Computational Physics Project 1: Pendulum

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The Simple Pendulum

1. Real space and nonlinear divergence

As k increases, the pendulum's trajectory becomes less linear. The numerical solutions also diverge after a number of periods. The 3rd order symplectic method diverges at k 15, and the semi-implicit Euler's method diverges at k 18.5.

2. Total energy over time

Although the total energy oscillates over time, the amplitude of the oscillation is small. More importantly, the energy does not diverge over time. All plots are with $\delta t = 0.1$

3. Comparing 3rd order symplectic integrator with semi-implicit Euler's in error over time

At first, the left of figure 3 appeared a surprise as the semi-implicit Euler's method seem to converge to the analytical method over time. However, with longer time, the right plot reveals that the semi-implicit Euler's has a periodic error, and the 3rd-order symplectic method has an error that grows much slower over time. We integrate over the square of error with regard to the analytical solution to measure the total error.

Route to Chaos

1. Phase space of nonlinear pendulum

As θ_0 approaches π, the trajectories go from an ellipse to a more "lemon" shape, as seen in figure 4.

For $\theta_0 = 0$, varying $\dot{\theta}_0 \in [0, \pi]$, the phase shows two different behaviors. For $\dot{\theta}_0$ between 0 and approximately 0.6π the phase space appears similar to the previous one, as seen in figure 5a. If $\dot{\theta}_0$ is greater than that, the motion is no longer periodic, and θ increases indefinitely, as seen in figure 5b.

2. Phase space of linear pendulum

For the linear pendulum, the phase space trajectory remains elliptical for all values of θ_0, as seen in figure 6

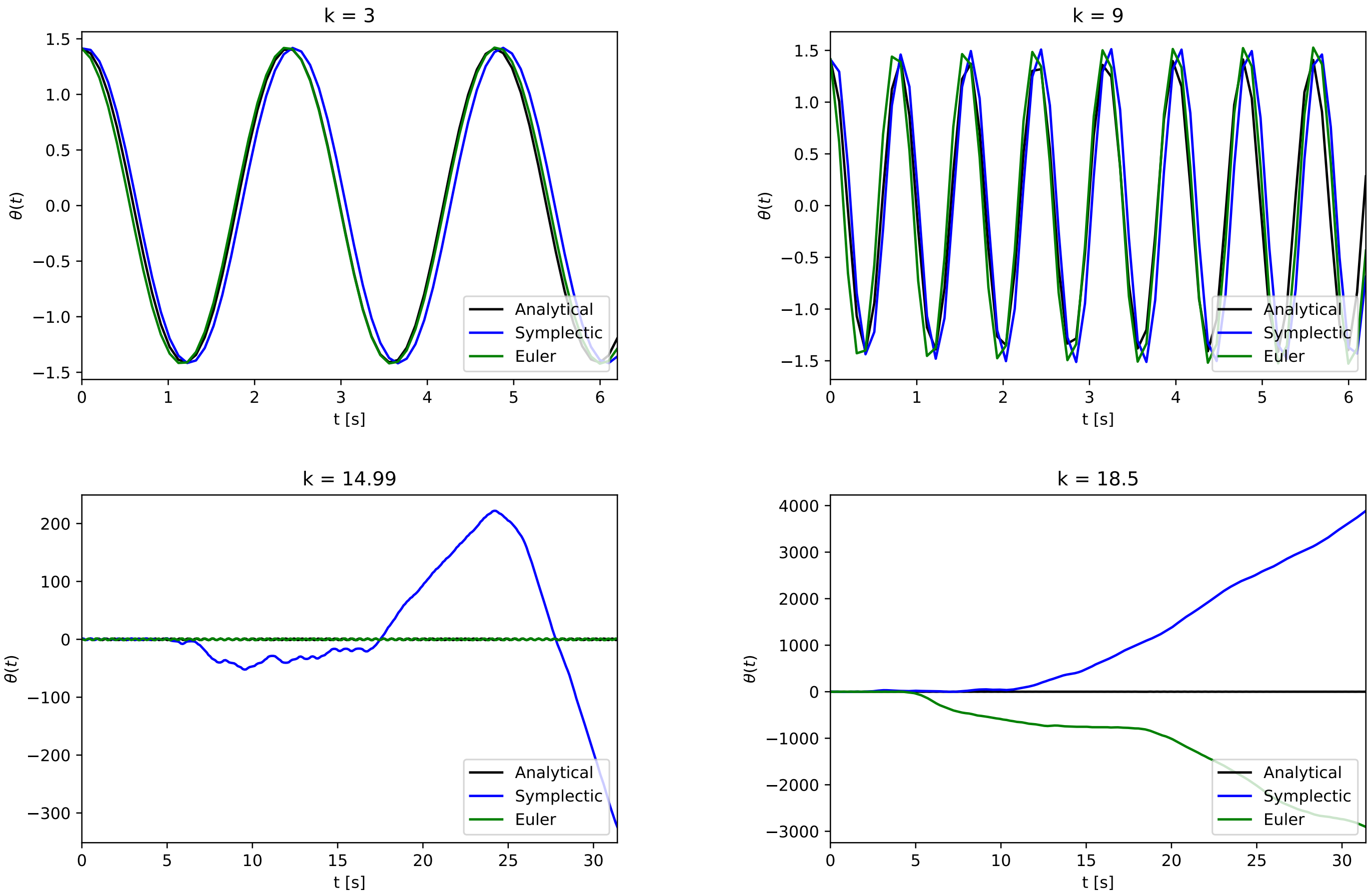

Figure 1: Real space for various k

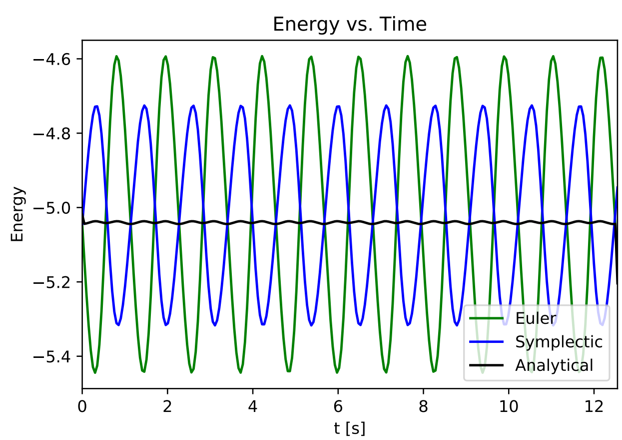

Figure 2: Real space for various k

3. Pendulum with driving force, $\gamma k^2 cos(\omega t)$

If a periodic driving force with $\omega = k$ is added, the frequency stays the same, but the ampltude varies periodically, as seen in figure 7.

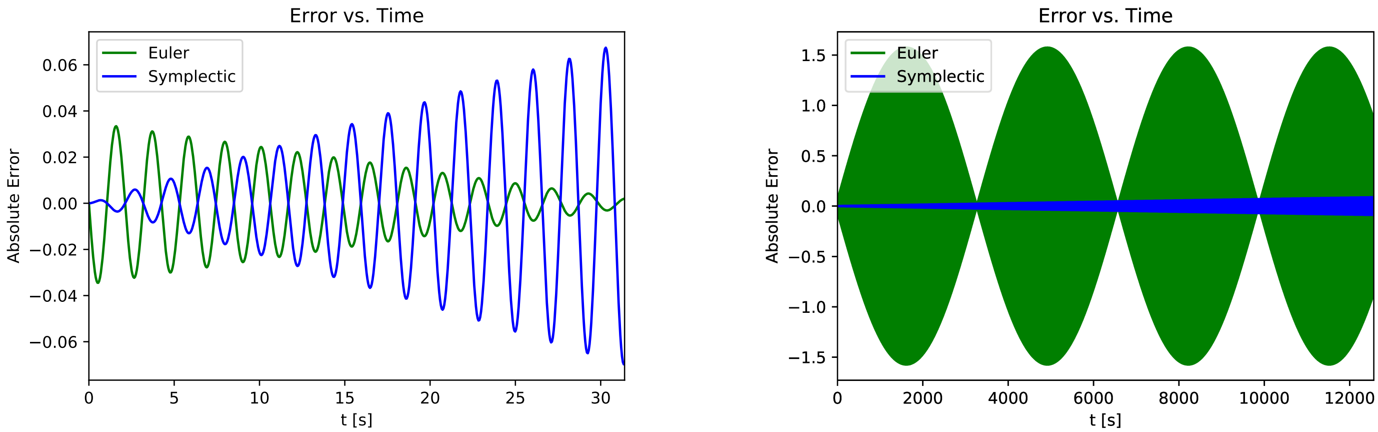

Figure 3: Real space for various k

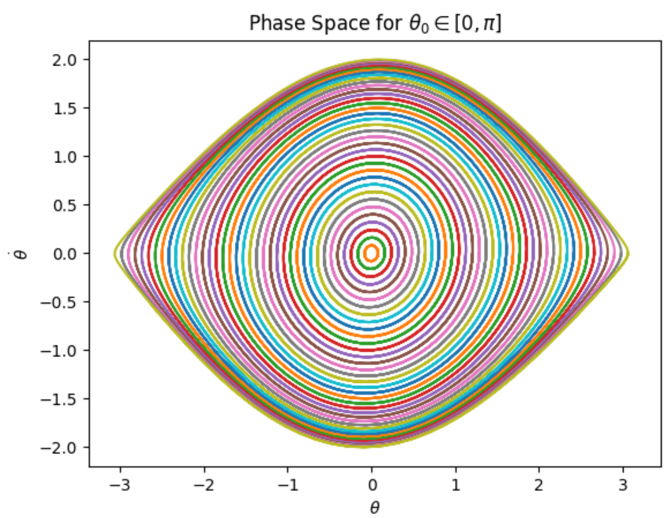

Figure 4: Plots of trajectory $(\theta, \dot{\theta})$, for many values of $\theta_0 \in [0, \pi]$

4. Exploration of driven system

Figure 8 shows how the trajectories vary with γ. The upper two panes identify values of γ that makes the pendulum display periodic behavior: for small γ, e.g. 0.0, 0.14, and 0.28, the pendulum's motion is not affected much and retains the same period; when $\gamma = 1.43$, the pendulum exhibits period doubling as shown by the red phase space trajectory. The lower two panes show chaotic behavior under different values of γ.

5. Identifying (θ_0, γ) for which the motion diverges

Figure 9 shows the phase plot for (θ_0, γ) for $\theta_0 \in [0, \pi]$, and $\gamma \in [0, 6]$, after a time interval of 8π seconds. The white regions indicate values for which the motion remained periodic. The blue regions indicate values for which the motion diverged. The darker the color, the greater the value of θ at the end of the time interval.

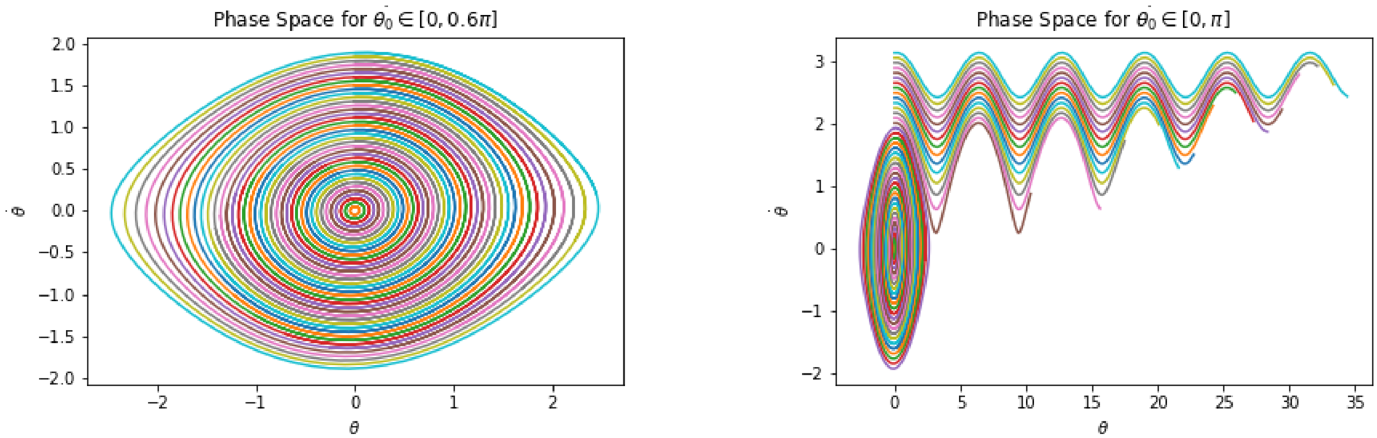

Figure 5: Trajectory for various $\dot{\theta}$

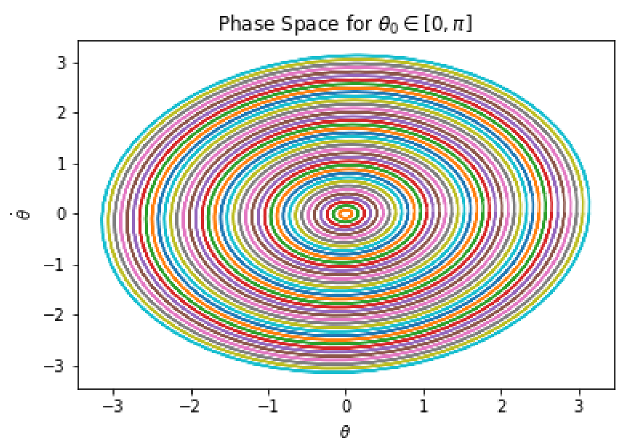

Figure 6: Plots of linearized trajectory $(\theta, \dot{\theta})$, for many values of $\theta_0 \in [0, \pi]$

6. Driven pendulum with damping $\ddot{\theta} + 2\beta\dot{\theta} + k^2 sin\theta = \gamma k^2 cos(\omega t)$

As γ increases, the pendulum transitions to chaos, as seen in figure 10. This transition is known as *period doubling*, which is becomes apparent from observing the plots. As γ increases, some trajectories diverge and become nonperiodic, while some trajectories remain stable and periodic, with a period twice as long as before. Figure 10 only shows trajectories that remained periodic.

7. Fourier analysis

By taking the fourier transform of a periodic trajectory, we can determine the relative contribution of the various frequencies that make up the trajectory. Figure 11 shows the fast fourier transform of the trajectory that is colored yellow in figure 10. Each spike corresponds to a frequency that is present in the trajectory, and the sizes of the spikes correspond to the amplitude of that frequency.

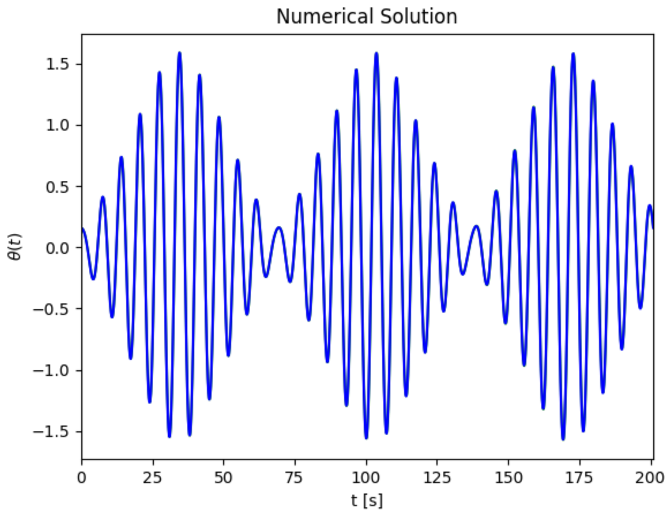

Figure 7: Solution for driven undamped pendulum

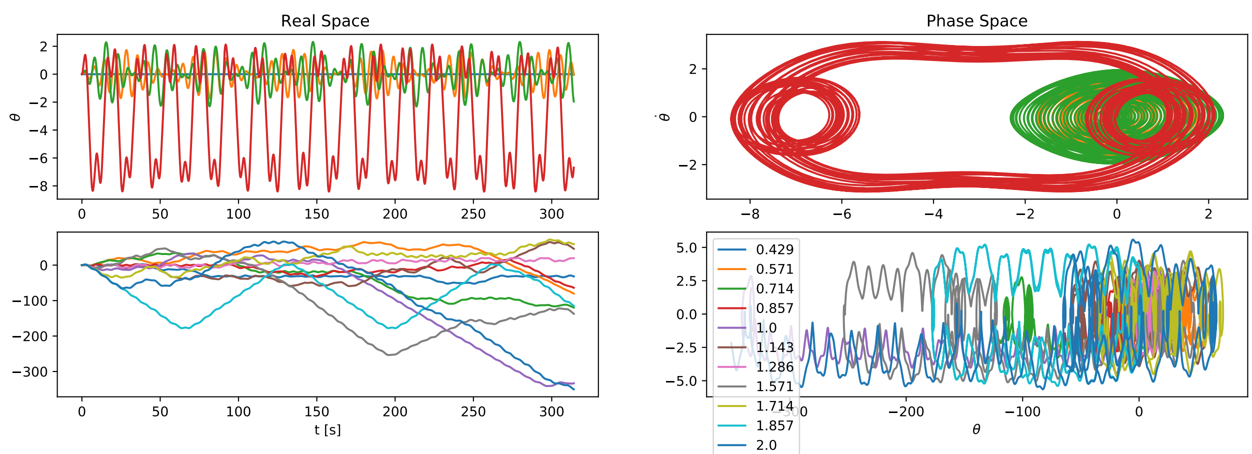

Figure 8: Trajectories for undamped driven pendulum for $\gamma \in [0, 2]$

The Double Pendulum

1. **Visualization**

2. **Lyapunov Exponent**

   The Lyapunov exponent is a measure of the chaos in a system.

3. **Transition to Chaos**

4. **Time for First Flip**

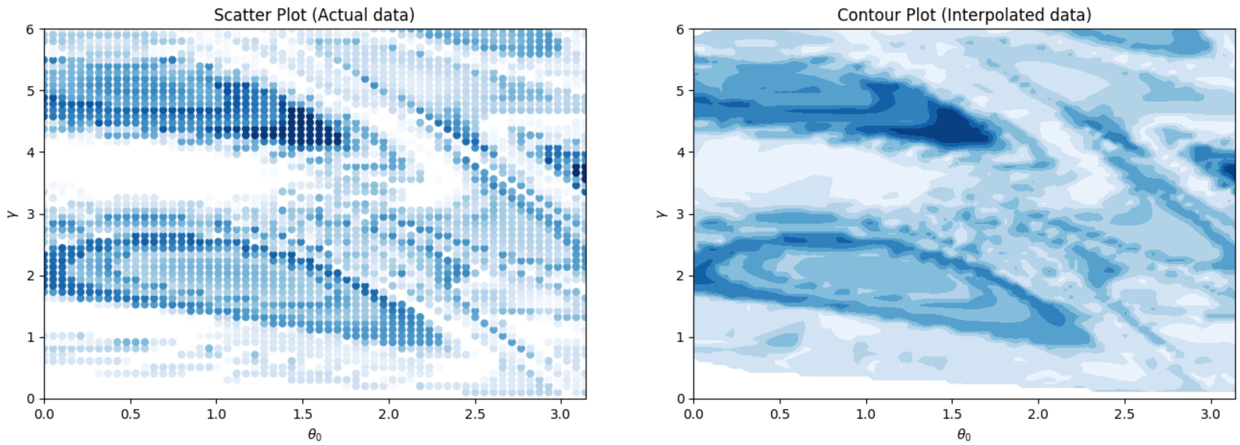

Figure 9: Phase plot for (θ_0, γ). On the left is the actual data that was calculated, and on the right is an interpolated contour plot.

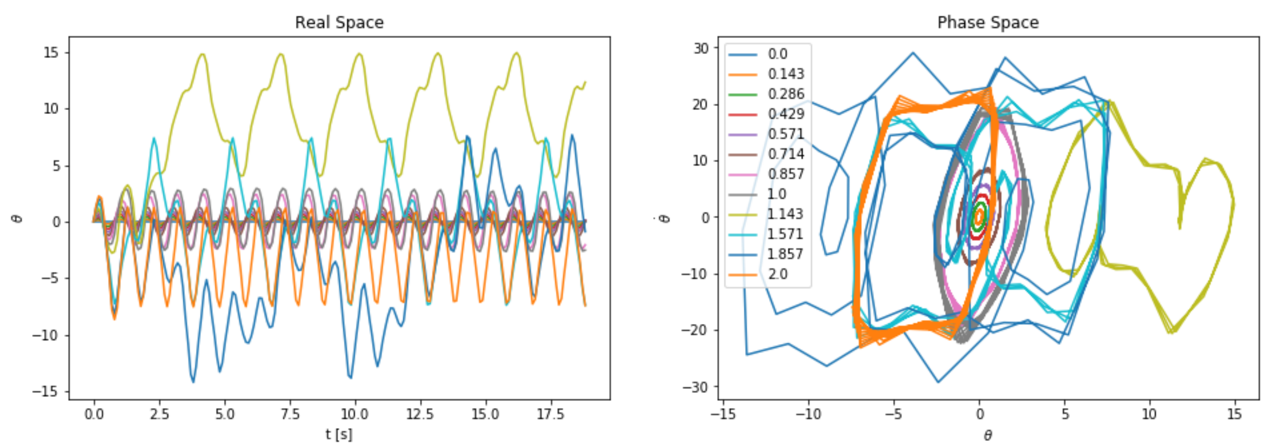

Figure 10: Plots of driven damped pendulum for trajectory various values of $\gamma \in [0, 2]$

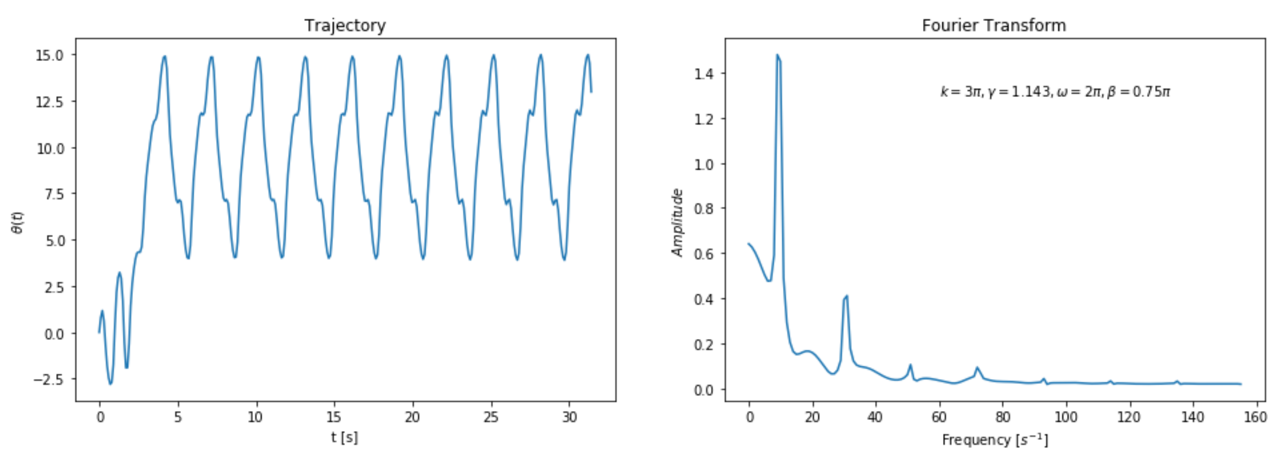

Figure 11: Fourier transform of damped driven pendulum with the parameters shown

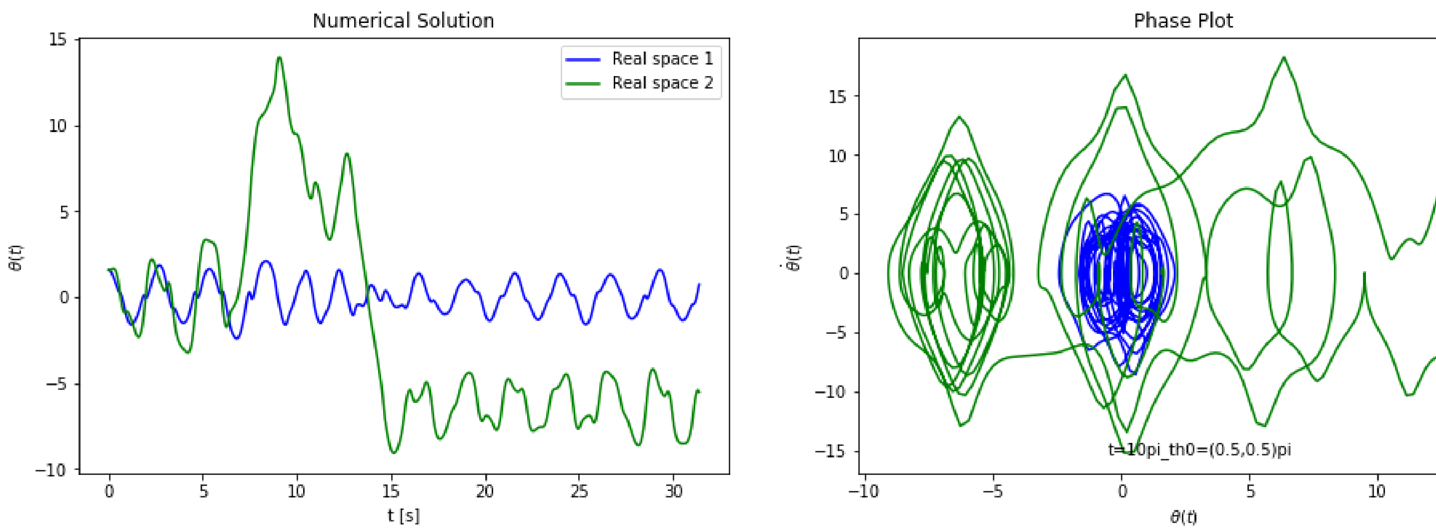

Figure 12: Trajectory of double pendulum

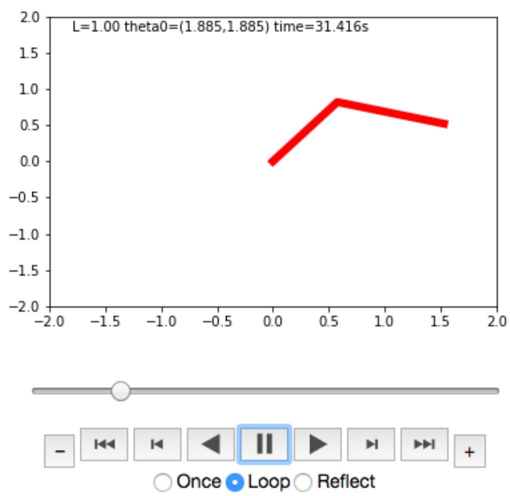

Figure 13: Screenshot of double pendulum animation

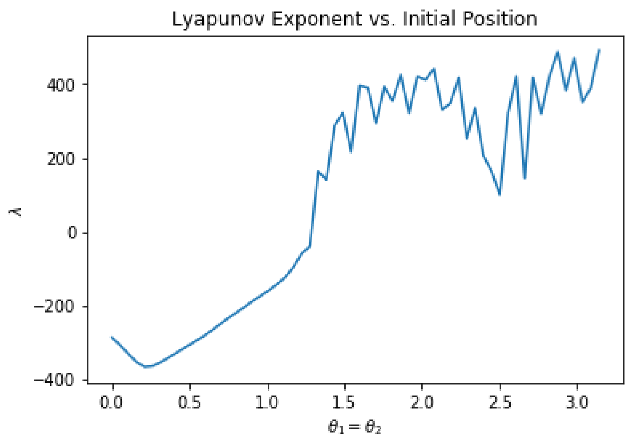

Figure 14: Plot of Lyapunov exponent vs. starting angle, where $\theta_1 = \theta_2$

Figure 15:

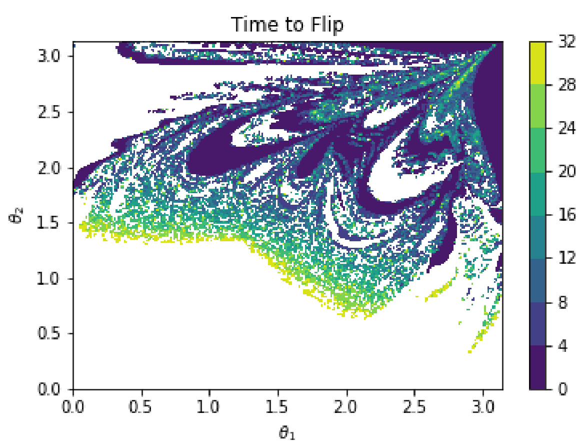

Figure 16: Color mapping of