Date

Older Axiomatic Proof Exercises :

14 $\vdash_D \Diamond(P \vee \sim P)$
(1) $P \vee \sim P$ (PL)
(2) $\Box(P \vee \sim P)$ (1, NEC)
(3) $\Box(P \vee \sim P) \rightarrow \Diamond(P \vee \sim P)$ (D)
(4) $\Diamond(P \vee \sim P)$ (2, 3, MP)

15 $\vdash_D \Box\Box P \rightarrow \Box\Diamond P$
(1) $\Box P \rightarrow \Diamond P$ (D)
(2) $\Box\Box P \rightarrow \Box\Diamond P$ (1, NEC, K, MP)

6a $\vdash_D \sim\Box(P \wedge \sim P)$ Negation ⇒ MT
(1) $\sim(P \wedge \sim P)$ (PL)
(2) $\Box\sim(P \wedge \sim P)$ (1, NEC)
~~(3) $\Diamond\sim(P \wedge \sim P)$ (2, T, MP)~~
~~(4)~~
(3) $\Box(P \wedge \sim P) \rightarrow \Diamond(P \wedge \sim P)$ (D)
(4) $\sim\Diamond(P \wedge \sim P)$ (2, PL DNI)
(5) $\sim\Box(P \wedge \sim P)$ (3, 4, MT)

b $\vdash_D \sim(\Box P \wedge \Box\sim P)$
(1) $\sim(P \wedge \sim P)$ (PL)
(2) $\Box\sim(P \wedge \sim P)$ (1, NEC)
(3) $\sim\Diamond(P \wedge \sim P)$ (2, PL DNI)
(4) $P \rightarrow (\sim P \rightarrow (P \wedge \sim P))$ (PL)
(5) $\Box P \rightarrow \Box(\sim P \rightarrow (P \wedge \sim P))$ (4, NEC, K, MP)
(6) $\Box(\sim P \rightarrow (P \wedge \sim P)) \rightarrow \Box\sim P \rightarrow \Box(P \wedge \sim P)$ (K)
(7) $\Box P \rightarrow (\Box\sim P \rightarrow \Box(P \wedge \sim P))$ (5, 6, PL syllogism)
(8) $(\Box P \wedge \Box\sim P) \rightarrow \Box(P \wedge \sim P)$ (7, PL import/export)
(9) $\Box(P \wedge \sim P) \rightarrow \Diamond(P \wedge \sim P)$ (T)
(10) $(\Box P \wedge \Box\sim P) \rightarrow \Diamond(P \wedge \sim P)$ (8, 9, PL syllogism)
(11) $\sim(\Box P \wedge \Box\sim P)$ (3, 10, PL MT)

c $\vdash_D \sim\Box[\Box(P \wedge \partial) \wedge \Box(P \rightarrow \sim\partial)]$
(1) $\sim(P \wedge \sim P)$ (PL)
(2) $\Box\sim(P \wedge \sim P)$ (1, NEC)
(3) $\sim\Diamond(P \wedge \sim P)$ (2, PL DNI)
(4) $\Box(P \wedge \sim P) \rightarrow \Diamond(P \wedge \sim P)$ (D)
(5) $\sim\Box(P \wedge \sim P)$ (3, 4, PL MT)
(6) $\Box\sim\Box(P \wedge \sim P)$ (5, NEC)
(7) $\sim\Diamond\Box(P \wedge \sim P)$ (6, PL DNI)
(8) $(P \wedge \partial) \rightarrow ((P \rightarrow \sim\partial) \rightarrow (P \wedge \sim P))$ (PL)
(9) $\Box(P \wedge \partial) \rightarrow \Box((P \rightarrow \sim\partial) \rightarrow (P \wedge \sim P))$ (8, NEC, K, MP)
(10) $\Box((P \rightarrow \sim\partial) \rightarrow (P \wedge \sim P)) \rightarrow (\Box(P \rightarrow \sim\partial) \rightarrow \Box(P \wedge \sim P))$ (K)
(11) $\Box(P \wedge \partial) \rightarrow (\Box(P \rightarrow \sim\partial) \rightarrow \Box(P \wedge \sim P))$ (9, 10, PL syllogism)
(12) $[\Box(P \wedge \partial) \wedge \Box(P \rightarrow \sim\partial)] \rightarrow \Box(P \wedge \sim P)$ (11, PL import/export)
(13) $\Box[\Box(P \wedge \partial) \wedge \Box(P \rightarrow \sim\partial)] \rightarrow \Box\Box(P \wedge \sim P)$ (12, NEC, K, MP)
(14) $\Box\Box(P \wedge \sim P) \rightarrow \Diamond\Box(P \wedge \sim P)$ (D)
(15) $\Box[\Box(P \wedge \partial) \wedge \Box(P \rightarrow \sim\partial)] \rightarrow \Diamond\Box(P \wedge \sim P)$
(13, 14, PL syllogism)
(16) $\sim\Box[\Box(P \wedge \partial) \wedge \Box(P \rightarrow \sim\partial)]$ (7, 15, PL MT)

T◇ $\vdash_T \phi \rightarrow \Diamond\phi$
(1) $\Box\sim\phi \rightarrow \sim\phi$ (T)
(2) $\sim\sim\phi \rightarrow \sim\Box\sim\phi$ (1, PL contraposition)
(3) $\phi \rightarrow \Diamond\phi$ (2, PL DNE)

D $\vdash_T \Box\phi \rightarrow \Diamond\phi$
(1) $\Box\sim\phi \rightarrow \sim\phi$ (T)
(2) $\phi \rightarrow \Diamond\phi$ (1, PL contraposition DNE)
(3) $\Box\phi \rightarrow \phi$ (T)
(4) $\Box\phi \rightarrow \Diamond\phi$ (2, 3, PL syllogism)

7a $\vdash_T \Diamond\Box P \rightarrow \Diamond(P \vee \partial)$
(1) $P \rightarrow (P \vee \partial)$ (PL)
(2) $\Box P \rightarrow P$ (T)
(3) $\Box P \rightarrow (P \vee \partial)$ (2, 3, PL syllogism)
(4) $\Diamond\Box P \rightarrow \Diamond(P \vee \partial)$ (3, NEC, K◇, MP)

b $\vdash_T [\Box P \wedge \Diamond\Box(P \rightarrow \partial)] \rightarrow \Diamond\partial$
(1) $P \rightarrow ((P \rightarrow \partial) \rightarrow \partial)$ (PL)
(2) $\Box(P \rightarrow \partial) \rightarrow (P \rightarrow \partial)$ (T)
(3) $P \rightarrow (\Box(P \rightarrow \partial) \rightarrow \partial)$ (1, 2, PL)
(4) $\Box P \rightarrow \Box(\Box(P \rightarrow \partial) \rightarrow \partial)$ (3, NEC, K, MP)
(5) $\Box(\Box(P \rightarrow \partial) \rightarrow \partial) \rightarrow (\Diamond\Box(P \rightarrow \partial) \rightarrow \Diamond\partial)$ (K◇)
(6) $\Box P \rightarrow (\Diamond\Box(P \rightarrow \partial) \rightarrow \Diamond\partial))$ (4, 5, PL syllogism)
(7) $[\Box P \wedge \Diamond\Box(P \rightarrow \partial)] \rightarrow \Diamond\partial$ (6, PL import/export)

c $\vdash_T \Diamond(P \rightarrow \Box\partial) \rightarrow (\Box P \rightarrow \Diamond\partial)$
(1) $\Box\partial \rightarrow \partial$ (T)
(2) $P \rightarrow ((P \rightarrow \Box\partial) \rightarrow \partial)$ (1, PL)
(3) $\Box P \rightarrow \Box((P \rightarrow \Box\partial) \rightarrow \partial)$ (2, NEC, K, MP)
(4) $\Box((P \rightarrow \Box\partial) \rightarrow \partial) \rightarrow (\Diamond(P \rightarrow \Box\partial) \rightarrow \Diamond\partial)$ (K◇)
(5) $\Box P \rightarrow (\Diamond(P \rightarrow \Box\partial) \rightarrow \Diamond\partial)$ (3, 4, PL syllogism)
(6) $\Diamond(P \rightarrow \Box\partial) \rightarrow (\Box P \rightarrow \Diamond\partial)$ (5, PL import/export)

B◇ $\vdash_B \phi \rightarrow \Box\Diamond\phi$
(1) $\Diamond\Box\sim\phi \rightarrow \sim\phi$ (B)
(2) $\sim\sim\phi \rightarrow \sim\Diamond\Box\sim\phi$ (1, PL contraposition)
(3) $\phi \rightarrow \Box\Diamond\phi$ (2, PL DNI DNE)

16 $\vdash_B [\Box P \wedge \Box\Diamond\Box(P \rightarrow \partial)] \rightarrow \Box\partial$
(1) $\Diamond\Box(P \rightarrow \partial) \rightarrow (P \rightarrow \partial)$ (B)
(2) $P \rightarrow ((\Diamond\Box(P \rightarrow \partial) \rightarrow \partial)$ (1, PL)
(3) $\Box P \rightarrow \Box(\Diamond\Box(P \rightarrow \partial) \rightarrow \partial)$ (2, NEC, K, MP)
(4) $\Box(\Diamond\Box(P \rightarrow \partial) \rightarrow \partial) \rightarrow (\Box\Diamond\Box(P \rightarrow \partial) \rightarrow \Box\partial)$ (K)
(5) $\Box P \rightarrow (\Box\Diamond\Box(P \rightarrow \partial) \rightarrow \Box\partial)$ (3, 4, PL syllogism)
(6) $[\Box P \wedge \Box\Diamond\Box(P \rightarrow \partial)] \rightarrow \Box\partial$ (5 PL import/export)

$\Box\sim\Box\Diamond\Box P \rightarrow \Box\sim\Box P$

B◇ $\vdash_B \Diamond\Box P \leftrightarrow \Diamond\Box\Diamond\Box P$
(1) $\Diamond\Box\Diamond\Box P \rightarrow \Diamond\Box P$ (B)
(2) $\Box P \rightarrow \Box\Diamond\Box P$ (B◇)
(3) $\sim\Box\Diamond\Box P \rightarrow \sim\Box P$ (2, PL contraposition)
(4) $\Box\sim\Box\Diamond\Box P \rightarrow \Box\sim\Box P$ (3, NEC, K, MP)
(5) $\Diamond\Box P \rightarrow \Diamond\Box\Diamond\Box P$ (4, PL contraposition)
(6) $\Diamond\Box P \leftrightarrow \Diamond\Box\Diamond\Box P$ (1, 5, PL)

6 $\vdash_B \Box\Box(P \to \Box\Diamond P) \to \Box(\sim P \to \Box\sim P)$

(1) $\Box(P \to \Box\Diamond P) \to (\Diamond P \to \Diamond\Box\Diamond P)$ (K◇)

(2) $\Diamond\Box\Diamond P \to P$ (B)

(3) $\Box(P \to \Box\Diamond P) \to (\Diamond P \to P)$ (1, 2, PL)

(4) $\Box\Box(P \to \Box\Diamond P) \to \Box(\Diamond P \to P)$ (3, NEC, K, MP)

SS $\vdash_{S4} \Diamond\Diamond\phi \to \Diamond\phi$

(1) $\Box\sim\phi \to \Box\Box\sim\phi$ (S4)

(2) $\Box\sim\phi \to \sim\sim\Box\sim\phi$ (PL DNI)

(3) $\Box\Box\sim\phi \to \Box\sim\sim\Box\sim\phi$ (1, NEC, K, MP)

(4) $\Box\sim\phi \to \Box\sim\sim\Box\sim\phi$ (1, 3, PL syllogism)

(5) $\Diamond\Diamond\phi \to \Diamond\phi$ (4, PL contraposition)

7 $\vdash_{S4} (\Diamond P \wedge \Box Q) \to \Diamond(P \wedge \Box Q)$

(1) $\Box Q \to (P \to (P \wedge \Box Q))$ (PL)

(2) $\Box\Box Q \to \Box(P \to (P \wedge \Box Q))$ (1, NEC, K, MP)

(3) $\Box(P \to (P \wedge \Box Q)) \to (\Diamond P \to \Diamond(P \wedge \Box Q))$ (K◇)

(4) $\Box\Box Q \to (\Diamond P \to \Diamond(P \wedge \Box Q))$ (2, 3, PL syllogism)

(5) $\Box Q \to \Box\Box Q$ (S4)

(7) $\Box Q \to (\Diamond P \to \Diamond(P \wedge \Box Q))$ (4, 5, PL syllogism)

(8) $(\Diamond P \wedge \Box Q) \to \Diamond(P \wedge \Box Q)$ (7, PL import/export)

9c $\vdash_{S4} \Box P \to \Box\Diamond\Box P$

(1) $\Box P \to \Box\Box P$ (S4)

(2) $\Box\Box P \to \Diamond\Box P$ (D)

(3) $\Box P \to \Diamond\Box P$ (1, 2, PL syllogism)

(4) $\Box\Box P \to \Box\Diamond\Box P$ (3, NEC, K, MP)

(5) $\Box P \to \Box\Diamond\Box P$ (1, 4, PL syllogism)

b $\vdash_{S4} \Box\Diamond\Box\Diamond P \to \Box\Diamond P$

(1) $\Box\Diamond P \to \Diamond\Diamond P$ (D)

(2) $\Diamond\Diamond P \to \Diamond P$ (S4◇)

(3) $\Box\Diamond P \to \Diamond\Diamond P$ (1, 2, PL syllogism)

(4) $\Box(\Box\Diamond P \to \Diamond\Diamond P)$ (3, NEC)

(5) $\Box(\Box\Diamond P \to \Diamond\Diamond P) \to (\Diamond\Box\Diamond P \to \Diamond\Diamond P)$ (K◇)

(6) $\Diamond\Box\Diamond P \to \Diamond\Diamond P$ (4, 5, MP)

(7) $\Diamond\Diamond P \to \Diamond P$ (S4◇)

(8) $\Diamond\Box\Diamond P \to \Diamond P$ (6, 7, PL syllogism)

(9) $\Box\Diamond\Box\Diamond P \to \Box\Diamond P$ (8, NEC, K, MP)

c $\vdash_{S4} \Diamond\Box P \to \Diamond\Box\Diamond\Box P$

(1) $\Box P \to \Diamond\Box P$ (T◇)

(2) $\Box\Box P \to \Box\Diamond\Box P$ (1, NEC, K, MP)

(3) $\Box P \to \Box\Box P$ (S4)

(4) $\Box P \to \Box\Diamond\Box P$ (2, 3, PL syllogism)

(5) $\Diamond\Box P \to \Diamond\Box\Diamond\Box P$ (4, NEC, K◇, MP)

SS $\vdash_{S5} \Diamond\phi \to \Box\Diamond\phi$

(1) $\Diamond\Box\sim\phi \to \Box\sim\phi$ (S5)

(2) $\Diamond\phi \to \sim\Diamond\Box\sim\phi$ (1, PL contraposition)

$= \Diamond\phi \to \sim\sim\Box\Diamond\phi$

(3) $\Diamond\phi \to \Box\Diamond\phi$ (2, PL DNE)

10 $\vdash_{S5} (\Box P \vee \Diamond Q) \leftrightarrow \Box(P \vee \Diamond Q)$

(1) $P \to (P \vee \Diamond Q)$ (PL)

(2) $\Box P \to \Box(P \vee \Diamond Q)$ (1, NEC, K, MP)

(3) $\Diamond Q \to (P \vee \Diamond Q)$ (PL)

(4) $\Box\Diamond Q \to \Box(P \vee \Diamond Q)$ (3, NEC, K, MP)

(5) $\Diamond Q \to \Box\Diamond Q$ (S5◇)

(6) $\Diamond Q \to \Box(P \vee \Diamond Q)$ (4, 5, PL syllogism)

(7) $\sim\sim P \to P$ (PL DNE)

(8) $\Box\sim\sim P \to \Box P$ (7, NEC, K, MP)

(9) $\sim\Box P \to \sim\Box\sim\sim P$ (8, PL contraposition)

$= \sim\Box P \to \Diamond\sim P$

(10) $\Box(\sim P \to \Diamond Q) \to (\Diamond\sim P \to \Diamond\Diamond Q)$ (K◇)

(11) $\Box(\sim P \to \Diamond Q) \to (\sim\Box P \to \Diamond\Diamond Q)$ (9, 10, PL)

(12) $\Diamond\Diamond Q \to \Diamond Q$ (S4◇)

(13) $\Box(\sim P \to \Diamond Q) \to (\sim\Box P \to \Diamond Q)$ (11, 12, PL)

$= \Box(P \vee \Diamond Q) \to (\Box P \vee \Diamond Q)$

(14) $(\Box P \vee \Diamond Q) \leftrightarrow \Box(P \vee \Diamond Q)$ (2, 6, 13, PL)

b $\vdash_{S5} \Diamond(P \wedge \Diamond Q) \leftrightarrow (\Diamond P \wedge \Diamond Q)$

(1) $(P \wedge \Diamond Q) \to P$ (PL)

(2) $\Diamond(P \wedge \Diamond Q) \to \Diamond P$ (1, NEC, K◇, MP)

(3) $(P \wedge \Diamond Q) \to \Diamond Q$ (PL)

(4) $\Diamond(P \wedge \Diamond Q) \to \Diamond\Diamond Q$ (3, NEC, K◇, MP)

(5) $\Diamond\Diamond P \to \Diamond P$ (S4◇)

(6) $\Diamond(P \wedge \Diamond Q) \to \Diamond Q$ (4, 5, PL syllogism)

(7) $\Diamond Q \to (P \to (P \wedge \Diamond Q))$ (PL)

(8) $\Box\Diamond Q \to \Box(P \to (P \wedge \Diamond Q))$ (7, NEC, K, MP)

(9) $\Box(P \to (P \wedge \Diamond Q)) \to (\Diamond P \to \Diamond(P \wedge \Diamond Q))$ (K◇)

(10) $\Diamond Q \to \Box\Diamond Q$ (S5◇)

(11) $\Diamond Q \to (\Diamond P \to \Diamond(P \wedge \Diamond Q))$ (8, 9, 10, PL syllogism)

(12) $(\Diamond P \wedge \Diamond Q) \to \Diamond(P \wedge \Diamond Q)$ (11, PL import/export)

(13) $\Diamond(P \wedge \Diamond Q) \leftrightarrow (\Diamond P \wedge \Diamond Q)$ (2, 6, 12, PL)

c $\vdash_{S5} \Box(\Box P \to \Box Q) \vee \Box(\Box Q \to \Box P)$

$= \sim\Box(\Box P \to \Box Q) \to \Box(\Box Q \to \Box P)$

(1) $\Box P \to (\Box Q \to \Box P)$ (PL)

(2) $\Box\Box P \to \Box(\Box Q \to \Box P)$ (1, NEC, K, MP)

(3) $\Box P \to \Box\Box P$ (S4)

(4) $\Box P \to \Box(\Box Q \to \Box P)$ (2, 3, PL syllogism)

(5) $(\Box P \wedge \sim\Box Q) \to \Box(\Box Q \to \Box P)$ (4, PL)

$= \sim(\Box P \to \Box Q) \to \Box(\Box Q \to \Box P)$

(6) $\Diamond\sim(\Box P \to \Box Q) \to \Diamond\Box(\Box Q \to \Box P)$ (5, NEC, K◇, MP)

(7) $\sim\Box(\Box P \to \Box Q) \to \Diamond\sim(\Box P \to \Box Q)$ (MN)

(8) $\Diamond\Box(\Box Q \to \Box P) \to \Box(\Box Q \to \Box P)$ (S5)

(9) $\sim\Box(\Box P \to \Box Q) \to \Box(\Box Q \to \Box P)$ (6, 7, 8, PL syllogism)

$= \Box(\Box P \to \Box Q) \vee \Box(\Box Q \to \Box P)$

d $\vdash_{S5} \Box[\Box(\Diamond P \to Q) \leftrightarrow \Box(P \to \Box Q)]$

(1) $P \to \Diamond P$ (T)

(2) $\Diamond P \to \Box\Diamond P$ (S5◇)

(3) $\Box(\Diamond P \to Q) \to (\Box\Diamond P \to \Box Q)$ (K)

(4) $\Box(\Diamond P \to Q) \to (P \to \Box Q)$ (2, 3, PL)

(5) $\Box\Box(\Diamond P \to Q) \to \Box(P \to \Box Q)$ (4, NEC, K, MP)

(6) $\Box(\Diamond P \to Q) \to \Box\Box(\Diamond P \to Q)$ (S4)

(7) $\Box(\Diamond P \to Q) \to \Box(P \to \Box Q)$ (5, 6, PL syllogism)

(8) $\Box(P \to \Box Q) \to (\Diamond P \to \Diamond \Box Q)$ (K◇)
(9) $\Diamond \Box Q \to Q$ (35, T, PL syllogism)
(10) $\Box(P \to \Box Q) \to (\Diamond P \to Q)$ (8, 9, PL)
(11) $\Box\Box(P \to \Box Q) \to \Box(\Diamond P \to Q)$ (10, NEC, K, MP)
(12) $\Box(P \to \Box Q) \to \Box\Box(P \to \Box Q)$ (S4)
(13) $\Box(P \to \Box Q) \to \Box(\Diamond P \to Q)$ (11, 12, PL syllogism)
(14) $\Box(\Diamond P \to Q) \leftrightarrow \Box(P \to \Box Q)$ (7, 13, PL)
(15) $\Box[\Box(\Diamond P \to Q) \leftrightarrow \Box(P \to \Box Q)]$