

$$\text{Let } u = \sum_{i=1}^L d_i \ln x_i, \quad d_i > 0, \quad \sum_{i=1}^L d_i = 1$$

$$\nabla u = (\partial_1 u, \partial_2 u, \dots, \partial_L u)$$

$$= (d_1/x_1, d_2/x_2, \dots, d_L/x_L)$$

$$b) D^2 u = \begin{pmatrix} \partial_1 \partial_1 u & \partial_1 \partial_2 u & \dots & \partial_1 \partial_L u \\ \partial_2 \partial_1 u & \partial_2 \partial_2 u & \dots & \partial_2 \partial_L u \\ \vdots & \vdots & \ddots & \vdots \\ \partial_L \partial_1 u & \partial_L \partial_2 u & \dots & \partial_L \partial_L u \end{pmatrix}$$

$$= \begin{pmatrix} -d_1 x_1^{-2} & 0 & \dots & 0 \\ 0 & -d_2 x_2^{-2} & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -d_L x_L^{-2} \end{pmatrix}$$

c) Let $D^2 u_{ij}$ denote the element in the i^{th} row and j^{th} column of $D^2 u$. By definition of $D^2 u$, $\forall i, j \in \{1, \dots, L\}$:

$$D^2 u_{ij} = \partial_i \partial_j u = \partial_i (d_j/x_j) = 0 \text{ for all } i \neq j. \text{ Then } D^2 u \text{ is a diagonal matrix so } D^2 u \text{ is a symmetric matrix. This is confirmed by inspection.}$$

Let $D^2 u^k$ denote the square submatrix of A with only the first k rows and columns retained. Then $D^2 u^1 = (-d_1 x_1^{-2})$

$$D^2 u^2 = \begin{pmatrix} -d_1 x_1^{-2} & 0 \\ 0 & -d_2 x_2^{-2} \end{pmatrix} \text{ and so on.}$$

$$\det (-1)^1 \det D^2 u^1 = d_1 x_1^{-2}$$

$$(-1)^2 \det D^2 u^2 = (d_1 x_1^{-2})(d_2 x_2^{-2})$$

$$(-1)^k \det D^2 u^k = (-1)^k \prod_{i=1}^k (-d_i x_i^{-2})$$

Since the determinant of a diagonal matrix is the product of its diagonal elements

$$= \prod_{i=1}^k (d_i x_i^{-2})$$

> 0 for all k

Given $\forall d_i > 0$, assuming that u is well-defined then

$\forall i: \ln x_i$ is well-defined then $\forall i: x_i \neq 0$ then $\forall i: x_i^{-2} > 0$.

By the determinant test, $D^2 u$ is negative definite.

d) $\vec{y}^T D^2 u \vec{y}$ is well-defined only if \vec{y} is a vector of length L . Let $\vec{y} = (y_1, \dots, y_L)^T$

$$D^2 u \vec{y} = \begin{pmatrix} -d_1 x_1^{-2} & 0 & \dots & 0 \\ 0 & -d_2 x_2^{-2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -d_L x_L^{-2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_L \end{pmatrix}$$

$$= \begin{pmatrix} -y_1 d_1 x_1^{-2} & -y_2 d_2 x_2^{-2} & \dots & -y_L d_L x_L^{-2} \end{pmatrix}$$

$$\vec{y}^T D^2 u \vec{y} = \begin{pmatrix} -y_1 d_1 x_1^{-2} \\ -y_2 d_2 x_2^{-2} \\ \vdots \\ -y_L d_L x_L^{-2} \end{pmatrix}$$

$$\vec{y}^T D^2 u \vec{y} = (y_1, y_2, \dots, y_L) \begin{pmatrix} -y_1 d_1 x_1^{-2} \\ -y_2 d_2 x_2^{-2} \\ \vdots \\ -y_L d_L x_L^{-2} \end{pmatrix}$$

$$= -y_1^2 d_1 x_1^{-2} - y_2^2 d_2 x_2^{-2} - \dots - y_L^2 d_L x_L^{-2}$$

$$= -\sum_{i=1}^L d_i y_i^2 x_i^{-2}$$

< 0 for all $\vec{y} \neq 0 \in \mathbb{R}^n$

Given $\forall d_i > 0$, assuming that u is well-defined hence

$\forall i: x_i^{-2} > 0$

By checking $\vec{y}^T D^2 u \vec{y}$, $D^2 u$ is negative definite.

$$e) u(\vec{x}') \approx u(\vec{x}) + \nabla u(\vec{x})(\vec{x}' - \vec{x}) + \frac{1}{2}(\vec{x}' - \vec{x})^T D^2 u(\vec{x})(\vec{x}' - \vec{x})$$

$$= 0 + (d_1/x_1, d_2/x_2, \dots, d_L/x_L) \begin{pmatrix} x_1' - x_1 \\ x_2' - x_2 \\ \vdots \\ x_L' - x_L \end{pmatrix} + \frac{1}{2} \begin{pmatrix} x_1' - x_1 & x_2' - x_2 & \dots & x_L' - x_L \end{pmatrix} D^2 u \begin{pmatrix} x_1' - x_1 \\ x_2' - x_2 \\ \vdots \\ x_L' - x_L \end{pmatrix}$$

$$\begin{aligned}
 &= \alpha_1(x'_1 - 1) + \alpha_2(x'_2 - 1) + \dots + \alpha_L(x'_L - 1) \\
 &\quad + \frac{1}{2} [-\alpha_1(x'_1 - 1)^2 - \alpha_2(x'_2 - 1)^2 + \dots + \alpha_L(x'_L - 1)^2] \\
 &= \sum_{i=1}^L \alpha_i (x'_i - 1) - \frac{1}{2} (x'_i - 1)^2
 \end{aligned}$$