Game Theory Paper 200604

1a Suppose $\tau = 1$

$\pi_i(s_i = 1; m, \tau = 1) = 10 - (1-\lambda)(1-m)^2 - \lambda(1-1)^2$
$= 10 - (1-\lambda)(1-m)^2$

$\pi_i(s_i = 0; m, \tau = 1) = 10 - (1-\lambda)(0-m)^2 - \lambda(0-1)^2$
$= 10 - (1-\lambda)m^2 - \lambda$

$s_i = 1$ is a BR iff
$\pi_i(s_i=1; m, \tau=1) \geq \pi_i(s_i=0; m, \tau=1) \iff$
$10 - (1-\lambda)(1-m)^2 \geq 10 - (1-\lambda)m^2 - \lambda \iff$
$(1-\lambda)(1-m)^2 \leq (1-\lambda)m^2 + \lambda \iff$
$(1-\lambda)(-2m+1) \leq \lambda \iff$
$1 - 2m \leq \lambda/(1-\lambda) \iff$ $\quad 1-2m \leq \lambda(2-2m)$
$m \geq (1-2\lambda)/(2-2\lambda)$ $\quad \lambda \geq (1-2m)/(2-2m)$

$s_i = 0$ is a BR iff
$\pi_i(s_i=0; m, \tau=0) \geq \pi_i(s_i=1; m, \tau=1) \iff$
$m \leq (1-2\lambda)/(2-2\lambda)$

$m=1$ corresponds to a NE iff $s_i = 1$ is a BR, ~~iff~~ which is iff $1 \geq (1-2\lambda)/(2-2\lambda) \iff 2-2\lambda \geq 1-2\lambda \iff 1 \geq 0$. ~~so~~ so $m=1$ is ~~a NE for all λ~~ corresponds to a NE for all λ.

$m=0$ corresponds to a NE iff $s_i = 0$ is a BR, which is iff $0 \leq (1-2\lambda)/(2-2\lambda) \iff 1-2\lambda \geq 0$ (given $\lambda \in (0,1)$) $\iff \lambda \leq 1/2$. So $m=0$ corresponds to a NE for $\lambda \leq 1/2$.

$m \in (0,1)$ corresponds to a NE iff both $s_i=1$ and $s_i=0$ are BRs, which is iff $m = (1-2\lambda)/(2-2\lambda) \iff$ ~~$1-2m = \lambda/(1-\lambda)$ given $\lambda, m \in (0,1)$,~~ $\lambda = (1-2m)/(2-2m)$. Given that $\lambda, m \in (0,1)$, $m < 1/2$ and $\lambda < 1/2$. So $m = (1-2\lambda)/(2-2\lambda)$ corresponds to a NE for $\lambda < 1/2$.

Suppose $\tau = 0$

$\pi_i(s_i = 1; m, \tau = 0) = 10 - (1-\lambda)(1-m)^2 - \lambda(1-0)^2$
$= 10 - (1-\lambda)(1-m)^2 - \lambda$

$\pi_i(s_i = 0; m, \tau = 0) = 10 - (1-\lambda)(0-m)^2 - \lambda(0-0)^2$
$= 10 - (1-\lambda)m^2$

$s_i = 1$ is a BR iff
$10 - (1-\lambda)(1-m)^2 - \lambda \geq 10 - (1-\lambda)m^2 \iff$
$(1-\lambda)(1-m)^2 + \lambda \leq (1-\lambda)m^2 \iff$
$\lambda \leq (1-\lambda)(2m-1) \iff$
~~$(1-2m)$~~ $(2m-1) \geq \lambda/(1-\lambda) \iff$
$m \geq 1/(2-2\lambda) \iff$
$2 - 2\lambda \geq 1/m \iff$
$2 - 1/m \geq 2\lambda \iff$
$\lambda \leq 1 - 1/(2m) = (2m-1)/(2m)$

$s_i = 0$ is a BR iff
~~$\lambda\geq$~~ $m \leq 1/(2-2\lambda) \iff \lambda \leq (2m-1)/(2m)$

$m=1$ corresponds to a NE iff $s_i=1$ is a BR, which is iff $\lambda \leq (2-1)/2 = 1/2$

$m=0$ corresponds to a ~~BR~~ NE iff $s_i = 0$ is a BR, which is iff $0 \leq 1/(2-2\lambda)$, which holds for all $\lambda \in (0,1)$, so $m=0$ always corresponds to a NE.

$m \in (0,1)$ corresponds to a NE iff both $s_i=0$ and $s_i=1$ are BR, which is iff $m = 1/(2-2\lambda)$, $\lambda = (2m-1)/(2m)$. Given that $\lambda, m \in (0,1)$, this is iff $\lambda < 1/2$ and $m = 1/(2-2\lambda)$.

b From (a), if $\tau = 1$, for all λ, $m=1$ is a NE, for $\lambda \leq 1/2$, $m=0$ is a NE, and for $\lambda < 1/2$, $m = (1-2\lambda)/(2-2\lambda)$ is a NE. If $\tau = 0$, for all λ, $m=0$ is a NE, for $\lambda \leq 1/2$, $m=1$ is a NE, and for $\lambda < 1/2$, $m = 1/(2-2\lambda)$ is a NE.

So for $\tau = 0$, regardless of λ, payoff $u = 0$. For $\tau = 1$, if $\lambda \leq 1/2$, payoff $u=0$, if $\lambda > 1/2$, payoff $u=1$. It is strictly profitable to change the commonly known value of τ from 0 to 1 iff $\lambda > 1/2$.

c $\pi_i'(s_i=1; m, \tau=0) = 11.5 - (1-\lambda)(1-m)^2 - \lambda$
$\pi_i'(s_i=0; m, \tau=0) = 10 - (1-\lambda)m^2$
$\pi_i'(s_i=1; m, \tau=0) \geq \pi_i'(s_i=0; m, \tau=0) \iff$
$11.5 - (1-\lambda)(1-m)^2 - \lambda \geq 10 - (1-\lambda)m^2 \iff$
$1.5 \geq (1-\lambda)(-2m+1) + \lambda = 1 - (1-\lambda)\cdot 2m$
which always holds given $m \in [0,1]$ and $\lambda \in (0,1)$ ~~because the RHS is some weighted average of 1 an~~
$s_i=1$ is strictly dominant for the modified players. Supposing that these players are rational, they always play $s_i=1$.

d $\tau = 0$, $\lambda = 1/4$

~~From (a), (a),~~

~~for $v <$~~ $s_i=1$ is a BR for unmodified players iff $m \geq 1/(2-2\lambda)$, and is strictly so iff $m > 1/(2-2\lambda) = 1/(2-2/4) = 7/12$, so only if $v > 7/12$ is the manager's gross (of modification cost) payoff 1. Otherwise, it is v ($s_i = 0$ is a BR for unmodified players, so the "worst" NE is such that only modified players play ~~$s=$~~ $s=1$). ~~In this~~ For $v > 7/12$, the manager has ~~gross~~ net payoff $1 - 1.5v < 1/8$. For $v \leq 7/12$, net payoff is $v - 1.5v = -0.5v < 0$. So modifying the payoffs of v players where

v is arbitrarily higher than $7/12$ is strictly profitable for the merger.

2a R's pure actions $A_R = \{S1, S2, S3\}$, where S1, S2, S3 respectively denote strike city 1, city 2, and city 3. C's pure actions are $A_C = \{$ ~~&~~ D1, D2, D3 $\}$, where D1, D2, D3 ~~re~~ respectively denote defend city 1, city 2, and city 3. Each player's mixed strategies are ~~that players~~ the probability distributions over that player's pure actions.

	D1	D2	D3
S1	$\underline{0}$, 0	$-V_1$, $\underline{V_1}$	$-V_1$, $\underline{V_1}$
S2	$-V_2$, $\underline{V_2}$	$\underline{0}$, 0	$-V_2$, V_2
S3	$-V_3$, V_3	$-V_3$, V_3	$\underline{0}$, 0

Best responses underlined

b ~~Supp~~ By inspection of the payoff matrix, there are no pure NE at which players play pure mutual best responses.

Suppose P1 plays pure S1 at NE, then by definition of NE, P2 plays BR D1, then P1 plays BR S2. By reductio, P1 does not play pure S1 at NE.

Similarly, ~~P1~~ if P1 plays pure S2 at NE, P2 plays BR D2, then P1 plays BR S1, so by reductio, P1 does not play pure S2 at NE. And if P1 plays pure S3 at NE, P2 plays BR D3, then P1 plays BR S1, so by reductio, P1 does not play pure S3 at NE.

So P1 does not play a pure strategy at NE, there is no pure NE.

c Suppose ~~p1=~~ $p_1 = 0$, then ~~$\pi_2(\sigma$ $\pi_2(D1, \sigma^*_1) = -p_2V_2 - p_3V_3 < \pi_2(\frac{1}{2}D_2 + \frac{1}{2}D_3, \sigma^*) = \frac{1}{2}(-p_3V_3) + \frac{1}{2}(-p_2V_2) = \frac{1}{2}(-p_2V_2 - p_3V_3)$~~, ~~i.e. so D1 is not a best response P2~~ $q_1 = 0$ because if $q_1 > 0$ then P2 has profitable deviation by reallocating ~~prob~~ probability mass from D1 ~~to~~ evenly between D2 and D3. Then $\pi_1(S1, \sigma^*_2) = V_1$, i.e. S1 yields V_1 with certainty because D1 is played with zero probability. $\pi_2(S2, \sigma^*_2) \le V_2$, i.e. S2 yields no more than V_2. So if, additionally, ~~P1~~ $p_2 > 0$, then P1 has profitable deviation by reallocating probability mass from S2 to S1. So $p_2 = 0$ (in fact, by analogous argument, if $p_1 = 0$ then $p_3 = 0$, so $p_1 \neq 0$).

Suppose $p_2 = 0$. Then $q_2 = 0$ because otherwise P2 has strictly profitable deviation by reallocating probability mass ~~evenly between~~ from D2 evenly to D1 and D3. Then ~~$\pi_2($~~ $\pi_1(S2, \sigma^*_2) = V_2$, i.e. ~~S2~~ S2 yields V_2 with certainty because city 2 is undefended. $\pi_1(S3, \sigma^*_2) \le V_3$. So if $p_3 > 0$, P1 has strictly profitable deviation by reallocating probability mass from S3 to S2. So if $p_2 = 0$ then also $p_3 = 0$.

d By the second result (if $p_2 = 0$ then $p_3 = 0$) ~~if~~ it is not the case that $p_1, p_3 > 0$ and $p_1 + p_3 = 1$ because this implies $p_2 = 0$ hence $p_3 = 0$, which is a contradiction.

By the first result (if $p_1 = 0$ then $p_2 = 0$), it is not the case that $p_2, p_3 > 0$ and $p_2 + p_3 = 1$ because this implies $p_1 = 0$ hence $p_2 = 0$, which is a contradiction.

e Suppose that ~~at NE~~, $p_1, p_2 > 0$ and ~~p1+~~ $p_1 + p_2 = 1$, then by definition of NE, P1 has no profitable deviation, which is iff

$\pi_1(S1, \sigma^*_2) = \pi_1(S2, \sigma^*_2) \ge \pi_1(S3, \sigma^*_2) \Leftrightarrow$

$(1-q_1)V_1 = (1-q_2)V_2 \ge (1-q_3)V_3 \Leftrightarrow$

$(1-q_1)V_1 = (1-q_2)V_2 \ge (q_1+q_2)V_3$

This also requires that P2 has no profitable deviation, which implies ~~$p_3=0$~~ $q_3 = 0$ (otherwise there is a profitable deviation ~~fo~~ for P2 by reallocating probability mass from D3 evenly to D1 and D2). So $q_1 + q_2 = 1$

By substitution,

$(1-q_1)V_1 = q_1V_2 \ge V_3 \Rightarrow$

$q_1 = V_1/V_1+V_2$, $q_2 = V_2/V_1+V_2$, $V_1V_2/V_1+V_2 \ge V_3$

Then, given $V_1, V_2 > 0$, ~~P2 plays ... and mixes between D1 and~~ $q_1, q_2 > 0$, P2 mixes over these at NE, ~~so~~ and has no profitable deviation, so

$\pi_2(D1, \sigma^*_1) = \pi_2(D2, \sigma^*_1) \ge \pi_2(D3, \sigma^*_1) \Leftrightarrow$

$-p_2V_2 - p_3V_3 = -p_1V_1 - p_3V_3 \ge -p_1V_1 - p_2V_2 \Leftrightarrow$

$p_1V_1 = p_2V_2$, $p_3 = 0 \Rightarrow p_1 = V_2/V_1+V_2$, $p_2 = V_1/V_1+V_2$

So if $V_1V_2/V_1+V_2 \ge V_3$ then $p_1^* = V_2/V_1+V_2$, $p_2 =$ ~~V_2/V_1~~ V_1/V_1+V_2, $p_3 = 0$, $q_1 = V_1/V_1+V_2$, $q_2 = V_2/V_1+V_2$, $q_3 = 0$ is a NE.

f $p_1, p_2, p_3 > 0$ ~~at NE~~ only if P1 is indifferent

$\pi_1(S1, \sigma^*_2) = \pi_1(S2, \sigma^*_2) = \pi_1(S3, \sigma^*_2) \Leftrightarrow$

$(1-q_1)V_1 = (1-q_2)V_2 = (1-q_3)V_3 \Leftrightarrow$

$(1-q_1)V_1 = (1-q_2)V_2 = (q_1+q_2)V_3$

$q_1, q_2 > 0$ (otherwise P1 has profitable deviation

$\pi_2(D_1, \sigma_1^*) = \pi_2(D_2, \sigma_2^*) \iff$

$-p_2 v_2 - p_3 v_3 = -p_1 v_1 - p_3 v_3 \iff$

$p_1 v_1 = p_2 v_2 \iff$

$p_1 = v_2/v_1 \, p_2$

let $r_m = 1 - q_m$

$r_1 v_1 = r_2 v_2 = (2 - r_1 - r_2) v_3 \iff$

$r_2 = r_1 v_1 / v_2 \, , \, (2 - r_1 - r_1 v_1/v_2) v_3 = r_1 v_1 \iff$

$r_2 = r_1 v_1 / v_2 \, , \, 2v_3 - r_1 v_3 - r_1 v_1 v_3 / v_2 = r_1 v_1 \iff$

$r_2 = r_1 v_1 / v_2 \, , \, r_1 = 2v_3 / v_3 + v_1 v_3 / v_2 + v_1 \cancel{= 2v_2 v_3/}$

$r_1 = 2v_2 v_3 / v_1 v_2 + v_2 v_3 + v_1 v_3 \iff$

$r_2 = 2v_1 v_3 / v_1 v_2 + v_2 v_3 + v_1 v_3 \iff$

$q_1 = v_1 v_2 - v_2 v_3 + v_3 v_1 / v_1 v_2 + v_2 v_3 + v_1 v_3$

$q_2 = v_1 v_2 + v_2 v_3 - v_3 v_1 / v_1 v_2 + v_2 v_3 + v_1 v_3$

$q_3 = -v_1 v_2 + v_2 v_3 + v_3 v_1 / v_1 v_2 + v_2 v_3 + v_1 v_3$

$q_1, q_2, q_3 \geq 0$, which is implied by $q_3 \geq 0$, which requires $v_2 v_3 + v_1 v_3 \geq v_1 v_2$

3a Players : $N = \{1, 2\}$, the two armies

Actions : $A_i = \{A, N\}$, ~~for Attack and Not~~ denoting Attack and Not Attack, for $i \in N$.

States : $\Omega = \{SS, SW, WS, WW\}$, denoting strong 1 strong 2, strong 1 weak 2, and so on

Signals (types): $t_1 = \tau_1(\omega) = \begin{cases} S & \text{if } \omega = SS \text{ or } SW \\ W & \text{if } \omega = WS \text{ or } WW \end{cases}$

$t_2 = \tau_2(\omega) = \begin{cases} S & \text{if } \omega = SS \text{ or } WS \\ W & \text{if } \omega = SW \text{ or } WW \end{cases}$

Beliefs: $P_1\{t_2 = S \mid t_1\} = P_1\{t_2 = W \mid t_1\} = 1/2$

$P_2\{t_1 = S \mid t_2\} = P_2\{t_1 = W \mid t_2\} = 1/2$

Payoffs: $\pi_i(s_i, s_{-i} \mid t_i, t_{-i})$

$$= \begin{cases} M & \text{if } s_i(t_i) = A \text{ and } s_{-i}(t_{-i}) = N \\ M-s & \text{if } s_i(t_i) = A,\ s_{-i}(t_{-i}) = A,\ t_i = S,\ t_{-i} = W \\ -s & \text{if } s_i(t_i) = A,\ s_{-i}(t_{-i}) = A,\ t_i = S,\ t_{-i} = S \\ -w & \text{if } s_i(t_i) = A,\ s_{-i}(t_{-i}) = A,\ t_i = W \\ 0 & \text{if } s_i(t_i) = N \end{cases}$$

b Each army's strategy is a type contingent plan of action. Each has pure strategies AA, AN, NA, NN, which respectively denote Attack if Strong and Attack if Weak, Attack if Strong and Not if Weak, ...

	~~AA~~	~~AN~~	~~NA~~	~~NN~~
~~AA~~	~~$M/2-s, -w$~~	~~$M/2-s, 0$~~	~~$0, -w$~~	~~$0,0$~~
~~AN~~	~~$\frac{M}{2}-\frac{s}{2}, \frac{M}{2}-\frac{w}{2}$~~	~~$\frac{M}{2}-\frac{s}{2}, 0$~~	~~$0, \frac{M-w}{2}\frac{}{2}$~~	~~$0,0$~~
~~NA~~	~~$M-\frac{s}{2}, \frac{M}{2}-\frac{w}{2}$~~	~~$M-\frac{s}{2}, 0$~~	~~$0, \frac{M}{2}-\frac{w}{2}$~~	~~$0,0$~~
~~NN~~	~~M, M~~	~~$M,0$~~	~~$0,M$~~	~~$0,0$~~

The game is symmetric, so we list only Row payoffs. Strong type payoffs listed first. Payoffs in interim expectation.

	AA	AN	NA	~~A~~ NN
NA	$\underline{\frac{M}{2}-s}, -w$	$\underline{\underline{\frac{M}{2}-\frac{s}{2}}}, \underline{\frac{M}{2}-\frac{w}{2}}$	$\underline{\underline{M-\frac{s}{2}}}, \underline{\frac{M}{2}-\frac{w}{2}}$	$\underline{\underline{M}}, \underline{\underline{M}}$
AN	$\underline{\frac{M}{2}-s}, \underline{\underline{0}}$	$\underline{\underline{\frac{M}{2}-\frac{s}{2}}}, \underline{0}$	$\underline{\underline{M-\frac{s}{2}}}, \underline{0}$	$\underline{\underline{M}}, 0$
NA	$0, -w$	$0, \underline{\frac{M}{2}-\frac{w}{2}}$	$0, \underline{\frac{M}{2}-\frac{w}{2}}$	$0, \underline{\underline{M}}$
NN	$0, \underline{\underline{0}}$	$0, \underline{0}$	$0, \underline{0}$	$0,0$

b Given $M > \max\{2s, w\} \Rightarrow M > 2s,\ M > w$

Best responses underlined. By inspection, the only pure BNE where players play pure mutual best responses in interim expectation ~~is (AN, AN)~~ are (AN, AA) and (AA, AN).

c The candidate pure symmetric BNE are (AA, AA), (AN, AN), (NA, NA), (NN, NN). ~~It is given~~

Given $w > 0$, we have $-w < 0$, so AA is never a best response against AA for weak types. (AA, AA) is never a BNE.

Given $M > 0$, ~~AA~~ NN is never a best ~~res~~ response against NN for either type, so (NN, NN) is ~~not~~ never a BNE.

NA is a best response against NA iff $0 \geq M - s/2 \Rightarrow M \leq s/2$, and $\frac{M}{2} - \frac{w}{2} \geq 0 \Rightarrow M \geq w$, which cannot both hold given $w > s$, so (NA, NA) is never a BNE.

There is a unique BNE and it is symmetric iff (AN, AN) is the unique BNE.

(AN, AN) is a BNE iff $M - s \geq 0$ and $M - w < 0$

By inspection of the ~~b~~ payoff matrix, given these conditions, we additionally require that (AA, NN) and (NN, AA) are not BNE, which is if, in addition, $\frac{M}{2} - s \geq 0$, $M > 2s$

So the conditions are $M - 2s > 0$, $M - w < 0$

d

4a Each firm's strategy is some history-contingent plan of action. The relevant actions for each firm are P (produce) and E (exit). Each firm's strategy is effectively some plan of which period ~~to~~ in which to exit contingent on the other firm's action.

So firm i's strategy has the following form: X_1 in $t=1$, then if firm j E in ~~period~~ $t=1$, P until t', otherwise X_2 in $t=2$, then if firm j E in $t=2$, P until t^2, otherwise X_3 in $t=3$, and so on.

Data

5a Given ~~a~~ a bargaining problem (U, d), where U is some set of possible agreement ~~payoffs~~ payoff ~~and~~ vectors and d is the disagreement payoff vector, the Nash bargaining ~~solution~~ solution is ~~$F(U,d)$~~

$$F(U, d) = \arg\max_{u \in U} \prod_i (u_i - d_i)$$

The axioms that characterise the Nash bargaining solution are weak Pareto efficiency (WPAR), symmetry (SYM), invariance to equivalent payoff representations (INV) and independence of irrelevant alternatives (IIA).

~~WPAR : for all $\nexists u' \in U$~~

~~G Given bargaining problem (U,d) and solution $u^* = F(U,d)$~~

Bargaining solution $F(U, d) = u^*$ satisfies:

WPAR iff $\nexists u' \in U : \forall i : u_i' > u^*$, i.e. there is no other agreement payoff vector that leaves every party better off.

SYM iff ~~supp~~ if U is symmetric and d is symmetric, then $\forall i,j$ $u_i = u_j$.

INV iff for all affine transformations f, ~~$\Leftarrow$~~ $F(U', d') = u'$ where $U' = \{f(u) : u \in U\}$, $d' =$ ~~$f(d)$~~ $(f(d_1), \ldots, f(d_n))$, and $u' = (f(u_1), \ldots, f(u_n))$.

IIA iff for all $U' \subseteq U$ such that $u^* \in U'$, $F(U', d) = u^*$

b $G(U, d) = \arg\max_{u \in U}$ ~~$(u_1 - d_1)$~~ u_1

satisfies all axioms except ~~SYM~~ SYM.

WPAR is satisfied because player 1 cannot be made strictly better off, INV is satisfied because ~~max~~ the maximum ~~of $u_1 - d_1$ is preserved~~ element of a set is unchanged by affine transformations, and IIA is satisfied because the maximum element of a set is ~~independent~~ the same in all subsets containing that element.

c $\max_{w,L} [(wL + R(\bar{L} - L)) - R\bar{L}][f(L) - wL]$

$= \max_{w,L} [(w - R)L][f(L) - wL]$

$=$ ~~max~~

FOC_w: $[(w-R)L](-L) + (L)[f(L) - wL] = 0 \Rightarrow$

$-wL^2 + RL^2 + Lf(L) - wL = 0 \Rightarrow$

$w(L + L^2) = RL^2 + Lf(L) \Rightarrow$

$w = (RL^2 + Lf(L)) / (L + L^2)$

FOC_L: $(w-R)[f(L) - wL] + [(w-R)L](f'(L) - w) = 0$

$\Rightarrow$

$f(L) - wL + Lf'(L) - wL = 0 \Rightarrow$

$2wL = f(L) + Lf'(L) \Rightarrow$

$w = \frac{1}{2}\left[f(L)/L + f'(L) \right]$

$FOC_w \Rightarrow$

$RL^2 = wL + wL^2 - Lf(L) \Rightarrow$

$R = w/L + w - f(L)/L$

Date

6a. In a two player symmetric game, strategy σ^* is an ESS iff (σ^*, σ^*) is a NE and there is no strategy $\sigma' \neq \sigma^*$ such that both (1) σ' is a BR against σ^*, and (2) ~~a~~ $\pi(\sigma', \sigma') \geq \pi(\sigma^*, \sigma')$.

This captures the idea of stability against invasion by mutants. A population of σ^* players is stable against invasion by mutant σ' players iff σ^* is a BR against σ^*, and ~~there is no mu~~ either σ' fares worse than σ^* against σ^* or σ' fares worse than σ^* against σ', so σ' fares worse than σ^* on avge upon invasion and invasion is unsuccessful.

b

	C	R
C	$\underline{a}$, $\underline{a}$	0, c
R	c, 0	$\underline{b}$, $\underline{b}$

$0<b<a$, $0<c<a$.

BRs underlined. By inspection, there are two pure NE, (C, C) and (R, R) where players play pure mutual BRs. Each of these is a strict symmetric NE, so each of these corresponds to an ESS. In other words, at each NE, only the strategy played is a BR, so no other strategy satisfies (1), so the condition for an ESS is satisfied. C and R are ESS.

There is no mixed ESS. At the mixed NE, by definition, each mixing player has no ~~profitable~~ π able devn and so must be indiff between the two pure actions, so C is a BR against any mixed candidate ~~ESS~~. (C, C) maximises each player's payoff, so condition (2) is satisfied for C. C is a ~~suc~~ successful mutant against any candidate mixed ESS, so there is no mixed ESS.

The ESS are C and R.

c Under the replicator dynamic, the ~~numb~~ number of X players ~~in~~ grows in each period by an amount proportionate to the number of X players in that period and the avge payoff of X players in that period. So the proportion of X players in each period grows by an amount that is proportionate to the proportion of X players and the difference between the avge payoff to an X player and the avge payoff in the population.

The replicator dynamic predicts the evolution of play well if ~~the same player's remaining in the~~ payoffs are a determinant of or correlated with persistence of a player in the game. Then, only players who fare well will remain in the game in the long run. The replicator dynamic is also useful in games played by large populations.

$$\begin{aligned}\dot p_C &= p_C([Ap]_C - p^T Ap)\\ &= p_C([Ap]_C - (p_C[Ap]_C + (1-p_C)[Ap]_H))\\ &= p_C(1-p_C)([Ap]_C - [Ap]_H)\\ &= p_C(1-p_C)(ap_C - (cp_C + b(1-p_C)))\\ &= p_C(1-p_C)((a+b-c)p_C - b)\end{aligned}$$

where $A = \begin{pmatrix} a & 0 \\ c & b \end{pmatrix}$ is the row payoff matrix and $p = \begin{pmatrix} p_C \\ p_R \end{pmatrix}$ is the player proportions vector.

$\dot p_C = 0 \iff p_C = 0$, $p_C = 1$, or $p_C = b/a+b-c$
$p_C \in (0, b/a+b-c) \Rightarrow \dot p_C < 0$
$p_C \in (1, b/a+b-c) \Rightarrow \dot p_C > 0$

There are three absorbing states, $p_C = 0$, $p_C =$ and $p_C = b/a+b-c$. Only the former two are stable. A displacement from the latter results in the system converging to one of the other two absorbing states. The $p_C = 0$ state has basin of attraction $[0, b/a+b-c)$ and the $p_C = 1$ state has basin of attraction $(b/a+b-c, 1]$. If the initial state is in ~~$\in$~~ $[0, b/a+b-c)$, the system converges to the $p_C = 0$ state. If it is in $(b/a+b-c, 1]$ it converges to the $p_C = 1$ state. The two stable absorbing states each corresponds to an ESS. Each absorbing state corresponds to a NE.

d Let $\sigma = p_C C + (1-p_C)R$ denote the strategy "played by" the population.

$\pi(C, \sigma) = ap_C$
$\pi(R, \sigma) = cp_C + b(1-p_C)$
$\pi(C, \sigma) \geq \pi(R, \sigma) \iff$
$ap_C \geq \cancel{b(1-} \; cp_C + b(1-p_C) \iff$
$\cancel{p_C \leq b/a+b-c} \; p_C \geq b/a+b-c$

C is a BR against the population iff p_C

$\geq b/a+b-c$. R is a BR against the population iff $p_C \leq b/a+b-c$.

From the $p_C=0$ state, the minimum number of non-best-responses necessary to evolve to the $p_C=1$ state is ~~b~~ $N\,b/a+b-c$, this ~~or~~ is when $N\,b/a+b-c$ R-players switch to C. From the $p_C=1$ state, the minimum number of non-best-response necessary to evolve to the $p_C=0$ state is $N(1-b/a+b-c) = N\,a-c/a+b-c$, this is when that many C players switch to R.

For small ε, the probability of such evolutions is dominated by the $(\varepsilon/2)^m$ term, where m is the minimum number of non-BRs that must be chosen.

So if $b > a-c$, then evolution from the $p_C=0$ state to the $p_C=1$ state is ~~more~~ less likely than in the other direction. The system ~~spends~~ is at or around one of these two absorbing states in the vast majority of periods and at or around the $p_C=1$ state ~~more~~ much more frequently than it is at or around the $p_C=0$ state. The two absorbing states correspond to ESS and the $p_C=1$ state is risk dominant. The reverse is true ~~for~~ if $b < a-c$. If $b = a-c$, then neither ESS is risk dominant, the system is at or around each ~~in an eq~~ equally frequently.

This game is a coordination game ~~that~~ where coordination on C is Pareto dominant. We might think that rational agents will settle on the Pareto dominant ~~eq~~ NE rather than the risk dominant NE as the stochastic BR dynamics predict.

7a Firm i wins the patent race iff $b_i > b_j$.

It is given that $b_j = \beta v_j^2$.

~~Then~~, ~~$b_i > b_j$~~

$b_i > b_j \iff$

$b_i > \beta v_j^2 \iff$

$v_j < \sqrt{b_i/\beta}$

Given that v_j is uniformly distributed on the unit interval, this has probability $\sqrt{b_i/\beta}$

If firm i wins the patent race, it ~~has~~ enjoys gross benefit v_i.

~~If~~ Regardless of whether firm i wins the patent race, it ~~has~~ pays the bid b_i.

So firm i's payoff is expected gross benefit less bid.

$u_i(b_i, v_i) = \sqrt{b_i/\beta}\, v_i - b_i$

b Each firm i maximises expected payoff.

$\max_{b_i} u_i(b_i, v_i)$

FOC: $\beta^{-1/2} v_i (\frac{1}{2} b_i^{-1/2}) - 1 = 0 \implies$

$\frac{1}{2}\beta^{-1/2} v_i b_i^{-1/2} = 1 \implies$

$b_i^{-1/2} = 2\beta^{1/2} v_i^{-1} \implies$

$b_i = \cancel{4\beta}\ 2^{-2}\beta^{-1} v_i^2 = \frac{1}{4\beta} v_i^2$

SOC: $\beta^{-1/2} v_i (\frac{1}{2})(-\frac{1}{2} b_i^{-3/2}) = -\frac{1}{4}\beta^{-1/2} v_i b_i^{-3/2} < 0$

$b_i = \frac{1}{4\beta} v_i^2$ uniquely solves each ~~firm's~~ firm i's ~~pro~~ expected payoff maximisation problem.

Given ~~that~~ $b_i^* = \beta v_i^2$.

$\beta v_i^2 = \frac{1}{4\beta} v_i^2 \implies$

$\beta = \frac{1}{4\beta} \implies$

$4\beta^2 = 1 \implies$

$\beta = \frac{1}{2}$ (reject $\beta = -\frac{1}{2}$)

At BNE, each firm plays $b_i^* = \frac{1}{2} v_i^2$

c By argument analogous to that in (a), ~~if~~ firm i wins the patent race iff $v_j < \sqrt{c_i/\gamma}$, then firm i captures gross value ~~$\frac{1}{2}$~~ $\frac{1}{2}(v_i + v_j)$, so the expected gross value to firm i is

$\int_0^{\sqrt{c_i/\gamma}} \frac{1}{2}(v_i + v_j)\, dv_j$

$= \frac{1}{2} v_i \int_0^{\sqrt{c_i/\gamma}} 1\, dv_j + \frac{1}{2}\int_0^{\sqrt{c_i/\gamma}} v_j\, dv_j$

$= \frac{1}{2} v_i [v_j]_0^{\sqrt{c_i/\gamma}} + \frac{1}{2}[\frac{1}{2} v_j^2]_0^{\sqrt{c_i/\gamma}}$

$= \frac{1}{2}\sqrt{c_i/\gamma}\, v_i + \frac{1}{2}[\frac{1}{2} c_i/\gamma]$

$= \frac{1}{2}\sqrt{c_i/\gamma}\, v_i + \frac{1}{4} c_i/\gamma$

Firm i's ~~bid cost~~ ~~expected~~ bid is c_i

so firm i's expected profit is equal to expected gross benefit less bid cost.

$\frac{1}{2}\sqrt{c_i/\gamma}\, v_i + \frac{1}{4} c_i/\gamma - c_i$

Firm i maximises expected payoff

FOC: $\frac{1}{2} v_i \gamma^{-1/2} (\frac{1}{2} c_i^{-1/2}) + \frac{1}{4}\gamma - 1 = 0 \implies$

$\frac{1}{4} v_i \gamma^{-1/2} c_i^{-1/2} = 1 - \gamma/4 \implies$

$c_i^{-1/2} = (4-\gamma) v_i^{-1} \gamma^{1/2} \implies$

$c_i = (4-\gamma)^{-2} v_i^2 \gamma^{-1}$

SOC: $-\frac{1}{8} v_i \gamma^{-1/2} c_i^{-3/2} < 0$

$c_i = (4-\gamma)^{-2}\gamma^{-1} v_i^2$ uniquely maximises ~~a~~ firm i's payoff.

At BNE, $c_i = \gamma v_i^2$ (by supposition).

$\gamma = (4-\gamma)^{-2}\gamma^{-1} \implies$

$\gamma^2 = (4-\gamma)^{-2} \implies$

$\gamma = (4-\gamma)^{-1} \implies$

$1/\gamma = 4 - \gamma \implies$

$\gamma^2 - 4\gamma + 1 = 0 \implies$

$\gamma = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{12}/2 = 2 \pm \sqrt{3} = 2 - \sqrt{3}$

(reject $\gamma = 2 + \sqrt{3}$)

d Under regional patenting, expected investment by each firm is $E\beta v_i^2 = \beta E v_i^2$. Under imperfect worldwide patenting, expected investment by each firm is $E\gamma v_i^2 = \gamma E v_i^2 < \beta E v_i^2$.

At ~~BNE~~ BNE, each firm's profit in interim expectation is $v_i^2 - b_i$ or $v_i^2 - c_i$ ~~in the two~~ under the two regimes.

If greater weight is put on investment than profit, then social surplus is increasing in investment, which is ~~de~~ higher under regional patenting, so regional patenting is preferable.

Date