Condorcet criterion, ordinality and reduction of coalitional manipulability

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Since any non-trivial voting system is susceptible to manipulation, we investigate how it is possible to reduce the probability that it is manipulable, that is, such that a coalition of voters, by casting an insincere ballot, may secure an outcome that is better from their point of view.

We show that, for a large class of voting systems, a simple modification allows to reduce manipulability. This modification is *Condorcification*: when there is a Condorcet winner, designate her; otherwise, use the original rule. If preferences are not strict total orders, a notion of Condorcet winner based on absolute majority is used.

When electors are independent, for any non-ordinal voting system (i.e. requiring information that is not included in the orders of preferences, for example grades), we prove that there exists an ordinal voting system whose manipulability rate is at most as high and which meets some other desirable properties. Furthermore, this result is also true when voters are not independent but the culture is *decomposable*, a weaker condition that we define.

Combining both results, we conclude that when searching for a voting system whose manipulability is minimal (in a large class of systems), one can restrict to voting systems that are ordinal and meet the Condorcet criterion.

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I Introduction

I.A Motivation

A voting system is said to be *manipulable* in a given configuration if and only if a coalition of voters, by misrepresenting their preferences, may secure an outcome that they all prefer to the result of sincere voting; in other words, if and only if sincere voting does not lead to a strong Nash equilibrium.

For example, the French presidential election of 2002 was held with the two-round system. In the first round, Jacques Chirac (right) received 19.9% of votes, Jean-Marie Le Pen (far right) 16.9%, Lionel Jospin (left) 16.2% and 13 miscellaneous candidates shared the rest. In the second round, Chirac won by 82.2% against Le Pen. However, according to some opinion surveys, Jospin would have won the second round against any contender.

So, there may have been a possibility of manipulation: if all voters who preferred Jospin to Chirac had voted for Jospin in the first round, then the second round might have been held between Jospin and Chirac, leading to a possible election of Jospin. But voters did not perform this manipulation, an essential observation which we will discuss.

It should be noticed that the term *manipulation*, widely employed in the academic community of social choice, must be taken in a neutral, technical sense, disregarding its negative moral connotation. In our opinion, *manipulation* (or *tactical voting*) is not intrinsically better or worse than *sincere voting*; but *manipulability*, which is the discrepancy between the results of the two, certainly is an undesirable property for a voting system.

Firstly, it challenges the outcome of the election. If we estimate that the result of sincere voting best represents the opinions of the voters, then manipulability is undesirable because it may lead to another outcome. On the contrary, if we estimate that a manipulated outcome may be better in terms of collective welfare, then manipulation itself is not undesirable[1], but manipulability still is: indeed, it makes this "better" outcome difficult to identify and produce. For example, if the whole population votes sincerely, it will not be achieved.

Secondly, manipulability leads to several problems for voters. Before the election, they face a dilemma: vote sincerely or try to vote tactically? In the later case, they need information about what the others will vote, which gives a questionable power to polling organizations. After the election, some sincere voters may experience regrets about the choice of their ballots, and also a feeling of injustice: since insincere ballots would have better defended their views, they may estimate that their sincere ballots did not have the impact they deserved.

Finally, it may be argued that resistance to manipulation is a prerequisite for the other desirable properties of a voting system. Indeed, most of such classical

1. About the French presidential election of 2002, proponents of Condorcet efficiency would argue that manipulation would have had a desirable effect, the victory of the presumed Condorcet winner.

properties[2] relate ballots and candidates; but if ballots do not reflect the true opinions of voters, then these properties become hard to interpret. On the opposite, when there is no possibility of manipulation, the practical relevance of other properties is perfectly clear.

Unfortunately, Gibbard (1973) proved that any non-dictatorial voting system with three eligible candidates or more is manipulable. Although this result is frequently cited under the form of Gibbard-Satterthwaite theorem (Satterthwaite, 1975), which deals only with *ordinal* voting systems (i.e. whose ballots are orders of preferences), it is worth remembering that Gibbard's fundamental theorem applies to any *game form*, where available strategies may be objects of any kind, including grades for example.

Once this negative result is known, the only hope is to limit the damage, by investigating in what extent classical voting systems are manipulable, and by identifying processes to design less manipulable voting systems.

To quantify the degree of coalitional manipulability of a voting system, several indicators have been defined and studied, for example by Lepelley and Mbih (1987), Saari (1990), Lepelley and Valognes (1999), Slinko (2004), Favardin and Lepelley (2006), Pritchard and Slinko (2006), Tideman (2006) and Reyhani et al. (2009). A very common one is the *manipulability rate*, which is the probability that a configuration is coalitionaly manipulable, under a given assumption on the probabilistic structure of the population (or *culture*). It is an important indicator because it is an upper bound for the others: if we could identify reasonable voting systems with close-to-zero manipulability rates in realistic cultures, then the practical impact of manipulability would be negligible.

Several authors have used a theoretical approach (Lepelley and Mbih, 1987, 1994; Kim and Roush, 1996; Lepelley and Valognes, 1999; Huang and Chua, 2000; Favardin et al., 2002; Lepelley and Valognes, 2003; Favardin and Lepelley, 2006; Pritchard and Wilson, 2007; Lepelley et al., 2008), computer simulations (Lepelley and Mbih, 1987; Pritchard and Wilson, 2007; Reyhani et al., 2009; Green-Armytage, 2011, 2014; Green-Armytage et al., 2014) or experimental results (Chamberlin et al., 1984; Tideman, 2006; Green-Armytage, 2014; Green-Armytage et al., 2014) to evaluate the manipulability rates of several voting systems, according to various assumptions about the structure of the population.

Among the studies above, some, like those of Chamberlin et al. (1984), Lepelley and Mbih (1994), Lepelley and Valognes (2003) or Green-Armytage (2011, 2014) suggest that Instant-Runoff Voting (IRV) is one of the least manipulable voting systems known. On the other hand, authors like Chamberlin et al. (1984), Smith (1999), Favardin et al. (2002), Lepelley and Valognes (2003), Favardin and Lepelley (2006) or Tideman (2006) emit the intuition that voting systems meeting the Condorcet criterion have a general trend to be less manipulable than others.

2. Tideman (2006) provides an overview of such classical desirable properties, such as independence of irrelevant alternatives, consistency, etc.

Combining both ideas, Green-Armytage et al. (2014) introduce an alteration of IRV that meets the Condorcet criterion. Then they prove, independently of us (Durand et al., 2012), that for a large class of voting systems, making them meet the Condorcet criterion cannot worsen their manipulability. The main difference between their approach and ours is that Green-Armytage et al. (2014) do not make perfectly clear what assumptions are made about the preferences of voters, despite the fact that their proof is valid only when preferences are strict total orders, whereas we provide the result in a wider framework that encompasses all kinds of profiles. We also improve this result by proving that for a large class of voting systems, making them meet the Condorcet criterion *strictly* reduces their manipulability.

Although most studies use an ordinal framework, some voting systems are not ordinal, especially Approval voting, Range voting and variants such as Majority Judgment. According to Balinski and Laraki (2010), one of the motivations for the latter is resistance to manipulation. However, simulations results by Durand et al. (2014) suggest that non-ordinal voting systems perform quite badly in terms of manipulation. In this paper, we will investigate this question from a theoretical point of view: given a non-ordinal voting system, is it always outperformed by a well-chosen ordinal voting system?

When studying manipulability rates, one problem is that we do not know the minimal manipulability rate achievable in a given class of voting systems (for the moment, let us say "reasonable" ones). So, we can compare voting systems with one another, but we are not able to tell whether a manipulability rate is far from minimal[3]. Ideally, it would be very interesting to identify a minimally manipulable voting system: even if it was too intricate to be used in practice, it could be used as a theoretical benchmark to evaluate the manipulability rates of other voting systems. The Condorcification and slicing theorems presented in this paper are a first step in this direction.

I.B Overview

Section II presents the first contribution of this paper: a very general class of models called *electoral spaces* where voters' preferences can embed not only orders over the candidates but also non-transitive binary relations, grades or any kind of information. In this framework, we define *state-based voting systems* and their manipulability.

In section III, we precise how we extend the usual notion of Condorcet winner when voters may have any kind of binary relation of preference. We also define the *informed majority coalition criterion*[4] (**InfMC**), which is met by a large class of common voting systems from literature and real life.

3. Whenever we mention minimal manipulability, it is always in a given class of "reasonable" voting systems, which we will define in the following. Indeed, if we considered all voting systems, the question would be trivial, since dictatorship is not manipulable at all.

4. This notion is based on the same idea as Peleg's third simple game associated to a voting system (Peleg, 1984).

For any voting system, we define its *Condorcification*[5], which simply adds a preliminary test to designate the Condorcet winner when she exists. This hybrid voting system might be seen as an artificial construction, and as such, it is not obvious *a priori* that it has good properties; but quite surprisingly, theorem III.8 states that if a voting system meets **InfMC**, then its Condorcification cannot be more manipulable.

We define the *resistant Condorcet winner* by a simple property about voters' preferences and we prove that it is characterized by an immunity to manipulation in all Condorcet voting systems. Exploiting this, we show that for all usual voting systems meeting **InfMC** but not the Condorcet criterion, Condorcification *strictly* decrease their manipulability.

In section IV, we study the effect of ordinality. For any voting system, we define its *slices*, each of them being an ordinal voting system[6]. For example, the Borda method is one among the infinity of slices derived from Range voting. The main result of this section is that any voting system has a slice whose manipulability rate is lower or equal, provided a condition on the culture that we call *decomposability*. This condition is met when voters are independent, but it is more general.

Remarking that slicing preserves the Condorcet criterion leads to combining both results. We deduce that in order to minimize manipulability in the class **InfMC**, we can restrict our search to ordinal voting systems meeting the Condorcet criterion. Finally, we remark that for any decomposable culture, there exists an "optimal" voting system in that sense.

In appendix A, we discuss questions of measurability for our probabilized sets. In appendix B, we extend the notion *state-based voting system* and justify its use throughout the paper. In appendices C, D and E, we prove some assertions from the main sections. In appendix F, we study the notion of *decomposability* in general. In appendix G, we discuss the assumptions of the slicing theorem.

II Framework

II.A Electoral space

Let V and C be positive integers. We note $\mathcal{V} = \{1, \ldots, V\}$ the set of the indexes of *voters* and $\mathcal{C} = \{1, \ldots, C\}$ the set of the indexes of the *candidates*. Candidates may be or not be voters themselves, without impact on our results.

We note $\mathcal{R}_\mathcal{C}$ the set of the binary relations over $\mathcal{C}$: an element of $\mathcal{R}_\mathcal{C}$ represents a voter's binary relation of preference over the candidates. $\mathcal{W}_\mathcal{C}$ denotes the

5. Such a process has already been used to define the Black method from the Borda method (Black, 1958).

6. As a language shortcut, the word *ordinal* refers to voters' binary relations of preference over the candidates, even if they are not transitive, but excludes other information such as grades or approval values.

set of *strict weak orders* over $\mathcal{C}$ (i.e. negatively transitive, irreflexive and anti-symmetric relations) and $\mathcal{L}_\mathcal{C}$ the set of *strict total orders* over $\mathcal{C}$ (i.e. transitive, irreflexive and weakly complete relations).

We note $\mathcal{R} = (\mathcal{R}_\mathcal{C})^V$. An element of $\mathcal{R}$ is called a *profile*; for each voter, it gives her binary relation of preference over the candidates.

Throughout this paper, we will illustrate our results with an example of model that allows to study most common voting systems. In this specific model, each voter v is capable of mentally establishing:

- A strict weak order of preference $p_v \in \mathcal{W}_\mathcal{C}$ over the candidates,
- A vector $u_v \in [0,1]^C$ of *grades* over the candidates,
- And a vector $a_v \in \{0,1\}^C$ of *approval values* over the candidates.

The triple $\omega_v = (p_v, u_v, a_v)$ will be called her sincere *state* and we will note P_v the function that extracts the first element of this triple: $\mathrm{P}_v(\omega_v) = p_v$.

We note Ω_v the set of possible states for voter v. In this first model, we have $\Omega_v = \mathcal{W}_\mathcal{C} \times [0,1]^C \times \{0,1\}^C$. This set should be seen as the analog of a universe in probability theory: for most problems, it is not necessary to specify exactly what this set is. The essential point is the possibility to define functions that extract specific information about voter v: for example, P_v for her binary relation of preference.

By analogy with the usual notations for random variables in probability theory, we will often write P_v as a shortcut for $\mathrm{P}_v(\omega_v)$: hence, expressions $c\ \mathrm{P}_v(\omega_v)\ d$ and $c\ \mathrm{P}_v\ d$ are synonyms meaning that voter v prefers candidate c to d.

A *configuration* is a V-tuple $\omega = (\omega_1, \ldots, \omega_V)$ giving the state of each voter. We denote $\Omega = \prod_{v \in \mathcal{V}} \Omega_v$ the set of all possible configurations and $\mathrm{P} = (\mathrm{P}_1, \ldots, \mathrm{P}_V)$ the multivariate function that maps a configuration ω to the corresponding profile $(\mathrm{P}_1(\omega_1), \ldots, \mathrm{P}_V(\omega_V))$.

Definition II.1 (electoral space)**.** An *electoral space*, or *ES*, is given by:

- Two positive integers V and C,
- For each voter $v \in \mathcal{V}$, a non-empty set Ω_v of her possible states,
- For each voter $v \in \mathcal{V}$, a function $\mathrm{P}_v : \Omega_v \to \mathcal{R}_\mathcal{C}$, whose output is her binary relation of preference.

Such an ES is denoted $(V, C, \Omega, \mathrm{P})$, or just Ω when there is no ambiguity.

Let us come back to our example, where $\omega_v = (p_v, u_v, a_v)$. For more realism, the social planner might assume that a voter's grading and approval vectors are coherent with her binary relation of preference: if $c\ p_v\ d$, then $u_v(c) > u_v(d)$ and $a_v(c) \geq a_v(d)$. It is easy to embed this assumption in the model, by defining Ω_v as the set of triples (p_v, u_v, a_v) meeting these conditions. Other assumptions can be included as well by choosing a suitable set Ω_v.

Another example of electoral space is of high theoretical importance: in traditional Arrovian social choice, it is common to represent each voter's opinion by only a strict total order over the candidates. We call this model the *electoral space of strict total orders for V and C*, where each Ω_v is the set $\mathcal{L}_\mathcal{C}$ and each P_v is the identity function.

The framework of electoral spaces allows also preferences that are not transitive. For example, let us consider an electoral space where each voter v mentally assigns three grades to each candidate: her state ω_v is the $3 \times C$ matrix of her grades. Let us assume, moreover, that she prefers candidate c to d if and only if (iff) c is better than d for at least two criteria: this defines the correspondence from her matrix of grades ω_v to her binary relation of preference P_v. Then, it is well known that P_v may contain a cycle.

Although we interpret P_v as a *strict* preference in our examples, antisymmetry is not requested in definition II.1. We will discuss a possible interpretation of this issue later. If the reader feels uncomfortable with this, she may read all the following with an additional assumption of antisymmetry in mind. However, in all generality, we will note:

- $c \,\mathrm{I}_v\, d$ iff not $c \,\mathrm{P}_v\, d$ and not $d \,\mathrm{P}_v\, c$ (indifference),
- $c \,\mathrm{PP}_v\, d$ iff $c \,\mathrm{P}_v\, d$ and not $d \,\mathrm{P}_v\, c$ (antisymmetric preference),
- $c \,\mathrm{MP}_v\, d$ iff $c \,\mathrm{P}_v\, d$ and $d \,\mathrm{P}_v\, c$ (mutual preference).

If relation P_v is antisymmetric, which is a common assumption, then there are only three mutually exclusive possibilities: $c \,\mathrm{P}_v\, d$ (which is equivalent to $c \,\mathrm{PP}_v\, d$ in this case), $d \,\mathrm{P}_v\, c$ and $c \,\mathrm{I}_v\, d$.

For some results, we will assume that voters have a minimal freedom of opinion (disregarding the possibility of expression in their ballot).

Definition II.2 (richness of an electoral space)**.** We say that:

1. Ω *comprises all strict total orders* iff any voter may have any strict total order as her binary relation of preference; that is, $\forall (v, p_v) \in \mathcal{V} \times \mathcal{L}_\mathcal{C}, \exists \omega_v \in \Omega_v$ s.t. $\mathrm{P}_v(\omega_v) = p_v$;
2. Ω *allows any candidate as most liked* iff any voter may strictly prefer any candidate to all the others; that is, $\forall (v, c) \in \mathcal{V} \times \mathcal{C}, \exists \omega_v \in \Omega_v$ s.t. $\forall d \in \mathcal{C} \setminus \{c\}, c \,\mathrm{PP}_v\, d$.

We trivially have the implication $1 \Rightarrow 2$.

For example, in our reference model where $\omega_v = (p_v, u_v, a_v)$, relation p_v may be any strict weak order; in particular, any strict total order is allowed. Hence, this electoral space meets properties 1 and 2.

Now we will endow Ω with a probability measure, or *culture*. In order to handle probabilistic notions rigorously, we need to consider sigma-algebras, measurable sets, measurable functions and probabilistic events. However, measurability is not a crucial problem in practice: for example, without the axiom of choice, any subset of $\mathbb{R}^C$ is Lebesgue-measurable. So, we will postpone the technical discussion about measurability to appendix A.

Definition II.3 (probabilized electoral space)**.** A *probabilized electoral space*, or *PES*, is given by an electoral space $(V, C, \Omega, \mathrm{P})$ and a probability measure π over Ω, called *culture*.

Such a PES is denoted $(V, C, \Omega, \mathrm{P}, \pi)$, or just (Ω, π).

We note μ the probability law of random variable P under culture π.

For example, let us consider our reference electoral space, where $\omega_v = (p_v, u_v, a_v)$. Independently for each voter v:

- Let us draw a vector of grades u_v uniformly in $[0,1]^C$;
- Let us define p_v as the strict weak order of preference naturally induced by u_v, in the sense that $c\, p_v\, d \Leftrightarrow u_v(c) > u_v(d)$;
- For each candidate c, let us define the approval value $a_v(c)$ as the rounding of $u_v(c)$ to the nearest integer, 0 or 1.

Then we have defined an example of culture π, i.e. a probability measure over the electoral space Ω. Implicitly, it defines a law μ for the profile P.

II.B State-based voting systems and manipulability

Generally, a voting system can be quite complex: states of opinion and ballots are not always represented by the same mathematical object, and voting can involve a multi-round process. However, it is shown in appendix B that, in order to reduce manipulability, we can restrict our study to what we call *state-based voting systems*[7].

Definition II.4 (state-based voting system)**.** A *state-based voting system* over Ω, or *SBVS*, is a function $f : \Omega \to \mathcal{C}$.

For example, let us consider one of the possible variants for the voting system called *Range voting*, in our reference electoral space where $\omega_v = (p_v, u_v, a_v)$.

- Each voter v communicates a state belonging to Ω_v;
- We say that she votes *sincerely* iff she communicates her true state ω_v;
- Function f takes into account only the vectors of grades communicated by the voters, then returns the candidate with highest total grade (and resolves ties in an arbitrary deterministic way).

When implementing such a voting system in practice, it is sufficient that ballots include only the information that is actually used by function f, such as grades in this example. But this state-based formalism significantly facilitates the analysis. Firstly, it avoids the need of a tedious additional function that maps states of opinion to ballots (as in appendix B). Secondly, it makes it easier to design transformations of voting systems, such as the Condorcification we will define in section III.

Definition II.5 (manipulability)**.** For $(\omega, \psi) \in \Omega^2$, we say that f is *manipulable in configuration* ω *towards* ψ iff:

$$\begin{cases} f(\psi) \neq f(\omega), \\ \forall v \in \mathcal{V}, \psi_v \neq \omega_v \Rightarrow f(\psi)\ \mathrm{P}_v\ f(\omega). \end{cases}$$

For $\omega \in \Omega$, we say that f is *manipulable in configuration* ω iff $\exists \psi \in \Omega$ s.t. f is manipulable in ω towards ψ.

7. This notion of *state-based voting system* is a generalization of what is called *elementary voting procedure* ("procédure de vote élémentaire") by Moulin (1978, chapter II, definition 2). The author already remarks that considering such a procedure only reduces the strategical possibilities of the voters.

We note M_f the set of configurations ω where f is manipulable, as well as its indicator function:

$$M_f : \left| \begin{array}{lcl} \Omega & \to & \{0,1\} \\ \omega & \to & \left| \begin{array}{l} 1 \text{ if } f \text{ is manipulable in } \omega, \\ 0 \text{ otherwise.} \end{array} \right. \end{array} \right.$$

For a culture π, we call *manipulability rate of f for π* (provided M_f is measurable, cf. appendix A):

$$\begin{aligned} \rho_\pi(f) &= \pi(f \text{ is manipulable in } \omega) \\ &= \int_{\omega \in \Omega} M_f(\omega)\pi(d\omega). \end{aligned}$$

When we consider two SBVS f and g, we say that f is *at most as manipulable as g* (in the sense of inclusion)[8] iff $M_f \subseteq M_g$. This powerful property implies[9] that for any culture π, we have $\rho_\pi(f) \leq \rho_\pi(g)$: voting system f guarantees a lower manipulability rate than g.

In the examples of electoral spaces we mentioned, relations P_v were supposed to be antisymmetric. Without this assumption, it may happen that $c \ \mathrm{MP}_v \ d$ (mutual preference); how can it be interpreted? Since relations P_v have only been exploited in definition II.5 so far, the assertion $c \ \mathrm{MP}_v \ d$ should be interpreted in terms of manipulation: it means that if c is the sincere winner, then voter v is susceptible to manipulate for d, and vice versa.

Some possible interpretations are that v has fuzzy preferences about candidates c and d, or that she is an anticonformist who likes to change the outcome of the vote when possible (at least between c and d).

Another interpretation is that v has no strong preference between these two candidates and is susceptible to accept a bribery to manipulate for either of them against the other. In this interpretation framework, $c \ \mathrm{I}_v \ d$ means that voter v is also indifferent between c and d, but not corruptible. When the model embeds such possibilities, the *manipulability* of configuration ω (as formally defined above) is interpreted as a vulnerability to the combined effects of strategic ballots by voters who really prefer the new winner to the sincere one and the *bribery* of some corruptible voters.

Again, we insist that for readers who might be disturbed by such models, all the following can be read with an assumption of antisymmetry and the usual interpretation of manipulability.

III Condorcification and manipulability

In this section, we define the *Condorcification* of a voting system and we prove that, for a large class of voting systems, it is at most as manipulable as

8. This notion is defined by Lepelley and Mbih (1994).

9. If Ω is endowed with a sigma-algebra such that any singleton is measurable, it is not only an implication but an equivalence.

the original system.

III.A Condorcet-related notions

For any two candidates c and d, we note $D_{cd}(\omega) = |c \mathrm{~P}_v~d|$, where the later expression is a notation shortcut for the number of voters who prefer c to d. The matrix $D(\omega)$ is called the *matrix of duels* of ω.

The relation of *absolute victory* $\mathrm{P}_M(\omega)$ is defined by: $c \mathrm{~P}_M~d$ iff $D_{cd} > \frac{V}{2}$. When this relation is met, we indifferently say that c has an *absolute victory* against d in ω, or that d has an *absolute defeat* against c in ω. In this paper, for conciseness, we will simply write *victory* or *defeat*: variants of these notions are discussed in appendix C. From P_M, we define relations I_M (absence of victory), PP_M (strict victory) and MP_M (mutual victories) as we did for individual preferences.

We will use several times the following (trivial) result.

Proposition III.1. $D_{cd} + D_{dc} = V + |c \mathrm{~MP}_v~d| - |c \mathrm{~I}_v~d|$.

From this, we deduce that if all preference relations P_v are antisymmetric (which is a common assumption), then relation P_M is antisymmetric: two distinct candidates c and d cannot have mutual victories.

On the other hand, if all relations P_v are complete and if the number of voters V is odd, then relation P_M is complete: between two distinct candidates c and d, there cannot be an absence of victory.

Definition III.2 (Condorcet winner)**.** Let $\omega \in \Omega$ and $c \in \mathcal{C}$.

We say that c is *absolute Condorcet winner* in ω, or simply *Condorcet winner*[10], iff c has a strict victory against any other candidate d; that is:

$$\forall d \in \mathcal{C} \setminus \{c\}, \begin{cases} |c \mathrm{~P}_v~d| > \dfrac{V}{2}, & (1) \\ |d \mathrm{~P}_v~c| \leq \dfrac{V}{2}. & (2) \end{cases}$$

If relations P_v are antisymmetric (which is a common assumption) then, as we noticed, any victory is strict. Hence, c is Condorcet winner in ω iff:

$$\forall d \in \mathcal{C} \setminus \{c\}, |c \mathrm{~P}_v~d| > \frac{V}{2}.$$

Definition III.3 (Condorcet-admissible candidate)**.** Let $\omega \in \Omega$ and $c \in \mathcal{C}$.

We say that c is *Condorcet-admissible*[11] in ω iff c has no defeat; that is:

$$\forall d \in \mathcal{C} \setminus \{c\}, |d \mathrm{~P}_v~c| \leq \frac{V}{2}.$$

10. When relations of preference are strict total orders, this notion coincides with the usual notion of *Condorcet winner*, hence our terminology. Cf. appendix C.

11. When relations of preference are strict total orders, this notion coincides with the usual notion of *weak Condorcet winner*. Cf. appendix C.

We have the following trivial but crucial property.

Proposition III.4. *If a candidate is Condorcet winner in ω, then she is Condorcet-admissible and no other candidate is Condorcet-admissible; in particular, she is the unique Condorcet winner.*

If all binary relations P_v are complete and if the number of voters V is odd, then we have remarked that relation P_M is complete. In that case, a Condorcet-admissible candidate is always a Condorcet winner and these two notions become equivalent.

Definition III.5 (Condorcet criterion)**.** We say that f *meets the Condorcet criterion* iff, for any configuration $\omega \in \Omega$ and candidate $c \in \mathcal{C}$, if c is Condorcet winner in ω, then $f(\omega) = c$.

Definition III.6 (Condorcification)**.** We call *Condorcification of* f the state-based voting system:

$$f^* : \left| \begin{array}{lll} \Omega & \to & \mathcal{C} \\ \omega & \to & \left| \begin{array}{l} \text{if } \omega \text{ has a Condorcet winner } c, \text{then } c, \\ \text{otherwise}, f(\omega). \end{array} \right. \end{array} \right.$$

By design, the Condorcification f^* meets the Condorcet criterion.

For example, let us consider the Condorcification of Range voting, in our reference electoral space where $\omega_v = (p_v, u_v, a_v)$.

- Each voter gives a full state.
- If there appears to be a Condorcet winner (computed with relations p_v that are communicated), then she is elected.
- Otherwise, the candidate with highest total grade (computed with vectors of grades u_v that are communicated) is elected.

To implement this system in practice, it is now necessary to include grades and weak orders of preference in the ballots. However, if the social planner has adopted the assumption $c\ p_v\ d \Leftrightarrow u_v(c) > u_v(d)$, then it is sufficient to ask the grades, because the weak order of preference can be immediately deduced from them.

III.B Condorcification theorem

Before stating the theorem, we need to define a criterion which is met by most common voting systems.

Definition III.7 (informed majority coalition criterion)**.** We say that f *meets the informed majority coalition criterion* iff any majority coalition, being informed of what the other voters do, may decide of the outcome. That is, $\forall \mathcal{M} \in \mathcal{P}(\mathcal{V})$, if $\mathrm{card}(\mathcal{M}) > \frac{V}{2}$, then $\forall c \in \mathcal{C}$, $\forall \omega_{\mathcal{V}\setminus\mathcal{M}} \in \Omega_{\mathcal{V}\setminus\mathcal{M}}$, $\exists \omega_{\mathcal{M}} \in \Omega_{\mathcal{M}}$ s.t. $f(\omega_{\mathcal{M}}, \omega_{\mathcal{V}\setminus\mathcal{M}}) = c$ (using standard notations where $\omega_{\mathcal{M}}$ is a vector of states for the members of coalition $\mathcal{M}$).

We indifferently note **InfMC** the set of voting systems (over space Ω) meeting this criterion, or the criterion itself.

It is easy to prove that most common voting systems meet **InfMC**: maximin[12], Ranked pairs (Tideman, 1987), Schulze method (Schulze, 2011), Kemeny-Young method (Kemeny, 1959; Young and Levenglick, 1978), Dodgson method, Baldwin method, Nanson method, plurality, two-round system, Bucklin method, approval voting (Brams and Fishburn, 1978), range voting with average, range voting with median, instant-runoff voting (IRV), Coombs method (and all other iterated scoring rules), Borda method. Veto (also called Antiplurality) is one of the few systems frequently under study and not meeting **InfMC**; however, it is hardly used in practice.

Among commonly used voting systems, even those whose usual rationale does not rely on the notion of majority (Approval voting, Range voting for example) meet **InfMC**, although it is clearly a majoritarian property.

In practice, these observations give a wide range of applications for the next theorem. From a theoretical point of view, we can wonder if there is a deep reason why most common voting systems meet **InfMC**[13]; we think that this question deserves further investigation.

In an electoral space allowing any candidate as most liked (which is a common assumption, cf. definition II.2), if f meets the Condorcet criterion, then it meets **InfMC**. The converse is false: plurality meets **InfMC** but not the Condorcet criterion.

Theorem III.8 (Condorcification)**.** *Let Ω be an ES and f an SBVS. We assume that f meets* **InfMC***.*

Then its Condorcification f^* is at most as manipulable as f: $M_{f^*} \subseteq M_f$.

We need the two following lemma, whose proofs are straightforward.

Lemma III.9. *Let $(\omega, \psi) \in \Omega^2$. We note $c = f(\omega)$ and $d = f(\psi)$.*

If c is Condorcet-admissible in ω and if f is manipulable in ω towards ψ, then d cannot have a victory against c in ψ; in particular, d is not a Condorcet winner in ψ.

Lemma III.9 extends a classical result by Moulin (1978, chapter I, theorem 1). It should be noticed that no assumption about the voting system f is needed. Contrarily to what can be read sometimes, this result does not protect a configuration where the sincere winner is Condorcet-admissible (or even Condorcet winner) from being manipulable.

Lemma III.10. *Let $\omega \in \Omega$. We assume that f meets* **InfMC***.*

If $f(\omega)$ is not Condorcet-admissible in ω, then f is manipulable in ω.

As an immediate consequence, if ω has no Condorcet-admissible candidate, then any voting system f meeting **InfMC** is manipulable in ω.

Now, we can prove theorem III.8.

12. Unless specific reference is mentioned, all voting systems used in this paper are described in details by Tideman (2006). Whatever tie-breaking rule is considered has no impact on our results.

13. At least, it it the case for commonly used voting systems that aim at treating all voters and all candidates symmetrically.

Proof. Let us suppose that f^* is manipulable in ω towards ψ, but f is not manipulable in ω.

Since f is not manipulable in ω, lemma III.10 ensures that $f(\omega)$ is Condorcet-admissible in ω. If she is a Condorcet winner in ω, then $f^*(\omega) = f(\omega)$; otherwise, there is no Condorcet winner in ω (proposition III.4) so, by definition of f^*, we have also $f^*(\omega) = f(\omega)$.

So, let us note $c = f^*(\omega) = f(\omega)$ and $d = f^*(\psi)$. Since c is Condorcet-admissible in ω, lemma III.9 (applied to f^*) ensures that d is not a Condorcet winner in ψ. So, by definition of f^*, we have $f^*(\psi) = f(\psi)$.

Hence, we have $f(\omega) = f^*(\omega)$ and $f(\psi) = f^*(\psi)$ so f is manipulable in ω: contradiction! □

In appendix C, we show that this theorem is tight in the following sense: it does not generalize when using extended versions of Condorcification involving "relative" Condorcet winners (instead of "absolute" ones), weak Condorcet winners or Condorcet-admissible candidates.

III.C Resistant Condorcet winner

In this section, we define the notion of *resistant Condorcet winner* and we prove that it is characterized by an immunity to manipulation. Then, using this property, we give a sufficient condition for the Condorcification f^* to be *strictly* less manipulable than the original system f.

If f meets the Condorcet criterion and if there is a Condorcet winner c, manipulators in favor of d need to prevent c from appearing as a Condorcet winner. So, they need to prevent a strict victory for c against some candidate $e \neq c$. However, this plan is doomed to failure if sincere voters (those who do not prefer d to c) ensure on their own: (1) a victory for c against e; and (2) a non-victory for e against c. This observation leads to the following definition.

Definition III.11 (resistant Condorcet winner)**.** Let $\omega \in \Omega$ and $c \in \mathcal{C}$.

We say that c is *resistant Condorcet winner* in ω iff $\forall (d, e) \in (\mathcal{C} \setminus \{c\})^2$:

$$\begin{cases} |\text{not}(d \mathrm{~P}_v~ c) \text{ and } c \mathrm{~P}_v~ e| > \dfrac{V}{2}, & (1) \\ |\text{not}(d \mathrm{~P}_v~ c) \text{ and not}(e \mathrm{~P}_v~ c)| \geq \dfrac{V}{2}, & (2) \end{cases}$$

If all binary relations P_v are antisymmetric (which is a common assumption), then c is resistant Condorcet winner iff $\forall (d, e) \in (\mathcal{C} \setminus \{c\})^2$:

$$|\text{not}(d \mathrm{~P}_v~ c) \text{ and } c \mathrm{~P}_v~ e| > \frac{V}{2}.$$

If, moreover, all binary relations P_v are complete, then the definition is even simpler:

$$|c \mathrm{~P}_v~ d \text{ and } c \mathrm{~P}_v~ e| > \frac{V}{2}.$$

In other words, for any pair of other candidates (d, e), there is a strict majority of voters who simultaneously prefer c to d and c to e.

It is clear that for a candidate c, being a resistant Condorcet winner implies being a Condorcet winner. Hence (proposition III.4), if there is a resistant Condorcet winner, then she is unique.

Theorem III.12 (characterization of the resistant Condorcet winner)**.** *Let $\omega \in \Omega$ and $c \in \mathcal{C}$. We consider the following conditions.*

1. *Candidate c is a resistant Condorcet winner in ω.*
2. *For any state-based voting system f meeting the Condorcet criterion, c is elected by sincere voting, i.e. $f(\omega) = c$, and f is not manipulable in ω.*

We have the implication 1 $\Rightarrow$ 2.

If the electoral space comprises all strict total orders (which is a common assumption, cf. definition II.2), then the converse 2 $\Rightarrow$ 1 is true.

Proof. 1 $\Rightarrow$ 2: since c is a resistant Condorcet winner, even after an attempt of manipulation for d, sincere voters guarantee that c still has a strict victory against any other candidate $e \neq c$; so, candidate c still appears as a Condorcet winner and she gets elected.

For the converse 2 $\Rightarrow$ 1, see appendix D. □

In condition 2, it is necessary to demand that all Condorcet voting systems f have the same output c; otherwise, the converse 2 $\Rightarrow$ 1 would not hold. Indeed, consider a configuration where all voters are indifferent between all candidates: then any voting system is obviously not manipulable, but there is no resistant Condorcet winner.

Definition III.13 (resistant-Condorcet criterion)**.** We say that f *meets the resistant-Condorcet criterion* iff, for any configuration $\omega \in \Omega$ and candidate $c \in \mathcal{C}$, if c is a resistant Condorcet winner in ω, then $f(\omega) = c$.

It is clear that meeting the Condorcet criterion implies meeting the resistant-Condorcet criterion.

Theorem III.14 (strict improvement by Condorcification)**.** *Let Ω be an ES and f an SBVS. We assume that f meets* **InfMC** *but not the resistant-Condorcet criterion.*

Then its Condorcification f^* is strictly less manipulable than f: $M_{f^*} \subsetneqq M_f$.

Proof. Condorcification theorem III.8 ensures the inclusion.

Since f does not meet the resistant-Condorcet criterion, there exists $\omega \in \Omega$ and $c \in \mathcal{C}$, resistant Condorcet winner in ω, such that $f(\omega) \neq c$. By proposition III.4, we know that $f(\omega)$ is not Condorcet-admissible in ω, so lemma III.10 ensures that f is manipulable in ω. But theorem III.12 ensures that f^* is not manipulable in ω. Thus, the inclusion is strict. □

Corollary III.15. *We consider electoral spaces that comprise all strict total orders. For each voting system f in the following list, there are values of V and C such that the Condorcification f^* is strictly less manipulable than f:*

approval voting, Borda method, Bucklin method, instant-runoff voting (IRV), Coombs method (and all other iterated scoring rules), range voting with median, range voting with average, plurality, two-round system.

Proof. It is sufficient to prove that none of these voting systems meet the resistant-Condorcet criterion. See appendix E for counterexamples. □

Up to this point, we were generally considering a given voting system f and we were showing that its Condorcification f^* was at most as manipulable, or even strictly less manipulable than f. These results might suggest to use voting systems such as the Condorcifications of Plurality, IRV, etc.

However, it seems to us that the previous results have a deeper implication, which concerns any social planner who would like to find an acceptable voting system whose manipulability is minimal.

Corollary III.16. *Let us consider the function:*

$$M : \left| \begin{array}{ccc} \textbf{InfMC} & \to & \mathcal{P}(\Omega) \\ f & \to & M_f, \end{array} \right.$$

which gives for each f the set of its manipulable configurations.

Let $A \in \mathcal{P}(\Omega)$ be a minimal value of M (if any), i.e. a subset of Ω such that at least one SBVS f meets $M_f = A$ but no SBVS f meets $M_f \subsetneq A$. Then:

- *Any SBVS f meeting $M_f = A$ must meet the resistant-Condorcet criterion;*
- *Among the SBVS meeting the Condorcet criterion, there exists f such that $M_f = A$.*

In other words, when searching a voting system in **InfMC** whose manipulability is minimal, investigation *must* be restricted to voting systems meeting the resistant-Condorcet criterion, and *can* be restricted to voting systems meeting the Condorcet criterion.

IV Slicing

In this section, we define the *slices* of a voting system: for example, the Borda method, which is an ordinal voting system, is a slice of Range voting. We introduce the notion of *decomposable* culture and we prove that independence of voters is a particular case of this condition. Then we prove the main result of this section: if the culture is decomposable, then for any voting system, one of its slices is at most as manipulable as the original system. Finally, we study the implications of Condorcification and slicing together.

IV.A Slices of a voting system

The idea of slicing is the following: each voter v communicates her binary relation of preference $p_v = \mathrm{P}_v(\omega_v)$. Then we use a predefined method, denoted y_v, to reconstitute a fictional state ω'_v that is coherent with p_v. Finally, we apply f to the fictional configuration $(\omega'_1, \ldots, \omega'_V)$. This whole process defines a voting system f_y that is *ordinal*: it depends only on binary relations of preference.

For example, in our reference electoral space where $\omega_v = (p_v, u_v, a_v)$, let us consider $y_v(p_v) = (p_v, u'_v, a'_v)$, where u'_v is the vector of Borda scores[14] associated to p_v and a'_v is an approval vector with 1 for each candidate. In this case, functions y_v are the same for each voter v, but it is not mandatory.

Let us examine the *slice* of Range voting by $y = (y_1, \ldots, y_V)$. Once voters have communicated binary relations of preference $p = (p_1, \ldots, p_V)$, we use y in order to reconstitute fictional states ω'_v: in particular, each u'_v is now a vector of Borda scores. Finally, we apply Range voting to this fictional configuration: the winner is the candidate with highest total Borda score. To sum up, the *slice* of Range voting by y is the Borda method. An infinity of other slices can be defined, depending on the choice of y.

We now give the formal definitions.

Notations IV.1 (space $\mathcal{Y}$)**.** For each voter v, we note:

$$\mathcal{Y}_v = \{y_v : \mathrm{P}_v(\Omega_v) \to \Omega_v \text{ s.t. } \mathrm{P}_v \circ y_v = \mathrm{Id}\}.$$

It is the set of functions y_v that, to each possible p_v, associate a fictional state $\omega'_v = y_v(p_v)$ that is coherent with p_v, i.e. such that $\mathrm{P}_v(\omega'_v) = p_v$.

We note $\mathcal{Y} = \prod_{v=1}^{V} \mathcal{Y}_v$. An function $y = (y_1, \ldots, y_V) \in \mathcal{Y}$ defines a slicing method for each voter: to each possible profile $p = (p_1, \ldots, p_V) \in \prod_{v \in \mathcal{V}} \mathrm{P}_v(\Omega_v)$, it associates a configuration $\omega' = y(p) = (y_1(p_1), \ldots, y_V(p_V))$ that is coherent with p.

Definition IV.2 (slice)**.** For $y \in \mathcal{Y}$, we call *slice of f by y* the voting system f_y defined as:

$$f_y : \left| \begin{array}{ccc} \Omega & \to & \mathcal{C} \\ \omega & \to & f(y(\mathrm{P}(\omega))). \end{array} \right.$$

Now, we present a lemma that gives a central idea of the slicing theorem: when in configuration $y(p)$, voting systems f and f_y give the same outcome; but for manipulators, their possibility of expression in f_y are included in those they have in f, so they have less power.

For example, let us consider a very specific configuration ω where each voter v's sincere vector of grades u_v is the vector of Borda scores associated to her binary relation p_v. Obviously, if voters cast sincere ballots, then Range

14. Borda score of candidate c (for voter v): c gets one point for each candidate d such that v prefers c to d, and half a point for each candidate that v judges incomparable or mutually preferable to c. Then we divide this score by $C - 1$, in order to have a value in $[0, 1]$.

voting and the Borda method give the same outcome. We simply remark that if the Borda method is manipulable in ω, then Range voting also is: manipulators can use the same strategies they would use in the Borda method.

Lemma IV.3. *For any profile $p \in \mathrm{P}(\Omega)$ and slicing method $y \in \mathcal{Y}$, if f_y is manipulable in $y(p)$, then f is manipulable in $y(p)$. In other terms, using the manipulability indicators:*

$$M_{f_y}(y(p)) \leq M_f(y(p)).$$

Proof. Suppose that the slice f_y is manipulable in configuration $\omega = y(p)$. By definition, there exists ballots $\psi \in \Omega$ such that $f_y(\psi) \neq f_y(\omega)$ and:

$$\forall v \in \mathcal{V}, \psi_v \neq \omega_v \Rightarrow f_y(\psi)\ \mathrm{P}_v(\omega_v)\ f_y(\omega).$$

The slice and the original voting system have the same sincere outcome in ω: indeed, expanding the definition of $f_y(\omega)$ and using $\mathrm{P} \circ y = \mathrm{Id}$, we have $f_y(\omega) = f(y(\mathrm{P}(y(p)))) = f(y(p)) = f(\omega)$.

Let us note $\phi = y(\mathrm{P}(\psi))$: these are the ballots that are actually taken into account by f_y (in our example, the conversion of strategic ballots ψ to Borda format). We rewrite:

$$\forall v \in \mathcal{V}, \psi_v \neq \omega_v \Rightarrow f(\phi)\ \mathrm{P}_v(\omega_v)\ f(\omega).$$

Now, we just have to deal with sincere voters. Let us remark that if $\psi_v = \omega_v$, then $\phi_v = y_v(\mathrm{P}_v(\psi_v)) = y_v(\mathrm{P}_v(\omega_v)) = y_v(\mathrm{P}_v(y_v(p_v))) = y_v(p_v) = \omega_v$: in other words, sincere voters in ψ are still sincere after the format conversion to ϕ. By contraposition, we have the implication $\phi_v \neq \omega_v \Rightarrow \psi_v \neq \omega_v$, which leads to:

$$\forall v \in \mathcal{V}, \phi_v \neq \omega_v \Rightarrow f(\phi)\ \mathrm{P}_v(\omega_v)\ f(\omega).$$

Hence, f is manipulable in ω towards ϕ. □

IV.B Decomposable electoral space

By choosing an appropriate pair (p, y), any configuration ω can be expressed as $\omega = y(p)$. For example, if the true state ω_v of each voter v corresponds to Borda scores, then it can be represented by her true relation p_v and a slicing fuction y_v that maps p_v to Borda scores. So, in this configuration $\omega = y(p)$, lemma IV.3 allows to compare the manipulability of f to the one of f_y.

The idea of decomposability is the following: by independently drawing p and y with suitable laws, we would like to reconstitute configuration ω with the correct probability measure π. If this property is met, then we will see (when proving the slicing theorem IV.9) that the manipulability rate of f can be compared to a well-chosen average of the ones of all possible slices f_y.

We will give the formal definition first, then an interpretation and an example. Let us recall that μ denotes the law of P (under culture π).

Definition IV.4 (decomposability)**.** We say that (Ω, π) is P-*decomposable*, or simply *decomposable*, iff there exists a probability law ν over $\mathcal{Y}$ such that for any event A on Ω:

$$\pi(\omega \in A) = (\mu \times \nu)\big(y(p) \in A\big).$$

In the following, when (Ω, π) is decomposable, we will always note ν an arbitrary measure over $\mathcal{Y}$, among those meeting this property.

This definition demands that π is the image measure of $\mu \times \nu$ by the operator that, to p and y, associates $y(p)$. In other words, by independently drawing p and y (with measures μ and ν), then considering $\omega = y(p)$, we draw ω with the correct probability measure π. Decomposability is linked to the notion of complementary information about the states of the voters: this idea is developed in appendix F.

Example IV.5. Let us consider $V = 2$ voters and $C = 2$ candidates named a and b. Let us assume that the state of a voter is the pair of a strict total order of preference and a complementary information, "apple" or "banana". In a real study case, these two fruits might have specific meanings, such as "strongly prefers" and "somehow prefers", but it does not matter for our present purpose.

Let π be the culture that draws with equal probability one of the two following configurations:

1. Each voter is in state $\mathcal{A} = (a \succ b, \text{apple})$;
2. Each voter is in state $\mathcal{B} = (b \succ a, \text{banana})$.

To prove that this PES is decomposable, let us consider the measure ν that surely draws two identical functions y_1 and y_2 such that for each voter v, $y_v(a \succ b) = \mathcal{A}$ and $y_v(b \succ a) = \mathcal{B}$.

Drawing the profile p with law μ, we have with equal probabilities $p = (a \succ b, a \succ b)$ or $p = (b \succ a, b \succ a)$. Then, drawing y with the (deterministic) law ν, we have with equal probabilities $y(p) = (\mathcal{A}, \mathcal{A})$ or $y(p) = (\mathcal{B}, \mathcal{B})$, which is exactly culture π.

In short, this PES (Ω, π) can be simulated by drawing the profile $p = (p_1, p_2)$ with law μ (which is directly defined by culture π), drawing $y = (y_1, y_2)$ with law ν (which we exhibited), then considering $\omega = y(p)$. This proves that this PES (Ω, π) is decomposable.

Generally, it is not straightforward to decide whether an electoral space is decomposable or not. For this reason, we will give some sufficient or necessary conditions.

Proposition IV.6. *If voters $(\omega_1, \ldots, \omega_V)$ are independent, then (Ω, π) is decomposable.*

This is a consequence of proposition F.3 (in appendix). However, independence is not necessary for decomposability. Indeed, in example IV.5, voters are not independent (either they are both in state $\mathcal{A}$, or in state $\mathcal{B}$), but we proved that the PES is decomposable.

Another sufficient condition is met by an important class of models. For example, let us consider an electoral space where each voter v's state is constituted by a strict total order of preference p_v over the candidates and an integer $k_v \in \{0, \ldots, C\}$. This integer may have the following interpretation: voter v "approves" the top k_v candidates in her order of preference.

Let us consider the following culture. We draw $(p_1, \ldots, p_V)$ according to a probability law μ over $(\mathcal{L}_\mathcal{C})^V$. Independently, we draw $(k_1, \ldots, k_V)$ according to a probability law ξ over $\{0, \ldots, C\}^V$.

Let us note that for μ as well as for ξ, voters may not be independent. But drawings by μ and ξ are independent by assumption. The following proposition proves that such a PES is decomposable.

Proposition IV.7. *For each $v \in \mathcal{V}$, let $\mathcal{P}_v$ be a non-empty subset of $\mathcal{R}_\mathcal{C}$, let $\mathcal{K}_v$ be a non-empty measurable set and let $\Omega_v = \mathcal{P}_v \times \mathcal{K}_v$. Let P_v be the function defined by $\mathrm{P}_v(p_v, k_v) = p_v$ and K_v the fonction defined by $K_v(p_v, k_v) = k_v$.*

Let π be a culture over $\Omega = \prod_{v \in \mathcal{V}} \Omega_v$.

If the two random variables $\mathrm{P} = (\mathrm{P}_1, \ldots, \mathrm{P}_V)$ and $K = (K_1, \ldots, K_V)$ are independent, then (Ω, π) is decomposable.

Proof. To $k_v \in \mathcal{K}_v$, we associate the function $\mathrm{concat}_v(k_v) \in \mathcal{Y}_v$ that consists of concatenating p_v and k_v in order to reconstitute a state ω_v:

$$\mathrm{concat}_v(k_v) : \left| \begin{array}{ccl} \mathrm{P}_v(\Omega_v) & \to & \Omega_v \\ p_v & \to & \big(\mathrm{concat}_v(k_v)\big)(p_v) = (p_v, k_v). \end{array} \right.$$

To $k \in \mathcal{K}$, we associate $\mathrm{concat}(k) = (\mathrm{concat}_1(k_1), \ldots, \mathrm{concat}_V(k_V)) \in \mathcal{Y}$: to each profile p, this function simply "appends" vector k to give a configuration.

Then, denoting ξ the law of K, the image measure ν of ξ by concat is suitable to prove decomposability. $\square$

However, the condition from proposition IV.7 is not necessary. When the sets Ω_v are defined as products $\mathcal{P}_v \times \mathcal{K}_v$ (where $\mathcal{P}_v \subseteq \mathcal{R}_\mathcal{C}$), it may be the case that random variables P and K are not independent, but the space is decomposable anyway.

Indeed, in example IV.5, if $\mathrm{P} = (a \succ b, a \succ b)$, then we know for sure that $K = (\mathrm{apple}, \mathrm{apple})$, whereas if $\mathrm{P} = (b \succ a, b \succ a)$, then $K = (\mathrm{banana}, \mathrm{banana})$; so, P and K are not independent. However, as we saw, the PES is decomposable.

Like the toy model IV.5 of apple and banana voters, the following example shows that even without meeting the sufficient conditions from propositions IV.6 and IV.7, the PES may be decomposable; but with a model that is more elaborate and may be of practical interest.

We draw $(p_1, \ldots, p_V)$ according to a law μ over $(\mathcal{W}_\mathcal{C})^V$. For each voter v, let k_v be the number of indifference classes in p_v; for example, if there is no tie, it is the number of candidates. We draw k_v grades according to a law $\xi_v(k_v)$ over $[0, 1]^{k_v}$. This law $\xi_v(k_v)$ is chosen such that the k_v grades are almost surely all different.

Given a strict weak order p_v and a vector of grades $(g_1, \ldots, g_{k_v})$, we build a state ω_v by attributing decreasing grades to candidates in the order of p_v.

For example, if $p_v = (1 \sim 2 \succ 3 \succ 4)$, then there are $k_v = 3$ indifference classes: $\{1, 2\}, \{3\}$ and $\{4\}$. So, we use the law $\xi_v(3)$ to draw 3 grades, for example $(0.1, 0.8, 0.2)$. Finally we have $\omega_v = (1 \sim 2 \succ 3 \succ 4, (0.8, 0.8, 0.2, 0.1))$.

To prove that this PES is decomposable, let us build measure ν. Let v be a voter. Simultaneously for each possible $k_v \in \{0, \ldots, C\}$, let us draw k_v grades according to $\xi_v(k_v)$. This perfectly defines y_v, a transformation that associates to any p_v a state ω_v. By definition, when drawing p with law μ and y this way, $\omega = y(p)$ is drawn with the correct probability measure.

Finally, we give a necessary condition for decomposability.

Proposition IV.8. *If (Ω, π) is decomposable, then for any subset of voters $\mathcal{V}'$, for any event A on $\prod_{v \in \mathcal{V}'} \Omega_v$ (i.e. concerning only voters in $\mathcal{V}'$), for any profile $p = (p_1, \ldots, p_V) \in \mathcal{R}$ of positive probability:*

$$\pi\big(\omega_{\mathcal{V}'} \in A \mid \mathrm{P} = p\big) = \pi\big(\omega_{\mathcal{V}'} \in A \mid \mathrm{P}_{\mathcal{V}'} = p_{\mathcal{V}'}\big).$$

This is a consequence of proposition F.4 (in appendix). The intuitive interpretation is that if one knows the relations p_v of a subset of voters $\mathcal{V}'$ and want to reconstitute their states ω_v in a probabilistic way, then knowing the relations p_v of other voters does not provide useful information. However, it should be noticed that knowing the whole states ω_v of other voters might do so.

In appendix F, we show that unfortunately, this condition is not sufficient to ensure decomposability.

IV.C Slicing theorem

Now we can state and prove the slicing theorem.

Theorem IV.9 (slicing)**.** *Let (Ω, π) be a PES and f an SBVS whose manipulability rate is well defined (i.e. M_f is measurable).*

If (Ω, π) is decomposable, then there exists $y \in \mathcal{Y}$ such that the slice of f by y has a manipulability rate that is at most as high as f:

$$\rho_\pi(f_y) \leq \rho_\pi(f).$$

Proof. First, we show that the manipulability rate of any slice f_y can be expressed by considering an ordinal electoral space (where each voter is only described by her binary relation of preference), endowed with the probability law μ. Indeed, by definition:

$$\rho_\pi(f_y) = \int_{\omega \in \Omega} M_{f_y}(\omega) \pi(d\omega).$$

Remarking that $M_{f_y}(\omega)$ depends only on the profile $\mathrm{P}(\omega)$ and that ω and $y(\mathrm{P}(\omega))$ have the same profile, we have:

$$\rho_\pi(f_y) = \int_{\omega \in \Omega} M_{f_y}(y(\mathrm{P}(\omega))) \pi(d\omega).$$

Hence, by substitution:

$$\rho_\pi(f_y) = \int_{p \in \mathrm{P}(\Omega)} M_{f_y}(y(p))\mu(dp).$$

Now, let us study the manipulability of f. Using decomposability, we have by substitution:

$$\rho_\pi(f) = \int_{(p,y) \in \mathrm{P}(\Omega) \times \mathcal{Y}} M_f(y(p))(\mu \times \nu)(dp, dy).$$

Fubini-Tonelli theorem gives:

$$\rho_\pi(f) = \int_{y \in \mathcal{Y}} \left(\int_{p \in \mathrm{P}(\Omega)} M_f(y(p))\mu(dp) \right) \nu(dy).$$

Lemma IV.3 ensures that $M_f(y(p)) \geq M_{f_y}(y(p))$, which leads to:

$$\begin{aligned} \rho_\pi(f) &\geq \int_{y \in \mathcal{Y}} \left(\int_{p \in \mathrm{P}(\Omega)} M_{f_y}(y(p))\mu(dp) \right) \nu(dy) \\ &\geq \int_{y \in \mathcal{Y}} \rho_\pi(f_y)\nu(dy). \end{aligned}$$

So, the manipulability rate of f is no less than the average of the manipulability rates of the slices f_y. Hence, there exists at least one slice f_y such that $\rho_\pi(f_y) \leq \rho_\pi(f)$. □

In appendix G, we discuss whether the decomposability assumption is necessary for this theorem.

From proposition IV.6, we deduce the following corollary.

Corollary IV.10. *If the voters $(\omega_1, \ldots, \omega_V)$ are independent, then there exists $y \in \mathcal{Y}$ such that $\rho_\pi(f_y) \leq \rho_\pi(f)$.*

Up to this point, we do not have a precise idea of what the voting system f_y in theorem IV.9 and corollary IV.11 looks like: we just know that is it a slice of f. But it could have undesirable properties: for example, it is not explicitly excluded that it is dictatorial, which would be an especially trivial and uninteresting way to decrease the manipulability rate. To avoid this kind of pitfall, we will now see that these slicing results show a particular relevance when combined with the Condorcification theorems.

IV.D Combining Condorcification and slicing

It is easily proved, but worth noticing, that if f meets the Condorcet criterion, then so does any slice f_y. Indeed, if there is a Condorcet winner in ω, then she is also Condorcet winner in $\omega' = y(\mathrm{P}(\omega))$, because the profile is the same in both configurations ω and ω'. This gives another corollary to theorem IV.9.

Corollary IV.11. *If (Ω, π) is decomposable and if f meets the Condorcet criterion, then there exists an ordinal SBVS f', meeting the Condorcet criterion, such that:*

$$\rho_\pi(f') \leq \rho_\pi(f).$$

Combining Condorcification theorem III.8 and corollary IV.11, we deduce the following theorem.

Theorem IV.12 (Condorcification and slicing)**.** *Let (Ω, π) be a PES and f an SBVS (such that M_f and M_{f^*} are measurable).*

We assume that:

- *(Ω, π) is decomposable;*
- *f meets* **InfMC***.*

Then there exists an ordinal SBVS f', meeting the Condorcet criterion, such that:

$$\rho_\pi(f') \leq \rho_\pi(f).$$

Proof. By Condorcification theorem III.8, we know that $\rho_\pi(f^*) \leq \rho_\pi(f)$. Applying slicing corollary IV.11 to f^*, we have a suitable f'. □

As an example, let us consider Range voting again. We have just proved that in any decomposable culture, there exists a voting system f' that does not use grades but only binary relations of preferences, meets the Condorcet criterion, and is at most as manipulable as Range voting (in the sense of the manipulability rate).

The power of the electoral space framework is that this result is not restricted to Range voting: it applies to voting systems where ballots are grades, approval values, multicriteria grades or any other kind of objects. Also, absolutely no assumption is needed on the binary relations of preferences: they may fail to be transitive or even antisymmetric.

Unfortunately, the proof of the slicing theorem is not constructive: *a priori*, we are not able to exhibit a suitable f'. But we think that the previous results have deeper implications. Indeed, if the electoral space is decomposable and if we look for a voting system in **InfMC** that is as little manipulable as possible, then we can restrict our research to ordinal voting systems meeting the Condorcet criterion. Moreover, since there are a finite number of such voting systems (for given values of V and C), this observation guarantees the existence of a voting system whose manipulability rate is minimal in **InfMC**. We are now going to formalize this.

Proposition IV.13. *Let $\Omega = (\mathcal{R}_\mathcal{C})^V$ be the electoral space of the binary relations for V and C. Let μ be a culture over Ω.*

Then there exists an SBVS g, meeting the Condorcet criterion, such that:

$$\rho_\mu(g) = \min\{\rho_\mu(g'),\ g' \text{ meets the Condorcet criterion}\}.$$

We say that g is ρ_μ-optimal among ordinal Condorcet voting systems.

Proof. There are a finite number of functions $g : \Omega \to \mathcal{C}$, *a fortiori* if we demand that they meet the Condorcet criterion. So, as least one of them minimizes $\rho_\mu(g)$. ☐

Theorem IV.14 (general optimality of an ordinal Condorcet voting system)**.** *Let* $(V, C, \Omega, \mathrm{P}, \pi)$ *be a PES. As usual, we note* μ *the law of* P.

In the electoral space of binary relations for V *and* C*, let* $g : (\mathcal{R}_\mathcal{C})^V \to \mathcal{C}$ *be an SBVS that is* ρ_μ*-optimal among ordinal Condorcet voting systems.*

If (Ω, π) *is decomposable, then:*

$$\rho_\pi(g \circ \mathrm{P}) = \min\{\rho_\pi(f),\ f \in \mathbf{InfMC}\}.$$

Proof. Let $f \in \mathbf{InfMC}$. By theorem IV.12 (Condorcification and slicing), there exists an ordinal SBVS f', meeting the Condorcet criterion, such that $\rho_\pi(f') \leq \rho_\pi(f)$. And since g is ρ_μ-optimal, $\rho_\pi(g \circ \mathrm{P}) \leq \rho_\pi(f')$. ☐

In other words, g is optimal, not only among ordinal Condorcet voting systems, but among the larger class **InfMC** of all voting systems that meet the informed majority coalition criterion and that may not be ordinal. As we saw, this includes most of common voting systems from literature and real life.

As a consequence, in order to find a voting system that minimizes the manipulability rate in **InfMC**, we can restrict our investigation to ordinal Condorcet voting systems, which come in finite number.

The work presented in this paper has been carried out at LINCS (`www.lincs.fr`).

A Measurability

In order to define probabilistic notions rigorously, we consider measurable sets, which are constituted by a set E and a sigma-algebra Σ_E over E. Such a measurable set is denoted (E, Σ_E), or just E when there is no ambiguity. An element of Σ_E is called an *event* on E.

We always endow the set $\mathcal{R}_\mathcal{C}$ of binary relations over $\mathcal{C}$ with its discrete sigma-algebra, which we denote $\Sigma_{\mathcal{R}_\mathcal{C}}$.

When we consider a Cartesian product E of measurable sets (E_v, Σ_{E_v}), we always endow it with its product sigma-algebra. For example, the set $\mathcal{R} = (\mathcal{R}_\mathcal{C})^V$ is endowed with the product sigma-algebra $\Sigma_\mathcal{R} = (\Sigma_{\mathcal{R}_\mathcal{C}})^V$, which is simply its discrete sigma-algebra.

We call Ω a *measurable electoral space* iff each Ω_v is endowed with a sigma-algebra Σ_{Ω_v} and each function $\mathrm{P}_v : \Omega_v \to \mathcal{R}_\mathcal{C}$ is measurable. When considering a probabilized electoral space, we always assume implicitly that it is a measurable electoral space.

For example, let us consider our reference electoral space where $\omega_v = (p_v, u_v, a_v)$. Let us endow each Ω_v with the product sigma-algebra of the discrete one on $\mathcal{R}_\mathcal{C}$, Lebesgue sigma-algebra on $[0, 1]^C$ and the discrete one on $\{0, 1\}^C$. Then each function P_v is obviously measurable. Hence, Ω is a measurable electoral space.

Each set $\mathcal{Y}_v$ of slicing methods has a canonical sigma-algebra. Indeed, it is the case for the set ${\Omega_v}^{\mathrm{P}_v(\Omega_v)}$ of functions from $\mathrm{P}_v(\Omega_v)$ to Ω_v: associating each function to the list of its values, consider the product sigma-algebra $\Sigma_{\Omega_v} \times \ldots \times \Sigma_{\Omega_v}$, with a number of factors equal to the cardinal of $\mathrm{P}_v(\Omega_v)$. Space $\mathcal{Y}_v$, as a subset of ${\Omega_v}^{\mathrm{P}_v(\Omega_v)}$, inherits from this sigma-algebra.

The following lemma solves some questions of measurability for slicing theorem IV.9 and optimality theorem IV.14.

Lemma A.1. *Let Ω be a measurable electoral space, E a measurable set and $g : \Omega \to E$.*

We assume that g depends only on binary relations of preference:

$$\forall(\omega, \psi) \in \Omega^2, \mathrm{P}(\omega) = \mathrm{P}(\psi) \Rightarrow g(\omega) = g(\psi).$$

Then g is measurable.

Proof. Since g depends only on $\mathrm{P}(\omega)$, we may define $h : (\mathcal{R}_\mathcal{C})^V \to E$ such that $g = h \circ \mathrm{P}$. Since $(\mathcal{R}_\mathcal{C})^V$ is endowed with the discrete measure, h is measurable; and by definition of a measurable electoral space, P is measurable. So, $g = h \circ \mathrm{P}$ is measurable. $\square$

In theorem IV.9, this lemma ensures that for any y, the manipulability indicator M_{f_y} is measurable. Hence, $\rho_\pi(f_y)$ is well defined.

Similarly, in theorem IV.14, this lemma ensures that $M_{g \circ \mathrm{P}}$ is measurable. Hence, $\rho_\pi(g \circ \mathrm{P})$ is well defined.

B General voting systems

We present here a general framework that allows to represent any kind of voting system. Then we prove that, for the purpose of diminishing manipulability, investigation can be restricted to *state-based voting systems*, which justifies that we did so throughout the paper.

Let us consider one of the possible variants for the voting rule called *Range voting*. Each voter v has at her disposal a set of strategies $\mathcal{S}_v = [0, 1]^C$: she has to attribute a grade to each candidate. Once these grades are communicated, we use a processing rule f that returns the candidate with highest total grade.

More generally, a *game form*[15] (for V and C) is given by:

- For each voter $v \in \mathcal{V}$, a set $\mathcal{S}_v$ whose elements are called *strategies*[16];
- A function $f : \mathcal{S}_1 \times \ldots \times \mathcal{S}_V \to \mathcal{C}$ that is called *processing rule*.

When voter v is in state $\omega_v \in \Omega_v$, one may wonder which strategy $S_v \in \mathcal{S}_v$ should be called a sincere ballot. As noticed by Gibbard (1973), there is no obvious general way to define sincere voting on the only basis of the game form itself[17]. So, sincerity needs to be defined extrinsically by functions $s_v : \Omega_v \to \mathcal{S}_v$ that, to each state of opinion ω_v, associate a ballot $S_v = s_v(\omega_v)$.

From a slightly different point of view, let us consider a voter who always chooses her ballot deterministically on the only basis of her state of opinion, without any information about the opinions and ballots of the other voters. It may be because she does not have access to this kind of information before the election, or because she refuses to depend on her sources of information, such as polling organizations. Then by definition, she precisely uses a function $s_v : \Omega_v \to \mathcal{S}_v$, which can be seen, in that case, as a heuristic way of voting without external information.

For the example of Range voting, we might choose $s_v(p_v, u_v, a_v) = u_v$. In the first interpretation, the social planner transmits the message to the voters that communicating their sincere vector u_v is considered an appropriate behavior. In the second interpretation, s_v is a heuristic way of voting for a voter without information about the other voters[18].

More generally, a *general voting system* F (over Ω) is given by:

- A game form $((\mathcal{S}_v)_{v \in \mathcal{V}}, f)$;
- For each voter $v \in \mathcal{V}$, a function $s_v : \omega_v \to \mathcal{S}_v$ that is called *sincerity function*.

For $\omega \in \Omega$ and $S = (S_1, \ldots, S_n) \in \mathcal{S}_1 \times \ldots \times \mathcal{S}_n$, we say that F is *manipulable*

15. The terminology *game form* is taken from Gibbard (1973).

16. Here, strategies are what is called "pure strategies" in game theory. They can simply be ballots, decision trees in a multi-stage process or, more generally, any kind of objects.

17. Gibbard (1973) provides an illuminating example, the *non-alcoholic party*.

18. Whatever the interpretation, other choices are possible: for example, each voter could be suggested to scale her vector of grades so that her minimum is 0 and her maximum is 1.

in ω towards S iff:

$$\begin{cases} f(S_1, \ldots, S_V) \neq f(s_1(\omega_1), \ldots, s_V(\omega_V)), \\ \forall v \in \mathcal{V}, S_v \neq s_v(\omega_v) \Rightarrow f(S_1, \ldots, S_V) \ \mathrm{P}_v \ f(s_1(\omega_1), \ldots, s_V(\omega_V)). \end{cases}$$

We say that F is *manipulable in ω* iff there exists $S \in \mathcal{S}_1 \times \ldots \times \mathcal{S}_V$ such that F is manipulable in ω towards S. The set of manipulable configurations is denoted M_F.

When F is not manipulable in ω, it means that with the heuristics s_v (that may be provided by the social planner, or chosen individually by each voter), voters are able to find a strong Nash equilibrium without any preliminary exchange of information. Hence, even if all the ballots are revealed after the election, they will have no regret about the ballots they chose.

Now, we shall prove that, for the purpose of diminishing manipulability, we can restrict our study to *state-based voting systems*. For example, in the electoral space of strict total orders, let us examine the usual *two-round system* with the most intuitive sincerity function: during each round, one votes for the candidate she prefers among the available candidates. If a voter votes for candidate c during the first round and for d in a second round between c and d, then obviously, she cannot be sincere.

To prevent this kind of behavior, we can modify the voting system: voters communicate their orders of preference; then, two-round system is emulated with the corresponding sincere strategies. We are going to generalize this idea.

Proposition B.1. *Let us consider the voting system $F' = ((\mathcal{S}'_v)_{v \in \mathcal{V}}, f', (s'_v)_{v \in \mathcal{V}})$, defined as follows.*

1. *Each voter communicates a state: $\forall v \in \mathcal{V}, \mathcal{S}'_v = \Omega_v$.*
2. *Sincerity consists of giving one's true state: $\forall v \in \mathcal{V}, s'_v = \mathrm{Id}$.*
3. *To get the result, the former rule f is used, considering that each voter uses the sincere strategy corresponding to the state she communicated: $f'(\omega) = f(s_1(\omega_1), \ldots, s_V(\omega_V))$.*

Then voting system F', called the state-based version *of F, is at most as manipulable as F: $M_{F'} \subseteq M_F$.*

Proof. In F', sincere voting leads to the same result as in F, but manipulators have access to at most the same strategic ballots. □

A voting system F is called a *state-based voting system* (SBVS) iff it is equal to its state-based version; that is, iff for any voter $v \in \mathcal{V}$, $\mathcal{S}_v = \Omega_v$ and $s_v = \mathrm{Id}$. Considering the state-based version of a voting system has several advantages:

- It may prevent from using strategies that are obviously not sincere, like in the example of the two-round system.
- It simplifies the framework by idenfying states and possible strategies, thus avoiding the need of sincerity functions s_v.
- In an SBVS, the ballot embeds the binary relation of preferences, even if it was not the case in the original voting system. For voting systems like Range voting, this step is necessary before defining Condorcification.

C Variants of Condorcet winner

In the paper, we focused on the notion of absolute Condorcet winner, based on strict absolute victories: in each duel, $|c \text{ P}_v\ d| > \frac{V}{2}$ and $|d \text{ P}_v\ c| \leq \frac{V}{2}$, which amounts to the first condition only when preferences are antisymmetric (which is a common assumption).

Instead, we could have considered weaker variants of Condorcet winner:

- Relative Condorcet winner: $|c \text{ P}_v\ d| > |d \text{ P}_v\ c|$ in each duel.
- Weak Condorcet winner: $|c \text{ P}_v\ d| \geq |d \text{ P}_v\ c|$ in each duel.
- Condorcet-admissible candidate: $|d \text{ P}_v\ c| \leq \frac{V}{2}$ in each duel.

We can wonder whether these notions lead to a Condorcification theorem similar to theorem III.8, but it is not the case.

Indeed, let us consider $V = 5$ voters, $C = 3$ candidates called a, b, c and the electoral space of strict weak orders. We define the voting system *Condorcet-dean*: if a candidate is an absolute Condorcet winner, then she is elected; otherwise, the dean a is declared the winner. Since this system meets the Condorcet criterion, it also meets **InfMC**.

Let $f^{\#}$ be a Condorcification variant of f based on any one of the following preliminary tests:

- If there is a relative Condorcet winner, elect her.
- If there is a unique weak Condorcet winner, elect her.
- If there is at least one weak Condorcet winner, choose one (according to some predefined rule).
- If there is a unique Condorcet-admissible candidate, elect her.
- If there is at least one Condorcet-admissible candidate, choose one (according to some predefined rule).

We will prove that whatever the variant chosen, $f^{\#}$ is not at most as manipulable as f.

Let us consider configuration ω as follows (where orders of preferences are represented with the most liked candidate on top) and its matrix of duels $D(\omega)$.

ω	Voter				
	1	2	3	4	5
Pref.	a	a, c	b	b	c
	b	b	a, c	c	a
	c			a	b

$D(\omega)$	a	b	c
a	–	3	1
b	2	–	3
c	2	2	–

There is no absolute Condorcet winner, so $f(\omega)$ is the dean a. It is easy to see that f is not manipulable in ω: if it were, that would be towards a configuration without a Condorcet winner (lemma III.9). But in all such configurations, a is elected!

Candidates b and c are not even Condorcet-admissible. So, whatever the Condorcification variant $f^{\#}$ mentioned above, we have $f^{\#}(\omega) = a$.

Now, let us consider configuration ψ as follows, where only voters 4 and 5 have been modified, in an attempt to make c win instead of a.

<table>
<tr><td rowspan="2">ψ</td><td colspan="5">Voter</td></tr>
<tr><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td></tr>
<tr><td rowspan="3">Pref.</td><td>a</td><td>a, c</td><td>b</td><td>c</td><td>c</td></tr>
<tr><td>b</td><td>b</td><td>a, c</td><td>b</td><td>b</td></tr>
<tr><td>c</td><td></td><td></td><td>a</td><td>a</td></tr>
</table>

$D(\psi)$	a	b	c
a	–	2	1
b	3	–	2
c	2	3	–

Now, c is relative Condorcet winner, weak Condorcet winner and Condorcet-admissible. Candidates a and b are not even Condorcet-admissible. So, whatever the variant $f^{\#}$, we have $f^{\#}(\psi) = c$.

In conclusion, $f^{\#}$ is manipulable in ω towards ψ, whereas f is not manipulable in ω. That is why the Condorcification we defined in the paper rely only on absolute Condorcet winners.

D Characterization of the resistant Condorcet winner

In this appendix, we finish the proof of theorem III.12. We will suppose that c is a Condorcet winner in configuration ω, but not a resistant Condorcet winner, and prove that there exists an SBVS f meeting the Condorcet criterion and manipulable in ω.

Since c is not a resistant Condorcet winner, at least one of conditions (1) or (2) from definition III.11 is not met. We distinguish three cases: condition (1) is not met for some $e = d$; condition (2) is not met; and condition (1) is not met with $e \neq d$.

In all three cases, the idea of the proof is the same: we exhibit a configuration ψ that has no Condorcet winner and that differs from ω only by voters preferring d to c. So, it is possible to choose an SBVS f that meets the Condorcet criterion and such that $f(\psi) = d$. From this, we deduce that f is manipulable in configuration ω towards ψ, in favor of candidate d.

Case 1 If condition (1) is not met for some $e = d$, it means that $|\text{not}(d \text{ P}_v\ c) \text{ and } c \text{ P}_v\ d| \leq \frac{V}{2}$. Let p be a strict total order of the form: $(d \succ c \succ$ other candidates). Since the electoral space comprises all strict total orders, for each manipulator v (voter who prefers d to c), we can choose a ballot ψ_v such that $\text{P}_v(\psi_v) = p$. For all other voters, let $\psi_v = \omega_v$. In the new configuration ψ, candidate c cannot appear as a Condorcet winner because $|c \text{ P}_v(\psi_v)\ d| = |\text{not}(d \text{ P}_v(\omega_v)\ c) \text{ and } c \text{ P}_v(\omega_v)\ d| \leq \frac{V}{2}$. Candidate d cannot appear as a Condorcet winner (lemma III.9) and neither can other candidates, because the number of voters who claim preferring c to them has not diminished.

Case 2 If condition (2) is not met for some d and e, it means that $|\text{not}(d \text{ P}_v\ c) \text{ and not}(e \text{ P}_v\ c)| < \frac{V}{2}$. Let us notice that $e \neq d$, otherwise c would not be a Condorcet winner. Up to switching roles between d and e, we can assume that e has no strict victory against d. Let p be a strict total order of the form: ($d \succ e \succ c \succ$ other candidates). For each manipulator v (voter who prefers d to c), we can choose a ballot ψ_v such that $\mathrm{P}_v(\psi_v)= p$. For all other voters, let $\psi_v = \omega_v$. In the new configuration ψ, candidate c cannot appear as a Condorcet winner because she has a defeat against e: indeed, $|\text{not}(e \ \mathrm{P}_v(\psi_v) \ c)| = |\text{not}(d \ \mathrm{P}_v(\omega_v) \ c) \text{ and not}(e \ \mathrm{P}_v(\omega_v) \ c)| < \frac{V}{2}$. Candidate d cannot appear as a Condorcet winner (lemma III.9), neither can candidate e because she still has no strict victory against d and neither can other candidates, because the number of voters who claim preferring c to them has not diminished.

Case 3 Remains the case where condition (1) is not met for some $e \neq d$. Denoting $B_{de} = |\text{not}(d \ \mathrm{P}_v \ c) \text{ and } c \ \mathrm{P}_v \ e|$, it means that $B_{de} \leq \frac{V}{2}$. Using the previous case, we may assume, however, that condition (2) is met.

In the final configuration ψ, we will ensure that there is neither a victory for c against e, nor for e against c.

Let p be a strict total order of the form: ($d \succ e \succ c \succ$ other candidates).

Let p' be a strict total order of the form: ($d \succ c \succ e \succ$ other candidates).

Since c is a Condorcet winner, we have $|c \ \mathrm{P}_v \ e| > \frac{V}{2}$ hence:

$$|d \ \mathrm{P}_v \ c \text{ and } c \ \mathrm{P}_v \ e| > \frac{V}{2} - B_{de} \geq 0.$$

As a consequence, we may choose $\lfloor \frac{V}{2} \rfloor - B_{de}$ voters among the manipulators (voters who prefer d to c); for each v of them, let us choose ψ_v such that $\mathrm{P}_v(\psi_v)= p'$. For the other manipulators, let us choose ψ_v such that $\mathrm{P}_v(\psi_v)= p$. Lastly, for the sincere voters (who do not prefer d to c), let $\psi_v = \omega_v$.

Then, we have:

$$D_{ce}(\psi) = B_{de} + \left(\lfloor \frac{V}{2} \rfloor - B_{de} \right) = \lfloor \frac{V}{2} \rfloor,$$

hence c has no victory against e.

By the way, (1) is not met for this pair (d, e) but (2) is met. Hence we have:

$$\begin{cases} |\text{not}(d \ \mathrm{P}_v \ c) \text{ and } c \ \mathrm{PP}_v \ e| + |\text{not}(d \ \mathrm{P}_v \ c) \text{ and } c \ \mathrm{MP}_v \ e| \leq \lfloor \dfrac{V}{2} \rfloor, \\ |\text{not}(d \ \mathrm{P}_v \ c) \text{ and } c \ \mathrm{PP}_v \ e| + |\text{not}(d \ \mathrm{P}_v \ c) \text{ and } c \ \mathrm{I}_v \ e| \quad \geq \lceil \dfrac{V}{2} \rceil, \end{cases}$$

so, by subtraction:

$$|\text{not}(d \ \mathrm{P}_v \ c) \text{ and } c \ \mathrm{MP}_v \ e| - |\text{not}(d \ \mathrm{P}_v \ c) \text{ and } c \ \mathrm{I}_v \ e| \leq \lfloor \frac{V}{2} \rfloor - \lceil \frac{V}{2} \rceil.$$

Using proposition III.1, we deduce:

$$\begin{aligned} D_{ec}(\psi) &= V + |c \ \mathrm{MP}_v(\psi_v) \ e| - |c \ \mathrm{I}_v(\psi_v) \ e| - D_{ce}(\psi), \\ &\leq V + \lfloor \frac{V}{2} \rfloor - \lceil \frac{V}{2} \rceil - \lfloor \frac{V}{2} \rfloor = \lfloor \frac{V}{2} \rfloor, \end{aligned}$$

hence e has no victory against c.

To sum up, neither c nor e can be a Condorcet winner. And, for the same reasons as in previous cases, neither can d nor the other candidates.

E Common voting systems and the resistant-Condorcet criterion

In this appendix, we finish the proof of corollary III.15 by showing that the voting systems mentioned do not respect the resistant-Condorcet criterion for some values of V and C. In the following examples, preferences are strict total orders, with the most liked candidate on top.

Counterexample 1 In the following table, each group has approximatively the same number of voters (slight differences can be introduced to avoid questions of tie-breaking).

	Group of voters				
	1	2	3	4	5
Preferences	d_1	d_2	d_3	d_4	d_5
	c	c	c	c	c
	Others	Others	Others	Others	Others
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Candidate c is preferred to any pair of other candidates (d_i, d_j) by about 60% of the voters, hence she is a resistant Condorcet winner. However, she is elected neither by IRV, nor by plurality, nor by the two-round system. For voting systems with grades (including approval voting), we consider the case where the sincere opinion of the voters is to put the maximum grade to their most liked candidate and the minimal grade to all the others; then c is neither elected by approval voting nor range voting (whether median or average grade is used).

Counterexample 2 In the following table, each group has approximatively the same number of voters, but group 0 is slightly larger than the others.

	Group of voters						
	0	1	2	3	4	5	6
Preferences	d_1	d_1	d_2	d_3	d_4	d_5	d_6
	d_2	c	c	c	c	c	c
	d_3	Others	d_1	d_1	d_1	d_1	d_1
	d_4		Others	Others	Others	Others	Others
	d_5	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	d_6						
	c	d_2	d_3	d_4	d_5	d_6	d_1

Candidate c is preferred to any pair of other candidates by about $4/7$ of the population, hence she is a resistant Condorcet winner. The average ranking of candidate d_1 is $(1\times2+3\times5)/7 = 17/7$ and the one of c is $(7\times1+2\times6)/7 = 25/7$ so c is not elected by the Borda method. Using the Coombs method, candidate c is eliminated in the first round therefore she is not elected.

Counterexample 3 The candidates are $\{a, c, d_1, d_2, e_1, \ldots, e_8\}$. The size of each group of voters is given in proportion of the whole population.

	Size of the group of voters						
	10%	10%	40%	10%	10%	10%	10%
Preferences	a	a	c	e_1	e_3	e_5	e_7
	d_1	d_2	a	e_2	e_4	e_6	e_8
	c	c	Others	c	c	c	c
	Others	Others		Others	Others	Others	Others

Candidate c is preferred to any pair $\{a, e_i\}$ by 70% of the voters and to any other pair of candidates by at least 80% of the voters, hence she is a resistant Condorcet winner. Candidate a has a median ranking of 2 and c has a median ranking of 3, hence c cannot be elected by the Bucklin method.

F Decomposability

In this appendix, we generalize the notion of decomposability to probabilized sets that are not necessarily electoral spaces and we prove some related results in this general case.

One dimension

We will prove the lemma of the *complementary random variable*, which shows that a decomposition as in definition IV.4 is always possible when there is only one voter.

Let us consider a probabilized space (Ω, π) and a random variable X with values in a finite measurable set $\mathcal{X}$ endowed with the discrete measure. We denote μ the law of X. One may picture $\omega \in \Omega$ as the state of one voter and X as her binary relation of preference.

When the random experience is realized, the state of the system is described by ω. The value $x = X(\omega)$ is a partial information about this state: if we know only x, we generally lack some information about ω. Let us imagine that there exists a space $\mathcal{Y}$ that allows to express this additional piece of information: it means that the pair (x, y) represents ω without ambiguity.

Let us imagine, moreover, that random variables x and y are independent: generally, if is a powerful property, because it allows to deal with the two variables separately.

The construction that we will consider is a generalization of this notion of complementary information. Indeed, we have a very important freedom: we can choose the set $\mathcal{Y}$. For the sake of generality, we will always choose the set of functions $y : X(\Omega) \to \Omega$ that are coherent with X, in the sense that $X \circ y = \mathrm{Id}$. Indeed, it is the general framework so that giving an x an a y perfectly defines a state ω that is coherent with x.

Lemma F.1 (complementary random variable)**.** *Let (Ω, π) be a probabilized set, X a random variable with value in a finite set $\mathcal{X}$ endowed with its discrete sigma-algebra, μ the law of X and $\mathcal{Y} = \{y : X(\Omega) \to \Omega \text{ s.t. } X \circ y = Id\}$.*

Then there exists a measure ν over $\mathcal{Y}$ such that for any event A on Ω:

$$\pi(\omega \in A) = (\mu \times \nu)\big(y(x) \in A\big).$$

Proof. For any $x \in X(\Omega)$:

- If $\pi(X = x) > 0$, we note π_x the measure of conditional probability knowing $X = x$ (restricted to $X^{-1}(x)$);
- If $\pi(X = x) = 0$, we choose an arbitrary $\omega_x \in X^{-1}(x)$ and we note π_x the probability measure that surely returns ω_x.

Identifying a function $y \in \mathcal{Y}$ to the list of its values for each possible argument x, we define ν as the product measure of all π_x.

Then for any event A on Ω:

$$\begin{aligned}(\mu \times \nu)\big(y(x) \in A\big) &= \sum_{x \in X(\Omega)} \mu(\{x\}) \cdot \nu(y(x) \in A) \\ &= \sum_{\pi(X=x)>0} \pi(X = x) \cdot \pi(\omega \in A \mid X = x) \\ &= \pi(\omega \in A).\end{aligned}$$

We thank Anne-Laure Basdevant and Arvind Singh for fruitful discussions about this lemma. □

Several dimensions

Now, we deal with decomposability in the general case.

Let $V \in \mathbb{N} \setminus \{0\}$. For each $v \in \{1, \dots, V\}$, let Ω_v be a measurable set and $X_v : \Omega_v \to \mathcal{X}_v$ a measurable function, where $\mathcal{X}_v$ is a finite set endowed with the discrete measure.

Let π be a probability measure over the universe $\Omega = \prod_{v=1}^{V} \Omega_v$. We note $X = (X_1, \dots, X_V)$ and μ the law of X.

For each v, we note $\mathcal{Y}_v = \{y_v : X_v(\Omega_v) \to \Omega_v \text{ s.t. } X_v \circ y_v = \mathrm{Id}\}$. We note $\mathcal{Y} = \prod_{v=1}^{V} \mathcal{Y}_v$. For $(x, y) \in X(\Omega) \times \mathcal{Y}$, we note $y(x) = (y_1(x_1), \dots, y_V(x_V))$.

Definition F.2 (decomposability in general)**.** We say that (Ω, π) is X-*decomposable* iff there exists a measure ν over $\mathcal{Y}$ such that for any event A on Ω:

$$\pi(\omega \in A) = (\mu \times \nu)\big(y(x) \in A\big).$$

The difficulty comes from our demand for complementary random variables y_v that are *individual*: y may not be any function from $X(\Omega)$ to Ω, but must be a V-tuple of functions, where each y_v is from $X_v(\Omega_v)$ to Ω_v. Indeed, in the proof of lemma IV.3, we need individual random variables (for the sincere voters). If we asked a *collective* random variable y that, from x, allows to reconstitute ω with the correct probability law, then it would always be possible, by a direct application of the lemma F.1 of the complementary random variable.

Proposition F.3. *If the random variables $(\omega_1, \dots, \omega_V)$ are independent, then (Ω, π) is X-decomposable.*

Proof. Simply apply lemma F.1 for each $v \in \{1, \dots, V\}$, which defines a measure ν_v over each set $\mathcal{Y}_v$. Then, define ν as the product measure of the ν_v's. ☐

Proposition F.4. *If (Ω, π) is X-decomposable then, for each subset $\mathcal{V}'$ of $\{1, \dots, V\}$, for each event A on $\prod_{v \in \mathcal{V}'} \Omega_v$, for each $x = (x_1, \dots, x_V)$ of positive probability:*

$$\pi\big(\omega_{\mathcal{V}'} \in A \mid X = x\big) = \pi\big(\omega_{\mathcal{V}'} \in A \mid X_{\mathcal{V}'} = x_{\mathcal{V}'}\big). \tag{1}$$

Proof. On one hand:

$$\begin{aligned} \pi\big(\omega_{\mathcal{V}'} \in A \mid X = x\big) &= (\mu \times \nu)\big(y_{\mathcal{V}'}(x_{\mathcal{V}'}) \in A \mid X = x\big) \\ &= \nu\big(y_{\mathcal{V}'}(x_{\mathcal{V}'}) \in A\big). \end{aligned}$$

On the other hand, we have similarly:

$$\pi\big(\omega_{\mathcal{V}'} \in A \mid X_{\mathcal{V}'} = x_{\mathcal{V}'}\big) = \nu\big(y_{\mathcal{V}'}(x_{\mathcal{V}'}) \in A\big).$$

☐

However, condition (1) does not ensure that (Ω, π) is X-decomposable. As a counterexample, let $V = 2$. The state ω_v (with $v = 1$ or $v = 2$) may take 4 values, noted ω_v^1 to ω_v^4. The variable X_v may take 2 values, x_v^a and x_v^b. The following table defines the correspondence between the states ω_v and the variables x_v, as well as measure π.

π	$\omega_1^1 \to x_1^a$	$\omega_1^2 \to x_1^a$	$\omega_1^3 \to x_1^b$	$\omega_1^4 \to x_1^b$
$\omega_2^1 \to x_2^a$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	0
$\omega_2^2 \to x_2^a$	$\frac{1}{16}$	$\frac{1}{16}$	0	$\frac{1}{8}$
$\omega_2^3 \to x_2^b$	$\frac{1}{8}$	0	0	$\frac{1}{8}$
$\omega_2^4 \to x_2^b$	0	$\frac{1}{8}$	$\frac{1}{8}$	0

It is tedious but easy to check that condition (1) is met. Indeed, for example:

$$\pi(\omega_1^1 \mid x_1^a \wedge x_2^a) = \frac{1}{2} = \pi(\omega_1^1 \mid x_1^a).$$

Now, let us suppose that (Ω, π) is X-decomposable. Let ν be a suitable measure for this decomposition. For $(\alpha, \beta, \gamma, \delta) \in \{1, 2\} \times \{3, 4\} \times \{1, 2\} \times \{3, 4\}$, let us note:

$$\nu(\alpha, \beta, \gamma, \delta) = \nu\Big(y_1(x_1^a) = \omega_1^\alpha \ \wedge \ y_1(x_1^b) = \omega_1^\beta \ \wedge \ y_2(x_2^a) = \omega_2^\gamma \ \wedge \ y_2(x_2^b) = \omega_2^\delta\Big).$$

For example, $\nu(\mathbf{1}, \mathbf{4}, \mathbf{2}, \mathbf{4})$ is a notation shortcut for the probability that, when drawing y with probability law ν, we have $y_1(x_1^a) = \omega_1^{\mathbf{1}}$ and $y_1(x_1^b) = \omega_1^{\mathbf{4}}$ and $y_2(x_2^a) = \omega_2^{\mathbf{2}}$ and $y_2(x_2^b) = \omega_2^{\mathbf{4}}$: this value of y corresponds to the choice of columns **1** and **4** and rows **2** and **4** in the table.

We notice the following facts.

- $0 = \pi(\omega_1^3 \wedge \omega_2^3 \mid x_1^b \wedge x_2^b) = \sum_{(\alpha,\gamma)} \nu(\alpha, 3, \gamma, 3)$. Since all terms in this sum are nonnegative, we have in particular $\nu(1, 3, 1, 3) = 0$.
- $0 = \pi(\omega_1^1 \wedge \omega_2^4 \mid x_1^a \wedge x_2^b) = \sum_{(\beta,\gamma)} \nu(1, \beta, \gamma, 4)$ hence $\nu(1, 3, 1, 4) = 0$ and $\nu(1, 4, 1, 4) = 0$.
- $0 = \pi(\omega_1^4 \wedge \omega_2^1 \mid x_1^b \wedge x_2^a) = \sum_{(\alpha,\delta)} \nu(\alpha, 4, 1, \delta)$ hence $\nu(1, 4, 1, 3) = 0$.

So, $\frac{1}{4} = \pi(\omega_1^1 \wedge \omega_2^1 \mid x_1^a \wedge x_2^a) = \sum_{(\beta,\delta)} \nu(1, \beta, 1, \delta) = \nu(1, 3, 1, 3) + \nu(1, 3, 1, 4) + \nu(1, 4, 1, 3) + \nu(1, 4, 1, 4) = 0$: this contradiction proves that (Ω, π) is not decomposable.

In this counterexample, it can be shown that there exists a *signed measure* ν that meets the usual relation for decomposition. Unfortunately, we need a positive measure: indeed, the proof of slicing theorem IV.9 uses the growth property of integration, which is based on its positivity property.

G Variants of the slicing theorem

In slicing theorem IV.9 and its consequence, theorem IV.12 of Condorcification and slicing, instead of having a voting system f' such that $\rho_\pi(f') \leq \rho_\pi(f)$ for a specific culture π, it would be stronger to have an inclusion: $M_{f'} \subseteq M_f$.

Since this generalization seems difficult, we will strengthen the assumptions and weaken the other demands from theorem IV.12.

- We assume that f meets the Condorcet criterion (instead of **InfMC**).
- We still demand that f' depends only on binary relations of preference.
- We only demand that f' meets **InfMC** (instead of the Condorcet criterion).

First, we remark that if such an f' exists, then by Condorcification theorem III.8, we can demand that it meets the Condorcet criterion.

We will prove that this generalization is not true.

Let us consider the following electoral space. There are $V = 3$ voters and $C = 3$ candidates a, b, c. The state of each voter is given by a strict total order of preference and a bit whose value is 0 or 1. Here is the non-ordinal voting system f that we consider:

- If there is a Condorcet winner, she is elected;
- If there is none, if at least two bits are 1 then c is elected, otherwise (i.e. if at least two bits are 0) b is elected.

Let us consider the three following configurations ω, ϕ and ψ.

Configuration	Voter			Condorcet winner
	1	2	3	
ω	a c , 0 b	b c , 1 a	c a , 1 b	c
ϕ	a b , 1 c	b a , 0 c	c a , 1 b	a
ψ	a b , 0 c	b c , 0 a	c b , 1 a	b

f is not manipulable in ω. On one hand indeed, to manipulate in favor of b, only the second voter may change her ballot. But then, b stays Condorcet loser; since there are three victories in the matrix of duels, there is a Condorcet winner who is not b. On the other hand, to manipulate in favor of a, the first voter may try to make the configuration appear as without Condorcet winner; but then, two bits are still equal to 1, so the winner is c and the manipulation fails.

Similarly, it can be shown that f is manipulable neither in ϕ nor in ψ.

Let us consider an ordinal f' meeting the Condorcet criterion, and the following family of configurations χ (voters' bits do not matter for f').

Configuration	Voter			Condorcet winner
	1	2	3	
χ	a b c	b c a	c a b	None

If $f'(\chi) = a$ (resp. b, c), then f' is manipulable in ω (resp. ϕ, ψ) towards χ. Hence, f' must be manipulable in at least one of the three configurations ω, ϕ, ψ, therefore it is impossible to have $M_{f'} \subseteq M_f$.

This counterexample also proves that slicing theorem IV.9 and its consequence, theorem IV.12, are not true when removing the assumption of decomposability (without another assumption to replace it).

Indeed, let us define the culture $\pi(\omega) = \pi(\phi) = \pi(\psi) = \frac{1}{3}$. Then, the initial voting system f has a manipulability rate equal to 0: though it is manipulable in some configurations, it is almost surely not manipulable. However, any ordinal f' meeting **InfMC** has a manipulability rate no lower than $\frac{1}{3}$.

Finally, it is an open question whether slicing theorems IV.9 and IV.12 hold true with an assumption that is weaker than decomposability, for example the condition presented in proposition IV.8.

References

L. Balinski and R. Laraki.
Majority Judgment: Measuring, Ranking, and Electing.
MIT Press, 2010.

Duncan Black.
The theory of committees and elections.
University Press, 1958.

Steven Brams and Peter Fishburn.
Approval voting.
American Political Science Review, 72:831–847, 1978.

John R. Chamberlin, Jerry L. Cohen, and Clyde H. Coombs.
Social choice observed: Five presidential elections of the american psychological association.
The Journal of Politics, 46:479–502, 1984.

François Durand, Fabien Mathieu, and Ludovic Noirie.
Manipulability of voting systems.
2012.
URL http://www.liafa.univ-paris-diderot.fr/~displexity/docpub/6mois/votes.pdf.

François Durand, Fabien Mathieu, and Ludovic Noirie.
New simulation results on coalitional manipulation.
2014.

Pierre Favardin and Dominique Lepelley.
Some further results on the manipulability of social choice rules.
Social Choice and Welfare, 26:485–509, 2006.

Pierre Favardin, Dominique Lepelley, and Jérôme Serais.
Borda rule, copeland method and strategic manipulation.
Review of Economic Design, 7:213–228, 2002.

Allan Gibbard.
Manipulation of voting schemes: A general result.
Econometrica, 41(4):pp. 587–601, 1973.

James Green-Armytage.
Four condorcet-hare hybrid methods for single-winner elections.
Voting matters, 29:1–14, 2011.

James Green-Armytage.
Strategic voting and nomination.
Social Choice and Welfare, 42(1):111–138, 2014.

James Green-Armytage, T. Nicolaus Tideman, and Rafael Cosman.
Statistical evaluation of voting rules.
2014.

HC Huang and Vincent CH Chua.
Analytical representation of probabilities under the iac condition.
Social Choice and Welfare, 17(1):143–155, 2000.

J.G. Kemeny.
Mathematics without numbers.
Daedalus, 88:575–591, 1959.

K. Kim and F. Roush.
Statistical manipulability of social choice functions.
Group Decision and Negotiation, 5:263–282, 1996.

Dominique Lepelley and Boniface Mbih.
The proportion of coalitionally unstable situations under the plurality rule.
Economics Letters, 24(4):311 – 315, 1987.

Dominique Lepelley and Boniface Mbih.
The vulnerability of four social choice functions to coalitional manipulation of preferences.
Social Choice and Welfare, 11:253–265, 1994.

Dominique Lepelley and Fabrice Valognes.
On the kim and roush voting procedure.
Group Decision and Negotiation, 8:109–123, 1999.

Dominique Lepelley and Fabrice Valognes.
Voting rules, manipulability and social homogeneity.
Public Choice, 116:165–184, 2003.

Dominique Lepelley, Ahmed Louichi, and Hatem Smaoui.
On ehrhart polynomials and probability calculations in voting theory.
Social Choice and Welfare, 30:363–383, 2008.

H. Moulin.
La Stratégie du vote.
Cahiers du CEREMADE. Université Paris IX-Dauphine Centre de recherche de mathématiques de la décision, 1978.

Bezalel Peleg.
Game theoretic analysis of voting in committees.
Number 7 in Econometric Society monographs in pure theory. Cambridge Univ. Press, 1984.

Geoffrey Pritchard and Arkadii Slinko.
On the average minimum size of a manipulating coalition.
Social Choice and Welfare, 27:263–277, 2006.

Geoffrey Pritchard and Mark Wilson.
Exact results on manipulability of positional voting rules.
Social Choice and Welfare, 29:487–513, 2007.

Reyhaneh Reyhani, Geoffrey Pritchard, and Mark Wilson.
A new measure of the diffculty of manipulation of voting rules, 2009.

Donald G. Saari.
Susceptibility to manipulation.
Public Choice, 64(1):pp. 21–41, 1990.

Mark Allen Satterthwaite.
Strategy-proofness and arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions.
Journal of Economic Theory, 10(2):187 – 217, 1975.

Markus Schulze.
A new monotonic, clone-independent, reversal symmetric, and condorcet-consistent single-winner election method.
Social Choice and Welfare, 36:267–303, 2011.

Arkadii Slinko.
How large should a coalition be to manipulate an election?
Mathematical Social Sciences, 47(3):289 – 293, 2004.

David A. Smith.
Manipulability measures of common social choice functions.
Social Choice and Welfare, 16:639–661, 1999.

N. Tideman.
Collective Decisions And Voting: The Potential for Public Choice.
Ashgate, 2006.

T. N. Tideman.
Independence of clones as a criterion for voting rules.
Social Choice and Welfare, 4:185–206, 1987.

H. P. Young and A. Levenglick.
A consistent extension of condorcet's election principle.
SIAM Journal on Applied Mathematics, 35(2), 1978.