Game Theory Problem Set 5

1a $u_A(x_A, m_A) = x_A + m_A$

$u_B(x_B, m_B) = 3x_B + m_B$

$x_A + x_B = 100 \Rightarrow x_B = 100 - x_A$

$m_A + m_B = 200 \Rightarrow m_B = 200 - m_A$

where $u_c$, $x_c$, and $m_c$ denote c's utility, c's share of farmland in acres and c's share of cash respectively for $c \in \{A, B\}$.

$X = \{(x_A, m_A) : x_A \in [0, 100], m_A \in [0, 200]\}$,

$D = (x_A = 50, m_A = 100)$

$U = \cancel{\{(u_A(x_A, m_A}$

$\{(u_A(x_A, m_A), u_B(x_B = 100 - x_A, m_B = 200 - m_A)) : (x_A, m_A) \in X\}$

$d = (u_A(x_A = 50, m_A = 100) = 150, u_B(x_B = 100 - x_A, m_B = 200 - m_A)$

$= u_B(x_B = 50, m_B = 100) = 250)$

where $X$ is the set of possible agreements, $D$ is the disagreement outcome, $U$ is the set of payoff pairs corresponding to $X$ and $d$ is the payoff pair corresponding to $D$.

~~Suppose $x_A = 100$, $m_A = 200$. then $u_A =$~~

~~Let~~

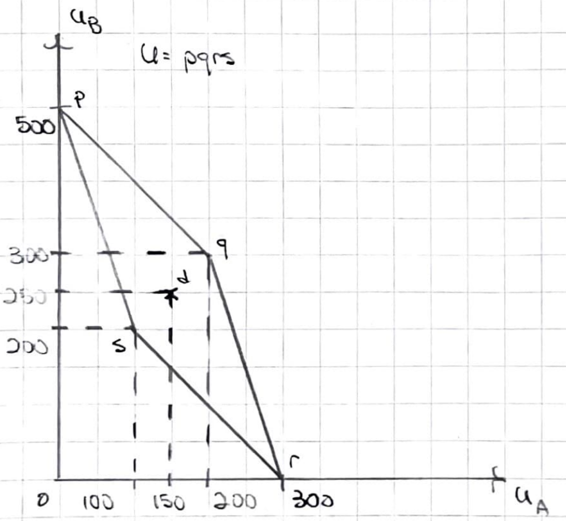
$u_A(x_A = 100, m_A = 200) = 300$, $u_B(x_A = 100, m_A = 200) = 0$

$u_A(x_A = 100, m_A = 0) = 100$, $u_B(x_A = 100, m_A = 0) = 200$

$u_A(x_A = 0, m_A = 200) = 200$, $u_B(x_A = 0, m_A = 200) = 300$

$u_A(x_A = 0, m_A = 0) = 0$, $u_B(x_A = 0, m_A = 0) = 500$

All other $x \in X$ are weighted averages of these possible agreements. Then, because utilities are linear, ~~utilit~~ for all other $x \in X$, the payoff pair corresponding to $x$ is a weighted average of these four possible agreements.




b Let $F(U, d) = u$ be a bargaining solution that satisfies the Nash axioms.

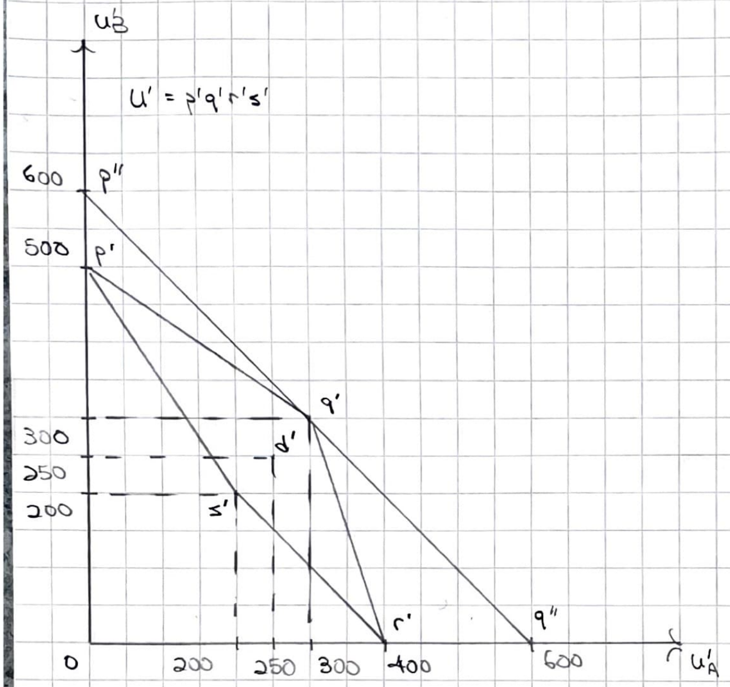
~~Let $u_A'(x_A, m_A) = 2u_A(x_A, m_A) - 100$, and $u_B'(x_B, m_B) = u_B(x_B, m_B)$~~

~~$U = \{(u_A'(x_A, m_A), u_B'(x_B = 100 - x_A, m_B = 200 - m_A) : (x_A, m_A) \in X\}$,~~

~~and $d = \{(u_A'(x_A = 50, m_A = 100), u_B'(x_B = 100 - x_A = 50, m_B = 200 - x_A = 100))$~~

$= ($

Let $u_A'(x_A, m_A) = u_A(x_A, m_A) + 100$, $u_B'(x_B, m_B) = u_B(x_B, m_B)$,
$U' = \{(u_A'(x_A, m_A), u_B'(x_B = 100 - x_A, m_B = 200 - m_A)) : (x_A, m_A) \in X\}$,
$d' = (u_A', u_B')|_0 = (250, 250)$




Consider the bargaining problem $(U'', d')$, where $U'' = \{(u_A', u_B') : u_A' + u_B' \in [0, 600]\}$. $U''$ is represented by area $Op''q''$. By inspection, $U'' \subset U'$.

Suppose that $(u_1', u_2') \in U''$, then $u_1' + u_2' \in [0, 600]$, then $(u_2', u_1') \in U''$. $d_1' = 250 = d_2'$, so the bargaining problem $(U'', d')$ is symmetric.

Let $u'' = F(U'', d')$. By symmetry, $u'' = (u_A^{*''}, u_B^{*''})$ is such that $u_A^{*''} = u_B^{*''}$. Suppose for reductio that $u_A'' + u_B'' < 600$. ~~Then~~ Let $\varepsilon = 600 - u_A'' - u_B''$. Then $u_A'' + \varepsilon/2 + u_B'' + \varepsilon/2 = 600$, $(u_A'' + \varepsilon/2, u_B'' + \varepsilon/2) \in U''$ and $u_A'' + \varepsilon/2 > u_A''$ and $u_B'' + \varepsilon/2 > u_B''$. By weak Pareto efficiency, $u'' \neq F(U'', d')$. By reductio $u_A'' + u_B'' = 600$. Solving simultaneously, $u_A'' = u_B'' = 300$. By independence of irrelevant alternatives, $u' = F(U', d') = u'' = (300, 300)$. By invariance to equivalent payoff representations, $u = F(U, d) = (u_A' - 100, u_B') = (200, 300)$. This corresponds to the distribution $x_A = 0$, $m_A = 200$, $x_B = 100$, $m_B = 0$.

c Suppose that A is the divider. ~~Sup~~ Denote the portions chosen by A as ~~$(x_A, m_A)$ or~~ $(x_1, m_1)$ and $(x_2 = 100 - x_1, m_2 = 200 - m_1)$. In the second stage subgame, B's maximisation problem is $\max_{i \in \{1,2\}} u_B(x_i, m_i) = 3x_i + m_i$.

B chooses $(x_1, m_1)$ ~~iff~~ if $3x_1 + m_1 \cancel{\not<}\, > 3x_2 + m_2 = 3(100 - x_1) + 200 - m_1$,
$3x_1 + m_1 \cancel{\not<}\, > 500 - 3x_1 - m_1$,
$3x_1 + m_1 \cancel{\not<}\, > 250$

In the first stage, A's payoff reduces to
$u_A(x_1, m_1) = x_1 + m_1$ iff $3x_1 + m_1 \not< 250$, $(100 - x_1) + (200 - m_1)$ otherwise

Suppose, without loss of generality, that A intends to induce B to choose $(x_2, m_2)$. Then A's maximisation problem is $\max_{x_1, m_1} x_1+m_1$ subject to $3x_1+m_1 \leq 250$, $x_1 \in [0,100]$, and $m_1 \in [0,200]$

~~By inspection~~

By inspection, A chooses $(x_1 = 50/3, m_1 = 200)$, then $(x_2 = 250/3, m_2 = 0)$, and B (weakly) prefers $(x_2, m_2)$ to $(x_1, m_1)$.

The SPE is the strategy profile such that A's strategy is to offer $(x_1 = 50/3, m_1 = 200)$ and $(x_2 = 250/3, m_2 = 0)$ and B's strategy is to choose $(x_1, m_1)$ iff $3x_1+m_1 > 250$.

Suppose that B is the divider. In the second stage subgame, A's maximisation problem is $\max_{i \in \{1,2\}} u_A(x_i, m_i) = x_i + m_i$. A chooses $(x_1, m_1)$ iff $u_A(x_1, m_1) = x_1+m_1 > u_A(x_2, m_2) = x_2+m_2 = (100-x_1)+(200-m_1)$ $x_1+m_1 > 150$. In the first stage, B's payoff function reduces to $u_B(x_1, m_1) = 3x_1+m_1$ iff $x_1+m_1 \leq 150$, $3(100-x_1)+(200-m_2)$ otherwise. Suppose wlog that B intends to induce A to choose $(x_2, m_2)$. Then B's maximisation problem is $\max_{x_1, m_1} 3x_1+m_1$ subject to $x_1+m_1 \leq 150$, $x_1 \in [0,100]$, $m_1 \in [0,200]$. ~~By inspec~~ inspection, ~~$x_1=100, m_1=50$~~ B chooses $(x_1 = 100, m_1 = 50)$, then $(x_2 = 100-x_1 = 0, m_2 = 200-m_1 = 150)$, and A (weakly) prefers $(x_2, m_2)$ to $(x_1, m_1)$. The SPE is the strategy profile such that B's strategy is to offer $(x_1 = 100, m_1 = 50)$ and $(x_2 = 0, m_2 = 150)$, and A's strategy is to choose $(x_1, m_1)$ iff $x_1+m_1 > 150$.

d Suppose that A is the divider, then at SPE, ~~$u_A = 50/3$~~
$u_A = 50/3 + 200$, $u_B = 250$

Suppose that B is the divider, then at SPE,
$u_A = 150$, $u_B = 350$

Expected utilities
$u_A = (50/3 + 200 + 150)/2 = 1100/6$,
$u_B = (\cancel{350+}250+350)/2 = 300$

This solution is not Pareto optimal because it is Pareto-dominated by $(x_A, m_A) = (0, 200)$, where $u_A = 200$ and $u_B = 300$.

Graphically, each of the "pure" divide and choose outcomes is some point on the pgr frontier that does not coincide with q since each first mover divides portions such as to advantage itself. By the convexity of U, any weighted average of the "pure" outcomes lies within the frontier, and is not Pareto optimal.

2a IIA: If $U' \subseteq U$ and $d' = d$ and $F(U,d) \in U'$, then ~~$F(U'$~~
$F(U',d) = F(U,d)$

consider $U' = \{(u_1,u_2) : u_1, u_2 \in [0,1]\}$, $U = U' \cup \{(0,10)\}$,
and $d = d' = (0,0)$. Then $u_1'^+ = u_2'^+ = u_1^+ = 1$ and $u_2^+ = 10$.

$\max u_1/u_1^+$ subject to $u_1/u_1^+ = u_2/u_2^+$ and $(u_1,u_2) \in U$
coincides with
$\max u_1$ subject to $u_1 = u_2/10$ and $(u_1,u_2) \in U$
~~$= \max u_1$ subject to $\neq$~~
~~$(u_1^*$~~
$= \max u_1$ subject to $(u_1, 10u_1) \in U$
$= 1/10$
$(u_1^*, u_2^*) = (1/10, \cancel{10}\ 1)$

$\max u_1'/u_1'^+$ subject to $u_1'/u_1'^+ = u_2'/u_2'^+$ and $(u_1',u_2') \in U'$
coincides with
$\max u_1'$ subject to $u_1' = u_2'$ and $(u_1',u_2') \in U'$
$= \max u_1'$ subject to $(u_1', u_1') \in U'$
$= 1$
$(u_1'^*, u_2'^*) = (1,1)$

Consequently, KS solution does not satisfy IIA.

b $u_1(x_1) = x_1$
$u_2(x_2) = x_2^\rho$ for $0 < \rho < 1$
~~$x_1 + x_2 = 1$~~
$U = \{(u_1(x_1), u_2(x_2)) : x_1 + x_2 = 1\}$
$u_1^+ = \max u_1$ st $(u_1,u_2) \in U = 1$
$u_2^+ = \max u_2$ st $(u_1,u_2) \in U = 1$

~~$\max u_1/u_1^+$ st $u_1/u_1^+ = u_2/u_2^+$, $(u_1,u_2) \in U$~~
~~$= \max u_1$ st $u_1 = u_2$, $(u_1,u_2) \in U$~~
~~$= \max u_1$ st $(u_1,u_1) \in U$~~

~~Suppose that $u_1/u_1^+ = u_2/u_2^+$, then $u_1 = u_2$, $x_1 = x_2^\rho$,~~
~~$x_1 = (1-x_1)^\rho$, $x_1^{1/\rho} + x_1 - 1 = 0$ $x_1 > x_2$ since $0 < \rho < 1$~~
~~and $x_1, x_2 \in [0,1]$~~
~~$\max u_1/u_1^+$ st $u_1/u_1^+ = u_2/u_2^+$, $(u_1,u_2) \in U$~~
~~$= \max u_1$ st $u_1 = u_2$, $(u_1,u_2) \in U$~~
~~$= \max x_1$ st $x_1^{1/\rho} + x_1 - 1 = 0$, $x_1 \in [0,1]$~~
~~EOE~~

Suppose that $u_1/u_1^+ = u_2/u_2^+$ and $(u_1,u_2) \in U$, then $u_1 = u_2$, $x_1 = x_2^\rho$.
Suppose for reductio that $x_2 = 1$, then $x_1 = 1$, then
$x_1 = 1$, then $(u_1,u_2) = (1,1) \notin U$. By reductio, $x_2 \neq 1$
Then $x_2 \in [0,1)$ ~~and so, since~~ consequently,
~~since 0~~ given $0 < \rho < 1$, $x_1 > x_2$.

c Let $U' = \{(u_1(x_1), u_2(x_2)) : x_1 + x_2 = s\}$, ~~$u_1'^+ = u_2'^+ = s$~~
$u_1'^+ = s$, $u_2'^+ = s^\rho$
~~Suppose that $u_1'^*/u_1'^+ = u_2'^*/u_2'^+$ and $(u_1'^*, u_2'^*) \in U$~~
~~$x_1 = x_2^\rho/s^\rho$ $(s - x_2) = (x_2/s)^\rho$~~
~~$x_1/s = (x_2/s)^\rho$, $1 - x_2/s = (x_2/s)^\rho$~~

Date

~~$\max u_1/u_1^\dagger$ subject to $u_1/u_1^\dagger = u_2/u_2^\dagger,\ (u_1, u_2) \in U$~~

~~$= \max x_1/s$ st $x_1/s = x_2^\rho/s^\rho,\ (x_1, x_2^\rho) \in U$~~

~~$= \max x_1/s$ st $x_2^\rho = x_1 s^{\rho-1},\ (x_1, x_2^\rho) \in U$~~

~~$= \max x_1/s$ st $(x_1, x_1 s^{\rho-1}) \in U$~~

~~coincides with~~

~~$\max x_1$ st $(x_2 s^{1-\rho}, x_2^\rho) \in U$~~

~~$= \max x_1$ st $x_2 s^{1-\rho} + x_2 = s$~~

~~$= \max x_1$~~

~~$\max x_2 s^{1-\rho}$ st $(x_2 s^{1-\rho}, x_2^\rho) \in U$~~

~~$= \max x_2 s^{1-\rho}$ st $x_2 s^{1-\rho} + x_2 = s$~~

~~$= \max x_2 s^{1-\rho}$ st $x_2 = s/(1+s^{1-\rho})$~~

$u_1^*/u_1^\dagger = u_2^*/u_2^\dagger$ and $(u_1^*, u_2^*) \in U$

$x_1/s = x_2^\rho/s^\rho$

$x_1 = x_2^\rho s^{1-\rho}$

$(x_1, x_2^\rho) \in U$

$\cancel{(x_1,}\ (x_2^\rho s^{1-\rho}, x_2^\rho) \in U$

$x_2^\rho s^{1-\rho} + x_2 = s$

$x_2^\rho s^{-\rho} + x_2^1 s^{-1} = 1$

~~$(x_2/s)^\rho + (x_2/s)^{-1} = 1$~~

Differentiating implicitly wrt $s$

$-\rho x_2^\rho s^{-\rho-1} + \rho x_2^{\rho-1}\, dx_2/ds\ s^{-\rho}$

$- x_2 s^{-2} + dx_2/ds\ s^{-1} = 0$

$(\rho x_2^{\rho-1} s^{-\rho} + s^{-1})\, dx_2/ds = \rho x_2^\rho s^{-\rho-1} + x_2 s^{-2}$

Given $0<\rho<1$, $x_2>0$, $s>0$,

$dx_2/ds > 0$

$dx_1/dx_2 = \rho x_2^{\rho-1} s^{1-\rho} > 0$

$dx_1/ds > 0$

consequently for $x_1$ and $x_2$ each increase when $s$ increases from 1 to some $s>1$

3a In the second stage subgame, given that P1 chose $x_2$, P2 chooses A (accept) or R (reject) to maximise

By backward induction

$u_2(a_2, x_2)$

$$u_2 = \begin{cases} x_2^\rho - s x_1^\rho = x_2^\rho - s(1-x_2)^\rho & \text{if } a_2 = A \\ 0 & \text{if } a_2 = R \end{cases}$$

where $a_2$ denotes P2's action. P2 chooses A iff $x_2^\rho - s x_1^\rho \geq 0$, $x_2^\rho \geq s x_1^\rho$, $x_2 \geq s^{1/\rho} x_1$, $x_2 \geq s^{1/\rho}(1-x_2)$, $(1+s^{1/\rho})x_2 \geq s^{1/\rho}$, $x_2 \geq s^{1/\rho}/(1+s^{1/\rho})$. Then P1's ~~maximi~~ payoff function in the first stage reduces to

$$u_1(x_2) = \begin{cases} (1-x_2)^\rho & \text{if } x_2 \geq s^{1/\rho}/(1+s^{1/\rho}) \\ 0 & \text{otherwise} \end{cases}$$

Then P1's maximisation problem is $\max_{x_2} u_1(x_2)$ ~~st~~ which has solution $x_2 = s^{1/\rho}/(1+s^{1/\rho})$.

The SPE is the strategy profile such that P1's strategy is to offer $x_2 = s^{1/\rho}/(1+s^{1/\rho})$ and P2's strategy is to A iff $x_2 \geq s^{1/\rho}/(1+s^{1/\rho})$ and R otherwise

b Supposing that s is known to P2, P2's payoff function, maximisation problem, and best response ~~are~~ function are unchanged from (a). P2 chooses A iff $x_2 \geq s^{1/\rho}(1-x_2)$, $s^{1/\rho} \leq x_2/(1-x_2)$, $s \leq (x_2/(1-x_2))^\rho$.

P1's expected payoff function reduces to

$$\pi_1(x_2) = (1-x_2)^\rho P(s \leq (x_2/(1-x_2))^\rho) + 0^\rho P(s > (x_2/(1-x_2))^\rho)$$
$$= (1-x_2)^\rho F((x_2/(1-x_2))^\rho)$$
$$= (1-x_2)^\rho (x_2/(1-x_2))^{2\rho}$$

P1's maximisation problem is $\max_{x_2} \pi_1(x_2)$ ~~that~~ ~~which coincides with~~ $\cancel{\max_{x_2} (1-x_2)(x_2/(1-x_2))^2}$ ~~since $0<\rho<1$ so raising to the power $1/\rho$ is a strictly increasing monotonic transformation.~~

~~FOC:~~ $\cancel{\partial/\partial x_2\ (1-x_2)(x_2/(1-x_2))^2}$

$\cancel{= \partial/\partial x_2\ x_2^2/(1-x_2)}$

$\cancel{= 2x_2/(1-x_2) + x_2^2/(1-x_2)^2 = 0}$

$\cancel{2x_2(1-x_2) + x_2^2 = 0}$

$\cancel{= 2x_2 - x_2^2 = 0}$

$\cancel{x_2 = 0 \text{ (reject) or } x_2 = 2 \text{ (reject)}}$

FOC: $\partial/\partial x_2\ x_2^{2\rho}/(1-x_2)^\rho$

$$= 2\rho x_2^{2\rho-1}/(1-x_2)^\rho + x_2^{2\rho}(-\rho)(1-x_2)^{-\rho-1}(-1)$$
$$= 2\rho x_2^{2\rho-1}/(1-x_2)^\rho + \rho x_2^{2\rho}/(1-x_2)^{\rho+1} = 0$$

$\cancel{2x_2^{-1}\ 2/(1-x_2) + x_2/}$

$\cancel{2 + x_2/(1-x_2) = 0}$

$$2/(1-x_2)^\rho + x_2/(1-x_2)^{\rho+1} = 0$$
$$2 + x_2/(1-x_2) = 0$$
$$x_2 = 2 \text{ (reject)}$$

There are no interior solutions. The constraints are $x_2 \in [0,1]$ and $F(s) \in [0,1] \Rightarrow (x_2/(1-x_2))^{2\rho} \in [0,1] \Rightarrow x_2 \in [0, 1/2]$

~~At $x_2 = 0$~~, $\pi_1(x_2=0) = 0$, $\pi_1(x_2 = 1/2) = (1/2)^\rho > 0$ ~~P1 chooses $x_2 = 1/2$~~

The SPE is such that P1's strategy is to offer $x_2 = 1/2$ and P2's strategy is to choose A iff $s \leq (x_2/(1-x_2))^\rho$. At SPE, P2 chooses A with probability 1.

c Expected total utility $U = (1-x_2)^\rho + E(x_2^\rho - s(1-x_2)^\rho)$

$$= x_2^\rho + (1-E(s))(1-x_2)^\rho = x_2^\rho + (1-2/3)(1-x_2)^\rho$$
$$= (1/2)^\rho + 1/3(1/2)^\rho = 4/3(1/2)^\rho$$