Modal Propositional Logic Metatheory Rough Notes v1

Propositional Logic

Proof of Soundness

- Lemma (PL-Validity of PL-Axioms)
  - Proof ($\vDash_{PL} PL1$)
    - Consider some arbitrary PL-interpretation $\mathcal{I}$ and some arbitrary PL-wffs ϕ, ψ.
    - Suppose for reductio that
      - (1) $V_{\mathcal{I}}(\phi \rightarrow (\psi \rightarrow \phi)) = 0$.
    - From (1), by definition of PL-valuation, $\rightarrow$ clause (by $\rightarrow$),
      - (2) $V_{\mathcal{I}}(\phi) = 1$,
      - (3) $V_{\mathcal{I}}(\psi \rightarrow \phi) = 0$.
    - From (3), by $\rightarrow$,
      - (4) $V_{\mathcal{I}}(\phi) = 0$.
    - From (2), (4), by reductio,
      - (5) $V_{\mathcal{I}}(\phi \rightarrow (\psi \rightarrow \phi)) = 1$.
    - From (5), by generalisation,
      - (6) for all PL-interpretations $\mathcal{I}$, PL-wffs ϕ, ψ, $V_{\mathcal{I}}(\phi \rightarrow (\psi \rightarrow \phi)) = 1$.
    - From (6), by definition of PL-validity,
      - (7) for all PL-wffs ϕ, ψ, $\vDash_{PL} \phi \rightarrow (\psi \rightarrow \phi)$.
    - From (7), by definition of a PL-instance of PL1,
      - (8) every PL-instance of PL1 is PL-valid.
  - Proof ($\vDash_{PL} PL2$)
    - Consider some arbitrary PL-interpretation $\mathcal{I}$ and some arbitrary PL-wffs ϕ, ψ, χ.
    - Suppose for reductio that
      - (1) $V_{\mathcal{I}}((\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi))) = 0$.
    - From (1), by $\rightarrow$,
      - (2) $V_{\mathcal{I}}(\phi \rightarrow (\psi \rightarrow \chi)) = 1$,
      - (3) $V_{\mathcal{I}}((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi)) = 0$.
    - From (3), by $\rightarrow$,
      - (4) $V_{\mathcal{I}}(\phi \rightarrow \psi) = 1$,
      - (5) $V_{\mathcal{I}}(\phi \rightarrow \chi) = 0$.
    - From (5), by $\rightarrow$,
      - (6) $V_{\mathcal{I}}(\phi) = 1$,
      - (7) $V_{\mathcal{I}}(\chi) = 0$.
    - From (4), (6), by $\rightarrow$,
      - (8) $V_{\mathcal{I}}(\psi) = 1$.
    - From (7), (8), by $\rightarrow$,
      - (9) $V_{\mathcal{I}}(\psi \rightarrow \chi) = 0$.
    - From (6), (9), by $\rightarrow$,
      - (10) $V_{\mathcal{I}}(\phi \rightarrow (\psi \rightarrow \chi)) = 0$.
    - From (2), (10), by reductio,
      - (11) $V_{\mathcal{I}}((\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi))) = 1$.
    - From (11), by generalisation,
      - (12) for all PL-interpretations $\mathcal{I}$, PL-wffs ϕ, ψ, χ, $V_{\mathcal{I}}((\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi))) = 1$.
    - From (12), by definition of PL-validity,
      - (13) for all PL-wffs ϕ, ψ, χ, $\vDash_{PL} (\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi))$.
    - From (13), by definition of a PL-instance of PL2,
      - (14) every PL-instance of PL2 is PL-valid.

- Proof ($\vDash_{PL} PL3$)
  - Consider some arbitrary PL-interpretation $\mathcal{I}$ and some arbitrary PL-wffs ϕ, ψ.
  - Suppose for reductio that
    - (1) $V_{\mathcal{I}}((\neg\psi \to \neg\phi) \to ((\neg\psi \to \phi) \to \psi)) = 0.$
  - From (1), by $\to$,
    - (2) $V_{\mathcal{I}}(\neg\psi \to \neg\phi) = 1,$
    - (3) $V_{\mathcal{I}}((\neg\psi \to \phi) \to \psi) = 0,$
  - From (3), by $\to$,
    - (4) $V_{\mathcal{I}}(\neg\psi \to \phi) = 1,$
    - (5) $V_{\mathcal{I}}(\psi) = 0.$
  - From (5), by $\neg$ clause,
    - (6) $V_{\mathcal{I}}(\neg\psi) = 1.$
  - From (2), (6), by $\to$,
    - (7) $V_{\mathcal{I}}(\neg\phi) = 1.$
  - From (4), (6), by $\to$,
    - (8) $V_{\mathcal{I}}(\phi) = 1.$
  - From (7), by $\neg$ clause,
    - (9) $V_{\mathcal{I}}(\phi) = 0.$
  - From (8), (9), by reductio,
    - (10) $V_{\mathcal{I}}((\neg\psi \to \neg\phi) \to ((\neg\psi \to \phi) \to \psi)) = 1.$
  - From (10), by generalisation,
    - (11) for all PL-interpretations $\mathcal{I}$, PL-wffs ϕ, ψ, $V_{\mathcal{I}}((\neg\psi \to \neg\phi) \to ((\neg\psi \to \phi) \to \psi)) = 1.$
  - From (11), by definition of PL-validity,
    - (12) for all PL-wffs ϕ, ψ, $\vDash_{PL} (\neg\psi \to \neg\phi) \to ((\neg\psi \to \phi) \to \psi).$
  - From (12), by definition of a PL-instance of PL3,
    - every PL-instance of PL3 is PL-valid.
- Lemma (PL-Rules Preserve PL-Validity)
  - Proof (MP Preserves PL-Validity)
    - Suppose that
      - (1) $\vDash_{PL} \phi,$
      - (2) $\vDash_{PL} \phi \to \psi.$
    - From (1), by definition of PL-validity,
      - (3) for all PL-interpretations $\mathcal{I}$, $V_{\mathcal{I}}(\phi) = 1.$
    - From (2), by definition of PL-validity,
      - (4) for all PL-interpretations $\mathcal{I}$, $V_{\mathcal{I}}(\phi \to \psi) = 1.$
    - Consider some arbitrary PL-interpretation $\mathcal{I}$.
    - From (3),
      - (5) $V_{\mathcal{I}}(\phi) = 1.$
    - From (4),
      - (6) $V_{\mathcal{I}}(\phi \to \psi) = 1$
    - From (5), (6), by $\to$,
      - (7) $V_{\mathcal{I}}(\psi) = 1.$
    - From (7), by generalisation,
      - (8) for all PL-interpretations $\mathcal{I}$, $V_{\mathcal{I}}(\psi) = 1.$
    - From (8), by definition of PL-validity,
      - (9) $\vDash_{PL} \psi.$
    - From the above, by conditional proof,
      - (10) if (1), (2), then (9), i.e. MP preserves PL-validity.
- Base Case
  - Suppose that there is an axiomatic proof (from $\emptyset$) in PL of arbitrary PL-wff ϕ which is a sequence of $n = 1$ PL-wffs. Then, by definition of an axiomatic proof in PL, this proof is the sequence ϕ and ϕ is a PL-axiom. Then, by the PL-validity of PL-axioms, $\vDash_{PL} \phi$. From the above, by conditional proof, if there is an axiomatic proof in PL of arbitrary PL-wff ϕ which is a sequence of $n = 1$ PL-wffs, then $\vDash_{PL} \phi$.

- Induction Hypothesis
  - Given n, suppose that for all $m < n$, if there is an axiomatic proof in PL of arbitrary PL-wff ϕ which is a sequence of m PL-wffs, then $\vDash_{PL} \phi$.
- Induction Step
  - Suppose that there is an axiomatic proof in PL of arbitrary PL-wff ϕ which is a sequence of n PL-wffs. Then, by definition of an axiomatic proof in PL, either ϕ is a PL-axiom, or ϕ follows from earlier wffs ψ, χ by a PL-rule.
  - Suppose that ϕ is a PL-axiom, then by the PL-validity of PL-axioms, $\vDash_{PL} \phi$.
  - Suppose that ϕ follows from earlier wffs ψ, χ by a PL-rule, then by the lemma that PL-rules preserve PL-validity, if $\vDash_{PL} \psi$ and $\vDash_{PL} \chi$, then $\vDash_{PL} \phi$. By definition of an axiomatic proof in PL, there is an axiomatic proof in PL of ψ which is a sequence of $m < n$ PL-wffs, namely the sequence obtained by eliminating all PL-wffs after ψ in the proof of ϕ. Then, by the Induction Hypothesis, $\vDash_{PL} \psi$. By an analogous argument, $\vDash_{PL} \chi$. Then $\vDash_{PL} \phi$.
  - By cases, if there is an axiomatic proof in PL of arbitrary PL-wff ϕ which is a sequence of n PL-wffs, $\vDash_{PL} \phi$.
  - By strong induction over the length of the axiomatic proof of ϕ, if there is an axiomatic proof of ϕ, then $\vDash_{PL} \phi$.

Proof of Cut

- [See Sider, 2010 - Logic for Philosophy, pp. 73-74]
- The rough idea of the (informal) proof is that the required (axiomatic) proof can be constructed by concatenating the given (axiomatic) proofs.

Proof of Deduction Theorem

- (See Sider, 2010 - Logic for Philosophy, pp. 74-76.)

Modal Propositional Logic

Proof of Soundness

- The proof of soundness in MPL can be adapted from the proof of soundness in PL.

Mathematical Methods

- Method (Conditional Proof)
  - Suppose A. Then B. By conditional proof, if A then B.
- Method (Biconditional Proof)
  - Suppose A. Then B. Suppose B. Then A. By biconditional proof, A iff B.
- Method (Proof by Cases)
  - Suppose A. Then C. Suppose B. Then C. By cases, if A or B then C.
- Method (Proof by Reductio)
  - Suppose A. Then B and not B. By reductio, not A.
- Method (Proof by Generalisation)
  - Consider arbitrary X, x. x has P. By generalisation, all X have P.