

Game Theory Problem Set 2

$$1. u_c(x, y) = \frac{x}{x+y} - x, u_L(y, x) = \frac{y}{x+y} - y$$

$$B_L(y) = \arg\max_x u_L(x, y)$$

Taking FOC for x ,

$$\frac{\partial u_L}{\partial x} = x(-)(x+y)^{-2} + (1)(x+y)^{-1} - 1 = 0,$$

$$x(x+y)^{-2} + (x+y)^{-1} - 1 = 0,$$

$$-x + (x+y) - (x+y)^2 = 0,$$

$$(x+y)^2 = y,$$

$$x+y = \sqrt{y} \text{ since } x, y > 0$$

$$x = \sqrt{y} - y$$

$$B_L(y) = \sqrt{y} - y$$

$$\text{By symmetry, } B_C(x) = \sqrt{x} - x$$

Let (x^*, y^*) be a NE. By definition of NE and BR,

$$x^* = B_L(y^*) \text{ and } y^* = B_C(x^*)$$

$$x^* = \sqrt{y^*} - y^*, y^* = \sqrt{x^*} - x^*$$

Suppose that (x^*, y^*) is a symmetric NE, then

$$x^* = y^*$$

$$y^* = \sqrt{x^*} - y^*$$

$$2y^* = \sqrt{x^*}$$

$$4y^* = y^*$$

$$4y^* = 1$$

$$y^* = \frac{1}{4}$$

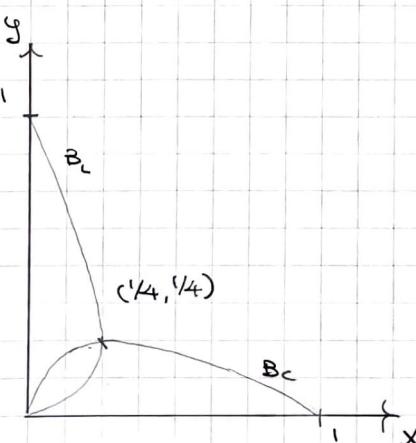
$$x^* = \frac{1}{4}$$

Then, $x^* = \sqrt{y^*} = y^*$ and $y^* = \sqrt{x^*} - x^*$, so this is a NE.
 $(x^* = \frac{1}{4}, y^* = \frac{1}{4})$ is indeed a NE.

Let v_L and v_C denote the vote shares of the Labour party and the Conservative party respectively.

$$\text{At } (x^* = \frac{1}{4}, y^* = \frac{1}{4}), v_L = \frac{x^*}{x^* + y^*} = \frac{1}{2}, v_C =$$

$$\frac{y^*}{x^* + y^*} = \frac{1}{2}.$$



At $(x=0, y=0)$, it is reasonable to suppose that $v_L = v_C = \frac{1}{2}$ since advertising expenditure is the sole determinant of vote share. Then, it is reasonable to suppose that $u_L(0, 0) = v_L(0, 0) - 0 = \frac{1}{2}$ and $u_C(0, 0) = v_C(0, 0) - 0 = \frac{1}{2}$. There is no equilibrium at these payoffs since candidate equilibrium $(0, 0)$ fails to deviate $(\varepsilon, 0)$ for

sufficiently small $\varepsilon > 0$ since the deviating player then has vote share 1 and payoff $1 - \varepsilon > \frac{1}{2}$ for $\varepsilon < \frac{1}{2}$

~~the payoff $(1, 1)$ at (x, y) if each player has payoff $\frac{1}{2}$ at $(x=0, y=0)$, this strategy profile is an NE since a strategy if some player instead plays z , his payoff is $1 - z < 1$, so there is no profitable deviation.~~ These payoffs do not make sense since they imply that each player has vote share 1 at $(x=0, y=0)$, but it is reasonable to accept the constraint $v_c + v_L = 1$.

$$2 \quad u_A(y_A, y_B) = \begin{cases} x - y_A & \text{if } y_A > y_B \\ \frac{x}{2} - y_A & \text{if } y_A = y_B \\ -y_A & \text{if } y_A < y_B \end{cases}$$

Suppose $y_B = x$, then

$$u_A(y_A=0, y_B)=0$$

$$u_A(y_A \in (0, y_B), y_B) = -y_A < 0$$

$$u_A(y_A=y_B, y_B) = \frac{x}{2} - y_A = -\frac{x}{2} < 0$$

$$u_A(y_A > y_B, y_B) = x - y_A < 0$$

$$\text{So } B_A(y_B=x) = \emptyset$$

Suppose $y_B > x$, then

$$u_A(y_A=0, y_B)=0$$

$$u_A(y_A \in (0, y_B), y_B) = -y_A < 0$$

$$u_A(y_A=y_B, y_B) = \frac{x}{2} - y_A < 0$$

$$u_A(y_A > y_B, y_B) = x - y_A < 0$$

$$\text{So } B_A(y_B > x) = \emptyset$$

Suppose $y_B < x$, then

$$B_A(y_B) \neq \emptyset \text{ since } u_A(y_A < y_B, y_B) = -y_A < 0.$$

$$u_A(y_A \in (y_B, x), y_B) = x - y_A > 0$$

$$B_A(y_B) \neq y_B \text{ since } u_A(y_A=y_B, y_B) = \frac{x}{2} - y_A = \frac{x}{2} - y_B$$

$$u_A(y_A=y_B+\epsilon, y_B) = x - y_B - \epsilon \text{ for sufficiently small } \epsilon \in (-\infty, \frac{x}{2})$$

$$B_A(y_B) \neq y_B \text{ since } u_A(y_A > y_B, y_B) = x - y_A <$$

$$u_A(y_A = y_A - y_B, y_B) = x - y_A, \text{ i.e. no } y_A > y_B$$

maximises u_A since there is always some profitable deviation $y_A' \in (y_A, y_B)$ that yields a higher u_A .

$$\text{So } B_A(y_B) = \emptyset$$

$$B_A(y_B) = \begin{cases} \emptyset & \text{if } y_B > x \\ \emptyset & \text{if } y_B < x \end{cases}$$

Suppose that I some pure NE $y^* = (y_A^*, y_B^*)$, then

by definition of NE and BR,

$$y_A^* = B_A(y_B^*) = \begin{cases} \emptyset & \text{if } y_B^* > x \\ \emptyset & \text{if } y_B^* < x \end{cases}$$

$$\text{By symmetry, } B_B(y_A^*) = \begin{cases} \emptyset & \text{if } y_A^* > x \\ \emptyset & \text{if } y_A^* < x \end{cases}$$

$$\text{then } y_B^* = B_B(y_A^*) = \begin{cases} \emptyset & \text{if } y_A^* > x \\ \emptyset & \text{if } y_A^* < x \end{cases}$$

Given $y_A, y_B \in \mathbb{R}^+$, $y_A^*, y_B^* \neq \emptyset$, so $y_A^* - y_B^* > 0$

Then $y_B^* < x$, so $y_A^* = \emptyset$.

By reductio, $\#$ pure NE $y^* = (y_A^*, y_B^*)$.

Suppose that each player plays some common y with positive probability p . Then, both play

both players play with probability p^2 , both players

play y . Then either player can profitably deviate by

reallocating probability mass from y to $y + \epsilon$ for

sufficiently small $\epsilon > 0$. ~~therefore~~ By so reallocating,

by reallocating all probability mass from y to

$y + \epsilon$, this player loses ϵ with probability

p and (when this action is played) but gains

$\frac{x}{2}$ with probability p^2 (when the deviating

It feels uncomfortable to work with ~~#~~ probability distributions where no action is played with positive probability because when a player does play some action, his doing so is entirely improbable, it had probability zero. So it seems we cannot say such things as "action right above the gap in the support is weakly dominant strictly dominated". What sort of language should we use?

player plays $y_1 + \epsilon$ and the other player plays y_1 .
 so the asset is awarded to the deviating player
 with probability 1 rather than with probability
 y_2 . So this deviation is profitable iff $\epsilon < p^x/2$, iff $\epsilon < p^x/2$.

Suppose that y^* is a mixed NE where each
 player plays the mixed strategy given by probability
 distribution $F(y)$.

Suppose that there is a gap $[y_1, y_2]$ in the support
 of $F(y)$, i.e. neither player plays any action in
 this interval.

Informally, then each player can profitably
 deviate by reallocating probability mass from the
 actions just above the interval to actions at
 the within and at the bottom of the interval.

Informally, the former actions are more costly than
 the latter, but both have the same probability of
 winning the contested asset since the other player
 never makes an intermediate bid. So y^* is not a
 Nash equilibrium if there is a gap in the support
 of $F(y)$.

Given that there are no gaps in the support of $F(y)$,
 the support of $F(y)$ is some interval $[y_1, y_2]$.

Suppose that 0 is not in the support of $F(y)$, then
 $y_1 > 0$. $u_x(y_1, y_{-x}^*) = -y_1$ since the other player
 plays y_1 with zero probability, so the player
 who plays y_1 never wins the asset. $u_x(0, y_{-x}^*) = 0$,
 so there is a profitable deviation, namely reallocating
 probability mass from y_1 to 0. Then
 y^* is not a NE. By reductio ad absurdum, if y^* is a NE,
 0 is in the support of $F(y)$. Given $y \in \mathbb{R}^+$, 0 is
 the lowest contribution in the support of $F(y)$.

y^* is a NE ^{only if} all actions in the support of y^*
 have equal payoff so there is no profitable
 deviation by reallocating probability mass from a
 less profitable action in the mix to a more
 profitable one. For each player x ,
~~then~~ $u_x(0, y_{-x}^*) = u_x(y_x, y_{-x}^*)$ for all y_x in
 the support of $F(y)$

$$0 = F(y_x)X - y_x \quad \text{for all } y_x \text{ in the support of } F(y)$$

$$F(y_x) = y_x/X$$

$F(y_x)$ is the cumulative distribution function of a
 uniform distribution $U(0, X)$

The equilibrium strategy of each player is the
 mixed strategy given by a uniform distribution
 over the actions $y_x \in [0, X]$.

3	L	R
T	6	8
B	6	2
	<u>2</u>	0
-8	0	

Best responses underlined.

By inspection, the pure strategy NE are (T, R) , (B, L) , where players play mutual best responses.

Suppose π_1^* mixed NE, then π_1 mixes T, B, so
 $\pi_2(T, \pi_1^*) = \pi_2(B, \pi_1^*)$,
 $6P_L + 2P_R = 2P_L, P_L = P_R, P_L + P_R = \frac{1}{2}$
 Then π_2 mixes L, R, so
 $\pi_2(L, \pi_1^*) = \pi_2(R, \pi_1^*)$,
 $6P_T + 2P_B = 2P_T, P_T = P_B = \frac{1}{2}$
 $\therefore \pi_1^* = (\frac{1}{2}T + \frac{1}{2}B, \frac{1}{2}L + \frac{1}{2}R)$ is the only mixed NE



By inspection, there are no hybrid NE.



Neither player has incentive to follow these instructions
~~If H, and π_1 is told to play T and π_2 is told to play R, π_1 has incentive to instead play~~

Each player has incentive to act as instructed given that the other player did so since (T, R) and (B, L) are ~~the~~ NE, so no player has incentive to deviate given that the other player plays his part in the NE. The game resembles a coordination game and the coin toss serves as a coordinating device. Expected payoffs are (5,5)



State	Strategy profile	Probability
x	(T, R)	$\frac{1}{3}$
y	(T, L)	$\frac{1}{3}$
z	(B, L)	$\frac{1}{3}$

Suppose π_1 is instructed to play T, then let π_1^* denote this correlated equilibrium

E(π_1^*, π_2) Suppose π_1 is instructed to play T, then π_1 has incentive to play T iff his expected payoff, given this instruction, from playing T is greater than from playing B.



$$\begin{aligned} & \frac{1}{2}6 + \frac{1}{2}3 \geq \frac{1}{2}8 \\ & \frac{1}{2}6 + \frac{1}{2}2 = 4 \geq \frac{1}{2}8 + \frac{1}{2}0 = 4 \\ & P(L|T) \cdot 6P(L|T) + 2P(R|T) = \frac{1}{2}6 + \frac{1}{2}2 = 4 \\ & \geq 8P(L|T) + 0P(R|T) = \frac{1}{2}8 + \frac{1}{2}0 = 4 \end{aligned}$$

Likewise if π_1 is instructed to play B
 $8P(L|B) = 4 \geq 6P(L|B) + 2P(R|B) = 6$

Likewise if π_1 is instructed to play L
 $6P(T|L) + 2P(B|L) = \frac{1}{2}6 + \frac{1}{2}2 = 4 \geq$

$$8P(T|R) = \frac{1}{2} \times 8 = 4$$

(Player 1 & P1.2 are instructed to play R
 $8P(T|R) = 8 \geq 6P(T|R) + 2P(B|R) = 6$

This is verified above that each player finds it weakly optimal to play as instructed.

If x_1 , ~~first~~ plays play (T, R) , $(u_1, u_2) = (2, 3)$
If y_1 plays play (T, L) , $(u_1, u_2) = (6, 6)$
If x_2 plays play (B, L) , $(u_1, u_2) = (3, 2)$
 $E(u_1, E(u_2)) = (\frac{1}{3} \times 2 + \frac{1}{3} \times 6 + \frac{1}{3} \times 8, \frac{1}{3} \times 8 + \frac{1}{3} \times 6)$
 $\frac{1}{3} \times 8 + \frac{1}{3} \times 6 + \frac{1}{3} \times 2) = (\frac{16}{3}, \frac{16}{3})$

Expected payoffs increase compared to the earlier correlated equilibrium since (T, L) which has higher average payoff is to each player than the average of (T, R) and (B, L) is mixed in the "mixed in" in this latter correlated equilibrium.

~~THIS PARAGRAPH READING~~ Each player's playing his part of (T, L) when so instructed is incentive compatible because each player, if he does not, risks receiving zero payoff due to miscoordination. In other words, because each player does not know what the other has been instructed to play, each player finds it weakly optimal to play his part in (T, L) when so instructed, even though this would not be optimal if each player knew that the other was ~~as~~ similarly instructed.

Correlated mixing of strategies seems inconsistent with the "individualism" of non-cooperative game theory. Correlated equilibrium can be interpreted as describing an equilibrium where players have access to some coordinating device like a traffic light or a ~~distributed~~ computer. 

$$4 P(C_j=L | C_i=L) = P(C_j=L \cap C_i=L) / P(C_i=L) = 2d/y_2 = 2d$$



$$P(C_j=H | C_i=L) = 1 - P(C_j=L | C_i=L) = 1 - 2d$$

By symmetry,
 $P(C_j=H | C_i=H) = 2d$, $P(C_j=L | C_i=H) = 1 - 2d$
 firm i believes, a posteriori, that firm j has the same cost with probability $2d$ and cost c_L

The strategy of each player i is the pair (σ_i^L, σ_i^H) where σ_i^x is the quantity player i chooses, or some probability distribution over quantities if firm i finds that it has cost c_X .

The (ex post) payoff of each firm is a function of its quantity, its opponents quantity, and its type

$$\pi_{ii} = (1 - q_i - q_j - c_i)q_i$$

The interim payoff of each firm is a function of its quantity and its type, its opponents

$$\pi_{iH} = 2d[1 -$$

$$\text{its quantity, its opponents quantity, and its type} \\ \pi_{iH} = 2d[(1 - q_{iH} - q_{jH} - c_H)q_{iH}] \\ + (1 - 2d)[(1 - q_{iH} - q_{jL} - c_H)q_{iH}]$$



$$(1 - q_{iH} - c_H)q_{iH} \geq 2d q_{jH} q_{iH} \geq (1 - 2d) q_{jL} q_{iH} \\ \pi_{iL} = 2d[(1 - q_{iL} - q_{jL} - c_L)q_{iL}] \\ + (1 - 2d)[(1 - q_{iL} - q_{jH} - c_L)q_{iL}] \\ = (1 - q_{iL} - c_L)q_{iL} - 2d q_{jL} q_{iL} - (1 - 2d) q_{jH} q_{iL}$$

Taking FOCs

$$\frac{\partial \pi_{iH}}{\partial q_{iH}} = (1 - q_{iH} - c_H) - q_{iH} - 2d q_{jH} - (1 - 2d) q_{jL} = 0,$$



$$q_{iH} = \frac{1}{2}[1 - c_H - 2d q_{jH} - (1 - 2d) q_{jL}]$$

$$\frac{\partial \pi_{iL}}{\partial q_{iL}} = (1 - q_{iL} - c_L) - q_{iL} - 2d q_{jL} - (1 - 2d) q_{jH} = 0,$$

$$q_{iL} = \frac{1}{2}[1 - c_L - 2d q_{jL} - (1 - 2d) q_{jH}]$$

$$\text{By symmetry, } q_{iH} = q_{jH}, q_{iL} = q_{jL} \\ q_{jH} = \frac{1}{2}[1 - c_H - 2d q_{iH} - (1 - 2d) q_{iL}] \\ q_{jL} = \frac{1}{2}[1 - c_L - 2d q_{iL} - (1 - 2d) q_{iH}]$$

This solution seems too convoluted, but would it have gotten the right answer?
 (Substituting BR into BR)



By substitution,

$$q_{iH} = \frac{1}{2}(1 - c_H) - \frac{2d}{2} \left[1 - c_H - 2d q_{iH} - (1 - 2d) q_{iL} \right] \\ - \frac{1 - 2d}{2} \left[1 - c_L - 2d q_{iL} - (1 - 2d) q_{iH} \right] \\ = \frac{1}{2}(1 - c_H) - \frac{2d}{2} \left(1 - c_H \right) - \frac{(1 - 2d)}{2} \left(1 - c_H \right) \\ + \frac{2d}{2} \left[2d q_{iH} + (1 - 2d) q_{iL} \right] \\ + \frac{(1 - 2d)}{2} \left[2d q_{iL} + (1 - 2d) q_{iH} \right] \\ = \frac{1}{2}(1 - c_H) + \frac{1}{4} \left[2d(2d q_{iH}) + 2d((1 - 2d) q_{iL}) \right] \\ + 2d(1 - 2d) q_{iL} + (1 - 2d)(1 - 2d) q_{iH} \\ = \frac{1}{4}(1 - c_H) + \frac{1}{4}$$



Suppose that the equilibrium is symmetric, then

$$q_{iH} = \frac{1}{2}[1 - c_H - 2d q_{iH} - (1 - 2d) q_{iL}]$$

$$(1 - 2d) q_{iH} = \frac{1}{2}[1 - c_H] - \frac{1 - 2d}{2} q_{iL}$$

$$(1 - 2d) q_{iL} = \frac{1}{2}[1 - c_L] - \frac{1 - 2d}{2} q_{iH}$$

$$= \frac{1}{2}[1 - c_L] - \frac{(1 - 2d)}{2} \left(\frac{1}{2}(1 - c_H) - \frac{1 - 2d}{2} q_{iH} \right)$$

why could it matter if interim BNE is equivalent to ex ante BNE?