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Towards less manipulable voting systems

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Note to the reader

This is a (rather quick) translation of the original French version of this memoir, which is entitled: "Vers des modes de scrutin moins manipulables". I apologize for the possible spelling and grammar mistakes in this English version.

Abstract

We investigate the coalitional manipulation of voting systems: is there a subset of voters who, by producing an insincere ballot, can secure an outcome that they strictly prefer to the candidate who wins if all voters provide a sincere ballot?

From a theoretical point of view, we develop a framework that allows us to study all kinds of voting systems: ballots can be linear orders of preferences over the candidates (*ordinal* systems), grades or approval values (*cardinal* systems) or even more general objects. We prove that for almost all voting systems from literature and real life, manipulability can be strictly diminished by adding a preliminary test that elects the Condorcet winner if one exists. Then we define the notion of *decomposable* culture and prove that it is met, in particular, when voters are independent. Under this assumption, we prove that for any voting system, there exists a voting system that is ordinal, has some common properties with the original voting system and is at most as manipulable. As a consequence of these theoretical results, when searching for a voting system whose manipulability is minimal (in a class of reasonable systems), investigation can be restricted to those that are ordinal and meet the Condorcet criterion.

In order to provide a tool to investigate these questions in practice, we present SVVAMP, a Python package we designed to study voting systems and their manipulability. We use it to compare the coalitional manipulability of several voting systems in a variety of cultures, i.e. probabilistic models generating populations of voters with random preferences. Then we perform the same kind of analysis on real elections. Lastly, we determine voting systems with minimal manipulability for very small values of the number of voters and the number of candidates and we compare them with classical voting systems from literature and real life. Generally speaking, we show that the Borda count, Range voting and Approval voting are especially vulnerable to manipulation. In contrast, we find an excellent resilience to manipulation for the voting system called *IRV* (also known as *STV*) and its variant *Condorcet-IRV*.

Thanks

Thanks to Sébastien, who accepted to be the director of this thesis. Thanks to Ludovic, who immersed himself with passion in this research area, which was not his main topic initially, and who brought contributions from fields as diverse as optics, quantum mechanics and telecommunications. Thanks to Fabien for his constant support during those years, the inestimable complementarity of our approaches, the thoroughness and the competence with which he lead my work and, most of all, the authenticity of his shirts.

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Thanks to Anne-Laure and Arvind for our discussions about the lemma of the complementary random variable. Thanks to Laurent for our discussions and for suggesting the voting system IRVD. Thanks to Benoit for accepting to have a look on a paper tablecloth to examine a question that I would have been unable to solve on my own.

Thanks to Displexity members, Carole, Hugues and all the others, who welcomed me and have listened carefully to my talks. Thanks to the organizers of Algotel 2012, to Nicolas and Pierre for contributing to organize our experiments. Thanks to Alix for communicating the database of my favorite brown website.

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Contents

Publications

Communications in a conference

François Durand, Benoît Kloeckner, Fabien Mathieu, and Ludovic Noirie. Geometry on the utility sphere. In *Proceedings of the 4th International Conference on Algorithmic Decision Theory (ADT)*, 2015.

François Durand, Fabien Mathieu, and Ludovic Noirie. Élection du best paper AlgoTel 2012: étude de la manipulabilité. In *AlgoTel 2014 – 16èmes Rencontres Francophones sur les Aspects Algorithmiques des Télécommunications*, 2014.

François Durand, Fabien Mathieu, and Ludovic Noirie. Élection d'un chemin dans un réseau: étude de la manipulabilité. In *AlgoTel 2014 – 16èmes Rencontres Francophones sur les Aspects Algorithmiques des Télécommunications*, 2014.

François Durand, Fabien Mathieu, and Ludovic Noirie. On the manipulability of voting systems: application to multi-operator networks. In *Proceedings of the 9th International Conference on Network and Service Management (CNSM)*, pages 292–297. IEEE, 2013.

Poster

François Durand, Fabien Mathieu, and Ludovic Noirie. Reducing manipulability. Poster presented during the 5th International Workshop on Computational Social Choice (COMSOC), 2014.

Communication in a work group

François Durand, Fabien Mathieu, and Ludovic Noirie. Manipulability of voting systems. Work group Displexity, http://www.liafa.univ-paris-diderot.fr/~displexity/docpub/6mois/votes.pdf, 2012.

Research reports

François Durand, Fabien Mathieu, and Ludovic Noirie. Making most voting systems meet the Condorcet criterion reduces their manipulability. https://hal.inria.fr/hal-01009134, 2014.

François Durand, Fabien Mathieu, and Ludovic Noirie. Making a voting system depend only on orders of preference reduces its manipulability rate. https://hal.inria.fr/hal-01009136, 2014.

Part I

Theoretical study of manipulability

Part II

Computer-assisted study of manipulability

Chapter 7

Simulations in spheroidal cultures

In this chapter and the two following ones, we study two issues that were among our core motivations to develop SVAMP.

1. Whereas manipulability is a theoretic necessity shown by the theorem proved by Gibbard (1973) and Satterthwaite (1975), is it a common phenomenon in practice?

2. What are the compared performances of different voting systems from this point of view?

These are the main motivations for this chapter and the two following ones. A secondary motivation is to illustrate the possibilities offered by SVAMP.

Generally, we focus on the coalitional manipulability (CM), which is the central subject of this memoir and which we call also *manipulability* with no more precision. However, since SVAMP does not have exact specific algorithm for the CM of each voting system, we will also use the trivial manipulation (TM) as a point of comparison between the voting systems. As we discussed in section 6.2.3, TM has the advantage of being easy to compute (in polynomial time, up the time needed to compute the winner). Moreover, when manipulation is possible, TM gives an indicative measure of the cases where manipulation is relatively cheap to implement in terms of communication.

Given the variety of the culture under study, reading these chapters 7 and 8 might be tedious for the impatient reader, but this an issue we did not manage to avoid. For a quicker overview, we suggest to read section 7.1 which defines the cultures under study in this chapter, section 7.2 which presents our reference scenario, section 7.3 which familiarizes with the conventions used for the curves, section 7.9 concluding this chapter, the introduction of chapter 8 and sections 8.6 and 8.7 concluding it, before going on with chapter 9, analyzing data from real-life elections.

7.1 Presentation of the spheroidal cultures

In voting theory, it is classic to consider the impartial culture (definition 1.14), which is defined for preferences that are strict total orders. Since we want to study together some ordinal and cardinal voting systems, the first question that naturally arises is the way to extend this model to voters that are characterized by their utilities over the candidates. In this section, we briefly describe our approach, which is presented in a more detailed and formal way in appendix B and in the paper by Durand et al. (2015).

Without adopting all assumptions from the model of expected utilites by Von Neumann and Morgenstern (1944), we will draw inspiration from it. We will not necessarily assume that voters' preferences are characterized by expected utilities, because there is no random element after the population is drawn: voting systems under study compute the winner in a deterministic way and manipulators use only pure strategies, not mixed strategies.

However, we keep an important property of this model: to begin with, a voter's utility vector $\mathbf{u}_v \in \mathbb{R}^C$ over the candidates is defined up to an additive constant and up to a positive multiplicative constant, which means that vector $\mathbf{u}_v$ represents the same preferences as a vector $\alpha\mathbf{u}_v + \beta\mathbf{1}$, where α is a positive real number, β any real number and where $\mathbf{1}$ denotes the vector whose all coordinates are equal to 1. This choice is motivated by the fact that each voter has no canonical way, neither to fix her point 0 of utility, nor to choose the "unit of measurement" that is used to evaluate utility values: she is only able to measure their respective intensities.

In a second time, to have a complete model for approval voting, we add the assumption that utility 0 is an *approval limit* (see section B.7): a voter approves of the candidates with a nonnegative utility and disapproves those with a negative utility. With this additional assumption, a voter's utility vector is defined only up to a positive multiplicative constant α. So, the utility space is the one of semi-lines in $\mathbb{R}^C$ and all the vectors in a given semi-line represent the same preferences in practice and are indistinguishable, even by the voter herself.

Intuitively, a natural way to represent this space consists of normalize each utility vector so that its Euclidean noram is equal to 1. So the utility space, i.e. the set of all semi-lines, is represented by the unit sphere of $\mathbb{R}^C$. In appendix B, we prove, by arguments of projective and differential geometry, that this representation is essentially the only one having good properties[1].

Then, we have a canonical way to generalize the impartial culture: we draw a vector on the surface of the unit sphere of $\mathbb{R}^C$ with a uniform probability law (in the sense of the usual Euclidean measure): this is what we call the *spherical culture*. Each voter has almost surely a strict total of preference, voters are independent and the culture is neutral: as a consequence, the ordinal image of this culture (definition 5.13) is the impartial culture indeed. So, for ordinal voting systems, this culture is simply equivalent to the impartial culture; but in addition, it makes it possible to study ordinal and cardinal systems in a common natural model.

After studying this model, it will be interesting to extend it further by introducing a correlation between voters. For that purpose, we the the Von Mises-Fisher or VMF model (Downs, 1966). This kind of culture makes it possible to model populations tending to have similar preferences, but with a certain dispersion. First, a unit vector $\mathbf{n}$ is fixed and called the *pole* of the distribution. Then, independently for each voter, a unit vector $\mathbf{u}$ is drawn in $\mathbb{R}^C$ according to a VMF

[1] More precisely, in the model with approval limit, the suitable representations are spheroids that are the image of the unit sphere by a stretching along direction $\mathbf{1}$, which constitue a family with one real parameter. The choice of this parameter has an impact only on approval voting: with exterme values of this stretching factor, we favor either utility vectors with a lot of elements having the same sign, or utility vectors whose sum of elements is zero and have therefore always elements of both signs. To simplify, we always consider the spherical model, which is intermediary between these two extreme cases.

distribution:

$$p(\mathbf{u}) = X_\kappa e^{\kappa\langle\,\mathbf{u}\,|\,\mathbf{n}\,\rangle},$$

where $\mathbf{n}$ is the pole of the distribution, κ its concentration, X_κ a normalization constant and where $\langle\,\mathbf{u}\,|\,\mathbf{n}\,\rangle$ denotes the canonical inner product.

Qualitatively, the VMF model is similar to Mallows' model (Mallows, 1957), which is used for ordinal preferences. Not only the first one is better adapted to cardinal preferences than the second one, but it has other advantages that are discussed in section B.5.

We can check that it extends the spherical culture: indeed, for $\kappa = 0$, the VMF model is equivalent to the spherical culture. For $\kappa = +\infty$, we obtain the other degenerate case of VMF culture: a Dirac peak on the pole $\mathbf{n}$.

We gather the VMF model and the spherical culture, which is one of its particular cases, under the name of *spheroidal* cultures, because of the underlying spherical model for the utility space.

7.2 Reference scenario

In this section, we present our methodology and our reference scenario. It is the spherical culture with $V = 33$ voters and $C = 5$ candidates. For this reference case, we choose an odd number of voters to avoid some questions of ties, for example the subtleties between the different Condorcet notions: in particular, the notions of Condorcet winner and Condorcet-admissible candidate are equivalent. In order to choose the values of V and C, we have explored the values $V = 2^k + 1$ for $k \in [\![1, 10]\!]$ (so, for V between 3 and 1025) and the values $C \in [\![3, 15]\!]$. In sections 7.3 and 7.4, we will see that, for most voting systems, in spherical culture, the manipulability rate seems to increase and tend to 1 when either one of these parameters increases. Our choice of $V = 33$ and $C = 5$ gives manipulability rates that are reasonably far from 1 and, as a consequence, facilitates the comparison between the voting systems under study.

Figure 7.1 illustrates the performances of several voting systems in this reference scenario. We can see CM and its variants TM, UM and ICM. For this figure, we have drawn 10 000 random populations with the culture under study.

The proportion of CM (resp. TM, UM, ICM) configurations for a voting system in these 10 000 random experiments gives an estimator of the CM (resp. TM, UM, ICM) rate for the culture. This induces an margin of uncertainty, which we will say *statistical*, of order $\frac{1}{\sqrt{10\,000}} = 1\%$ on the CM (resp. TM, UM, ICM) rate. By convention, this part of the uncertainty is not represented, neither in figure 7.1 nor in the following ones.

By the way, for some of the voting systems, the algorithm used to determine CM is not exact. Histograms in figure 7.1 represent a lower bound of the proportion of manipulable configurations, which is given by the cases where S∀AMP was able to prove CM. The uncertainty bar (for example, for Baldwin's method) indicates an upper bound of this proportion, which is given by the cases where S∀AMP was able to prove that CM is impossible. We will call the difference between these bounds the *algorithmic uncertainty*.

Let us seize the occasion to examine the performances of the algorithms that are implemented in S∀AMP for CM.

For Maximin, Schulze and Borda, we use an approximate algorithm but most of time, it seems that S∀AMP can decide wether these systems are manipulable or not: in figure 7.1, the algorithmic uncertainty is 1,8 % for Schulze's method and equal to zero for Maximin and Borda.

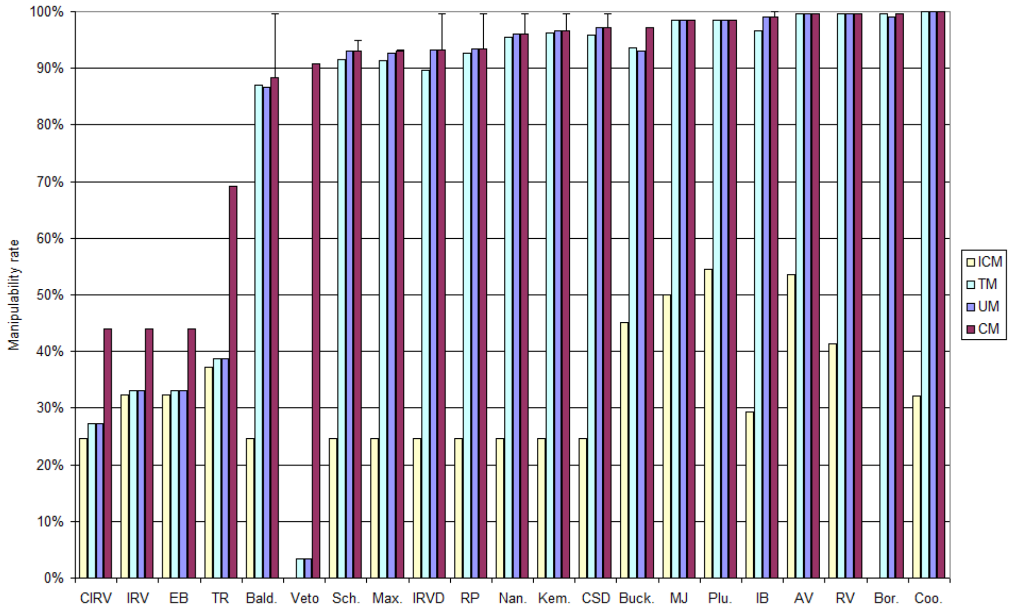

Figure 7.1: Coalitional manipulation (CM) and variants. Spherical culture, $V = 33$, $C = 5$.

For Baldwin, VTID, PO, Nanson, CSD, Kemeny and IB[2], SVAMP uses only its generic manipulation methods, essentially based on preliminary tests and TM. In that case, we have essentially a lower bound of manipulability, which will be sufficient in many cases to prove that these systems are more manipulable that some others. In this reference scenario of figure 7.1, for example, we know that these systems are more manipulable than CIRV and IRV.

For EB, we will always use the exact algorithm. For IRV, we use the exact algorithm also, except in section 7.3, where we use the slow option (section 6.5.2) when the number of candidates is greater or equal to 9 in order to limit the computation time. For CIRV, we use the heuristic presented in section 6.5.3: despite its not being an exact algorithm, it appears in the specific case of figure 7.1 that the algorithmic uncertainty is equal to zero. It means also that in practice, CIRV has a manipulability rate that is very close to IRV, even if we know that in general, it is strictly lower by the strong Condorcification theorem 2.20.

In figure 7.1, however, we can remark a non-negligible difference: CIRV is significantly less vulnerable to ICM, TM and UM than IRV, even if is CM rate is very close. So, manipulation looks more difficult in CIRV than in IRV.

By contrast, IRV and EB present very similar performances, even if the example of table 1.1 (section 1.4.2) proves that EB is strictly more manipulable than IRV (since they are respectively equivalent to TR and ITR in this example with 3 candidates).

[2]In this chapter and the following ones, we do not include Kim-Roush and IRVA, which were implemented in SVAMP after realizing these simulations and use also generic algorithms for the manipulation problem. For reasons of computation time and in order to avoid last-minutes mistakes in this memoir, we decided to stick to what we had, and we apologize to the reader for that. That being said, based on the first tests we did with these systems, the performances of Kim-Roush method for the manipulability seem to be quite bad. As for IRVA, by contrast, results are quite good and we will discuss this point when concluding this memoir.

Veto is the only voting system for which the TM rate (3%) is very different from the CM rate (91 %). It is easy to explain since the trivial strategy is very bad in Veto: it is certainly optimal to avoid a victory for the sincere winner w, but manipulators forfeit any kind of control on the winning candidate. On the contrary, the optimal strategy is to coordinate in order to balance the total number of negative votes on all candidates, except the one we wish to get elected.

Lastly, we note that IRVD is significantly more manipulable than CIRV, IRV and EB. However, at first glance, this voting system is based on a similar mechanism. We can try to understand why with an example where there are 3 candidates (possibly, after some eliminations took place). If the sincere winner w is not a Condorcet winner, then the profile is manipulable anyway, so the interesting case is when w is a Condorcet winner. Let us note c the candidate for whom we want to manipulable and d the third candidate.

In IRVD, c must never be confronted to w in duel, otherwise she loses. So, here is the only possible scenario: w has a duel against d and loses, then c has a duel against d and wins. So, it is necessary that w and d are selected for the first electoral duel: for this purpose, the best strategy is to defend c by placing her on top of each manipulator's ballot. By the way, w must lose against d: for that purpose, the best strategy is to attack w by placing her in the bottom. So, the different imperatives, i.e. defending c and attacking the dangerous candidate w, are perfectly compatible.

Now, let us examine the situation in EB or IRV. In each elimination round, some of the manipulators must defend c in order to prevent her elimination. Moreover, as long as w is not eliminated, no sincere voter votes for c: manipulators can only rely on themselves for this purpose. By the way, manipulators must avoid a last round between c and w, for the same reasons as before. So, they must attack w to eliminate her immediately. But for this, they may need to share their votes between c and d, which is more difficult, *a priori*, than in VTID where defending c is sufficient.

This leads us to another remark: although IRV is the PSR-SE for Plurality, it present a similarity with Veto (and, conversely, Commbs' method has a similarity with Plurality). Indeed, the choice of the eliminated candidate in IRV is done in Veto: each voter, by voting for one candidate, emits a veto against her elimination and the candidate with least vetos against her elimination (i.e. with most votes) is designated (i.e. eliminated). Now let us recall that the maximal family in Veto is, up to questions of tie, the family of threshold $V\frac{C-1}{C}$ (proposition 4.28): so, it is especially difficult to choose the eliminated candidate in IRV, which may explain the low manipulability of this voting system, although it meets **InfMC** if we consider the system in whole and not each round separately.

7.3 Spherical culture: number of candidates C

In this section and the following one, we will consider variants for the reference scenario, by varying each time a specific parameter. To begin with, we keep the spherical culture of the reference scenario, i.e. the particular case where the concentration κ is equal to zero, we leave the number of voters $V = 33$ constant and we represent in figure 7.2 the CM rate as a function of the number of candidates C.

These first curves are the occasion to precise our graphic conventions for the rest of this chapter and the following one. When the name of a voting system is accompanied by a star in the legend, it means that, for some points of the curve,

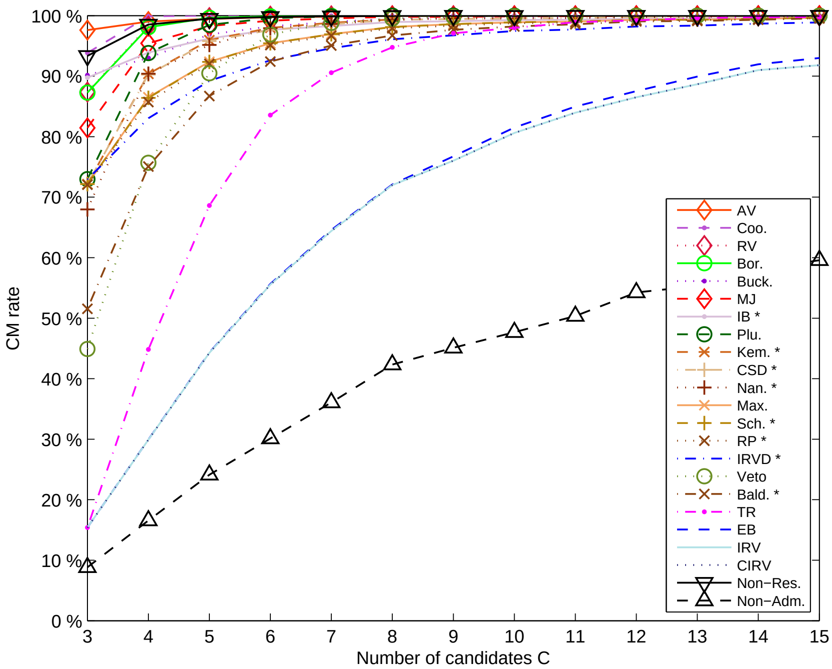

Figure 7.2: CM rate as a function of the number of candidates C. Spherical culture, $V = 33$.

the approximate algorithm used by S∀AMP found a difference between the lower and the upper bounds that is greater than 1 %. In that case, given the number of voting systems under study, we draw only the CM rate found by S∀AMP, i.e. the lower bound. For example, it is so for Baldwin's method in figure 7.2.

When there is no star in the legend, it does not necessarily mean that the algorithm is exact but that, for each point of the curve, S∀AMP found an algorithmic uncertainty that is lower than 1 %. So, for CIRV or Maximin, although approximate algorithms are used, the algorithmic uncertainty is lower than 1 % for each point in figure 7.2.

In order to facilitate the comparison of voting systems in all the figures, we assign to each voting system a unique graphic style. These styles are gathered in families: systems based on grades use tones of red with diamond markers; PSR use tones of green with circle markers; "natural" Condorcet systems use tones of brown with cross markers; systems of IRV family use tones of blue without marker; other systems use tones of rose and violet with markers the shape of big points.

In order to facilitate the correspondence with the curves, the legend presents the voting systems by decreasing order or the manipulability rate found (on average, on the points of the curve).

As a reference, we represent the proportion of non-resistant configurations (i.e. where there is no resistant Condorcet winner), in black with triangles oriented downward; according to theorem 2.17, this gives an upper bound for the manipulability rate for all voting systems meeting the Condorcet criterion. Similarly, we represent the proportion of non-admissible configurations (i.e. where there is no Condorcet-admissible candidate), in black with triangles oriented upward, which gives a lower bound of the manipulability rate for all voting systems meeting **InfMC** (lemma 2.7).

When the number of voters is even, we will represent also the proportion of non-Condorcet configurations (i.e. where there is no Condorcet winner), in black dotted line with the same triangle marker. However, when the number of candidates is odd, as in figure 7.2 and most of the others, it is useless to draw this curve: since we consider cultures where preferences are almost surely strict total orders, the notions of Condorcet-admissible candidate and Condorcet winner are equivalent, which is not the case when the number of candidates is even.

The first observation drawn from figure 7.2 is the following conjecture.

Conjecture 7.1
In spherical culture, for $V \geq 3$, the manipulability rate of each voting system under study here is an increasing function of the number of candidates C.

In order to understand this fact qualitatively, let us note also that the probability to have a non-admissible configuration, i.e. a non-Condorcet configuration since V is odd, seems to increase with the number of candidates.

Kelly (1974) conjectured that, for $V = 3$ or $V \geq 5$ voters, the probability that there exists a weak Condorcet winner (i.e. a Condorcet-admissible candidate) is a decreasing function of C; this has been proved by Fishburn et al. (1979) for $V = 3$ voters. Similarly, Kelly (1974) conjectured that, for $V \geq 3$, the probability that there exists a Condorcet winner decreases; this has been proved for $V \to +\infty$ (Gehrlein, 2006). To the best of our knowledge, others cases remain conjectures. In the same kind of idea, figure 7.2 leads us to the following conjecture and proposition.

Conjecture 7.2
In spherical culture, for $V \geq 3$, the probability that there exists a resistant Condorcet winner is a decreasing function of the number of candidates C.

Proposition 7.3
In spherical culture, for $V \geq 3$ and $C \to +\infty$, the probability that there exists a resistant Condorcet winner tends to 0.

It is easy to prove this proposition with a previous result: indeed, in impartial culture et with a constant number of voters, the probability that there exists a Condorcet winner tends to 0 when C tends to $+\infty$ (May, 1971). So, it is also true for the resistant Condorcet winner.

In figure 7.2, we also observe the following phenomenon, which we will be able to demonstrate partially.

Conjecture 7.4
In spherical culture, for $V \geq 3$ and $C \to +\infty$, the manipulability rate of any voting system meeting **InfMC** *tends to* 1.

The assumption **InfMC** concerns all voting systems studied here, except Veto, which we will discuss soon. It is easy to prove conjecture 7.4 for an odd number of voters. Indeed, in impartial culture and with a constant number of voters, we have also recalled that the probability that there exists a Condorcet winner tends to 0 when C tends to $+\infty$ (May, 1971). But, for V off, the notions of Condorcet winner and Condorcet-admissible candidate are equivalent. So, the sincere winner w is not Condorcet-admissible with high probability. For a voting system meeting **InfMC**, lemma 2.7 ensures that the configuration is manipulable.

For an even number of voters, to the best of our knowledge, it is not proven that the probability to have a Condorcet-admissible candidate (i.e. a weak Condorcet winner, given the assumptions) tends to 0 when C tends to $+\infty$. If it is true, then our proof of conjecture 7.4 is also valid for an even number of voters.

By contract, although the CM rate of Veto seems also to tend to 1 in figure 7.2, we can prove that this limit behavior is not true for Veto, when it uses the tie-breaking rule implemented in S∀AMP.

Proposition 7.5
In impartial culture, for V constant and $C \to +\infty$, the manipulability rate of Veto (with the lexicographical tie-breaking rule on candidates) does not tend to 1.

Proof. For $C > V$, let us consider the restriction of the impartial culture for C to the $V + 1$ candidates with lowest indexes: it is also an impartial culture. As a consequence, with a probability equal to $\frac{1}{((V+1)!)^V}$, all the voters have the following order of preference over the $V+1$ first candidates: $(1 \succ 2 \succ \ldots \succ V+1)$. Their relations of preferences over the other candidates has no impact on our demonstration.

With such preferences, no voter votes against candidate 1, so she is declared the winner. There is no possible manipulation for candidates $2, \ldots, V+1$ because no voter is interested. Lastly, because of the tie-breaking rule, no candidate among $V+2, \ldots, C$ can be the winner: even if each of the first V candidates receives a veto, candidate $V+1$ is elected. So, manipulation is impossible.

As a consequence, with a probability at least $\frac{1}{((V+1)!)^V}$, the profile is not manipulable. □

In figure 7.2, we also note that CIRV, IRV and EB are significantly less manipulable than the others and that the two-round system presents performances that are intermediary between these three IRV-like systems and the other voting systems, except for large values of C. Indeed, the theoretical results above prove that TR is (slightly) more manipulable than Veto for a large enough number of candidates.

We also observe that the curves for CIRV, IRV and EB are almost the same. The difference of manipulability between CIRV and IRV is not detectable in this figure: the difference in the proportion of observed manipulable configurations is always equal to zero. As for the difference between IRV and EB, S∀AMP find slightly less manipulations for IRV for $C = 9$ and higher: this corresponds to the value where we started to use the approximate algorithm for IRV, whereas we kept on using the exact algorithm for EB. The algorithmic uncertainty for IRV, of order 1 %, corresponds precisely to the difference between the curves of IRV and EB. This difference should not be over-interpreted: it is a combination, in unknown proportions, of a (slight) difference of manipulability between IRV and EB and a (slight) loss of performance for the approximate algorithm for IRV when the number of candidates increases.

So, even if we proved theoretically that CIRV is strictly less manipulable than IRV (by corollary 2.21 of the Condorcification theorems) and that IRV is itself strictly less manipulable that EB (section 1.4.2), the differences are, in fact, of very small amplitude, especially when we compare them with the other voting systems under study here.

In order to explain all these phenomena, we can propose the explanation that they are essentially linked to the degradation of the "quality" of the population, to its growing disunity. In particular, for the voting systems meeting **InfMC**

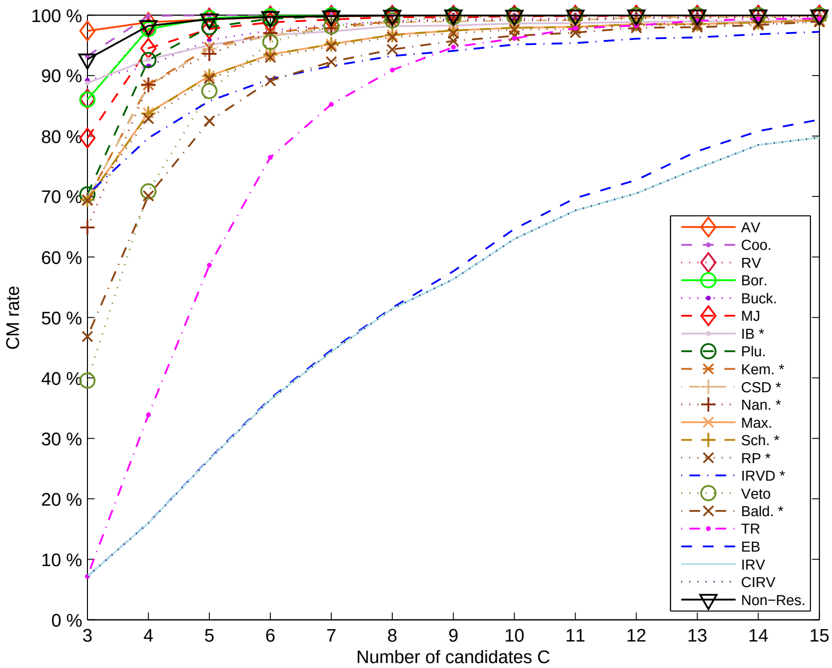

Figure 7.3: Normalized CM rate (relative to admissible configurations) as a function of the number of candidates C. Spherical culture, $V = 33$.

(all those under study here, except Veto), we know, by lemma 2.7, that the non-admissible configurations are doomed to be manipulable anyway. So, we can wonder if the increase of the rate of non-admissible configurations is the only factor explaining the increase of the manipulability rates. In order to test this idea, we have indicated in figure 7.3 the *normalized CM rate*, i.e. relative to the proportion of non-admissible configurations. For the voting systems meeting **InfMC**, it is their manipulability rate for the restriction of the impartial culture to the admissible configurations. The interesting observation is that, even so, the phenomena seen above remain true, which leads us to the following conjecture.

Conjecture 7.6

We consider the spherical culture with $V \geq 3$.

The normalized manipulability rate of each voting system under study here is an increasing function of the number of candidates C.

For $C \to +\infty$, the normalized manipulability rate of each voting system under study here, except Veto, tends to 1.

This conjecture does not invalid our intuitive explanation concerning the increase of the manipulability rates. The fast that Condorcet-admissible candidates are less frequent is, in our opinion, only a symptom of a more general phenomenon of increasing disunity, which give an intuitive picture explaining that some voting systems that does not meet **InfMC**, such as Veto, are also concerned.

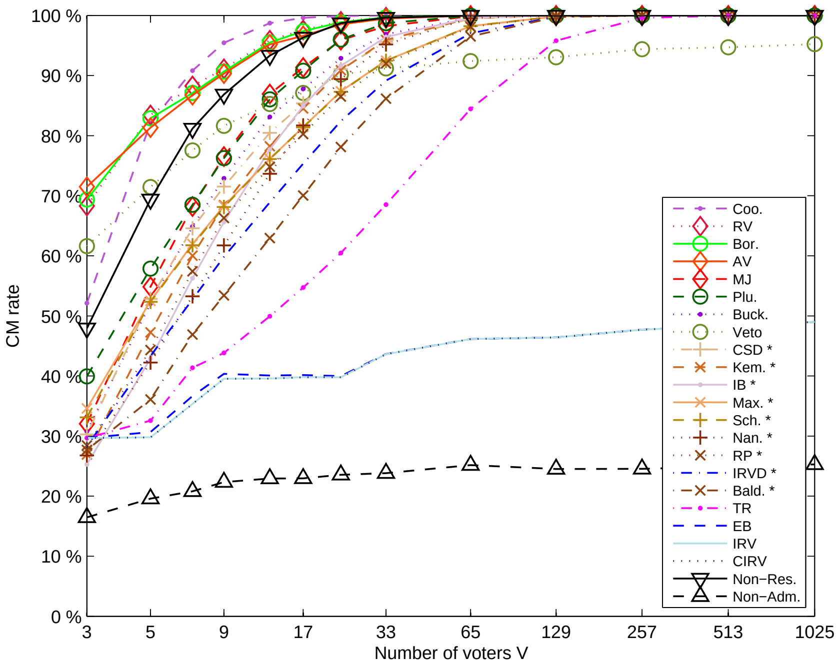

Figure 7.4: CM rate as a function of the number of voters V. Spherical culture, $C = 5$, large odd values of V.

7.4 Spherical culture: number of voters V

7.4.1 Odd number of voters V

In figure 7.4, we draw the CM rates for odd numbers of voters V. The parity of the number of voters can create particular phenomena, which we will examine in section 7.4.2.

For most voting systems, the CM rate seems to be an increasing function of the number of voters V (on the set of odd values). In the particular case of EB, the seeming non-monotony between $V = 9$ and $V = 13$ is less than the statistical uncertainty of 1 % so, even in that case, it is not excluded that the CM rate is an increasing function of the number of voters.

For CIRV, IRV and SE, we observe in figure 7.4 that the CM rate are, once again, significantly lower that for the other systems and that the differences between these three systems are very small. This time, we used the exact algorithm for IRV. Since we know that IRV is at most as manipulable as EB in the set-theoretic sense (section 1.4.2), the small difference between IRV and EB (at its maximum, slightly more than 1 %) is only due to an actueal difference of manipulability, i.e. to profile where EB is manipulable but where IRV is not. This difference seems to tend to 0 when the number of voters tends to $+\infty$. As for the difference between CIRV and IRV, it is, once again, extremely small[3].

[3] Our approximate algorithm for CIRV was not able to decide manipulability for 2 cases out of 10 000 for $V = 5$ and 1 case out of 10 000 for $V = 13$ and $V = 17$. In all other cases, the manipulability, true or false, could be decided and it is equal to the one of IRV.

Apart from these systems, the two-round system behaves better thanb the other systems, except for about a hundred voters or more, where Veto becomes less manipulable.

For all voting systems except CIRV, IRV, EB and Veto, the CM rate seems to tend to 1 when the number of voters V tends to $+\infty$. But, unlike the case $C \to +\infty$, the probability of existence for a Condorcet winner does not tend to 0 when V tends to $+\infty$ (Gehrlein, 2006), hence we cannot prove this observation by the same means. However, there exists theoretical results on this matter. On one hand, Kim and Roush (1996) show the following results.

- For Veto, if $C = 3$ and $V \to +\infty$, the CM rate tends to $\frac{1}{2}$.
- For Veto, if $C > 3$ and $V \to +\infty$, the CM rate tends to a limit that is strictly between 0 and 1.
- For all PSR except Veto, if $C \geq 3$ and $V \to +\infty$, the CM rate tends to 1.
- For Maximin, if $C \geq 3$ and $V \to +\infty$, the CM rate tends to 1.
- For Coombs' method, if $C = 3$ and $V \to +\infty$, the CM rate tends to 1.

On the other hand, Lepelley and Valognes (1999) show that for EB, if $C = 3$ and $V \to +\infty$, the CM rate tends to 0.16887 (approximate value): in particular, it is a value strictly between 0 and 1.

With these theoretical results in mind and observing figure 7.4, we propose the following conjectures.

Conjecture 7.7
In impartial culture, for $C \geq 3$ and $V \to +\infty$, the CM rate tends to 1 for Baldwin, Bucklin, Coombs, CSD, IB, IRVD, Kemeny, Nanson, RP, Schulze and the two-round system.

In spherical culture, this is also true for range voting, approval voting and majority judgment.

Let us recall that this result is proven for Maximin and all PSR except Veto..

Conjecture 7.8
In impartial culture, for $C \geq 3$ and $V \to +\infty$, the CM rate tends to a limit that is strictly between 0 and 1 for CVTI, VTI and SE.

Let us recall that this result is proven for Veto.

In spherical culture, it seems that CIRV, IRV, EB and Veto are the only voting systems, among those under study here, whose manipulability rate does not tend to 1 when the number of voters V tends to $+\infty$.

7.4.2 Parity of V

If we use odd and even values of V, we can observe non-monotone phenomena for the CM rate of some of the voting systems. In figure 7.5, we consider all values of the number of voters from 3 to 33 and we represent the CM rate of the affected voting systems; other ones are omitted in order to lighten the figure.

First of all, let us examine the majority judgment or Bucklin's method, whose CM rate has oscillations without algorithmic uncertainty between even and odd values of V.

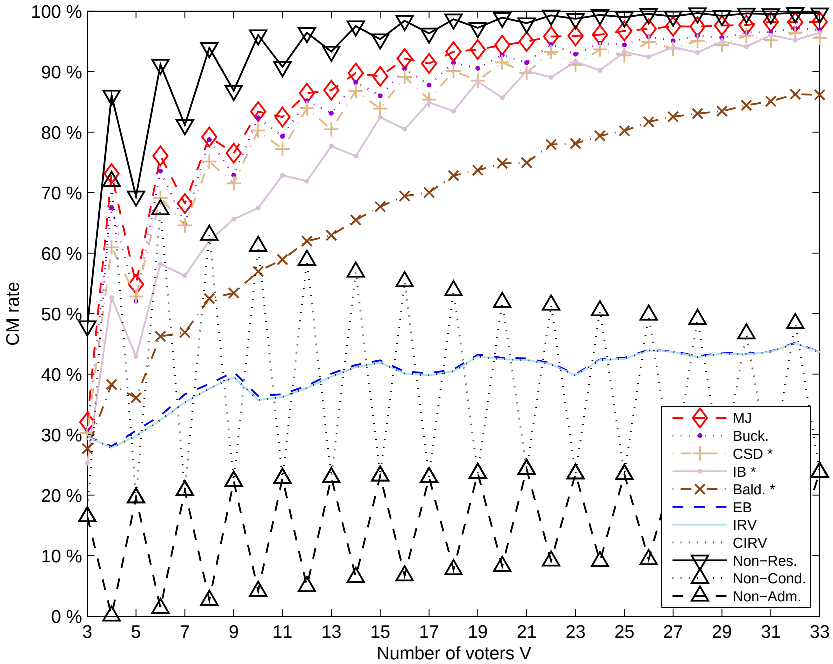

Figure 7.5: CM rate as a function of the number of voters V for voting systems with oscillatory behaviors. Spherical culture, $C = 5$, small values of V.

This oscillatory behavior could be linked to fact that the probability of existence for a Condorcet winner or a Condorcet-admissible candidate oscillate also between even and odd values of V. However, as proven by Gehrlein (2006) with theoretical arguments, these oscillations are in antiphase. When V is odd, the two notions are equivalents. When V go from an odd value to an adjacent even value, the probability to have at least a Condorcet-admissible candidate increases, whereas the probability to have a Condorcet winner decreases. In the curves we obtained by computer simulation, we observe also oscillations for the probability of existence of a resistant Condorcet winner, in phase with the Condorcet winner[4]. But, as we saw, the probability of existence of a Condorcet-admissible candidate, of a Condorcet winner (which was equivalent in previous curves) or a resistant Condorcet winner are indicators with an impact on manipulability.

However, if this explanation is correct, it is surprising that other voting systems meeting **InfMC** do not present the same kind of oscillatory behavior. We can propose the idea that there are two concurrent effects: the larger probability to have a Condorcet-admissible candidate is a better protection against manipulation for V even, but the probability to have a Condorcet winner or even a resistant one is a better protection for V odd. So, the two effects may cancel each other partially, which may explain that a few voting systems present oscillations. We will se that this phenomenon is far more frequent in a Gaussian well culture and we will explain why (section 8.2.2).

[4]It is also known (Gehrlein, 2006) that for a given parity, the probability of existence of a Condorcet winner is monotone. At first glance, it does not seem true in figure 7.5 between $V = 28$, $V = 30$ and $V = 32$. However, the observed difference between $V = 30$ and $V = 32$ is less than 2 % and it is purely statistical. Let us recall that this uncertainty, of order 1 % for one value and $\sqrt{2} \times 1$ % for the difference between two values, is only an order of magnitude, which may sometimes be overcome.

$\frac{V}{2}$ (sincères)	$\frac{V}{2}$ (manipulateurs)
Divers : 1	c : 1
w : 0,9	$\vdots$
Divers : 0,5	Divers : 0,5
c : 0,2	$\vdots$
Divers : 0	w : 0

Table 7.1: Explanatory example for the majority judgment and Bucklin's method.

This explanation leaves an additional question. For the voting systems meeting **InfMC**, the usual lower bound on the manipulability rate is given by the proportion of non-admissible configurations; so, we might think that the oscillations of the curves will be in phase. But we observe the opposite: oscillations of majority judgment and Bucklin's method are in phase with the proportion of Condorcet configurations.

For these reasons, we propose another explanation, which does not exclude the previous one. The majority judgment and Bucklin's method rely on the notion of median, which has a slightly different definition depending on the parity of the number of voters. More precisely, let us examine what may happen with an even number of voter, if there are exactly $\frac{V}{2}$ voters who prefer a candidate c to the sincere winner w. Generally, the typical sincere voter gives w neither the maximal rank nor the maximal grade; and she gives c neither the minimal rank nor the minimal grade. After the manipulation, we obtain the kind of profile represented in table 7.4.2.

In majority judgment or in Bucklin's method, by convention, the unfavorable median is used, i.e. the lower median grade in majority judgment and the upper median rank in Bucklin's method. So the median grade considered by the majority judgment (resp. the median rank considered by Bucklin's method) for w is -0 (resp. C), whereas the median grade (resp. the median rank) for c is 0.2 (resp. $C - 1$ for example). Then, a coalition having half of the voters may generally manipulate. In contract, if we consider a profile similar to the above simplified example but with an odd number of voters, a strict majority of voters is necessary to make c win. This may explain the fact that the manipulability rate is higher for an even number of voters.

For CSD, the oscillatory phenomenon remains to be confirmed, because the curve is only a lower bound of the MC rate (in practice, it is the TM rate). However, we can note that the rule of CSD has a behavior depending on the parity of V: indeed, the penalty for a defeat of c against d is $1 + D_{dc} - \frac{V}{2}$ when V is even, but only $\frac{1}{2} + D_{dc} - \frac{V}{2}$ when V is odd.

For CIRV, IRV and EB, we observe only effects of non-monotony, whose amplitude is more than the margin of uncertainty. However, they do not seem to have an obvious pseudo-period. Given the way these systems are counted, we propose the explanation that several effects combine with one another, depending on the modulo of V by all integers from 2 to C.

For IB, as for CSD, the surve is only a lower bound, but it can be commented upon as a curve of the TM rate. This time, there seems to be a transition about $V = 8$ or $V = 9$: oscillations remain but the phase of the curve gets reversed! For smaller values of V, the manipulability is greater for V even (than for adjacent odd values); for larger values, it is greater for V odd. Since it is a multiround voting systems, it is possible to imagine effects of modulo as for CIRV, IRV and EB and parity effects as for Bukclin's method. Anyway, the complex rule of this voting systems makes the perspective of a simple explanation quite optimistic.

Lastly, we remark that the non-monotony phenomena have a decreasing amplitude when the number of voters V increases. This is easy to explain: when the number of voters is large, phenomena of ties, implying exactly half of the voters, are quite rare. As a joke, we can sum up this observation by the saying: "infinity is odd". Or: « when infinity is even, she does not notice ».

7.5 Monopolar culture: concentration κ

Now, we consider non-degenerated VMF cultures, i.e. with a concentration κ that is not equal to zero. Let us recall that we first choose a unit vector $\mathbf{n}$, called the *pole* of the distribution. Then, independently for each voter, we draw a unit vector $\mathbf{u}$ in $\mathbb{R}^C$ according to a VMF distribution:

$$p(\mathbf{u}) = X_\kappa e^{\kappa \langle \mathbf{u} \mid \mathbf{n} \rangle},$$

where X_κ is a normalization constant.

Since we have already studied the impact of the number of candidates or the number of voters, we will study the effect of concentration and position of the pole. Then, we will extend the model to several poles, in order to represent several social groups, each one with a typical opinion.

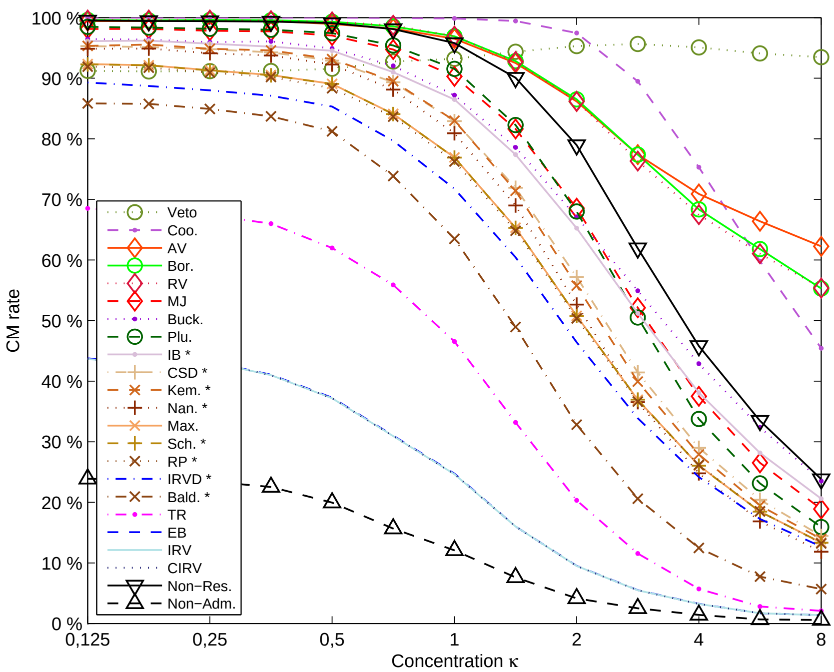

Figure 7.6: CM rate as a function of concentration κ. VMF culture, $V = 33$, $C = 5$, one pole of random position.

In figure 7.6, for each population, the pole **n** is drawn at random, then voters are drawn with a given concentration κ. Let us note that voters are not independent. More precisely, once the pole is chosen, they are independent; but, on the set of all possible populations (with all possible positions of the pole), voters are not independent. Indeed, *a priori*, a voter has a uniform probability on the unit sphere of $\mathbb{R}^C$; but if we know, for example, that the first $V-1$ voters are close to a given point of the utility space, then the last voters has a higher probability to be close to this point too.

As we could expect, the higher the concentration κ, the lower the CM rates for most voting systems. In the degenerated case $\kappa = +\infty$, all voters always have the same utility vector, hence any unanimous voting system (section 1.2.2) is non-manipulable: it is the case for all systems under study here, except Veto (see below).

However, some voting systems seem to be significantly less reactive than others to an increase of κ: Veto, approval voting, Borda's method, range voting and, to a lesser extent, Coombs' method. It means that, even when voters have relatively similar opinions, these voting systems keep a high manipulability.

For Veto, the CM is not even monotone. And we will show that it does not tend to 0, unlike what happens for unanimous voting systems.

Proposition 7.9

We consider Veto, with a tie-breaking rule that uses no other information about preferences but the candidate against which each voter emits her ballot.

We assume $V \geq C-1$.

In VMF culture with a pole drawn uniformly at random, when $\kappa \to +\infty$*, the manipulability rate tends to* $1 - \dfrac{1}{C-1}$.

The assumption on the tie-breaking rule is met, in particular, when ties are broken by lexicographical order on candidates, as in SWAMP. By the way, in figure 7.6, since $C = 5$, the limit CM rate is $\dfrac{3}{4}$; but it is far from reached: convergence seems to be quite slow.

Proof. In the limit case where all voters have the same order of preference, they all vote against the same candidate and only the tie-breaking rule designates the winner among all the other candidates. Since the pole is drawn in a neutral way, there is one chance out of $C-1$ that the elected candidate is the one preferred by the voters. In all other cases, all the voters may form a coalition and manipulate in order to make their favorite candidate win: indeed, since $V \geq C-1$, it is possible that at least one voter votes against each other candidate. $\square$

Once again, we remark the small and extremely similar CM rates for CIRV, IRV and EB, followed by intermediary performances for TR, which is less manipulable than the other voting systems.

7.6 Monopolar culture: position of the pole

Now, we will work with a constant concentration κ and vary the position of the voters. We keep the same order of preference over candidates $1 \succ 2 \succ \ldots \succ 5$ but we vary the shape of the opinion for the typical voter (i.e. located exactly on the pole of the distribution). First, we consider the limit case where the typical voter prefers candidate 1 and where she is indifferent between the other candidates; and we perform a transition to a relatively balanced state where her utilities are Borda scores (figure 7.7). Then, we move the typical voter from a Borda-like state to another limit case where she hates candidate 5 and where she is indifferent between the other candidates (figure 7.8).

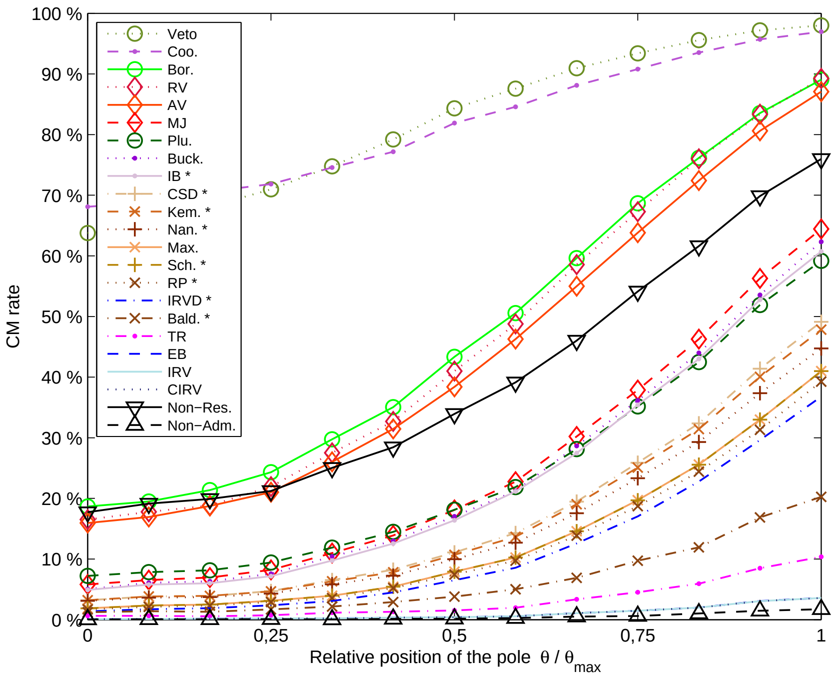

Figure 7.7: CM rate as a function of the position of the pole $\mathbf{n}$. VMF culture, $V = 33$, $C = 5$, one pole of concentration $\kappa = 2$. For $\theta = 0$, we have $\mathbf{n} = \mathbf{n}_0 = \frac{1}{\sqrt{20}}(4, -1, -1, -1, -1)$. For $\theta = \theta_{\max}$, we have $\mathbf{n} = \mathbf{n}_1 = \frac{1}{\sqrt{10}}(2, 1, 0, -1, -2)$.

So, in figure 7.7, concentration $\kappa = 2$ is constant, but the position of the pole $\mathbf{n}$ is imposed and not drawn at random. Let us consider two unit utility vectors $\mathbf{n}_0 = \frac{1}{\sqrt{20}}(4, -1, -1, -1, -1)$ and $\mathbf{n}_1 = \frac{1}{\sqrt{10}}(2, 1, 0, -1, -2)$. Denoting $\theta_{\max}$ the angle between $\mathbf{n}_0$ and $\mathbf{n}_1$, we have $\theta_{\max} = 45°$. We explore the geodesic of the unit sphere going from $\mathbf{n}_0$ to $\mathbf{n}_1$, taking as parameter the polar angle θ since $\mathbf{n}_0$. When $\theta = 0$, we have $\mathbf{n} = \mathbf{n}_0$: the typical voters prefers candidate 1 and she is indifferent between the other candidates. When $\theta = \theta_{\max}$, we have $\mathbf{n} = \mathbf{n}_1$: the typical voter has a clear order of preference $1 \succ 2 \succ 3 \succ 4 \succ 5$, with utilities that are Borda scores, up to normalization.

We observe that all voting systems become more manipulablt when the pole of the distribution gets closer to $\mathbf{n}_1$ (with very different amplitudes for this phenomenon, depending on the voting system). We propose the following explanation. With a given value of κ, when the pole is $\mathbf{n}_0$, the population is strongly polarized in favor of candidate 1, hence it is unlikely that a voter prefers another candidate to candidate 1. But, when the pole is $\mathbf{n}_1$, it is closer to points of the hypersphere where voters prefer candidate 2 to candidate 1. As a consequence, coalitions for candidate 2 have more members and get better chances to succeed in manipulation.

Like before, we note the good performances of CIRV, IRV and EB, followed by the two-round system.

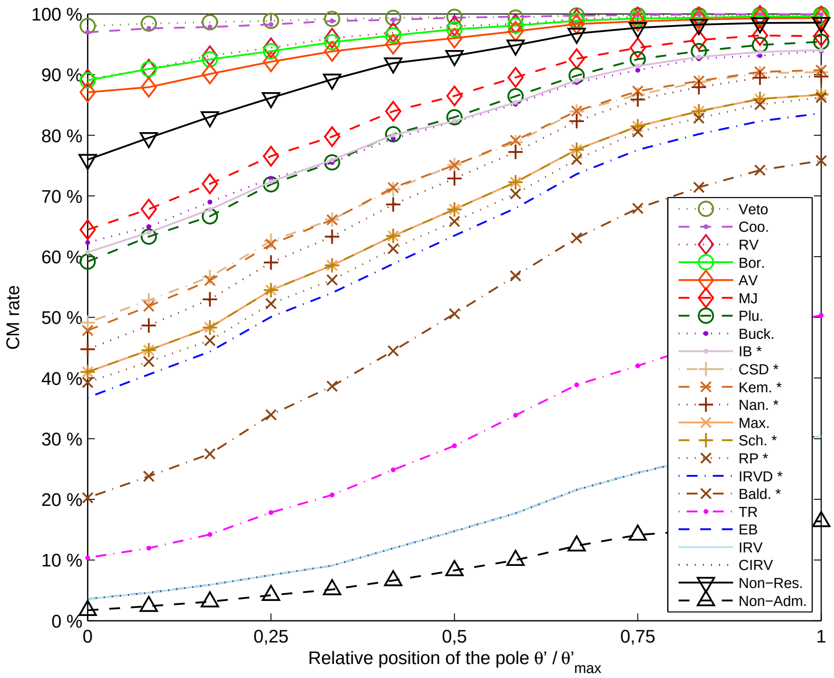

Figure 7.8: CM rate as a function of the position of the pole $\mathbf{n}$ (continued). VMF culture, $V = 33$, $C = 5$, one pole of concentration $\kappa = 2$. For $\theta' = 0$, we have $\mathbf{n} = \mathbf{n}_1 = \frac{1}{\sqrt{10}}(2, 1, 0, -1, -2)$. For $\theta' = \theta'_{\text{max}}$, we have $\mathbf{n} = \mathbf{n}_2 = \frac{1}{\sqrt{20}}(1, 1, 1, 1, -4)$.

In figure 7.8, we continue our investigation by moving toward the pole $\mathbf{n}_2 = \frac{1}{\sqrt{20}}(1, 1, 1, 1, -4)$. Denoting θ'_{max} the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$, we have again $\theta'_{max} = 45°$. We take θ' as parameter, the polar angle since $\mathbf{n}_1$. For $\theta' = 0$, we have $\mathbf{n} = \mathbf{n}_1$: the typical voter has a well-established order of preference $1 \succ 2 \succ 3 \succ 4 \succ 5$. For $\theta' = \theta'_{max}$, we have $\mathbf{n} = \mathbf{n}_2$: the typical voter dislikes candidate 5 and she is indifferent between the other candidates.

One may notice that the path from $\mathbf{n}_0$ to $\mathbf{n}_1$ then from $\mathbf{n}_1$ to $\mathbf{n}_2$ is not a geodesic of the sphere[5]: indeed, as shown in appendix B, geodesics in the utility space are unanimity zones. But in $\mathbf{n}_0$ and $\mathbf{n}_2$, the typical voter is indifferent between candidates 2, 3 and 4, which is not the case in $\mathbf{n}_1$. So, the path followed by the pole $\mathbf{n}$ on the sphere in figure 7.7 then in figure 7.8 is a broken line from $\mathbf{n}_0$ to $\mathbf{n}_1$ then from $\mathbf{n}_1$ to $\mathbf{n}_2$.

In figure 7.8, we observe that the CM rates and the rates of non-admissible configurations and non-resistant configurations keep on increasing. The culture becomes similar to a uniform culture over the four reasonable candidates (all of them except candidate 5), except the fact that there is an additional candidate, less liked by all voterws. This candidate has no chance of winning in any reasonable voting system (even Veto) but she may disturb voting systems that are the most sensitive to adding an irrelevant candidate, for example Borda's method: for a manipulation, this candidate makes it possible to add an additional point in the difference between the desired candidate c and the sincere winner w, so she

[5] For information, if we continue the arc from $\mathbf{n}_0$ to $\mathbf{n}_1$ until $\theta = 90°$, we obtain $\mathbf{n} = \frac{1}{\sqrt{20}}(0, 3, 1, -1, -3)$, which does not represent the same order of preference.

increases the possibility of manipulation, compared to a uniform culture for only 4 candidates.

7.7 Monopolar culture: number of poles

Until now, we have considered culture with only one pole $\mathbf{n}$. However, in practice, a population of agents generally has preference that are not centered around one typical opinion. Especially, we can imagine that there are several social groups, each of them having some kind of cohesion. We use the words *social group* with a very comprehensive sense: for example, in a political election, it may designate a socio-professional group or a community of interest; in other contexts, such as a professional or associative organization, it may designate a group of individuals with a certain cohesion, whether it relies on ideological or personal reasons.

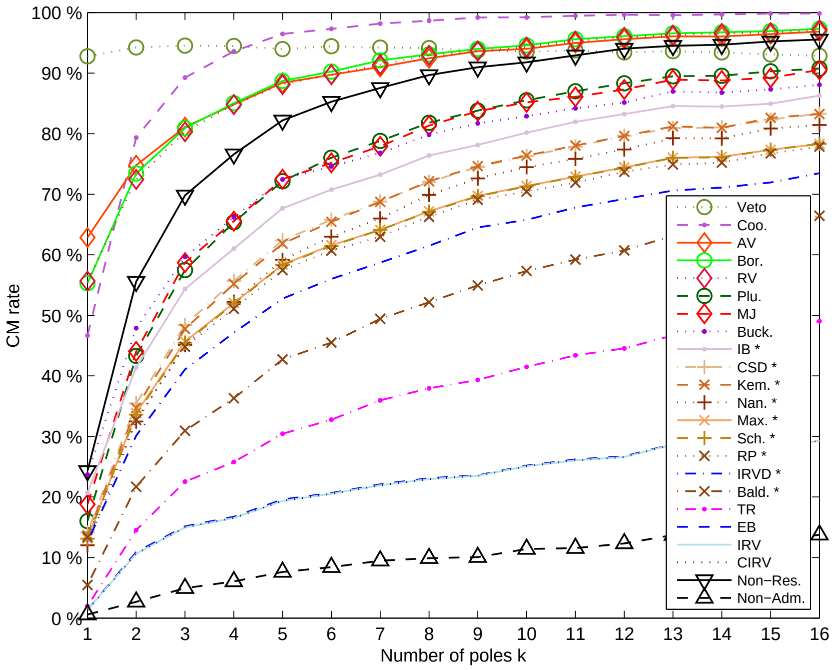

Figure 7.9: CM rate as a function of the number of poles. VMF culture, $V = 33$, $C = 5$, equiprobable poles with random positions, $\kappa = 2$.

In figure 7.9, we extend VMF model by considering a multipolar culture composed of several social groups. Formally, we take a parameter k representing the number of groups. Here is how we draw a population. First, we draw k unit vectors $\mathbf{n}_1, \ldots, \mathbf{n}_k$ independently and uniformly, which will be the poles of the distribution, i.e. the utility vector of the typical voter for each social group. Then, for each voter independently, we draw an integer i between 1 and k with equiprobability, which represents her social group, then we draw her utility vector according to a VMF distribution of pole $\mathbf{n}_i$ and concentration κ. In order to simplify, we consider only social group with comparable size (since i is drawn with equiprobability) and with the same constant concentration κ, even if SVAMP makes it possible to vary these parameters also.

As we could expect, we observe in figure 7.9 that the CM rate increase with the number of poles (except for Veto).

With V and C fixed, when the number of poles k tends to $+\infty$, the probability that there exists a couple of voters who are in the same social group tends to 0, so the distribution becomes uniform on the hypersphere: hence, the limit distribution is simply the spherical culture. The curves confirm this theoretical remark: for $k = 16$, CM rate are slightly smaller but already comparable to those observed in a spherical culture for $V = 33$ et $C = 5$ (cf. figure 7.2 for example).

7.8 Multipolar culture: relative positions of the poles

Lastly, we will investigate the relative positions of the social groups in the utility space. The most simple example to study this phenomenon consists in considering two social groups with comparable sizes (each voter is equiprobably in one or the other) and the same concentration κ.

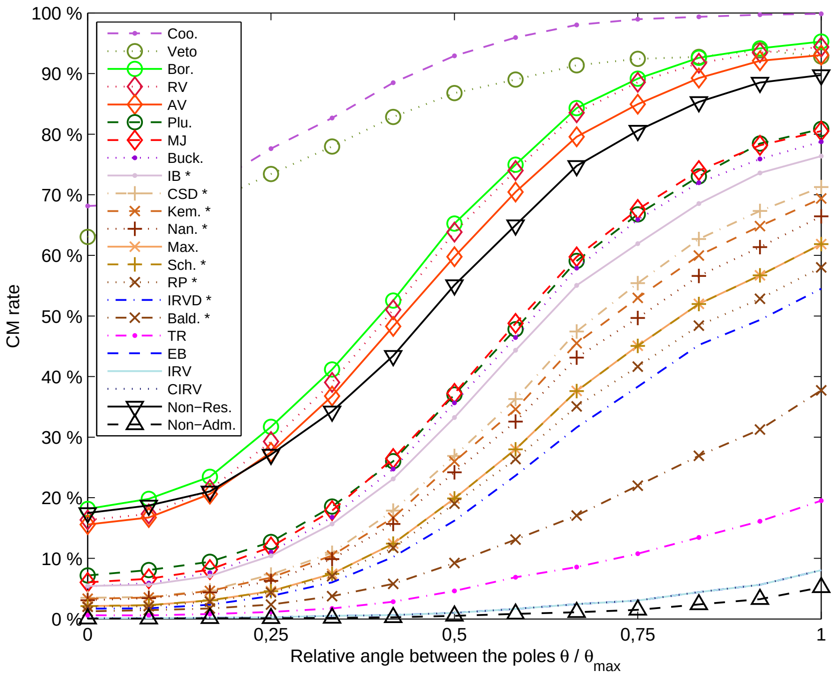

Figure 7.10: CM rate as a function of the angle between poles. VMF culture, $V = 33$, $C = 5$, two equiprobable poles, $\kappa = 2$. The first pole is $\mathbf{n}_0 = \frac{1}{\sqrt{20}}(4, -1, -1, -1, -1)$. For $\theta = 0$, the second pole is the same. For $\theta = \theta_{max}$, the second pole is $\mathbf{n}_1 = \frac{1}{\sqrt{20}}(-1, 4, -1, -1, -1)$.

Let us consider the two utility vectors $\mathbf{n}_0 = \frac{1}{\sqrt{20}}(4, -1, -1, -1, -1)$ and $\mathbf{n}_1 = \frac{1}{\sqrt{20}}(-1, 4, -1, -1, -1)$. Denoting $\theta_{\max}$ the angle between $\mathbf{n}_0$ and $\mathbf{n}_1$, we have $\cos(\theta_{\max}) = -\frac{1}{4}$ hence $\theta_{max} \simeq 104°$.

The first pole under consideration is always $\mathbf{n}_0$. The second pole explores the arc of circle from $\mathbf{n}_0$ to $\mathbf{n}_1$, using the angle θ as parameter. When $\theta = 0$, both poles coincide in $\mathbf{n}_0$: the typical voter prefers candidate 1 and she is indifferent between the other candidates. When $\theta = \theta_{\max}$, the second pole is $\mathbf{n}_1$; the typical voter in the second group prefers candidate 2 and she is indifferent between the other candidates

We observe in figure 7.10 that the CM rate increases for all voting systems under study when θ increases. This is not surprising: when disunity grows in the population, motivation to manipulate grows also.

Once again, we note the remarkable performances for CIRV, IRV and EB, followed by the two-round system.

7.9 Meta-analysis in spheroidal cultures

In order not to make this study longer, we show mercy to the meritorious reader and we do not make other parameters vary: so, we will not vary the relative sizes of the social groups and we will not study the case where concentrations κ are different from one group to another. The results in these cases are similar and each one can do such experiments with SVVAMP.

From all the previous curves, a general trend seems to appears. Some of the voting systems are almost always less manipulable than the others: CIRV, IRV and EB. The only exception was for $V = 3$ voters (figure 7.4), where the lower bound found for some voting systems is smaller than the rate found for CIRV, IRV and EB. However, with 3 voters, the tie-breaking rule plays a role that is so important that it is difficult to draw meaningful conclusions about the voting systems themselves. So, we choose to ignore the particular case $V = 3$ in the rest of this section.

In this chapter and the following ones, we will often perform a *meta-analysis*, which makes it possible to present in a compact way the results we obtained for a set of cultures that may be heterogeneous, and to examine the compared performances of different voting systems.

7.9.1 Meta-analysis of CM in spheroidal cultures

The graph in figure 7.11 illustrates this method for the CM rate. Each vertex represents a voting system. An edge from a voting system f to another one g means that, for all cultures observed in this chapter (except $V = 3$), i.e. for each point of each curve of this chapter, voting system f is proven at most as manipulable as g.

More precisely, for a voting system f and a given culture (i.e. a fixed set of parameters), let us note $\underline{\tau_{\text{CM}}}(f)$ the lower bound found by SVVAMP and $\overline{\tau_{\text{CM}}}(f)$ the upper bound. For two distinct voting systems f and g, we draw an edge from f to g iff we always have $\overline{\tau_{\text{CM}}}(f) \leq \underline{\tau_{\text{CM}}}(g)$.

In all our meta-analyses, we exclude the statistical uncertainty, i.e. we consider the *proportion of manipulable configurations* in each drawing of 10 000 experiments, and not the exact *manipulability rate* in the underlying culture. The motivation is the following. Imagine that a voting system is generally far less manipulable than another one; but that in a specific culture, both systems have a rate close to 100 %. Because of this last case only, since there is a statistical uncertainty of order 1 %, it is impossible to conclude that the first system always has a lower manipulability rate than the second one in the cultures under study. By considering the proportion of manipulable configurations in the actual drawings, we can conclude more often. So, one need to stay aware that this meta-analysis gives results on a set of random experiments, not on a set of cultures. Despite

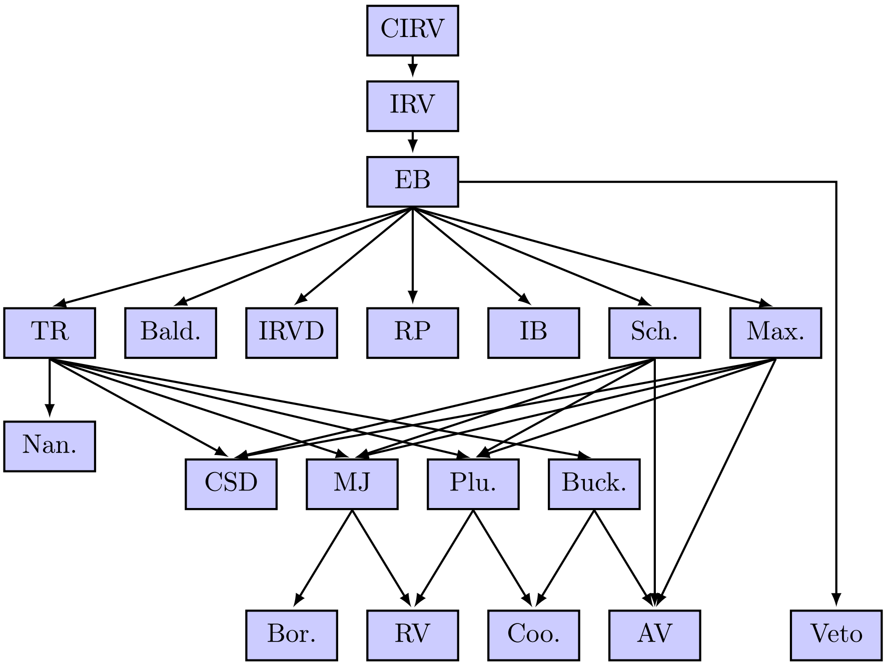

Figure 7.11: Meta-analysis of CM in all spheroidal cultures under study (except $V = 3$).

this limitation, it gives a compact representation of the results, which provides a qualitative indication about what happens in the cultures under study.

The other advantage of ignoring the statistical uncertainty is that the meta-analysis graph we obtain is necessarily transitive (otherwise, we could have a binary relation of the same type as in example 1.7). For the sake of readability, we represent only the minimal set of edges that makes it possible to deduce the graph in whole by transitivity (i.e. its *transitive reduction*).

For some voting systems, such as Baldwin's method, the approximate algorithm gives essentially a lower bound of manipulability, so it is not possible to establish that the voting system is at most as manipulable as another one. On the opposite, it is possible to show that is is more manipulable: for this reason, Baldwin's method, for example, has incoming edges by no outgoing edges, although its lower bound of manipulability, represented in the curves, is generally quite small in general. For Maximin, Schulze and Borda, the upper bound is precise enough to establish their eventual superiority compared to some other systems, even if that case does not happen in practice ofr Borda's method.

The main conclusion of figure 7.11 is that CIRV, IRV and EB are generally less manipulable that the other voting systems. Although we know, by theoretical arguments, that CIRV is strictly less manipulable than IRV, which is strictly less manipulable than EB, the observed difference are generally very small: for each group of 10 000 experiment drawn for the curves of this chapter, the difference between the proportions of manipulable profiles is 0,02 % or less between CIRV and IRV and 1,3 % or less between CIRV and EB.

In these simulations, these three voting systems are followed by the two-round system, which is generally (but not always) less manipulable than the other voting systems under study. For example, the difference of manipulability in favor of

IRVD against the two-round system is never more than 1,3 %; on the opposite, the difference of manipulability in favor of the two-round system against IRVD may by more than 57 %. The two-round system can only be clearly dominated by Veto, and only for a large number of voters. Then, we can informally distinguish several groups, gathered by line in the figure.

For Balwin, IRVD, RP and IB, the manipulability rates are generally quite similar, by they are only lower bounds that permit only to establish their defeats against EB and their quasi-domination by the two-round system. As for Nanson's method, it is always more manipulable than the two-round system in all the curves of this chapter. Maximin and Schulze's method have performances that are very similar (the observed difference is always less than 1 %) and better than Plurality, CSD and the majority judgment. They are often better than Bucklin's method (the difference of manipulability, when in favor of the later, is never more than 1 %).

The approximate algorithm for CSD does not make it possible to establish whether it is less manipulable than the voting systems in the group below; we simly notice that it was not proven more manipulable than Plurality, the majority judgment or Bucklin's method, hence its indicative position in the figure.

Voting systems with the worst performances are Borda's method (more manipulable than the majority judgment), range voting (more manipulable than the majority judgment or Plurality), Coombs's method (more manipulable than Plurality or Bucklin's method) and approval voting (more manipulable than Bucklin's method). In many figures of this chapter, the manipulability curves of these four voting systems are higher that the proportion of non-resistant configurations: in these cases, this means that any Condorcet voting system is less manipulable.

Veto may be considered as a group of its own, since it behaves very differently according to the cases. On one hand, it has a rare advantage: in impartial culture, its manipulability rate does not tend to 1 when the number of voters V tends to $+\infty$, unlike all PSR and Maximin. Based on our simulations, we even conjecture that, among the voting systems under study here, it shares this desirable property only with CIRV, IRV and EB. On the other hand, Veto has a very unfortunate disadvantage: in a popultion where all the voters have the same preferences, its manipulability rate does not tend to 0, unlike all voting systems under study here and more generally all unanimous ones.

7.9.2 Meta-analysis of MT in spheroidal cultures

Figure 7.11 has the virtue of dealing with CM in all generality, but with the drawback not to be able to give complete conclusions for some of the voting systems using the approximate algorithm. In order to have a complementary indication of manipulability for all voting systems, we represent a similar graph for TM in figure 7.12. Moreover, TM gives an indicative measure of realistic manipulations, which can be performed with a limited exchange of information. For TM, we do not give the detailed curves and we skip directly to the meta-analysis.

First, let us notice that most of the edges in figure 7.11 (explicitly drawn or implied by transitivity) are included in the ones in figure 7.12: so, TM gives a reasonable indicator about the hierarchy of manipulability between voting systems.

We observe that Veto constitutes an isolated connected component: it is sometimes more TM and sometimes less TM than any other voting system here.

For TM as well as for CM, voting systems CIRV, IRV, EB (in that order) are always less manipulable than all the order voting systems under study, except

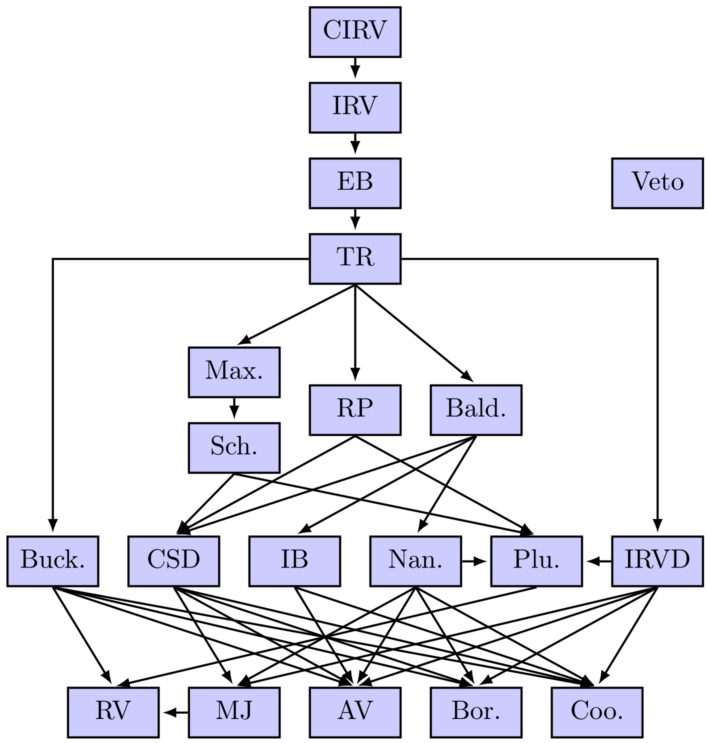

Figure 7.12: Meta-analysis of TM in all spheroidal cultures under study (except $V = 3$).

Veto. There are always followed by the two-round system, which presents intermediary performances. Then, we may indicatively distinguish three groups, by increasing TM:

- Maximin and Schulze's method (with very similar performances), RP and Baldwin's method;
- CSD, IB, Nanson's method and Plurality (even if Nanson's method is less TM than Plurality);
- Range voting, the majority judgment, approval voting, Borda and Coombs (while noticing that majority judgment is less TM than range voting).

Bucklin's method is a bit isolated in the figure: it is always always more TM than the two-round system and always less than range voting, but it is comparable to no voting system in the intermediary groups. The case of IRVD is similar: always more TM than the two-round system, less TM than Plurality and all voting systems in the last group, but not comparable to the others.

Chapter 8

Simulations in cultures based on a political spectrum

In this chapter, we will study cultures based on a *political spectrum*, which we had mentioned in section 6.1.3. Our reference model will be the *Gaussian well* and we will discuss rapidly another similar model, the *Euclidean box*, in section 8.5.

In these models, we use a n-dimensional space (for a given integer n), which we call the *political spectrum*. In Gaussian well, we take as parameter a vector of nonnegative real numbers $(\sigma_1, \dots, \sigma_n)$: each number σ_i is called the *characteristic length* (in short, *length*) of the political spectrum along axis i. For each voter v (resp. each candidate c), we draw at random a position $\mathbf{x}_v = (x_v^1, \dots, x_v^n)$ (resp. $\mathbf{y}_c = (y_c^1, \dots, y_c^n)$). Each coordinate x_v^i (resp. y_c^i) is drawn independently according to a normal law of standard deviation σ_i. The utility of a voter v for a candidate c is $A - \delta(\mathbf{x}_v, \mathbf{y}_c)$, where δ denotes the usual Euclidean distance and A is a constant such that the average utility is equal to zero. This constant has an impact only on approval voting.

When the political spectrum is unidimensional ($n = 1$), the culture is obviously single-peaked (definition 1.11). If the number of voters is odd, there is always a Condorcet winner, who is the most liked candidate of the median voter (on the political spectrum). It the number of voters is even, then Condorcet-admissible candidates are the most liked candidates of the two median voters; if it is the same candidate, she is Condorcet winner. As a consequence, the proportion of non-admissible configurations is equal to zero and it cannot give us an interesting lower bound of manipulability, unlike in previous chapter. As a reminder, and for the sake of homogeneity, we will keep on representing this proportion in the figures anyway.

Metaphorically, in a unidimensional political spectrum, Condorcet-admissible candidates will be called *centrists*[1]. Similarly, candidates with a higher (resp. lower) abscissa in the political spectrum will be called *rightist* (resp. *leftist*) candidates.

By the way, in the unidimensional case, it is not needed to give the value of σ: indeed, up to changing the unit of length in the political spectrum, all models are equivalent (because utility vectors are defined up to a multiplicative constant, cf.

[1]It is only a convention of language, based on an analogy. Even if we accept the assumption that voters' preferences in a political election are approximately single-peaked, the possible Condorcet winner is not necessarily a centrist candidate, in the usual political sense of this word. By the way, we will have the opportunity to discuss the experimental validity of this assumption in 9.

appendix B). In contract, as soon as there are two dimensions, the relative lengths of the political spectrum along different axes are important. The experimental validity of such multidimensional models for political elections is notably shown by Laslier (2004, 2006).

8.1 Unidimensional Gaussian well: number of candidates C

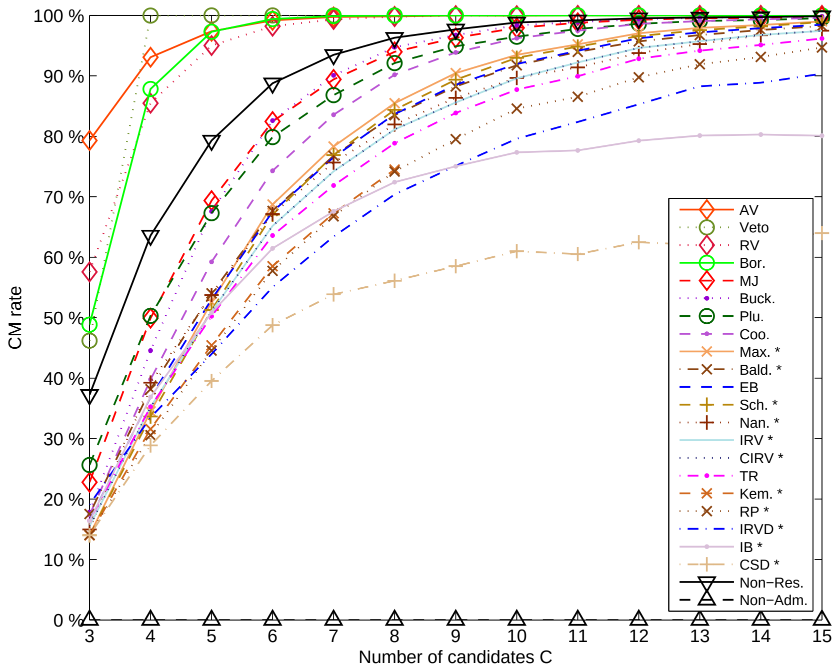

Figure 8.1: CM rate as a function of the number of candidates C. Gaussian well, $V = 33$, $n = 1$.

In figure 8.1, we draw the CM rate as a function of the number of candidates C in a unidimensional Gaussian well.

The first conclusion from figure 8.1 is that the manipulability rates seem to tend to 1, like in spherical culture. As for CSD and IB, it is difficult to give a conclusion about this possible limit, since we have only lower bounds of manipulability that does not tend obviously to 1. Generally speaking, the convergences we observe to a manipulability rate of 1 seem slower than in spherical culture. In the special case of Veto, the manipulability seem also to tend to 1 but again, we can prove that it is not true (with the lexicographic tie-breaking rule): indeed, we can reason the same way as in proposition 7.5 because the restriction of a Gaussian well culture to candidates $1, \dots, V+1$ is also a Gaussian well culture.

We observe that the relative performances of the different voting systems are not the same as in chapter 7. In particular, now, it is not clear whether CIRV, IRV and EB have a lower manipulability rate than the other voting systems. In order to confirm this observation, figure 8.2 presents the same curve, with a focus on these three voting systems and those for which the algorithmic uncertainty

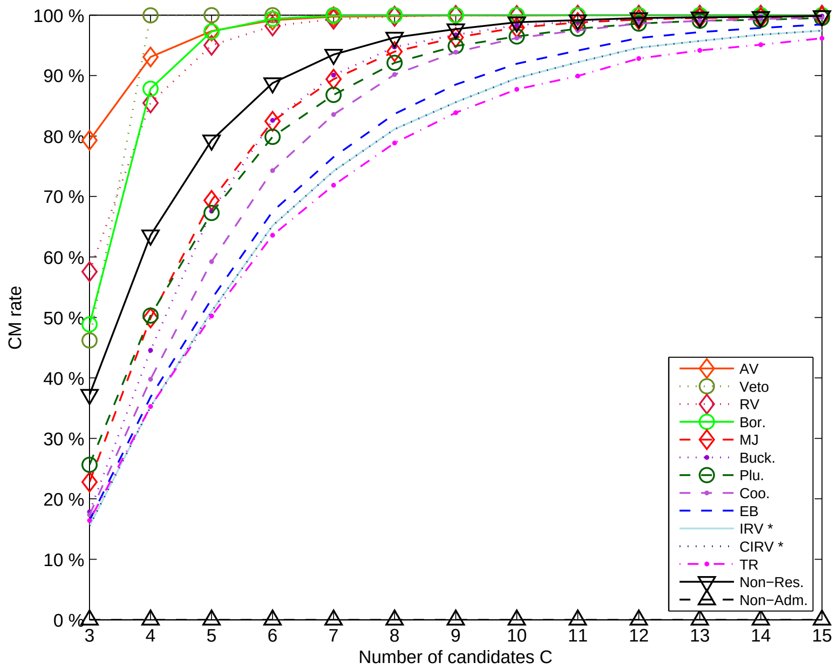

Figure 8.2: CM rate as a function of the number of candidates C, for a selection of voting systems. Gaussian well, $V = 33$, $n = 1$.

is less than 1 % for each point (which includes Borda's method, although the algorithm is not exact).

This time, the margins of uncertainty for CIRV and IRV are sometimes greater than 1 % because we used the option `fast` in SVAMP when the number of candidates C is greater or equal to 9 (hence the star in the legend of figure 8.2). However, we used the exact algorithm for EB, so the curve for EB gives an upper bound for those of CIRV and IRV.

In figure 8.2, we observe that the two-round system is (slightly) better than CIRV, IRV and EB, unlike what happens in the spheroidal cultures (spherical or VMF). We can propose a qualitative explanation. In IRV, once the candidates of extreme right and left are eliminated, their proponents note for moderate leftist and rightist candidates, and the centrist has the risk of being eliminated because she receives only a few transferred votes. Since IRV meets **InfMC**, this implies that the configuration is manipulable. In contrast, in the two-round system, moderate rightist and leftist candidates have no transferred vote at the moment when the selection for the final round (i.e. the second round) is performed, so the centrist has more chances to get to the second round.

However, we notice in figure 8.2 that CIRV, IRV and EB remain less manipulable than Coombs' method, which is less manipulable than Plurality, Bucklin's method and the majority judgment, which are themselves less manipulable than Borda's method, range voting, Veto and approval voting. So, the hierarchy that we have already observed for these voting systems seems to be confirmed, except for Coombs' method, which seems to perform better here than in the spheroidal cultures. This could be linked to the fact that in a *single-peaked* culture, Coombs' method meets the Condorcet criterion (Grofman and Feld, 2004).

For the comparison between CIRV, IRV, EB and the other voting systems, the question remains open. In future works, it would be interesting to develop more efficient algorithms for these voting systems, in order to know whether they are less manipulable than CIRV, IRV and EB in such a culture. In the meta-analyses of sections 8.6 and 8.7, using TM will allow us to say more about this comparison.

8.2 Unidimensional Gaussian well: number of voters V

8.2.1 Number of voters V odd

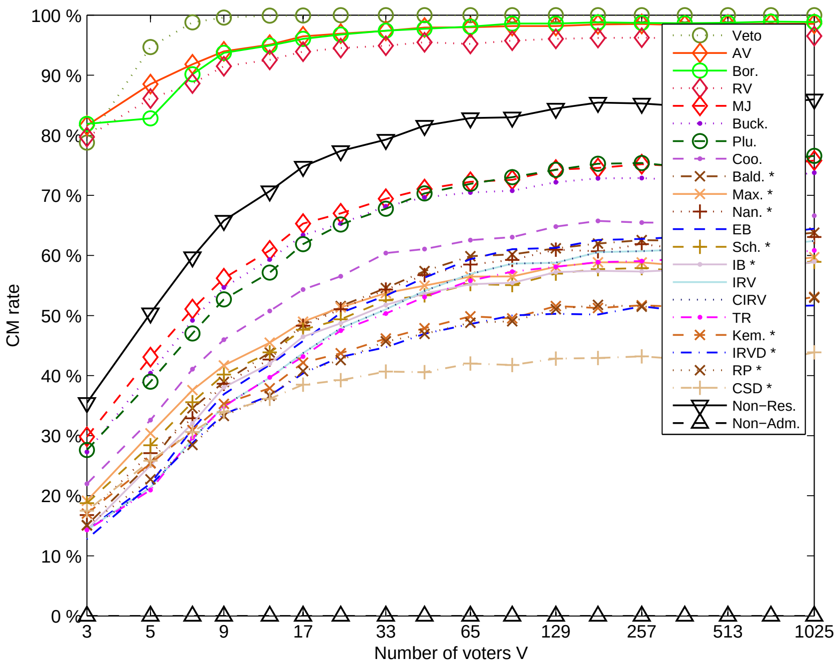

Figure 8.3: CM rate as a function of the number of voters V. Gaussian well, $C = 5$, $n = 1$, large odd values of V.

Like we did in spherical culture, figure 8.3 represents the CM rate as a function of the number of voters V for large odd values, for a unidimensional Gaussian well. Like in spherical culture, manipulability rates seem to be increasing functions of the number of voters V (on the set of odd values). However, they do not seem to tend to 1 for most voting systems, which lead us to the following conjecture.

Conjecture 8.1

In unidimensional Gaussian well, for $C \geq 3$ and $V \to +\infty$, the CM rate tends to 1 for Veto.

In unidimensional Gaussian well, for $C \geq 3$ and $V \to +\infty$, the CM rate tends to a limit that is strictly between 0 and 1 for all other voting systems under study here.

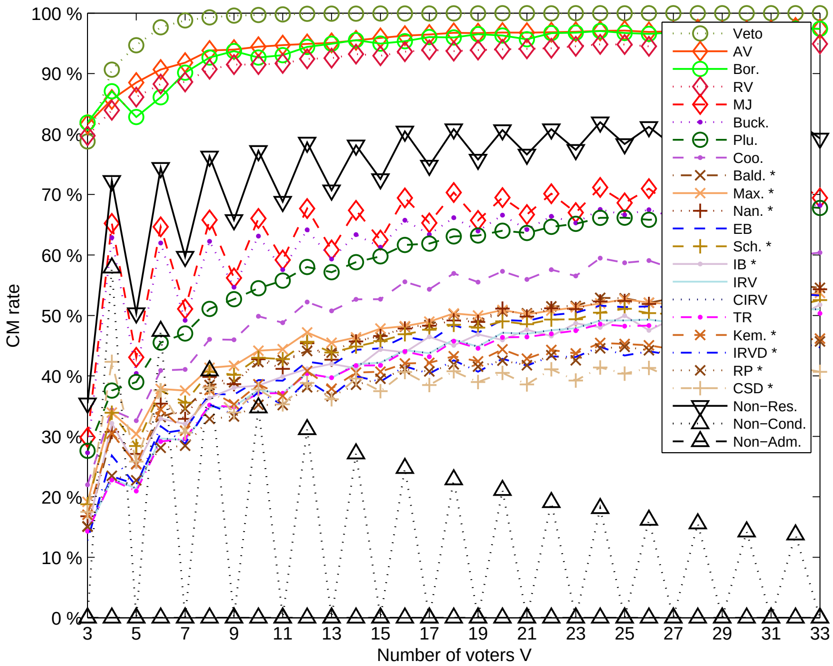

Figure 8.4: CM rate as a function of the number of voters V. Gaussian well, $C = 5$, $n = 1$, small values of V.

8.2.2 Parity of V

In spherical culture, we have seen in figure 7.5 that, for some voting systems, the manipulability rate present oscillations depending on the parity of V (and sometimes more complicated non-monotony phenomena). In order to examine this kind of phenomenon in Gaussian well, figure 8.4 represents the CM rate as a funciton of the number of voters V for all values between 3 and 33. The oscillatory phenomenon is far more frequent than in spherical culture: it concerns all voting systems under study except Veto, approval voting, Borda's method and range voting.

We can explain it by the fact that, unlike in spherical culture, there is essentially no concurrent effects anymore: the probability of existence for a Condorcet winner or a resistant Condorcet winner, presenting in phase oscillations, make the odd values of V less favorable to manipulation. In particular, for V odd, less us recall that there is always a Condorcet winner. For the majority judgment and Bucklin's method, their specific mechanisms generate an effect of parity which goes in the same direction, as we explained in section 7.4.2. As for the probability of existence for a Condorcet-admissible candidate, it is constant (and equal to 1), unlike in spherical culture where its oscillations, in antiphase with the other effects, were partially canceling them.

In single-peaked culture, by the way, we can notice that the notion of resistant Condorcet winner is equivalent to the notion of majority favorite. Indeed, if it is not an extreme candidate, it is necessary that a majority of voters prefer her simultaneously to the two candidates on her immediate left and right, hence to all other candidates. If it is a extreme-left candidate (for example), simply being a Condorcet winner demands that a majority of voters prefer her to the candidate on her immediate right, hence to all other candidates.

For V even, we also observe that the probability of existence for a Condorcet winner increases with V, which is easy to understand: with a constant number of candidates, the more voters there are, the less probable it is that the frontier between two most-liked candidates is located precisely between the two median voters. Generally, and with no surprise, parity effects diminish when the number of voters increases, like in spherical culture.

8.3 Unidimensional Gaussian well: shift y_0

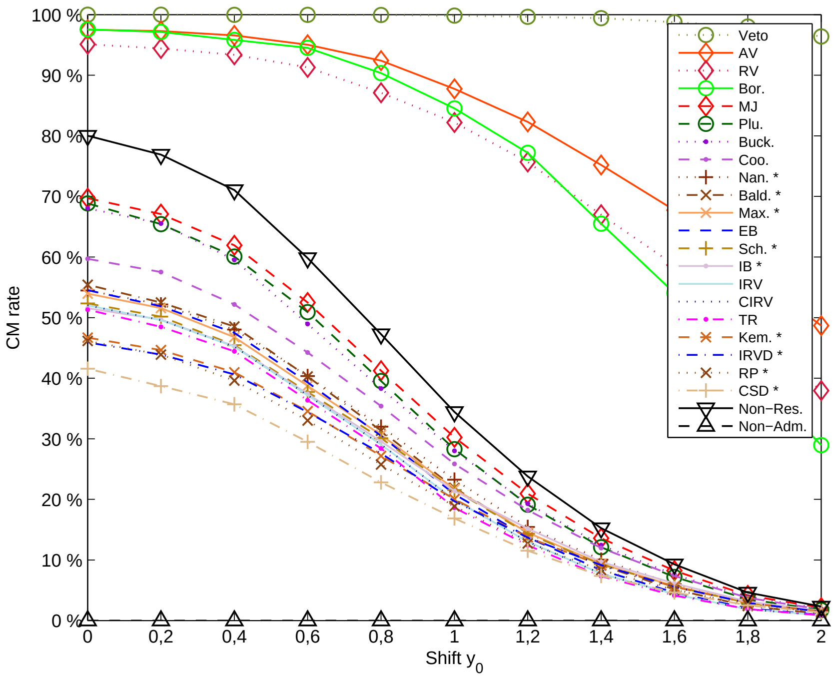

Figure 8.5: CM rate as a function of the shift between voters and candidates. Gaussian well, $V = 33$, $C = 5$, $n = 1$, $\sigma = 1$.

In figure 8.5, we consider also a unidimensional Gaussian well of characteristic length $\sigma = 1$. But we add a shift y_0 between the distribution of the candidates and the one of the voters: the normal law that is used for the candidates is not centered in 0 anymore but in y_0. When this shift tends to $+\infty$, each candidate is on the right of each voter. In particular, the candidate who is most on the right is most liked by each voter[2]. Such a situation is not very realistic because, in that case, some candidates have a strategic interest to move their political offer in order to attract more voters; while being aware of this limitation, we study this case to explore qualitatively the impact of all the parameters for this model.

We explore what happens for a shift varying from 0 to 2. As we could expect, voting systems tend to be less manipulable when the shift increases, i.e. when voters agree more.

The interesting point is the similarity with what we observed in figure 7.6, describing a VMF culture where the concentration κ increases. In particular, some voting systems are significantly less reactive to the increasing agreement between the voters, in the sense that their manipulability rates decrease more

[2] There is no hidden message in this paragraph. We could have used the opposite convention, but we needed to choose.

slowly: approval voting, range voting, Borda's method and especially Veto. It is the same list of voting systems as for figure 7.6, except for Coombs' method, which behaves better in Gaussian well.

From a theoretical point of view, the arguments we exposed in section 7.5 remain valid: since the culture tends to a perfect agreement between the voters, the manipulability rate of each unanimous voting system tends to 0 and the manipulability rate of Veto tends (slowly) to $1 - \frac{1}{C-1}$.

8.4 Multidimensional Gaussian well: number of dimensions n

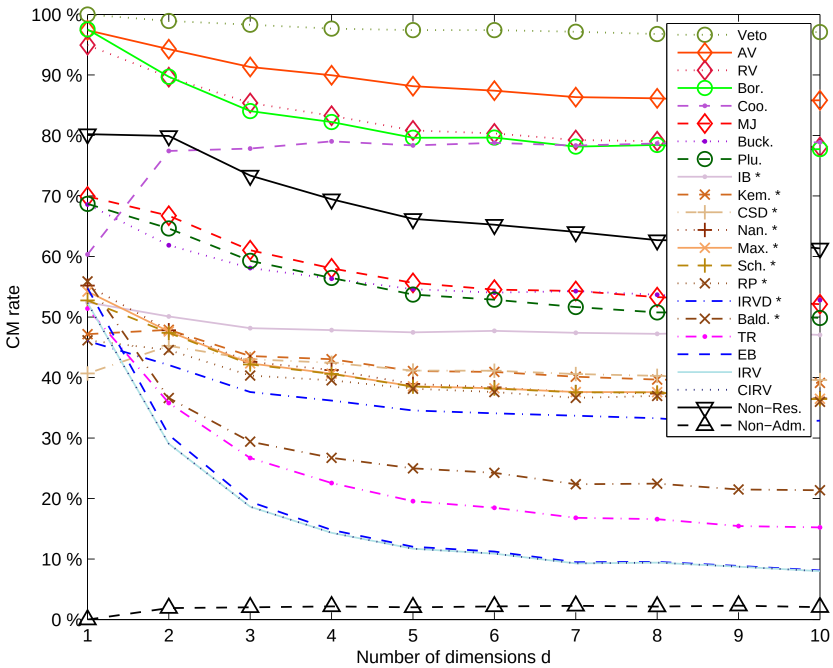

Figure 8.6: CM rate as a function of the number of dimensions n. Gaussian well, $V = 33$, $C = 5$, $\sigma = (1, \ldots, 1)$.

Until now, we have considered political spectra with one dimension only. In figure 8.6, we extend this model by considering a multidimensional Gaussian well, with a dimension n varying from 1 to 10.

At first glance, one might think that if n increases, then the culture tends to a spherical culture, like for a VMF culture with an infinite number of poles (section 7.7). If that was true, then the CM rates should increase. However, such a belief would be erroneous. Indeed, even with a multidimensional political spectrum, a candidate whose position is close to the origin has more chances to be liked by the voters, hence voters are not independent, even in the limit $n \to +\infty$: this is sufficient to prove that the limit culture is not the spherical culture.

As a matter of fact, we observe in figure 8.6 that when n increases, the CM rate decreases for most voting systems (except for Coombs' method, and for IB whose variation is small).

Starting from $n = 2$, CIRV, IRV and EB become less manipulable than the other voting systems. As a consequence, it seems that the (relatively) bad performance of these three voting systems is deeply connected to having a unidimensional political spectrum.

They can even present performances that are better than those observed in the spheroidal cultures from chapter 7. For comparison, in the reference scenario (figure 7.1), the manipulability rates of these three voting systems were 44 %. With the same numbers of voters and candidates, in Gaussian well with $n = 10$, the rates are about 8 %.

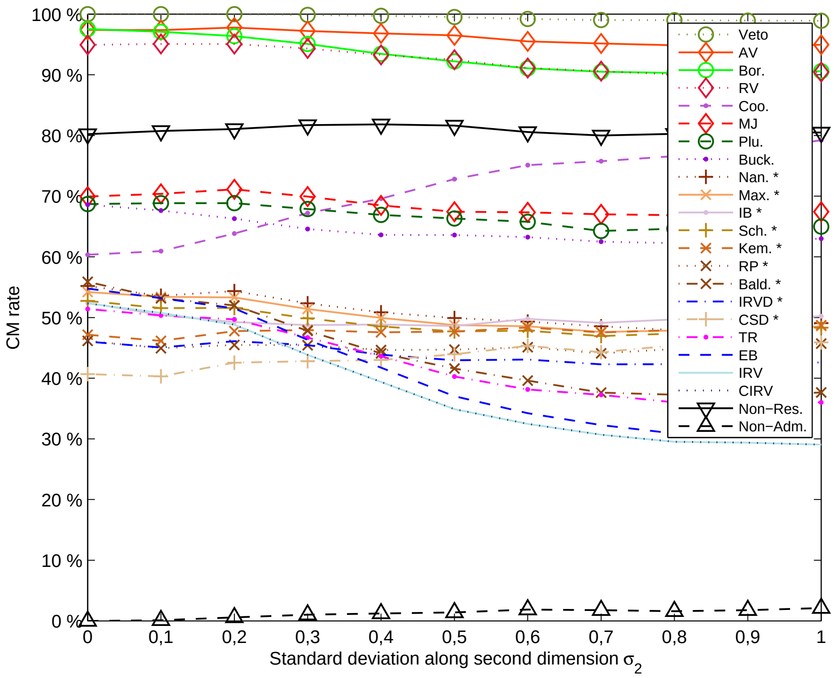

Figure 8.7: CM rate as a funciton of σ_2. Gaussian well, $V = 33$, $C = 5$, $\sigma_1 = 1$.

In order to enrich the previous observation, we explore in figure 8.7 the transition from a unidimensional culture to a bidimensional culture. For this purpose, we consider a bidimensional Gaussian well whose characteristic lengths are $(1, \sigma_2)$. For $\sigma_2 = 0$, the culture is unidimensional. For $\sigma_2 = 1$, we have a "square" culture: both dimensions are equally important.

It is no use continuing the figure for $\sigma_2 \in [1, +\infty]$ because we would get the exact symmetry of the curves represented here. Indeed, up to changing to unit of length and to inverting the axes of the political spectrum, the culture $\sigma = (1, \sigma_2)$ is equivalent to the culture $\sigma' = (1, \frac{1}{\sigma_2})$.

These curves seem to interpolate naturally what happens between dimensions 1 et 2 in figure 8.6: the manipulability rates decrease. Like in figure 8.6, only the one of Coombs' method is clearly increasing. For $\sigma_2 \geq 0{,}4$ approximately, CIRV, IRV and EB become less manipulable than the other voting systems, like in the spheroidal cultures. Similarly, for $\sigma_2 \geq 0{,}5$ or so, the two-round system has an intermediary manipulability between these three voting systems and the others.

8.5 Comparison with a Euclidean box

We can wonder if the phenomena we observed in Gaussian well are a qualitative consequence of a culture based on a unidimensional politicall spectrum or if they depend on the Gaussian distribution we used fo candidates and voters.

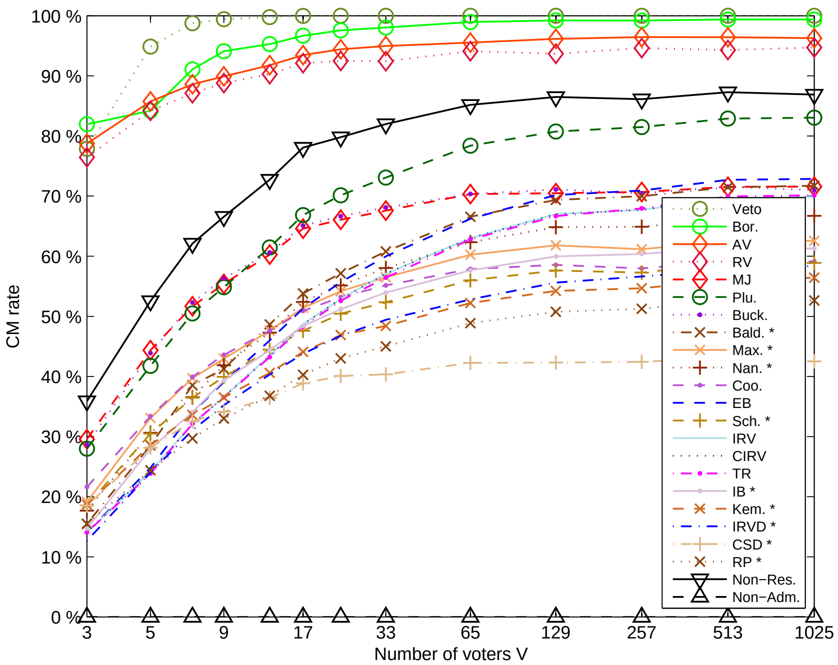

Figure 8.8: CM rate as a function of the number of voters V. Euclidean box, $C = 5$, $n = 1$, large odd values of V.

As an example, figure 8.8 presents the manipulability rates in a unidimensional Euclidean box, for odd values of the number of voters: instead of using a normal law, positions for the voters and the candidates are drawn uniformly in a segment $[-1, 1]$.

We observe a great similarity with figure 8.3, which seems to indicate that our findings have a certain general validity on the cultures based on a unidimensional political spectrum. The main difference concerns the relative performances of TR and IRV (or CIRV), which are even closer than is Gaussian well.

8.6 Meta-analysis in unidimensional culture

8.6.1 Meta-analysis of CM in unidimensional culture

Like figure 7.11 for spheroidal cultures, figure 8.9 presents the comparison of CM between the voting systems in a culture of unidimensional political spectrum, whether in Gaussian well or in Euclidean box. For the moment, we focus on this case because we have seen in section 8.4 that in a multidimensional culture, the observed behaviors are relatively similar to those observed in chapter 7. In section 8.7, we will do a synthesis of all cultures under study, including those from previous chapter.

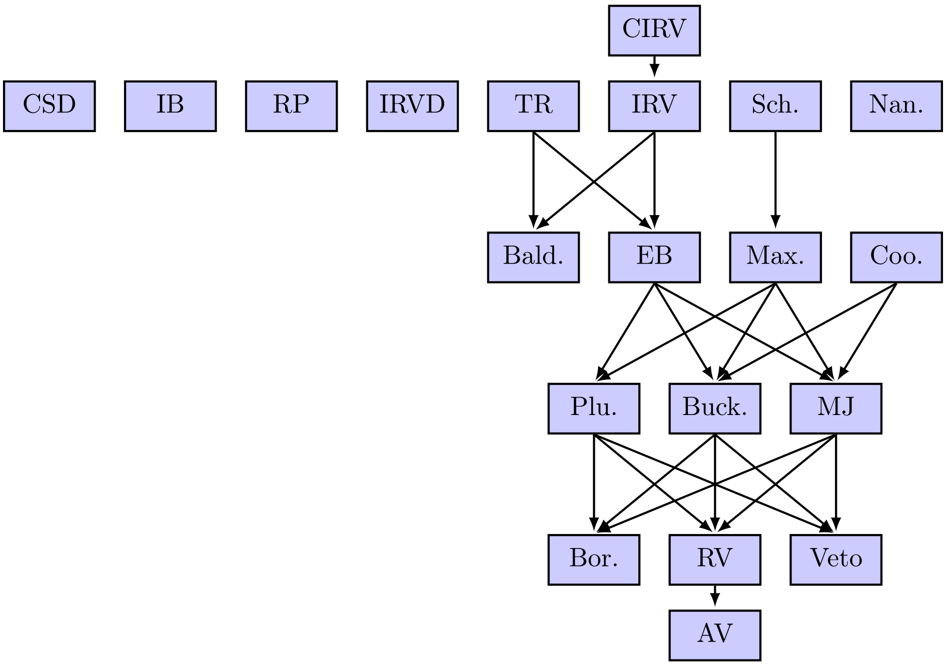

Figure 8.9: Meta-analysis of CM in all unidimensional cultures under study.

In the graph of figure 8.9, we see that in unidimensional culture, the two-round system becomes competitive with CIRV and IRV, and even dominates EB in the cultures we studied. As for CSD, IB, RP, IRVD, Schulze and Nanson, the approximate algorithms do not make it possible to conclude in all cases but these voting systems present promising results. Once again, Schulze's method and Maximin present performances that are very close (the gap is always 2 % or less). Baldwin's method is more manipulable than the two-round system.

The exhaustive ballot (which is more manipulable than the two-round system and IRV), Maximin (which is more manipulable than Schulze's method) and Coombs' method serve as references of manipulability in order to compare to the next two groups, by increasing order of manipulability:

- Plurality, Bucklin's method and the majority judgment;
- Borda's method, range voting, Veto and approval voting (which is more manipulable than range voting).

Like in spheroidal culture, these three last voting systems have often a manipulability rate that is more than the proportion of non-resistant configurations, which makes them more manipulable than any Condorcet voting system in those cases. Coombs' method, which seems to behave better in a unidimensional political spectrum than in spheroidal culture, is not concerned by this bad behavior anymore. This is not obvious *a priori*: even if we recalled that it meets the Condorcet criterion for single-peaked preferences, in particular those obtained by sincere voting in this model, it does not necessarily so for configurations obtained from a manipulation.

Generally speaking, some Condorcet voting systems, especially Schulze's method for which the algorithmic uncertainty is low, seem to behave better in a unidimensional political spectrum than in a spherical culture. Intuitively,

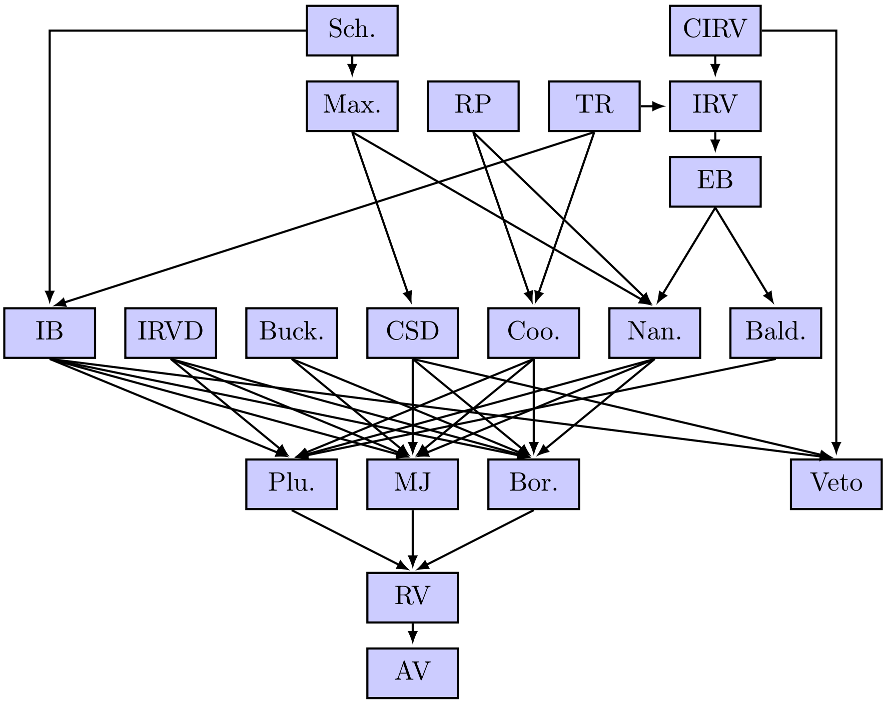

Figure 8.10: Meta-analysis of TM in all unidimensional cultures under study

we can understand why: in spherical culture, the probability of existence for a Condorcet-admissible candidate is relatively low and we know that it tends to 0 for V odd and $C \to \infty$ (Gehrlein, 2006). On the opposite, in a single-peaked culture and in particular in a unidimensional Gaussian well, this probability (which is equal to the probability that there exists a Condorcet winner for V odd) is equal to 1. A non-Condorcet voting system starts the race with a major handicap: for V odd, each time the winner is a non-Condorcet candidate, the configuration is manipulable.

8.6.2 Meta-analysis of TM in unidimensional culture

Like we did for spheroidal cultures, we present in figure 8.10 a similar graph for TM. Since there is no algorithmic uncertainty, we are able to conclude in all cases. Hence, if there is no edge (explicit or implied by transitivity) between two voting systems, it means that one is sometimes less TM than the other, and sometimes the opposite.

Voting systems with no incoming edge are the most performing from a TM point of view: it is Schulze's method (with performances very close to Maximin, but slightly better), RP, the two-round system, CIRV, IRVD and Bucklin's method.

Voting systems with the worst performances are Plurality, the majority judgement and Borda's method, which are less TM than range voting, which is less TM than approval voting.

Once again, Veto is a particular case: it is dominated by some voting systems, such as Schulze's method, the two-round system or CIRV, but it is not comparable to many others.

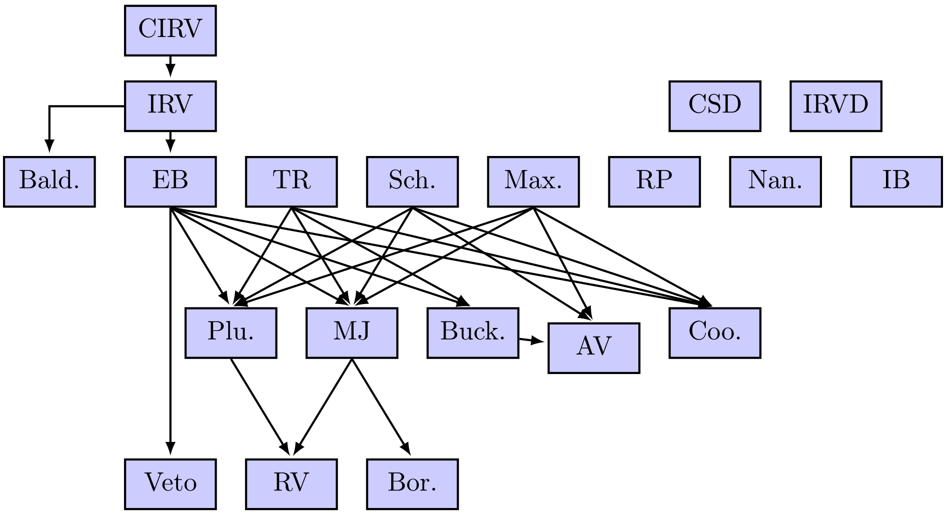

Figure 8.11: Meta-analysis of CM in all cultures from chapters 7 and 8 (except spherical culture with $V = 3$).

8.7 Meta-analysis in all cultures under study

8.7.1 Meta-analysis of CM in all cultures under study

In order to do a synthesis of this chapter and previous one, we can wonder whether some voting systems are less manipulable than others in all cultures under study in chapters 7 and 8. This is the subject of the graph in figure 8.11, which relies on all previous curves, except the spherical culture with $V = 3$, for the reasons mentioned in section 7.9.

Voting systems with no incoming edge are CIRV, the two-round system, Schulze's method and Maximin (with very close performances from each other), CSD, IRVD, RP, Nanson's method and IB. As for these last five, the approximate algorithm gives essentially a lower bound of manipulability, which makes them incomparable on the set of all experiments. As usual, we will give a complementary point of view with TM.

On the opposite, we have noticed several times the very poor performances of Borda's method, range voting and approval voting, whose manipulability rates are often more than the proportion of non-resistant configurations. In all such cases, it means that these voting systems are more manipulable than any voting system meeting the Condorcet criterion.

Among the voting systems that are wide spread in practice, it is worth noticing that Plurality is dominated by voting systems such as CIRV, IRV, EB, the two-round system, Schulze's method or Maximin. From the point of view of manipulability, this seems to plead for a limited use of Plurality in actual elections and its replacement by one of these voting systems.

8.7.2 Meta-analysis of TM in all cultures under study

Similarly, figure 8.12 makes it possible to compare TM of the different voting systems in all cultures from this chapter and previous one (except the spherical culture with $V = 3$). So, an edge from a voting system to another one means

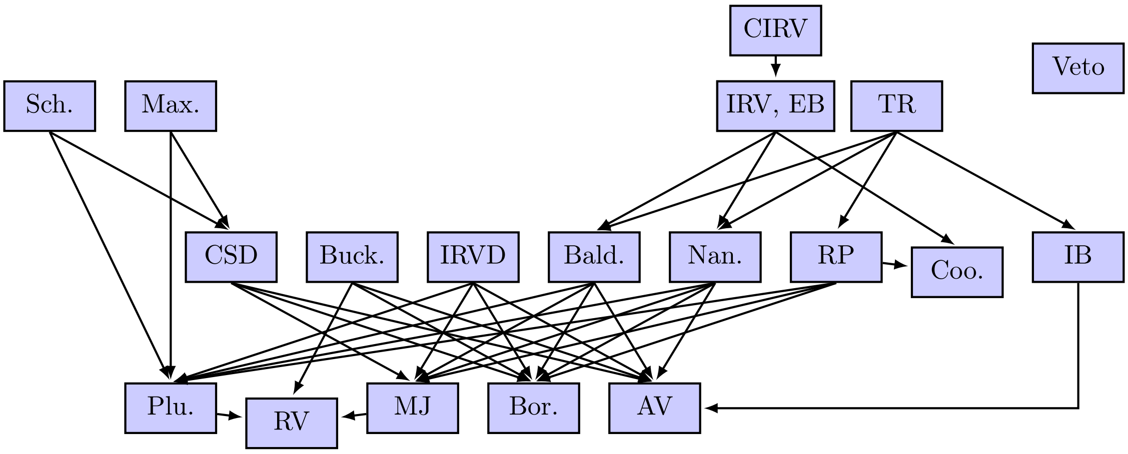

Figure 8.12: Meta-analysis of TM in all cultures from chapters 7 and 8 (except spherical culture with $V = 3$).

that the first one is better than the second one from a TM point of view. Once again, we note that the voting systems with the best performances are CIRV, the two-round system, Schulze's method, Maximin, Bucklin's method. IRVD has no incoming edge either, but its difference of manipulability with the two round system is never favorable with a great amplitude in all cultures under study (it is always less than 2 %), whereas we have seen that TR is sometimes far less TM than IRVD.

All other voting systems, except Veto, are dominated by at least one of these voting systems. Veto is a connected component on its own, which confirms its being not comparable with all other voting systems under study, amplified by its resistance to trivial manipulation.

At last, we note the poor performances of the following voting systems: Coombs' method, Plurality, range voting, the majority judgment, Borda's method and approval voting.

However, as we discussed in the introduction of this memoir, approval voting has the advantage of proposing a strategy that is quite natural in practice, the *Leader rule* (Laslier, 2009), which makes it possible to reach equilibria with a limited exchange of information. In contrast, the complexity to establish a strategic ballot in IRV seems, experimentally, discourage voters from undertaking such a computation and incite them to vote sincerely (Van der Straeten et al., 2010), which can be seen as an advantage or a drawback, depending on the point of view.

Chapter 9

Analysis of experimental data

In previous chapters, we studied the manipulability of voting systems in artificial culture, either spheroidal (chapter 7) or based on a political spectrum (chapter 8).

An important conclusion of these chapters was the low manipulability of CIRV, IRV and EB, except in cultures based on a unidimensional political spectrum, where some other voting systems, such as the two-round system or Schulze's method, present interesting performances. So, it is natural to wonder what happens in the real world, and this is the object of this chapter.

We rely on a corpus of 168 experiments coming for different contexts. In order to be able to use the exact algorithm for IRV while keeping a reasonable computation time, we limit ourselves to elections with 3 to 14 candidates. Kemeny's method will be excluded from this chapter, because of the computation time needed to decide the winner.

9.1 Presentation of the experiments

9.1.1 Realized experiments

The three following datasets were obtained according to relatively similar modalities, with our direct participation to establishing the modalities of the election and the recollection of the ballots.

Algotel During Algotel 2012 conference[1], the program committee has pre-selected 5 papers, named here arbitrarily A, B, C, D and E, for the election of the best paper of the conference. In order to make the final choice, all participants were asked to attribute a grade between 0 and 10 to each paper, evaluating the quality and presentation. It was possible to grade only some of the papers, the absence of note being considered as 0. Out of 72 people participating to the conference, there were 57 valid ballots, 1 blank vote and 2 null ballots.

Participants were told that their ballots would be tested on several voting systems, but they did not know which one would be used to designate the winning paper. In case of disagreement between the voting systems, we had the possibility to award all the papers designated by at least one voting system. However, in practice, we will see that all voting systems were declaring the same winner, so the question did not arise.

[1] http://algotel2012.ens-lyon.fr/

Grade	French appreciation	English translation
5 stars	Culte!	Classic!
4 stars	Franchement bien	Really good
3 stars	Pas mal	Quite good
2 stars	Bof, sans plus	Blah, neither good nor bad
1 star	Vraiment pas aimé!	I really disliked!

Table 9.1: Scale of grades and appreciations in the website `www.bdtheque.com`.

The conclusions of this experiment, which are partially exposed in this chapter, are also available in the article by Durand et al. (2014a).

Bordeaux During the special day of the Doctoral School of Mathematics and Computer Science in Bordeaux[2] in November 2014, an poll was organized to elect the best poster among 11 posters present by students in their last year of PhD. The modalities were similar to Algotel 2012, but authorized grades ranged from 0 to 20. There were 86 valid ballots for as many participants.

Paris VII In April 2015, an internal poll was organized to give an orientation for choosing the new name of the research department of computer science in university Paris VII–Paris Diderot[3]. 10 possible names were proposed. This time, voters did not attribute grades but appreciations: *Good*, *Quite good*, *Neither good nor bad*, *Quite bad* or *Bad*. There were 95 valid ballots for as many participants.

We thank the organizers of these events for making this experiments possible.

9.1.2 Website www.bdtheque.com

The website `www.bdtheque.com` is a collaborative site dedicated to comics. Users are invited to give their opinion on series of comics according to the scale presented in table 9.1, which is, in the same time, a scale of grades and a scale of appreciations (most of which are idiomatic French expressions and difficult to translate exactly in English).

In June 2012, the webmaster, Alix Bergeret, whom we thank here, was kind enough to give us the database of the site, under the condition that we preserve the anonymity of the users.

From this, we drew 12 experiments in the following way. For each integer $C \in [\![3, 14]\!]$, we wished to choose C candidate series and select the intersection of all users who attributed grades to all of them. By the way, all goal was to have a significant number of voters, or even maximize it if possible. But this problem, called the *Maximum Subset Intersection Problem*, was recently proven $\mathcal{NP}$-complete by Xavier (2012).

As an approximation, we used a greedy algorithm. We start with $C = 0$ (i.e. $\mathcal{C} = \varnothing$) and we initialize the set $\mathcal{V}$ to the set of all users. For each incrementation of C, we select the series with most grades by users still being in $\mathcal{V}$ and we add this series to the set of candidates $\mathcal{C}$, then we eliminate from $\mathcal{V}$ all users who did not attribute a grade to the last series we added. Table 9.2 show the correspondence

[2] `http://www.math.u-bordeaux1.fr/ED/ecole_doctorale/`
[3] `http://www.univ-paris-diderot.fr/`

C	3	4	5	6	7	8	9	10	11	12	13	14
V	33	24	21	19	18	15	14	14	13	13	12	12

Table 9.2: Bdtheque: number of candidates and voters

between the number of candidate series and the number of users obtained by this algorithm[4].

The particularity of this dataset is that, at the moment a user attributes a grades, she is not in a context of election: indeed, the aim is not to select collectively one option among a set of candidate options, as it is the case when it comes to award a prize or elect a person for a position. Moreover, this evaluation has no important stake and has limited practical consequence on each voter's life, unlike a political election for example. Finally, there is a way to limit and in some way control strategic voting: indeed, the policy of the website demands that any grade comes with a comment that is developed enough to justify it. If a user wishes to attribute an artificially low grade to a series for purely strategic reasons, she needs to pay a cognitive cost and a time cost in order to do it. For all these reasons, we can hope — but we can only hope — than this dataset is hardly affected by strategic voting.

9.1.3 Judgment of Paris

In 1976, 11 experts (9 from France and 2 from the USA) met for two blind wine tastings: first 10 white wines of Chardonnay grape, then 10 red wines based on Cabernet Sauvignon. Both tastings led to a grading by experts and a ranking of the wines.

Grades were ranging from 0 to 20, with the possibility of half-points. Experts knew in advance that range voting was to be used, which is a difference between the experiments mentioned above. Indeed, for Algotel, Bordeaux and Paris VII, we had informed voters that their ballots "would be tested on several voting systems", with no more information. And for bdtheque, as we said, it is not really a context of election.

Surprisingly, in each of the two categories, a Californian wine got the first prize, although the jury was French in great majority. This victory lead to numerous comments in the press about the modalities of the election and had a important impact on the development of American wines.

The dataset we use for the Judgment of Paris come from the website www.liquidasset.com/lindley.htm.

9.1.4 PrefLib dataset

PrefLib (http://www.preflib.org/) is a collaborative database collecting datasets of collective preferences in order to make them accessible for social choice specialists (Mattei and Walsh, 2013). Unlike the experiments above, it contains only ordinal data: it can be weak orders (with ties) or strict orders, complete

[4]For the curious reader: the series we obtained by this algorithm are, in this order, *XIII*, *Lanfeust of Troy*, *Blacksad*, *The Quest for the Time-Bird*, *Universal War One*, *The Third Testament*, *Largo Winch*, *De Cape et de Crocs*, *The adventures of Tintin*, *Thorgal*, *Asterix*, *Peter Pan* (by Loisel), *Wake* and *Lanfeust of the Stars*.

or incomplete. When the orders are incomplete, we assume that all non-ranked candidates are placed after all ranked candidates[5].

For the present study, we consider *a priori* all election files from PrefLib, with the following exceptions.

- To be able to use the exact algorithm for IRV while keeping a reasonable computation time, we exclude all elections with strictly more than 14 candidates.
- We exclude elections whose preferences derive from cardinal preferences (grades or approval values), since this cardinal information is not provided in PrefLib. Indeed, we would need to extrapolate arbitrarily grades from ordinal preferences, as we will see in section 9.2, and these grades would not be conform to the original experiment *a priori*.
- We exclude elections obtained by random sampling from large datasets (PrefLib 4, 11 and 15), considering that the particular realization of such a random drawing is specific to its author.

In PrefLib, files are organized in datasets gathering elections held in similar contexts and coming from a given source. For example, the dataset named PrefLib 1 comprises political elections held in Dublin in 2002.

In this study, we use 151 experiments coming from the political field (PrefLib 1, 5, 8, 16 to 23), professional or associative field (PrefLib 2, 7, 9 and 12) or cognition experiments (PrefLib 24 and 25). PrefLib 1 is a donation by Jeffrey O'Neill, who administrates the website `http://www.openstv.org/`. PrefLib 5, 8 et 16 à 23 come from this website. PrefLib 2 comes from `http://www.debian.org/vote/`. PrefLib 7, 9 et 12 are respectively donations by Nicolaus Tideman, Piotr Faliszewski and Carleton Coffrin. PrefLib 24 and 25 come from Mao et al. (2013). For more information about these datasets, we encourage the reader to visit the website `http://www.preflib.org/`.

Table 9.3 presents an overview of all the experiments we use in this chapter. In total, we rely on 168 experiments, from 10 to almost 300 000 voters and from 3 to 14 candidates.

9.2 Methodology

Let us consider a given experiment, such as the election of the best paper in Algotel. As illustrated in figure 9.1, to the grades of the initial ballots, we add a random noise whose amplitude is negligible compared to the differences between grades. Once this noise is added, we obtain a configuration for the whole population, which we call a *realization*. This way, if a voter was putting a candidate strictly before another one in her original ballot (such as D before E in figure 9.1), this order is still respected because the amplitude of noise is negligible. In contract, if a voter was putting several candidate in ex aequo positions (such as A and B in figure 9.1), they are placed in a random order once the noise is added.

For each experiment, we draw several random realizations. This serve two purposes: evaluate the coherence of our results on the space made richer than

[5] When only incomplete preference are available, another possibility, studied by Konczak and Lang (2005), consists of considering the set of all possible Condorcet winner, the set of all possible winners for a given voting system, etc. However, the authors show that it leads to difficult problem of algorithmic complexity, with important exceptions like the possible winner of a PSR or the possible Condorcet winners.

Data	Experiments	V	C	Ballots
Algotel	1	57	5	Grades 0–10
Bordeaux	1	86	11	Grades 0–20
Paris VII	1	95	10	Appreciations
Bdtheque	12	12–33	3–14	Grades 1–5
Judgment of Paris	2	11	10	Grades 0–20 *
PrefLib 1: Dublin	3	29 988–64 081	9–14	SOI
PrefLib 2: Debian	8	143–504	4–9	SOI
PrefLib 5: Burlington	2	8 980–9 788	6	TOI
PrefLib 7: ERS	75	32–3 419	3–14	SOI
PrefLib 8: Glasgow	21	5 199–12 744	8–13	SOI
PrefLib 9: AGH Course	2	146–153	7–9	SOC
PrefLib 12: T-shirt	1	30	11	SOC
PrefLib 16: Aspen	2	2 487–2 528	5–11	TOI
PrefLib 17: Berkley	1	4 189	4	TOI
PrefLib 18: Minneapolis	2	32 086–36 655	7–9	SOI
PrefLib 19: Oakland	7	11 358–145 443	4–11	TOI
PrefLib 20: Pierce	4	40 031–299 664	4–7	TOI
PrefLib 21: San Francisco	11	24 180–184 046	4–10	TOI
PrefLib 22: San Leandro	3	22 539–25 564	4–7	TOI
PrefLib 23: Takoma	1	204	4	TOI
PrefLib 24: MT Dots	4	794–800	4	SOC
PrefLib 25: MT Puzzle	4	793–797	4	SOC
Total	168	10–299 664	3–14	

Table 9.3: Sum-up of all experiments. SOC: strict orders, complete list. SOI: strict orders, incomplete list. TOI: orders with ties, incomplete list. * Half-points are authorized.

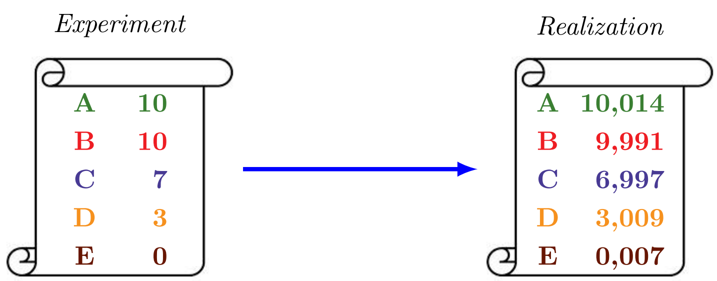

Figure 9.1: Addition of a random noise to the ballots.

the unique original experiment, and break ties in order to simplify the analysis of voting systems based on orders of preference. Hence, when we mention, for example, the CM rate of Plurality for the experiment Algotel, it is the CM rate of Plurality for a culture consisting of drawing a random configuration in a small neighborhood of this experiment.

For the purely ordinal data of each PrefLib experiment, we perform an additional pre-treatment step: first of all, ordinal preferences are converted into cardinal preferences by using Borda scores. Let us notice that this particular choice has no impact but for approval voting and range voting. Indeed, for ordinal voting systems, our technique is anyway equivalent to keep voters' strict preferences and break their indifferences in an impartial way, i.e. symmetric relatively to the candidates. As for the majority judgment, the topological order on the space of grades or appreciation is the only thing that matters; numerical values have no impact.

For the experiment Paris VII, a pre-treatment step is also added. Each of the five possible appreciations is converted to an integer value ranging from -2 to $+2$. Here again, this specific choice has an impact on approval voting and range voting only.

In this chapter, we work at two levels:

- We use the Algotel experiment as a *recurrent example*. When we analyze this example, we use 10 000 realizations.

- We also perform *meta-analyses* on the set of all experiments: in that case, we rely on 100 realizations for each of the 168 experiments.

In both cases, the random aspect of the noise induces a statistical uncertainty of order 1 % on the measured rates ($\sqrt{1/10\,000}$ or $\sqrt{1/16\,800}$).

For the meta-analyses, we will give, on one hand, histograms representing average rates on the 168 experiments. These results must not be interpreted as precise quantitative conclusions but rather as qualitative indications: indeed, conferring the same weight to all 168 experiments is an arbitrary choice. In particular, it gives a relatively large weight (45 %) to the experiments for the dataset PrefLib 7. On the opposite, giving an equal weight to each dataset from PrefLib would lead to give as much importance to datasets with only one experiment, such as PrefLib 12 or PrefLib 17, as we would give to PrefLib 7, from which we draw 75 elections. So, there is no perfect solution.

On the other hand, we will establish graphs of meta-analysis (see section 7.9). They will indicate that a given voting system is less manipulable than another one in all the experiments, which is independent of all weighting on them. Then, we will see that similar trends appear on the whole set of experiments.

Lastly, it is impossible to determine *a posteriori* whether a ballots corresponds to a sincere opinion or it is a more sophisticated strategy. So, we have no choice but assuming that participants were not too far from sincere voting and that ballots (with noise) are sincere. Without this assumption, it would be very difficult to analyze quantitatively the impact of manipulations. That being said, as we have already noticed, some datasets are more likely to be free from strategic voting, in particular the experiments drawn from the bdtheque website.

	A	B	C	D	E	Borda score	Top of ballot
A	—	**35,5**	**36,0**	**36,5**	**39,0**	147,0	16,5
B	21,5	—	**29,5**	**33,0**	**33,0**	117,0	14,7
C	21,0	27,5	—	**32,0**	**31,5**	111,9	9,8
D	20,5	24,0	25,0	—	**29,5**	99,0	10,3
E	18,0	24,0	25,5	27,5	—	95,1	5,7

Table 9.4: Algotel: average matrix of duels, Borda scores and tops of ballots.

9.3 “Sincere” results and Condorcet notions

For the Algotel experiment, table 9.4 gives the matrix of duels, the Borda scores, and the number of ballots where each candidate is on top, on average on the ballots with noise. In the matrix of duels, victories are represented in bold police. We observe, in particular, that A wins all its duels on average: so, it is Condorcet winner in average. In fact, we even observe a stronger property: A is Condorcet winner in all realizations we tested.

As for the other candidates, A is followed by B which loses only against A, then C which loses only against the two first candidates, and so on. So, we have on average a *Condorcet strict total order*: $A \succ B \succ C \succ D \succ E$. In practice, this strict total order was observed in 99 % of the realizations.

So, it is tempting to say that A should be declared the winner, and it happens in almost all cases. The main exception is for Plurality, which elects B in 18% of the cases: since B is on top of almost as many ballots as A (table 9.4), the random noise can change the result. Much more rarely, other exceptions concern Bucklin's method, the two-round system and IRV, which may elect B or C with a very low probability (less than 1‰ of the realizations tested).

These results are relatively unexpected: since it was the first time we made such an experiment, we initially thought that there would be more diversity in the possible winners[6]. They show that if ballots are sincere, A seem to be a canonical winner, attesting a clear choice from the participants.

So, we can wonder whether these phenomena are exceptional. To address this question, let us switch to meta-analysis: as a reminder, we study all 168 experiments, each one with 100 realizations. As shown in figure 9.2, there is, actually, almost always at least a Condorcet-admissible candidate (99 %) and it is very frequent to have a Condorcet winner (96 %). These results confirm and extend those by Tideman (2006), based on dataset that are now gathered in PrefLib 7 and, for this reason, are included in our own study[7].

We have also computed the rate of apparition of stronger phenomena (in the sense that each of them implies the existence of a Condorcet winner). In 79 % of the realizations, there is a Condorcet strict total order, which makes it a quite

[6]For example, in impartial culture with 3 candidates, Merlin et al. (2000) show that the probability that a certain collection of usual voting systems return the same result is about 50 %. Since the assumption $C = 3$ seem rather favorable from this point of view, the difference between this theoretical result and our observation show to which extent the impartial culture should be considered as a worst case.

[7]Tideman's results are obtained with a different methodology, without adding a random noise. Our results, leading to the same qualitative conclusions, are complementary. For the sake of comparison, if we exclude the experiments from PrefLib 7 in order to have an independent study, the existence rates are similar: 98 % for a Condorcet-admissible candidate and 95 % for a Condorcet winner.

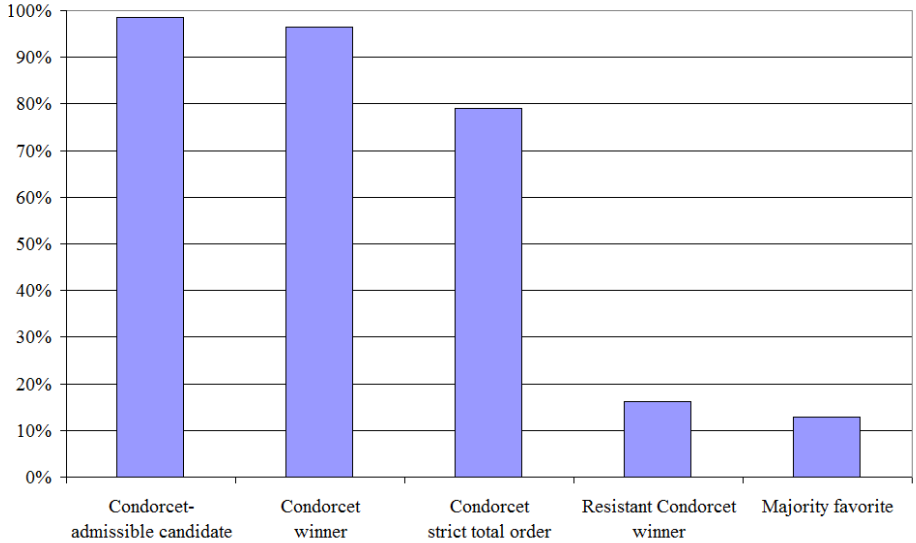

Figure 9.2: Meta-analysis: rate of apparition for certain structures in the preferences of the population.

frequent phenomenon, although this property gives a very rigid structure to the population[8]. The rate of existence of a resistant Condorcet winner is not negligible (16 %), which ensure that in all such cases, any Condorcet voting system is not manipulable. Lastly, there is a majority favorite in 13 % of the cases: most of resistant Condorcet winners we observe in practice are majority favorites, but not all of them.

Like in the Algotel experiment, proponents of the Condorcet criterion will estimate, as a consequence, that most of time, there is a canonical winner who should be elected. So, we can wonder to what extent each voting system is susceptible to violate the Condorcet criterion. Figure 9.3 shows the probability that a Condorcet winner exists but is not elected by a given voting system (in "sincere" voting). Since this rate is equal to zero for all Condorcet voting system, they are not represented in this figure.

Among the non-Condorcet voting systems, IRV (equivalent to EB in sincere voting) presents the best performances: in the dataset under study, it violates the Condorcet criterion in 2 % of the cases only. It is followed by IB and the two-round system (5 %). The worst performances are reached by approval voting (15 %), Plurality (22 %) and Veto (31 %).

[8]The probability of existence for an intransitivity paradox (absence of Condorcet winner, absence of Condorcet strict total order) was intensely studied by theory of computer simulations. In addition to the reference book by Gehrlein (2006), one can notably cite Ruben (1954); Campbell and Tullock (1965); Garman and Kamien (1968); Niemi and Weisberg (1968); DeMeyer and Plott (1970); Pomeranz and Weil Jr (1970); Tullock and Campbell (1970); Gehrlein and Fishburn (1976); Fishburn et al. (1979); Gehrlein (1981); Jones et al. (1995); Gehrlein (1999); Maassen and Bezembinder (2002); Merlin et al. (2002); Tsetlin et al. (2003). Except in single-peaked models, these works generally conclude that the absence of Condorcet winner comes with a relatively important probability. The study of these probabilities in actual experiments, while not being totally new, is now developing thanks to a better access to datasets and seem, in contract, indicate that it is very frequent to have a Condorcet winner, as our study confirms.

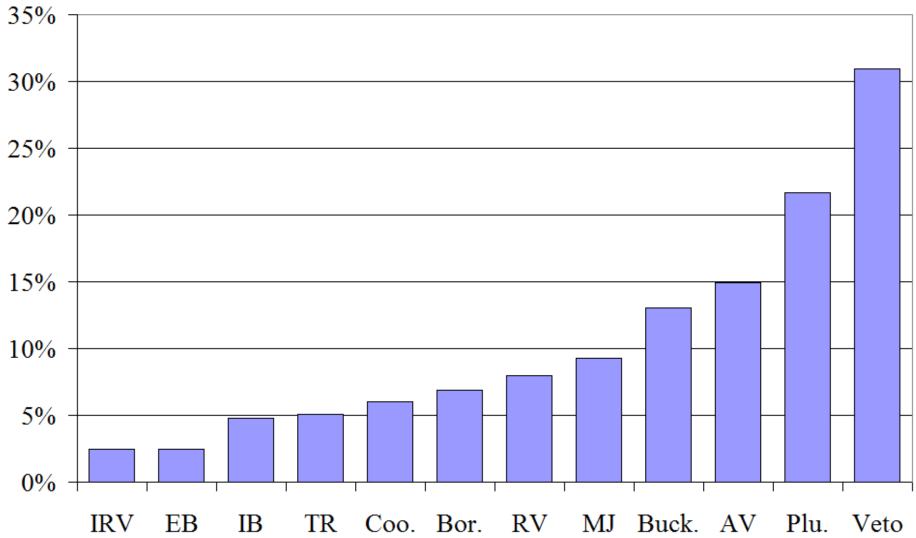

Figure 9.3: Meta-analysis: rate of violation of the Condorcet criterion

For all the voting systems under study except Veto, each case of violating the Condorcet criterion is also a case of trivial manipulability in favor of the Condorcet winner: so, we already know, for example, that the TM rate for Plurality is at least 22 %, if only for that reason.

9.4 Coalitional manipulation

9.4.1 Average CM rates

Let us switch back to the Algotel experiment. In figure 9.4, we represent the CM rate for each voting system. It reads this ways: for example, the two-round system (TR) is manipulable in 30 % of the realizations. Like we said before, it is the CM rate of this voting system in a culture consisting of drawing a random configuration in a small neighborhood of the actual experiment in the space of preferences.

Since some algorithms are approximate, an uncertainty bar is indicating for the corresponding voting systems: for example, the CM rate of Schulze's method is between 25 % and 37 %. For some other voting systems, such as Maximin or Borda's method, the algorithm is not exact[9], but we see that the uncertainty bar has a length equal to zero: in the particular case of this experiment, the algorithm was able to decide manipulability for all the realizations.

For this Algotel experiment, we see that CIRV, IRV and EB are better than all the other voting systems, with a CM rate lower than 1 % (the statistical uncertainty). On the opposite, voting systems with the worst proven performances are Borda's method, Plurality, approval voting, Coombs' method, the majority judgment and range voting, which CM rates close to 100 %.

Now, let us switch to meta-analysis. Average CM rates are represented in figure 9.5. For the voting systems without dedicated algorithm (Baldwin, IRVD,

[9] For these voting systems, we used the `fast` option in SVVAMP.

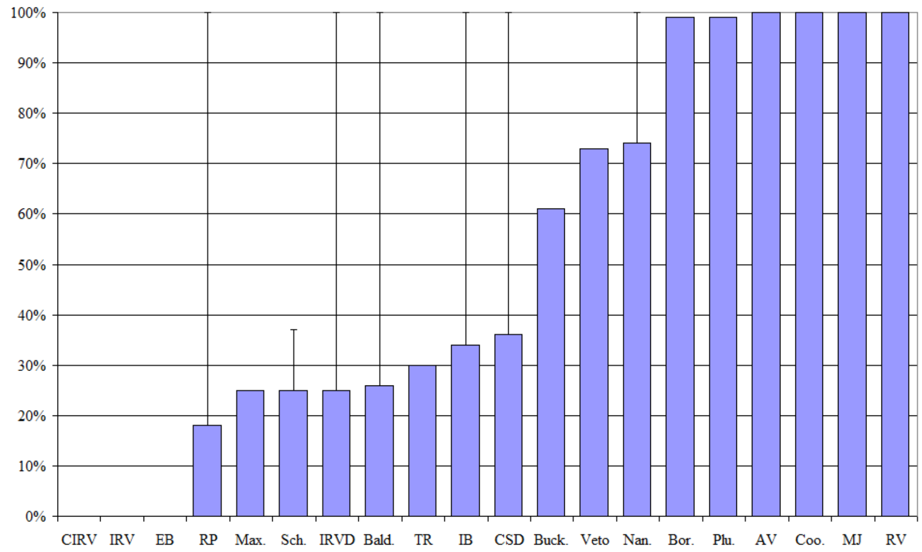

Figure 9.4: Algotel: CM rates

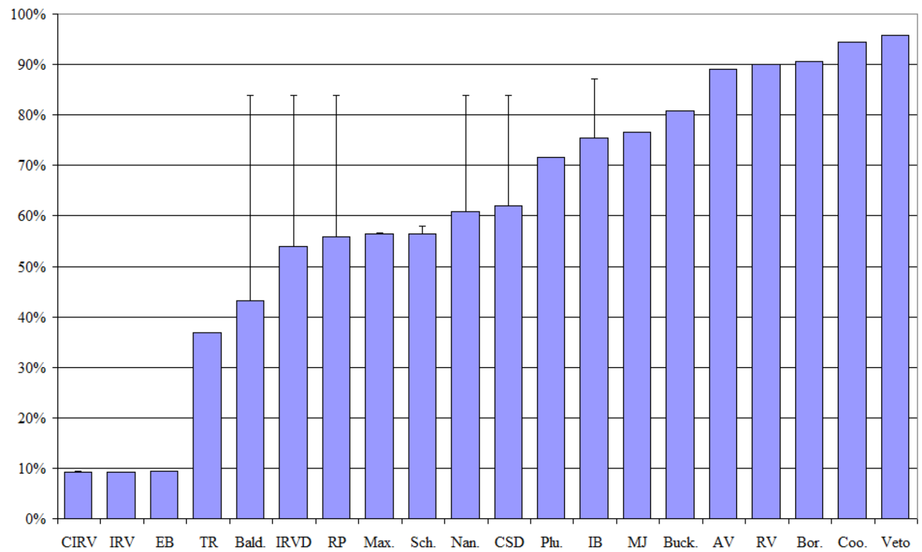

Figure 9.5: Meta-analysis: CM rates.

etc.), the upper bound found by S∀AMP are significantly less than 100 %. To explain it, let us recall that there is a resistant Condorcet winner in 16 % of the realizations: in all these cases, S∀AMP knows that a Condorcet voting system is not manipulable (section 6.3.4). For IB, S∀AMP uses a similar result involving the majority favorite. Let us notice that the algorithms used for CIRV, Maximin and Borda's method, despite being approximate, fail to decide manipulability in less than 1 % of realizations studied here.

Concerning the average CM rate, once again, there is a strikingly good performance of CIRV, IRV and EB (9%). Even if we know, by theory, that CIRV is strictly less manipulable than IRV (corollary 2.21 of Condorcification theorems), which is strictly less manipulable than EB (section 1.4.2), the difference is less than 1 % in this corpus of experiments[10]. This result is similar with what we saw in artificial culture from chapters 7 and 8, except than in unidimensional cultures, there was more frequently a slight difference between IRV and EB.

All other voting systems are significantly more manipulable, even the two-round system with a CM rate of 37 %. Qualitatively, this difference is more similar with what we saw in spheroidal cultures or in multidimensional political spectrum than in unidimensional cultures.

The most manipulable voting systems on average, with rate close to 90 % and even more, are approval voting, range voting, Borda's method, Coombs' method and Veto. These results are similar to those from chapters 7 and 8, except for Coombs' method whose bad behavior is closer to the results obtained in spheroidal cultures or in multidimensional political spectrum than in unidimensional cultures.

We know that, for all voting systems under study except Veto, if the winner is not Condorcet-admissible (whether because there is not one, or because the voting does not designate her), the configuration is necessarily manipulable (lemma 2.7). This lead some authors (such as Lepelley and Valognes, 2003) to think that Condorcet voting systems might be less manipulable. While this seems to be confirmed globally, we must nuance this idea. Indeed, there are great discrepancies, whereas CIRV has a CM rate of 9%, Maximin and Schulze's method have a CM rate between 56 % and 58 %, which makes them significantly more manipulable than IRV (9 %), which does not respect the Condorcet criterion. About this, once again, results are closer to the results obtained in spheroidal cultures or in multidimensional political spectrum than in unidimensional cultures.

[10] Over all the tested realizations, S∀AMP finds an average CM rate of 9,32 % for EB, 9,28 % for IRV and between 9,20 % and 9,28 % for CIRV. So, differences are detectable but their amplitude, lower than the statistical uncertainty, must only be considered as indications.

9.4.2 Comparing the CM of all voting systems

	CIRV	IRV	EB	TR	Max.	Sch.	Bald.	IRVD	RP	Nan.	Plu.	CSD	MJ	IB	Buck.	AV	Bor.	Veto	RV	Coo.
CIRV	—	0 (4)	**0 (4)**	**40**	**62**	**62**	**49 (81)**	**60 (81)**	**61 (81)**	**63 (81)**	**70**	**64 (81)**	**76**	**78 (83)**	**80**	**86**	**88**	**92**	**86**	**90**
IRV	0	—	**3**	**40**	**62**	**62**	**49 (81)**	**60 (81)**	**61 (81)**	**63 (81)**	**70**	**64 (81)**	**76**	**78 (83)**	**80**	**86**	**88**	**92**	**86**	**90**
EB	0	0	—	**40**	**62**	**62**	**49 (81)**	**60 (81)**	**61 (81)**	**63 (81)**	**70**	**64 (81)**	**76**	**78 (83)**	**80**	**86**	**88**	**92**	**86**	**90**
TR	0	0	0	—	**35**	**35 (37)**	**18 (55)**	**30 (55)**	**33 (55)**	**37 (55)**	**45**	**39 (55)**	**50**	**51 (57)**	**55**	**60**	**62**	**67**	**60**	**64**
Max.	0	0	0	5	—	0 (14)	1 (38)	0 (38)	1 (38)	**14 (38)**	**27**	**19 (38)**	**33**	**31 (40)**	**38**	**43**	**45**	**49**	**43**	**47**
Sch.	0	0	0	3 (5)	0 (4)	—	1 (38)	0 (38)	0 (38)	**10 (38)**	**26 (27)**	**13 (38)**	**32 (33)**	**27 (40)**	**33 (38)**	**42 (43)**	**44 (45)**	**49**	**42 (43)**	**46 (47)**
Bald.	0	0	0	1 (10)	0 (30)	0 (31)	—	0 (50)	0 (50)	0 (50)	2 (39)	0 (50)	4 (45)	0 (52)	3 (49)	**16 (55)**	**18 (57)**	**23 (61)**	**16 (55)**	**20 (59)**
IRVD	0	0	0	1 (8)	0 (17)	0 (18)	0 (40)	—	0 (40)	0 (40)	2 (29)	0 (40)	4 (35)	0 (42)	3 (39)	**16 (45)**	**18 (46)**	**23 (51)**	**16 (45)**	**20 (49)**
RP	0	0	0	1 (7)	0 (11)	0 (15)	0 (38)	0 (38)	—	0 (38)	2 (27)	0 (38)	4 (33)	0 (40)	3 (38)	**16 (43)**	**18 (45)**	**23 (49)**	**16 (43)**	**20 (47)**
Nan.	0	0	0	1 (2)	0 (5)	0 (10)	0 (34)	0 (34)	0 (34)	—	2 (23)	0 (34)	4 (28)	0 (36)	3 (33)	**16 (39)**	**18 (40)**	**23 (45)**	**16 (39)**	**20 (43)**
Plu.	0	0	0	0	0	0 (2)	0 (23)	0 (23)	0 (23)	0 (23)	—	1 (23)	**15**	13 (24)	**15**	**26**	**29**	**34**	**27**	**32**
CSD	0	0	0	1	0	0 (6)	0 (33)	0 (33)	0 (33)	0 (33)	2 (20)	—	4 (27)	0 (35)	3 (31)	**16 (38)**	**17 (40)**	**23 (45)**	**16 (38)**	**20 (42)**
MJ	0	0	0	1	0	0	0 (17)	0 (17)	0 (17)	1 (17)	4	1 (17)	—	9 (19)	**11**	**23**	**23**	**27**	**23**	**27**
IB	0	0	0	0 (3)	0 (3)	0 (7)	0 (21)	0 (21)	0 (21)	0 (21)	0 (16)	0 (21)	2 (17)	—	0 (19)	**10 (26)**	**8 (29)**	**14 (33)**	**10 (26)**	**11 (31)**
Buck.	0	0	0	1	0	0 (3)	0 (13)	0 (13)	0 (13)	1 (13)	10	2 (13)	**11**	1 (15)	—	**17**	**20**	**25**	**18**	**23**
AV	0	0	0	1	0	0	0 (6)	0 (6)	0 (6)	0 (6)	3	0 (6)	1	2 (7)	4	—	**8**	**11**	**5**	**10**
Bor.	0	0	0	1	0	0	0 (1)	0 (1)	0 (1)	0 (1)	1	0 (2)	2	0 (5)	0	5	—	**10**	**4**	**9**
Veto	0	0	0	2	4	4	3 (8)	4 (8)	4 (8)	5 (8)	5	4 (8)	8	5 (8)	5	10	9	—	**10**	**10**
RV	0	0	0	1	0	0	0 (2)	0 (2)	0 (2)	0 (2)	2	0 (2)	1	1 (3)	1	2	**4**	8	—	**6**
Coo.	0	0	0	0	0	0	0 (1)	0 (1)	0 (1)	0 (1)	0	0 (1)	1	0 (3)	1	5	1	7	4	—

Table 9.5: Meta-analysis: Comparison of CM by couple of voting systems (in percents).

Like we did in the meta-analyses of chapters 7 and 8, we now compare the voting systems by pairs on the corpus of experiments. Since it is difficult to give detailed results for each experiment, unlike previous chapters where curves made it possible to visualize and compare results for the different voting systems, we provide the table 9.5 that establishes the meta-analysis graph.

This table reads this way. In cell (U2T, Sch.) for example, the first percentage means that it is *certain* that TR has a CM rate strictly lower that Schulze's method in 35 % of experiments (in the sense that the upper bound for the first one is strictly lower than the lower bound for the second one). The percentage inside parenthesis means that it is *possible* that it is so in 37 % of the experiments (in the sense that the lower bound for the first one is strictly lower than the upper bound for the second).

In contract, in cell (Sch., U2T), one can read that it is certain that Schulze's method has a CM rate lower than TR in 3 % of the experiments, and that it is possible in 5 % of the experiments.

With a uniform weighting over the 168 experiments, we can conclude that the duel is won by TR. Indeed, even in the less favorable assumption for TR, it has a CM rate lower to Schulze's method in 35 % of the experiments; whereas Schulze's method has a lower CM rate than TR in 5 % of the experiment. For information, such victories, based on uniform weighting, are indicated in bold police in the table. Because of the uncertainty for some voting systems, we are not able to give the results for all duels between voting systems and we will give complementary results thanks to TM.

Like we did in previous chapter, we present in figure 9.6 the meta-analysis graph, based on victories in all the experiments: an edge from a voting system f to another one g means that it is certain that g has a CM rate stricly lower than f in no experiment. So, this property is independent of any kind of weighting over the 168 experiments.

The most striking conclusion is that CIRV dominates IRV, which dominates EB (which we already knew by theory), and that the later dominates all other voting systems under study in all the experiments. Maximin and Schulze's method, whose approximate algorithm provide a limited uncertainty, constitute a good reference point to compare with other voting systems. Notably, we see that the poorest performances are obtained for Coombs and Borda, CSD, the majority judgment, approval voting and range voting. Poor performances for Borda's method, majority judgment, approval voting and range voting are similar to those in chapters 7 and 8. The following points are closer to behaviors observed in spheroidal cultures and multidimensional political spectra: the supremacy of CIRV, IRV and EB; intermediary performances for Maximin and Schulze's method (whereas they are quite good in unidimensional cultures); and the poor performances for Coombs' method and CSD (whereas they are not so bad in unidimensional cultures).

9.4.3 CM by candidate

To finish this study of CM, we will investigate the manipulability by candidate for the Algotel experiment. In figure 9.7, we represent the manipulability rate in favor of each candidate.

Without doing an exhaustive analysis, some observations seem especially relevant on this figure. First, the low manipulability for A simply comes from the fact that A is almost always the sincere winner, so its proponents have nothing special to do. The cases of manipulability for A, especially in Plurality, correspond to the cases where B, or even C, is the sincere winner. Then, there is always a manipulation for A.

The worst results are obtained for two relatively natural voting systems, approval voting and range voting, which are manipulable in favor of the four opponents of A in all the realizations! The majority judgment, Borda's method, Plurality and Coombs' method present also a high risk of manipulation for a variety of candidates.

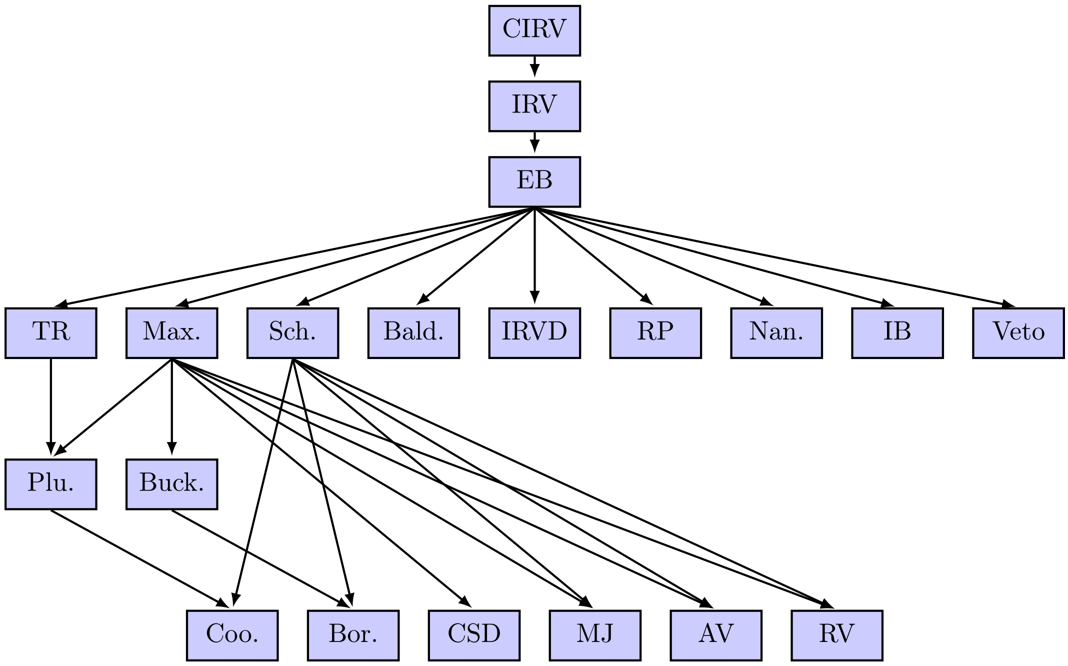

Figure 9.6: Meta-analysis of CM in the experiments.

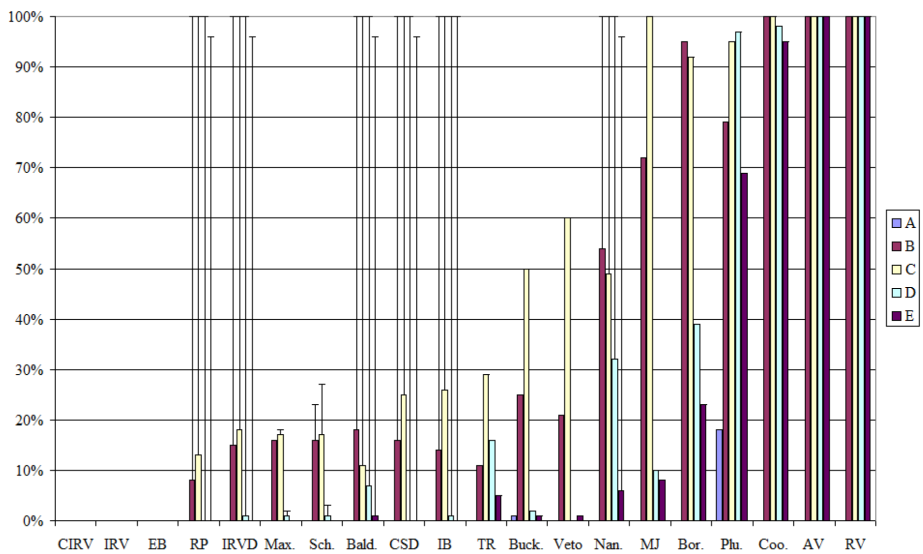

Figure 9.7: Algotel: CM rate in favor of each candidate

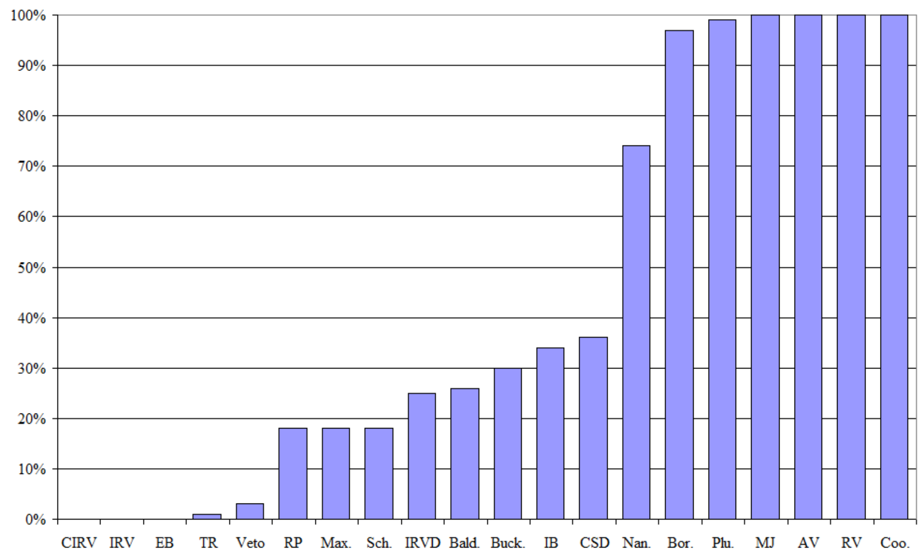

Figure 9.8: Algotel: TM rates

9.5 Trivial manipulation

As we frequently mentioned, trivial manipulability (TM) is an interesting criterion for two reasons. Firstly, it keeps only the cases where manipulation is relatively easy to identify and coordinate, which makes it occurrence more likely in practive. Secondly, its algorithmic simplicity makes it possible to perform exact computation for all voting systems studied here, so we get complementary information to what we saw for CM.

9.5.1 Average TM rates

In figure 9.8, we represent the TM rate for each voting system in the Algotell experiment. Since conclusions from figures 9.8 and 9.9 are qualitatively similar, we are going to comment directly on the meta-analysis from figure 9.9. Four voting systems distinguish themselves clearly, with rates lower than 10 %: CIRV (3 %), IRV and EB[11] (5 %) and the two-round system (8 %). Then, we have Veto (17 %) and Baldwin's method (43 %), then a great deal of voting systems between 50 % and 80 %. Finally, four voting systems have a TM rate close to 90 % or more: approval voting, range voting, Borda and Coombs' method.

As we have already noticed and explained in section 7.2, Veto is the only voting system whose TM rate (17%) is very different from the CM rate (96 %).

[11] Let us recall that IRV and EB are equivalent for the sake of trivial manipulation.

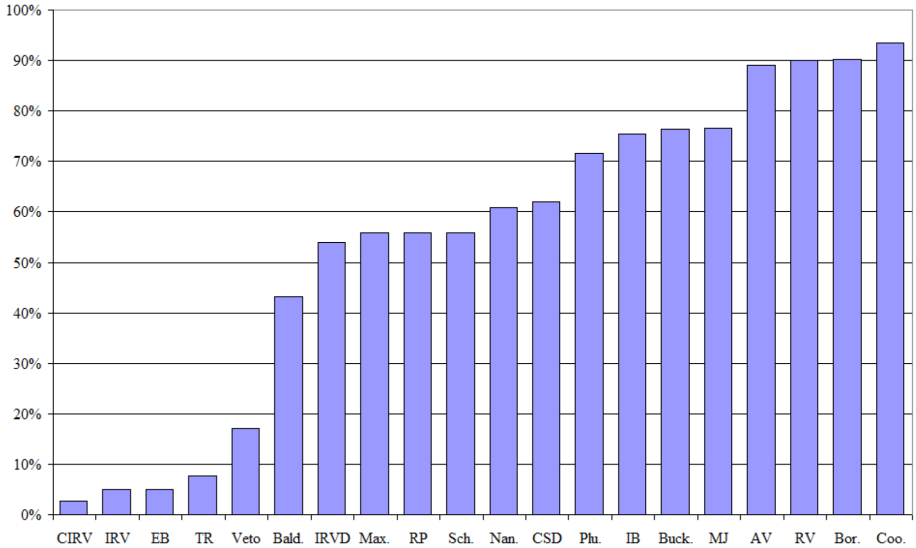

Figure 9.9: Meta-analysis: TM rates

9.5.2 Comparing the TM rates of all voting systems

	CIRV	IRV	EB	TR	Veto	Bald.	IRVD	RP	Max.	Sch.	Nan.	CSD	IB	Plu.	Buck.	MJ	AV	Bor.	Coo.	RV
CIRV	—	**14**	**14**	**16**	**38**	**53**	**64**	**65**	**66**	**66**	**67**	**68**	**82**	**74**	**82**	**80**	**90**	**92**	**94**	**90**
IRV	0	—	**0**	**12**	**38**	**51**	**63**	**64**	**64**	**64**	**65**	**67**	**80**	**73**	**80**	**78**	**88**	**90**	**92**	**88**
EB	0	**0**	—	**12**	**38**	**51**	**63**	**64**	**64**	**64**	**65**	**67**	**80**	**73**	**80**	**78**	**88**	**90**	**92**	**88**
TR	0	1	1	—	**36**	**50**	**61**	**63**	**63**	**63**	**64**	**65**	**79**	**71**	**79**	**77**	**87**	**89**	**91**	**87**
Veto	5	8	8	10	—	**46**	**57**	**57**	**58**	**58**	**60**	**61**	**73**	**68**	**73**	**73**	**80**	**81**	**84**	**80**
Bald.	0	0	0	0	30	—	**25**	**28**	**29**	**29**	**32**	**33**	**46**	**39**	**47**	**45**	**55**	**57**	**59**	**55**
IRVD	0	0	0	1	25	5	—	**15**	**16**	**16**	**20**	**21**	**37**	**29**	**38**	**35**	**45**	**46**	**49**	**45**
RP	0	0	0	1	25	1	3	—	**8**	**8**	**18**	**20**	**34**	**27**	**36**	**33**	**43**	**45**	**47**	**43**
Max.	0	0	0	1	24	2	1	5	—	**2**	**19**	**20**	**35**	**28**	**37**	**34**	**44**	**46**	**48**	**44**
Sch.	0	0	0	1	24	2	1	4	0	—	**18**	**19**	**34**	**27**	**36**	**33**	**43**	**45**	**47**	**43**
Nan.	0	0	0	0	23	0	1	0	2	2	—	**14**	**24**	**23**	**30**	**28**	**39**	**40**	**43**	**39**
CSD	0	0	0	0	23	0	0	0	0	0	2	—	**23**	**20**	**27**	**27**	**38**	**40**	**42**	**38**
IB	0	0	0	1	17	1	0	1	1	1	7	8	—	**16**	**17**	**17**	**26**	**29**	**31**	**26**
Plu.	0	0	0	0	20	0	0	0	0	0	0	1	13	—	**13**	**15**	**26**	**29**	**31**	**27**
Buck.	0	0	0	0	17	1	1	1	1	1	2	5	7	11	—	**13**	**21**	**24**	**27**	**23**
MJ	0	0	0	0	17	0	0	0	0	0	1	1	9	4	8	—	**23**	**23**	**25**	**23**
AV	0	0	0	0	5	0	0	0	0	0	0	0	2	3	2	1	—	**8**	**10**	**5**
Bor.	0	0	0	0	7	0	0	0	0	0	0	0	0	1	0	2	6	—	**8**	**5**
Coo.	0	0	0	0	7	0	0	0	0	0	0	0	0	0	1	1	8	5	—	**7**
RV	0	0	0	0	5	0	0	0	0	0	0	0	1	2	0	1	2	4	6	—

Table 9.6: Meta-analysis: Comparison of TM by couple of voting systems (in percents)

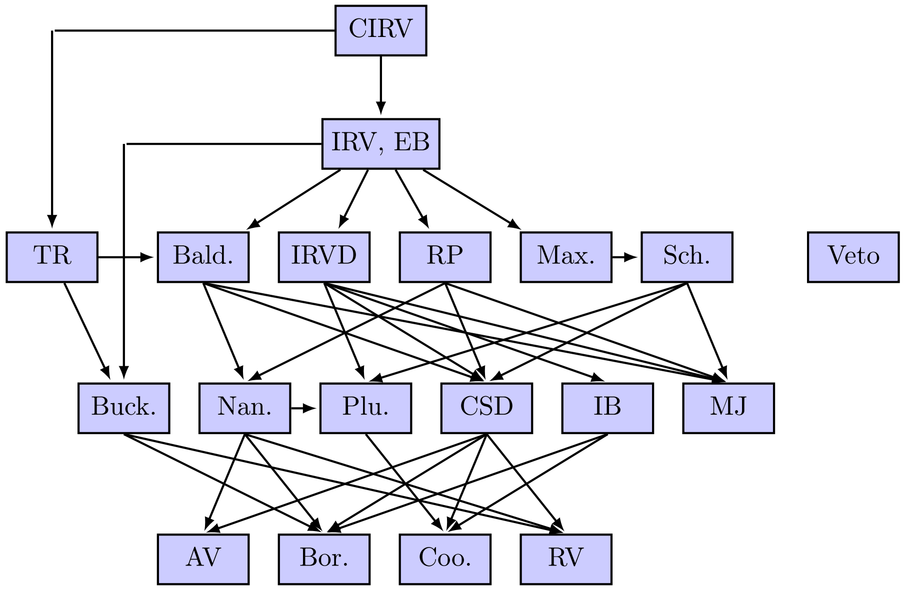

Figure 9.10: Meta-analysis of TM in the experiments.

Like for CM, we provide table 9.6 of meta-analysis for TM. This time, there is no algorithmic uncertainty, which facilitates reading the table and provides more precise results.

In particular, the most performing voting system, CIRV, is never outperformed by any other voting system, except Veto. IRV and EB present similar performances, although they are outperformed by the two-round system in 1 % of the experiments (however, they are better in 12 % of the cases). Among the other voting systems, the two-round system is the one with the best results. Although it is strictly outperformed by CIRV (for TM) in 16 % of the experiments and strictly outperformed by IRV or EB in 12 % of the experiments (versus 1 % for the converse), it is strictly better than Veto in 36 % of the experiments (versus 10 % for the converse) and compared to any other voting system under study, it is only outperformed in 0 to 1 % of the experiments.

Like before, the meta-analysis graph in figure 9.10 is based on dominations in all the experiments, so it is independent of any weighting. We will notably notice that, in our corpus of experiments, CIRV is strictly better that all other voting systems under study, except Veto, which constitutes, by itself, an isolated connected component. In particular, IRV, EB and TR are more manipulable than CIRV. Then we can, as a indication, distinguish three groups, by increasing order of TM.

1. Baldwin's method, IRVD, RP, Maximin and Schulze's method. Once again, these two later present very similar performances (with a slight advantage for Maximin).

2. Bucklin and Nanson's method, Plurality, CSD, IB and the majority judgment. Among these, Nanson is less MT than Plurality.

3. Approval voting, Borda, Coombs and range voting.

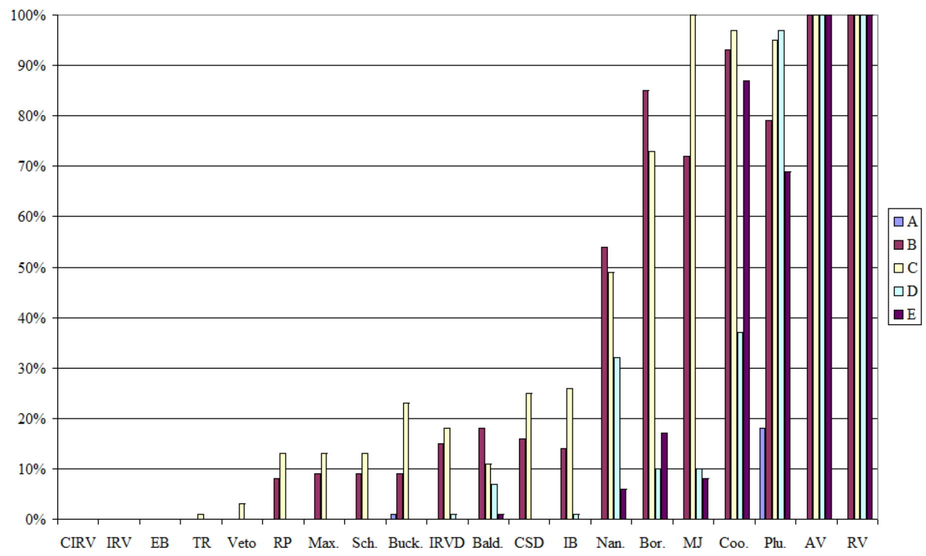

Figure 9.11: Algotel: TM rate in favor of each candidate.

9.5.3 TM by candidate

Like for CM, we examinate in figure 9.11 the manipulability by candidate for the Algotel experiment. Globally, results are similar to those obtained for CM in figure 9.7, in particular the superiority of CIRV, IRV and EB. However, we observe the following differences.

Whereas TR could sometimes be CM for a variety of candidates, it is TM (and quite rarely) only in favor of candidate C. Hence, manipulations in the two-round system need, generally, relatively sophisticated strategies.

Veto is far less TM than CM, which we have already noticed and explained, since the trivial strategy is very sub-optimal in this voting system.

With no surprise, results are the same in CM and TM for the majority judgment, Plurality, approval voting and range voting: for these voting systems, the trivial strategy is optimal in the sens that if a manipulation is possible, then this strategy achieves it. As we discussed as early as the introduction of this memoir, this can be seen as a drawback or as an advantage for these voting systems: manipulation being simple to implement makes it possible to research equilibria with a limited exchange of information. However, we think that the property of being non-manipulable is even more interesting (which is the case for CIRV, IRV and EB in the Algotel experiment, cf. figure 9.7 for example), since it makes it possible to reach an equilibrium with an exchange of information equal to zero, as we discussed notably in chapter 3.

9.6 Synthesis of results

In this chapter, we have analyzed the results from 168 experiments from the real worlds by adding a random noise, which makes it possible to explore the space of preference in the neighborhood of each experiment.

We we able to establish that there is almost always a Condorcet-admissible candidate, very often a Condorcet winner and that the presence of a Condorcet strict total order, which is a very structuring property for the population, is actually a quite frequent phenomenon.

Among the non-Condorcet voting systems, IRV and EB are those violating the Condorcet criterion least often, followed by IB and the two-round system. Those violating it most often are approval voting, Plurality and Veto. Despite these discrepancies, we observe that even non-Condorcet voting systems designate the possible Condorcet winner in a majority of cases (whose size depends significantly on the voting system).

We have seen that manipulability is not only a theoretical concept, but a very concrete reality which is susceptible to happen even in a context where the result seems obvious *a priori*: indeed, a great of voting systems under study are manipulable in a large proportion of experiments.

Even if the voting sytems designate relatively often the same winner in "sincere" voting (the Condorcet winner, when she exists, even for the non-Condorcet voting systems), the choice of the voting system is also crucial for another reason: observed CM rates range from less than 10 % to more than 90 %, depending on the voting system.

CIRV, IRV and EB are the least manipulable ones. By theory, we know that CIRV is strictly less manipulable than IRV, which is strictly less manipulable than EB; but in practice, the gap is very small. For practical applications, it is worth considering preferring IRV to CIRV, because the counting in IRV is easier if there is no computer means. Among the other voting systems, the two-round system is generally the least manipulable. The most manipulable ones are approval voting, Borda, Coombs, range voting and Veto.

The results we obtained are globally closer to those we got in the spheroidal cultures from chapter 7 or in the cultures based on a multidimensional political spectrum from chapter 8 than in unidimensional cultures (chapitre 8 also), althoug our corpus of experiments includes elections from the political field, where the assumption of single-peakedness might seem more realistic *a priori*, or at least a first interesting approximation of reality. While not validating totally these spheroidal or multidimensional models, this conclusion encourages to pursue their study in future works.

Chapter 10

Optimal voting systems

In this chapter, we focus on voting systems whose manipulability rate is minimal in the class of those meeting **InfMC**: by commodity of language, we will say that such a system is *optimal*.

Thanks to corollary 2.23 of Condorcification theorems, we know to for the research of such a voting system, we can restrict our investigation to Condorcet systems. By optimality theorem 5.15, we know that if the culture is decomposable, in particular if voters are independent, then there exists an optimal system and we restrict the investigation to systems that are ordinal and meet the Condorcet criterion.

In section 10.1, we present our technique to treat this question. In section 10.1.1, we define the *opportunity graph* of an electoral space. Independent of the voting system under study, this graph indicates whether a configuration is susceptible of being manipulable towards another one, depending on the candidate who is declared the winner in the two configurations by the voting system we consider using. In the representation, a culture is simply represented by a weighting on the vertices of the graph.

In sections 10.1.2, 10.1.3 and 10.1.4, we show that the research of an optimum gets simplified if we restrict to Condorcet systems, if we assume that semi-admissible configurations come with zero probability and if we restrict to ordinal systems.

In section 10.1.5, we present a greedy algorithm which makes it possible to search for a approximate optimum, but we will see that it is not exact. In section 10.1.6, we show that searching a optimum, with previous assumptions, is equivalent to an integer linear programming optimization problem, which the dedicated software CPLEX can treat for moderate values of the parameters.

In section 10.2, we restrict our investigation to the impartial culture with an odd number of voters, which meets all previous assumptions. We discuss the solutions we find for $V = 3$, 5 or 7 and $C = 3$, as weel as for $V = 3$ and $C = 4$ (sections 10.2.1 to 10.2.4).

In section 10.3, we conclude this chapter by comparing the minimal manipulability rates (in the class **InfMC**) to those obtained for usual voting systems.

10.1 Opportunity graph

10.1.1 Definition

As usual, we work in an electoral space Ω. The *opportunity graph* we define in this section is entirely defined by Ω: it express a structure of the electoral space, which is independent of the voting system that might be considered. In this framework, we will see that a culture is simply represented by weights on the vertices of the graph.

Before defining the opportunity graph, we need to define the (w, c)*-pointing*.

Definition 10.1

Let ω and ψ be two distinct configurations, w and c two distinct candidates. We say that ω (w, c)*-points* to ψ if and only if:

$$\forall v \in \mathrm{Sinc}_\omega(\mathrm{w} \to c), \omega_v = \psi_v.$$

In other words:

$$\forall v \in \mathcal{V}, \omega_v \neq \psi_v \Rightarrow c \ \mathrm{P}_v(\omega_v) \ \mathrm{w}.$$

If a voter presents a different state in ω and ψ, she prefers c to w in ω.

Intuitively, this property means that if w is the winner in ω and if c is the winner in ψ, then ω is manipulable is favor of c towards ψ: indeed, voters who prefer c to w can change their ballots in order to produce configuration ψ.

For example, let us consider the following configurations.

$$\omega: \begin{array}{c|c|c} a & b & c \\ b & c & b \\ c & a & a \end{array} \qquad\qquad \psi: \begin{array}{c|c|c} a & b & c \\ b & c & \mathbf{a} \\ c & a & \mathbf{b} \end{array}$$

Configuration ω (b, c)-points to ψ: indeed, the only voter whose state changes prefers c to b in ω. So, it would be bad for a voting system to designate b in ω and c in ψ: indeed, since ω (b, c)-points to ψ, it would imply that configuration ω is manipulable.

In the above example, configuration ω (a, c)-points and (a, b)-points also to ψ. This (a, b)-pointing is the occasion to stress on the fact that each modified voter (the unique modified voter, in this example) prefers by assumption b to a in the starting configuration ω, but not necessarily in the arrival configuration ψ. So, if a voting designated a in ω and b in ψ, than the last voter on the right would be able to manipulable configuration ω in favor of b by lowering b in her order of preference! Such a situation is not to be excluded *a priori*: some wide spread voting systems, such as the two-round system or IRV, present such non-monotony features.

More generally, we do not have only an implication from pointing to manipulability but the following equivalence, which is a simple translation of the definition of manipulability.

Proposition 10.2

Let $f : \Omega \to \mathcal{C}$ be an SBVS, ω and ψ two distinct configurations.

System f is manipulable in ω towards ψ if and only if ω $(f(\omega), f(\psi))$-points to ψ.

This remark will allow us to reword the questions of manipulability into graph questions.

Definition 10.3

Let us consider the labeled multigraph $(\Omega, E, e : E \to \mathcal{C}^2)$, defined the following way.

- Vertices are the states $\omega \in \Omega$.
- E is the set of edges.
- e is a function that, to each edge, associates a label, which is a pair of candidates.
- A vertex ω has an edge with label (w, c) to a vertex ψ if and only if configuration ω (w, c)-points to ψ.

This labeled multigraph is called the *opportunity graph* of electoral space Ω.

It is a multigraph: *a priori*, it is possible that a vertex ω has several edges to a vertex ψ with different labels, as we have already seen.

Then, a SBVS $f : \Omega \to \mathcal{C}$ is seen as a function that, to each vertex ω of the opportunity graph, associates an element in $\mathcal{C}$. The manipulability indicator of f in ω, which we have already denoted $\mathrm{CM}_f(\omega)$, is a boolean function over the vertices whose value is 1 in a vertex ω if and only if there exists at least one vertex ψ in direction of which ω has an edge with label $(f(\omega), f(\psi))$.

The opportunity graph is a representation that makes it possible to study the manipulability rate: if we consider a probability law π over the electoral space, it is sufficient to give each vertex ω a weight $\pi(\omega)$. Then, the manipulability rate of f is the total weight of manipulable vertices according to f:

$$\tau_{\mathrm{CM}}(f) = \sum_{\omega \in \Omega} \mathrm{CM}_f(\omega)\pi(\omega).$$

Searching for a voting system whose manipulability rate is minimal and respecting a given constraint (for example **Cond**) means searching for a function f that minimizes the total weight of manipulable vertices among those meeting this constraint.

The problem gets simpler at least in the three following cases, which make it possible to "clean up" the graph, i.e. to remove irrelevant edges without altering its validity for researching an optimum.

- If the constraint under study implies that a configuration ω designates a certain winner w_0, then for $\mathrm{w} \neq \mathrm{w}_0$, it is useless to consider edges (w, c) that are outgoing from ω.
- If we know that a configuration ω is necessarily manipulable (because of the constraint under study), then it is useless to consider the outgoing edges from ω.
- If an event $A \subset \Omega$ has zero probability, then for any configuration $\omega \in A$, it is useless to consider its outgoing edges: indeed, making ω manipulable does increase the manipulability rate. In contrast, its incoming edges are important, because ω could make manipulable some configurations ω' that come with nonzero probability and point to ω.

	ψ Condorcet	ψ semi-Condorcet	ψ non-admissible
ω Condorcet (winner w_0)	no	for $\mathrm{w} = \mathrm{w}_0$	
ω semi-Condorcet	pour any w		
ω non-admissible	no		

Table 10.1: Searching an optimal Condorcet voting system: (w, c)-pointings to consider from ω to ψ.

10.1.2 Restriction aux modes de scrutin Condorcet

By the corollary 2.23 of Condorcification theorem, we know that it is interesting to search a voting system that, in the class of those meeting **Cond**, has a minimal manipulability. Indeed, such a voting system will also also a minimal manipulability in the larger class of voting systems meeting **InfMC**.

For this specific constraint (**Cond**), the two last simplifications seen above become the following ones.

- If ω has a Condorcet winner w_0, we know that w_0 is necessarily the winner in ω hence for $\mathrm{w} \neq \mathrm{w}_0$, it is useless to consider outgoing edges (w, c).
- If ω is a non-admissible configuration, we know that it is necessarily manipulable, hence it is useless to consider its outgoing edges.

Moreover, if ω and ψ both have a Condorcet winner, then ω cannot be manipulable toward ψ, hence it is useless to consider edges from ω to ψ.

Table 10.1 sums up which edges should be considered in order to search for a Condorcet voting system whose manipulability is minimal.

In short, Condorcet configurations get simplified because we already know the winner; and non-admissible configurations, because we already know their manipulability (equal to True). By the way, we can notice that any resistant configuration is also a simple case, because we also know its manipulability (equal to False); in practice, such a configuration has no outgoing edge, once the graph is cleaned up as indicated.

Semi-Condorcet configurations (with at least one Condorcet-admissible candidate, but with no Condorcet winner) are the most complicated ones: indeed, neither their winner nor their manipulability is known *a priori*.

10.1.3 Semi-Condorcet configurations with zero probability

Now, we add the assumption that the set of semi-Condorcet configurations has zero probability. It is the case, in particular, if the number of voters is odd and preferences are almost surely strict total orders. About this last assumption, it may be so because other kinds of binary relations are considered impossible in the electoral space under study, like the one of strict total orders, or because they are authorized but come with zero probability in the culture under study: especially, it is the case if preferences derive from utilities and if the probability that two utilities are equal is zero, like in all cultures from chapters 7 and 8. So, it is an important particular case.

As we noticed in section 10.1.1, this implies that we can ignore outgoing edges from semi-Condorcet configurations, because their manipulability has no impact on the manipulability rate.

	ψ Condorcet	ψ semi-Condorcet	ψ non-admissible
ω Condorcet (winner w_0)	no		for $\mathrm{w} = \mathrm{w}_0$
ω semi-Condorcet	no		
ω non-admissible	no		

Table 10.2: Searching an optimal Condorcet voting system: (w, c)-pointings to consider from ω to ψ, if semi-Condorcet configurations have zero probability. Without altering the optimality, we impose that in each semi-Condorcet configuration, the winner is an arbitrary Condorcet-admissible candidate.

However, *a priori*, a semi-Condorcet configuration ψ might make manipulable a configuration ω having a nonzero probability. In order to avoid this risk, we add the following constraint: for any semi-Condorcet configuration, the winner is an arbitrary Condorcet-admissible candidate. So, a semi-Condorcet configuration cannot make manipulable neither a non-admissible configuration (which is manipulable anyway) nor a Condorcet configuration (because the candidate who would benefit from such a manipulation would still have a defeat against the Condorcet winner, so she could not be Condorcet-admissible, lemma 2.6). By imposing this additional property, we can also ignore the incoming edges of the semi-Condorcet configurations, and finally we can totally ignore these configurations.

For the reseach of a voting system with minimal manipulability rate, this assumption does not alter the optimality of the voting system we will obtain: indeed, for any optimal voting system, previous remarks show that, if we modify it by designating a Condorcet-admissible candidate in any semi-Condorcet configuration, then the modified system has exactly the same manipulability rate.

Table 10.2 sums up the edges that need to be considered with these new assumptions. Now, it is a bipartite labeled graph, because edges comes only from Condorcet configurations and og only to non-admissible configurations.

Now, when ω (w, c)-points to ψ, it is useless to precise w, since in the cleaned-up graph, the candidate considered is necessarily the Condorcet winner in ω. As a consequence, we will simply say that ω c-points to ψ (which implies that ω is a Condorcet configuration and ψ a non-admissible configuration).

Figure 10.1 gives a representation of such an opportunity graph. It is a simplified example, which does not necessarily correspond to an electoral space in particular. Each configuration ω^i is Condorcet, each configuration ψ^j is non-admissible. Semi-Condorcet or resistant configurations are not represented because they have no impact on our problem. In order to represent graphically the edge with label c from a configuration ω to a configuration ψ, we give ψ^j a "plug" denoted $\psi^j.c$ and we connect configuration ω^i to the plug $\psi^j.c$.

This graph reads this way: for example, in configuration ψ^2, if candidate 2 is declared the winner by a given voting system, then configurations ω^4 and ω^5 become manipulable: we will also say that they are *contaminated*. The problem is equivalent to choosing exactly one plug for each configuration ψ, while minimizing the total weight of the contaminated configurations (or more simply their cardinal, if Condorcet configurations are in finite number and endowed with uniform probability).

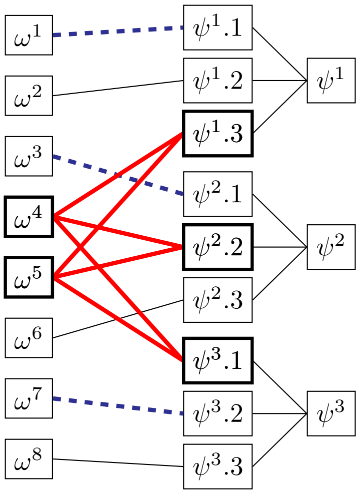

Figure 10.1: Type of graph used to search an optimal Condorcet voting system, in a discrete electoral space where semi-Condorcet configurations have zero probability. Dashed blue line: a possible result of the greedy algorithm. Bold red line: the global optimum.

10.1.4 Restriction to ordinal voting systems

Sling theorems (5.9 and 5.10) and their combinations with Condorcification theorems (5.12 and 5.15) suggest to search an optimum among voting systems that, not only respect the Condorcet criterion, but are also ordinal (in the sense that they depend only on binary relations of preferences, even if these are not orders). Indeed, if the culture is decomposable, such an optimum will have a minimal manipulability rate in the larger class of voting system that meet **InfMC** and are not necessarily ordinal (optimality theorem 5.15). So, we can add the assumption that the opportunity graph is finite. In that case, there exists a finite number of possible voting systems, so there is necessarily at least one voting system whose manipulability rate is minimal, as we have already noticed in proposition 5.14.

10.1.5 Greedy algorithm

It is easy to conceive a greedy minimization algorithm proceeding by local optimization: for each non-admissible configuration ψ^j, we choose the plug $\psi^j.c$ with least incoming edges. Such an approximate solution is represented in dashed blue line in figure 10.1. In this example, we would choose the candidate 1 or 2 for configuration ψ^1 (which contaminates ω^1 or ω^2), 1 or 3 for configuration ψ^2 (which contaminates ω^3 or ω^6) and 2 or 3 for configuration ψ^3 (which contaminates ω^7 or ω^8). This way, 3 configurations would be contaminated.

However, this algorithm is not optimal for a general graph of this type: in figure 10.1, we represented in red bold line the optimal solution, which is strictly better. By choosing candidate 3 for ψ^1, 2 for ψ^2 and 1 for ψ^3, we contaminate only 2 configurations, ω^4 and ω^5.

In the general case of a labeled multigraph, such as the one present in figure 10.1, it is likely that the problem is $\mathcal{NP}$-difficult. This does not exclude, *a priori*, that the problem can simplify itself, for example, for the subset of graphs obtaines as opportunity graphs for all electoral spaces of strict total orders (for any V odd and any C). But it would be surprising that we avoid such pathological situation, like in figure 10.1, and we can already expect that this greedy algorithm is not optimal.

10.1.6 Exact approach: integer linear programming optimization

To sum up, we are looking for voting systems meeting **InfMC** and whose manipulability rate is minimal. We can restrict the investigation to ordinal voting systems (if the culture is decomposable) and meeting the Condorcet criterion. By the way, we assume that semi-Condorcet configurations have zero probability and, in these configurations, we impose that the winner is an arbitrary Condorcet-admissible candidate, with no impact on the optimality of the voting system we will obtain.

In this framework, our problem is equivalent to an integer linear programming optimization problem, in the following way.

- For each pair (ψ, c), where ψ is a non-admissible configuration and c a candidate, we declare the integer variable $W(\psi, c)$ whose value is 1 if c is declared the winner in ψ and 0 otherwise.
- For each Condorcet configuration ω, we declare the integer variable $\mathrm{CM}(\omega)$ whose value is 1 if ω is manipulable and 0 otherwise.
- For each non-admissible configuration ψ, there is only one winner, which translates to the constraint $\sum_c W(\psi, c) = 1$.
- For each triple (ω, ψ, c) such that ω c-points to ψ, we know that if $W(\psi, c) = 1$, then $\mathrm{CM}(\omega) = 1$, which translates to the constraint $\mathrm{CM}(\omega) \geq W(\psi, c)$.
- The objective is to minimize $\sum_\omega \pi(\omega)\mathrm{CM}(\omega)$.

The advantage of this formulation is that there exists generic softwares implementing efficient algorithms to solve integer linear programming problems. During different stages of our work, we used AIMMS first, then IBM ILOG CPLEX Optimization Studio, which both use CPLEX engine.

The drawback of our problem is that it need a large number of variables. For example, in the electoral space of strict total orders with V odd, it is between $(C!)^V$ and $C \times (C!)^V$. Indeed, there is a variable for each Condorcet configuration and C variables for each non-admissible configuration (there is no semi-Condorcet configuration in that case). So, the problem is reasonably treatable only for very small values of the parameters. However, we will see that the case we can exploit in practice are already very instructive about optimal voting systems in general.

10.2 Optimal voting systems for small values of V and C

Now, we consider the impartial culture: voters have almost surely a strict total order of preference, they are independent, and the culture is neutral (and anonymous). This culture meets all previous assumptions. In this particular case, there is no semi-Condorcet configuration, hence non-admissible configurations are exactly non-Condorcet configurations. By the way, configurations are equiprobable, hence minimizing the total weight of the contaminated configurations amounts to minimizing their cardinal.

10.2.1 $V = 3$ and $C = 3$: a lot of optima

For $V = 3$ voters and $C = 3$ candidates, there are $(C!)^V = 216$ configurations in the electoral space and $C^{\text{card}(\Omega)} \simeq 10^{103}$ possible voting systems in total (including those not meeting the Condorcet criterion). However, it is possible to deal with the question manually, which is interesting to give intuition and will allow us to prove a more general result than only finding an optimum by the brute force of CPLEX.

Lemma 10.4

Out of the $(C!)^V = 216$ possible profiles, there are 12 non-Condorcet profiles. It it the following one:

$$\psi : \begin{array}{c|c|c} a & b & c \\ b & c & a \\ c & a & b \end{array}$$

and its variants by permutation of voters and/or candidates.

Configuration ψ is a minimal example of the classic Condorcet paradox: a defeats b, who defeats c, who defeats a.

Proof. For a profile to be non-Condorcet, it is necessary that candidates on tops of ballots are distinct: otherwise the candidate who is on top twice is Condorcet winner. Up tot exchanging b and c, let us assume that the voter who places a on top prefers b to c. When counting also the voter who puts b on top, b has already a victory against c; so, it is necessary to have a victory for a against b and one for c against a. Then, it is easy to conclude that the configuration is ψ. □

To define a Condorcet voting system, one only has to choose the winner in each non-Condorcet profile. So, there are "only" $C^{12} = 531\,441$ Condorcet voting systems.

Before going further on the case $V = 3$ and $C = 3$, we remark the following easy lemma, which is true in general, not only for these values of the parameters.

Lemma 10.5

In order for a configuration ω, with a Condorcet winner w*, c-points to a configuration ψ, it is necessary that* w *is preferred to c in ψ by a strict majority of voters.*

Proof. It is simply a rewording of lemma 2.6. In ω, a majority of voter prefer w to c. But these voters cannot change their state in ψ. □

Now, we are going to use a series of quite simple lemmas in order to deal with the case $V = 3$ and $C = 3$.

Lemma 10.6

Configuration ψ below is a-pointed only by the following profile:

$$\omega : \begin{array}{c|c|c} a & b & c \\ c & c & a \\ b & a & b \end{array} ,$$

whose Condorcet winner is c.

Proof. Let ω be a profile that a-points to ψ and w its Condorcet winner. By lemma 10.5, it is necessary that w has a victory against a in ψ: so, this candidate w is necessarily c. In ω, there are at least two voters who prefer c to a, and by a-pointing, their ballots do not change in ψ. Since only the two voters on the right prefer c to a in ψ, they are necessarily the ones: so, they have the same ballot in ω. Remains to be determined the ballot for the voter on the left, knowing that, by assumption, she prefers a to c. If she puts b on top, then b is Condorcet winner (excluded) and if she places b in the middle of herr list, then it is configuration ψ. So, she puts b in the end of her ballot. □

We deduce immediately that ψ is b-pointed or c-pointed by one profile only, obtained from ω by permutation of the roles.

Lemma 10.7

With previous notations, configuration ω, whose Condorcet winner is c, points only to configuration ψ.

Proof. Let us assume that ω b-points to a certain configuration. In this one, the first and the last voter, who prefer c to b, are unmodified. Since their ballots are not circular permutations of each other, it is impossible to obtain a non-Condorcet profile this way (lemma 10.4).

Now, let us assume that ω a-points to a certain configuration. In this one, the two last voters, who prefer c to a, are unmodified. But we know (lemma 10.4) that, in order to obtain a non-Condorcet configuration, ballots must be circular permutations of one another, hence the only possible configuration is ψ. □

Proposition 10.8

In impartial culture, for $V = 3$ and $C = 3$, all SBVS meeting the Condorcet criterion have the same manipulability rate: $\frac{24}{216} = \frac{1}{9} \simeq 11{,}11$ % .

Proof. The connected component of profile ψ contains exactly three Condorcet profiles, ω, ω' and ω'', as represented in figure 10.2. Indeed, only these three profiles point to ψ (lemma 10.6) and none of them points to another profile (lemme 10.7).

As a consequence, the graph contains exactly 12 non-singleton connected components. Other non-trivial components are obtained from figure 10.2 by permutating voters and/or candidates.

In this component, whatever winner we choose for ψ, it contaminates one, and exactly one, Condorcet profile. Hence, manipulable profiles are the 12 non-Condorcet profiles and the 12 contaminated Condorcet profiles: the manipulability rate is $\frac{24}{216}$. □

In section 2.8, we had written without proof that, in the electoral space of strict total orders, their is no Condorcet SBVS that reached the upper bound of manipulability we gave, i.e. that is manipulable in all non-resistant configurations. Figure 10.2 proves this fact: indeed, profile ψ cannot contaminate in the same time ω, ω' and ω''. Since these profiles have no other way to be contaminated (lemma 10.7), they cannot be manipulable in the same time. By the way, it is easy to prove that these profiles are non-resistant, since by the graph in figure 10.2, each of them can be manipulable.

10.2.2 $V = 5$ and $C = 3$: CVTI and that's all, folks!

For $V = 5$ voters and $C = 3$ candidates, it is difficult to guarantee the global optimum manually, but we can, at least, perform the greedy algorithm manually.

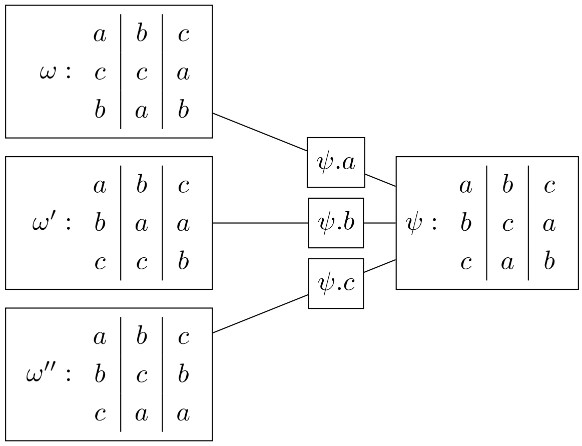

Figure 10.2: A connected component of the graph for $V = 3$ and $C = 3$.

In fact, we will see that, in this case, the result is the same. The manual approach will allow us to have a better understanding of the optimal voting system.

Among $(C!)^V = 7\,776$ possible profiles, it is easy to show, by manual enumeration up to symmetries, that 540 profiles are non-Condorcet. On one hand, there is the following profile:

$$\psi: \begin{array}{c|c|c|c|c} a & a & b & b & c \\ b & b & c & c & a \\ c & c & a & a & b \end{array}$$

and its 180 variants by permuting voters and/or candidates. On the other hand, there is the following profile:

$$\chi: \begin{array}{c|c|c|c|c} a & a & b & b & c \\ b & c & c & c & a \\ c & b & a & a & b \end{array}$$

and its 360 variants by permuting voters and/or candidates. The first profile ψ has half as much variants as the second one χ because it has an additional symmetry, consisting of exchanging the two voters on the left.

With the same techniques as for $V = 3$ voters, it is humanly possible to prove that there exists only one profile that a-points to ψ, only one profile that a-points to χ, and that it is the same profile ω:

$$\omega: \begin{array}{c|c|c|c|c} a & a & b & b & c \\ c & c & c & c & a \\ b & b & a & a & b \end{array},$$

whose Condorcet winner is c.

It is not more difficult to show that in contast, there are strictly more than one profile that b-point (or c-point) to ψ (or χ). Then, when using the greedy algorithm, one must have $f(\psi) = f(\chi) = a$, which makes only profile ω manipulable. By the way, this profile ω is also made manipulable by the profile obtained by exchanging the two voters on the lelft in χ. As a consequence, the greedy algorithm contaminates ω and its variants by permutation, i.e. 180 Condorcet profiles. In total, there are $360 + 180 + 180 = 720$ manipulable profiles and the manipulability rate is $\frac{720}{7\,776} = \frac{5}{54} \simeq 9{,}26$ %.

Using CPLEX, we establish that in fact, it is a global optimum. Then, in order to know whether this optimum is unique, we require CPLEZ to find the optimum either with the additional constraint $W(\psi, a) = 0$, or with the additional constraint $W(\chi, a) = 0$, i.e. we forbid that a is winner in ψ or in χ. In both cases, we see that the optimal manipulability rate is strictly more than $\frac{720}{7\,776} = \frac{5}{54}$. So, it is necessary, for an optimal voting system, to designate a in ψ and in χ. By an argument of symmetry, the possible winner is also unique, for any optimal voting system, in any profile obtained from ψ or χ by permuting voters and/or candidates. So, the optimum we exhibited is unique.

By the way, we observe that the voting system we obtained is CIRV: indeed, it is sufficient to check that, in profiles ψ and χ above, the voting systems returns the same result as VTI, which is true. For $V = 5$ and $C = 3$, since there is never a tie between several candidates in CIRV (or IRV), the question of the tie-breaking rule vanishes: CIRV defines perfectly the solution in a unique way, and this solution is anonymous and neutral. We sup up all these observations in the following proposition and its immediate corollary.

Proposition 10.9

We consider the electoral space of strict total orders for $V = 5$ and $C = 3$, endowed with the impartial culture.

Among SBVS meeting the Condorcet criterion, CIRV is the only system having a minimal manipulability rate.

Corollary 10.10

We consider a probabilized electoral space with $V = 5$ and $C = 3$. We assume that the culture is decomposable (for example, because voters are independents) and that the probability law induced on the profile P *is the impartial culture.*

Among the voting systems meeting **InfMC**, *CIRV has the minimal manipulability rate, which is equal to $\frac{720}{7\,776} = \frac{5}{54} = \simeq 9{,}26$ %. It is the only system that meets this property, is an SBVS and respects the Condorcet criterion. Moreover, it is anonymous and neutral (cf. section 1.2.2).*

By the way, we are going to see that the optimal voting system cannot, in this case, rely only on the matrix of duels. Indeed, profile χ seen above has the following matrix of duels:

$D(\chi)$	a	b	c
a	–	3	2
b	2	–	3
c	3	2	–

This matrix is invariant when applying to the candidates the circular permutation $(a \to b \to c \to a)$. In particuler, it is the same matrix of duels for this profile:

$$\chi' : \begin{array}{c|c|c|c|c} b & b & c & c & a \\ a & c & a & a & b \\ c & a & b & b & c \end{array},$$

which is obtained from χ by this permutation.

The optimal voting system, which we proved to be CIRV for these parameters, cannot depend only on the matrix of duels: it designates a in χ and b in χ'.

More formally, let us consider a voting system f whose result depends only on the matrix of duels. This voting system may use a tie-breaking rule based on the identity of the candidates and/or the matrix of duels, but not on voters' detailed preferences: for example, it can break ties by lexicographical order on the candidates, as in SVAMP. Then, such a voting system f must designate the same winner in χ and in χ', so it is not optimal.

We think that an important consequence of this case study is that in the class **InfMC**, no voting system whose result depends only on the matrix of duels can be optimal for $V = 5$ and $C = 3$. Indeed, we have just proved this affirmation for Condorcet voting systems. For a non-Condorcet voting system that depends only on the matrix of duels, we just have to notice that its Condorcificaiton (which is at most as manipulable as the original system) depends only on the matrix of duels too; so it cannot be optimal. Notably, this observation excludes Condorcet-dean, Baldwin, Borda, Black (Condorcet-Borda), CSD, Kemeny, Maximin, Nanson, RP and Schulze's method.

10.2.3 $V = 7$ and $C = 3$: choose the tie-breaking rule for CIRV, and choose wisely

For $V = 7$ voters and $C = 3$ candidates, it becomes extremely painful to deal with the problem manually. There are $(C!)^V = 279\,936$ profiles in total. And more importantly, there are 7 types of non-Condorcet profiles (up to permuting voters and/or candidates), instead of one or two types in previous cases. Let us recall that, for each of these non-Condorcet profiles, we need to consider all Condorcet profiles pointing to it.

So, we use a computer to do the trick. The greedy algorithm finds a voting system with a manipulabilty rate of 13,46 %. CPLEX finds an optimal solution with a rate equal to $\frac{31\,920}{279\,936} = \frac{665}{5832} \simeq 11{,}40$ %. So, the greedy algorithm is not optimal in that case.

In order to study the set of solutions, we can use CPLEX and impose additional constraints. For example, we can impose the value $W(\psi, c) = 0$ for a given ψ and a given c: it the optimum is not reachable in that case, it means that c is necessarily the winner in ψ.

We can also search for voting systems that are optimal in the smaller class of those that are anonymous, neutral or both. For example, in order to find an anonymous optimum, we quotient the opportunity graph by the equivalence relation consisting, for two configurations, to be deduced from each other by permuting some voters. In order to find a neutral optimum, we quotient the opportunity graph by permuting candidates; then, we need to be careful to modify c-pointing accordingly.

The minimal manipulability rate found by CPLEX for an anonymous and neutral voting system is $\frac{31\,920}{279\,936} = \frac{665}{5832} \simeq 11{,}40$ %: hence, it is an optimum, even among voting systems that are not necessarily anonymous and/or neutral.

Table 10.3 sums up the main conclusions of this study. For each of the 7 types of non-Condorcet profiles (up to permuting voters and/or candidates), we give a necessary condition on the winner so that the voting system is optimal, and we do the same with the additional constraint of anonymity and neutrality.

Let us take profile ψ_4 as an example. For an optimal voting system, we read in column f from table 10.3 that the winner is necessarily a or b. If, moreover, we demand that the voting system is anonymous and neutral, then we read in column $f_{\text{a\&n}}$ from table 10.3 that the winner must be b. It is far from obvious,

Non-Condorcet profile	Matrix of duels	f	$f_{\text{a\&n}}$	CIRV
$\psi_1 : \begin{array}{c\|c\|c\|c\|c\|c\|c} a & a & a & b & c & c & c \\ c & c & c & a & b & b & b \\ b & b & b & c & a & a & a \end{array}$	$\begin{array}{\|c\|c\|c\|c\|} \hline D(\psi_1) & a & b & c \\ \hline a & - & 3 & 4 \\ \hline b & 4 & - & 1 \\ \hline c & 3 & 6 & - \\ \hline \end{array}$	a	a	a
$\psi_2 : \begin{array}{c\|c\|c\|c\|c\|c\|c} a & a & a & b & c & c & c \\ b & c & c & a & b & b & b \\ c & b & b & c & a & a & a \end{array}$	$\begin{array}{\|c\|c\|c\|c\|} \hline D(\psi_2) & a & b & c \\ \hline a & - & 3 & 4 \\ \hline b & 4 & - & 2 \\ \hline c & 3 & 5 & - \\ \hline \end{array}$	a	a	a
$\psi_3 : \begin{array}{c\|c\|c\|c\|c\|c\|c} a & a & a & b & c & c & c \\ b & b & c & a & b & b & b \\ c & c & b & c & a & a & a \end{array}$	$\begin{array}{\|c\|c\|c\|c\|} \hline D(\psi_3) & a & b & c \\ \hline a & - & 3 & 4 \\ \hline b & 4 & - & 3 \\ \hline c & 3 & 4 & - \\ \hline \end{array}$	a	a	a
$\psi_4 : \begin{array}{c\|c\|c\|c\|c\|c\|c} a & a & a & b & b & c & c \\ c & c & c & a & c & b & b \\ b & b & b & c & a & a & a \end{array}$	$\begin{array}{\|c\|c\|c\|c\|} \hline D(\psi_4) & a & b & c \\ \hline a & - & 3 & 4 \\ \hline b & 4 & - & 2 \\ \hline c & 3 & 5 & - \\ \hline \end{array}$	a, b	b	a, b
$\psi_5 : \begin{array}{c\|c\|c\|c\|c\|c\|c} a & a & a & b & b & c & c \\ b & c & c & a & c & b & b \\ c & b & b & c & a & a & a \end{array}$	$\begin{array}{\|c\|c\|c\|c\|} \hline D(\psi_5) & a & b & c \\ \hline a & - & 3 & 4 \\ \hline b & 4 & - & 3 \\ \hline c & 3 & 4 & - \\ \hline \end{array}$	a, b	b	a, b
$\psi_6 : \begin{array}{c\|c\|c\|c\|c\|c\|c} a & a & a & b & b & c & c \\ c & c & c & a & a & b & b \\ b & b & b & c & c & a & a \end{array}$	$\begin{array}{\|c\|c\|c\|c\|} \hline D(\psi_6) & a & b & c \\ \hline a & - & 3 & 5 \\ \hline b & 4 & - & 2 \\ \hline c & 2 & 5 & - \\ \hline \end{array}$	a, b	a, b	a, b
$\psi_7 : \begin{array}{c\|c\|c\|c\|c\|c\|c} a & a & a & b & b & c & c \\ b & c & c & a & a & b & b \\ c & b & b & c & c & a & a \end{array}$	$\begin{array}{\|c\|c\|c\|c\|} \hline D(\psi_7) & a & b & c \\ \hline a & - & 3 & 5 \\ \hline b & 4 & - & 3 \\ \hline c & 2 & 4 & - \\ \hline \end{array}$	a, b	a, b	a, b

Table 10.3: Optimal solutions for $V = 7$ and $C = 3$. f: any optimum. $f_{\text{a\&n}}$: anonymous and neutral optimum. Read this way: in order for f to be an optimum, it is necessary that $f(\psi_4)$ is a or b, but the conditions presented here are not sufficient. In order for $f_{\text{a\&n}}$ to be an anonymous and neutral optimum, it is necessary and sufficient to meet the conditions in column $f_{\text{a\&n}}$ (and their variants by anonymity and neutrality).

because profile ψ_4 has no symmetry relatively to the candidates. However, if we impose a as winner for an anonymous and neutral voting system, then we observe, by using CPLEX, that it is no more possible to reach the minimal manipulability. By the way, we observe that the conditions in column $f_{\mathrm{a\&n}}$ are, in fact, sufficient conditions to have an optimum.

For an any optimum (without any requirement of symmetry), conditions in column f from table 10.3 are only necessary: it is possible that all these conditions are met, but the voting system is not optimal. It is the case, for example, for CIRV with the lexicographical tie-breaking rule on candidates used in S$\forall$AMP, as we will see in table 10.4 from section 10.3. These necessary conditions are tight in the following sense: for each of them, there exists an optimum meeting it. For example, there exists at least one optimum f such that $f(\psi_4) = a$ and there exists at least one such that $f(\psi_4) = b$. These necessary conditions allow us to establish to following proposition.

Proposition 10.11
We consider the electoral space of strict total orders for $V = 7$ and $C = 3$, endowed with the impartial culture.

Whatever the tie-breaking rule, the following voting systems are not optimal among Condorcet voting systems: Condorcet-Bucklin, Condorcet-Borda (Black's method), Condorcet-Coombs and Condorcet-IB.

No voting system based only on the matrix of duels is optimal.

Any optimal Condorcet voting system has a manipulability rate equal to $\frac{31\,920}{279\,936} = \frac{665}{5832} \simeq 11{,}40\ \%$ *and can be seen as a variant of CIRV, with an adequate tie-breaking rule.*

Proof. According to table 10.3, in order for the voting system to be optimal, it is necessary that candidate a is declared the winner in ψ_1. But in Condorcet-Bucklin or Condorcet-Borda (Black's method), c is declared the winner.

Similarly, a must win in ψ_2. But in Condorcet-Coombs or IB, c is declared the winner.

Still from table 10.3, it is necessary that candidate a wins in ψ_3. This profile has a matrix of duels that is symmetric by the permutation $(a \to b \to c \to a)$, but the profile itself presents no symmetry relatively to the candidates. Let us note ψ'_3 the profile obtained by applying this circular permutation to ψ_3. If the voting systems depends only on the matrix of duels, then it must also designate a in ψ'_3. But if it is optimal, then the winner must be the image of a by this permutation, namely b. So, these two assumptions are incompatible.

In table 10.3, we see that the possible winners for an optimal voting systems are always exactly the same as in CIRV, the actual winner depending on the tie-breaking rule. This proves that any optimum is a variant of CIRV (but not the converse). □

For any anonymous and neutral optimum, CPLEX makes it possible to ensure that the conditions in column $f_{\mathrm{a\&n}}$ from table 10.3 are not only necessary but also sufficient. From this, we deduce the following proposition.

Proposition 10.12
We consider the electoral space of strict total orders for $V = 7$ and $C = 3$, endowed with the impartial culture.

For any anonymous and neutral tie-breaking rule, the following voting systems are not optimal among the Condorcet systems: Baldwin, Condorcet-Plurality, CSD, IRVD, Kemeny, Maximin, Nanson, RP or Schulze's method.

Among the Condorcet SBVS, there exists four that are optimal, anonymous and neutral. Each of them can be seen as CIRV, with an adequate tie-breaking rule.

Proof. According to table 10.3, in order for an anonymous and neutral voting system to be optimal, it is necessary that b wins in ψ_4. But this is not the case neither in IRVD (b faces c in the first duel and loses), nor in RP (the first validated victory is the one by c against b), nor in CSD or Maximin (b has the lowest score), nor in Condorcet-Plurality (a wins), nor in Baldwin or Nanson's method (b is eliminated during the first round), nor in Schulze's method (c is *better* than b), nor in Kemeny's method (b is necessarily lower than c).

In order to define an optimal, anonymous and neutral voting system, it is necessary and sufficient to choose the winner among two possibilities for profiles ψ_6 and ψ_7. So, four voting systems are possible. All other non-Condorcet profile are deduced from those in table 10.3 by anonymity and neutrality. The fact that they are all variants of CIRV is a simple particular case of proposition 10.11, which we recall for the sake of exhaustiveness. □

Propositions 10.11 and 10.12 exclude all Condorcet voting systems we studied except CIRV, whatever the tie-breaking rule if it is anonymous and neutral. CIRV, with an anonymous and neutral tie-breaking rule, is optimal if and only if the tie-breaking rule meets the conditions indicated in column $f_{\text{a\&n}}$ from table 10.3.

It would be interesting to reword such a tie-breaking rule, not by an exhaustive enumeration of all cases, but by a property that we could try to use for other values of the parameters V and C (intensive definition). This could make it possible to identify a version of CIRV that would be especially resistant to manipulation in the general case.

Let us give it a first try with the following rule: in case of tie between two candidates in an elimination round, we organize a virtual duel and we eliminate the losing candidate[1]. But this rule does not work. Indeed, in profiles ψ_4, ψ_5, ψ_6 and ψ_7, candidates b and c are tied for elimination in the first round and b, the loser of the duel, is eliminated; then a is declared the winner. By permuting voters and/or candidates, we deduce that this tie-breaking rule determines a unique winner in all non-Condorcet configurations and that the voting system we obtain is anonymous and neutral. But, in that case, the winner in ψ_4 should be b and not a (table 10.3). So, the voting system cannot be optimal.

Let us consider another tie-breaking rule: if two winners are possible by CIRV, then we organize a virtual duel to choose the final winner of the election[2]. In ψ_4, ψ_5, ψ_6 and ψ_7, the winner we obtain is b, so the voting system is optimal indeed.

However, this solution is not perfectly satisfactory. In order for an anonymous and neutral voting system to be optimal, b must win in ψ_4 and ψ_5; but in ψ_6 or ψ_7, choosing a is also possible. So, the tie-breaking rule we exhibited is sufficient but not necessary: in order to cover exactly the set of anonymous and neutral solution, it would be necessary to identify a "natural" tie-breaking rule (i.e. defined in an intensive way, and as simple as possible) that would choose candidate b in profiles ψ_4 and ψ_5 but stays indecisive between a and b in profiles ψ_6 and ψ_7. We leave this question open for future works.

[1] In case of a tie between more than two candidates, we can choose the eliminated candidate by using a tie-breaking rule of which the electoral duel is a particular case: for example, we eliminate the candidate with the lowest Borda score in the matrix of duels restricted to the candidates tied for elimination

[2] If there is a tie between more than two candidates, we can choose a tie-breaking rule of which the electoral duel is a particular case: for example, we designate the candidate with highest Borda score in the matrix of duels restricted to the candidates tied for victory.

10.2.4 $V = 3$ and $C = 4$: a complicated set of optima

For $V = 3$ candidates and $C = 4$ candidates, one might believe that the problem is simpler, because there are "only" $(C!)^V = 13\,824$ profiles in total. However, there are 12 types of non-Condorcet profiles (up to permuting voters and/or candidates), instead of 7 in previous case. Moreover, we will see that there is no anonymous and neutral solution and that, as a consequence, the set of solutions is more difficult to explore.

The greedy algorithm find a voting system with a manipulability rate of 21,09 %. CPLEX finds an optimal solution with a rate of $\frac{2\,688}{13\,824} = \frac{7}{36} \simeq 19{,}44$ %. So, the greedy algorithm is not optimal in that case either. By a detailed study of the solution that CPLEX exhibits by default, we could show that it was neither anonymous nor neutral. As a consequence, the solution is not unique: indeed, any voting system that is deduced by a permutation of candidates is distinct from it and is also optimal.

If we impose that the solution is neutral, we find the same manipulability rate. If we impose that the solution is anonymous, we obtain a rate equal to $\frac{2\,712}{13\,824} = \frac{113}{576} \simeq 19{,}62$ %. If we impose that the solution is anonymous and neutral, we obtain a rate equal to $\frac{3\,264}{13\,824} = \frac{17}{72} \simeq 23{,}61$ %.

It is difficult to study the set of all solutions in that case (with no symmetry assumption). Indeed, because of their multiplicity, there are generally 2 or even 3 possible winners for each non-Condorcet profile[3].

However, we can say more about the neutral solutions. For this purpose, let us consider the following profiles.

$$\psi_1 : \begin{array}{c|c|c} a & c & b \\ b & a & c \\ c & b & a \\ d & d & d \end{array} \qquad \psi_2 : \begin{array}{c|c|c} a & b & c \\ b & c & a \\ c & a & b \\ d & d & d \end{array}$$

By a permutation of candidates, these two profiles cover all the cases where voters place the same candidate in last position and realize a minimal example of Condorcet paradox between the three other candidates. These profiles are deduced from each other by a permutation of the two voters on the right.

By using CPLEX, we see that an optimum cannot elect d in ψ_1, which is quite intuitive because it is the candidate placed in last position by all the voters. By symmetry of this profile, any candidate a, b or c can be designated in an optimal voting system. This amounts to give a privilege to voter 1, 2 or 3 in this profile ψ_1 and those deduced by permuting the candidates. In get the idea, let us assume that a is declared the winner. This designates voter 1 as having a privilege in ψ_1 and breaks also the symmetry between the two other voters: voter 2 (resp. 3) is the one for which the winning candidate is in second (resp. third) position.

In profile ψ_2, CPLEX informs us that now, there are only two possible winners: a or b. In other words, either we favor the same voter as in profile ψ_1, or the voter whose second most liked candidate is elected in ψ_1. When we fix one option or the other, CPLEX ensure that the neutral optimum is unique.

Since there a 3 possible choices for ψ_1 then 2 choices for ψ_2, there are 6 possible solutions. Seeing this number, one might think that it is only one voting system and its variants for the $3! = 6$ permutations of the voters. But it is not the case.

[3] In an optimal voting system, there are 2 possible winners in 8 types of non-Condorcet profiles and 3 possible winner in 4 types of non-Condorcet profiles. This remains true, even if we impose that the solution is neutral.

Indeed, let us consider the unique neutral optimum f obtained with $f(\psi_1) = a$ and $f(\psi_2) = a$. If we exchange the two last voters, then the optimal voting system we obtain is the same. Hence, the orbit of f by permuting the voters has only 3 elements, and not 6. Similarly, if we consider the unique neutral optimum g obtained with $g(\psi_1) = a$ and $g(\psi_2) = b$, its orbit has 3 elements by permuting the voters. So, there are exactly two distinct neutral solutions, up to permuting the voters.

Profiles ψ_1 and ψ_2 above allow us also to understand intuitively why the anonymous and neutral optimum is significantly more manipulable than the general optimum. Indeed, if we impose anonymity and neutrality, then candidate d must win in both these profiles, whereas she is the least liked by all voters. This idea can be linked to the fact that, for these values of the parameters, there exists no neutral, anonymous and efficient voting system (cf. section 1.2.2).

The following proposition will sum up the observations we have made until now.

Proposition 10.13

We consider the electoral space of strict total orders for $V = 3$ and $C = 4$, endowed with the impartial culture.

Any optimal Condorcet voting system has a manipulability rate equal to $\frac{2\,688}{13\,824} = 19{,}44\ \%$. *There exists at least a non-neutral optimum (and there it is not unique).*

There exists exactly 6 optimal voting systems that are neutral. Up to permuting the voters, there are 2 distinct solutions. Each one has 3 variants by permuting the voters.

There is no optimum that is anonymous.

Now, we will present some results that link these solutions to the usual voting systems.

Proposition 10.14

We consider the electoral space of stric total orders for $V = 3$ and $C = 4$, endowed with the impartial cultre.

For any neutral tie-breaking rule, the following voting systems are not optimal among Condorcet systems: Baldwin, Condorcet-Borda (Black's method), Condorcet-Coombs, CIRV, Condorcet-Plurality, CSD, IRVD, Kemeny and Nanson's method.

Proof. As above, let us note f the neutral optimum obtained with $f(\psi_1) = a$ and $f(\psi_2) = a$ and g the neutral optimum obtained for $g(\psi_1) = a$ and $g(\psi_2) = b$.

Let us consider the following profile.

$$\psi : \begin{array}{c|c|c} b & a & d \\ c & c & a \\ d & b & b \\ a & d & c \end{array}$$

$D(\psi)$	a	b	c	d
a	–	2	2	1
b	1	–	2	2
c	1	1	–	2
d	2	1	1	–

Thanks to the optimization realized by CPLEX, we could determine that the winner must be c, for f and also for g.

But she is elected neither in CIRV (c is eliminated during the first round), nor in Condorcet-Plurality (c has no vote), nor in CSD (c has the lowest score), nor in Condorcet-Borda or Nanson's method (c has the lowest Borda score), nor in Kemeny's method (if we rearrange the matrix of duels with c in first position, there are two defeats with an amplitude of one vote above the diagonal, instead of only one defeat of one vote if we keep the alphabetical order on the candidates).

In IRVD, since c has no vote and all others have at least one, c participates to all duels of elimination. Since she is not a Condorcet winner, she cannot win all of them. Hence c cannot be elected.

In Baldwin's method or in Condorcet-Coombs, in order for c to win, she must face d during last round because it is the only candidate she can defeat in electoral duel. Hence, during the first round, it is necessary to eliminate neither c nor d.

But, in Condorcet-Coombs, there is a tie between a, c and d during the first round; but since neither c nor d can be eliminated, a must be eliminated. In that case, d is eliminated, then c and finally, b is declared the winner. So, c cannot win.

In Baldwin's method, c or d are the only one who can be eliminated during the first round. So, c cannot win. □

Proposition 10.15

We consider the electoral space of strict total orders for $V = 3$ and $C = 4$, endowed with the impartial culture.

Any optimal Condorcet voting system can be seen as Condorcet-Bucklin, Condorcet-IB, Schulze's method, RP or Maximin, with an adequate tie-breaking rule.

Proof. For the sake of conciseness, we simply explain the method we used. For each of the 12 types of non-Condorcet profiles (up to permuting voters and/or candidates), we determine by CPLEX which candidates can be winners in an optimal voting system, as we did in table 10.3 for $V = 7$ and $C = 3$.

In each case, we see in practice that the set of possible winners is the same with or without the assumption of neutrality, and that it is equal to the set of candidates who can win in Condorcet-Bucklin. Manually, we check that this set is inclued (strictly, by the way) in the set of candidates who can win in Condorcet-IB (resp. Schulze's method, RP, Maximin). We stress on the fact that it means that any optimum is Condorcet-Bucklin with an adequate tie-breaking rule, but not necessarily that any tie-breaking rule makes Condorcet-Bucklin optimal (as we will see in table 10.4). □

Previous proposition can be interpreted as an argument in favor of the five voting systems we mentioned. However, some of them lead to more ties than the others. Condorcet-Bucklin leaves least choice to the tie-breaking rule: out of 12 types of non-Condorcet profiles (up to permuting voters and/or candidates), it causes a 3-candidate tie in 4 profiles and 2-candidate tie in 8 profiles. IB causes always a 3-candidate tie. Schulze's method and RP cause a 4-candidate tie (i.e. a tie between all candidates!) in 6 profiles and a 3-candidate tie in 6 profiles. Lastly, Maximin causes a 4-candidate tie in 7 profiles and 3-candidate tie in 5 profiles. So, it is not such a big surprise that any optimal voting system can be seen as Maximin with an adequate tie-breaking rule, because Maximin is especially irresolute in the case $V = 3$ and $C = 4$.

So, we could determine the minimal manipulability rate for a Condorcet voting system, and more generally for a voting system meeting **InfMC**, in impartial culture, in four cases: $V = 3, 5$ or 7 voters and $C = 3$ candidates; $V = 3$ voters and $C = 4$ candidates. For more voters or candidates, the problem is too large to be treated on the machine we use[4].

[4] Dell Precision M6600, Intel Core I7-2820QM à 2,30 GHz, 8 Mo de cache, 16 Go de RAM à 1,33 GHz en DDR3.

10.3 Comparison between the optimum and the usual voting systems

Now, we will compare the optimal manipulability rate (in class **InfMC**) to the one of several voting systems. Let us notice that, when we have the opportunity graph, we can easily determine the manipulability rate of any Condorcet voting system: indeed, it is sufficient to write down the winners of all non-Condorcet configurations and to contaminate the corresponding Condorcet configurations.

In table 10.4, we have indicated the exact manipulability rates of a variety of voting systems.

- The optimum is obtained by integer linear programming in CPLEX, as well as the optimum in the smaller class of Condorcet systems that are anonymous, neutral or both. For $V = 3$ and $C = 3$, it it not possible to have an anonymous and neutral voting system (see proposition 1.16).
- The approximate optimum is obtained with the greedy algorithm.
- For the voting systems meeting the Condorcet criterion, we use the opportunity graph to get their exact manipulability. For those implemented in SVVAMP, we can also use it with the option `CM_option = 'exact'`, which provides an additional check of the results.
- For the other voting systems, such as Borda or Veto, we use SVVAMP with the option `CM_option = 'exact'`.

In this table, we stress on the fact that tie-breaking rules are important, because we consider a small number of voters. For that reason, it is exaggerated to say that we have represent the manipulability rate for IRV (for example): it is the one of IRV, with the tie-breaking rule used in SVVAMP, i.e. by lexicographical order on candidates. So, we have seen above that some voting systems we already knew (especially CIRV for $V = 7$ and $C = 3$; Condorcet-Bucklin for $V = 3$ and $C = 4$) could be optimal when using an adequate tie-breaking rule, but this does not appear in table 10.4. For future works, it would be interesting to vary this tie-breaking rule.

For each value of the parameters, we have also indicated the rate of non-Condorcet profiles, which was giving a first lower bound of the manipulability rate for voting systems meeting **InfMC**, before we knew the optimal rate.

Each manipulability rate is written in bold green police if it is equal to the minimal manipulability rate in **InfMC**.

For $V = 3$ and $C = 3$, many voting systems are optimal, in particular all those meeting the Condorcet criterion, as we showed in section 10.2.1.

For $V = 5$ and $C = 3$, the optimal voting systems are CIRV (which coincides with the result of the greedy algorithm and is the only Condorcet optimum, as we showed in section 10.2.2), and also IRV and EB (which coincides with the two-round system, because there are 3 candidats). This does not contradict what we saw in section 10.2.2: indeed, we showed that, among the voting systems meeting **InfMC**, CIRV is the unique optimum that is a Condorcet SBVS. But VTI and SE do not meet the Condorcet criterion and EB is not an SBVS. So, it it impossible that they are less manipulable than CIRV, but it is not forbidden that they have the same manipulability. By the way, Green-Armytage et al. (2014) proved that, when there are 3 candidates, CIRV and IRV have the same manipulability.

Populations				
Number of voters V	3	5	7	3
Number of candidates C	3	3	3	4
Non-Condorcet profiles (%)	5,56	6,94	7,50	11,11
Taux de MC (%)				
Optimum	**11,11**	**9,26**	**11,40**	**19,44**
Neutral optimum	**11,11**	**9,26**	**11,40**	**19,44**
Anonymous optimum	**11,11**	**9,26**	**11,40**	19,62
Anonymous neutral optimum	—	**9,26**	**11,40**	***23,61***
Approximate optimum (greedy)	**11,11**	**9,26**	13,46	21,09
CIRV	**11,11**	**9,26**	12,38	21,35
IRV	**11,11**	**9,26**	12,38	21,35
EB	**11,11**	**9,26**	12,53	21,35
TR	**11,11**	**9,26**	12,53	21,35
RP	**11,11**	18,52	26,56	21,09
Bald.	**11,11**	17,75	24,26	22,27
CBuck.	**11,11**	18,52	24,98	22,22
CDean	**11,11**	20,06	27,31	22,83
Nan.	**11,11**	***20,83***	***27,86***	***22,92***
CDict.	**11,11**	***20,37***	27,28	***23,96***
Kem.	**11,11**	***21,60***	***29,26***	***22,92***
CSD	**11,11**	***21,60***	***29,26***	***22,92***
Sch.	**11,11**	***21,60***	***29,26***	***23,09***
IRVD	**11,11**	***22,38***	***30,76***	22,57
CCoo.	**11,11**	***22,38***	***36,16***	22,22
Max.	**11,11**	***21,60***	***29,26***	***24,05***
CPlu.	**11,11**	***22,38***	***30,76***	***24,83***
IB	**11,11**	***22,38***	***36,16***	***23,26***
CBor.	**11,11**	***27,01***	***37,76***	***27,78***
Plu.	***16,67***	***26,23***	***33,01***	***30,21***
Buck.	***13,89***	***28,16***	***40,13***	***23,61***
Veto	***26,39***	***32,83***	***35,54***	***50,20***
Coo.	***22,22***	***38,58***	***53,41***	***39,84***
Bor.	***23,61***	***44,37***	***53,36***	***51,39***

Table 10.4: Exact CM rates, in percents (impartial culture). We use the lexicographical tie-breaking rule on candidates, which is implemented in SVVAMP.

For $V = 7$ and $C = 3$, we showed that there are 4 voting systems that are optimal among Condorcet voting systems and that are also anonymous and neutral. Each of them can be seen as a variant of CIRV, with an adequate tie-breaking rule.

For $V = 3$ and $C = 4$, we saw that CIRV cannot be optimal, even with a well-chosen tie-breaking rule. Any optimal Condorcet voting system can be seen as Condorcet-Bucklin, Condorcet-IB, Schulze's method, RP or Maximin, with an adequate tie-breaking rule.

Since the system Condorcet-dean is intuitively the simplest of Condorcet systems (with Condorcet-dictatorship), it seems a good reference for manipulability: indeed, if a voting system is more manipulable than this one, it has the bad property to have worst performances than a very unsophisticated voting system. As an indication, manipulability rates that are higher than the one of Condorcet-dean are written in italic red police in the table. Many voting systems (Nanson, Condorcet-dictatorship, Kemeny, CSD, Schulze, IRVD, Condorcet-Coombs, Maximin, Condorcet-Plurality, IB, Condorcet-Borda, Plurality, Bucklin, Veto, Coombs and Borda) are generally more manipulable than Condorcet-dean for the parameters under consideration. The only voting systems that always have a lower manipulability rate in this table are the one obtained by the greedy algorithm, CIRV, IRV, EB, the two-round system, RP, Baldwin's method and Condorcet-Bucklin.

Appendices

Notations

Non-alphabetical symbols

$[\alpha, \beta[$	Real interval from α included to β excluded (French convention).
$[\![j, k]\!]$	Integer interval from j to k included.
$\lfloor \alpha \rfloor$	Floor function of real number α.
$\lceil \alpha \rceil$	Ceiling function of real number α.
$\lvert \mathcal{A}(v) \rvert$	Number of voters v meeting assertion $\mathcal{A}(v)$.
$\pi(A \mid B)$	Conditional probability of event A knowing B.

Greek alphabet

μ	The law of variable P (unless otherwise stated).
π	A *culture* over electoral space Ω. More generally, a probability measure.
$\tau^{\pi}_{\mathrm{CM}}(f)$	Coalitional manipulability rate of voting system f in culture π.
Ω	Set $\prod_{v \in \mathcal{V}} \Omega_v$ of possible configurations ω. Also used as a notation shortcut for an electoral space $(V, C, \Omega, \mathrm{P})$.
Ω_M	Set of possible states ω_M for voters in a set M.
Ω_v	Set of possible states ω_v for voter v.

Latin alphabet

$C \in \mathbb{N} \setminus \{0\}$	Number of candidates.
$\mathcal{C}$	Set $[\![1, C]\!]$ of indexes for the candidates.
$\mathrm{card}(E)$	Cardinal of set E.
$D(\omega)$	Matrix of duels in ω. The coefficient of indexes c and d is denoted $D_{cd}(\omega)$ or, in short, D_{cd}.

$\mathcal{F}_\mathcal{C}$	Set of strict weak orders over $\mathcal{C}$.
f	A state-based voting system (SBVS), i.e. a function $\Omega \to \mathcal{C}$. In the case of a general voting system, f denotes its processing function $\mathcal{S}_1 \times \ldots \times \mathcal{S}_V \to \mathcal{C}$.
f^*	Condorcification of f.
$f^{\text{adm}}, f^{!\text{adm}}$	Condorcification variants of f based on the notion of Condorcet-admissible candidate.
$f^{\text{faible}}, f^{!\text{faible}}$	Condorcification variants of f based on the notion of weak Condorcet winner.
f^{rel}	Relative Condorcification of f.
$f^{\mathcal{M}}$	$\mathcal{M}$-Condorcification of f.
f_y	Slice of f by a slicing method y.
$c \mathrm{I}_v d$	Voter v is indifferent between c and d.
Id	The identity function (the context precises in which set).
$\mathcal{L}_\mathcal{C}$	Set of strict total orders over $\mathcal{C}$.
$\mathcal{M}$	A family of collections of coalitions.
$\mathcal{M}_c \in \mathcal{P}(\mathcal{P}(\mathcal{V}))$	A collection of coalitions that are said *winning* for candidate c.
$\mathrm{Manip}_\omega(\mathrm{w} \to c)$	Set of voters preferring c to w. In short, $\mathrm{Manip}(\mathrm{w} \to c)$.
CM_f	Set of configurations ω where f is manipulable (or indicator function of this set).
$\mathrm{mean}(x_1, \ldots, x_k)$	Arithmetical average of $x_1, \ldots, x_k$.
P	Function $\Omega \to \mathcal{R}$ that, to state ω of the population, associates profile $\mathrm{P}(\omega) = (\mathrm{P}_1(\omega_1), \ldots, \mathrm{P}_V(\omega_V))$.
$c \mathrm{P}_v d$	Voter v prefers c to d.
$c \mathrm{P}_{\text{abs}} d$	c has an absolute victory against d: $\lvert c \mathrm{P}_v d \rvert > \frac{V}{2}$.
$c \mathrm{P}_{\text{rel}} d$	c has a relative victory against d: $\lvert c \mathrm{P}_v d \rvert > \lvert d \mathrm{P}_v c \rvert$.
$c \mathrm{P}_{\mathcal{M}} d$	c has an $\mathcal{M}$-victory against d: $\{v \text{ s.t. } c \mathrm{P}_v d\} \in \mathcal{M}_c$.
$c \mathrm{MP}_v d$	Voter v prefers c to d and vice versa (impossible if P_v is antisymmetric).
$c \mathrm{PP}_v d$	Voter v prefers c to d but not d to c (synonym of $c \mathrm{P}_v d$ if P_v is antisymmetric).
$\mathcal{R}$	Set ${\mathcal{R}_\mathcal{C}}^V$ whose an element (profile) represents binary relations of preference for the whole population of voters.
$\mathcal{R}_\mathcal{C}$	Set of binary relations over $\mathcal{C}$.
$\mathrm{Sinc}_\omega(\mathrm{w} \to c)$	Set of voters who do not prefer c to w. In short, $\mathrm{Sinc}(\mathrm{w} \to c)$.
$V \in \mathbb{N} \setminus \{0\}$	Number of voters.
$(V, C, \Omega, \mathrm{P})$	An electoral space. In short, Ω.
$\mathcal{V}$	Set $[\![1, V]\!]$ of indexes for the voters.
$\mathrm{vect}(E)$	Linear span of E, where E is a part of a vector space.
$\mathcal{Y}$	Set $\prod_{v \in \mathcal{V}} \mathcal{Y}_v$ of slicing methods y for the whole population of voters.

$\mathcal{Y}_v$	Set $\{y_v : \mathrm{P}(\Omega_v) \to \Omega_v$ s.t. $\mathrm{P}_v \circ y_v = \mathrm{Id}\}$ of slicing methods y_v for voter v.

Acronyms and abbreviations

AV	Approval voting.
Bald.	Baldwin's method.
Bor.	Borda's method.
Buck.	Bucklin's method.
CIRV	Condorcification of IRV.
CM	Coalition manipulation / manipulable.
Cond	Condorcet criterion.
Coo.	Coombs' method.
CSD	Condorcet's method with sum of defeats.
EB	Exhaustive ballot.
IB	Iterated Bucklin's method.
ICM	Ignorant-coalition manipulation / manipulable.
iff	If and only if.
IgnMC	Ignorant majority coalition criterion.
IIA	Independence of irrelevant alternatives.
IM	Individual manipulation / manipulable.
InfMC	Informed majority coalition criterion.
IRV	Instant-runoff voting.
IRVA	Instant-runoff voting based on the average.
IRVD	Instant-runoff voting with duels.
ITR	Instant two-round system.
Kem.	Kemeny's method.
KR	Kim-Roush's method.
MajBal	Majority ballot criterion.
MajFav	Majority favorite criterion.
MajUniBal	Majority unison ballot criterion.
Max.	Maximin.
MJ	Majority Judgement.
Nan.	Nanson's methodd.
Plu.	Plurality.
RP	Ranked Pairs method.
RV	Range voting.
s.t.	Such that.
SBVS	State-based voting system.

Sch.	Schulze's method.
SVVAMP	Simulator of Various Voting Algorithms in Manipulating Populations.
TM	Trivial manipulation / manipulable.
TR	Two-round system.
UM	Unison manipulation / manipulable.
VMF	Von Mises–Fisher.

Bibliography

Fuad Aleskerov and Eldeniz Kurbanov. Degree of manipulability of social choice procedures. In Society for the Advancement of Economic Theory, editor, *Current trends in economics: theory and applications*, Studies in economic theory, pages 13–27. Springer, 1999.

Fuad Aleskerov, Daniel Karabekyan, Remzi Sanver, and Vyacheslav Yakuba. Computing the degree of manipulability in the case of multiple choice. *Computational Social Choice (COMSOC-2008)*, page 27, 2008.

Kenneth Arrow. A difficulty in the concept of social welfare. *The Journal of Political Economy*, 58(4):328–346, 1950.

Navin Aswal, Shurojit Chatterji, and Arunava Sen. Dictatorial domains. *Economic Theory*, 22(1):45–62, 2003.

Michèle Audin. *Geometry*. Universitext (Berlin. Print). Springer Berlin Heidelberg, 2003.

Haris Aziz, Serge Gaspers, Nicholas Mattei, Nina Narodytska, and Toby Walsh. Ties matter: Complexity of manipulation when tie-breaking with a random vote. In *AAAI*, 2013.

Michel Balinski and Rida Laraki. *Majority Judgment: Measuring, Ranking, and Electing*. MIT Press, 2010.

Salvador Barberá. Strategy-proofness and pivotal voters: A direct proof of the Gibbard-Satterthwaite theorem. *International Economic Review*, 24(2):413–417, 1983.

Salvador Barberà. An introduction to strategy-proof social choice functions. *Social Choice and Welfare*, 18:619–653, 2001.

Salvador Barberá and Bezalel Peleg. Strategy-proof voting schemes with continuous preferences. *Social Choice and Welfare*, 7:31–38, 1990.

John Bartholdi and James Orlin. Single transferable vote resists strategic voting. *Social Choice and Welfare*, 8:341–354, 1991.

John Bartholdi, Craig Tovey, and Michael Trick. The computational difficulty of manipulating an election. *Social Choice and Welfare*, 6:227–241, 1989a.

John Bartholdi, Craig Tovey, and Michael Trick. Voting schemes for which it can be difficult to tell who won the election. *Social Choice and welfare*, 6(2): 157–165, 1989b.

Eugenio Beltrami. Résolution du problème de reporter les points d'une surface sur un plan, de manière que les lignes géodésiques soient représentée par des lignes droites. *Annali di Matematica*, 1866.

Eugenio Beltrami. Essai d'interprétation de la géométrie noneuclidéenne. Trad. par J. Hoüel. *Ann. Sci. École Norm. Sup.*, 6:251–288, 1869.

Jean-Pierre Benoît. The Gibbard-Satterthwaite theorem: a simple proof. *Economics Letters*, 69(3):319–322, 2000.

Douglas Bernheim, Bezalel Peleg, and Michael Whinston. Coalition-proof Nash equilibria I. Concepts. *Journal of Economic Theory*, 42(1):1–12, 1987.

Nadja Betzler, Jiong Guo, and Rolf Niedermeier. Parameterized computational complexity of Dodgson and Young elections. *Information and Computation*, 208(2):165–177, 2010.

Nadja Betzler, Rolf Niedermeier, and Gerhard Woeginger. Unweighted coalitional manipulation under the Borda rule is NP-hard. In *Proceedings of the 22th International Joint Conference on Artificial Intelligence (IJCAI '11)*, 2011.

Duncan Black. *The theory of committees and elections*. University Press, 1958.

Jean-Marie Blin and Mark Satterthwaite. Strategy-proofness and single-peakedness. *Public Choice*, 26:51–58, 1976.

Kim Border and J.S. Jordan. Straightforward elections, unanimity and phantom voters. *The Review of Economic Studies*, 50(1):153–170, 1983.

Steven Brams. Voting procedures. In R.J. Aumann and S. Hart, editors, *Handbook of Game Theory with Economic Applications*, volume 2 of *Handbook of Game Theory with Economic Applications*, chapter 30, pages 1055–1089. Elsevier, 1994.

Steven Brams. Approval voting. In C.K. Rowley and F. Schneider, editors, *The Encyclopedia of Public Choice*, pages 344–346. Springer US, 2003.

Steven Brams and Peter Fishburn. Approval voting. *American Political Science Review*, 72:831–847, 1978.

Felix Brandt and Markus Brill. Necessary and sufficient conditions for the strategyproofness of irresolute social choice functions. In *Proceedings of the 13th Conference on Theoretical Aspects of Rationality and Knowledge*, pages 136–142. ACM, 2011.

Markus Brill and Vincent Conitzer. Strategic voting and strategic candidacy. In *Proceedings of the Twenty-Ninth AAAI Conference on Artificial Intelligence, Austin, USA*, 2015.

Markus Brill and Felix Fischer. The price of neutrality for the ranked pairs method. *COMSOC 2012*, page 95, 2012.

Colin Campbell and Gordon Tullock. A measure of the importance of cyclical majorities. *The Economic Journal*, 75(300):853–857, 1965.

Ioannis Caragiannis, Jason Covey, Michal Feldman, Christopher Homan, Christos Kaklamanis, Nikos Karanikolas, Ariel Procaccia, and Jeffrey Rosenschein. On the approximability of Dodgson and Young elections. In *Proceedings of the twentieth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1058–1067. Society for Industrial and Applied Mathematics, 2009.

John Chamberlin, Jerry Cohen, and Clyde Coombs. Social choice observed: Five presidential elections of the american psychological association. *The Journal of Politics*, 46:479–502, 1984.

Graciela Chichilnisky. Von Neumann-Morgenstern utilities and cardinal preferences. *Mathematics of Operations Research*, 10(4):633–641, 1985.

Tom Coleman and Vanessa Teague. On the complexity of manipulating elections. In Joachim Gudmundsson and Barry Jay, editors, *Thirteenth Computing: The Australasian Theory Symposium (CATS2007)*, volume 65 of *CRPIT*, pages 25–33. ACS, 2007.

Marie Jean Antoine Nicolas de Caritat, marquis de Condorcet. *Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix.* Imprimerie royale, 1785.

Vincent Conitzer and Tuomas Sandholm. Complexity of manipulating elections with few candidates. In *Eighteenth national conference on Artificial intelligence*, pages 314–319. American Association for Artificial Intelligence, 2002.

Vincent Conitzer and Tuomas Sandholm. Universal voting protocol tweaks to make manipulation hard. In *Proceedings of the 18th international joint conference on Artificial intelligence*, pages 781–788. Morgan Kaufmann Publishers Inc., 2003.

Vincent Conitzer and Tuomas Sandholm. Nonexistence of voting rules that are usually hard to manipulate. In *Proceedings of the 21st national conference on Artificial intelligence - Volume 1*, pages 627–634. AAAI Press, 2006.

Vincent Conitzer, Jérôme Lang, and Tuomas Sandholm. How many candidates are needed to make elections hard to manipulate? In *Proceedings of the 9th conference on Theoretical aspects of rationality and knowledge*, pages 201–214. ACM, 2003.

Vincent Conitzer, Tuomas Sandholm, and Jérôme Lang. When are elections with few candidates hard to manipulate? *J. ACM*, 54, June 2007.

Jessica Davies, George Katsirelos, Nina Narodytska, and Toby Walsh. Complexity of and algorithms for Borda manipulation. *AAAI*, 11:657–662, August 2011.

Jessica Davies, George Katsirelos, Nina Narodytska, Toby Walsh, and Lirong Xia. Complexity of and algorithms for the manipulation of Borda, Nanson's and Baldwin's voting rules. *Artificial Intelligence*, 217:20–42, 2014.

Frank DeMeyer and Charles Plott. The probability of a cyclical majority. *Econometrica*, 38(2):345–354, 1970.

Amogh Dhamdhere and Constantine Dovrolis. The Internet is flat: modeling the transition from a transit hierarchy to a peering mesh. In *Proceedings of the 6th International Conference*, Co-NEXT '10, page 21. ACM, 2010.

Thomas Downs. Some relationships among the Von Mises distributions of different dimensions. *Biometrika*, 53(1/2):269–272, 1966.

John Duggan and Thomas Schwartz. Strategic manipulability without resoluteness or shared beliefs: Gibbard-Satterthwaite generalized. *Social Choice and Welfare*, 17:85–93, 2000.

François Durand, Fabien Mathieu, and Ludovic Noirie. Manipulability of voting systems. Groupe de travail Displexity, http://www.liafa.univ-paris-diderot.fr/~displexity/docpub/6mois/votes.pdf, 2012.

François Durand, Fabien Mathieu, and Ludovic Noirie. On the manipulability of voting systems: application to multi-operator networks. In *Proceedings of the 9th International Conference on Network and Service Management (CNSM)*, pages 292–297. IEEE, 2013.

François Durand, Fabien Mathieu, and Ludovic Noirie. Élection du best paper AlgoTel 2012: étude de la manipulabilité. In *AlgoTel 2014 – 16èmes Rencontres Francophones sur les Aspects Algorithmiques des Télécommunications*, 2014a.

François Durand, Fabien Mathieu, and Ludovic Noirie. Making most voting systems meet the Condorcet criterion reduces their manipulability. https://hal.inria.fr/hal-01009134, 2014b.

François Durand, Fabien Mathieu, and Ludovic Noirie. Élection d'un chemin dans un réseau: étude de la manipulabilité. In *AlgoTel 2014 – 16èmes Rencontres Francophones sur les Aspects Algorithmiques des Télécommunications*, 2014c.

François Durand, Fabien Mathieu, and Ludovic Noirie. Reducing manipulability. Poster présenté au 5th International Workshop on Computational Social Choice (COMSOC), 2014d.

François Durand, Fabien Mathieu, and Ludovic Noirie. Making a voting system depend only on orders of preference reduces its manipulability rate. https://hal.inria.fr/hal-01009136, 2014e.

François Durand, Benoît Kloeckner, Fabien Mathieu, and Ludovic Noirie. Geometry on the utility sphere. In *Proceedings of the 4th International Conference on Algorithmic Decision Theory (ADT)*, 2015.

Bhaskar Dutta, Matthew Jackson, and Michel Le Breton. Strategic candidacy and voting procedures. *Econometrica*, 69(4):1013–1037, 2001.

Bhaskar Dutta, Matthew Jackson, and Michel Le Breton. Voting by successive elimination and strategic candidacy. *Journal of Economic Theory*, 103(1):190–218, 2002.

Lars Ehlers, Hans Peters, and Ton Storcken. Threshold strategy-proofness: on manipulability in large voting problems. *Games and Economic Behavior*, 49 (1):103–116, 2004.

Edith Elkind and Helger Lipmaa. Small coalitions cannot manipulate voting. In Andrew Patrick and Moti Yung, editors, *Financial Cryptography and Data Security*, volume 3570 of *Lecture Notes in Computer Science*, pages 578–578. Springer Berlin / Heidelberg, 2005a.

Edith Elkind and Helger Lipmaa. Hybrid voting protocols and hardness of manipulation. In Xiaotie Deng and Ding-Zhu Du, editors, *Algorithms and Computation*, volume 3827 of *Lecture Notes in Computer Science*, pages 206–215. Springer Berlin / Heidelberg, 2005b.

Piotr Faliszewski and Ariel Procaccia. AI's war on manipulation: Are we winning? *AI Magazine*, 31(4):53–64, 2010.

Pierre Favardin and Dominique Lepelley. Some further results on the manipulability of social choice rules. *Social Choice and Welfare*, 26:485–509, 2006.

Pierre Favardin, Dominique Lepelley, and Jérôme Serais. Borda rule, Copeland method and strategic manipulation. *Review of Economic Design*, 7:213–228, 2002.

Allan Feldman. *Welfare economics and social choice theory*. Kluwer Nijhoff Publishing. Martinus Nijhoff Pub., 1980.

Peter Fishburn. *Utility Theory for Decision Making*. Wiley, New York, 1970.

Peter Fishburn. *Nonlinear preference and utility theory*. Johns Hopkins series in the mathematical sciences. Johns Hopkins University Press, 1988.

Peter Fishburn and Steven Brams. Efficacy, power and equity under approval voting. *Public Choice*, 37(3):425–434, 1981.

Peter Fishburn and William Gehrlein. Borda's rule, positional voting, and Condorcet's simple majority principle. *Public Choice*, 28(1):79–88, 1976.

Peter Fishburn, William Gehrlein, and Eric Maskin. Condorcet proportions and Kelly's conjectures. *Discrete Applied Mathematics*, 1(4):229–252, 1979.

Mark Garman and Morton Kamien. The paradox of voting: Probability calculations. *Behavioral Science*, 13:306–316, 1968.

Serge Gaspers, Thomas Kalinowski, Nina Narodytska, and Toby Walsh. Coalitional manipulation for Schulze's rule. In *Proceedings of the 2013 international conference on Autonomous agents and multi-agent systems*, pages 431–438. International Foundation for Autonomous Agents and Multiagent Systems, 2013.

John Geanakoplos. Three brief proofs of Arrow's impossibility theorem. *Economic Theory*, 26(1):211–215, 2005.

William Gehrlein. The expected probability of Condorcet's paradox. *Economics Letters*, 7(1):33–37, 1981.

William Gehrlein. Approximating the probability that a Condorcet winner exists. 1999.

William Gehrlein. *Condorcet's Paradox*. Theory and Decision Library C. Springer, 2006.

William Gehrlein and Peter Fishburn. The probability of the paradox of voting: A computable solution. *Journal of Economic Theory*, 13(1):14–25, 1976.

Allan Gibbard. Manipulation of voting schemes: A general result. *Econometrica*, 41(4):587–601, 1973.

Allan Gibbard. Manipulation of schemes that mix voting with chance. *Econometrica*, 45(3):665–681, 1977.

Allan Gibbard. Straightforwardness of game forms with lotteries as outcomes. *Econometrica*, 46(3):595–614, 1978.

Allan Gibbard. Social choice and the Arrow conditions. *Economics and Philosophy*, 30(03):269–284, 2014.

Peter Gärdenfors. Manipulation of social choice functions. *Journal of Economic Theory*, 13:217–228, 1976.

Peter Gärdenfors. A concise proof of theorem on manipulation of social choice functions. *Public Choice*, 32:137–142, 1977.

James Green-Armytage. Four Condorcet-Hare hybrid methods for single-winner elections. *Voting matters*, 29:1–14, 2011.

James Green-Armytage. Strategic voting and nomination. *Social Choice and Welfare*, 42(1):111–138, 2014.

James Green-Armytage, Nicolaus Tideman, and Rafael Cosman. Statistical evaluation of voting rules. 2014.

Bernard Grofman and Scott Feld. If you like the alternative vote (a.k.a. the instant runoff), then you ought to know about the Coombs rule. *Electoral Studies*, 23 (4):641–659, 2004.

François Guénard and Gilbert Lelièvre. *Compléments d'analyse*. Number 1 in Compléments d'analyse. E.N.S., 1985.

Peter Hammond. Interpersonal comparisons of utility: Why and how they are and should be made. In *Interpersonal Comparisons of Well-Being*, pages 200–254. University Press, 1991.

Lane Hemaspaandra, Rahman Lavaee, and Curtis Menton. Schulze and ranked-pairs voting are fixed-parameter tractable to bribe, manipulate, and control. In *Proceedings of the 2013 international conference on Autonomous agents and multi-agent systems*, pages 1345–1346. International Foundation for Autonomous Agents and Multiagent Systems, 2013.

Andrew Jennings. *Monotonicity and Manipulability of Ordinal and Cardinal Social Choice Functions*. BiblioBazaar, 2011.

Bradford Jones, Benjamin Radcliff, Charles Taber, and Richard Timpone. Condorcet winners and the paradox of voting: Probability calculations for weak preference orders. *The American Political Science Review*, 89(1):137–144, March 1995.

Nathan Keller. A tight quantitative version of Arrow's impossibility theorem. *Journal of the European Mathematical Society*, 14(5):1331–1355, 2012.

Jerry Kelly. Voting anomalies, the number of voters, and the number of alternatives. *Econometrica*, pages 239–251, 1974.

Jerry Kelly. Almost all social choice rules are highly manipulable, but a few aren't. *Social Choice and Welfare*, 10:161–175, 1993.

John Kemeny. Mathematics without numbers. *Daedalus*, 88:575–591, 1959.

K.H. Kim and F.W. Roush. Statistical manipulability of social choice functions. *Group Decision and Negotiation*, 5:263–282, 1996.

Benoît Kloeckner. *Un bref aperçu de la géométrie projective*. Calvage & Mounet, 2012.

Kathrin Konczak and Jérôme Lang. Voting procedures with incomplete preferences. In *Proc. IJCAI-05 Multidisciplinary Workshop on Advances in Preference Handling*, 2005.

David Kreps. *A Course in Microeconomic Theory.* Princeton University Press, 1990.

Craig Labovitz, Scott Iekel-Johnson, Danny McPherson, Jon Oberheide, and Farnam Jahanian. Internet inter-domain traffic. In *Proceedings of the ACM SIGCOMM 2010 conference*, pages 75–86, 2010.

Jérome Lang, Nicolas Maudet, and Maria Polukarov. New results on equilibria in strategic candidacy. In *Algorithmic Game Theory*, pages 13–25. Springer, 2013.

Jean-Francois Laslier. Spatial approval voting. *Political Analysis*, 14:160–185(26), 2006.

Jean-François Laslier. The leader rule: A model of strategic approval voting in a large electorate. *Journal of Theoretical Politics*, 21(1):113–136, 2009.

Jean-François Laslier and Karine Van der Straeten. A live experiment on approval voting. *Experimental Economics*, 11(1):97–105, 2008.

Jean-François Laslier. *Le vote et la règle majoritaire.* CNRS science politique. CNRS Editions, 2004.

Stephane Laveau and Olivier Faugeras. Oriented projective geometry for computer vision. In *ECCV96*, pages 147–156. Springer-Verlag, 1996.

Dominique Lepelley and Boniface Mbih. The proportion of coalitionally unstable situations under the plurality rule. *Economics Letters*, 24(4):311–315, 1987.

Dominique Lepelley and Boniface Mbih. The vulnerability of four social choice functions to coalitional manipulation of preferences. *Social Choice and Welfare*, 11:253–265, 1994.

Dominique Lepelley and Vincent Merlin. Choix social positionnel et principe majoritaire. *Annales d'Economie et de Statistique*, pages 29–48, 1998.

Dominique Lepelley and Fabrice Valognes. On the Kim and Roush voting procedure. *Group Decision and Negotiation*, 8:109–123, 1999.

Dominique Lepelley and Fabrice Valognes. Voting rules, manipulability and social homogeneity. *Public Choice*, 116:165–184, 2003.

Dominique Lepelley, Ahmed Louichi, and Hatem Smaoui. On Ehrhart polynomials and probability calculations in voting theory. *Social Choice and Welfare*, 30:363–383, 2008.

Ramon Llull. *De arte electionis.* 1299.

Ramon Llull. *Blanquerna.* c. 1285.

Hans Maassen and Thom Bezembinder. Generating random weak orders and the probability of a Condorcet winner. *Social Choice and Welfare*, 19(3):517–532, 2002.

Colin Mallows. Non-null ranking models. *Biometrika*, pages 114–130, 1957.

Andrew Mao, Ariel Procaccia, and Yiling Chen. Better human computation through principled voting. In *Proceedings of of the 27th Conference on Artificial Intelligence (AAAI'13)*, 2013.

K. V. Mardia. Statistics of directional data. *Journal of the Royal Statistical Society. Series B (Methodological)*, 37(3):349–393, 1975.

Andreu Mas-Colell, Michael Whinston, and Jerry Green. *Microeconomic Theory*. Oxford University Press, 1995.

Nicholas Mattei and Toby Walsh. Preflib: A library of preference data. In *Proceedings of Third International Conference on Algorithmic Decision Theory (ADT 2013)*, Lecture Notes in Artificial Intelligence. Springer, 2013.

Kenneth May. A set of independent, necessary and sufficient conditions for simple majority decision. *Econometrica*, 20(4):680–684, 1952.

Robert May. Some mathematical remarks on the paradox of voting. *Behavioral Science*, 16:143–151, 1971.

Iain McLean. The Borda and Condorcet principles: three medieval applications. *Social Choice and Welfare*, 7(2):99–108, 1990.

Vincent Merlin. The axiomatic characterizations of majority voting and scoring rules. *Mathématiques et sciences humaines*, (163), 2003.

Vincent Merlin, Maria Tataru, and Fabrice Valognes. On the probability that all decision rules select the same winner. *Journal of Mathematical Economics*, 33 (2):183–207, 2000.

Vincent Merlin, Monica Tataru, and Fabrice Valognes. On the likelihood of Condorcet's profiles. *Social Choice and Welfare*, 19:193–206, 2002.

Elchanan Mossel. A quantitative Arrow theorem. *Probability Theory and Related Fields*, 154(1-2):49–88, 2012.

Hervé Moulin. *La Stratégie du vote*. Cahiers du CEREMADE. Université Paris IX-Dauphine, Centre de recherche de mathématiques de la décision, 1978.

Hervé Moulin. On strategy-proofness and single peakedness. *Public Choice*, 35 (4):437–455, 1980.

Hervé Moulin. Condorcet's principle implies the no show paradox. *Journal of Economic Theory*, 45(1):53–64, June 1988.

Eitan Muller and Mark Satterthwaite. The equivalence of strong positive association and strategy-proofness. *Journal of Economic Theory*, 14(2):412–418, 1977.

Roger Myerson. Theoretical comparisons of electoral systems. *European Economic Review*, 43(4):671–697, 1999.

Nina Narodytska, Toby Walsh, and Lirong Xia. Manipulation of Nanson's and Baldwin's rules. In *Workshop on Social Choice and Artificial Intelligence*, page 64, 2011.

Richard Niemi and Herbert Weisberg. A mathematical solution for the probability of the paradox of voting. *Behavioral Science*, 13(4):317–323, 1968.

Noam Nisam, Tim Roughgarden, Éva Tardos, and Vijay Vazirani. *Algorithmic Game Theory*. Cambridge University Press, 2007.

Shmuel Nitzan. The vulnerability of point-voting schemes to preference variation and strategic manipulation. *Public Choice*, 47:349–370, 1985.

Svetlana Obraztsova and Edith Elkind. On the complexity of voting manipulation under randomized tie-breaking. *COMSOC 2012*, page 347, 2012.

Svetlana Obraztsova, Edith Elkind, and Noam Hazon. Ties matter: Complexity of voting manipulation revisited. In *The 10th International Conference on Autonomous Agents and Multiagent Systems-Volume 1*, pages 71–78. International Foundation for Autonomous Agents and Multiagent Systems, 2011.

David Parkes and Lirong Xia. A complexity-of-strategic-behavior comparison between Schulze's rule and ranked pairs. In *Proceedings of the 26th AAAI Conference on Artificial Intelligence (AAAI'12)*. American Association for Artificial Intelligence, 2012.

Bezalel Peleg. *Game theoretic analysis of voting in committees*. Number 7 in Econometric Society monographs in pure theory. Cambridge Univ. Press, 1984.

Joaquín Pérez. The strong no show paradoxes are a common flaw in Condorcet voting correspondences. *Social Choice and Welfare*, 18(3):601–616, 2001.

John Pomeranz and Roman Weil Jr. The cyclical majority problem. *Commun. ACM*, 13:251–254, April 1970.

Geoffrey Pritchard and Arkadii Slinko. On the average minimum size of a manipulating coalition. *Social Choice and Welfare*, 27:263–277, 2006.

Geoffrey Pritchard and Mark Wilson. Exact results on manipulability of positional voting rules. *Social Choice and Welfare*, 29:487–513, 2007.

Ariel Procaccia and Jeffrey Rosenschein. Junta distributions and the average-case complexity of manipulating elections. In *Proceedings of the fifth international joint conference on Autonomous agents and multiagent systems*, AAMAS '06, pages 497–504. ACM, 2006.

Ariel Procaccia and Jeffrey Rosenschein. Average-case tractability of manipulation in voting via the fraction of manipulators. In *Proceedings of the 6th international joint conference on Autonomous agents and multiagent systems*, AAMAS '07, page 105. ACM, 2007.

Philip Reny. Arrow's theorem and the Gibbard-Satterthwaite theorem: a unified approach. *Economics Letters*, 70:99–105, January 2001.

Reyhaneh Reyhani. *Strategic manipulation in voting systems*. PhD thesis, 2013.

Reyhaneh Reyhani, Geoffrey Pritchard, and Mark Wilson. A new measure of the difficulty of manipulation of voting rules, 2009.

Harold Ruben. On the moments of order statistics in samples from normal populations. *Biometrika*, 41(1/2):200–227, June 1954.

Donald Saari. Susceptibility to manipulation. *Public Choice*, 64(1):21–41, 1990.

Donald Saari. *Geometry of voting*, volume 3. Springer Science & Business Media, 2012.

Donald Saari and Vincent Merlin. A geometric examination of Kemeny's rule. *Social Choice and Welfare*, 17(3):403–438, 2000.

Mark Satterthwaite. Strategy-proofness and Arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions. *Journal of Economic Theory*, 10(2):187–217, 1975.

Markus Schulze. A new monotonic, clone-independent, reversal symmetric, and Condorcet-consistent single-winner election method. *Social Choice and Welfare*, 36:267–303, 2011.

Arunava Sen. Another direct proof of the Gibbard-Satterthwaite theorem. *Economics Letters*, 70(3):381–385, 2001.

Murat Sertel and Remzi Sanver. Strong equilibrium outcomes of voting games are the generalized Condorcet winners. *Social Choice and Welfare*, 22:331–347, 2004.

Arkadii Slinko. How large should a coalition be to manipulate an election? *Mathematical Social Sciences*, 47(3):289–293, 2004.

Arkadii Slinko and Shaun White. Nondictatorial social choice rules are safely manipulable. In *COMSOC'08*, pages 403–413, 2008.

David Smith. Manipulability measures of common social choice functions. *Social Choice and Welfare*, 16:639–661, 1999.

John Smith. Aggregation of preferences with variable electorate. *Econometrica: Journal of the Econometric Society*, pages 1027–1041, 1973.

Michael Spivak. *A comprehensive introduction to differential geometry. Vol. III.* Publish or Perish Inc., second edition, 1979a.

Michael Spivak. *A comprehensive introduction to differential geometry. Vol. IV.* Publish or Perish Inc., second edition, 1979b.

Jorge Stolfi. Oriented projective geometry. In *Proceedings of the third annual symposium on Computational geometry*, SCG '87, pages 76–85. ACM, 1987.

Alan Taylor. *Social choice and the mathematics of manipulation.* Outlooks Series. Cambridge University Press, 2005.

Nicolaus Tideman. Independence of clones as a criterion for voting rules. *Social Choice and Welfare*, 4:185–206, 1987.

Nicolaus Tideman. *Collective Decisions And Voting: The Potential for Public Choice*. Ashgate, 2006.

Ilia Tsetlin, Michel Regenwetter, and Bernard Grofman. The impartial culture maximizes the probability of majority cycles. *Social Choice and Welfare*, 21: 387–398, 2003.

Gordon Tullock and Colin Campbell. Computer simulation of a small voting system. *The Economic Journal*, 80(317):97–104, 1970.

Gary Ulrich. Computer generation of distributions on the m-sphere. *Applied Statistics*, pages 158–163, 1984.

Karine Van der Straeten, Jean-François Laslier, Nicolas Sauger, and André Blais. Strategic, sincere, and heuristic voting under four election rules: an experimental study. *Social Choice and Welfare*, 35(3):435–472, 2010.

John Von Neumann and Oskar Morgenstern. *Theory of games and economic behavior*. Princeton University Press, 1944.

John Von Neumann, Oskar Morgenstern, Harold Kuhn, and Ariel Rubinstein. *Theory of Games and Economic Behavior (Commemorative Edition)*. Princeton Classic Editions. Princeton University Press, 2007.

Toby Walsh. Manipulability of single transferable vote. In F. Brandt, V. Conitzer, L. Hemaspaandra, J.-F. Laslier, and W. Zwicker, editors, *Computational Foundations of Social Choice*, number 10101 in Dagstuhl Seminar Proceedings. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, Germany, 2010a.

Toby Walsh. An empirical study of the manipulability of single transferable voting. In *ECAI*, volume 10, pages 257–262, 2010b.

Tiance Wang, Paul Cuff, and Sanjeev Kulkarni. Condorcet methods are less susceptible to strategic voting. 2014.

G.S. Watson and E.J. Williams. On the construction of significance tests on the circle and the sphere. *Biometrika*, 43(3/4):344–352, 1956.

Tjark Weber. Alternatives vs. outcomes: A note on the Gibbard-Satterthwaite theorem. Technical report, 2009.

Andrew Wood. Simulation of the Von Mises Fisher distribution. *Communications in Statistics-Simulation and Computation*, 23(1):157–164, 1994.

Eduardo Xavier. A note on a maximum k-subset intersection problem. *Information Processing Letters*, 112(12):471–472, 2012.

Lirong Xia and Vincent Conitzer. Generalized scoring rules and the frequency of coalitional manipulability. In *Proceedings of the 9th ACM Conference on Electronic Commerce*, pages 109–118. ACM, 2008.

Lirong Xia, Michael Zuckerman, Ariel Procaccia, Vincent Conitzer, and Jeffrey Rosenschein. Complexity of unweighted coalitional manipulation under some common voting rules. In *International Joint Conference on Artificial Intelligence*, pages 348–353, 2009.

Peyton Young and Arthur Levenglick. A consistent extension of Condorcet's election principle. *SIAM Journal on Applied Mathematics*, 35(2), 1978.

Michael Zuckerman, Ariel Procaccia, and Jeffrey Rosenschein. Algorithms for the coalitional manipulation problem. *Artificial Intelligence*, 173(2):392–412, 2009.

Michael Zuckerman, Omer Lev, and Jeffrey Rosenschein. An algorithm for the coalitional manipulation problem under Maximin. In *The 10th International Conference on Autonomous Agents and Multiagent Systems - Volume 2*, AAMAS '11, pages 845–852, 2011.