

Microeconomic Analysis Problem Set 3

1a. $x, y \in \{1, 2, 3, \dots\}$: $x \geq y \Leftrightarrow x \geq y+1$

Suppose \succeq is not connected. Then by definition of connectedness, $\exists x, y \in \{1, 2, 3, \dots\}$: $x \not\succeq y$ and $y \not\succeq x$, then $x \not\succeq y+1$, $x > y+1$, $y < x-1$, $y \leq x+1$, $y \geq x$. By reductio, \succeq is connected.

Suppose \succeq is not reflexive. Then by definition of reflexivity, $\exists x \in \{1, 2, 3, \dots\}$: $x \not\succeq x$, then $x \not\succeq y$, $x+1 \succ x$. By reductio, \succeq is reflexive.

Suppose \succeq is not complete. Then by definition of completeness, $\exists x, y \in \{1, 2, 3, \dots\}$: $x \not\succeq y$ and $y \not\succeq x$, then $x \not\succeq y+1$, $x > y+1$, $y < x-1$, $y \leq x+1$, $y \geq x$. By reductio, \succeq is complete.

Suppose \succeq is not transitive. Then by definition of transitivity, $\exists x, y, z \in \{1, 2, 3, \dots\}$: $x \succ y$ and $y \succ z$ and $x \not\succ z$, then $x \succ y+1$, $y \succ z+1$, $x \succ y+1$

\succeq is not transitive since $x \succ y$ and $y \succ z$ but $x \not\succ z$ for $x=3, y=2, z=1$, by definition of transitivity

Since \succeq is reflexive, it is not asymmetric. Suppose that \succeq is asymmetric. Since \succeq is reflexive, and $\succeq \neq \emptyset$, $\exists x \in \{1, 2, 3, \dots\}$: $x \succ x$. supposing that \succeq is asymmetric, by definition of asymmetry, $x \not\succ x$. By reductio, \succeq is not asymmetric.

\succeq is not antisymmetric since $x \succ y$ and $y \succ x$ but $x \succ y$ for $x=2, y=1$, by definition of antisymmetry.

b) Ann's preferences are rational iff they are complete and transitive. Since Ann's preferences are not transitive, they are not rational.

\succeq can be represented by a utility index only if \succeq is rational, so \succeq cannot be represented by a utility index.

2 $\vdash \neg R \rightarrow \neg P$

By construction, $(x,y) \in R \leftrightarrow (y,x) \notin P$ ①

Suppose R is complete, then by definition of completeness, since R is a binary relation on X ,
 $\forall x, y \in X : xRy \text{ or } yRx$, then by ① $(y,x) \notin P$ or
 ~~$(x,y) \notin P$~~ $\forall x, y \in X : (y,x) \notin P \text{ or } (x,y) \in P$, i.e.
 $\forall x, y \in X : \text{not both } (x,y) \notin P \text{ and } (y,x) \in P$, by
definition of transitivity, P is ~~not transitive~~ (on X).
asymmetric (on X).

Suppose R is not complete, then by definition of completeness, since R is a binary relation on X ,
 $\exists x, y \in X : \text{neither } xRy \text{ nor } yRx$, then by ① $\exists x, y \in X : yPx \text{ and } xPy$, by definition of asymmetry, P is not asymmetric.

So R is complete iff P is asymmetric.

Suppose that R is transitive, then by definition of transitivity, $\forall x, y, z \in X : \text{if } xRy \text{ and } yRz \text{ then } xRz$.
Then by ①, $\forall x, y, z \in X : \text{if not } yPx \text{ and not } zPy \text{ then not } zPx$, by definition of negative transitivity, P is negative transitive.

Suppose that R is not transitive, then by definition of transitivity, $\exists x, y, z \in X : xRy \text{ and } yRz \text{ but not } xRz$.
Then by ①, $\exists x, y, z \in X : \text{not } yPx \text{ and not } zPy \text{ but not } zPx$, by definition of negative transitivity, P is not negative transitive.

So R is transitive iff P is negative transitive.

3a Let $u(x_1, x_2) = x_1^3 x_2$

Then $\forall \vec{x}, \vec{y} \in \mathbb{R}_{++}^2$: if $\vec{x} \succcurlyeq \vec{y}$, then by definition given $\vec{x} \succcurlyeq \vec{y} \Leftrightarrow x_1^3 x_2 \geq y_1^3 y_2$,
 $x_1^3 x_2 \geq y_1^3 y_2$, then by construction of $u(\cdot)$, $u(\vec{x}) \geq u(\vec{y})$.
 Then, by definition of utility representation, \succcurlyeq is represented by $u(\cdot)$. So \succcurlyeq can be represented by a (continuous) utility function.

b Let $u'(x_1, x_2) = x_1^3 x_2$. From (a), \succcurlyeq is represented by u' .

For all $f: \mathbb{R} \rightarrow \mathbb{R}$, ~~if~~ strictly increasing $f: \mathbb{R} \rightarrow \mathbb{R}$, then $v = f \circ u'$ is a utility function that represents \succcurlyeq .
 $u(x_1, x_2) = x_1^{3/4} x_2^{1/4} = f \circ u'(x_1, x_2)$

By definition of representation by a utility function,
 $\forall \vec{x}, \vec{y} \in \mathbb{R}_{++}^2$: if $\vec{x} \succcurlyeq \vec{y}$ then $u'(\vec{x}) \geq u'(\vec{y})$, then
~~if~~ since f is ~~if~~ strictly increasing, $f \circ u'(\vec{x}) \geq f \circ u'(\vec{y})$, so $f \circ u'$ is a utility function that represents \succcurlyeq .

$$u = f \circ u' \text{ where } f(x) = x^{1/4}$$

$$v = f \circ u' \text{ where } f(x) = \ln x$$

$$w = f \circ u' \text{ where } f(x) = x^{-1}$$

f_u and f_v are strictly increasing in the range of u' , so u and v represent \succcurlyeq .

f_w is not strictly increasing (and is in fact strictly decreasing in the range of u'), so w does not represent \succcurlyeq .

c Let u be the uniform

Let $u(x_1, x_2) = x_1^3 x_2$ and v be the uniform consumption equivalent utility representation, i.e. $v(x_1, x_2) = x$ s.t. $(x_1, x_2) \sim (x, x)$

Given that $(x_1, x_2) \succcurlyeq (y_1, y_2) \Leftrightarrow x_1^3 x_2 \geq y_1^3 y_2$, by definition of u , $\vec{x} \succcurlyeq \vec{y}$ iff $\vec{x} \succ \vec{y}$ and $\vec{y} \succcurlyeq \vec{x}$, then $\vec{x} \succ \vec{y}$ iff $x_1^3 x_2 = y_1^3 y_2$. ~~iff~~ $(x_1, x_2) \sim (x, x)$ iff $x_1^3 x_2 = x^4$, $x = x_1^{3/4} x_2^{1/4}$, $v(x_1, x_2) = x_1^{3/4} x_2^{1/4}$.

The uniform consumption equivalent utility representation is the Cobb-Douglas function corresponding Cobb-Douglas.

4a Let $\vec{x} = (x_1, x_2, x_3)$ denote the observed student's choices, where x_1 is her choice on Monday, x_2 on Tuesday, and x_3 on Wednesday. Let \succ denote the observed student's preferences. Let \succsim denote the observed student's \Rightarrow rational strict preference relation.

Suppose $\vec{x} = (F, P, M)$, then $F \succsim M$, $F \succsim P$, $P \succsim V$ and $M \succsim V$. We cannot infer whether $M \succ P$ or $P \succ M$, so the observed student did not choose (F, P, M) .

Suppose $\vec{x} = (F, P, V)$, then $F \succsim M$, $F \succsim P$, $P \succsim V$ and $V \succsim M$, then $F \succsim P \succsim V \succsim M$, i.e. we can infer \succ , so the observed student could have chosen (F, P, V) .

By symmetry, the observed student could have chosen (F, V, M) .

Suppose $\vec{x} = (F, V, V)$, then we cannot infer whether $P \succ M$ or $M \succ P$, so the observed student did not choose (F, V, V) .

Suppose $\vec{x} = (M, P, M)$, then we cannot infer whether $F \succsim V$ or $V \succsim F$, so $\vec{x} \notin (M, P, M)$. Suppose $\vec{x} = (M, P, V)$, then we cannot infer whether $M \succsim P$, $P \succsim V$, and $V \succsim M$, so \succ is not transitive hence not rational. By reductio, $\vec{x} \notin (M, P, V)$. Suppose $\vec{x} = (M, V, M)$, then we cannot infer whether $F \succsim P$ or $P \succsim F$, so $\vec{x} \notin (M, V, M)$. Suppose $\vec{x} = (M, V, V)$, then we cannot infer whether $F \succsim P$ or $P \succsim F$, so $\vec{x} \notin (M, V, V)$. So $x_1 \neq M$. By symmetry, $x_1 \neq P$.

$$\vec{x} = (F, P, V) \text{ or } (F, V, M)$$

b Let $X'_i = \{x_i\} / \{x_i\}$ for $i \in \{1, 2, 3\}$

c (A) \succ directly reveals $x_i \succsim x_i'$ for all $x_i' \in X'_i$, i.e. each choice reveals that the chosen item is element is preferred by the observed student to each element other element in that day's menu.
 All other preferences are revealed indirectly by the transitivity of rational preferences, from the directly revealed preferences.

Suppose $\vec{x} = (F, P, V)$, then the following preferences are revealed directly: $F \succsim M$, $F \succsim P$, $P \succsim V$, $V \succsim M$, and the following preferences are revealed indirectly: $P \succsim M$, $F \succsim V$.

Suppose $\vec{x} = (F, V, M)$, then $F \succsim M$, $F \succsim P$, $V \succsim P$, and $M \succsim V$ are revealed directly and $F \succsim V$ and $M \succsim P$ are revealed indirectly.

c $\vec{x} = (P, V, V) \Rightarrow P \succsim F$, $P \succsim M$, $V \succsim P$, $V \succsim M$. (directly)
 $\Rightarrow V \succsim F$ (indirectly)
 So whether $M \succsim F$ or $F \succsim M$ cannot be inferred from $\vec{x} = (P, V, V)$. So \succ is fully mapped iff $\Delta_4 = \{F, M\}$.

- (P, V, M)
- $\Rightarrow P \succ F, P \succ M, M \succ V, V \succ P$ (directly)
 - $\Rightarrow V \succ F, V \succ M, V \succ V$ (indirectly). We conclude that the observed student's preferences are irrational.
 - Since it is irrational to strictly prefer some element to itself.

e Denote the Monday meat dish MM. Then $\Delta = \{F, MM, P\}$, $\Delta_2 = \{P, V\}$, $\Delta_3 = \{M, V\}$, $\Delta_4 = \{M, F, V, P\}$

 $c(\Delta_1, \succ) = P, c(\Delta_2, \succ) = V, c(\Delta_3, \succ) = M \Rightarrow$
 $P \succ F, P \succ MM, V \succ P, M \succ V$ (directly) \Rightarrow
 $V \succ F, V \succ MM, M \succ P, M \succ F, M \succ MM$ (indirectly) \Rightarrow
 $c(\Delta_4, \succ) = M$

5a By inspection, $U(L)$ is a function of \vec{p} and N , and is independent of \vec{x} . So if $\exists u: \mathbb{R} \rightarrow \mathbb{R}$ such that $U(L) = \sum_{i=1}^N p_i u(x_i)$, $u(x_i) = c$ for all x_i , where c is some constant. Then $U(L) = \sum_{i=1}^N c p_i = c \sum p_i = c$, then $U(L)$ is ~~a function~~ not a function of \vec{p} and N .
By reductio, $\nexists u: \mathbb{R} \rightarrow \mathbb{R}$ such that $U(L) = \sum_{i=1}^N p_i u(x_i)$, i.e. $U(L)$ is not consistent with the EU hypothesis.

b By an analogous argument to that in (a), since $U(L)$ is ~~not~~ independent of \vec{x} , $U(L) = c$, where c is some if the EU hypothesis holds, $U(L) = c$ where c is some constant, and $U(L)$ is not a function of \vec{p} .
By reductio, the EU hypothesis does not hold, $U(L)$ is not consistent with the EU hypothesis.

c $U(L) = \sum_{i \geq G} p_i = \sum_{i=1}^N p_i u(x_i, \vec{x})$, where $u(x_i, \vec{x}) = \begin{cases} 1 & \text{if } x_i \geq x_G \\ 0 & \text{otherwise.} \end{cases}$
Let $L' = [p_2, \dots, p_N, p_{N+1}; x_2, \dots, x_N, x_{N+1}]$ where $x_{N+1} \neq x_N$.
 $U(L) = p_G + \dots + p_N$, $U(L') = p_{G+1} + \dots + p_N = U(L) - p_G + p_1$.
Suppose that $U(L)$ is consistent with the EU hypothesis, then $\exists u: \mathbb{R} \rightarrow \mathbb{R}$ such that $U(L) = \sum_{i=1}^N p_i u(x_i)$, then $U(L') = U(L) - p_1 u(x_1) + p_1 u(x_{N+1})$, then $p_1 - p_G = p_1 u(x_{N+1}) - p_1 u(x_1)$. Let $L'' = [p_2, \dots, p_{G-1}, p'_G = p_G - \epsilon, p_G + \epsilon, \dots, p_N, p_1; x_2, \dots, x_N, x_{N+1}]$.
By an analogous argument, $p_1 - p'_G = p_1 u(x_{N+1}) - p_1 u(x_1)$,
By reductio, since then $p_1 - p'_G = p_1 - p_G$. By reductio, since $p_G \neq p'_G$, $U(L)$ is not consistent with the EU hypothesis.

d Suppose that $U(L)$ is consistent with the EU hypothesis, then $\exists u: \mathbb{R} \rightarrow \mathbb{R}$ such that $U(L) = \sum_{i=1}^N p_i u(x_i)$.
Let i^* denote argmax $u(x_i)$ such that $p_i > 0$.
~~By inspection, $U(L)$ is independent of p_i for all $i \neq i^*$, then $u'(x_i) = 0$ for all $i \neq i^*$.~~ Let L' be the lottery identical to L except in reallocating all probability from x_{i^*} to some other result with non-zero probability in L . Then $U(L') = 0$. By inspection consider L such that $p_i: p_i > 0$. By inspection, $U(L)$ is independent of \vec{p} , so $U'(x_i) = 0$ for all i , so $U(L) = 0$ for all such L , so $U'(x_i) = 0$ for all x_i , so $U(L) = 0$ for all L . $U(L)$ is consistent with the EU hypothesis only if $u(x_i) = 0$ for all x_i . It is trivially true that ~~the~~ $U(L)$ is consistent with the EU hypothesis if $u(x_i) = 0$ for all x_i .

e Consider $k = \max_{i \neq k} p_i / p_k$ max k st $p_k / p_i > p_k$.
Let L' be the lottery obtained by reallocating probability mass from some x_i where $i \neq k$ to x_k . Suppose that $U(L)$ is consistent with the EU hypothesis. By construction of $U(L)$, $U(L) = U(L')$, $U(L) = \sum_{i=1}^N p_i u(x_i)$, then $p_k u(x_k) = p_i u(x_k)$. Since i and k are chosen arbitrarily, $u(x_i) = u(x_k)$ for all x_i, x_k , $U(x)$ is a constant, c , $U(L) = c$ for all L . By reductio, $U(L)$ is not consistent with the EU hypothesis.

6 In the first arrangement, each agent has compound lottery

$$L_i = \left[\frac{1}{N}, \frac{N-1}{N}; w+L, w \right]$$

which reduces to

$$L_i^R = \left[\frac{1}{N}, \frac{N-1}{N}; p_1, \frac{N-1}{N}p_1 + \dots + \frac{N-1}{N}p_k; \dots \right]$$

$$L_i^R = \left[p_1, \frac{p_1}{N}, \dots, p_k, \frac{p_k}{N}; w+x_1, w+x_2, \dots, w+x_k, w \right]$$

In the second arrangement, each agent has compound lottery

$$L_2 = \left[p_1, p_2, \dots, p_k; w+\frac{x_1}{N}, w+\frac{x_2}{N}, \dots, w+\frac{x_k}{N} \right]$$

Each agent has strictly risk averse EU preferences

$$U(L) = \sum_{i=1}^k p_i U(w+x_i), \text{ where } U \text{ is strictly concave}$$

$$U(L_i) = U(L_i^R)$$

$$= p_1 \frac{1}{N} U(w+x_1) + p_2 \frac{1}{N} U(w+x_2) + \dots + p_k \frac{1}{N} U(w+x_k) + \frac{N-1}{N} U(w)$$

$$= \frac{1}{N} \left[p_1 U(w+x_1) + p_2 U(w+x_2) + \dots + p_k U(w+x_k) + (N-1)U(w) \right]$$

$$= \frac{1}{N} \sum_{i=1}^k p_i U(w+x_i) + \frac{N-1}{N} U(w)$$

$$U(L_2) = \sum_{i=1}^k p_i U(w+\frac{x_i}{N})$$

$$= U(w) + \sum_{i=1}^k p_i [U(w+\frac{x_i}{N}) - U(w)]$$

$$= \sum_{i=1}^k p_i [U(w+\frac{x_i}{N}) - U(w)]$$

$$U(L_1) = U(w) + \sum_{i=1}^k p_i \frac{1}{N} [U(w+x_i) - U(w)]$$

since U is strictly concave, $U(w+\frac{x_i}{N}) - U(w) > \frac{1}{N} [U(w+x_i) - U(w)]$, so $U(L_2) > U(L_1)$.

Risk averse agents prefer the second arrangement under which risk is shared across agents.