Microeconomic Analysis Paper 190530

1a i $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

Let λ and $\vec{v}$ be an eigenvalue and an ~~eigen~~ eigenvector of A. By definition, $A\vec{v} = \lambda\vec{v}$.

~~[illegible]~~ $\iff (A - \lambda I)\vec{v} = \vec{0}$ ~~$\Rightarrow \det(A - \lambda I) = 0$~~

$\iff \det(A - \lambda I) = 0$

$\det(A - 4I) = \det \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$ ~~$= \det$~~

$= -2 \det\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} - 1 \det\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} + 1 \det\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$

$= -2(3) - 1(-3) + 1(3) = 0$

So $\lambda_1 = 4$ is an eigenvalue of A

$\det(A - I) = \det\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ ~~$= \det$~~

$= \det\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \det\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \det\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$= 0 - 0 + 0 = 0$

So $\lambda_2 = 1$ is an eigenvalue of A.

$A\vec{v_1} = \lambda_1\vec{v_1} \iff \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} v_1^1 \\ v_1^2 \\ v_1^3 \end{pmatrix} = 4\begin{pmatrix} v_1^1 \\ v_1^2 \\ v_1^3 \end{pmatrix} \iff$

$2v_1^1 + v_1^2 + v_1^3 = 4v_1^1,\ v_1^1 + 2v_1^2 + v_1^3 = 4v_1^2,\ v_1^1 + v_1^2 + 2v_1^3 = 4v_1^3 \iff$

$v_1^1 = v_1^2 = v_1^3 \implies$

Any $\vec{v_1}$ with $v_1^1 = v_1^2 = v_1^3$ is an eigenvalue with $\lambda_1 = 4$.

$A\vec{v_2} = \lambda_2\vec{v_2} \iff \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} v_2^1 \\ v_2^2 \\ v_2^3 \end{pmatrix} = \begin{pmatrix} v_2^1 \\ v_2^2 \\ v_2^3 \end{pmatrix} \iff$

$2v_2^1 + v_2^2 + v_2^3 = v_2^1,\ v_2^1 + 2v_2^2 + v_2^3 = v_2^2,\ v_2^1 + v_2^2 + 2v_2^3 = v_2^3 \iff$

$v_2^1 + v_2^2 + v_2^3 = 0$

Any $\vec{v_2}$ such that $v_2^1 + v_2^2 + v_3^3$ is an eigenvector with eigenvalue $\lambda_2 = 1$

ii ~~$A = VDV^{-1}$, where $V =$~~

Consider the following eigenvalues and eigenvectors.

$\lambda_1 = 4,\ \vec{v_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\ \lambda_2 = 1,\ \vec{v_2} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},\ \lambda_3 = 1,\ \vec{v_3} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

Let V be an "eigenbasis" of A. $V = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$

~~Set~~

Let $D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Then $A = VDV^{-1}$, $D = V^{-1}AV$, $A^6 = VD^6V^{-1}$.

Compute V^{-1} by Gauss-Jordan elimination.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{} \begin{array}{l} R_2 \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & -1 & 1 & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array}\right) \xrightarrow{} \begin{array}{l} R_2 \leftarrow R_2 - 2R_3 \\ R_3 \leftarrow R_3 - 2R_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 1 & -2 \\ 0 & 0 & 3 & 1 & -2 & 1 \end{array}\right) \xrightarrow{} \begin{array}{l} R_2 \leftarrow \tfrac{1}{3}R_2 \\ R_3 \leftarrow \tfrac{1}{3}R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \tfrac{1}{3} & \tfrac{1}{3} & -\tfrac{2}{3} \\ 0 & 0 & 1 & \tfrac{1}{3} & -\tfrac{2}{3} & \tfrac{1}{3} \end{array}\right) \xrightarrow{} R_1 \leftarrow R_1 - R_2 - R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \tfrac{1}{3} & \tfrac{1}{3} & \tfrac{1}{3} \\ 0 & 1 & 0 & \tfrac{1}{3} & \tfrac{1}{3} & -\tfrac{2}{3} \\ 0 & 0 & 1 & \tfrac{1}{3} & -\tfrac{2}{3} & \tfrac{1}{3} \end{array}\right)$$

$V^{-1} = \tfrac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \\ 1 & -2 & 1 \end{pmatrix}$

$A^6 = VD^6V^{-1}$

$= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^6 \tfrac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \\ 1 & -2 & 1 \end{pmatrix}$

$= \tfrac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 4096 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \\ 1 & -2 & 1 \end{pmatrix}$

$= \tfrac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 4096 & 4096 & 4096 \\ 1 & 1 & -2 \\ 1 & -2 & 1 \end{pmatrix}$

$= \tfrac{1}{3}\begin{pmatrix} 4098 & 4095 & 4095 \\ 4095 & 4098 & 4095 \\ 4095 & 4095 & 4098 \end{pmatrix}$

$= \begin{pmatrix} 1366 & 1365 & 1365 \\ 1365 & 1366 & 1365 \\ 1365 & 1365 & 1366 \end{pmatrix}$

b The four vectors form a basis of $\mathbb{R}^4$ iff they span $\mathbb{R}^4$. 4 linearly independent vectors are necessary and sufficient to span $\mathbb{R}^4$. Suppose that some non-null vector is in the intersection of span$[\vec{u_1}, \vec{u_2}]$ and span$[\vec{v_1}, \vec{v_2}]$. ~~Deno~~ Denote this vector $\vec{w}$. Then, supposing that $\vec{v_1}, \vec{v_2} \notin$ span$[\vec{u_1}, \vec{u_2}]$, then by ~~definition~~ $\vec{w}$ is a linear combination of $\vec{v_1}$ and $\vec{v_2}$, so $\vec{v_2}$ is a linear combination of $\vec{w}$ and $\vec{v_1}$, so $\vec{v_2}$ is a linear combination of $\vec{v_1}, \vec{u_1}, \vec{u_2}$, and the four vectors are not independent. If $\vec{v_1}$ ~~[illegible]~~ $\in$ span$[\vec{u_1}, \vec{u_2}]$ then the four are also not linearly independent. Likewise for $\vec{v_2}$. So if the spans intersect (except $\vec{0}$), the four are not linearly independent and do not form a basis.

If the spans do not intersect (except $\vec{0}$), then no linear combination of $\vec{u_1}, \vec{u_2}$ is a linear combination of $\vec{v_1}, \vec{v_2}$, so $\vec{v_1}$ is not a linear combination of $\vec{u_1}, \vec{u_2}, \vec{v_2}$, likewise for the rest, they are linearly independent, so they are a basis of $\mathbb{R}^4$.

c i $f(x,y) = \begin{cases} xy(x^2-y^2)/x^2+y^2 & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$

$$\begin{aligned} f_x(0,y) &= \lim_{h\to 0} f(h,y) - f(0,y)/h \\ &= \lim_{h\to 0} (hy(h^2-y^2)/h^2+y^2 - 0)/h \\ &= \lim_{h\to 0} y(h^2-y^2)/h^2+y^2 \\ &= \cancel{} -y \end{aligned}$$

$$\begin{aligned} f_y(x,0) &= \lim_{h\to 0} f(x,h) - f(x,0)/h \\ &= \lim_{h\to 0} (xh(x^2-h^2)/x^2+h^2 - 0)/h \\ &= \lim_{h\to 0} (x(x^2-h^2)/x^2+h^2) \\ &= x \end{aligned}$$

$$\begin{aligned} f_x(0,0) &= \lim_{h\to 0} f(h,0) - f(0,0)/h \\ &= \lim_{h\to 0} 0/h \\ &= 0 \end{aligned}$$

$$\begin{aligned} f_y(0,0) &= \lim_{h\to 0} f(0,h) - f(0,0)/h \\ &= \lim_{h\to 0} 0/h \\ &= 0 \end{aligned}$$

ii
$$\begin{aligned} f_{xx}(0,0) &= \lim_{h\to 0} f_x(h,0) - f_x(0,0)/h \\ &= \lim_{h\to 0} 0-0/h \\ &= 0 \end{aligned}$$

$$\begin{aligned} f_{yy}(0,0) &= \lim_{h\to 0} f_y(0,h) - f_y(0,0)/h \\ &= \lim_{h\to 0} 0-0/h \\ &= 0 \end{aligned}$$

$$\begin{aligned} f_{xy}(0,0) &= \lim_{h\to 0} f_y(h,0) - f_y(0,0)/h \\ &= \lim_{h\to 0} h-0/h \\ &= 1 \end{aligned}$$

$$\begin{aligned} f_{yx}(0,0) &= \lim_{h\to 0} f_x(0,h) - f_x(0,0)/h \\ &= \lim_{h\to 0} -h/h \\ &= -1 \end{aligned}$$

$$D^2f(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The Hessian is asymmetric iff f is not C^2 at the origin. the second-order partial derivatives are not continuous at the origin.

2a An optimisation problem is concave iff it has the following form.

$\max_{\vec{x}} f(\vec{x})$ s.t. $\vec{g}(\vec{x}) \leq \vec{b}$, where $\vec{g}(\vec{x}) = (g_1(\vec{x}), \ldots, g_m(\vec{x}))$, f is concave, and each of $g_1, \ldots, g_m$ is convex.

~~Then, the if an or~~ An optimisation problem is convex iff it has the following form.

$\min_{\vec{x}} f(\vec{x})$ s.t. $\vec{g}(\vec{x}) \geq \vec{b}$, where f is convex and each of $g_1, \ldots, g_m$ is concave.

If an optimisation problem is concave (convex), then the KT-FOCs are sufficient for a maximum (minimum). If ~~the~~, in addition, the constraint set is non-empty, then the KT-FOCs are also necessary.

The above does not apply if f is not differentiable then the KT-FOCs are neither necessary nor sufficient.

b $f(x,y) = x^2+y^2-cxy$

$Df(x,y) = (2x-cy \quad 2y-cx)$

$D^2f(x,y) = \begin{pmatrix} 2 & -c \\ -c & 2 \end{pmatrix}$

$\text{tr } D^2f(x,y) = 4 > 0$

$\det D^2f(x,y) = 4-c^2$

$\text{tr } D^2f(x,y) > 0 \implies$ at least one eigenvalue is strictly positive $\implies D^2f(x,y)$ is not negative definite or negative semi-definite $\implies$ ~~$D^2f(x,y)$~~ f is not (even weakly) concave.

If $\det D^2f(x,y) \geq 0 \iff c \in [-2,2]$, then both eigenvalues of the Hessian are weakly positive, the Hessian is positive definite or positive semi-definite, then f is convex.

Otherwise, $c \notin [-2,2]$, $\det D^2f(x,y) < 0$, one eigenvalue is positive the other is negative, so the Hessian is indefinite, f is neither concave nor convex.

c $c=1$, f is strictly convex, the FOCs are necessary and sufficient for a minimum.

$FOC_x: 2x-y=0$

$FOC_y: 2y-x=0$

$\implies x=y=0$

The unique global minimum is at $(x,y)=(0,0)$.

f remains strictly convex and the ~~unique solution to the f~~ global minimum is in the constraint set, so there is no other local minimum.

The new constraint set is the perimeter of a square. ~~this is~~

Constraint set

$\{(x,y): x=4, y \in [-4,4]\} \cup$
$\{(x,y): x=-4, y \in [-4,4]\} \cup$
$\{(x,y): y=4, x \in [-4,4]\} \cup$
$\{(x,y): y=-4, x \in [-4,4]\}$

Along the first edge, $f(4,y) = 16+y^2-4y$, which has a minimum at $y=2$.

Along the second edge, $f(-4,y) = 16+y^2+4y$, which has a minimum at $y=-2$

Along the third edge, $f(x,4) = x^2+16-4x$ which has a minimum at $x=2$.

Along the fourth edge, $f(x,-4) = x^2+16+4x$ which has a minimum at $x=-2$.

The local minima are $(4,2), (-4,-2), (2,4), (-2,-4)$.

d $\max_{x,y}$ ~~$f(x,y)=$~~ x^2+y^2-xy s.t.

$C_x: x^2 \leq 16$

$C_y: y^2 \leq 16$

$\mathcal{L} = x^2+y^2-xy - \lambda_x(x^2-16) - \lambda_y(y^2-16)$

$FOC_x: 2x-y-2\lambda_x x = 0$

$FOC_y: 2y-x-2\lambda_y y = 0$

$CS_x: \lambda_x \geq 0, x^2 \leq 16, \lambda_x(x^2-16)=0$

$CS_y: \lambda_y \geq 0, y^2 \leq 16, \lambda_y(y^2-16)=0$

Suppose $\lambda_x, \lambda_y = 0$, then by FOC_x, FOC_y, $x=y=0$, then the remaining FOC are satisfied so $(x,y)=(0,0)$ is a candidate ~~maxim~~ maximum.

Suppose $\lambda_x > 0, \lambda_y = 0$, then by CS_x, $x^2=16$, $x=\pm 4$. Suppose $x=4$. Then by FOC_y, $y=2$, and the remaining FOCs are satisfied. Suppose $x=-4$. Then by FOC_y, ~~$y=2$~~ $y=-2$ and the remaining FOCs are satisfied. So $(x,y)=(4,2), (-4,-2)$ are candidate maxima

By symmetry, the two candidate maxima supposing that $\lambda_x=0, \lambda_y>0$ are $(x,y)=(2,4)$ and $(-2,-4)$

Suppose that $\lambda_x, \lambda_y > 0$, then by CS_x, CS_y $x=\pm 4, y=\pm 4$.
If $x=y=4$, then by FOC_x, FOC_y, $\lambda_x=\lambda_y=1/2$
If $x=y=-4$, then by FOC_x, FOC_y, $\lambda_x=\lambda_y=1/2$

if $x=4$, $y=-4$, $\lambda_x = \lambda_y = 3/2$
if $x=4$, $y=-4$, $\lambda_x = \lambda_y = 3/2$
Each of ~~the~~ $(4,4)$, $(4,-4)$, $(-4,4)$, $(-4,-4)$ satisfies the FOCs and is a candidate maximum.

The candidate maxima are $(0,0)$, $(2,4)$, $(-2,-4)$, $(4,2)$, $(-4,-2)$, $(4,4)$, $(4,-4)$, $(-4,4)$, $(-4,-4)$. The first five of these are local minima (from earlier) so are not maxima.

The remaining four are local maxima. Both constraints bind so there are zero degrees of freedom, the SOC for a maximum at each point is vacuously satisfied.

4a The lottery faced by the agent A given coverage amount β is $L(\beta)$.

$L(\beta) = [\pi, 1-\pi; Y-\beta M - L + \beta L, Y - \beta M]$

$= [\pi, 1-\pi; (Y-M) + M - \beta M - L + \beta L, (Y-M) + M - \beta M]$

$= [\pi, 1-\pi; (Y-M) + (1-\beta)(M-L), (Y-M) + (1-\beta)M]$

$= [\pi, 1-\pi; w + \alpha(M-L), w + \alpha M]$

So A has final wealth $w + \alpha\tilde{x}$, where $\tilde{x}$ takes value $M-L$ w.p. π and value M ~~with value~~ w.p. $1-\pi$.

A has expected utility $V(\alpha) = E[u(w + \alpha\tilde{x})]$

A maximises expected utility. A has ~~exp~~ the following expected utility maximisation problem.

$\max_\alpha V(\alpha)$

b FOC: $V'(\alpha) = 0 \iff$

$E[u'(w+\alpha\tilde{x})\tilde{x}] = 0$

SOC: $V''(\alpha) < 0 \iff$

$E[u''(w+\alpha\tilde{x})\tilde{x}^2] < 0$

Given that A is risk averse, Bernoulli utility function u is concave, so $u''(y) < 0$ for all y ~~and~~ hence $E[u''(w+\alpha\tilde{x})x^2] < 0$ and the SOC is satisfied for all α.

$V'(0) = E[u'(w)\tilde{x}] = u'(w)E[\tilde{x}]$

$E[\tilde{x}] = \pi(M-L) + (1-\pi)M = M - \pi L > 0$ (given $M > \pi L$)

~~$V'(0)$~~ Given that u is strictly increasing, $u'(y) > 0$ for all y.

$V'(0) = u'(w)E[\tilde{x}] > 0$

~~So at the~~ At the optimum, $\alpha \neq 0$ because deviation to marginally lower α is always strictly profitable. So at the optimum, $\beta \neq 1$. Then, given ~~$0 \leq \alpha$~~ $0 \leq \beta \leq 1$, $\beta^* < 1$.

Intuitively, $\beta^* < 1$ ~~bec~~, i.e. a risk-averse ~~ag~~ expected utility maximiser always under insures because a risk-averse expected utility ~~m~~ maximiser always takes a positive share of a favourable gamble. Where insurance is actuarially unfair, under insurance is a favourable gamble. Expected value of this gamble is increasing in direct proportion to the share α. Risk premium for a small gamble is approximately proportionate to the variance which in turn is directly proportionate to ~~the square of the share~~ α^2. So for small α, EV initially increases more rapidly than ~~for large α~~ RP, so CE increases and ~~taking some~~ EU increases, so some positive α is optimal.

c $E[u'(w + \alpha^*\tilde{x})\tilde{x}] = 0$

Differentiate wrt x

$E[u''(w + \alpha^*\tilde{x})(1 + \partial\alpha^*/\partial w\,\tilde{x})\tilde{x}] = 0 \iff$

$E[u''(w+\alpha^*\tilde{x})\tilde{x} + u''(w+\alpha^*\tilde{x})\tilde{x}^2\,\partial\alpha^*/\partial w] = 0 \iff$

$E[u''(w+\alpha\tilde{x})\tilde{x}] + \partial\alpha^*/\partial w\, E[u''(w+\alpha\tilde{x})\tilde{x}^2] = 0 \iff$

$\partial\alpha^*/\partial w = -E[u''(w+\alpha\tilde{x})\tilde{x}] / E[u''(w+\alpha\tilde{x})\tilde{x}^2]$

$\partial\beta^*/\partial Y = E[u''(w+\alpha\tilde{x})\tilde{x}] / E[u''(w+\alpha\tilde{x})\tilde{x}^2]$

Given concave u, the denominator is always negative.

$\text{sign}(\partial\beta^*/\partial Y) = -\text{sign}(E[u''(w+\alpha\tilde{x})\tilde{x}])$

Assume decreasing absolute risk aversion, i.e. $A(y) = -u''(y)/u'(y)$ is decreasing in y. Then, ~~$A(w+\alpha\tilde{x})$~~ presumably, the ~~denom~~ numerator will be positive, so optimal coverage is decreasing in Y.

Experimental and empirical evidence is mostly consistent with DARA.

Date

5a For given known θ, the firm F optimally chooses some wage effort pair w, e to maximise net profit $\pi(e) - w$ subject to the worker W's participation constraint PC: ~~$g(e,\theta) \geq \bar{u} = 0$. At any optimum~~ $w - g(e,\theta) \geq \bar{u} = 0$. At any optimum, the participation constraint binds. Any candidate optimum such that PC does not bind fails to deviation by reducing w by small amount ε (such that PC remains satisfied), which increases net profit. Then at any optimum, $w = g(e,\theta)$, F maximises $\pi(e) - g(e,\theta)$.

FOC: $\pi'(e) - g_e(e,\theta) = 0 \iff \pi'(e) = g_e(e,\theta)$.

~~SOC: $\pi''(e)$~~

And PC binds: $w = g(e,\theta)$.

$$\pi'(e_H) = g_e(e_H,\theta_H), \quad w_H = g(e_H,\theta_H)$$
$$\pi'(e_L) = g_e(e_L,\theta_L), \quad w_L = g(e_L,\theta_L)$$
$$\iff$$
$$e_H^{*-1/2} = 1/\theta_H, \quad w_H^* = e_H^*/\theta_H$$
$$e_L^{*-1/2} = 1/\theta_L, \quad w_L^* = e_L^*/\theta_H$$
$$\iff$$
$$e_H^* = \theta_H^2, \quad w_H^* = \theta_H$$
$$e_L^* = \theta_L^2, \quad w_L^* = \theta_L$$

bi According to the revelation principle, ~~it is~~ F can restrict attention to contracts of the form $(w_H, e_H), (w_L, e_L)$ that satisfy PC for each type and incentive compatibility constraint (IC) for each type. the outcome of Any contract, however complex, can be replicated by a contract with the above simple structure. This is because ~~the~~ W of each type ultimately chooses ~~the effort the wage~~ some wage-effort pair that satisfies PC and IC.

$$\max E[\pi(e) - w] \;\; \cancel{= \lambda(\pi(e_H) - w_H)}$$
$$= \lambda(\pi(e_H) - w_H) + (1-\lambda)(\pi(e_L) - w_L) \text{ s.t.}$$
$$PC_L: \; w_L - g(e_L,\theta_L) \geq \bar{u} = 0$$
$$PC_H: \; w_H - g(e_H,\theta_H) \geq \bar{u} = 0$$
$$IC_L: \; w_L - g(e_L,\theta_L) \geq w_H - g(e_H,\theta_L)$$
$$IC_H: \; w_H - g(e_H,\theta_H) \geq w_L - g(e_L,\theta_H)$$

ii $\bar{u} \leq_1 w_L - g(e_L,\theta_L) <_2 w_L - g(e_L,\theta_H) \leq_3 w_H - g(e_H,\theta_H)$

$\leq_1$ holds iff PC_L is satisfied, $<_2$ holds by the single-crossing property, i.e. given that $\theta_H > \theta_L$, and $\leq_3$ holds iff IC_H is satisfied. So PC_H ($w_H - g(e_H,\theta_H) \geq \bar{u}$) is redundant when PC_L and IC_H are satisfied.

At any optimum, PC_L binds. Any candidate optimum such that PC_L does not bind fails to deviation by decreasing both w_H and w_L by small amount ε (such that PC_L ~~remains satisfied)~~. ~~IC~~ remains satisfied) IC_L and IC_H remain satisfied because the change on the RHS and LHS ~~of~~ are equal for each. PC_H remains satisfied because PC_L and IC_H remain satisfied. And expected wage decreases hence expected net profit increases.

At any optimum IC_H binds. Any candidate optimum such that IC_H ~~fails~~ does not bind fails to deviation by decreasing w_H by small amount ε such that IC_H remains satisfied. PC_L is unaffected, IC_L is "loosened"; PC_H remains satisfied because PC_L and IC_H remain satisfied. Expected wage decreases hence expected net profit increases.

iii
$$PC_L \implies w_L = g(e_L,\theta_L) = e_L/\theta_L$$
$$IC_H \implies w_H - g(e_H,\theta_H) = w_L - g(e_L,\theta_H) \iff$$
$$w_H = w_L - g(e_L,\theta_H) + g(e_H,\theta_H) \iff$$
$$w_H = e_L/\theta_L - e_L/\theta_H + e_H/\theta_H \iff$$
$$w_H = e_L/\theta_L + (e_H - e_L)/\theta_H$$

Neglect IC_L

~~objective function~~ F's objective function reduces as follows

$$E[\pi(e) - w]$$
$$= \lambda\left(\pi(e_H) - \left(e_L/\theta_L + (e_H - e_L)/\theta_H\right)\right) + (1-\lambda)(\pi(e_L) - e_L/\theta_L)$$
$$= \lambda\pi(e_H) - \lambda\left((e_H - e_L)/\theta_H\right) + (1-\lambda)\pi(e_L) - e_L/\theta_L$$

$$\text{FOC } e_H: \; \lambda\pi'(e_H) - \lambda/\theta_H = 0$$
$$\text{FOC } e_L: \; \lambda/\theta_H + (1-\lambda)\pi'(e_L) - 1/\theta_L = 0$$

FOC $e_H \implies \pi'(e_H) = 1/\theta_H = g_e(e_H,\theta_H) \implies$ (from (a)) $\hat{e}_H = e_H^* = \theta_H^2$. The optimal effort level to require of H types is unchanged.

$$\text{FOC } e_L \implies (1-\lambda)\pi'(e_L) = 1/\theta_L - \lambda/\theta_H \iff$$
$$\pi'(e_L) = \frac{1}{1-\lambda}\frac{1}{\theta_L} - \frac{\lambda}{1-\lambda}\theta_H$$
$$= 1/\theta_L + \frac{\lambda}{1-\lambda}\frac{1}{\theta_L} - \frac{\lambda}{1-\lambda}\frac{1}{\theta_H}$$
$$= 1/\theta_L + \frac{\lambda}{1-\lambda}\left[1/\theta_L - 1/\theta_H\right]$$
$$= g_e(e_L,\theta_L) + \frac{\lambda}{1-\lambda}\left[g_e(e_L,\theta_L) - g_e(e_L,\theta_H)\right] \iff$$
$$\hat{e}_L^{-1/2} = 1/\theta_L + \frac{\lambda}{1-\lambda}\left[1/\theta_L - 1/\theta_H\right]$$

Given $\lambda \in (0,1)$, $\theta_H > \theta_L$, we have that $\frac{\lambda}{1-\lambda}\left[1/\theta_L - 1/\theta_H\right] > 0$, hence $\hat{e}_L^{-1/2} > \theta_L^{-1} \implies \hat{e}_L < \theta_L^2 = e_L^*$. Optimal effort to require from L types is lower than before.

$$\hat{w}_H = e_H/\theta_H - \hat{e}_L/\theta_H + \hat{e}_L/\theta_L$$
$$= e_H^*/\theta_H + \hat{e}_L(1/\theta_L - 1/\theta_H)$$
$$= w_H^* + \hat{e}_L(1/\theta_L - 1/\theta_H) > w_H^*$$

Optimal wage for H types is higher than before.

$\hat{w}_L = \hat{e}_L / \theta_L < e_L^* / \theta_L = w_L^*$

Optimal wage for L types is lower than before.

iv IC_L: $\hat{e}_H / \theta_L - \hat{e}_L / \theta_L \geq \hat{e}_H / \theta_H - \hat{e}_L / \theta_H$ (given). $\Leftrightarrow$

~~Given that~~

$1/\theta_L \, (\hat{e}_H - \hat{e}_L) \geq 1/\theta_H \, (\hat{e}_H - \hat{e}_L)$

Given that $\theta_L > \theta_H$, ~~it is t~~ it trivially follows that this is satisfied with strict equality.

When types are unobservable, it is optimal to demand undistorted "first-best" effort from H types and distort the effort demanded of L types (adjusting wage for L types accordingly so their PC remains satisfied) so high effort remains optimal for H types. H types are also offered higher wage to preserve incentive compatibility, so H types have positive surplus, L types have zero surplus.