Date

Game Theory Problem Set 2

1 $u_L(x,y) = x/_{x+y} - x$, $u_C(y,x) = y/_{x+y} - y$

~~$B_L(y) = \arg\max_x u_L(x,y)$~~

Taking FOC for x,

$\partial u_L/\partial x = x(-1)(x+y)^{-2} + (1)(x+y)^{-1} - 1 = 0$,

$-x(x+y)^{-2} + (x+y)^{-1} - 1 = 0$,

$-x + (x+y) - (x+y)^2 = 0$,

$(x+y)^2 = y$,

$x+y = \sqrt{y}$ since $x, y > 0$

$x = \sqrt{y} - y$

$B_L(y) = \sqrt{y} - y$

By symmetry, $B_C(x) = \sqrt{x} - x$

Let (x^*, y^*) be a NE. By definition of NE and BR,

$x^* = B_L(y^*)$ and $y^* = B_C(x^*)$,

$x^* = \sqrt{y^*} - y^*$, $y^* = \sqrt{x^*} - x^*$

Suppose that (x^*, y^*) is a symmetric NE, then

$x^* = y^*$,

$y^* = \sqrt{y^*} - y^*$,

$2y^* = \sqrt{y^*}$

$4{y^*}^2 = y^*$,

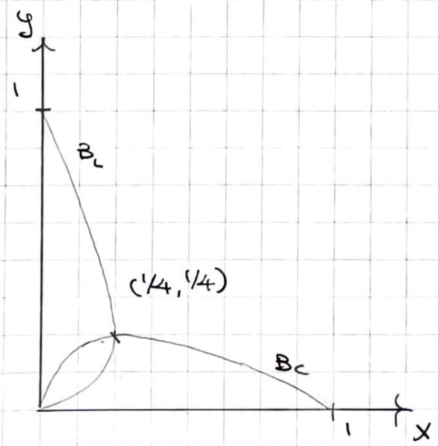
$4y^* = 1$

$y^* = 1/4$.

$x^* = 1/4$

Then, $x^* = \sqrt{y^*} - y^*$ and $y^* = \sqrt{x^*} - x^*$, ~~so this is~~ so $(x^* = 1/4, y^* = 1/4)$ is indeed a NE.

Let V_L and V_C denote the vote shares of the Labour party and the conservative party respectively. At $(x^* = 1/4, y^* = 1/4)$, $V_L = x^*/_{x^*+y^*} = 1/2$, $V_C = y^*/_{x^*+y^*} = 1/2$.



At $(x=0, y=0)$, it is reasonable to suppose that $V_L = V_C = 1/2$ since advertising expenditure is the sole determinant of vote share. Then, it is reasonable to suppose that $u_L(0,0) = V_L(0,0) - 0 = 1/2$ and $u_C(0,0) = V_C(0,0) - 0 = 1/2$. There is no ~~equi~~ equilibrium at these payoffs since candidate equilibrium $(0,0)$ fails to deviation $(\varepsilon, 0)$ for sufficiently small $\varepsilon > 0$ since the deviating player then has vote share 1 and payoff $1-\varepsilon > 1/2$ for $\varepsilon < 1/2$

~~The payoffs (1,1) at (are~~ If each player has payoff 1 at $(x=0, y=0)$, this strategy profile is an NE since ~~a play~~ if some player instead plays z, his payoff is $1-z<1$, so there is no profitable deviation. These payoffs do not make sense since they imply that each player has vote share 1 at $(x=0, y=0)$, but it is reasonable to accept the constraint $V_c + V_L = 1$.

2 $u_A(y_A, y_B) = \begin{cases} X - y_A & \text{if } y_A > y_B \\ X/2 - y_A & \text{if } y_A = y_B \\ -y_A & \text{if } y_A < y_B \end{cases}$

Suppose $y_B = X$, then
$u_A(y_A = 0, y_B) = 0$,
$u_A(y_A \in (0, y_B), y_B) = -y_A < 0$
$u_A(y_A = y_B, y_B) = X/2 - y_A = -X/2 < 0$
$u_A(y_A > y_B, y_B) = X - y_A < 0$
So $B_A(y_B = X) = 0$

Suppose $y_B > X$, then
$u_A(y_A = 0, y_B) = 0$
$u_A(y_A \in (0, y_B), y_B) = -y_A < 0$
$u_A(y_A = y_B, y_B) = X/2 - y_A < 0$
$u_A(y_A > y_B, y_B) = X - y_A < 0$
So $B_A(y_B > X) = 0$

Suppose $y_B < X$, then
$B_A(y_B) \neq y_B$ since ~~these~~ $u_A(y_A < y_B, y_B) = -y_A < 0 <$.
$u_A(y_A \in (y_B, X), y_B) = X - y_A > 0$
$B_A(y_B) \neq y_B$ since $u_A(y_A = y_B, y_B) = X/2 - y_A = X/2 - y_B <$
$u_A(y_A = y_B + \varepsilon, y_B) = X - y_B - \varepsilon$ for sufficiently small
ε $(\varepsilon < X/2)$
$B_A(y_B) \neq y_B$ since $u_A(y_A > y_B, y_B) = X - y_A <$
$u_A(y_A' = y_A - \frac{y_A - y_B}{2}, y_B) = X - y_A'$, i.e. no $y_A > y_B$
maximises u_A since there is always some ~~profitable~~
~~deviat~~ $y_A' \in (y_A, y_B)$ that yields a higher u_A.
So $B_A(y_B) = \emptyset$

$B_A(y_B) = \begin{cases} 0 & \text{if } y_B \geq X \\ \emptyset & \text{if } y_B < X \end{cases}$

Suppose that $\exists$ some pure NE $y^* = (y_A^*, y_B^*)$, then
by definition of NE and BR,
$y_A^* = B_A(y_B^*) = \begin{cases} 0 & \text{if } y_B^* \geq X \\ \emptyset & \text{if } y_B^* < X \end{cases}$
By symmetry, $B_B(y_A) = \begin{cases} 0 & \text{if } y_A \geq X \\ \emptyset & \text{if } y_A < X \end{cases}$
then $y_B^* = B_B(y_A^*) = \begin{cases} 0 & \text{if } y_A^* \geq X \\ \emptyset & \text{if } y_A^* < X \end{cases}$
Given $y_A, y_B \in \mathbb{R}^+$, $y_A^*, y_B^* \neq \emptyset$, so $y_A^* = y_B^* = 0$
Then $y_B^* < X$, so $y_A^* = \emptyset$.
By reductio, $\nexists$ pure NE $y^* = (y_A^*, y_B^*)$.

Suppose that each player plays some common y with positive probability p. Then, ~~so both play~~ ~~both players play~~ with probability p^2, both players play y. Then either player can profitably deviate by reallocating probability mass from y to $y + \varepsilon$ for sufficiently small $\varepsilon > 0$. ~~Then, By so reallocating,~~ By reallocating all probability mass from y to $y + \varepsilon$, this player ~~incurs~~ loses ε with probability p ~~and~~ (when this action is played) but gains $X/2$ with probability p^2 (when the deviating

It feels uncomfortable to work with probability distributions where no action is played with positive probability because when a player does play some action, his doing so is entirely improbable, it had probability zero. So it seems we cannot say such things as "action right above the gap in the support is ~~weakly domin~~ strictly dominated". What sort of language should we use?

player plays $y+\varepsilon$ and the other player plays y, so the asset is awarded to the deviating player with probability 1 rather than with probability $1/2$. So this deviation is profitable iff $p\varepsilon < p^2X/2$, iff $\varepsilon < pX/2$.

Suppose that y^* is a mixed NE where each player plays the mixed strategy given by ~~probab~~ probability distribution $F(y)$.

Suppose that there is a gap $[y_1, y_2]$ in the support of $F(y)$, i.e. neither player plays any action in this interval.

~~Informally~~, Then each player can profitably deviate by reallocating probability mass from the actions ~~just~~ just above the interval to actions ~~at the~~ within and at the bottom of the interval. Informally, the former actions are more costly than the latter, but both have the same probability of winning the contested asset since the other player never makes an intermediate bid. So y^* is not a Nash equilibrium if there is a gap in the support of $F(y)$.

Given that there are no gaps in the support of $F(y)$, the support of $F(y)$ is some interval $[\underline{y}, \bar{y}]$. Suppose that 0 is not in the support of $F(y)$, then $\underline{y} > 0$. $u_x(\underline{y}, y^*_{-x}) = -\underline{y}$ since the other player plays $\underline{y}$ with zero probability, so ~~the~~ the player who plays $\underline{y}$ never wins the asset. $u_x(0, y^*_{-x}) = 0$, so there is a profitable deviation, namely ~~realloc~~ reallocating probability mass from $\underline{y}$ to 0. Then y^* is not a NE. ~~By reduct~~ So if y^* is a NE, 0 is in the support of $F(y)$. Given $y \in \mathbb{R}^+$, 0 is the lowest contribution in the support of $F(y)$.

y^* is a NE ~~iff~~ only if all actions in the support of y^* have equal payoff so there is no profitable deviation by reallocating probability mass from a less profitable action in the mix to a more profitable one. For each player X,

~~$u_x(0$~~ $u_x(0, y^*_{-x}) = u_x(y_x, y^*_{-x})$ for all y_x in the support of $F(y)$

$0 = F(y_x)X - y_x$ ~~for a~~ for all y_x in the support of $F(y)$

$F(y_x) = y_x/X$

$F(y_x)$ is the cumulative distribution function of a uniform distribution $U(0, X)$

The equilibrium strategy of each player X is the mixed strategy given by a uniform distribution over the actions $y_x \in [0, X]$.

3	L	R
T	6	$\underline{8}$
	6	$\underline{2}$
B	$\underline{2}$	0
	$\underline{8}$	0

Best responses underlined.

By inspection, the pure strategy NE are (T,R), (B,L); where players play mutual best responses.

Suppose $\exists$ mixed NE s^*, then Pl.1 mixes T, B, so ~~iff~~ $\pi_1(T, s_2^*) = \pi_1(B, s_2^*)$,

$6p_L + 2p_R = 8p_L$, $2p_R = 2p_L$, $p_L = p_R$, $p_L + p_R = 1$, $p_L = p_R = \frac{1}{2}$

Then Pl.2 mixes L, R, so

$\pi_2(L, s_1^*) = \pi_2(R, s_1^*)$,

$6p_T + 2p_B = 8p_T$, $p_T = p_B = \frac{1}{2}$

$\therefore s^* = (\frac{1}{2}T + \frac{1}{2}B, \frac{1}{2}L + \frac{1}{2}R)$ is the only mixed NE

By inspection, there are no hybrid NE.

~~Neither player has incentive to follow these instructions if H, and Pl.1 is told to play T and Pl.2 is told to play R, Pl.1 has incentive to instead play~~

Each player has incentive to act as instructed given that the other player did so since (T,R) and (B,L) are ~~the~~ NE, so no player has incentive to deviate given that the other player plays his part in the NE. The game resembles a coordination game and the coin toss serves as a coordinating device. Expected payoffs are $(5,5)$

~~Coin~~ State	Strategy Profile	Probability
x	(T,R)	$\frac{1}{3}$
y	(T,L)	$\frac{1}{3}$
z	(B,L)	$\frac{1}{3}$

~~Suppose Pl.1 is instructed to play T, then let s^* denote this correlated equilibrium~~

~~$E(\pi_1(T$~~ Suppose Pl.1 is instructed to play T, then Pl.1 has incentive to play T iff his expected payoff, given this instruction, from playing T is greater than from playing B.

~~$\frac{1}{2}6 + \frac{1}{2}2 \geq \frac{1}{2}8$~~

~~$\frac{1}{2}6 + \frac{1}{2}2 = 4 \geq \frac{1}{2}8 + \frac{1}{2}0 = 4$~~

~~$P(L|T)$~~ $6P(L|T) + 2P(R|T) = \frac{1}{2}\times 6 + \frac{1}{2}\times 2 = 4$

$\geq 8P(L|T) + 0P(R|T) = \frac{1}{2}\times 8 + \frac{1}{2}\times 0 = 4$

Likewise if Pl.1 is instructed to play B

$8P(L|B) = 8 \geq 6P(L|B) + 2P(R|B) = 6$

Likewise if Pl.2 is instructed to play L

$6P(T|L) + 2P(B|L) = \frac{1}{2}\times 6 + \frac{1}{2}\times 2 = 4 \geq$

$8P(\cancel{\#}\, T|L) = \frac{1}{2}\times 8 = 4$

Likewise if Pl.2 is instructed to play R
$8P(T|R) = 8 \geq 6P(T|R) + 2P(B|R) = 6$

~~Ver~~ Verified above that each player finds it weakly optimal to play as instructed.

If x, ~~the, it~~ players play (T,R), $(u_1,u_2)=(2,8)$
If y, players play (T,L), $(u_1,u_2)=(6,6)$
If z, players play (B,L), $(u_1,u_2)=(8,2)$
$(E(u_1), E(u_2)) = (\frac{1}{3}\times 2+\frac{1}{3}\times 6+\frac{1}{3}\times 8,\ \cancel{\frac{1}{3}\times 8+\frac{1}{3}\times 6}$
$\frac{1}{3}\times 8+\frac{1}{3}\times 6+\frac{1}{3}\times 2) = (\frac{16}{3}, \frac{16}{3})$

Expected payoffs increase compared to the earlier correlated equilibrium since (T,L) which has higher ~~average~~ payoff ~~to~~ to each player than the average of (T,R) and (B,L) is ~~mixed in th~~ "mixed in" in this latter correlated equilibrium. ~~This mixing remains playing~~ Each player's playing his part of (T,L) when so instructed is incentive compatible because each player, if he does not, risks receiving zero payoff due to miscoordination. In other words, because each player does not know what the other has been instructed to play, each player finds it weakly optimal to play his part in (T,L) when so instructed, even though this would not be optimal if each player knew that the other was ~~is~~ similarly instructed.

Correlated mixing of strategies seems inconsistent with the "individualism" of non-cooperative game theory. Correlated equilibrium can be interpreted as describing an equilibrium where players have access to some coordinating device like a traffic light or a ~~distributed~~ computer. ~~Ex~~

4 $P(c_j=L \mid c_i=L) = P(c_j=L \cap c_i=L)/P(c_i=L) = \alpha/\tfrac{1}{2} = 2\alpha$

$P(c_j=H \mid c_i=L) = 1-P(c_j=L \mid c_i=L) = 1-2\alpha$

By symmetry,

$P(c_j=H \mid c_i=H) = 2\alpha$, $P(c_j=L \mid c_i=H) = 1-2\alpha$

~~Firm i believes, a posteriori, that firm j has the same cost with probability~~ 2α and a ~~differ~~

The strategy of each player i is the pair (σ_i^L, σ_i^H) where σ_i^x is the quantity player i chooses, or some probability distribution over quantities if firm i finds that it ~~is a f~~ has cost c_x.

The (ex post) payoff of each firm is a function of its quantity, its opponents quantity, and its type

$\pi_i = (1-q_i-q_j-c_i)q_i$

The interim payoff of each firm is a function of ~~its quantity and its type, its opponents c.~~

~~$\pi_{iL} = 2\alpha[(1-$~~

its quantity, its opponents quantity, and its type

$\pi_{iH} = 2\alpha[(1-q_{iH}-q_{jH}-c_H)q_{iH}]$
$\quad + (1-2\alpha)[1-q_{iH}-q_{jL}-c_H)q_{iH}]$
$\quad = \cancel{[(1-q_{iH}-c_H)q_{iH}]}$
$\quad (1-q_{iH}-c_H)q_{iH} - 2\alpha q_{jH}q_{iH} - (1-2\alpha)q_{jL}q_{iH}$

$\pi_{iL} = 2\alpha[(1-q_{iL}-q_{jL}-c_L)q_{iL}]$
$\quad + (1-2\alpha)[(1-q_{iL}-q_{jH}-c_L)q_{iL}]$
$\quad = (1-q_{iL}-c_L)q_{iL} - 2\alpha q_{jL}q_{iL} - (1-2\alpha)q_{jH}q_{iL}$

Taking FOCs

$\partial\pi_{iH}/\partial q_{iH} = (1-q_{iH}-c_H) - q_{iH} - 2\alpha q_{jH} - (1-2\alpha)q_{jL} = 0$,
$q_{iH} = \tfrac{1}{2}[1-c_H-2\alpha q_{jH}-(1-2\alpha)q_{jL}]$
$\partial\pi_{iL}/\partial q_{iL} = (1-q_{iL}-c_L) - q_{iL} - 2\alpha q_{jL} - (1-2\alpha)q_{jH} = 0$,
$q_{iL} = \tfrac{1}{2}[1-c_L-2\alpha q_{jL}-(1-2\alpha)q_{jH}]$

~~By symmetry, $q_{iH}=q_{jH}$, $q_{iL}=q_{jL}$~~
~~$q_{jH} = \tfrac{1}{2}[1-c_H-2\alpha q_{iH}-(1-2\alpha)q_{iL}]$~~
~~$q_{jL} = \tfrac{1}{2}[1-c_L-2\alpha q_{iL}-(1-2\alpha)q_{iH}]$~~

By substitution,
~~$q_{iH} = \tfrac{1}{2}(1-c_H) - \tfrac{2\alpha}{4}[1-c_H-2\alpha q_{iH}-(1-2\alpha)q_{iL}]$~~
~~$\quad - \tfrac{1-2\alpha}{4}[1-c_L-2\alpha q_{iL}-(1-2\alpha)q_{iH}]$~~
~~$\quad = \tfrac{1}{2}(1-c_H) - \tfrac{2\alpha}{4}(1-c_H) - \tfrac{1-2\alpha}{4}(1-c_H)$~~
~~$\quad + \tfrac{2\alpha}{4}[2\alpha q_{iH}+(1-2\alpha)q_{iL}]$~~
~~$\quad + \tfrac{1-2\alpha}{4}[2\alpha q_{iL}+(1-2\alpha)q_{iH}]$~~
~~$\quad = \tfrac{1}{4}(1-c_H) + \tfrac{1}{4}[2\alpha(2\alpha q_{iH}) + 2\alpha(1-2\alpha)q_{iL}$~~
~~$\quad + 2\alpha(1-2\alpha)q_{iL} + (1-2\alpha)(1-2\alpha)q_{iH}]$~~
~~$\quad = \tfrac{1}{4}(1-c_H) + \tfrac{1}{4}[$~~

This solution seems too convoluted, but would it have gotten the right answer? (substituting BR into BR)

~~Suppose that the equilibrium is symmetric, then~~
~~$q_{iH} = \tfrac{1}{2}[1-c_H-2\alpha q_{iH}-(1-2\alpha)q_{iL}]$~~
~~$(1+\alpha)q_{iH} = \tfrac{1}{2}[1-c_H] - \tfrac{1-2\alpha}{2}q_{iL}$~~
~~$(1+\alpha)q_{iL} = \tfrac{1}{2}[1-c_L] - \tfrac{1-2\alpha}{2}q_{iH}$~~
~~$\quad = \tfrac{1}{2}[1-c_L] - (\tfrac{1-2\alpha}{2})(\tfrac{1}{1+\alpha})[\tfrac{1}{2}(1-c_H) - \tfrac{1-2\alpha}{2}q_{iL}]$~~

why would α matter if interim BNE is equivalent to ex ante BNE?