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Towards less manipulable voting systems

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Note to the reader

This is a (rather quick) translation of the original French version of this memoir, which is entitled: "Vers des modes de scrutin moins manipulables". I apologize for the possible spelling and grammar mistakes in this English version.

Abstract

We investigate the coalitional manipulation of voting systems: is there a subset of voters who, by producing an insincere ballot, can secure an outcome that they strictly prefer to the candidate who wins if all voters provide a sincere ballot?

From a theoretical point of view, we develop a framework that allows us to study all kinds of voting systems: ballots can be linear orders of preferences over the candidates (*ordinal* systems), grades or approval values (*cardinal* systems) or even more general objects. We prove that for almost all voting systems from literature and real life, manipulability can be strictly diminished by adding a preliminary test that elects the Condorcet winner if one exists. Then we define the notion of *decomposable* culture and prove that it is met, in particular, when voters are independent. Under this assumption, we prove that for any voting system, there exists a voting system that is ordinal, has some common properties with the original voting system and is at most as manipulable. As a consequence of these theoretical results, when searching for a voting system whose manipulability is minimal (in a class of reasonable systems), investigation can be restricted to those that are ordinal and meet the Condorcet criterion.

In order to provide a tool to investigate these questions in practice, we present SVVAMP, a Python package we designed to study voting systems and their manipulability. We use it to compare the coalitional manipulability of several voting systems in a variety of cultures, i.e. probabilistic models generating populations of voters with random preferences. Then we perform the same kind of analysis on real elections. Lastly, we determine voting systems with minimal manipulability for very small values of the number of voters and the number of candidates and we compare them with classical voting systems from literature and real life. Generally speaking, we show that the Borda count, Range voting and Approval voting are especially vulnerable to manipulation. In contrast, we find an excellent resilience to manipulation for the voting system called *IRV* (also known as *STV*) and its variant *Condorcet-IRV*.

Thanks

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Contents

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Part I

Theoretical study of manipulability

Chapter 3

Majoritarian criteria

In previous chapters, we have recalled the Condorcet criterion (**Cond**), then we have presented the informed majority coalition criterion (**InfMC**) and the resistant-Condorcet criterion (**rCond**). Now, we will see how these properties fit in a larger family of majoritarian criteria and develop links with concepts from game theory, such as the set of strong Nash equilibria (SNE) and the ability to reach them.

In section 3.1, we define some other criteria related to the notion of majority. In voting theory, it is usual to consider the majority criterion, whose definition will be recalled, and which we call the *majority favorite criterion* (**MajFav**) in order to distinguish it from other majoritarian criteria. We also introduce the *ignorant majority coalition criterion* and the *majority ballot criterion*. The initial motivation of these definitions is simply practical: they give criteria that are easy to test and make it possible to prove that a given voting system meets **InfMC**.

Indeed, we show in section 3.2 that all the other criteria under study imply **InfMC**. Moreover, she show that they form a chain of implications[1], from the strongest criterion (**Cond**) to the weakest one (**InfMC**).

In the introduction, we have already mentioned strong Nash equilibria (SNE) and we have discussed the interest of such a notion[2]. In particular, the non-manipulability of a configuration means precisely that sincere voting is an SNE for the corresponding preferences. Brill and Conitzer (2015) showed that, for a voting system meeting **InfMC**, if there exists a Condorcet winner for voters' sincere preferences, then she is the only one who can win in a strong Nash equilibrium. This connection between a majority criterion and a notion of equilibrium leads us in section 3.3 to consider several equilibrium criteria for a voting system: the fact that the existence of an SNE is guaranteed by the existence of a Condorcet winner (**XSNEC**) or by the existence of a Condorcet-admissible candidate (**XSNEA**) and the restriction of SNE to the Condorcete winners (**RSNEC**) or to the Condorcet-admissible candidates (**RSNEA**). If the two first ones are existence criteria, the two last ones may be seen as a weak version of unicity criteria. We reveal the relations of implication between these equilibrium criteria and the majoritarian criteria. In particular, not only defining the criteria makes it possible to extend the result by Brill and Conitzer by showing that **InfMC** implies **RSNEA**, but we show also that these two criteria are actually equivalent.

[1] In all rigor, we will see that this chain of implications is complete if the electoral space allows any candidate as most liked, which is a common assumption (definition 1.10).

[2] About SNE and several variants of this concept, one may consult Bernheim et al. (1987).

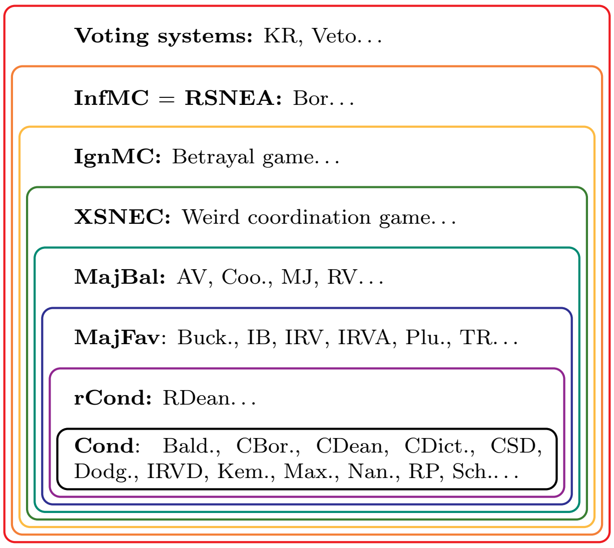

Figure 3.1: Inclusion diagram of majoritarian criteria. We assume that the preferences are antisymmetric and that the electoral space allows any candidate as most liked.

In proposition 2.3, we wrote without proof that almost all classic voting system meet **InfMC**. In section 3.4, we prove this result and we precise it, by studying which criteria are met by classic voting systems and we detail this study, depending on the number of candidates. This will allow us to demonstrate progressively the inclusion diagram of figure 3.1, which has the drawback to spoil a dramatic suspense but the advantage to provide the reader with a road map for this chapter. This diagram reads this way. For example, the set of voting systems meeting **MajFav** is included (in general, stricly) in the set of those meeting **MajBal**; Coombs' method belong to the later, but generally not to the former (except in particular electoral spaces, such as the one of strict total orders with 2 candidates).

This section 3.4, dedicated to criteria met by different voting systems is a synthesis of classic results from the literature and original contributions. To the best of our knowledge, results about **InfMC**, **IgnMC**, **MajBal** and **rCond** are original, since we defined those criteria; that being said, results about **IgnMC** and **MajBal** are quite easy from the definition, since these criteria were precisely conceived to be easily testable. Results about **InfMC** and **rCond** require usually more effort. Result about **MajFav** and **Cond** are classic, with the following nuances. Firstly, we found no trace of exhaustive study of **MajFav** for all IPSR-EA in the literature; that being said, it would be surprising that the results we present had not been formulated, because it is a classic criterion and classic voting systems. Secondly, results about IB are all original, since this voting system is a contribution of this memoir.

Lastly, in section 3.5, we propose a reflection about the criteria under study, in terms of quantity of information necessary to coordinate the manipulation strategies and to reach SNE. This way, we discuss why these criteria may be considered as desirable for a voting system.

3.1 Definition of majoritarian criteria

In previous chapters, we have already recalled the definition of the Condorcet criterion (definition 1.32) and we introduced the informed majority coalition criterion (definition 2.1) and the resistant-Condorcet criterion (definition 2.18).Now, we define three other criteria, which will prove convenient tools to prove that a given voting meets **InfMC** (and the it is, as a consequence, concerned by the weak Condorcification theorem 2.9).

Before defining the criteria themselves, let us define the notion of *majority favorite*.

Definition 3.1 (majority favorite)
For a configuration $\omega \in \Omega$ and a candidate $c \in \mathcal{C}$, we say that c is a *majority favorite* in ω iff a strict majority of voters prefer strictly c to any other candidate: $|\forall d \in \mathcal{C} \setminus \{c\}, c \, \mathrm{PP}_v \, d| > \frac{V}{2}$. When preferences are strict orders (weak or total), it simply means that more than half the voters put c in first position in their order of preference, without tie with other candidates.

If c is a majority favorite, than c is obviously a resistant Condorcet winner.

Definition 3.2 (majority criteria)
We say that that f meets the *majority favorite criterion* (**MajFav**) iff, for any configuration $\omega \in \Omega$ and for any candidate $c \in \mathcal{C}$, if c is majority favorite in ω, then $f(\omega) = c$.

We say that f meets the *majority ballot criterion* (**MajBal**) iff, for any candidate c, there exists an assignment of ballots to the voters that meets the following property: if it is respected by a strict majority of voters, then c is declared the winner. Formally, this condition reads: $\forall c \in \mathcal{C}, \exists \psi^c \in \Omega$ s.t. $\forall \omega \in \Omega, \left[|\omega_v = \psi_v^c| > \frac{V}{2} \Rightarrow f(\omega) = c\right]$.

We say that f meets the *ignorant majority coalition criterion* (**IgnMC**) iff any majority coalition can choose the result, whatever the other voters do. Formally, $\forall M \in \mathcal{P}(\mathcal{V})$, if $\mathrm{card}(M) > \frac{V}{2}$ then: $\forall c \in \mathcal{C}, \exists \omega_M \in \Omega_M$ s.t. $\forall \omega_{\mathcal{V}\setminus M} \in \Omega_{\mathcal{V}\setminus M}, f(\omega_M, \omega_{\mathcal{V}\setminus M}) = c$.

Like for **InfMC**, **Cond** or **rCond**, each notation **MajFav**, **MajBal** or **IgnMC** designates the criterion itself or the set of SBVS (over Ω) meeting it.

It is quite easy to extend the majoritarian criteria to general voting systems defined in section 1.4. In that case, it appears that **InfMC**, **IgnMC** and **MajBal** are properties for the *game form*: these criteria describe the power given by the processing rule to a strict majority of voters. In contrast, **MajFav**, **rCond** and **Cond** are properties for the *voting system*: they establish a link between voter's preferences and the sincere result. For these last criteria, it is necessary to explicit the electoral space and the sincerity functions.

In an anonymous electoral space (section 1.2.2), we can also define the *majority unison ballot criterion* (**MajUniBal**) by the following property: for any candidate c, there exists a ballot ψ_0^c (belonging to any Ω_v, since they are all identical) such that, if it is used by a strict majority of voters, then c is declared the winner. We preferred not including this notion in the inclusion diagram of figure 3.1 because it is not defined in all electoral spaces.

The more general definition of **MajBal** avoids the need to have an anonymous electoral space. This wording makes it also possible to apply this criterion to non-anonymous voting systems (even if the electoral space itself is so): indeed, it is possible to give a different instruction to each voter; the criterion only requires that, if the instructions are followed by a strict majority of voters, candidate c is elected. In the case of a general voting system (section 1.4), this definition has also the advantage that it is stable by *isomorphism of voting system*, in the following sense: if we change the labels of the ballots while changing the processing rule accordingly, it has no impact on the fact that the criterion is met or not.

Criterion **IgnMC** is very similar to **InfMC** (definition 2.1): the only difference resides in the exchange of quantifiers "$\exists \omega_M \in \Omega_M$" and "$\forall \omega_{\mathcal{V}\setminus M} \in \Omega_{\mathcal{V}\setminus M}$". In practice, if quantifiers are in this order, which is the one used in **IgnMC**, manipulation is more difficult because manipulators vote first and other voters can reply; in the inverse order, which is the one used in **InfMC**, sincere voters' ballots are known first, and manipulators can choose their vote depending on them. So, **IgnMC** is a more demanding criterion than **InfMC**.

3.2 Implications between majoritarian criteria

The following propostion establishes a hierarchy between the six majoritarian criteria we have defined. We present them in the form of inclusions in order to show the connection with diagramme 3.1 but it is equivalent to present them as implications. For example, the inclusion **Cond** $\subseteq$ **rCond** is another way to say that for any SVBE f, we have: $f \in$ **Cond** $\Rightarrow f \in$ **rCond**.

Proposition 3.3

We have the following inclusions.

1. **Cond** $\subseteq$ **rCond** $\subseteq$ **MajFav**.
2. **MajBal** $\subseteq$ **IgnMC** $\subseteq$ **InfMC**.

If the electoral space allows any candidate as most liked (which is a common assumption, cf. definition 1.10), then we have also **MajFav** $\subseteq$ **MajBal**. *Under this assumption, we have:*

$$\textbf{Cond} \subseteq \textbf{rCond} \subseteq \textbf{MajFav} \subseteq \textbf{MajBal} \subseteq \textbf{IgnMC} \subseteq \textbf{InfMC}.$$

All inclusions are strict in general, i.e. all converse implications are false.

Proof. **Cond** $\subseteq$ **rCond** $\subseteq$ **MajFav**: this chain of inclusions is immediately deduced from the facts that a majority favorite is a resistant Condorcet winner and that a resistant Condorcet winner is a Condorcet winner.

MajBal $\subseteq$ **IgnMC**: for any candidate c, **MajBal** ensure that there exists a assignment of ballots ψ^c that makes it possible to elect c if it is used by a strict majority of voters. If a majority coalition wishes to elect c, they just have to use these ballots so that c wins, whatever the other voters reply. So, the voting system meets **IgnMC**.

IgnMC $\subseteq$ **InfMC**: this inclusion is immediately deduced from the remark we did about the order of quantifiers in the definition of these criteria.

MajFav $\subseteq$ **MajBal**: let us consider an assignment of ballots where each voter claims that she strictly prefers a certain candidate c to all other candidates, which is possible because the electoral space allows any candidate as favorite. If the voting system under consideration meets **MajFav**, it is sufficient that a strict majority of voters respects this assignment so that c wins. So, the voting system meets **MajBal**.

The fact that all converse implications are false will be proven when studying the classic voting systems in section 3.4. As for now, we can have an overview of counter-examples in figure 3.1. □

In an anonymous electoral space, it is clear that we have also **MajUniBal** $\subseteq$ **MajBal**. If, moreover, the electoral space allows any candidate as most liked, then we have also **MajFav** $\subseteq$ **MajUniBal**.

3.3 Connection with the strong Nash equilibria (SNE)

As Gibbard (1973) notices, the pair of a voting system and a configuration of preferences ω for the voters defines a game, in the usual sense of game theory: each player-voter has a set of strategies, well-determined objectives and there exists a rule that makes it possible to decide the result, depending on players' strategies. So, it is immediate to adapt the usual notion of *strong Nash equilibrium* in this context.

By the way, we can remark that we using the framework of general voting systems (section 1.4), any game can be expressed as a voting system and a state of preferences for the voters. Hence the terminology of *game form*: it is a proto-game defining the procedure to follow, and lacking only the information of players' preferences about the possible outcomes of the game.

Definition 3.4 (strong Nash equilibrium)

Let f be an SBVS and $(\omega, \psi) \in \Omega^2$.

We say that ψ is a *strong Nash equilibrium* (ENF) for preferences ω in system f iff ψ is a strong Nash equilibrium in the game defined by (f, ω). I.e., there exists no configuration ϕ such that:

$$\begin{cases} f(\phi) \neq f(\psi), \\ \forall v \in \mathrm{Sinc}_\omega(f(\psi) \to f(\phi)), \phi_v = \psi_v. \end{cases}$$

In the case of a general voting system, using notations from section 1.4, we would adapt the definition by saying that a vector $S = (S_1, \ldots, S_V)$ of strategies for the voters (i.e. ballots) is an SNE for a configuration of preference ω. As we have already noticed, the fact that a configuration ω is not manipulable simply means that sincere voting $s(\omega)$ is an SNE for ω.

For an SBVS, the formalism is simplified because notions of authorized ballot and possible state of preference are identified. The fact that a configuration ω is not manipulable means that configuration ω (seen as ballots) is an SNE for ω (seen as preferences).

As we mentioned in the introduction of this chapter, Brill and Conitzer (2015) showed that, in a system meeting **InfMC**, if a configuration is Condorcet, then only the Condorcet winner can be the winner of an SNE. This result suggests a deep connection between some of the majoritarian criteria, not only with manipulability, but with the notion of SNE in general. After reading this article and thanks to fruitful discussions in front of a probably non-alcoholic beverage with Markus Brill, whom we thank here, we examined a few questions connected to this issue. So, we have defined the four following equilibrium criteria and we have studied their connection with the majoritarian criteria.

Definition 3.5 (equilibrium criteria)

We say that f meets the criterion of *restriction of possible SNE to Condorcet-admissible candidates* (**RSNEA**) iff for any $(\omega, \psi) \in \Omega^2$: if ψ is an SNE for preferences ω, then $f(\psi)$ is Condorcet-admissible in ω.

We say that f meets the criterion of *restriction of possible SNE to Condorcet winners* (**RSNEC**) iff for any $(\omega, \psi) \in \Omega^2$: if ψ is an SNE for preferences ω, then $f(\psi)$ is Condorcet winner in ω.

We say that f meets the criterion of *existence of an SNE for any Condorcet winner* (**XSNEC**) iff $\forall(\omega, c) \in \Omega \times \mathcal{C}$: if c is Condorcer winner in ω, then there exists ψ that is an SNE for ω and such that $f(\psi) = c$.

We say that f meets the criterion of *existence of an SNE for any Condorcet-admissible candidate* (**XSNEA**) iff $\forall(\omega, c) \in \Omega \times \mathcal{C}$: if c is Condorcet-admissible in ω, then there exists ψ that is an SNE for ω and such that $f(\psi) = c$.

As we noticed in the introduction of this chapter, the criterion **RSNEC** can be seen as a unicity criterion, not for the SNE, but for the possible winner in a SNE. In an electoral space where there exists at least a semi-Condorcet configuration (i.e. if it is possible to have a candidate who is Condorcet-admissible but not Condorcet winner, as in the electoral space of strict total orders with an even number of voters), criterion **RSNEA** is a weaker version, which we will proved met by most usual voting systems. If there is no semi-Condorcet configuration (for example, in the electoral space of strict total orders with an odd number of voters), the two notions are equivalent. Similarly, in that case, existence criteria **XSNEC** and **XSNEA** are equivalent.

Criterion **RSNEA** implies, in particular, Brill and Conitzer's criterion: when it is met, if ω is a Condorcet configuration, then the Condorcet winner is the only Condorcet-admissible candidate (proposition 1.31), so she is the only possible winner of an SNE.

Now, we study the connections between these equilibrium notions and the majoritarian criteria. We focus first on **RSNEA** and **XSNEC**, which will show more naturally integrable in the chain of inclusions of proposition 3.3, which justifies their position in the inclusion diagram of figure 3.1.

Proposition 3.6

We assume that the electoral space allows any candidate as most liked.
Then **InfMC** $=$ **RSNEA**.

In the same kind of idea, Sertel and Sanver (2004) show that in the electoral space of strict weak orders, if a voting system meets **MajFav**, then the winner of an SNE is necessarily a weak Condorcet winner (which is equivalent to a Condorcet-admissible candidate in this context): in other words, **MajFav** $\subseteq$ **RSNEA**. SO, proposition 3.6 extends this result. As for the authors of this paper, they generalize this result in another direction by considering a family of variants for criteria **MajFav** and **RSNEA**.

Proof. If a voting system f meets **InfMC**, then it meets **RSNEA**: indeed, if the winner candidate of a configuration ψ is not Condorcet-admissible in ω, then a strict majority of voters prefer another candidate (in the sense of ω) and they can make her win, by virtue of **InfMC**. Hence ψ cannot be an SNE for ω.

Now, let us consider a voting system that does not meet **InfMC** and let us show that it does not meet **RSNEA**. By definition, there exists a candidate c, a coalition with a strict majority M, a configuration $\psi_{\mathcal{V}\setminus M}$ of other voters such that, whatever the ballots ψ_M chosen by the coalition, $f(\psi_M, \psi_{\mathcal{V}\setminus M}) \neq c$.

Let us consider the state $\psi_{\mathcal{V}\setminus M}$ mentioned above (such that the minority makes the election of c impossible) and an arbitrary state ψ_M of the majority coalition M. Let us note $a = f(\psi_M, \psi_{\mathcal{V}\setminus M})$, which is by assumption distinct of c.

Now, let us consider the following profile of preferences ω.

Voters in M (majority)	Voters in $\mathcal{V} \setminus M$ (minority)
c	a
a	c
Others	Others

Clearly, c is Condorcet winner, hence she is the only Condorcet-admissible candidate.

If ballots are ψ, then candidate a is winner by assumption. The only possible deviations come from voters in M, and their only wish is to make c win, but it is impossible. As a consequence, ψ is an SNE for ω whose winner is not Condorcet-admissible in ω. Hence, the voting system under consideration does not meet **RSNEA**. □

Proposition 3.7

We assume that the binary relations of preference are antisymmetric.

We have: **MajBal** $\subseteq$ **XSNEC**. *The converse is false in general.*

Proof. Let f be an SBVS meeting **MajBal** and ω a configuration. If there exists a Condorcet winner c in ω, then consider configuration ψ^c whose existence is granted by **MajBal**: if a strict majority of vtoers respect this assignment of ballots, then c wins. It is easy to see that it is an SNE for ω: indeed, if a subset of voters want to make win some candidate d instead of c, then since c is Condorcet winner and since preferences are antisymmetric, these voters are a strict minority. So, by definition of ψ^c, candidate c remains the winner. Hence, f meets **XSNEC**.

In order to show that the converse is false, let us consider the electoral space of strict total orders for $V = 3$ voters and $C = 3$ candidates. We will define a voting system we call the *weird coordination game* and prove that it meets **XSNEC** but not **MajBal**.

Here is the rule.

1. If there exists at least a doubleton of voters whose voters put the same candidate on top of their ballots but their two last candidates in different orders, then their top candidate wins.

2. In all other cases, the dean (i.e. a candidate fixed in advance) is declared the winner.

So, if two voters want to elect some candidate c who is not the dean, they need to coordinate cautiously to put c on top and especially to ensure that their ballots are not identical.

Let us show that this system meets **XSNEC**. Consider a Condorcet configuration ω. Since $V = 3$ and $C = 3$, a Condorcet winner is necessarily the most liked candidate for at least one voter (otherwise, another candidate is on top of at least two ballots, so she is majority favorite and *a fortiori* Condorcet winner).

If the Condorcet winner is the most liked for at least two voters, then they just need to desynchronize the bottom of their ballots so that she is elected, and the configuration is an SNE: they are fully satisfied, and if the third voter is not, well, too bad for her.

If the Condorcet winner is the most liked candidate for exactly one voter, then it is easy to show that, up to permuting some voters and/or candidates, the profile ω is of the following type.

$$\begin{array}{c|c|c} c & a & b \\ \text{Autres} & c & c \\ & b & a \end{array}$$

Candidate c is Condorcet winner. Up to exchanging a and b and the two last voters, we can assume that the dean is not b: so, it is c or a.

Then, consider the following profile ψ.

$$\begin{array}{c|c|c} c & a & \mathbf{c} \\ a & c & \mathbf{b} \\ b & b & a \end{array}$$

Candidate c wins and this is an SNE for ω: indeed, the second voter cannot change the result and the third voter cannot make b win (she cannot exploit neither rule 1 because b cannot become a majority favorite, nor rule 2 because b is not the dean). Hence, the weird coordination game meets **XSNEC**.

Now, let us prove that it does not meet **MajBal**. Consider a candidate c, distinct from the dean, and let us try to conceive an assignment of ballots ψ^c ensuring a victory of c as soon as it is used by a strict majority of voters. For each pair of voters, there must be exactly one ballot $c \succ a \succ b$ and a ballot $c \succ b \succ a$, which is impossible: indeed, we authorize only two different ballots for three voters, so there exists two voters with the same ballots. If two voters use this identical ballot, the other voter just needs not to put c on top so that the dean is elected, which contradicts the definition of ψ^c. Hence, the weird coordination game does not meet **MajBal**. □

Proposition 3.8

We assume that the electoral space allows any candidate as most liked.

Then **XSNEC** $\subseteq$ **IgnMC**. *The converse is false in general.*

Proof. Let us consider a voting system that does not meet **IgnMC**. By definition, there exists a candidate c and a coalition with a strict majority M such that for any assignment ψ_M of ballots for the coalition, there exists a response $\psi_{\mathcal{V}\setminus M}$ from other voters such that $f(\psi_M, \psi_{\mathcal{V}\setminus M}) \neq c$.

Consider a profile of preference ω of the following type.

Voters in M (majority)	Voters in $\mathcal{V} \setminus M$ (minority)
c	Others
Others	c

Clearly, c is the Condorcet winner.

In a state ψ that is an SNE for ω, it is impossible that c wins: indeed, members of $\mathcal{V} \setminus M$ can prevent a victory for c and they want to do it, whatever the alternate result. So, the voting system does not meet **XSNEC**, which proves the inclusion **XSNEC** $\subseteq$ **IgnMC** by contraposition.

In order to prove that the converse is false, we are going to define a voting system we call the *betrayal game* and show that it meets **IgnMC** but not **XSNEC**. We will use an non-ordinal voting system to reveal the underlying intuition, then we will show that, with a bit of effort, this counterexample can be adapted to an ordinal system.

Here is the rule. Each voter v emits a ballot of the type $(M_v, c_v) \in \mathcal{P}(\mathcal{V}) \times \mathcal{C}$: she announces a coaltion, i.e. a set of voters, and the index of a candidate. In order to get the idea, we can interpret M_v as the set of voters that v considers as her "friends". We could demand that M_v contains v herrself, but this additional assumption has no impact on our demonstration. Now, here is how we determine the winning candidate.

1. If there exists a coalition with a strict majority of voters M and a candidate c such that each voter in M emits the ballot (M, c), then c wins (there is no ambiguity because such a coalition is necessarily unique).

2. In all other cases, the result is $\sum_{v \in \mathcal{V}} c_v$, moved by modulo C to interval $[\![1, C]\!]$.

Clearly, this voting system meets **IgnMC**: indeed, if a coalition with a strict majority M want to make a certain candidate c win, they can simply coordinate to announce the ballot (M, c), which makes c win, whatever the others voters do.

In order to show that the betrayal game does not meet **XSNEC**, consider the following profile, where candidate c is Condorcet winner.

$$\begin{array}{c|c|c}
c & a & b \\
a & c & c \\
b & b & a
\end{array}$$

Can there be an SNE where c is elected? If c is elected by virtue of rule 1, then the second voter, the third voter or both participate to coalition M since it is a majority. Any one of them can betray coalition M by naming the coalition consisting of only herseld and make a or b win by choosing the adequate modulo.

The other possibility is that c is elected by virtue of rule 2. Then the second voter can name the coalition consisting of herself and make a win by choosing the adequate modulo.

As a consequence, there is no SNE where c is elected[3], which proves that the betrayal game does not meet **XSNEC**.

If one prefers a purely ordinal example, it is sufficient to consider additional candidates $d_1, \ldots, d_4$ and to coder coalition M in the order given on the four last candidates of a ballot (4 candidates provide a sufficient expressiveness because $4! = 24 > 2^3 = 8$), in a way similar to what we did in section 2.8 to define a very manipulable Condorcet system. Then, we consider a variant of the above profile, where candidates $d_1, \ldots, d_4$ are last for all voters and we show similarly that there is no SNE where c is elected. □

Until now, we focused on criteria **RSNEA** and **XSNEC** and we have proved their position in the inclusion diagram of figure 3.1. Criteria **RSNEC** and **XSNEA** are obviously stronger versions (respectively), in the sense that **RSNEC** $\subseteq$ **RSNEA** and **XSNEA** $\subseteq$ **XSNEC**. If the electoral space contains no semi-Condorcet configuration (such as the electoral space of strict total orders with an odd number of voters), then it is immediate that these inclusions are actually equalities: **RSNEC** = **RSNEA** et **XSNEA** = **XSNEC**.

As a consequence, for all criteria inside the zone **RSNEA** (resp. **XSNEC**), in particular **Cond**, there exists some SBVS that also meet **RSNEC** (resp.

[3]In fact, with this configuration of preference, it is even possible to prove that there is no SNE at all.

XSNEA): indeed, we can simply consider an electoral space with no semi-Condorcet configuration. In that case, any system meeting a criterion stronger than **RSNEA** (resp. **XSNEC**) meets also **RSNEC** (resp. **XSNEA**), since these notions are equivalent.

On the other side, for all criteria inside the zone **RSNEA** (resp. **XSNEC**), in particular **Cond**, there exists some SBVS that do not meet **RSNEC** (resp. **XSNEA**). Indeed, consider the electoral space of strict total orders with an even number of voters and the voting system Condorcet-dean, where the dean is denoted a.

1. If half of the voters prefer a candidate c, distinct from the dean, to all other candidates and if the other half prefer all the other candidates to c, then c is Condorcet-admissible but there is no SNE where she is elected (there is always a possible deviation in favor of a). So, this system does not meet **XSNEA**.

2. If half of the voters prefer the dean a to all other candidates and if the other half have a last in their order of preference, then sincere voting is a SNE where a is elected, although she is only Condorcet-admissible but not Condorcet winner. So, this system does not meet **RSNEC**.

As a consequence, **XSNEA** is included in **XSNEC** but it has, in general, no simple relation of inclusion with the criteria inside the zone **XSNEC**: for example, it may have a non-empty intersection with **Cond** but also with its complement. It is similar for **RSNEC**, which is included in **RSNEA** but has no simple relation of inclusion with criteria stronger than **RSNEA**.

By the way, we can remark than, in an electoral space where there exists at least one semi-Condorcet configuration, criteria **XSNEA** and **RSNEC** are incompatible: indded, in a semi-Condorcet configuration, **XSNEA** demands that there exists a SNE where a Condorcet-admissible candidate is elected, but since this candidate is not a Condorcet winner, this contradicts **RSNEC**.

For all these reasons, it seems that criteria **XSNEA** and **RSNEC** are less "natural" than their respective weaker versions **XSNEC** and **RSNEA**; and we have chosen not to include them in the inclusion diagram of figure 3.1 for the sake of readability.

To sum up this section, we have studied the "unicity" and the existence of an SNE, in connection with Condorcet notions.

For the "unicity" of the equilibrium, we have defined the criterion **RSNEC**, which demands than only a Condorcet winner can benefit from an SNE. But eventually, it seems that **RSNEA** is a more natural property than its weak version **RSNEC**, in the sense that it is more frequently met by usual systems, knowing that these notions are of course equivalent when there is no semi-Condorcet configuration. Whereas it was quite clear that **InfMC** implies **RSNEA**, we have shown that actually, these two criteria are equivalent.

We also wanted to know what guarantees the existence of an SNE when preferences are Condorcet, i.e. the criterion **XSNEC**. We have seen that this criterion is more natural than its strong version **XSNEA**. We have proved that **MajBal** is sufficient but not necessary (as illustrated by the weird coordination game) and that **IgnMC** is necessary but not sufficient (as shown by the betrayal game).

3.4 Majoritarian criteria met by the usual voting systems

Now, we understand better the position of majoritarian and equilibrium criteria in the inclusion diagram of figure 3.1. Before proposing a reflection about the motivation and the consequences of these criteria, we will study which criteria are met by the usual voting systems. In this whole section, we will consider a fixed number of candidates C but a variable number of voters V: so, we will say that a voting system[4] meets a given criterion for a given value of C iff it meets it for any number of voters; and that it does not meet it if there exists a configuration that contradicts the criterion, whatever the number of voters.

The motivation of this choice is two-sided. One one hand, when an election is organized in the real life, the number of candidates is almost always known (at least at the moment where ballots must be chosen by the voters), whereas it is very frequent that the number of voters in unknown in advance. On the other hand, fixing the number of voters may lead to difficult questions of quantification: so, it may happen that a given system meets some criterion for a value of (V, C) for backpacking-type reasons, whereas it does not meet the criterion in general for this value of C (and especially for a large enough number of voters). In other words, it is practical and technically more tractable to let the number of voters vary. For this, our approach is the same as Smith (1973), who studied some of these questions for PSR with a variable number of voters.

3.4.1 Cardinal voting systems

Proposition 3.9

1. *Range voting, the majority judgment and approval voting meet* **MajBal**.

2. *Let* $C \geq 3$. *Assume that, for any voter* v *and for any strict total order of preference* p_v *over the candidates, voter* v *has at least at her disposal:*

   - *One state* ω_v *such that* $\mathrm{P}_v(\omega_v) = p_v$ *and where she attributes the maximal grade to her most-liked candidate and the minimal grade to all the others;*
   - *And one state* ω'_v *such that* $\mathrm{P}_v(\omega'_v) = p_v$ *and where she attributes the maximal grade to her two most-liked candidates and the minimal grades to all the others.*

   Then these voting system do not meet **MajFav**.

Proof. 1. Let c be a candidate. Consider a ballot consisting of attributing the maximal grade to c and the minimal grade to other candidates. If a strict majority of voters use this ballot, then c is elected. So, these voting systems meet **MajBal**. If we define these voting systems reasonably, in an anonymous electoral space, then we have just proved that they even meet **MajUniBal**.

2. Consider the following configuration where candidates are denoted $c, d_1, \ldots, d_{C-1}$ and where minimal and maximal grades are respectively 0 and 1

[4] To respect a mathematically rigorous terminology, we should say a *meta-voting system*, because by varying the number of voters, we consider systems that are defined over several different electoral spaces. However, we will avoid this coquetry of language and we will keep on speaking about *voting systems*, even in that case.

by convention.

2	1
$c : 1$	$d_1 : 1$
$d_1 : 1$	$d_2 : 0$
$d_2 : 0$	$\vdots$
$\vdots$	$d_{C-1} : 0$
$d_{C-1} : 0$	$c : 0$

Then candidate c is majority favorite but d_1 is elected. $\square$

Property 2 is also true for $C = 2$ but we have preferred not to mention it in the proposition because the assumption, in that case, is debatable. Indeed, if there are only two candidates and if the binary relations of preference are strict total orders, then there is a natural sincerity function, since each voter has a dominant strategy: attribute the maximal grade to her most-liked candidate and the minimal grade to the other. With this sincerity function, these voting systems become equivalent to the simple majority vote (mentioned in the introduction) and they trivially meet **MajFav** (and even **Cond**). In contract, as soon as $C \geq 3$, the assumption that it is possible to sincerely vote for one's most-liked candidate or for one's two most-liked candidates is quite natural in practice, and compliant with the spirit of approval voting, range voting and the majority judgment.

In all the rest of this section 3.4, dedicated to usual voting systems, we work in the electoral space of strict total orders.

As a reminder, Baldwin, Condorcet-Borda (Black's method), Condorcet-dean, Condorcet-dictatorship, CSD, Dodgson, IRVD, Kemeny, Maximin, Nanson, RP and Schulze's method meet **Cond**, which immediately determines their positions in the inclusion diagram of figure 3.1.

3.4.2 Plurality with one or several rounds

Even if we will see Plurality again as a particular case of proposition 3.13 about PSR, IRV as a particular case of proposition 3.16 about IPSR-SE and IRVA as a particular case of proposition 3.21 about IPSR-EA, we wil first study these voting systems on their own and take the occasion to examine the two-round system also. On one hand, these voting systems are widely spread in practice (except IRVA) and they all present some similarities of principle. On the other hand, this will allow us to prove, in a simple context, some result that will be useful afterwards.

Proposition 3.10

1. *Plurality, the two-round system (TR or ITR), IRV and IRVA meet* **MajFav**.
2. *They meet* **rCond** *iff* $C \leq 4$.
3. *They do not meet* **Cond** *(except in the trivial case* $C \leq 2$*).*

In order to prove this proposition, we will use the following lemma.

Lemma 3.11

We consider the electoral space of strict total orders and we assume that some candidate c is resistant Condorcet winner.

If $C \leq 4$, then the score for c in the sense of Plurality is strictly the highest one.

If $C \geq 5$, then the score for c in the sense of Plurality can be strictly the lowest one.

Proof. For $C \leq 3$, a resistant Condorcet winner is, by definition, a majority favorite, hence the affirmation is obvious. So, let us assume $C = 4$. Let the candidates be named c, d_1, d_2, d_3 and let us denote γ (resp. δ_1, δ_2, δ_3) the number of voters who have c (resp. d_1, d_2, d_3) on top of their order of preference. So, we have the following profile.

γ	δ_1	δ_2	δ_3
c	d_1	d_2	d_3
Others	Others	Others	Others

Since c is resistant Condorcet winner, a strict majority of voters strictly prefer her to d_1 and d_2 simultaneously. But we know that at least $\delta_1 + \delta_2$ voters do not meet this condition. As a consequence, we have $\gamma + \delta_3 > \delta_1 + \delta_2$. Similar relations are obtained by permuting the δ_i's.

Assume that a candidate, for example d_1, has a score greater or equal than the score for c in the sense of Plurality, i.e. we have $\delta_1 \geq \gamma$. Combining this relation with previous ones, we obtain $\delta_1 + \delta_2 \geq \gamma + \delta_2 > \delta_1 + \delta_3 \geq \gamma + \delta_3 > \delta_1 + \delta_2$, which is a contradiction.

For $C \geq 5$, consider a profile of the following type.

1	2	2	2	2
c	d_1	d_2	d_3	d_4
	c	c	c	c
Autres	Autres	Autres	Autres	Autres

Candidate c is preferred to any other pair of opponents by at least 5 voters out of 9, hence she is resistant Condorcet winner. But her Plurality score is the lowest one. □

Lemma 3.12

We consider the electoral space of strict total orders and we assume that some candidate c is Condorcet winner.

If $C \geq 3$, the score for c in the sense of Plurality can be the lowest one.

Proof. This classic property is a particular case of a lemma by Smith (1973), showing the same result not only for Plurality but for all PSR except Borda. We will come back to it in lemma 3.17. For the moment, it is sufficient to consider the following profile.

1	2	2
c	d_1	d_2
	c	c
Others	Others	Others

□

Now, we have all the elements to prove proposition 3.10.

Proof. 1. It is clear that Plurality, the two-round system, IRV and IRVA meet **MajFav**.

2. Lemma 3.11 proves that with 4 candidates or less, a resistant Condorcet winner always has the best score in Plurality. In the two-round system, it implies that she is selected for the second round; since she is Condorcet winner, she wins

this last duel. In IRV or IRVA, it implies that she has the best score during each round; in particular, she wins the election.

Lemma 3.11 says also that with 5 candidates or more, a resistant Condorcet winner can have the strictly worst score in Plurality. So, she can lose in Plurality and be eliminated during the first round in the two-round system, IRV or IRVA.

3. Lemma 3.12 proves that with 3 candidates or more, a Condorcet winner can have the strictly worst score in Plurality, and as a consequence, lose also in the two-round system, IRV or IRVA. □

3.4.3 Positional scoring rules (PSR)

We already know the criteria met by Plurality. Now, we generalize our investigation to PSR, which comprise notably Borda's method and Veto.

In order to study PSR and later IPSR, it will often be convenient to enunciate the result by using the arithmetical average of all the weight, or of some of them. We will use the following convention: for some real numbers $x_1, \ldots, x_n$, we note $\mathrm{mean}(x_1, \ldots, x_n)$ their arithmetical average.

Proposition 3.13

Let f be a PSR with vector of weights $\mathbf{x}$.

1. *In order for f to meet* **InfMC***, it is necessary that for any $k \in [\![2, C]\!]$, we have:*

$$\mathrm{mean}(x_1, \ldots, x_k) + \mathrm{mean}(x_{C-k+1}, \ldots, x_C) \leq x_1 + x_C.$$

2. *In order for f to meet* **InfMC***, it is sufficient that for any $k \in [\![2, C]\!]$, we have:*

$$x_k + x_{C-k+2} \leq x_1 + x_C$$

   and that at least $j - 1$ such inequalities are strict, where j is the greatest integer such that $x_j = x_1$.

3. *f meets* **IgnMC** *iff it is Plurality. In that case, it also meets* **MajFav**.

4. *f meets* **rCond** *iff it is Plurality with $C \leq 4$.*

5. *f does not meet* **Cond** *(except in the trivial case $C \leq 2$).*

First of all, let us notice that the particular case $k = 1$ is not mentioned in property 1. Indeed, it would be useless because the condition becomes $x_1 + x_C \leq x_1 + x_C$, which is always met.

For property 2, we consider the sum of x_k and x_{C-k+2}, i.e. we add x_2 and x_C, or x_3 and x_{C-1}, etc. The particular case $k = 1$ would not even be defined, since the weight x_{C+1} does not exist.

Proof. 1. We will deal with the particular case $k = C$ last. Assume that there exists $k \in [\![2, C - 1]\!]$ such that:

$$\mathrm{mean}(x_1, \ldots, x_k) + \mathrm{mean}(x_{C-k+1}, \ldots, x_C) > x_1 + x_C.$$

Let α be a natural integer. Consider α sincer voters who always put some candidates $d_1, \ldots, d_k$ on top (not necessarily in the same order) and candidate c last. If $\alpha + 1$ additional voters wish to make c win, it is necessary that c has a better score than candidates $d_1, \ldots, d_k$. To achieve this necessary condition, manipulators cannot do best than put c on top and $d_1, \ldots, d_k$ in the bottom of their ballots (not necessarily in the same order). But then, we have:

$$\begin{aligned}
&\mathrm{mean}(\mathrm{score}(d_1), \ldots, \mathrm{score}(d_k)) - \mathrm{score}(c) \\
&= \alpha\, \mathrm{mean}(x_1, \ldots, x_k) + (\alpha+1)\, \mathrm{mean}(x_{C-k+1}, \ldots, x_C) - (\alpha+1)x_1 - \alpha x_C \\
&= (\alpha+1)\left[\frac{\alpha}{\alpha+1}\, \mathrm{mean}(x_1, \ldots, x_k) + \mathrm{mean}(x_{C-k+1}, \ldots, x_C) - x_1 - \frac{\alpha}{\alpha+1} x_C\right].
\end{aligned}$$

For α large enough, this quantity is positive, hence at least one candidate d_i has a better score than c, who therefore cannot be elected. Hence, f does not meet **InfMC**.

Remains the particular case $k = C$. Let us assume that f meet **InfMC**. Now, we know that it means the condition we gave for $k = C - 1$, which we can reword this way:

$$x_1 + 2x_2 + \ldots + 2x_{C-1} + x_C \leq (C-1)(x_1 + x_C).$$

By adding the trivial inequality $x_1 + x_C \leq x_1 + x_C$, we obtain $2\sum_i x_i \leq C(x_1 + x_C)$, which proves that the condition we gave is also met for $k = C$.

We can notice that, in order to prove that a given voting system meets all the necessary conditions we mentioned, it is not necessary to test the particular case $k = C$, which is directly deduced of the condition for $k = C - 1$. At the opposite, to prove that a given system does not meet **InfMC**, it can be convenient to test first the condition for $k = C$, in case this one is not met.

2. Let us prove that with the conditions we gave, f meets **InfMC**. If there are α sincere voters and β manipulators in favor of some candidate c, with $\beta > \alpha$, let us separate the β manipulators in two groups: the α first of them will be used to give c a score that is greater or equal to other candidates' and the $\beta - \alpha$ other manipulators will make this inequality strict.

The α first manipulators pu themselves in bijection with the sincere voters and each of them uses as ballot the inverse order of her alter ego, while moving c on top of her ballot. Then, the inequalities that are assumed ensure that the other candidates have at most the same score as c and that at least $j - 1$ of them have a strictly lower score.

When an additional manipulator is added, she can attribute x_1 points to c and to these $j - 1$ candidates and strictly less to the others, so candidate c has strictly the highest score. If $\beta - \alpha > 1$, the remaining manipulators simply put c on top and the inequality remains strict.

3. Let us show that the only PSR meeting **IgnMC** is Plurality. Up to adding a constant to the weights and multiplying them by a positive constant, we can assume $x_1 = 1$ and $x_C = 0$.

Consider $\frac{V}{2} + \varepsilon$ voters who want to make c win and $\frac{V}{2} - \varepsilon$ voters who vote after them and whose only goal is to prevent them from electing c, with $\varepsilon > 0$. As for the position of c in the ballots, her proponents cannot do better than putting her on top and the others cannot do better than putting her in the bottom of their ballots. So, we have $\mathrm{score}(c) = \frac{V}{2} + \varepsilon$.

Each proponent of c distributes at least x_2 points to the other candidates, hence at least $\frac{x_2}{C-1}$ points in average to each of them. Among these candidates, let d be the one receiving most points from the proponents of c. In order to prevent c from winning, other voters' best strategy is to put d on top of their ballots. Then we have:

$$\mathrm{score}(d) \geq \left(\frac{V}{2} + \varepsilon\right)\frac{x_2}{C-1} + \left(\frac{V}{2} - \varepsilon\right).$$

In order for f to meet **IgnMC**, it is necessary that even for ε negligible compared to V, we have $\text{score}(d) < \text{score}(c)$. Considering the limit $\varepsilon \to 0$, we obtain after simplification $x_2 \leq 0$, hence $x_2 = \ldots = x_C = 0$ (because weights are always decreasing, cf. definition 1.33). So, f is Plurality.

Conversely, Plurality meets **IgnMC** and even **MajFav** (proposition 3.10).

4. In order for f to meet **rCond**, it is necessary that it meets **MajFav** hence, according the previous point, it is Plurality. But we know (proposition 3.10) that Plurality meets **rCond** iff $C \leq 4$.

5. Finally, it is a classic property that f does not meet **Cond** (except if $C \leq 2$). A possible proof is the following. In order for f to meet **Cond**, it must meet **MajFav**, hence f is necessarily Plurality. But we know (proposition 3.10) that Plurality does not meet **Cond** (except in the trivial case $C \leq 2$). ☐

The fact that the only PSR meeting **MajFav** is Plurality is already proved by Lepelley and Merlin (1998). Point 3 shows that in fact, it is the only PSR meeting the criterion **IgnMC** (less demanding, *a priori*).

In order to complete proposition 3.13 about PSR, ideally, it would be desirable to find a necessary and sufficient condition such that a PSR meets **InfMC**. According to our ongoing research on this topic, we conjecture that it is possible to find a sufficient condition whose form is close to the necessary condition we gave. We leave this question open for future works.

From proposition 3.13, we immediately deduce the two following corollaries for Borda's method and Veto.

Corollary 3.14

1. *Borda's method meets* **InfMC**.
2. *It does not meet* **IgnMC** *(except in the trivial case $C \leq 2$).*

Corollary 3.15
Veto does not meet **InfMC** *(except in the trivial case $C \leq 2$).*

In both cases, it is possible to prove the result directly, with using proposition 3.13. For Veto, we can give an especially short proof. Let $C \geq 3$ and $V = 7$. Assume that 3 sincere voters vote against candidate c. Then c has more vetos than the average (which is $\frac{V}{C}$), hence she cannot win, whatever the other 4 voters do (although they constitute a strict majority).

3.4.4 Iterated PSR with simple elimination

We have already studied the case of IRV in proposition 3.10. Now, we will extend this result to the class of IPSR-SE, which comprise, notably, Baldwin and Coombs' method.

Proposition 3.16
Let f be an IPSR-SE, with vectors of weights $(\mathbf{x}^k)_{k \leq C}$.

1. *f meets* **MajUniBal**.
2. *f meets* **MajFav** *iff for any $k \in [\![3, C]\!]$:*

$$\text{mean}(x_1^k, x_k^k) \geq \text{mean}(x_1^k, \ldots, x_k^k).$$

If this condition is met and if $C \leq 4$, then f meets also **rCond**.

3. *f meets* **Cond** *iff it is Baldwin's method.*

The condition we gave such that an IPSR-SE meets **MajFav** is proven by Lepelley and Merlin (1998). The condition we gave to meet **Cond** is proven by Smith (1973). We are going to give a sketch of proof in order to have a complete overview of proposition 3.16.

In order to prove these results and, later, to study IPSR-EA, we will use several times the following lemma.

Lemma 3.17 (Smith)
Let f be a PSR (for a fixed C and variable V).
If f is Borda's method, then a Condorcet winner has necessarily a score that is strictly greater than the average score.
Otherwise, a Condorcet winner can have a score that is strictly the lowest.

The first affirmation above is mentioned as a reminder: it comes from the fact that a candidate's Borda score is the sum of the score of her row in the matrix of duels. The second one is proven by Smith (1973).

The following lemma shows a similar result for the majority favorite.

Lemma 3.18
Let f be a PSE with vector of weights $\mathbf{x}$ (with $C \geq 2$).
We consider the following condition:

$$\text{mean}(x_1, x_C) \geq \text{mean}(x_1, \ldots, x_C).$$

If it is met, then a majority favorite has necessarily a score that is strictly greater than the average score.
Otherwise, a majority favorite can have a score that is strictly the lowest.

Proof. The minimal score that a majority favorite c can reach is $\frac{V}{2}(x_1 + x_C) + \varepsilon(x_1 - x_C)$, where $\varepsilon > 0$ can be made negligible compared to V. And the average score is $V \text{mean}(x_1, \ldots, x_C)$. In order for c's score to be always strictly greater than the average score, it is necessary and sufficient that $\frac{1}{2}(x_1 + x_C) \geq \text{mean}(x_1, \ldots, x_C)$.

If the condition is not met, let us remark that it is possible that the score for a majority favorite c is not only lower or equal, but actually strictly lower than the average score (by taking ε small enough). Consider such a profile. Up to make $C - 1$ copies by a circular permutation of other candidates and join the profiles we obtain, all opponents have the same score, whereas c's score is strictly lower than the average: so, her score is strictly the lowest. But c remains a majority favorite in the profile we obtain. □

Finally, for $C \leq 4$, the following lemma shows that the same condition gives a similar result for a resistant Condorcet winner.

Lemma 3.19
Let f a PSR with vector of weights $\mathbf{x}$.
Consider the following condition:

$$\text{mean}(x_1, x_C) \geq \text{mean}(x_1, \ldots, x_C).$$

If it is met and if $C \leq 4$, then a resistant Condorcet winner has necessarily a score that is strictly greater than the average score.
If it is not met (for any C), then a resistant Condorcet winner can have a score that is strictly the lowest[5]*.*

[5] The condition can be violated only if $C \geq 3$.

Proof. The second affirmation is easier: if the condition is not met, then according to lemma 3.18, a majority favorite (who is, *a fortiori*, a resistant Condorcet winner) can have a score that is strictly the lowest. As for the first affirmation, If $C \leq 3$, it directly comes from lemma 3.18 because in that case, notions of majority favorite and resistant Condorcet winner are equivalent.

We just need to prove the first affirmation for $C = 4$. Up to subtract x_4 from each weight, we can assume $x_4 = 0$. Let us reword the condition that is met by assumption:

$$\frac{x_1 + x_2 + x_3 + 0}{4} \leq \frac{x_1 + 0}{2},$$

i.e. $x_2 + x_3 \leq x_1$.

For v a voter and $\{d, e\}$ a doubleton of distinct candidates, we say that $(v, \{d, e\})$ gives a *supremacy* to c iff v puts simultaneously c before d and e. To be a resistant Condorcet winner, it is necessary to have strictly more than $\frac{3}{2}V$ supremacies, because a strict majority is necessary against each doubleton of opponents.

If a voter puts c on top, it gives three supremacies to c: one over the doubleton of opponents $\{d_1, d_2\}$, one over $\{d_1, d_3\}$ and one over $\{d_2, d_3\}$. If a voter puts c in second position, it gives her only one supremacy, over the two candidates in the bottom of the ballot. If a voter puts c in third or fourth position, its gives her no supremacy. So, the "unit price" of a supremacy (in points of score) is $\frac{1}{3}x_1$ when c is in first position and x_2 when c is in second position.

Case 1 Assume $\frac{1}{3}x_1 \leq x_2$. Then, to be a resistant Condorcet winner while having as few points as possible, the optimum is to be in first position for a strict majority of voters (never in second position); in other words, to be a majority favorite. So, according to lemma 3.18, c's score is strictly greater than the average score.

Case 2 Assume $x_2 < \frac{1}{3}x_1$. Now, to be a resistant Condorcet winner while having as few points as possible, the optimum is to be in second position as often as possible. However, being always in second position is not enough, because it gives only V supremacies (instead of the $\frac{3}{2}V$ supremacies we need). To have enough supremacies, it is necessary to be in first position for $\frac{V}{4} + \varepsilon$ voters, for some $\varepsilon > 0$. With this notation, it is necessary to be in second position for at least $\frac{3V}{4} - 3\varepsilon$ voters (so, there remains 2ε voters who can do whatever they like). So we have (denoting mean(score) the average score of all candidates):

$$\begin{aligned}
\text{score}\,(c) - \text{mean}(\text{score}) &\geq \frac{V}{4}x_1 + \frac{3V}{4}x_2 + \varepsilon\,(x_1 - 3x_2) - V\,\text{mean}(x_1, \ldots, x_4) \\
&\geq \frac{V}{4}x_2 + \frac{V}{4}(x_2 - x_3) + \varepsilon\,(x_1 - 3x_2)\,.
\end{aligned}$$

But the two first terms are nonnegative by definition of a PSR and the third one is positive by assumption. Hence, c's score is strictly greater than the average score. □

The first affirmation in lemma 3.19 cannot be extended to $C \geq 5$: in that case, a resistant Condorcet winner can even have a score that is strictly lower than the average score. Indeed, let us consider Plurality: $\mathbf{x} = (1, 0, \ldots, 0)$. Condition $\text{mean}(x_1, x_C) \geq \text{mean}(x_1, \ldots, x_C)$ is clearly met. However, we have already shown in lemma 3.11 that a resistant Condorcet winner can have a Plurality score that is strictly the lowest.

Now, we have at our disposal all elements to prove proposition 3.16 about IPSR-SE.

Proof. 1. Let us prove that f meets **MajUniBal**. If a strict-majority coalition wished to make candidate c win, it is sufficient that they put c on top of their ballots and all other candidates in the same arbitrary order. During the counting round where k candidates remain, let us note d_k the candidate put last by these proponents of c. Even if all other voters put d_k first and c last in their ballots, we have $\text{score}_k(c) > \frac{V}{2}(x_1^k + x_k^k) > \text{score}_k(d_k)$, hence c cannot be eliminated. So, she wins the election.

2. From lemma 3.18, we immediately deduce that f meets **MajFav** iff for any $k \in [\![3, C]\!]$, $\text{mean}(x_1^k, x_k^k) \geq \text{mean}(x_1^k, \ldots, x_k^k)$. If this condition is met and if $C \leq 4$, lemma 3.19 ensures that a resistant Condorcet winner cannot be eliminated, hence f meets **rCond**.

3. The fact that the only IPSR-SE meeting **Cond** is Baldwin's method is proven by Smith (1973): it is a direct consequence of 3.17. □

In order to complete proposition 3.16 about IPSR-SE, there remains to find a general criterion for **rCond**, but we think that it is optimistic to hope for a simple relation that would be valid for any number of candidates. We have a first taste of the problem in the proof of lemma 3.19: the more candidates there are, the more ways to gain supremacies. Depending on the values of weights, it can be cheaper (in points of score) to be more often in first position, in second position, etc. Moreover, if it is cheaper to be more often in k-th position for some $k > 1$, we may have to complete with voter that put the candidate in highest positions, as we did in the second case of the proof. Then, there is a sub-case distinction, depending on the unit price of supremacy in these different positions. So, we can expect (and fear) having to distinguish all non-empty subsets of the first $C - 2$ first positions as possible sources of supremacies, which may lead, in the worst case, to $2^{C-2} - 1$ inequalities.

Anyway, we know that for $C \geq 5$, it is possible that an IPSR-SE meets **MajFav** without meeting **rCond**, since it is the case for IRV (proposition 3.10).

From proposition 3.16, we immediately deduce the following corollary about Coombs' method.

Corollary 3.20

1. *Coombs' method meets* **MajBal***.*

2. *It does not meet* **MajFav** *(except in the trivial case $C \leq 2$).*

As well as for corollary 3.15, we can prove the last result in a more concise way than using proposition 3.16. Even if it is well known that Coombs' method does not meet **MajFav**, here is a counter-example as a reminder.

2	2	3
a	a	b
b	c	c
Autres	Autres	Autres
c	b	a

Candidate a is majority favorite but she is eliminated during the first round.

3.4.5 Iterated PRS with elimination based on the average

We have already studied IRVA in proposition 3.10 and now, we will study IPSR-EA in general, which comprise notably Nanson and Kim-Roush methods. For this purpose, we will exploit some of the lemma we gave when studying IPSR-SE.

Proposition 3.21

Let f_C be an IPSR-EA, with vectors of weights $(\mathbf{x}^k)_{k\leq C}$. For any $C' < C$, we note $f_{C'}$ the IPSR-EA of vectors of weights $(\mathbf{x}^k)_{k\leq C'}$.

1. *The following conditions are equivalent.*
   (a) *For any $C' \leq C$, $f_{C'}$ meets* **InfMC***.*
   (b) *For any $k \in [\![3, C]\!]$, we have:*
   $$\text{mean}(x_1^k, x_k^k) \geq \text{mean}(x_1^k, \ldots, x_k^k).$$
   When they are met, each $f_{C'}$ meets also **MajFav** *and, for $C' \leq 4$,* **rCond***.*
2. *f_C meets* **Cond** *iff it is Nanson's method.*

It should be noticed that, if we weaken condition 1a by demanding only that f_C meets **InfMC**, then condition 1b is not necessary anymore. Indeed, let $C = 4$, $\mathbf{x}^4 = (1, 0, 0, 0)$ (Plurality) and $\mathbf{x}^3 = (1, 1, 0)$ (Veto). So, condition 1b is not met for $k = 3$.

However, we will show that f meets **MajFav**, hence **InfMC**. Assume that a strict majority of voters put c on top of their ballots. Since there are 4 candidates in the beginning of the counting process, a candidate must have at least $\frac{V}{4}$ votes to get to the following round: since c has a strict majority of votes, at most one opponent of c may reach that threshold, hence at least two opponents are eliminated during the first round. As a consequence, there is no round for $k = 3$ and c is ensured to win. Hence, f_C meets **MajFav** and weaker criteria, in particular **InfMC**, although condition 1b from previous proposition is not met.

Proof. not 1b $\Rightarrow$ not 1a: Let k such that $\text{mean}(x_1^k, x_k^k) < \text{mean}(x_1^k, \ldots, x_k^k)$. Consider the voting system $f_{C'}$, with $C' = k$. Assume that $\frac{V}{2} - \varepsilon$ sincere voters put candidate c in last position, with $\varepsilon > 0$. In order to avoid an immediate elimination of c, the $\frac{V}{2} + \varepsilon$ manipulators cannot do better than putting her on top of their ballots. Then (denoting $\text{mean}(\text{score}_k)$ the average score of all candidates):

$$\begin{aligned}\text{mean}(\text{score}_k) - \text{score}_k(c) &= V\,\text{mean}(x_1^k, \ldots, x_k^k) - \left[\left(\frac{V}{2} - \varepsilon\right) x_k^k + \left(\frac{V}{2} + \varepsilon\right) x_1^k\right] \\ &= V\big[\text{mean}(x_1^k, \ldots, x_k^k) - \text{mean}(x_1^k, x_k^k)\big] - \varepsilon\big[x_1^k - x_k^k\big],\end{aligned}$$

If ε is small enough compared to V, then the above quantity is strictly positive, so c is eliminated. Hence, $f_{C'}$ does not meet **InfMC**.

1b $\Rightarrow$ 1a: thanks to lemma 3.18, we know that condition 1b implies that a majority favorite has always a score that is strictly more than the average, hence she cannot be eliminated. Therefore, each $f_{C'}$ meets **MajFav** and, as a consequence, weaker criteria in particular **InfMC**.

Now, assume that these conditions are met. For $C' \leq 4$, lemma 3.19 ensures that a resistant Condorcet winner has always a score that is strictly greater than the average, so she cannot be eliminated. Hence, $f_{C'}$ meets **rCond**.

2. The fact that the only IPSR-EA meeting **Cond** is Nanson's method is a direct consequence of lemma 3.17 proven by Smith (1973). □

There remains to find a general criterion for **rCond** but, like for IPSR-SE, we think that it is optimistic to hope for a simple relation that would be valid for any number of candidates C. Here again, we know anyway that an IPSR-EA can meet **MajFav** without meeting **rCond**, since it is the case of IRVA for $C \geq 5$ (proposition 3.10).

To finish our study of ISPR-EA, let us examine the particular case of Kim-Roush method.

Proposition 3.22
Kim-Roush method does not meet **InfMC** *(except in the trivial case $C \leq 2$).*

It is not really a corollary of proposition 3.21 (which would only allow us to conclude that for a given $C \geq 3$, there exists $C' \leq C$ such that Kim-Roush method C' candidates does not meet **InfMC**). So, we have to prove this result directly.

Proof. We can use the same counterexample as for Veto (corollary 3.15). For $C \geq 3$ and $V = 7$ voters, assume that 3 voters vote against some candidate c. Then, whatever the majority consisting of the other 4 voters do, c is eliminated during the first round because she receives more vetos than the average, which is $\frac{V}{C} \leq \frac{7}{3}$. □

3.4.6 Simple or iterated Bucklin's method

Proposition 3.23

1. *Bucklin's method meetd* **MajFav***.*
2. *It meets* **rCond** *iff $C \leq 3$.*
3. *It does not meet* **Cond** *(except in the trivial case $C \leq 2$).*

Proof. 1, 2. It is easy to prove the cases where criteria are met. For **MajFav**, it is sufficient to notice that when there is a majority favorite, her median rank is 1, whereas for any other candidate, her median rank is strictly more. For **rCond**, it is sufficient to notice that, for $C \leq 3$, a resistant Condorcet winner is necessarily a majority favorite (in the electoral space of strict total orders). Now, let us prove the cases where some criteria are not met.

2. Consider the following profile, where $C \geq 4$.

49	11	6	6	14	14
c	a	d_1	d_2	a	a
a	c	c	c	d_1	d_2
d_1	d_1	a	a	c	c
d_2	d_2	d_2	d_1	d_2	d_1
Others	Others	Others	Others	Others	Others

Candidate c is preferred to any pair (a, d_i) by 55 voters and to the pair (d_1, d_2) by 60 voters out of 100, hence she is resistant Condorcet winner. But, in the sense of Bucklin, we have score$(c) = (2\ ;\ 72)$ and score$(a) = (2\ ;\ 88)$, so a wins. Hence, for $C \geq 4$, Bucklin's method does not meet **rCond**.

3. It is well known that Bucklin's method does not meet **Cond** (except in the trivial case $C \leq 2$) but we are going to give a counter-example as a reminder.

40	15	15	30
c	a	b	a
a	c	c	b
b	b	a	c
Others	Others	Others	Others

Candidate c is the Condorcet winner, but we have $\text{score}(c) = (2\ ;\ 70)$ et $\text{score}(a) = (2\ ;\ 85)$. Hence, a is elected. □

Proposition 3.24

1. *Iterated Bucklin's method meets* **MajFav***.*
2. *It meets* **rCond** *iff $C \leq 4$.*
3. *It does not meet* **Cond** *(except in the trivial case $C \leq 2$).*

To prove this proposition, we will use the following lemma.

Lemma 3.25
We consider the electoral space of strict total orders with $C \leq 4$ and we assume that some candidate c is resistant Condorcet winner.

Then there exists a candidate d whose median rank (in the sense of Bucklin) is strictly worse than c's median rank. In particular, d's score in the sense of Bucklin is strictly worse than c's score.

Proof. If $C \leq 3$, this is deduced from the fact that a resistant Condorcet winner is necessarily a majority favorite (in the electoral space of strict total orders): so, her median rank is 1 and she is the only one candidate with this property.

If $C \leq 4$, let us note the candidates c, d_1, d_2 et d_3. Since c is resistant Condorcet winner, she is preferred by a strict majority of voters to d_1 and d_2 simultaneously: so, this majority of voters put c with the rank 1 or 2. As a consequence, c's median rank in the sense of Bucklin is 2 at worst (i.e. 1 or 2).

Let us assume that no candidate has a strictly worse median rank. For this, it is necessary that each of the 4 candidates (including c) occupies strictly more than $\frac{V}{2}$ positions in ranks 1 or 2 of the V voters: so, there are strictly more than $2V$ pigeons for $2V$ holes, which is contradictory. □

Now, we can prove proposition 3.24.

Proof. 1, 2. It is clear that IB meets **MajFav**. Lemma 3.25 proves that for $C \leq 4$, it meets **rCond**. Now, let us show the cases where criteria are not met.

2. For $C = 5$, let us note the candidates $\{c, d_1, \ldots, d_4\}$. Let us note $\alpha = 18$, $\beta = 4$ and $\gamma = 15$ and consider the following profile. For the first column, for example, our notation means that, for any permutation σ of integers from 1 to 4, there are α voters who prefer c then $d_{\sigma(1)}$, $d_{\sigma(2)}$, $d_{\sigma(3)}$ and finally $d_{\sigma(4)}$. Overall, for the first column, there are $4! \times \alpha = 24\alpha$ voters who put c in first position. In

total, there are $24(\alpha + \beta + \gamma) = 888$ voters.

24α	24β	24γ
c	$\bullet$	$\bullet$
$\bullet$	$\bullet$	$\bullet$
$\bullet$	c	$\bullet$
$\bullet$	$\bullet$	$\bullet$
$\bullet$	$\bullet$	c

Candidate c is preferred to any pair (d_i, d_j) by $24\alpha + 4\beta = 448$ voters, so there is a resistant Condorcet winner. In the sense of Bucklin, we have score$(c) = (3\,;\,24\alpha + 24\beta) = (3\,;\,528)$ and for any other candidate d_i, we have score$(d_i) = (3\,;\,12\alpha + 12\beta + 18\gamma) = (3\,;\,534)$. Hence, c is eliminated. Therefore, IB does not meet **rCond**.

3. For $C = 3$, let us prove that IB does not meet **Cond**.

24	24	4	4	22	22
c	c	d_1	d_2	d_1	d_2
d_1	d_2	c	c	d_2	d_1
d_2	d_1	d_2	d_1	c	c

Candidate c is preferred to any other candidate d_i by 52 voters, hence she is Condorcet winner. But, in the sense of Bucklin, we have score$(c) = (2\,;\,56)$ and score$(d_i) = (2\,;\,72)$, so c is eliminated.

To adapt these counterexamples to more candidates, it is sufficient to add other candidates in the same order in the bottom of each order of preferences: these dummy candidates will be eliminated during the first counting rounds and we will be in the same situation as in the above counterexamples. □

3.5 Informational aspect of the majoritarian criteria

To conclude this chapter, we propose a qualitative interpretation of the criteria we studied. Here, it is a casual discussion, whose goal is simply to propose a thought experiment allowing us to apprehend the question of information and communication that may arise to vote strategically in a given voting system, and possibly find an SNE or, at least, to find the same candidate as if we had found an SNE.

For the sake of simplicity, we consider an electoral space where there is no semi-Condorcet configuration, such as the electoral space of strict total orders with an odd number of voters: it allows us to identify **RSNEA** and **RSNEC** on one hand, **XSNEA** and **XSNEC** on the other.

We limit our discussion to voting systems meeting **InfMC**, since they seem to be favored by practice in application cases where is it desired to have some kind of equality between voters and between candidates (which does not meet that we are not interested in other kinds of systems, which we will study in the next chapter). With this assumption, we have seen that **RSNEA** (which is, by the way, equivalent to **RSNEC** in that case) is also met. So, the only configurations ω that might have an SNE are the Condorcet configurations, and the only possible winner of an SNE is the Condorcet winner.

	InfMC	IgnMC	XSNEC	MajBal	MajFav	rCond	Cond
Property of SNE							
Maximize the set of all ω with an SNE			✓	✓	✓	✓	✓
There might exist a system with minimal manipulability						✓	✓
There exists a system with minimal manipulability							✓
Any SNE winner coincides with the sincere winner							✓
Information and communication issues							
Target candidate c	✗	✗	✗	✗	✗	✗	
Assignments of ballots	✗	✗	✗	✗			
Members of the coalition	✗	✗	✗				
Other voters' ballots	✗						

Table 3.1: Informational aspect of the majoritarian and equilibrium criteria. Each column illustrates the properties of the set of voting systems meeting a criterion but not more demanding ones.

Lastly, we assume that the electoral space is finite, which ensures the existence of a voting system whose manipulability is minimal (in the set-theory sense) in any class of voting systems, in particular class **InfMC**[6].

The objective of our thought experiment is to illustrate the following fact: the more criteria are met by a given voting system in the inclusion diagram of figure 3.1, the more we progress on the following issues:

- More often, there is at least an SNE;
- It is easier to obtain the same result as in an SNE (i.e. elect the Condorcet winner if there is one);
- il is easier to reach an SNE.

When we use the expression "easy", it it from the point of view of the quantity of information exchanged between the agents: the higher it is, the easier the task.

Here is our thought experiment, whose conclusions are summed up in table 3.1. Let us imagine that an external coordinator wants to help voters to find the only possible result of an SNE if it exists (i.e. the Condorcet winner with our assumptions). We assume that there is a strict majority of voters, which we call here *the* coalition, who want to collaborate with the coordinator to achieve this goal. Using criterion **InfMC**, we can imagine the following protocol, where we exploit some supernatural gifts the coordinator has.

[6] We will see in chapter 5 another important sufficient condition that guarantees the existence of an optimum (in the probabilistic sense, this time) in some particular cultures, even in infinite electoral spaces, typically those that are used for certain cardinal systems.

1. The coordinator has the magic power to know voters' preferences, which allows her to identify which candidate c is the Condorcet winner. So, this is the target candidate, which we want to get elected.

2. The members of the coalition send a message to the coordinator to inform her that they will follow her instructions.

3. The coordinator has the magic power to know the ballots emitted by the other voters.

4. Using **InfMC**, the coordinator computes ballots for the members of the coalition that make c win and she send this information to them.

To use this protocol, there are four aspects of information to deal with: on one hand, the coordinator must know the target candidate c, the identity of the members of the coalition and the ballots emitted by the other voters; on the other hand, she must send to the members of the coalition the assignment of their ballots.

Of course, we do not pretend that such a flow of actions is actually possible. The purpose of this thought experiment is simply to illustrate which issues of information and communication can arise for a population of strategic voters searching for a strong Nash equilibrium or, at least, trying to elect the same candidate as if they reached an equilibrium.

Now, let us see how the situation improves if we demand voting systems meeting stronger criteria than **InfMC**. The ideas below are summed up in table 3.1.

If the voting system meets **IgnMC**, then the coordinator does not need to know the ballots emitted by the voters who do not belong to the coalition.

If the voting system meets **XSNEC**, the exchanges of information are the same *a priori* (lower part of the table). But the system maximizes the set of configurations of preference where an SNE exists, since we guarantee that any Condorcet configuration has an SNE and since the criterion **InfMC**, equivalent to **RSNEC** with our assumptions, impose that these are the only ones. So, **XSNEC** does no modify the protocol we use, but maximizes the set of configurations of preference where it makes it possible to obtain a result that is the same as in an SNE.

If the voting system meets **MajBal**, then up to diffusing an assignation of ballots to all voters and not only to the members of the coalition, the coordinator does not need to know in advance which voters will follow her instructions. In practice, all usual voting systems meeting **MajBal** meet also **MajUniBal** (such as the IPSR-SE, for example Coombs' method), so the coordinator can simply *broadcast* one ballot only and not use personalized messages for each voter. It is the case, for example, for approval voting, for which Laslier (2009) shows, by a different approach, that an equilibrium can be reached relatively cheaply.

If the voting system meets **MajFav**, then it is useless to diffuse an assignment of ballots to the voters, the coordinator can simply tell them the name of candidate c instead: it is sufficient that obedient voters simply put c on top of their ballots.

If we restrict ourselves to the class **rCond**, information issues are the same *a priori* (lower part of the table). But it is not excluded that one of this voting systems has a minimal manipulability (in the set-theory sense), whereas it is impossible for a voting system no meeting **rCond**, as mentioned in corollary 2.22 of the Condorcification theorems. Such a system with minimal manipulability maximizes the set of non-manipulable configurations, i.e. the set of configurations where an SNE can be found with absolutely no exchange of information, simply by sincere voting.

If the voting system meets **Cond**, then when there is an SNE, its result coincides with the one obtained by sincere voting. Then, the coordinator can exchange no information with the voters: if they vote sincerely, they can always find the same result as an SNE (if one exists) without any exchange of information.

Moreover, in class **Cond**, the existence of a voting system with minimal manipulability is not only possible but guaranteed by corollary 2.22. If such an optimal system is used, then voters have the ability to find an SNE as often as it is possible to do so without any exchange of information.

Chapter 4

Generalized Condorcification

Until now, we have focused on voting systems meeting **InfMC**, which is a criterion connected to the notion of majority. As we discussed in the introduction of this memoir, the majoritarian principle comes from the simple majority vote and, as formalized by May's theorem (May, 1952), it comes directly form principles of anonymity and neutrality.

However there are practical applications where a voting system violates the anonymity, the neutrality or both, for reasons that can be seen as legitimate. For example, a meeting of part owners in a building of an assembly of shareholders in a business company are generally not symmetric between voters. When the French Constitution is revised, the new version must get two thirds of the votes, which generates an asymmetry between candidates: indeed, the other candidate is the old version of the Constitution, which needs only strictly more than one third of the votes.

In the general case, criteria such as **InfMC** are not necessarily met by the voting system and we can not use the weak Condorcification theorem 2.9: so, we have no guarantee not to increase manipulability with the usual Condorcification, based on the notion of majority. However, we can hope that a similar theorem is valid with a notion inspired by Condorcification. For that purpose, we use an approach that is inspired by the theory of *simple games* and we define the *generalized Condorcification*, then we explore its connections with manipulability.

In section 4.1, we define the notion of *family*, which is key for this chapter. To each candidate c, a family $\mathcal{M}$ associates a set $\mathcal{M}_c$ of coalitions. Later, this kind of object will describe, for each candidate c, a set of coalitions having the power to make c win, in a specific sense that we will choose (i.e., either in an ignorant or informed way). We use this notion to extend the notions of Condorcet winner, Condorcet-admissible candidate and majority favorite.

In section 4.2, we extend some majority criteria, limiting ourselves to those that will be useful for the theorems of this chapter; in particular, we extend **InfMC** to a criterion denoted $\mathcal{M}$**InfC**. Moreover, we define two brand new criteria, $\mathcal{M}$**InfC-A** and $\mathcal{M}$**IgnC-A**, which demand that a coalition belonging to $\mathcal{M}_c$ (resp. ignorant or informed) can not only make candidate c win, but can also do so while ensuring that c is *admissible* in the sense of the family $\mathcal{M}$ under consideration. Later, these technical criteria will allow us to compose generalized Condorcifications using different families. By the way, like we did for majoritarian criteria, we establish relations of implication between the criteria under study.

In section 4.3, we naturally define the $\mathcal{M}$-Condorcification of a voting system f, which we note $f^{\mathcal{M}}$. Then, it is easy to adapt the proof of the weak

Condorcification theorem 2.9 to show that, if a voting system meets $\mathcal{M}$**InfC**, then $f^{\mathcal{M}}$ is at most as manipulable as f: it is the *generalized Condorcification theorem* 4.18.

For the sake of conciseness, we do not go further in the correspondence with chapter 2: in particular, we do not generalize the notion of resistant Condorcet winner and the strong Condorcification theorem 2.20. Instead, we prefer focusing of the new possibilities that are offered by the generalized Condorcification.

In particular, it opens a possibility of choice for the family $\mathcal{M}$ that we use to define $f^{\mathcal{M}}$. In section 4.4, we compare Condorcifications obtained by the means of two families $\mathcal{M}$ and $\mathcal{M}'$ and we introduce the *compared Condorcification theorem* 4.21 which makes it possible to prove, under certain assumptions, that $f^{\mathcal{M}'}$ is at most as manipulable as $f^{\mathcal{M}}$.

In section 4.5, we address the following natural question: among the families $\mathcal{M}$ that make it possible to benefit from the generalized Condorcification theorem 4.18, is there one of them that diminished manipulability most? For that purpose, we define the *maximal family* of f: for each candidate c, we consider all coalitions that have the ability to make c when they are informed of the ballots emitted by the other voters. We call *maximal Condorcification* of f the system $f^{\mathcal{M}'}$, where $\mathcal{M}'$ is the maximal family. Then we prove the *maximal Condorcification theorem* 4.25 which states, under certain assumptions, that among the Condorcifications meeting the assumptions of theorem 4.18, the maximal Condorcification is the least manipulable.

In section 4.6, we investigate several examples of applications for the generalized Condorcification. For classic voting systems meeting **InfMC**, we show that the maximal Condorcification is the usual Condorcification, i.e. the majoritarian one (section 4.6.1). Then, we study the maximal Condorcification for Veto. Lastly, we study generalized Condorcification for several voting systems violating anonymity, neutrality or both.

4.1 Family of collections of coalitions

As a reminder, the theory of *simple games* is used to study some cooperative games in a framework that is abstract and relatively simplified. We have a subset of players who, in our case, are voters. A *simple game* is defined by a set of coalitions $\mathcal{M}$, i.e. of subsets of voters. If a coalition M belongs to $\mathcal{M}$, we say that M is a *winning* coalition; otherwise, we say that it is a *losing* coalition.

The spirit of this model is that, if members of a winning coalition M manage to coordinate (according to some kind of modality), then they win the game, generally for the disadvantage of other players. Depending on the set of coalitions $\mathcal{M}$ defining the simple game under study, one can address a variety of question, such as the respective powers of the players and the way rewards can be shared between the winners (which leads to define notions like the *Shapley value*).

In the case of voting systems, coalitions that have the ability to make a given candidate c win depend on c *a priori*: in the example of changing the French Constitution, one third of voters is enough to keep the old version, but two thirds are necessary to switch to the new version. These observations lead to define the notion of *famille*, which is the key object of this chapter.

4.1.1 Definition of a family and basic properties

Definition 4.1 (family of subsets of coalitions)
We call *family of subsets of coalitions*, or *family*, a function:

$$\mathcal{M} : \left| \begin{array}{lll} \mathcal{C} & \to & \mathcal{P}(\mathcal{P}(\mathcal{V})) \\ c & \to & \mathcal{M}_c. \end{array} \right.$$

Intuitively, we suggest to see a family as an object giving the following information: for each candidate c, it describes what coalitions of voters can make c win, in several senses that we will give later. So, for a coalition of voters $M \in \mathcal{P}(\mathcal{V})$, we will say that M is an *$\mathcal{M}$-winning coalition for c* iff $M \in \mathcal{M}_c$. For the moment, it is only a convention of language; we will see why it is convenient in the following.

Definition 4.2 (basic properties of a family)
We say that $\mathcal{M}$ is *monotone* iff $\forall c \in \mathcal{C}, \forall (M, M') \in \mathcal{P}(\mathcal{V})^2$:

$$M \in \mathcal{M}_c \text{ and } M \subseteq M' \Rightarrow M' \in \mathcal{M}_c.$$

We say that $\mathcal{M}$ is *exclusive* iff $\forall (c, d) \in \mathcal{C}^2, \forall (M, M') \in \mathcal{P}(\mathcal{V})^2$, if $c \neq d$ then:

$$M \in \mathcal{M}_c \text{ and } M' \in \mathcal{M}_d \Rightarrow M \cap M' \neq \varnothing.$$

We say that $\mathcal{M}$ is *neutral* iff $\forall (c, d) \in \mathcal{C}^2, \mathcal{M}_c = \mathcal{M}_d$.
We say that $\mathcal{M}$ is *anonymous* iff $\forall c \in \mathcal{C}, \forall \sigma \in \mathfrak{S}_{\mathcal{V}}, \forall M \in \mathcal{P}(\mathcal{V}) : M \in \mathcal{M}_c \Leftrightarrow \sigma(M) \in \mathcal{M}_c$, where $\sigma(M)$ denoted the coalition we obtain by considering the images of the members of coalition M by a permutation σ of the voters.

Monotony means that if M is a winning coalition for c, then any coalition M' containing M is a winning coalition for c. With the interpretation we gave, it is a quite natural assumption.
Exclusivity means that, if two coalitions M and M' are disjoint, then they cannot be winning, respectively, for two distinct candidates c and d. This notion is similar to the notion of *proper* simple game, which demands that a coalition and its complement cannot be both winning. If we assume that voters in M and those in M' have the same powers, then it is natural to consider that both coalitions cannot make c and d win simultaneously, which is the case for an ignorant manipulation. In contract, if we talk about informed manipulation, then exclusivity is not obvious: indeed, an informed coalition might manipulate for some candidate a; then a disjoint informed coalition, knowing these ballots, might change their own ballots to make another candidate b win; but then, the first coalition, knowing these new ballots, might change their ballots to make a win again, etc. We will see an example of this phenomenon with parity voting.
Finally, the meaning of anonymity and neutrality is obvious. About anonymity, it is clearly equivalent to say that for a coalition M, belonging to $\mathcal{M}_c$ depends only on its cardinal.

In order to have a convenient example for the following, we will define a simple particular case: *threshold families*.

Definition 4.3 (threshold family)
For $\alpha \in [0, V]$, we call *family of threshold α* the neutral family consisting of all coalitions with strictly more than α voters. Formally:

$$\mathcal{M} : \left| \begin{array}{lll} \mathcal{C} & \to & \mathcal{P}(\mathcal{P}(\mathcal{V})) \\ c & \to & \{M \in \mathcal{P}(\mathcal{V}) \text{ s.t. } \mathrm{card}(M) > \alpha\}. \end{array} \right.$$

For $x \in [0, 100]$, we will also say *family of x %* to designate the family of threshold $V \cdot x$ %.

We call *majoritarian family* the family of 50 %, i.e. the one consisting of all coalitions having strictly more than half of the voters.

It is easy to see that threshold families are the only neutral, anonymous and monotone families. By the way, the majoritarian family is exclusive; among threshold families, it is the maximal exclusive family (maximal in the sense of inclusion, i.e. with the minimal threshold). For example, the family of 90 % (which is also exclusive) does not contain a coalition of 70 % of the voters, whereas the majoritarian family does.

4.1.2 Victories and generalized Condorcet notions

Now, we will quickly follow the same step as in previous chapters, where we were focusing on the majoritarian family. Having authorized preferences that can fail to be antisymmetric lead us, in that particular case, to adopt already definitions that are quite general and can be easily generalized to any family.

We have seen in chapter 2, and especially in section 2.5, that the notion of absolute victory $\mathrm{P}_{\mathrm{abs}}$ is the one naturally leading to Condorcification theorems, unlike relation $\mathrm{P}_{\mathrm{rel}}$. So, we will focus on generalizing relation $\mathrm{P}_{\mathrm{abs}}$.

Definition 4.4 ($\mathcal{M}$-victory, $\mathcal{M}$-defeat)

For $\omega \in \Omega$, we note $\mathrm{P}_{\mathcal{M}}(\omega)$, or in short $\mathrm{P}_{\mathcal{M}}$, the binary relation on candidates defined by $\forall (c, d) \in \mathcal{C}^2$:

$$c \, \mathrm{P}_{\mathcal{M}} \, d \Leftrightarrow \{v \in \mathcal{V} \text{ s.t. } c \, \mathrm{P}_v \, d\} \in \mathcal{M}_c.$$

In other words, this relation means that the set of voters preferring c to d are a winning coalition for c (in the sense of family $\mathcal{M}$).

When this relation is met, we say that c has an *$\mathcal{M}$-victory* against d in ω, or that d has an *$\mathcal{M}$-defeat* against c in ω.

We say that c has a *strict $\mathcal{M}$-victory* (resp. *strict $\mathcal{M}$-defeat*) against d in ω ssi she has an $\mathcal{M}$-victory and no $\mathcal{M}$-defeat (resp. an $\mathcal{M}$-defeat and no $\mathcal{M}$-victory) against d.

In the case of the majoritarian family, we have seen (as a consequence of proposition 1.25) that if preferences are antisymmetric, then the relation of victory also is. This is not necessarily the case in general. Indeed, consider the family of 40 % and the following profile:

45	10	45
c	c, d	d
d		c

Then c has a $\mathcal{M}$-victory against d but it is not strict, since d has also an $\mathcal{M}$-victory against c. However, it is easy to give a sufficient condition such that the relation of victory is antisymmetric.

Proposition 4.5

Let $\omega \in \Omega$. We assume that:

- *Family $\mathcal{M}$ is exclusive;*
- *Relations P_v are antisymmetric.*

Then relation $\mathrm{P}_{\mathcal{M}}$ is antisymmetric. In other words, between two distinct candidates c and d, there cannot be mutual $\mathcal{M}$-victories.

Proof. If two distinct candidates c and d have mutual $\mathcal{M}$-victories, then by definition:

$$\begin{cases} \{v \in \mathcal{V} \text{ s.t. } c \mathrm{P}_v d\} \in \mathcal{M}_c, \\ \{v \in \mathcal{V} \text{ s.t. } d \mathrm{P}_v c\} \in \mathcal{M}_d. \end{cases}$$

Hence, by exclusivity, $\{v \in \mathcal{V} \text{ s.t. } c \mathrm{P}_v d\} \cap \{v \in \mathcal{V} \text{ s.t. } d \mathrm{P}_v c\} \neq \varnothing$, which contradicts the antisymmetry of relations P_v. □

For the majoritarian family, we have already noticed that the relation of victory is not necessarily complete. On one hand, with an even number of voters, it can be the case that exactly half of the voters prefer c to d and the other half prefer d to c (even if preferences meet quite strong assumption, such as being strict total orders). On the other hand, if preferences are strict weak orders, for example, it is possible that 45 % of voters prefer c to d and that the same number of voters have the opposite opinion: then, none of the two candidates has a victory against the other. The following proposition gives a sufficient condition such that the relation of victory is complete.

Proposition 4.6

Let $\omega \in \Omega$. We assume that:

- *Family $\mathcal{M}$ is such that for any pair (c, d) of distinct candidates and for any pair $(M, M') \in \mathcal{P}(\mathcal{V})^2$, if $M \cup M' = \mathcal{V}$, then $M \in \mathcal{M}_c$ or $M' \in \mathcal{M}_d$;*
- *Relations P_v are complete.*

Then relation $\mathrm{P}_{\mathcal{M}}$ is complete; in other words, between two distinct candidates c and d, there cannot be an absence of $\mathcal{M}$-victory.

In this proposition, the assumption on family $\mathcal{M}$ is in the same spirit as the notion of *strong* simple game, which impose that between a coalition and its complement, at least of them must be winning.

Proof. Let c and d be two distinct candidates. Let us note $M = \{v \in \mathcal{V} \text{ s.t. } c \mathrm{P}_v d\}$ and $M' = \{v \in \mathcal{V} \text{ s.t. } d \mathrm{P}_v c\}$. Since relations P_v are complete, we have $M \cup M' = \mathcal{V}$. So, from the assumption on $\mathcal{M}$, we have $M \in \mathcal{M}_c$ or $M' \in \mathcal{M}_d$. □

From proposition 4.6, we have that for any family containing the majoritarian family, then the relation of victory $\mathrm{P}_{\mathcal{M}}$ is complete. But it is not necessarily antisymmetric. For example, for the family of 30 %, if preferences are complete, then in any duel, at least one candidate has a victory against the other, but it is possible that the converse is simultaneously true.

Since we have extended the notion of victory, it is immediate to do so with the Condorcet winner and Condorcet-admissible candidates.

Definition 4.7 ($\mathcal{M}$-Condorcet, $\mathcal{M}$-admissible)

Let $\omega \in \Omega$ and $c \in \mathcal{C}$.

We say that c is *$\mathcal{M}$-Condorcet* in ω iff c has an $\mathcal{M}$-strict victory against any candidate d, i.e.:

$$\forall d \in \mathcal{C} \setminus \{c\}, \begin{cases} \{v \in \mathcal{V} \text{ s.t. } c \mathrm{P}_v d\} \in \mathcal{M}_c, & (4.1) \\ \{v \in \mathcal{V} \text{ s.t. } d \mathrm{P}_v c\} \notin \mathcal{M}_d. & (4.2) \end{cases}$$

We say that c is $\mathcal{M}$-*admissible* in ω iff c has no $\mathcal{M}$-defeat, i.e.:

$$\forall d \in \mathcal{C} \setminus \{c\}, \{v \in \mathcal{V} \text{ s.t. } d \mathrel{\mathrm{P}_v} c\} \notin \mathcal{M}_d.$$

If family $\mathcal{M}$ is exclusive and if relations P_v are antisymmetric, then any victory is strict (proposition 4.5) hence condition 4.2 may be omitted: it is implied by condition 4.1. In the particular case of the majoritarian family, we had already noticed this simplification in the definition 1.26 of the Condorcet winner.

The following proposition extends proposition 1.31, and its proof is immediate from the definitions.

Proposition 4.8
Soit $\omega \in \Omega$. If a candidate is $\mathcal{M}$-Condorcet in ω, then:

- *She is $\mathcal{M}$-admissible;*
- *No other candidate is $\mathcal{M}$-admissible.*

In particular, if there is an $\mathcal{M}$-Condorcet candidate, she is unique.

Actually, the motivation of condition (4.2) in definition 4.7 is precisely to ensure that in the general case, on one hand, any $\mathcal{M}$-Condorcet candidate is also $\mathcal{M}$-admissible, and on the other hand, that the $\mathcal{M}$-Condorcet candidate, where she exists, is unique.

4.1.3 $\mathcal{M}$-favorite candidate

Now, we extend the notion of majority favorite. It will not be directly used in the generalized Condorcification theorem 4.18 but it will be a convenient tool to establish a connection between generalized Condorcet notions and manipulation. We will see that this notion requires additional caution, compared to the particular case of the majority favorite.

Definition 4.9 ($\mathcal{M}$-favorite)
For $\omega \in \Omega$ and $c \in \mathcal{C}$, we say that c is $\mathcal{M}$-*favorite* in ω iff:

$$\{v \in \mathcal{V} \text{ s.t. } \forall d \in \mathcal{C} \setminus \{c\}, c \mathrel{\mathrm{PP}_v} d\} \in \mathcal{M}_c.$$

If we consider the family of 40 %, then it is clear that the $\mathcal{M}$-favorite is not always unique: indeed, 45 % of the voters may have a given candidate c as most liked, and as many voters may have another candidate d as most liked. The following proposition gives a sufficient condition so that the $\mathcal{M}$-favorite is unique and shows that, under quite natural assumption, this condition is necessary.

Proposition 4.10
We consider the following conditions.

1. *$\mathcal{M}$ is exclusive.*
2. *In any situation with an $\mathcal{M}$-favorite candidate, she is unique.*

We have 1 $\Rightarrow$ 2.
If we assume that the electoral space allows any candidate as most liked and that $\mathcal{M}$ is monotone, then 2 $\Rightarrow$ 1.

Proof. Implication 1 $\Rightarrow$ 2 being easy, we shall only prove not 1 $\Rightarrow$ not 2.

Let us suppose that $\mathcal{M}$ is not exclusive. Then there are two distinct candidate c and d, two disjoint coalitions M and M', such that $M \in \mathcal{M}_c$ and $M' \in \mathcal{M}_d$. Since $\mathcal{M}$ is monotone, we have $\mathcal{V} \setminus M \in \mathcal{M}_d$. Since the electoral space allows any candidate as most liked, there exists a situation where members of M claim that c is their favorite and members of $\mathcal{V} \setminus M$ claim that d is their favorite. Then, candidate c and d are both $\mathcal{M}$-favorite. $\square$

As we have already mentioned in previous chapters, it is cleat that a majority favorite is necessarily a Condorcet winner. The following proposition, whose proof is immediate from the definitions, extends this observation if the family under consideration is monotone and exclusive, as is the majoritarian family.

Proposition 4.11

We assume that $\mathcal{M}$ is monotone and exclusive.

Pour $\omega \in \Omega$ et $c \in \mathcal{C}$, if c is $\mathcal{M}$-favorite in ω, then c is $\mathcal{M}$-Condorcet in ω.

4.2 Criteria associated to a family

4.2.1 Definitions

We will extend criteria **Cond**, **MajFav**, **IgnMC** and **InfMC**, which we had defined in the particular case of the majoritarian family. In addition, we will define two brand new criteria, $\mathcal{M}$**IgnC-A** et $\mathcal{M}$**InfC-A**. Later, we will show that they make it possible to combine and compare generalized Condorcifications achieved with different families $\mathcal{M}$ and $\mathcal{M}'$ for one given voting system. For the sake of conciseness, we will note mention the generalization of other majoritarian criteria.

Definition 4.12 (criteria associated to a family $\mathcal{M}$)

We say that f *meets the $\mathcal{M}$-Condorcet criterion* ($\mathcal{M}$**Cond**) iff, for any $\omega \in \Omega$ and $c \in \mathcal{C}$, if c is $\mathcal{M}$-Condorcet in ω, then $f(\omega) = c$.

We say that f *meets the $\mathcal{M}$-favorite criterion* ($\mathcal{M}$**Fav**) iff, for any $\omega \in \Omega$ and $c \in \mathcal{C}$, if c is $\mathcal{M}$-favorite in ω, then $f(\omega) = c$.

We say that f *meets the ignorant $\mathcal{M}$-coalition criterion* ($\mathcal{M}$**IgnC**) iff $\forall c \in \mathcal{C}, \forall M \in \mathcal{M}_c, \exists \omega_M \in \Omega_M$ s.t.:

$$\forall \omega_{\mathcal{V}\setminus M} \in \Omega_{\mathcal{V}\setminus M}, f(\omega_M, \omega_{\mathcal{V}\setminus M}) = c.$$

We say that f *meets the admissible ignorant $\mathcal{M}$-coalition criterion* ($\mathcal{M}$**IgnC-A**) iff $\forall c \in \mathcal{C}, \forall M \in \mathcal{M}_c, \exists \omega_M \in \Omega_M$ s.t.:

$$\forall \omega_{\mathcal{V}\setminus M} \in \Omega_{\mathcal{V}\setminus M}, \begin{cases} f(\omega_M, \omega_{\mathcal{V}\setminus M}) = c, \\ c \text{ is } \mathcal{M}\text{-admissible in } (\omega_M, \omega_{\mathcal{V}\setminus M}). \end{cases}$$

We say that f *meets the informed $\mathcal{M}$-coalition criterion* ($\mathcal{M}$**InfC**) iff $\forall c \in \mathcal{C}, \forall M \in \mathcal{M}_c, \forall \omega_{\mathcal{V}\setminus M} \in \Omega_{\mathcal{V}\setminus M}$:

$$\exists \omega_M \in \Omega_M \text{ s.t. } f(\omega_M, \omega_{\mathcal{V}\setminus M}) = c$$

We say that f *meets the admissible informed $\mathcal{M}$-coalition criterion* ($\mathcal{M}$**InfC-A**) ssi $\forall c \in \mathcal{C}, \forall M \in \mathcal{M}_c, \forall \omega_{\mathcal{V}\setminus M} \in \Omega_{\mathcal{V}\setminus M}$:

$$\exists \omega_M \in \Omega_M \text{ s.t. } \begin{cases} f(\omega_M, \omega_{\mathcal{V}\setminus M}) = c, \\ c \text{ is } \mathcal{M}\text{-admissible in } (\omega_M, \omega_{\mathcal{V}\setminus M}). \end{cases}$$

$$\begin{tikzcd}
\mathcal{M}\textbf{Cond} \arrow[r, "\text{me}"] & \mathcal{M}\textbf{Fav} \arrow[r, "\text{mp}"] & \mathcal{M}\textbf{IgnC-A} \arrow[r] \arrow[d] & \mathcal{M}\textbf{InfC-A} \arrow[d] \\
 & & \mathcal{M}\textbf{IgnC} \arrow[r] & \mathcal{M}\textbf{InfC}
\end{tikzcd}$$

Figure 4.1: Implications between criteria associated to a family $\mathcal{M}$ (m: monotone family, e: exclusive family, a: the electoral space allows any candidate as most liked).

Criteria $\mathcal{M}$**Cond**, $\mathcal{M}$**Fav**, $\mathcal{M}$**IgnC** and $\mathcal{M}$**InfC** extend quite naturally the corresponding criteria for the majoritarian family. Criterion $\mathcal{M}$**InfC-A** (resp. $\mathcal{M}$**IgnC-A**) is a strengthening of $\mathcal{M}$**InfC** (resp. $\mathcal{M}$**IgnC**) demanding that the informed (resp. ignorant) coalition, in addition to succeeding in making a given candidate c win, is able to ensure that c appears as $\mathcal{M}$-admissible. For usual voting systems, it is a natural condition: with the same notations as in the definition, if members of M can make c win, they can generally do so while using a ballot where c is strictly preferred to all other candidates; in that case, c is $\mathcal{M}$-favorite in the configuration we obtain and, under the common assumption that $\mathcal{M}$ is monotone and exclusive, it implies that c is $\mathcal{M}$-Condorcet (proposition 4.11) and *a fortiori* $\mathcal{M}$-admissible.

As we have already noticed informally, the exclusivity assumption is natural when it comes to ignorant manipulation. The following proposition precises this observation.

Proposition 4.13
If a voting system meets $\mathcal{M}$**IgnC***, then family* $\mathcal{M}$ *is exclusive.*

Proof. If two disjoint coalitions have the respective power to make two distinct candidates c et d win in an ignorant way, then they can make c and d simultaneously, which contradict the uniqueness of the output of f. □

In contrast, the exclusivity assumption is not obvious when it comes to informed manipulation: when a voting system meets $\mathcal{M}$**InfC**, it may be the case that $\mathcal{M}$ is not exclusive.

Indeed, let us consider parity voting: if there are an odd (resp. even) of black balls, then candidate a (resp. b) is declared the winner. Let us consider the neutral family $\mathcal{M}$ consisting of all non-empty coalitions: the parity voting meets $\mathcal{M}$**InfC**. Denoting v and v' two distinct voters (we assume $V \geq 2$), coalitions $\{v\}$ and $\{v'\}$ are distinct and both of them are winner, hence the family is not exclusive.

In this voting system, if voter v votes last and knows the other ballots, then she can choose the output; and so does voter v'. So, the non-exclusivity is not contradictory with the uniqueness of the output.

4.2.2 Implications between the criteria associated to a family

The following proposition gives the relations of implications between the different criteria, i.e. inclusions between the corresponding sets of voting systems. All these results are summed up in the implication graph in figure 4.1.

Proposition 4.14

*If $\mathcal{M}$ is monotone and exclusive, then $\mathcal{M}$**Cond** $\subseteq$ $\mathcal{M}$**Fav**.*

*If $\mathcal{M}$ is monotone and if the electoral space allows any candidate as most liked, then $\mathcal{M}$**Fav** $\subseteq$ $\mathcal{M}$**IgnC-A**.*

We have the following inclusions:

- $\mathcal{M}$**IgnC-A** $\subseteq$ $\mathcal{M}$**IgnC**,
- $\mathcal{M}$**IgnC-A** $\subseteq$ $\mathcal{M}$**InfC-A**
- $\mathcal{M}$**IgnC** $\subseteq$ $\mathcal{M}$**InfC**,
- $\mathcal{M}$**InfC-A** $\subseteq$ $\mathcal{M}$**InfC**.

However, in general, we have neither one of the two following inclusions:

- $\mathcal{M}$**IgnC** $\subseteq$ $\mathcal{M}$**InfC-A**,
- $\mathcal{M}$**InfC-A** $\subseteq$ $\mathcal{M}$**IgnC**.

Proof. $\mathcal{M}$**Cond** $\subseteq$ $\mathcal{M}$**Fav**: Assume that a voting system meets $\mathcal{M}$**Cond**. If c is $\mathcal{M}$-favorite in ω, then since $\mathcal{M}$ is monotone and exclusive, c is $\mathcal{M}$-Condorcet in ω (proposition 4.11) hence she is elected.

$\mathcal{M}$**Fav** $\subseteq$ $\mathcal{M}$**IgnC-A**: Assume that a voting system meets $\mathcal{M}$**Fav**. Let $c \in \mathcal{C}$ and $M \in \mathcal{M}_c$. It is sufficient that members of M claim that they strictly prefer c to all other candidates, which is possible because the electoral space allows any candidate as most liked. Whatever the other voters do, the set of voters claiming that c is their most liked candidate contains M, so it belongs to $\mathcal{M}_c$ (by monotony), hence c is $\mathcal{M}$-favorite and gets elected. This proves, in the same time, that the system meets $\mathcal{M}$**IgnC**, hence (proposition 4.13) family $\mathcal{M}$ is exclusive.

Let us continue our manipulation for c: she is $\mathcal{M}$-favorite, we know now that family $\mathcal{M}$ is exclusive and it is also monotone by assumption, so c is $\mathcal{M}$-Condorcet (proposition 4.11) and, *a fortiori*, $\mathcal{M}$-admissible. Therefore, the system meets $\mathcal{M}$**IgnC-A**.

The other implications are immediately deduced from the definitions.

To show that in general, we have neither $\mathcal{M}$**IgnC** $\subseteq$ $\mathcal{M}$**InfC-A** nor $\mathcal{M}$**InfC-A** $\subseteq$ $\mathcal{M}$**IgnC**, let us consider the majoritarian family.

In corollary 3.14 of proposition 3.13 about PSR, we have already shown that Borda's method meets $\mathcal{M}$**InfC**. But in the proof, ballots emitted by the manipulators put always candidate c on top of their ballots, hence Borda's method meets $\mathcal{M}$**InfC-A**. However, corollary 3.14 also states that in general, it does not meet $\mathcal{M}$**IgnC**.

Lastly, in the electoral space of strict total orders, with $C = 2$ and V odd, consider the following voting system: the inversed majority vote. Each voter communicates an order of preference, but candidate a wins iff b is on top of a majority of ballots (and vice versa). If a majority coalition wants to make a candidate win, it is sufficient that they put always this candidate on the bottom of their ballots; but, by definition of this weird voting system, a winning candidate never appears as a Condorcet-admissible candidate. $\square$

4.3 Generalized Condorcification theorem

Now, we will generalize the weak Condorcification theorem 2.9 in our general framework using families of collections of coalitions. The approach is essentially the same, so we will give only short explanations.

The following lemma generalizes lemma 2.4. As in the initial lemma, no assumption is made on the possible criteria that f meets.

Lemma 4.15

Let $(\omega, \psi) \in \Omega^2$. We note $\mathrm{w} = f(\omega)$ and $c = f(\psi)$ and we assume that f is manipulable in ω toward ψ.

We assume that $\mathcal{M}$ is monotone.

If w is $\mathcal{M}$-admissible in ω, then c cannot have an $\mathcal{M}$-victory against w in ψ; in particular, c is not $\mathcal{M}$-Condorcet in ψ.

Proof. Since w is $\mathcal{M}$-admissible in ω, we have: $\{v \in \mathcal{V}, c\ \mathrm{P}_v(\omega_v)\ \mathrm{w}\} \notin \mathcal{M}_c$. But, by definition of manipulability, $\{v \in \mathcal{V}, c\ \mathrm{P}_v(\psi_v)\ \mathrm{w}\} \subseteq \{v \in \mathcal{V}, c\ \mathrm{P}_v(\omega_v)\ \mathrm{w}\}$. Hence, since family $\mathcal{M}$ is monotone, $\{v \in \mathcal{V}, c\ \mathrm{P}_v(\psi_v)\ \mathrm{w}\} \notin \mathcal{M}_c$. $\square$

By the way, lemma 4.15 leads to extending lemma 2.5: if family $\mathcal{M}$ is monotone and if the voting system meets the $\mathcal{M}$-Condorcet criterion, then a configuration with a $\mathcal{M}$-Condorcet candidate cannot be manipulable to another configuration with an $\mathcal{M}$-Condorcet candidate.

The following lemma extends lemma 2.7.

Lemma 4.16

*Let $\omega \in \Omega$. We assume that f meets $\mathcal{M}$***InfC***.*

If $f(\omega)$ is not $\mathcal{M}$-admissible in ω, then f is manipulable in ω.

Proof. Let us note $\mathrm{w} = f(\omega)$. Since w is not $\mathcal{M}$-admissible, then there exists another candidate c with a victory against w:

$$\exists c \in \mathcal{C} \setminus \{\mathrm{w}\} \text{ s.t. } \{v \in \mathcal{V} \text{ s.t. } c\ \mathrm{P}_v\ \mathrm{w}\} \in \mathcal{M}_c.$$

Let us note $M = \mathrm{Manip}(\mathrm{w} \to c)$ the coalition for c against w. Using $\mathcal{M}$**InfC**, we have:

$$\exists \psi_M \in \Omega_M \text{ s.t. } f(\omega_{\mathcal{V}\setminus M}, \psi_M) = c.$$

Hence, f is manipulable in ω (in favor of candidate c). $\square$

As with lemma 2.7, we can deduce that, if a configuration ω is not $\mathcal{M}$-admissible, then any voting system meeting $\mathcal{M}$**InfC** is manipulable in ω.

Now, we can extend the definition 2.8 of Condorcification.

Definition 4.17 ($\mathcal{M}$-Condorcification)

We call *$\mathcal{M}$-Condorcification of f* the SBVS:

$$f^{\mathcal{M}} : \left| \begin{array}{ccl} \Omega & \to & \mathcal{C} \\ \omega & \to & \left| \begin{array}{l} \text{if } \omega \text{ has an } \mathcal{M}\text{-Condorcet candidate } c, \text{then } c, \\ \text{otherwise, } f(\omega). \end{array} \right. \end{array} \right.$$

By definition, the $\mathcal{M}$-Condorcification of f meets the $\mathcal{M}$-Condorcet criterion. In particular, if family $\mathcal{M}$ is monotone and exclusive and if the electoral space allows any candidate as most liked, then it meets $\mathcal{M}$**InfC** (proposition 4.14).

In the context of this chapter, we will call *majoritarian Condorcification* the usual Condorcification f^* (definition 2.8), i.e. based on the majoritarian family.

Now, we have all the tools needed to extend the weak Condorcification theorem 2.9.

Theorem 4.18 (generalized Condorcification)
Let f be an SBVS and $\mathcal{M}$ a family. We assume that:

- *$\mathcal{M}$ is monotone;*
- *f meets $\mathcal{M}$***InfC***.*

Then $f^{\mathcal{M}}$ is at most as manipulable as f:

$$\mathrm{CM}_{f^{\mathcal{M}}} \subseteq \mathrm{CM}_f.$$

Proof. Let us assume that $f^{\mathcal{M}}$ is manipulable in ω toward ψ, but that f is not manipulable in ω.

Since f is not manipulable in ω, lemma 4.16 ensures that $f(\omega)$ is $\mathcal{M}$-admissible in ω. If she is $\mathcal{M}$-Condorcet in ω, then $f^{\mathcal{M}}(\omega) = f(\omega)$; otherwise, there is no $\mathcal{M}$-Condorcet candidate in ω (proposition 4.8) hence, by definition of $f^{\mathcal{M}}$, we also have $f^{\mathcal{M}}(\omega) = f(\omega)$.

Let us note $\mathrm{w} = f^{\mathcal{M}}(\omega) = f(\omega)$ and $c = f^{\mathcal{M}}(\psi)$. Since w is $\mathcal{M}$-admissible in ω, lemma 4.15 (applied to $f^{\mathcal{M}}$) ensures that w is not $\mathcal{M}$-Condorcet in ψ. Hence, by definition of $f^{\mathcal{M}}$, we have $f^{\mathcal{M}}(\psi) = f(\psi)$.

Therefore, we have $f(\omega) = f^{\mathcal{M}}(\omega)$ and $f(\psi) = f^{\mathcal{M}}(\psi)$, so f is manipulable en ω: this contradicts the assumptions. □

4.4 Compared Condorcification theorem

Like we have just extended the weak Condorcification theorem 2.9, we could also extend the strong Condorcification theorem 2.20 by defining a notion of resistant $\mathcal{M}$-Condorcet candidate that would extend the notion of resistant Condorcet winner (definition 2.16). For the sake of conciseness, we will not discuss this point further. Instead, we will focus on an issue that is specific to generalized Condorcification: choosing the family $\mathcal{M}$ we use to condorcify.

Indeed, it is generally possible that a voting system meets $\mathcal{M}$**InfC** for several possible choices of family $\mathcal{M}$. In this case, we can think of using the generalized Condorcification theorem 4.18 with one family or another. Then, natural questions arise: under what conditions can we "condorcify a Condorcification" while going on diminishing (in the large sense) manipulability? Does one Condorcification diminish manipulability more than (or at least as much as) another one? Finally, is there a family that makes it possible to diminish manipulability more than (or at least as much as) all the others?

Unless otherwise stated, $\mathcal{M}$ and $\mathcal{M}'$ will denote two family, with no specific assumption *a priori*.

Definition 4.19 (more condorcifying family)
We say that $\mathcal{M}'$ is *more condocifying* than $\mathcal{M}$ (in the large sense) iff an $\mathcal{M}$-Condorcet candidate is always also $\mathcal{M}'$-Condorcet, i.e.:

$$\forall \omega \in \Omega, \forall c \in \mathcal{C} : c \text{ is } \mathcal{M}\text{-Condorcet in } \omega \Rightarrow c \text{ is } \mathcal{M}'\text{-Condorcet in } \omega.$$

This notion leads to a particular case for composing two Condorcifications: if $\mathcal{M}'$ is more condorcifying than $\mathcal{M}$, then we have $(f^{\mathcal{M}})^{\mathcal{M}'} = f^{\mathcal{M}'}$. Indeed, for configurations where $f^{\mathcal{M}}$ and f have distinct winners, there is an $\mathcal{M}$-Condorcet candidate, who therefore is $\mathcal{M}'$-Condorcet; hence, when we condorcify with family $\mathcal{M}'$, the winner is that candidate, whatever the initial system is.

The following lemma will make it possible that the first Condorcification, achieved with family $\mathcal{M}$, meets the assumptions that will allow it to benefit from the second Condorcificaiton, achieved with family $\mathcal{M}'$. This lemma is actually the main motivation for defining criterion $\mathcal{M}$**InfC-A**.

Lemma 4.20 (of the admissible informed coalition)
We assume that:

- *$\mathcal{M}'$ is more condorcifying than $\mathcal{M}$;*
- *f meets $\mathcal{M}'$***InfC-A**.

*Then $f^{\mathcal{M}}$ meets $\mathcal{M}'$***InfC-A**.

Let us interpret this lemma: we assume that f meets a good property ($\mathcal{M}'$**InfC-A**) for $\mathcal{M}'$, i.e. the second family we will use, and we conclude that $f^{\mathcal{M}}$ meets the same property ($\mathcal{M}'$**InfC-A**), still for this second family $\mathcal{M}'$. Intuitively, the motivation is the following: later, we will consider a fixed family $\mathcal{M}'$, that is very condorcifying, and we will just have to check that f meets $\mathcal{M}'$**InfC-A** for this fixed family, that is, once and for all. Afterward, we will choose for $\mathcal{M}$ any family that is less condorcifying than $\mathcal{M}'$: this lemma will allow us to show that, in all cases, $f^{\mathcal{M}}$ meets the property which makes it possible to continue with $\mathcal{M}'$-Condorcification while going on diminishing manipulability; and we will not need to test conditions depending on $\mathcal{M}$ for each family $\mathcal{M}$ we might consider. Actually, it would be sufficient that $f^{\mathcal{M}}$ meets $\mathcal{M}'$**InfC** as a conclusion of the lemma for the following application; the fact that $f^{\mathcal{M}}$ meets also $\mathcal{M}'$**InfC-A** is a bonus gift.

Proof. Let $\omega \in \Omega, c \in \mathcal{C}, M \in \mathcal{M}'_c$. Since f meets $\mathcal{M}'$**InfC-A**, there exists a ballot $\psi_M \in \Omega_M$ for the coalition such that:

$$\begin{cases} f(\omega_{\mathcal{V}\setminus M}, \psi_M) = c, \\ c \text{ is } \mathcal{M}'\text{-admissible in } (\omega_{\mathcal{V}\setminus M}, \psi_M). \end{cases}$$

Let us denote $d = f^{\mathcal{M}}(\omega_{\mathcal{V}\setminus M}, \psi_M)$. If d is $\mathcal{M}$-Condorcet in $(\omega_{\mathcal{V}\setminus M}, \psi_M)$, then since $\mathcal{M}'$ is more condorcifying than $\mathcal{M}$, candidate d is also $\mathcal{M}'$-Condorcet, hence she is the only $\mathcal{M}'$-admissible candidate (proposition 4.8), so $d = c$. Alternately, if d is not $\mathcal{M}$-Condorcet in $(\omega_{\mathcal{V}\setminus M}, \psi_M)$, then by definition of $f^{\mathcal{M}}$, we also have $d = c$. As a consequence, M can make c win in an informed way in system $f^{\mathcal{M}}$, while guarantying that c is $\mathcal{M}'$-admissible. So, $f^{\mathcal{M}}$ meets $\mathcal{M}'$**InfC-A**. □

The examples we will see in section 4.6 will show that the assumptions of lemma 4.20 are actually quite commonly met. With these assumptions, we can now compare the Condorcifications we obtain by using families $\mathcal{M}$ and $\mathcal{M}'$. This is the subject of the *compared Condorcification theorem.*

Theorem 4.21 (compared Condorcification)
Let f be an SBVS, $\mathcal{M}$ and $\mathcal{M}'$ two families.
We assume that:

- *$\mathcal{M}'$ is monotone;*
- *$\mathcal{M}'$ is more condorcifying than $\mathcal{M}$;*
- *f meets $\mathcal{M}'$***InfC-A***.*

Then $f^{\mathcal{M}'}$ is at most as manipulable as $f^{\mathcal{M}}$:

$$\mathrm{CM}_{f^{\mathcal{M}'}} \subseteq \mathrm{CM}_{f^{\mathcal{M}}}.$$

Proof. By lemma 4.20, we know that $f^{\mathcal{M}}$ meets $\mathcal{M}'$**InfC**. Then, it is sufficient to apply generalized Condorcification theorem 4.18 to $f^{\mathcal{M}}$ and $\mathcal{M}'$, remembering that $(f^{\mathcal{M}})^{\mathcal{M}'} = f^{\mathcal{M}'}$. □

While it is not obvious at first glance, this theorem implicitly contains the fact that $f^{\mathcal{M}'}$ is less manipulable than f (which we already know, by generalized Condorcification theorem 4.18). Indeed, let us consider the particular case of family $\mathcal{M}$ such that for any candidate c, $\mathcal{M}_c = \varnothing$. Then, a candidate is always $\mathcal{M}$-admissible and never $\mathcal{M}$-Condorcet. In particular, any family is more condorcifying than $\mathcal{M}$ and we have $f^{\mathcal{M}} = f$. Hence, the conclusion of the theorem becomes: $\mathrm{CM}_{f^{\mathcal{M}'}} \subseteq \mathrm{CM}_f$.

With only the assumptions of this theorem, we are not guaranteed that $f^{\mathcal{M}}$ is less manipulable than f: indeed, we have not assumed that $\mathcal{M}$ meets the assumptions of generalized Condorcification theorem 4.18. That being said, we know the most important: $f^{\mathcal{M}'}$ is at most as manipulable as f and as any system $f^{\mathcal{M}}$, where $\mathcal{M}$ is less condorcifying than $\mathcal{M}'$.

When reading this theorem, a question naturally arises: is the conclusion still true if we assume only that f meets $\mathcal{M}'$**InfC** instead of $\mathcal{M}'$**InfC-A**? We will see that it is not the case.

Let us consider the electoral space of strict total orders with $V = 3$ voters and $C = 3$ candidates named a, b and c. We use the following voting system f.

1. If all voters put the same candidate on top of their ballot, then c wins.
2. If at least one voter puts a on top and at least one voter puts c on top, then a wins.
3. In all other cases, b wins.

We note $\mathcal{M}$ the *unanimous* family, i.e. the family of coalitions comprising all voters, and $\mathcal{M}'$ the majoritarian family. Since $\mathcal{M}'$ is a threshold family, it is monotone, and it is easily checked that it is more condorcifying than $\mathcal{M}$: indeed, a candidate who is $\mathcal{M}$-Condorcet is preferred by all voters, hence she is Condorcer winner, i.e. $\mathcal{M}'$-Condorcet.

Let us show that f meets $\mathcal{M}'$**InfC**.

- To make c win, it is sufficient that 2 manipulators vote like the third voter (whether the later is sincere or not)
- To make a win, it is sufficient that a manipulator puts a on top and that another manipulator puts c on top.
- To make b win, if the sincere winner puts b on top, it is sufficient that manipulators put a and b on top; and if the sincere voter puts a or c on top, it is sufficient that manipulations both put b on top.

It should be noticed that the manipulations we proposed are not admissible. By the way, the manipulation for c necessarily generates a unanimous favorite (i.e. apparently most liked by all voters) and it is not possible to choose that it is c: in particular, we can not always ensure that c is Condorcet-admissible, so f does not meet $\mathcal{M}'$**InfC-A**.

Consider the following profile ω, which we know well: it is a minimal example of Condorcet paradox.

$$\begin{array}{c|c|c} a & b & c \\ b & c & a \\ c & a & b \end{array}$$

In f, candidate a win by virtue of rule 2. Since there is neither $\mathcal{M}$-Condorcet (unanimously preferred candidate) nor $\mathcal{M}'$-Condorcet (usual Condorcet winner), candidate a wins also in $f^{\mathcal{M}}$ and in $f^{\mathcal{M}'}$.

In $f^{\mathcal{M}'}$, i.e. the usual majoritarian Condorcification of f, configuration ω is clearly manipulable because it is not admissible (lemma 2.7). By the way, it can be noticed that the manipulation for c is not done in the same way in f and $f^{\mathcal{M}'}$. In f, manipulators need to put a on top to benefit from rule 1. In $f^{\mathcal{M}'}$, they must put c on top so that c is Condorcet winner.

On the opposite, in $f^{\mathcal{M}}$ (using the unanimous family), we will show that the configuration is not manipulable.

- To manipulate for c, the two last voters need that c in on top of all ballots, which they cannot achieve.
- To manipulate for b, only the second voter is interested. For her to succeed, it is necessary that either 1) b is on top of all ballots in the final configuration (to benefit from $\mathcal{M}$-Condorcification), or 2) there are not two ballots with respectively a and c on top (to benefit from the original rule f by avoiding the case 2); but both are impossible.

Let us sum up: all the assumptions of the theorem are meet, except that instead of meeting $\mathcal{M}'$**InfC-A**, system f meets only $\mathcal{M}'$**InfC**. And we have exhibited a configuration where $f^{\mathcal{M}'}$ is manipulable but where $f^{\mathcal{M}}$ is not manipulable. So, the conclusion of the theorem is not valid anymore if we only assume that f meets $\mathcal{M}'$**InfC**. This counter-example motivates the use of assumption $\mathcal{M}'$**InfC-A**.

4.5 Maximal Condorcification theorem

Now, we want to know if there exists a "best" family to apply Condorcification with, i.e. to diminish manipulability with generalized Condorcification theorem 4.18. The natural idea is to use the family that, to each candidate c, associates all coalition having the ability to make c win when they are informed of the ballots emitted by the other voters: that is what we will call the *maximal family* of a voting system. In this section, we examine under what assumptions this family leads to a generalized Condorcificaiton that, in some sense, is optimal.

Definition 4.22 (maximal family of informed winning coalitions)
We call *maximal family of informed winning coalitions* of f, or *maximal family* of f, the largest family $\mathcal{M}'$ (in the sense of inclusion) such that f meets $\mathcal{M}'$**InfC**, i.e. such that $\forall c \in \mathcal{C}, \forall M \in \mathcal{P}(\mathcal{V})$:

$$M \in \mathcal{M}'_c \Leftrightarrow \left[\forall \omega_{\mathcal{V}\setminus M} \in \Omega_{\mathcal{V}\setminus M}, \exists \omega_M \in \Omega_M \text{ s.t. } f(\omega_M, \omega_{\mathcal{V}\setminus M}) = c\right].$$

With this notation, the voting system $f^{\mathcal{M}'}$ is called the *maximal Condorcification* of f.

By definition, the maximal family is the largest family making it possible to apply generalized Condorcification theorem 4.18. So, we can wonder whether it is the one that diminishes most manipulability by generalized Condorcification.

Another possible convention would be to call maximal family the largest family $\mathcal{M}'$ such that f meets $\mathcal{M}'$**InfC-A**, which is, by definition, included in the maximal family in the above sense. As we will see, maximal Condorcification theorem 4.25 will deal with reasonable voting system, where these two notions are the same. But in practice, it is easier to identify the maximal family $\mathcal{M}'$ (in the sense above) then to check that f meets also $\mathcal{M}'$**InfC-A**, rather than identifying the largest family such that f meets $\mathcal{M}'$**IgnC-A**, then checking that it is also maximal for the notion $\mathcal{M}'$**InfC**.

The following proposition is immediate from the definition: indeed, if an informed coalition is always able to make c win, then any larger coalition (in the sense of inclusion) can also make c win.

Proposition 4.23

The maximal family of f is monotone.

As a consequence, we can apply generalized Condorcification theorem 4.18 to the maximal family $\mathcal{M}'$, which proves that $f^{\mathcal{M}'}$ is less manipulable than f. We will show that, under certain assumptions, the maximal Condorcification $f^{\mathcal{M}'}$ is the least manipulable of all $f^{\mathcal{M}}$, where $\mathcal{M}$ is a family such that f meets $\mathcal{M}$**InfC**.

By proposition 4.23, we know that the maximal family is monotone. But it is not necessarily exclusive. Indeed, for parity voting (with $V \geq 2$), it is easy to show that the maximal family is the one comprising all non-empty coalitions; but we have already noticed that it is not exclusive.

In order to apply comparted Condorcification theorem 4.21, we will show in the following proposition that, under certain assumptions, any other family $\mathcal{M}$ that can benefit from generalized Condorcification theorem 4.18 is less condorcifying than the maximal family $\mathcal{M}'$.

Proposition 4.24

Let $\mathcal{M}'$ be the maximal family of f and $\mathcal{M}$ a family.
We assume that:

- *Relations P_v are always antisymmetric;*
- *$\mathcal{M}'$ is exclusive.*

*If f meets $\mathcal{M}$**InfC**, then $\mathcal{M}'$ is more condorcifying than $\mathcal{M}$.*

Proof. Let $\phi \in \Omega$ and $c \in \mathcal{C}$. Assume that c is $\mathcal{M}$-Condorcet in ϕ and let us show that she is also $\mathcal{M}'$-Condorcet, i.e. in the sense of the maximal family of f.

For any $d \in \mathcal{C} \setminus \{c\}$, candidate c has an $\mathcal{M}$-victory against d:

$$M = \{v \in \mathcal{V} \text{ s.t. } c \mathrm{\ P}_v\ d\} \in \mathcal{M}_c.$$

Since $M \in \mathcal{M}_c$ and f meets $\mathcal{M}$**InfC**, we have:

$$\forall \omega_{\mathcal{V}\setminus M} \in \Omega_{\mathcal{V}\setminus M}, \exists \omega_M \in \Omega_M \text{ s.t. } f(\omega_M, \omega_{\mathcal{V}\setminus M}) = c.$$

By definition of the maximal family $\mathcal{M}'$ of f, this implies that $M \in \mathcal{M}'_c$. So, c has an $\mathcal{M}'$-victory against d.

Since $\mathcal{M}'$ is exclusive and since preferences are antisymmetric, proposition 4.5 ensures that this $\mathcal{M}'$-victory is strict. □

In practice, we often consider antisymmetric preferences, so the assumption of proposition 4.24 that has more chances not to be satisfied is the exclusivity of the maximal family $\mathcal{M}'$. Let us show that, in that case, the conclusion of proposition 4.24 is not met.

For that purpose, consider parity voting (black and white balls). It is easy to see that the maximal family $\mathcal{M}'$ is neutral and contains all non-empty coalitions. This family is not exclusive (assuming $V \geq 2$) because two singletons of distinct voters are winning coalitions for candidates a and b respectively. In order for the candidate a to be $\mathcal{M}'$-Condorcet, it is necessary and sufficient that she has a victory, i.e. that at least one voter prefers her to b, and that she has no defeat, i.e. that no voter prefers b to a; in other words (assuming that preferences are strict total orders), it is necessary and sufficient that a is the most liked candidate for each voter. Now, if we consider the majoritarian family $\mathcal{M}$, in order for a to be $\mathcal{M}$-Condorcet, it is necessary and sufficient that she is preferred by a strict majority of voters, which is a less demanding condition. So, it is false that $\mathcal{M}'$ is more condorcifying than $\mathcal{M}$. In this example, it is even the opposite: $\mathcal{M}$ is more condorcifying than $\mathcal{M}'$.

Now, we just have to gather the properties we know for the maximal family in the following theorem.

Theorem 4.25 (maximal Condorcification)
Let f be an SBVS and $\mathcal{M}'$ its maximal family.
*We assume that f meets $\mathcal{M}'$***InfC-A***.*

1. *The maximal Condorcification $f^{\mathcal{M}'}$ is at most as manipulable as f:*
$$\mathrm{CM}_{f^{\mathcal{M}'}} \subseteq \mathrm{CM}_f.$$

2. *For any family $\mathcal{M}$ that is less condorcifying than $\mathcal{M}'$ (in the large sense), the maximal Condorcification $f^{\mathcal{M}'}$ is at most as manipulable as $f^{\mathcal{M}}$:*
$$\mathrm{CM}_{f^{\mathcal{M}'}} \subseteq \mathrm{CM}_{f^{\mathcal{M}}}.$$

3. *We assume that moreover, preferences P_v are always antisymmetric and that the maximal family $\mathcal{M}'$ is exclusive. Then, for any family $\mathcal{M}$ such that f meets $\mathcal{M}$***InfC***, the maximal Condorcification $f^{\mathcal{M}'}$ is at most as manipulable as $f^{\mathcal{M}}$:*
$$\mathrm{CM}_{f^{\mathcal{M}'}} \subseteq \mathrm{CM}_{f^{\mathcal{M}}}.$$

Proof. 1. By proposition 4.23, we know that the maximal family $\mathcal{M}'$ is monotone. And since f meets $\mathcal{M}'$**InfC-A**, il meets *a fortiori* $\mathcal{M}'$**InfC** (proposition 4.14). So, we can use generalized Condorcification theorem 4.18.

2. Since the maximal family $\mathcal{M}'$ is monotone, we can use compared Condorcification theorem 4.21.

3. By proposition 4.24, family $\mathcal{M}'$ is less condorcifying $\mathcal{M}$ hence we can use point 2 of this very theorem. □

4.6 Examples of generalized Condorcification

Now, we will apply generalized Condorcification several voting systems in order to show how this notion behaves in practice.

4.6.1 Usual voting systems meeting InfMC

In the case of usual voting systems meeting **InfMC**, we already know that they can benefit from the majoritarian Condorcification (theorem 2.9). The objective of this section is to examine whether it is their maximal Condorcification.

First of all, let us notice that most usual voting systems meet $\mathcal{M}$**IgnC** and even $\mathcal{M}$**IgnC-A** for the majoritarian family. Indeed, in these reasonable voting systems, if some manipulators (in particular, those belonging to an ignorant majority coalition) can make some candidate c win, then they can do so while putting c on top of their ballots.

To deal with these voting systems, the following proposition will be useful. It key point is conclusion 1, which establishes an implication between $\mathcal{M}'$**IgnC-A** and the maximality of $\mathcal{M}'$ in the case of the majoritarian family. Conclusions 2, 3 and 4 simply reword the conclusions from the maximal Condorcification theorem in this particular case.

Proposition 4.26

Let f be an SBVS. We assume that:

- *Relations P_v are always antisymmetric;*
- *The number of voters V is odd;*
- *f meets the admissible ignorant majority coalition criterion.*

1. *Then, the maximal family $\mathcal{M}'$ of f is the majoritarian family.*
2. *The majoritarian coalition $f^{\mathcal{M}'} = f^*$ is at most as manipulable as f:*
$$\mathrm{CM}_{f^*} \subseteq \mathrm{CM}_f.$$
3. *For any family $\mathcal{M}$ that is less condorcifying than the majoritarian family $\mathcal{M}'$ (in the large sense), the majoritarian Condorcification f^* is at most as manipulable as $f^{\mathcal{M}}$:*
$$\mathrm{CM}_{f^*} \subseteq \mathrm{CM}_{f^{\mathcal{M}}}.$$
4. *For any family $\mathcal{M}$ such that f meets $\mathcal{M}$**InfC**, the majoritarian Condorcification f^* is at most as manipulable as $f^{\mathcal{M}}$:*
$$\mathrm{CM}_{f^*} \subseteq \mathrm{CM}_{f^{\mathcal{M}}}.$$

Proof. Since f meets the ignorant majority coalition criterion, it is easy to see that its maximal family is the majoritarian one: with more than a majority, a coalition can always choose the winner; with less, other voters can always do so (because V is odd, so they have a strict majority). This proves point 1.

Since f meets $\mathcal{M}'$**IgnC-A** by assumption, it meets also $\mathcal{M}'$**InfC-A** (proposition 4.14), hence we can use maximal Condorcification theorem 4.25. Then, we can conclude immediately for points 2 and 3.

Since preferences are always antisymmetric by assumption and since the majoritarian family $\mathcal{M}'$ is exclusive (definition 4.3), maximal Condorcification theorem 4.25 proves also point 4. ☐

Corollary 4.27

We assume that relations P_v are always antisymmetric and that V is odd.

Then the conclusions of proposition 4.26 are true for approval voting, Baldwin, Borda, Bucklin, Condorcet-Borda (Black's method), Coombs, CSD, Dodgson, IB, IRV, IRVD, IRVA, ITR, Kemeny, the majority judgment, Maximin, Nanson, Plurality, range voting, RP and Schulze's method.

Proof. All the systems mentioned here, except Borda's method, meet the admissible ignorant majority coalition criterion. For those ones, proposition 4.26 does the job.

If system f is Borda's method, we cannot apply proposition 4.26 because this voting system does not meet **IgnMC** in general (corollary 3.14 of proposition 3.13). However, we will see that the result remains true, as a corollary of the maximal Condorcification theorem 4.25.

Indeed, f meets the admissible informed majority coalition criterion. Let us show that the majoritarian family is maximal: if a coalition is not majoritarian, then the other voters have a strict majority (because V is odd); if they put some candidate d first and c last in their ballots, then c cannot be elected, whatever the manipulators in the minority coalition do. So, we can apply maximal Condorcification theorem 4.25. □

For all systems meeting the Condorcet criterion, Condorcification obviously does not modify the voting system. Proposition 4.26 and corollary 4.27 mean that, even in their original state, we cannot hope to diminish their manipulability by applying generalized Condorcification theorem 4.18. In other words, they are their own maximal Condorcification.

4.6.2 Veto

The system Veto, which does not meet **InfMC**, cannot benefit from the weak (majoritarian) Condorcification theorem 2.9. Now, we can determine its maximal family and apply the maximal Condorcification theorem 4.25.

Proposition 4.28

The voting system f under study is Veto. We assume that preferences are always antisymmetric.

We assume $\mathrm{mod}(V, C) = C - 1$.

1. *Then the maximal family $\mathcal{M}'$ of Veto is the threshold family whose coalitions have a cardinal greater or equal to* $(C-1)\left\lceil \frac{V}{C} \right\rceil$.
2. *Veto meets* $\mathcal{M}'$**IgnC-A**.
3. *The maximal family $\mathcal{M}'$ is exclusive.*

So, all the conclusions of the maximal Condorcification theorem 4.25 are met.

Just like we assumed an odd number of voters in proposition 4.26 and corollary 4.27 to avoid questions of ties in voting system that are linked to the notion of majority, proposition 4.28 uses a modulo assumption in order to simplify the question: $\mathrm{mod}(V, C) = C - 1$. We can remark that for $C = 2$, Veto becomes equivalent to simple majority vote; then, this equation simply means that V is odd.

In the limit where V is large compared to C, we can consider in first approximation that the maximal family of Veto is the one of the $\left(1 - \frac{1}{C}\right) V$, up to questions of ties and rounding. In other words, to choose the winner in Veto, a large coalition is necessary, larger when C is larger.

Proof. Let us note $\alpha = (C-1)\left\lceil \frac{V}{C} \right\rceil$ and $\mathcal{M}'$ the threshold family whose coalitions have a cardinal greater or equal to α. We will show that $\mathcal{M}'$ meets the above properties and, especially, that it is the maximal family of f. During all the beginning of this proof, we will not need the assumption $\mathrm{mod}(V, C) = C - 1$ but only the weaker assumption that C does not divide V.

3. This weaker assumption is sufficient to have $\alpha > (C-1)\frac{V}{C} \geq \frac{V}{2}$. Hence, family $\mathcal{M}'$ is exclusive.

2. If a coalition of cardinal $(C-1)\left\lceil \frac{V}{C} \right\rceil$ wants to make some candidate c win, it is sufficient that $\left\lceil \frac{V}{C} \right\rceil$ members of the coalition vote against each other candidate. Then, since C does not divide V, each of these opponent candidates receives a number of vetos that is stricly greater than the average number of vetos, hence only c can be elected. Therefore, Veto meets $\mathcal{M}'$**IgnC**.

Members of the coalition can achieve this manipulation while putting c on top of their ballors. Since family $\mathcal{M}'$ is exclusive, this ensures that c appears as an $\mathcal{M}'$-admissible candidate (as for any other candidate d, she cannot have a position that is better than c in α ballots, hence she cannot have an $\mathcal{M}'$-victory against c). So, Veto meets $\mathcal{M}'$**IgnC-A**.

1. Since Veto meets $\mathcal{M}'$**IgnC-A**, we already know that it meets $\mathcal{M}'$**InfC-A** (proposition 4.14). To show that $\mathcal{M}'$ is maximal, we just have to show that a coalition that does not belong to $\mathcal{M}'$ does not have the ability to choose the winner (in an informed way). This is the moment where we use the assumption $\mathrm{mod}(V, C) = C - 1$. Noticing that in the general case:

$$\left\lceil \frac{V}{C} \right\rceil = \frac{V + (C - \mathrm{mod}(V, C))}{C},$$

we deduce $\alpha = \frac{(C-1)(V+1)}{C}$.

Let us consider a coalition that does not belong to $\mathcal{M}'$, i.e. whose size is strictly lower than α. Then the cardinal of its complement is at least:

$$V + 1 - \alpha = \frac{V+1}{C} > \frac{V}{C}.$$

So, if all voters in the complement vote against candidate c, then she has a number of vetos that is greater than the average, so she cannot be elected. Hence, the considered coalition is not always able to make c win (in an informed way), which proves that $\mathcal{M}'$ is the maximal family of Veto. ☐

We will see that the maximal Condorcification of Veto can be strictly less manipulable than Veto. For that purpose, consider $V = 7$ voters and $C = 4$ candidates. We have $\mathrm{mod}\ (V, C) = C - 1$, hence the assumptions of proposition 4.28 are met and the maximal family $\mathcal{M}'$ is the one of coalitions with 6 voters or more. Let us examine the following configuration ω.

3	3	1
c	c	a
a	a	b
b	d	d
d	b	c

$D(\omega)$	a	b	c	d
a	–	7	1	7
b	0	–	1	4
c	6	6	–	6
d	0	3	1	–

In Veto, candidate a is elected and the configuration is manipulable for c: it is sufficient that the 6 manipulators emit 2 vetos against each of candidates a, b and d. In the maximal Condorcification of Veto, candidate c is elected and it is easy to see that it is not manipulable: indeed, c is not only $\mathcal{M}'$-Condorcet but she is even $\mathcal{M}'$-favorite; as a consequence, voters putting c on top, and therefore never manipulate, are sufficient to ensure that she is $\mathcal{M}'$-Condorcet.

For the sake of curiosity, we can also show that the usual majoritarian Condorcification of Veto does not work, in the sense that it can be manipulable in some configurations where Veto is not. For that purpose, consider the following configuration ω, again with $V = 7$ voters and $C = 4$ candidates.

1	2	1	1	2
c	c	d	a	a
a	a	c	b	b
d	b	a	c	d
b	d	b	d	c

$D(\omega)$	a	b	c	d
a	–	7	3	6
b	0	–	3	5
c	4	4	–	4
d	1	2	3	–

Candidate a is the only one without veto, so she is declared the winner. Let us show that Veto is not manipulable in ω. No voter prefer b to a. Candidates c and d receive 2 vetos each, which is more than the average number of vetos $\frac{V}{C} = \frac{7}{4}$, hence it is impossible to manipulate for them. As a consequence, Veto is not manipulable in ω.

Now, let us examine the majoritarian Condorcification of Veto. Candidate c is Condorcet winner in ω, hence she is elected. Let us consider the following configuration ψ, which is an attempt of manipulation for a.

1	2	1	1	2
c	c	d	a	a
a	a	c	b	b
d	b	a	**d**	d
b	d	b	**c**	c

$D(\psi)$	a	b	c	d
a	–	7	3	6
b	0	–	3	5
c	4	4	–	3
d	1	2	4	–

Then, there is no Condorcet winner anymore so a is elected. Hence, the majoritarian Condorcification of Veto is manipulable in ω toward ψ in favor of a, whereas Veto is not manipulable in ω.

4.6.3 Parity voting

Now, we will examine a case where it is not possible to use the maximal Condorcification theorem 4.25. Let us consider parity voting: if there is a odd (resp. even) number of black balls, then candidate a (resp. b) is declared the winner. In addition to her ball, each voters communicates also her order of preference, but it has no impact on the outcome.

We have already noticed that the maximal family of this system is the one of all non-empty coalitions and that this family is not exclusive. So, it is not possible to use the maximal Condorcification theorem 4.25.

However, we can notice that any monotone family $\mathcal{M}$ such that f meets $\mathcal{M}$**InfC**, f meets also $\mathcal{M}$**InfC-A**. By compared Condorcification theorem 4.21, if one of these families $\mathcal{M}'$ is more condorcifying than another of these families $\mathcal{M}'$, then $f^{\mathcal{M}'}$ is at most as manipulable as $f^{\mathcal{M}}$.

The relation "be more condorcifying" is clearly a (partial) order, in particular over the set of monotones families $\mathcal{M}$ such that f meets $\mathcal{M}$**InfC**. The maximal elements of this order lead to optimal Condorcifications (in terms of manipulability), but we do not know, *a priori*, how to compare between them from the point of view of manipulability.

Let us examine two examples.

Assuming V odd, let us consider the majoritarian family. Then, the generalized Condorcification is the simple majority vote, which is not manipulable.

Now, let $\mathcal{M}_a$ be the set of coalitions with strictly more than a third of the voters and $\mathcal{M}_b$ the set of coalition with at least two thirds of the voters. In the generalized Condorcification, b wins if she has at least two thirds of the votes; otherwise, it is a. In that case also, the generalized Condorcificaiton is not manipulable.

Both these Condorcifications are ways to diminish manipulability, obtained with two families that are not comparable, i.e. such that none of them is more condorcifying than the other.

By the way, we can examine the Condorcifications of parity voting that use threshold families (i.e. neutral, anonymous and monotone).

First of all, let us notice that for generalized Condorcet, using for example the family of 30 % or the one of 70 % is equivalent (assuming that V is not divisible by 10 to avoid distinction between strict and large inequalities). Indeed, in the family of 30 %, some candidate c has a strict absolute victory against some d iff more than 30 % prefer c to d (victory) and less than 30 % prefer d to c (no defeat); since preferences are strict total orders, this amounts to saying that more than 70 % of voters prefer c to d, i.e. that c has a strict absolute victory against d in the sense of the family of 70 %.

Still with the assumption that binary relations are strict total orders, we observe that the majoritarian family is, among threshold families, the one accepting most victories (since even in the family of 30 %, it is necessary to have victories with more than 50 % of voters). So, it is the most condorcifying of monotone, neutral and anonymous families.

As a consequence, we can retrieve the simple majority vote as the optimal Condorcification of parity voting among those obtained with monotone, anonymous and neutral families.

To illustrate the above points, consider the following configuration ω, where all voters use a white ball.

60	39
a	b
b	a

Since there is an even number of black balls (equal to 0), candidate b is elected. If we use the family $\mathcal{M}$ of 70 %, then there is no $\mathcal{M}$-Condorcet candidate: no candidate has a victory against the other. If we use the family of 30 %, then there is no $\mathcal{M}$-Condorcet candidate either: candidates a and b have mutual victories against each other (so, they are not even $\mathcal{M}$-admissible). In both these systems, it is sufficient that a candidate preferring a replaces her white ball with a black ball, without modifying her order of preference, to manipulate in favor of a.

In contract, if we use the majoritarian family, then the Condorcification of parity voting is simply the simple majority vote: a is elected and this system is never manipulable (sincere voting, consisting of voting for one's most liked candidate, is a dominant strategy).

4.6.4 Vote of a law

Now, we will focus on voting systems violating anonymity, neutrality or both. In all the rest of this chapter, we note $\mathcal{M}'$ the maximal family of the system f under study. To simplify, we assume that binary relations of preference are strict total orders.

We will often use implicitly proposition 4.13: if a system meets $\mathcal{M}$**IgnC**, then $\mathcal{M}$ is exclusive. Since preferences are antisymmetric, we recall that the definition 4.7 of an $\mathcal{M}$-Condorcet candidate amounts to relation (4.1) only: it is sufficient to check that the candidate has a victory against any other candidate, and it is not need to check that no other candidate has a victory against her.

Let us examine a first system that can be used to vote a new law. One can think of revising the French Constitution, for example. Three options are the candidates: two versions a and b of the law project, and $\varnothing$ which represents the status quo, i.e. the fact that none of the versions is chosen. The voting system is the following:

1. Voters vote between a and $\varnothing$. If a receives at least two thirds of the votes, then a is declared the winner and the process is over.

2. Otherwise, voters vote between b and $\varnothing$. If b receives at last two thirds of the votes, then b is declared the winner; otherwise, $\varnothing$ wins.

It is a *general voting sytem* in several rounds (section 1.4). The *game form* that is uses leads to a question, *a priori*, to define the sincerity function[1]: if a voter has the order of preference $(b \succ a \succ \varnothing)$, it is not clear what her ballot should be during the first round because actually, the choice is not between a and $\varnothing$ but rather between a and the fact of doing an election between b and $\varnothing$. The *voting system* we will study uses the sincerity function implied by our initial wording: for the first round, sincere voting consists of voting for a iff she prefers a to $\varnothing$.

First of all, we consider instead the state-based version of this system, which is at most as manipulable (proposition 1.4.2): each voters communicates her order of preference, then the original system is emulated.

Now, let us examine its maximal family.

For any coalition M, we have:

$$M \in \mathcal{M}'_\varnothing \Leftrightarrow \operatorname{card}(M) > \frac{V}{3}.$$

For any coalition M and any candidate $c \neq \varnothing$, we have:

$$M \in \mathcal{M}'_c \Leftrightarrow \operatorname{card}(M) \geq \frac{2V}{3}.$$

It is easy to check that the voting system meets $\mathcal{M}'$**IgnC-A**. So, the maximal family $\mathcal{M}'$ is exclusive.

To compare the manipulability of f and $f^{\mathcal{M}'}$, let us consider the following profile ω.

70	30
b	a
a	b
$\varnothing$	$\varnothing$

In the original system (or its state-based version), candidate a wins during the first round, at least if voters are sincere. But the configuration is manipulable in favor of b: indeed, its proponents can exchange a and $\varnothing$ in their order of preference; then, the project of law a is rejected, then b is accepted.

[1] This example is very similar to the one of the *non-alcoholic party* by Gibbard (1973), which is precisely conceived to illustrate the issue of defining sincere voting.

In the maximal Condorcification, b is $\mathcal{M}'$-Condorcet, so she is elected and it is not manipulable: indeed, candidate b is $\mathcal{M}'$-favorite, hence voters having b as most liked ensure, on their own, that b is $\mathcal{M}'$-Condorcet.

In this example, the maximal Condorcification leads to a voting system that is strictly less manipulable than the original system.

4.6.5 Plurality with an imposition power

Renaud[2] and his friends consider going to the ball (b), to the temple (t), to visit Germaine (g) or the Pépette (p), or staying in the same place and just discussing ($\varnothing$). Plurality is used, with the following exception. Renaud, who owns the car, has the right to impose candidate $\varnothing$: if he votes for $\varnothing$, then this candidate is automatically elected.

For any coalition M, we have:

$$M \in \mathcal{M}'_{\varnothing} \Leftrightarrow \text{Renaud} \in M \text{ or } \operatorname{card}(M) > \frac{V}{2}.$$

For any coalition M and any candidate $c \neq \varnothing$, we have:

$$M \in \mathcal{M}'_{c} \Leftrightarrow \text{Renaud} \in M \text{ and } \operatorname{card}(M) > \frac{V}{2}.$$

It is easy to check that the voting system meets $\mathcal{M}'$**IgnC-A**. Hence, the maximal family $\mathcal{M}'$ is exclusive.

Let us examine the following profile ω, with $V = 9$ voters.

1 (Renaud)	1	1	1	1	1	1	1	1
g	$\varnothing$	b	b	b	p	p	t	t
$\varnothing$	g	$\varnothing$	$\varnothing$	t	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
t	t	g	t	$\varnothing$	g	g	g	g
p	p	p	g	g	t	t	b	b
b	b	t	p	p	b	b	p	p

$D(\omega)$	$\varnothing$	b	t	g	p
$\varnothing$	–	6*	6*	8	7*
b	3	–	3	3	5
t	3	6*	–	4	6*
g	1*	6*	5*	–	7*
p	2	4*	3	2	–

In the matrix of duels above, we conventionally mark with a star each set of voters containing Renaud.

In the original voting system, we have $f(\omega) = b$, but Renaud can manipulate in favor of $\varnothing$ by using his imposition power.

If we use the majoritarian Condorcification f^*, then the sincere winner is $\varnothing$ because it is the Condorcet winner. But the profile remains manipulable: indeed, let us consider the following configuration ψ, which is an attempt to manipulate for b.

1 (Renaud)	1	1	1	1	1	1	1	1
g	$\varnothing$	b	b	b	p	p	t	t
$\varnothing$	g	$\mathbf{g}$	$\mathbf{t}$	t	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
t	t	$\mathbf{t}$	$\boldsymbol{\varnothing}$	$\varnothing$	g	g	g	g
p	p	$\boldsymbol{\varnothing}$	g	g	t	t	b	b
b	b	$\mathbf{p}$	p	p	b	b	p	p

$D(\psi)$	$\varnothing$	b	t	g	p
$\varnothing$	–	6*	4*	6	7*
b	3	–	3	3	5
t	5	6*	–	4	7*
g	3*	6*	5*	–	7*
p	2	4*	2	2	–

[2]Translator's note (i.e. author's note): Renaud is a popular anarchist French singer. Among his songs, in "Tu vas au bal?", he narrates a humorous conversation with a friend where they consider going to the ball, to the church or to the prostitutes, but mostly stay in the same place, talking about it for hours. Finally, he ends up dancing in the church with the prostitutes. In other songs, he mentions recurring characters named Germaine (in the song of the same name and in "Mon H.L.M.") and the Pépette (in "Près des auto-tamponneuses" and "Le Retour de la Pépette").

There is no Condorcet winner and b wins, hence the majoritarian Condorcification f^* is manipulable in ω toward ψ in favor of b.

Lastly, if we use the maximal Condorcification, then the sincere winner in ω is also $\varnothing$ because it is $\mathcal{M}'$-Condorcet. If we wish to manipulate in favor of g, then only Renaud is interested, but he can not avoid that $\varnothing$ stays $\mathcal{M}'$-Condorcet because it will still have a strict majority against any other candidate; so, manipulation fails. If we wish to manipulate in favor of a candidate distinct from g, then Renaud does not participate to the manipulation hence his vote, by itself, ensures that $\varnothing$ keeps an $\mathcal{M}'$-victory against t, p and b. The only hope for the manilators is to prevent $\varnothing$ from having an $\mathcal{M}'$-victory against g. If we consider a manipulation for b (resp. t, p), then we can read in the matrix of duels that there are 3 manipulators (resp. 3, 2), hence the number of votes for $\varnothing$ against g, which is 8 in sincere voting, will be, after the manipulation, greater or equal to $8 - 3 = 5$ (resp. 5, 6), hence $\varnothing$ keeps a victory against g and manipulation fails. There, $f^{\mathcal{M}'}$ is not manipulable in ω, whereas f and f^* are.

4.6.6 Election of Secretary-General of the United Nations

Now, we will study an example that is a bit more complex but coming from international politics. We will only give the maximal family, with discussing in details the voting system that is obtained by maximal Condorcification.

The election of the Secretary-General of the United Nations can be modeled in the following way. It uses range voting, where authorized grades are -1, 0 and 1; but each nation that is a member of the Security Council, a subset of $\mathcal{V}$ denoted CS, has a power of reject about any person who is a candidate. So, there are actually as many candidates as persons aspiring to the position, plus one special candidate $\varnothing$ meaning: “no physical person is elected” (which has several consequences in practice).

So, each member of SC has a right of reject over any candidate, except the special candidate $\varnothing$. This is a stronger power that Renaud's power of imposition seen in section 4.6.5: indeed, a member of the SC can not only impose candidate $\varnothing$ as winner (as Renaud can), but she can also, in a more nuanced way, choose to put her veto over some of the candidates but not all of them. For example, she can put her veto over all candidates except a given physical person c and the special candidate $\varnothing$, which restricts the choice of other voters to these two candidates: Renaud does not have such a power.

For any coalition M, we have:

$$M \in \mathcal{M}'_{\varnothing} \Leftrightarrow \exists v \in \text{CS s.t. } v \in M.$$

For any coalition M and any candidate $c \neq \varnothing$, we have:

$$M \in \mathcal{M}'_c \Leftrightarrow \text{CS} \subseteq M \text{ and } \operatorname{card}(M) > \frac{V}{2}.$$

It is easy to check that the voting system meets $\mathcal{M}'$**IgnC-A**, hence the maximal family $\mathcal{M}'$ is exclusive.

So, we can use the maximal Condorcification theorem 4.25. For the sake of conciseness, we will give no example of application this time. We notice that, in order for a physical candidate to be $\mathcal{M}'$-Condorcet, she must not only be Condorcet winner in the usual majoritarian sense but also the most liked candidate for each member of the Security Council. So, for physical candidates, being $\mathcal{M}'$-Condorcet is a very demanding condition. On the opposite, in order for the special candidate $\varnothing$ to be $\mathcal{M}'$-Condorcet, it is sufficient that for any physical candidate c, there exists a member of the Security Council who prefers $\varnothing$ to c, which is an especially weak condition. Analyzing the maximal family shows that, in such a system, the special outcome $\varnothing$ is much easier to obtain than any physical candidate, in the initial system and in its maximal Condorcification as well.

If the actors involved have diverse other reasons to avoid the special outcome $\varnothing$, we can imagine, and that is what happens in real life, that there exists processes that are external to the voting system and taht allow to pre-select the candidate in order not to have a deadlock during the vote itself. As it happens, members of the Security Council lead preliminary negotiations in order to identify a candidate who is consensual enough and then, they submit this candidate to the vote of the General Assembly (whose they are members also). To the best of our knowledge, during each election of the Secretary-General of the United Nations until now, the Security Council pre-selected one candidate, and only one, for the final election with all members of the General Assembly.

Part II

Computer-assisted study of manipulability

Appendices

Notations

Non-alphabetical symbols

$[\alpha, \beta[$	Real interval from α included to β excluded (French convention).
$[\![j, k]\!]$	Integer interval from j to k included.
$\lfloor \alpha \rfloor$	Floor function of real number α.
$\lceil \alpha \rceil$	Ceiling function of real number α.
$\lvert \mathcal{A}(v) \rvert$	Number of voters v meeting assertion $\mathcal{A}(v)$.
$\pi(A \mid B)$	Conditional probability of event A knowing B.

Greek alphabet

μ	The law of variable P (unless otherwise stated).
π	A *culture* over electoral space Ω. More generally, a probability measure.
$\tau_{\mathrm{CM}}^{\pi}(f)$	Coalitional manipulability rate of voting system f in culture π.
Ω	Set $\prod_{v \in \mathcal{V}} \Omega_v$ of possible configurations ω. Also used as a notation shortcut for an electoral space $(V, C, \Omega, \mathrm{P})$.
Ω_M	Set of possible states ω_M for voters in a set M.
Ω_v	Set of possible states ω_v for voter v.

Latin alphabet

$C \in \mathbb{N} \setminus \{0\}$	Number of candidates.
$\mathcal{C}$	Set $[\![1, C]\!]$ of indexes for the candidates.
$\mathrm{card}(E)$	Cardinal of set E.
$D(\omega)$	Matrix of duels in ω. The coefficient of indexes c and d is denoted $D_{cd}(\omega)$ or, in short, D_{cd}.

$\mathcal{F}_\mathcal{C}$	Set of strict weak orders over $\mathcal{C}$.
f	A state-based voting system (SBVS), i.e. a function $\Omega \to \mathcal{C}$. In the case of a general voting system, f denotes its processing function $\mathcal{S}_1 \times \ldots \times \mathcal{S}_V \to \mathcal{C}$.
f^*	Condorcification of f.
$f^{\mathrm{adm}}, f^{\mathrm{!adm}}$	Condorcification variants of f based on the notion of Condorcet-admissible candidate.
$f^{\mathrm{faible}}, f^{\mathrm{!faible}}$	Condorcification variants of f based on the notion of weak Condorcet winner.
f^{rel}	Relative Condorcification of f.
$f^{\mathcal{M}}$	$\mathcal{M}$-Condorcification of f.
f_y	Slice of f by a slicing method y.
$c \mathrm{~I}_v d$	Voter v is indifferent between c and d.
Id	The identity function (the context precises in which set).
$\mathcal{L}_\mathcal{C}$	Set of strict total orders over $\mathcal{C}$.
$\mathcal{M}$	A family of collections of coalitions.
$\mathcal{M}_c \in \mathcal{P}(\mathcal{P}(\mathcal{V}))$	A collection of coalitions that are said *winning* for candidate c.
$\mathrm{Manip}_\omega(\mathrm{w} \to c)$	Set of voters preferring c to w. In short, $\mathrm{Manip}(\mathrm{w} \to c)$.
CM_f	Set of configurations ω where f is manipulable (or indicator function of this set).
$\mathrm{mean}(x_1, \ldots, x_k)$	Arithmetical average of $x_1, \ldots, x_k$.
P	Function $\Omega \to \mathcal{R}$ that, to state ω of the population, associates profile $\mathrm{P}(\omega) = (\mathrm{P}_1(\omega_1), \ldots, \mathrm{P}_V(\omega_V))$.
$c \mathrm{~P}_v d$	Voter v prefers c to d.
$c \mathrm{~P}_{\mathrm{abs}} d$	c has an absolute victory against d: $\lvert c \mathrm{~P}_v d\rvert > \frac{V}{2}$.
$c \mathrm{~P}_{\mathrm{rel}} d$	c has a relative victory against d: $\lvert c \mathrm{~P}_v d\rvert > \lvert d \mathrm{~P}_v c\rvert$.
$c \mathrm{~P}_{\mathcal{M}} d$	c has an $\mathcal{M}$-victory against d: $\{v \text{ s.t. } c \mathrm{~P}_v d\} \in \mathcal{M}_c$.
$c \mathrm{~MP}_v d$	Voter v prefers c to d and vice versa (impossible if P_v is antisymmetric).
$c \mathrm{~PP}_v d$	Voter v prefers c to d but not d to c (synonym of $c \mathrm{~P}_v d$ if P_v is antisymmetric).
$\mathcal{R}$	Set ${\mathcal{R}_\mathcal{C}}^V$ whose an element (profile) represents binary relations of preference for the whole population of voters.
$\mathcal{R}_\mathcal{C}$	Set of binary relations over $\mathcal{C}$.
$\mathrm{Sinc}_\omega(\mathrm{w} \to c)$	Set of voters who do not prefer c to w. In short, $\mathrm{Sinc}(\mathrm{w} \to c)$.
$V \in \mathbb{N} \setminus \{0\}$	Number of voters.
$(V, C, \Omega, \mathrm{P})$	An electoral space. In short, Ω.
$\mathcal{V}$	Set $[\![1, V]\!]$ of indexes for the voters.
$\mathrm{vect}(E)$	Linear span of E, where E is a part of a vector space.
$\mathcal{Y}$	Set $\prod_{v \in \mathcal{V}} \mathcal{Y}_v$ of slicing methods y for the whole population of voters.

$\mathcal{Y}_v$	Set $\{y_v : \mathrm{P}(\Omega_v) \to \Omega_v$ s.t. $\mathrm{P}_v \circ y_v = \mathrm{Id}\}$ of slicing methods y_v for voter v.

Acronyms and abbreviations

AV	Approval voting.
Bald.	Baldwin's method.
Bor.	Borda's method.
Buck.	Bucklin's method.
CIRV	Condorcification of IRV.
CM	Coalition manipulation / manipulable.
Cond	Condorcet criterion.
Coo.	Coombs' method.
CSD	Condorcet's method with sum of defeats.
EB	Exhaustive ballot.
IB	Iterated Bucklin's method.
ICM	Ignorant-coalition manipulation / manipulable.
iff	If and only if.
IgnMC	Ignorant majority coalition criterion.
IIA	Independence of irrelevant alternatives.
IM	Individual manipulation / manipulable.
InfMC	Informed majority coalition criterion.
IRV	Instant-runoff voting.
IRVA	Instant-runoff voting based on the average.
IRVD	Instant-runoff voting with duels.
ITR	Instant two-round system.
Kem.	Kemeny's method.
KR	Kim-Roush's method.
MajBal	Majority ballot criterion.
MajFav	Majority favorite criterion.
MajUniBal	Majority unison ballot criterion.
Max.	Maximin.
MJ	Majority Judgement.
Nan.	Nanson's methodd.
Plu.	Plurality.
RP	Ranked Pairs method.
RV	Range voting.
s.t.	Such that.
SBVS	State-based voting system.

Sch.	Schulze's method.
S∀AMP	Simulator of Various Voting Algorithms in Manipulating Populations.
TM	Trivial manipulation / manipulable.
TR	Two-round system.
UM	Unison manipulation / manipulable.
VMF	Von Mises–Fisher.

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