Philosophical Logic Paper 190604

1a i Connectives: $\sim$, $\vee$, $\Box$

Model: $\langle W, R, I \rangle$, where
$W \neq \emptyset$, $R \subseteq$ reflexive and transitive on W, ~~#~~
$I(\alpha, w) \in \{0, 1, \#\}$ for all α, w, and
$\forall w, w' \in W$: if Rww', then (1) if $I(\alpha, w) = 1$ then $I(\alpha, w') = 1$, and (2) if $I(\alpha, w) = 0$ then $I(\alpha, w) = 0$.

An A-model is a trivalent S4 model such that each world w' accessible from each world w agrees with w on definite sentence letters.

Prove by induction that for all wff ϕ ~~contain~~ containing no occurrences of $\Box$, ~~if AVM(for all worlds w and for all worlds w' accessible AVM(φ,w)~~ if $AVm(\phi, w) \neq \#$, then $AVm(\phi, w) = AVm(\phi, w')$.

Consider arbitrary ~~too~~ A-model $M = \langle W, R, I \rangle$. Consider arbitrary world $w \in W$, and arbitrary world w' such that Rww'.

Base case
Consider arbitrary wff ϕ ~~conta~~ such that ϕ contains no occurrences of $\Box$ and ϕ has complexity, i.e. number of connectives, 0, i.e. $c(\phi) = 0$. Then ϕ is a sentence letter α. Suppose for conditional proof that ~~AVm(φ)≠# then then AVm(φ)=1~~ $AVm(\phi, w) \neq \#$. Then $AVm(\phi, w) = I(\alpha, w) = I(\alpha, w') = AVm(\phi, w')$ (where the second equality follows from a given property of A-model M.

Induction Hypothesis
Given n, ~~suppose~~ for all $m < n$, for all wffs ϕ such that no $\Box$, $c(\phi) = m$, ~~AVm~~ if $AVm(\phi, w) \neq \#$, then $AVm(\phi, w) = AVm(\phi, w')$.

Induction step
Consider arbitrary wff ϕ such that ϕ contains no occurrences of $\Box$ and $c(\phi) = n$. Suppose for conditional proof that $AVm(\phi, w) \neq \#$. Then $\phi = \sim\psi$ or $\phi = \psi \vee \chi$.

Suppose $\phi = \sim\psi$. Then $AVm(\phi, w) = 1$ iff $AVm(\psi, w) = 0$ iff (by IH) $AVm(\psi, w') = 0$ iff $AVm(\phi, w') = 1$. And $AVm(\phi, w) = 0$ iff $AVm(\psi, w) = 1$ iff (by IH) $AVm(\psi, w') = 1$ iff $AVm(\phi, w') = 0$. So, given that $AVm(\phi, w) \neq \#$, $AVm(\phi, w) = AVm(\phi, w')$

Suppose $\phi = \psi \vee \chi$. Then $AVm(\phi, w) = 1$ iff $AVm(\psi, w) = 1$ or $AVm(\chi, w) = 1$ iff (by IH) $AVm(\psi, w') = 1$ or $AVm(\chi, w') = 1$ iff $AVm(\phi, w') = 1$. And $AVm(\phi, w) = 0$ iff $AVm(\psi, w) = 0$ and $AVm(\chi, w) = 0$ iff (by IH) $AVm(\psi, w') = 0$ and $AVm(\chi, w') = 0$ iff $AVm(\phi, w') = 0$. Given that $AVm(\phi, w) \neq \#$, $AVm(\phi, w) = AVm(\phi, w')$.

By cases, conditional proof, generalisation, for all ϕ such that $c(\phi) = n$ and ϕ contains no occurrence of $\Box$, if $AVm(\phi, w) \neq \#$, then $AVm(\phi, w) = AVm(\phi, w')$.

By induction over complexity, generalisation, for all $M = \langle W, R, I \rangle$, $w, w' \in W$ such that Rww', ~~#~~ ϕ containing no occurrences of $\Box$, if $AVm(\phi, w) = 1$ then $AVm(\phi, w') = 1$ and if $AVm(\phi, w) = 0$ then $AVm(\phi, w') = 0$

From the above, for all $M = \langle W, R, I \rangle$ ~~and~~, ~~w~~ $w \in W$, and ϕ containing no $\Box$, if $AVm(\phi, w) = 1$ then for all $w' \in W$ such that Rww', $AVm(\phi, w') = 1$ then by $\Box$ clause, $AVm(\Box\phi, w) = 1$. So ~~#~~ for all such ϕ, ~~#~~ for all M, w, if $AVm(\phi, w) = 1$ then $AVm(\Box\phi, w) = 1$, so $\phi \vDash_A \Box\phi$.

ii No. Consider the following counterexample.

$\phi = \Diamond P = \sim\Box\sim P$
$M = \langle W, R, I \rangle$
$W = \{0, 1, 2\}$
$R = \{\langle 0, 1 \rangle, \langle 0, 2 \rangle, \ldots\}$
$I(P, 0) = \#$, $I(P, 1) = 1$, $I(P, 2) = 0$.
$I(\alpha, w) = 0$ for all other sentence letters and worlds α, w.
$AVm(\phi, 0) = 1$ (Because $AVm(P, 1) = 1$ and $R01$)
$AVm(\Box\phi, 0) = 0$ (Because $AVm(\Diamond P, 2) = 0$ and $R02$)
$\Rightarrow \phi \nvDash_A \Box\phi$

iii $\nvDash_A \Box P \vee \Box\sim P$

Consider the following countermodel.

$M = \langle W, R, I \rangle$
$W = \{0, 1, 2\}$
$R = \{\langle 0, 1 \rangle, \langle 0, 2 \rangle, \ldots\}$
$I(P, 0) = \#$, $I(P, 1) = 1$, $I(P, 0) = 0$
$I(\alpha, w) = 0$ for all other (α, w)
$AVm(\Box P \vee \Box\sim P, 0) = 0$
$\Rightarrow \nvDash_A \Box P \vee \Box\sim P$

$\nvDash_A \Box(P \vee \sim P)$

Consider the following countermodel.

$M = \langle W, R, I \rangle$
$W = \{0\}$
$R = \{\langle 0, 0 \rangle\}$
$I(P, 0) = \#$

$I(\alpha, w) = 0$ for all other α, w.
$AVm(\Box(P \vee \sim P), 0) = \#$
$\Rightarrow \nvDash_A \Box(P \vee \sim P)$

di Yes.

Consider arbitrary A-model $M = \langle W, R, I \rangle$ and arbitrary $w \in W$. Consider arbitrary wff ϕ containing no $\Box$. Suppose $BVm(\phi, w) = 1$ then, because ~~BVm and~~ and AVm ~~have identical~~ are identical apart from the $\Box$ clause, BVm and AVm agree for all ϕ containing no $\Box$. Then $AVm(\phi, w) = 1$, then by the result in (ai), $AVm(\phi, w') = 1$ for all $w' \in W$ such that Rww', then $AVm(\phi, w'') = 1$ for all ~~w''~~ complete $w'' \in W$ such that Rww'', then $BVm(\phi, w'') = 1$ for all such w'', then $BVm(\Box\phi, w) = 1$. ~~By genera~~ By conditional proof, generalisation, definition of B-semantic consequence, $\phi \vDash_B \Box\phi$ for all ϕ containing no $\Box$.

ii $\nvDash_B \Box P \vee \Box \sim P$

The countermodel from (aiii, I) applies.

$\vDash_B \Box(P \vee \sim P)$

Consider arbitrary model $M = \langle W, R, I \rangle$ and world $w \in W$. Suppose for reductio that ~~BV~~
(1) $BVm(\Box(P \vee \sim P), w) \neq 1$
  (1), $\Box \Rightarrow$
(2) $\exists w' \in W$: w' is complete, Rww', and $BVm(P \vee \sim P, w') \neq 1$
  (2), $\vee \Rightarrow$
(3) $\exists w' \in W$: … and $BVm(P, w') \neq 1$ and $BVm(\sim P, w') \neq 1$
  (3), $\sim \Rightarrow$
(4) $\exists w' \in W$: … and $BVm(P, w') \neq 1$ and $BVm(P, w') \neq 0$
  (4), reductio (completeness and sentence letter clause) $\Rightarrow$
(5) $BVm(\Box(P \vee \sim P), w) = 1$
  (5), generalisation, definition of B validity $\Rightarrow$
(6) $\vDash_B \Box(P \vee \sim P)$

c According to both A and B, "it will rain … 2100" is a semantic consequence of "it rains … 2100". This is potentially ~~count~~ counterintuitive because we think, for example, that "the World Trade Center will not be standing tomorrow" is not flatly true, but in some sense indeterminate when spoken on September 10th even though "the World Trade Center is not standing on September 11th" is true. This is issue (1).

According to both A and B, "it will rain … 2100 or it will not rain …" is not a logical truth, but we think that this is a logical truth. This is issue (2).

One solution to (2) that is available to B but not to A is to formalise "it will rain … 2100 or it will not rain …" not as $\Box R \vee \Box \sim R$ but as $\Box(R \vee \sim R)$, then, according to B, the English disjunction is a logical truth.

This alternative formalisation is ~~AA~~ less close to the structure of the English sentence so ~~it seems~~ one worries that this solution is suspicious or ad hoc. The alternative formalisation is at least not entirely idiosyncratic. We formalise such modal conditionals as "if he is unmarried then he must be a ~~B~~ bachelor" as $\Box(U \to B)$ rather than as $U \to \Box B$ (because we do not think some person's actually being unmarried implies that ~~he is necessarily~~ his being a bachelor is some sort of metaphysical necessity).

The alternative formalisation remains unmotivated. We would say, in English "either it is true that it will rain … 2100 or it is true that it will not rain … 2100". In saying this, we seem to reject that there is anything wrong with the $\Box P \vee \Box \sim P$ formalisation. In contrast, we would not say "if he is unmarried, then it is a metaphysical necessity that he is a bachelor".

Issue (1) ~~seems to m~~ is a mistake. The sort of indeterminacy in "the WTC will not be standing tomorrow" when spoken on September 10th is an epistemic indeterminacy, not a future contingent indeterminacy. ~~If A fact the WTC is not~~ The WTC's not ~~sta~~ standing on September 11th ~~fact, if the~~ from an epistemically unconstrained standpoint, does seem to logically imply ~~AA~~ "the WTC ~~is not st~~ will not be standing on September 11th".

So issue (1) is a mistake but issue (2) is not. A has no apparent solution to (2) but B's solution seems ad hoc, and an explanation of why we should favour the alternative formalisation is necessary.

2a i A PC= model is some ordered pair $\langle D, I \rangle$ where D, the domain, is some non-empty set, and I, the interpretation function, is some function that assigns to each constant $(a, b, a_1, b_1, \ldots)$ some element of D, and to each ~~predicate some~~ n-place predicate some n-ary relation over D.

A SOL= model is ~~identical to a~~ defined identically.

ii A PC= variable assignment g given a PC= model $M = \langle D, I \rangle$, is some function that assigns to each variable $(x, y, x_1, y_1, \ldots)$ some element of D.

A SOL= variable assignment g is defined identically, except in that g also assigns to each ~~predicate~~ n-place predicate variable $(X, Y, X_1, Y_1, \ldots)$ some n-ary relation over D.

iii A PC= valuation function given PC= model M $= \langle D, I \rangle$ and ~~SOL=~~ variable assignment g, is some function $V_{M,g}$ from ~~sentence lett~~ wffs ϕ to truth values $(0, 1)$.

$V_{M,g}(\pi\alpha_1 \ldots \alpha_n) = 1$ iff $\langle |\alpha_1|_{M,g}, \ldots, |\alpha_n|_{M,g} \rangle \in |\pi|_{M,g}$ where ~~to~~ each of $\alpha_1, \ldots, \alpha_n$ is a term (i.e. a constant or a variable), π is a n-place predicate, $|\alpha|_{M,g} = I(\alpha)$ if α is a constant, $g(\alpha)$ if α is a variable, and $|\pi|_{M,g} = I(\pi)$.

Clauses for $\sim$ and $\rightarrow$ are the obvious.

$V_{M,g}(\forall\alpha\phi) = 1$ iff $\forall u \in D$: $V_{M,g^{\alpha}_{u}}(\phi) = 1$, where α is some variable, ϕ is some wff, and g^{α}_{u} is the variable assignment that differs from g only in assigning u to α.

$V_{M,g}(\alpha = \beta) = 1$ iff $|\alpha|_{M,g} = |\beta|_{M,g}$, where each of α, β is a term.

For SOL= valuation function, the following modifications are necessary.

The basic wff clause is modified to accommodate predicate variables. This ~~=~~ requires only that ~~th~~ $|\pi|_{M,g} = I(\pi)$ if π is a predicate constant, $g(\pi)$ if π is a predicate variable.

The universal quantifier clause is modified to accommodate quantification over predicate variables.

$V_{M,g}(\forall\pi\phi) = 1$ iff $\forall U \in D^n$ (where D^n is the set of n-ary relations over D), : $V_{M,g^{\pi}_{U}}(\phi) = 1$ where g^{π}_{U} is the variable assignment that differs from g only in assigning U to π.

The identity clause is modified to accommodate predicate terms.

$V_{M,g}(\pi = \rho) = 1$ iff $|\pi|_{M,g} = |\rho|_{M,g}$, where each of π and ρ is a predicate term (predicate constant or predicate variable).

b $\exists xFx$ formalises "there is at least one F thing". $\exists x_1 \exists x_2 (Fx_1 \wedge Fx_2 \wedge x_1 \neq x_2)$ formalises "there are at least two F things" ($x_1 \neq x_2$ abbreviates $\sim(x_1 = x_2)$). $\exists x_1 \exists x_2 \exists x_3 (Fx_1 \wedge Fx_2 \wedge Fx_3 \wedge x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge x_2 \neq x_3)$ formalises "there are at least ~~F t~~ three F things". This ~~gene~~ generalises in the obvious way. Let $\exists_{\geq n}F$ abbreviate the formalisation of "there are at least n F things". Let $\exists_n F$ abbreviate $\exists_{\geq n}F \wedge \sim\exists_{\geq n+1}F$

Suppose that "there are infinitely many F things" is formalisable in PC=. Denote this formalisation ωF. Such a formalisation exists iff "there are finitely many F things" is formalisable.

$\{\exists_n F : n \in \mathbb{N}\} \cup \{\neg\omega F\}$ is finitely satisfiable but not satisfiable, so if $\neg\omega F$ exists in PC=, it violates compactness. So no formalisation in PC= of "there are finitely many earthlings" or "there are infinitely many aliens" exists.

Suppose that "there are more F things than G things" is formalisable in PC=, denote this formalisation $F > G$.

$\{\exists_n F, \exists_n G : n \in \mathbb{N}\} \cup \{F > G\}$, is satisfiable in an infinite domain but not in a countably infinite one. By reductio, $F > G$ does not exist in PC=.

"There is a one-to-one mapping of all F things to G things" is formalisable in SOL= as
$\exists R\, [\forall x \forall y (Rxy \rightarrow Fx \wedge Gy) \wedge$
$\forall x (Fx \leftrightarrow \exists y Rxy) \wedge$
$\sim\exists x \exists y_1 \exists y_2 (Rxy_1 \wedge Rxy_2 \wedge y_1 \neq y_2)$
$\sim\exists x_1 \exists x_2 \exists y (Rx_1y \wedge Rx_2y \wedge x_1 \neq x_2)]$

This reads "there exists R such that (1) R is a relation from F things to G things, (2) ~~every F thing~~ the domain of R is the set of F things, (3) R is functional, and (4) R is one-to-one". If such a mapping exists, then there are weakly more G things than F things. Let $F \preccurlyeq G$ abbreviate that formalisation. Let $F \prec G$ abbreviate $(F \preccurlyeq G) \wedge \sim(G \preccurlyeq F)$.

"The F things are a subset of the G things" is formalisable as $\forall x (Fx \rightarrow Gx)$, abbreviate this as $F \subseteq G$. Let $F \subset G$ abbreviate $(F \subseteq G) \wedge \sim (G \subseteq F)$.

Then "there are infinitely many F things" is formalisable as $\exists F' [F' \subset F \wedge F \preccurlyeq F'$ and $F' \preccurlyeq F]$ which reads "there is a strict subset of F of the same size as F.

So the argument is formalisable as
$P_1 : \sim \infty A$
$P_2 : \infty E$
$C : A \prec E$

That the conclusion is a semantic consequence of the premises is obvious.

The ~~argument~~ English argument in (b) ~~or~~ seems to be logically valid. Its logical validity can be recognised by SOL= but not PC=. If we think that ~~a logic~~ one important function of logical theory is to ~~recog~~ systematically recognise logical validities, then this is reason ~~to the~~ consider SOL= as logic (rather than set theory). ~~The arg~~ ~~argum~~

The argument in (b) is not an idiosyncratic case. Consider the set of sentences "not: there is exactly one alien", "not: there are exactly two aliens", ... "there are finitely many aliens". This set of sentences is logically inconsistent but ~~not in a way the~~ this logical inconsistency can be ~~recept~~ recognised by SOL= but not PC=.

Other cases to do with ancestral also exist. For example, consider the set of sentences "not: Jones's child has blue eyes", "not: Jones's grandchild has blue eyes", ..., "Jones's descendant has blue eyes". This set of sentences is logically inconsistent, but again this logical inconsistency cannot be recognised in first-order logic.

But if ~~the~~ one of the ~~essen~~ functions of a logical theory is to recognise logical truths and validities, we might prefer a complete system of logic, for which any logical truth can be established or validated in an axiomatic proof system. SOL= is incomplete, but Gödel has shown that PC= is complete, so it seems we should ~~favour~~ consider SOL= to ~~be~~ ~~it~~ not be logic but something like set theory.

The demand for completeness cannot be consistently motivated. If we think a ~~theory~~ logical language ~~should~~ should be complete because it is a function of such a theory to identify logical validities and consistencies, then it is not clear why we should not also demand decidability. A logic is decidable iff there exists some effective method for determining logical truth that, roughly speaking, can be performed mechanically and without ingenuity, and terminates in a finite number of steps. So for example, PC is decidable because truth tables are such an effective method, but PC= is not. But we would certainly think that PC= is logical, so we cannot demand decidability, then we cannot consistently demand completeness.

SOL= naturally extends PC=. The definition of a model is unchanged. The definition of a variable assignment is extended in the natural way to assign extensions to predicate variables, and the only changes to the valuation function are ~~to allow for the~~ ~~stata~~ ~~interpret the extensions of predicate~~ also entirely natural extensions. The syntax is unchanged except in including predicate variables $X, Y, \ldots$. Definitions and semantics of SOL are recognisably logical.

So first order logic is not all of logic.

4ai Valuation function V_M given PTC model $M = \langle T, \preceq, I \rangle$ is the two piece function from PC-wffs and times (ϕ, t) with $t \in T$ to truth values, $\{0, 1\}$, such that:

$V_M(\alpha, t) = I(\alpha, t)$ for all sentence letters α, for all times $t \in T$.

$V_M(\sim\phi, t) = 1$ iff $V_M(\phi, t) = 0$

$V_M(\phi \to \psi, t) = 1$ iff $V_M(\phi, t) = 0$ or $V_M(\psi, t) = 1$

$V_M(G\phi, t) = 1$ iff $\forall t', t \preceq t' : V_M(\phi, t') = 1$

$V_M(H\phi, t) = 1$ iff $\forall t', t' \preceq t : V_M(\phi, t') = 1$

PC-wff ϕ is PTC-valid, i.e. $\vDash_{PTC} \phi$, iff for all PTC-models $M = \langle T, \preceq, I \rangle$, for all times $t \in T$, $V_M(\phi, t) = 1$

ii Given $\preceq$, let $\preceq^* = \{\langle t_1, t_2 \rangle : \langle t_2, t_1 \rangle \in \preceq\}$. Given $M = \langle T, \preceq, I \rangle$, let $M^* = \langle T, \preceq^*, I \rangle$.

Consider arbitrary PC-wff ϕ. ~~Suppose that consider arbitrary~~ ~~PC-model M and $= \langle T, \preceq, I \rangle$~~ and ~~time $t \in T$. $V_M(G\psi, t) = 1$ iff $\forall t', t \preceq t' : V_M(\psi, t') = 1$ iff $\forall t', t' \preceq t : V_{M^*}(\psi, t') = 1$ iff $V_M(G\psi, t') = 1$ $V_{M^*}(H\psi, t) = 1$~~

Prove by induction that for all PTC-wff ϕ, for all PTC-models $M = \langle T, \preceq, I \rangle$, ~~$V_M(\phi, t)$~~ for all $t \in T$, $V_M(\phi, t) = V_{M^*}(\phi^*, t)$.

Base case
Consider arbitrary PTC-wff ϕ ~~st~~ such that complexity, i.e. number of connectives, $C(\phi) = 0$. Then ϕ is some sentence letter α, as is ϕ^*. $V_M(\phi, t) = I(\alpha, t) = V_{M^*}(\phi, t)$. By ~~generat~~ generalization, this holds for all ϕ such that $C(\phi) = 0$.

Induction Hypothesis.
Given n, for all $m < n$, for all ϕ such that $C(\phi) = m$ $V_M(\phi, t) = V_{M^*}(\phi^*, t)$.

Induction step
Consider arbitrary PTC-wff ϕ such that $C(\phi) = n$. Then ϕ = $\sim\psi$, $\psi \to \chi$, $G\psi$, or $H\psi$.

Suppose $\phi = \sim\psi$, then $V_M(\phi, t) = 1$ iff $V_M(\psi, t) = 0$ iff by IH $V_{M^*}(\psi^*, t) = 0$ iff $V_{M^*}(\phi^*, t) = 1$. ~~$C(\phi) = n$~~ IH applies because $C(\psi) = n - 1 < n$. So $V_M(\phi, t) = V_{M^*}(\phi^*, t)$.

Suppose $\phi = \psi \to \chi$, then $V_M(\phi, t) = 1$ iff $V_M(\psi, t) = 0$ or $V_M(\chi, t) = 1$ iff by IH $V_{M^*}(\psi^*, t) = 0$ or $V_{M^*}(\chi^*, t) = 1$ iff $V_{M^*}(\phi^*, t) = 1$, so $V_M(\phi, t) = V_{M^*}(\phi^*, t)$.

Suppose $\phi = G\psi$, then $V_M(\phi, t) = 1$ iff $\forall t', t \preceq t'$: $V_M(\psi, t') = 1$ iff $\forall t', t' \preceq^* t : V_M(\psi, t') = 1$ iff by IH $\forall t', t' \preceq^* t : V_{M^*}(\psi^*, t') = 1$ iff $V_{M^*}(H\psi^*, t) = 1$ iff $V_{M^*}(\phi^*, t) = 1$, so $V_M(\phi, t) = V_{M^*}(\phi^*, t)$. The case is similar for $\phi = H\psi$.

By cases, by generalisation, for all ϕ such that $C(\phi) = n$, $V_M(\phi, t) = V_{M^*}(\phi^*, t)$. ~~By induction~~

By induction, for all M, t, ϕ, $V_M(\phi, t) = V_{M^*}(\phi^*, t)$.

For all PTC-models M, M^* is a PTC model, and this mapping is one-to-one. ~~So if~~ Then, $\vDash_{PTC} \phi$ iff for all M, for all t, $V_M(\phi, t) = 1$ iff for all M^*, for all t, $V_{M^*}(\phi, t^*) = 1$ iff for all M, for all t, $V_M(\phi, t) = 1$ iff $\vDash_{PTC} \phi^*$.

bi (1) $G(\phi \to \psi) \to (F\phi \to F\psi)$ (G◇)
(2) $(F\phi \to F\psi) \to GP(F\phi \to F\psi)$ (GP-conv)
(3) $G(\phi \to \psi) \to GP(F\phi \to F\psi)$ (1, 2, PC syllogism)

G◇: $\vdash_{PTC} G(\phi \to \psi) \to (F\phi \to F\psi)$
(1) $(\phi \to \psi) \to (\sim\psi \to \sim\phi)$ (PC contraposition)
(2) $G(\phi \to \psi) \to G(\sim\psi \to \sim\phi)$ (1, GNEC, GDIST, MP)
(3) $G(\sim\psi \to \sim\phi) \to (G\sim\psi \to G\sim\phi)$ (GDIST)
(4) $(G\sim\psi \to G\sim\phi) \to (\sim G\sim\phi \to \sim G\sim\psi)$
$= (G\sim\psi \to G\sim\phi) \to (F\phi \to F\psi)$
(PC contraposition)
(5) $G(\phi \to \psi) \to (F\phi \to F\psi)$ (2, 3, 4, PC syllogism)

(1) $(\phi \wedge \psi) \to \phi$ (PC)
(2) $G(\phi \wedge \psi) \to G\phi$ (1, GNEC, GDIST, MP)
(3) $(\phi \wedge \psi) \to \psi$ (PC)
(4) $G(\phi \wedge \psi) \to G\psi$ (3, GNEC, GDIST, MP)
(5) $G(\phi \wedge \psi) \to (G\phi \wedge G\psi)$ (2, 4, PC)
(6) $\phi \to (\psi \to (\phi \wedge \psi))$ (PC)
(7) $G\phi \to G(\psi \to (\phi \wedge \psi))$ (6, GNEC, GDIST, MP)
(8) $G(\psi \to (\phi \wedge \psi)) \to (G\psi \to G(\phi \wedge \psi))$ (GDIST)
(9) $G\phi \to (G\psi \to G(\phi \wedge \psi))$ (7, 8, PC syllogism)
(10) $(G\phi \wedge G\psi) \to G(\phi \wedge \psi)$ (9, PC import)
(11) $G(\phi \wedge \psi) \leftrightarrow (G\phi \wedge G\psi)$ (5, 10, PC)

ii $\preceq$ is transitive on T.

Consider arbitrary frame $F = \langle T, \preceq \rangle$. Suppose that $\preceq$ is transitive on T. Consider ~~every~~ arb PTC-model $M = \langle T, \preceq, I \rangle$ based on F. ~~Suppose for~~ consider arbitrary $t \in T$.

Suppose for reductio that

(1) $V_M(GP \to GGP, t) = 0$

(1), $\to$ $\Rightarrow$

(2) $V_M(GP, t) = 1$

(3) $V_M(GGP, t) = 0$

(2), G $\Rightarrow$

(4) $\forall t', t \preceq t' : V_M(P, t') = 1$

(3), G $\Rightarrow$

(5) $\exists t', t \preceq t' : V_M(GP, t') = 0$

(5), G $\Rightarrow$

(6) $\exists t', t \preceq t' : \exists t'', t' \preceq t'' : V_M(P, t'') = 0$

(6), transitivity $\Rightarrow$

(7) $\exists t''$, $t \preceq t'' : V_M(P, t'') = 0$

(4), (7), reductio

(8) $V_M(GP \to GGP, t) = 1$

(8), generalisation

(9) for all frames $F = \langle T, \preceq \rangle$, if $\preceq$ is transitive on T, then $GP \to GGP$ is valid in F.

ii $\preceq$ is weakly connected on T.

Consider arbitrary $F = \langle T, \preceq \rangle$. Suppose for conditional proof that $\preceq$ is weakly connected on T, i.e. for all $x, x' \in T$, if $\exists y \in T$ such that $x \preceq y$ and $x' \preceq y$ or $y \preceq x$ or $y \preceq x'$, then $x \preceq x'$ or $x' \preceq x$. Consider arbitrary model M ~~bea~~ $= \langle T, \preceq, I \rangle$ based on F and arbitrary time $t \in T$.

Suppose for reductio that

(1) $V_M(PFQ \to (PQ \vee FQ), t) = 0$

(1), $\to$ $\Rightarrow$

(2) $V_M(PFQ, t) = 1$

~~$\Leftrightarrow$~~

(3) $V_M(PQ \vee FQ, t) = 0$

(3), derived $\vee$ $\Rightarrow$

(4) $V_M(PQ, t) = 0$

(5) $V_M(FQ, t) = 0$

(2), derived P $\Rightarrow$

(6) $\exists t', t' \preceq t : V_M(FQ, t') = 1$

(6), derived F $\Rightarrow$

(7) $\exists t', t' \preceq t : \exists t'', t' \preceq t'' : V_M(Q, t'') = 1$

(7), weak connectivity $\to$

(8) $\exists t''$, $t \preceq t''$ or $t'' \preceq t : V_M(Q, t'') = 1$

(4), derived P $\Rightarrow$

(9) ~~tau~~ $\nexists t', t' \preceq t : V_M(Q, t') = 1$

(5), derived P $\Rightarrow$

(10) $\nexists t'$, ~~$t' \preceq$~~ $t \preceq t' : V_M(Q, t') = 1$

(9), (10) $\Rightarrow$

(11) $\nexists t', t \preceq t'$ or $t' \preceq t : V_M(Q, t') = 1$

(8), (11), reductio $\Rightarrow$

(12) $V_M(PFQ \to (PQ \vee FQ), t) = 1$

(12), generalisation, conditional proof

(13) for all F ~~st~~ $= \langle T, \preceq \rangle$ if $\preceq$ is weakly connected on T, then $PFQ \to (PQ \vee FQ)$ is valid in F.

c In general, ~~in a P~~ for such an ~~int~~ interpretation, we require that $\preceq$ is a ~~s~~ reflexive, transitive, antisymmetric, and weakly (or strictly) connected relation on T.

F can be so interpreted ~~iff~~ iff, by symmetry, P can be interpreted as "it either is or has been the case that". This is iff we interpret $t \preceq t'$ as "time t occurs weakly before time t'".

On such an interpretation, we must also ~~≠~~ interpret G and H respectively as "it is and always will be the case that" and "it is and has always been the case that".

~~On such an interpretation~~, ~~we~~ We would want ~~≠~~ such English sentences as "if it is and always will be that Socrates is mortal, then it is and always will be that it is and always will be that Socrates is mortal", so we would want $GP \to GGP$ to be a logical truth. From above, ~~this is the~~ the most natural condition that implies this is the transitivity of $\preceq$ on T.

Similarly, we would want such English sentences as "~~it is or has been the~~ if it is or has been that it is or will be that Socrates dies, then it is or has been that Socrates dies or it is or will be that Socrates dies". So we want $PFQ \to (PQ \vee FQ)$ to be a logical truth. This ~~requires~~ is most naturally so if $\preceq$ is weakly connected on T.

Reflexivity is required for the logical validity of such sentences as "if I am a student then it either is or will be that I am a student".

Density, i.e. there is some time t ~~between~~ between any two times t' and t" and eternality, i.e. there is no last time, are controversial.