\ & \vdots & & \vdots & \\ \cdots & \mathrm{k}_{21}^e & \cdots & \mathrm{k}_{22}^e & \cdots \\ & \vdots & & \vdots & \end{bmatrix}} \begin{Bmatrix} \vdots \\ u_i \\ \vdots \\ u_j \\ \vdots \end{Bmatrix} \tag{15}$$

This reasoning will have to be expanded to all degrees of freedom of the structure, namely for elements with two degrees of freedom per node.

3.4 Solving the system of equations

For the types of structures described in the previous sections, the global stiffness matrices are always singular. This fact leads to the system of algebraic equations having no single solution. Thus, for the structure to be in static equilibrium and the corresponding system to have a defined a solution, it is necessary to impose boundary conditions. These will have to be suitable for the structure under review and, at least, in sufficient number to avoid all rigid body motion.

Boundary conditions used in this example are either loads or imposed displacement constraints. For the case of constraints (null imposed displacements), the resolution of the system is simplified, where it is merely enough to delete the lines and columns corresponding to the degrees of freedom where the displacements are forced to be zero, solving a reduced (non-indeterminate) system of equations. The complete system can be used *a posteriori* to calculate the reactions at the supports of the structure. For the loads, they just have to be added to the load vector at the respective degrees of freedom.

4 Reference structural solution

As a comparison basis for the initial tasks of model implementation and evaluation, this section provides a reference solution. This was calculated using both Fortran and python codes developed by the authors. Taking the values defined in Section 1, an initial solution was obtained. Figure 3 shows the deformed structure. Tables 3 and 4 show the calculated values for nodal displacements and element responses, respectively.

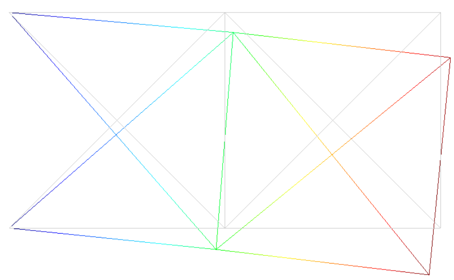


Figure 3. Reference deformed shape of a 10-bar plane truss structure.

Table 3. Reference nodal displacements u_x and u_y [m].

node	u_x	u_y
1	0	0
2	0.2065×10^{-2}	-0.4915×10^{-2}
3	0.2489×10^{-2}	-0.1114×10^{-1}
4	0	0
5	-0.2163×10^{-2}	-0.5290×10^{-2}
6	-0.2795×10^{-2}	-0.1156×10^{-1}

5 Reference optimisation solution

To solve these two variations of the 10-bar plane truss optimisation problem, the same Particle Swarm Opti-

Table 4. Reference element axial loads and stresses.

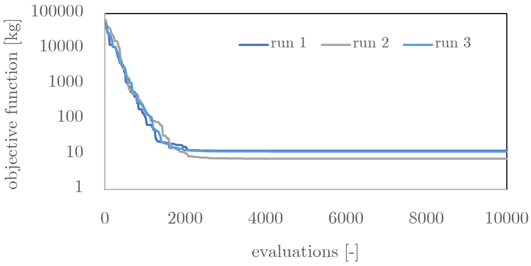
element	f [N]	σ [Pa]
1	1.95×10^5	1.55×10^8
2	4.01×10^4	3.19×10^7
3	-2.05×10^5	-1.63×10^8
4	-5.99×10^4	-4.77×10^7
5	3.55×10^4	2.82×10^8
6	4.01×10^4	3.19×10^7
7	1.48×10^5	1.18×10^8
8	-1.35×10^5	-1.07×10^8
9	8.47×10^4	6.74×10^7
10	-5.67×10^4	-4.52×10^7

misation (PSO) algorithm that students programmed in class was used, albeit implemented in python (adapted from [6]). For both problems population size was double the number of variables and the maximum number of iterations was 500. The following parameters were used: cognitive and social control $c_1 = c_2 = 1.5$, inertia $w = 0.725$ and no inertia damping ($w_\mathrm{d} = 1.0$). Constraints were imposed using an external penalty method, with a constant weight of 1.0×10^2. Three runs were made for each problem and the calculated solutions are presented in Table 5.

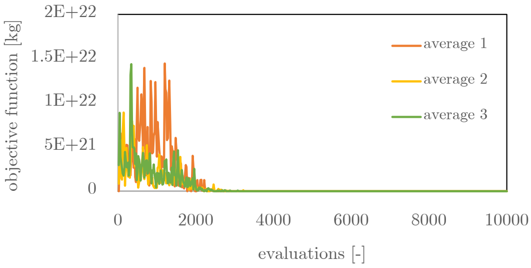
Table 5. Solutions for the two forms of the benchmark.

variables	solution	objective function
area	[7.73778419e-04 1.96349541e-05 1.60711476e-04 1.60886723e-04 1.96349541e-05 1.96349541e-05 5.79589100e-04 1.63665600e-04 5.72276975e-04 1.96349541e-05]	7.51647706
coordinates	[0.75840104 0.29066951 0.85071106 0.10238311 0.84718942 0.1]	24.8710087

The evolution of the iterative process is presented for both approaches. Figures 4 show the evolution of the objective function for the best solution of each of the three runs, for the approach using the areas as variables, as well as the average for the population on each run. The same is presented for approach 2 on Figures 5. In both cases the influence of the penalties is clear on the average curves, namely because of the nature of the stress constraint, which was not normalised. As an illustration of the iterative process, Figure 6 shows the evolution of the coordinate variables of the second approach.

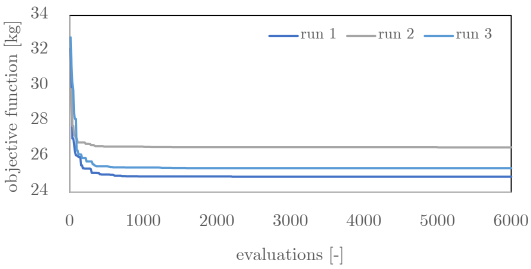


(a)

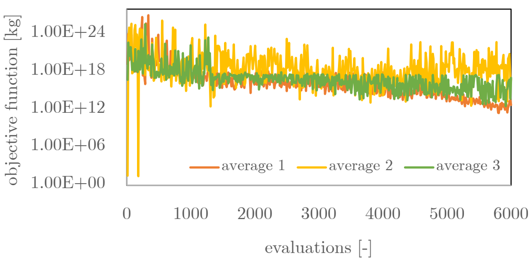


(b)

Figure 4. Optimisation problem with area variables: evolution of the objective function over three PSO runs for (a) the best solution and (b) an average solution.



(a)



(b)

Figure 5. Optimisation problem with area variables: evolution of the objective function over three PSO runs for (a) the best solution and (b) an average solution.

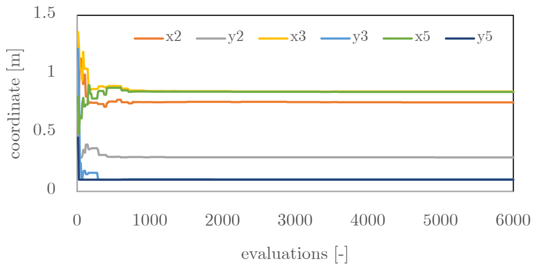


Figure 6. Evolution of the coordinate variables for the best solution of approach 2.

The obtained solution for both problems are illustrated on Figure 7. For the first case, line thicknesses are proportional to the calculated element area. Figure 8, in turn, shows some random solutions obtained during the iterations.

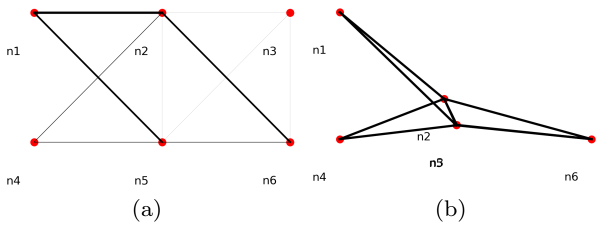


Figure 7. Best solution obtained for the problems with (a) area variables and (b) coordinate variables.

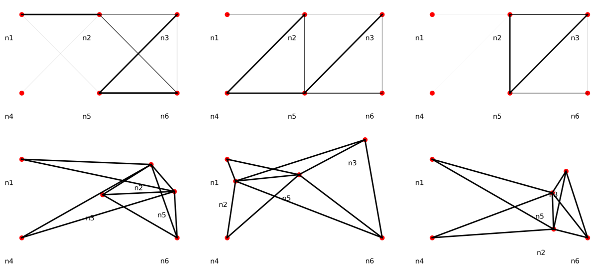


Figure 8. Randomly selected intermediate solutions for the optimisation problems.

6 Final remarks

This second version of the guidelines for this benchmark expand the previous document with a reference solution to the optimisation problem. This was obtained using a basic version of an algorithm programmed by students in class, clearly not global optima for either problem. Students should present their own results, as well as the developed algorithms, and discuss relevant aspects of their development process and obtained solutions. Assessment elements for this *benchmark* must follow the format provided in the respective template.

Finally, as an optional bonus task to test the robustness of the different layers of implementation, a second structure is presented. Figure 9 shows structure and boundary conditions, considering the same reference dimensions and values used for the precious problems.

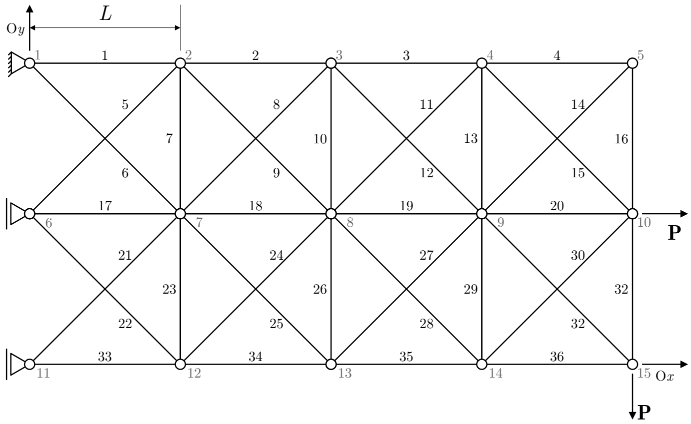


Figure 9. 36-bar plane truss structure.

References

[1] V.B. Venkayya. "Design of optimum structures". In: *Computers & Structures* 1.2 (1971), pp. 265–309.

[2] A. Alimoradi, C.M. Foley, and S. Pezeshk. "*Benchmark Problems in Structural Design and Performance Optimization: Past, Present, and Future – Part I*". In: *Proceedings of 19th Analysis and Computation Specialty Conference*. Vol. 1. 2010, pp. 1–12.

[3] A. Ning J. R. A. Martins. *Engineering Design Optimization*. Cambridge University Press (online at https://mdobook.github.io/), 2021.

[4] MatWeb: Online Materials Information Resource. https://www.matweb.com/ (access March 2023).

[5] F. Teixeira-Dias *etal. Método dos Elementos Finitos: Técnicas de Simulação Numérica em Engenharia*. 2a Ed., ETEP, 2018.

[6] S. M. K. Heris. *Implementation of Particle Swarm Optimization in Python (project YPEA127)*. open source code from http://yarpiz.com, retreived in 2017.