Predicate Logic Philosophical Rough Notes

Predicate Logic with Identity

- "Logical truths" of predicate logic with identity (that Quine refers to as truths of identity theory) are falsifiable by substituting the identity predicate with other predicates. In contrast, we think that (truly) logical truths are not so falsifiable. For example, substituting some sentence (not sentence letter) for another in propositional logic, does not falsify the truths of propositional logic. Such a substitution involves only considering a different "instantiation" of a sentence letter. Similarly, substituting some predicate (not predicate letter) or some constant (not constant term) for another in simple predicate logic does not falsify the truths of simple predicate logic. Again, such a substitution involves only considering different "instantiations" of predicate letters and constant terms (which schematise over the predicates and constants that instantiate them).
- The contrast between generalities that can be expressed in the object language (English, for our purposes) and generalities that "call for" semantic ascent "marks a conspicuous and tempting place at which to draw the line between the other sciences and logic". The generality "all men are mortal" in English, generalises on "Tom is mortal", "Dick is mortal", and so on. This generalisation can be expressed in English. In contrast, generalisation on "Tom is mortal or Tom is not mortal" and "Snow is white or snow is not white" requires semantic ascent. We so generalise by saying "Every sentence of the form p or not p is true", this generalisation involves talk of the truth of sentences, while the earlier generalisation does not. Among the sentences generalised on, in the former case, it is names that are changed, whereas in the latter case, it is sentences or clauses that are changed. We generalise on "Tom is Tom", "Dick is Dick", and so on in English by saying "everything is itself". If the identity predicate is "proper to" logic, then it is unique in that its generalities can be expressed in English without semantic ascent.
- Godel proved that complete proof procedures are available for predicate logic with identity, but also that complete proof procedures are not available for elementary number theory.
- A "serviceable facsimile" of identity can be constructed in a language with truth functions ($\neg$, $\rightarrow$, etc.) and quantification ($\forall$, $\exists$) and a finite lexicon of predicates. The idea is to construct a facsimile of $x = y$ as a sentence that "reads" x As iff y As, and Bs iff y Bs, ..., and for all a, x Ps a iff y Ps a, ..., and so on. This "simulation" of identity "means" that x and y are indistinguishable by any sentences that can be phrased in the language. If x and y are indeed distinct, then this simulation fails to capture such distinctness, but such failure is "unobservable from within the language". In predicate logic, schematic predicate letters (F, G) are used in place of actual predicates ("is red", "is the parent of"). In the same spirit, the notation $x = y$ can be viewed schematically as standing in place of the compound sentence (in the object language, English), whose counterpart is the construct above.

Second-Order Logic

- Quine rejects that quantification over predicates is appropriate because predicates are not names (of either sets or attributes) and it is predicates that predicate letters stand in place of. In contrast, the variables ordinarily quantified over stand in place of names. So, for example, in $\exists x(\text{x walks})$, the thing that walks is the thing named by the name that x stands in the place of. To extend quantification to predicate letters, it seems, is to say that some or all entities of the sort named by predicates are thus and so. For example, in "$\forall X : \text{Aristotle is } X$" seems to treat X as standing in place of some name of something that Aristotle is or has. But "X" stands in the place of a predicate, not a name for some attribute (that Aristotle could have).
  - Boolos responds that to quantify over X is not to treat X as standing in place of a name (being in a name position) but to treat X as having some range. "$\exists X$" does not have to be taken as saying that some entities of the sort named by predicates are thus and so; it can be taken to say that some of the entities (extensions) had by predicates contain thus and such." Quine hastily supposes that quantification over predicate variables is necessarily exactly analogous to quantification over ordinary variables.
- Then, Quine argues further that the sentences of second-order logic are better expressed in set-theoretic notation that makes the set-theoretic presuppositions of second-order logic explicit. For example, Quine argues that we should write "$\forall \alpha : a \in \alpha$" rather than "$\forall X : Xa$".
  - Boolos responds that this results in a loss of validity and implication. $\exists X \forall x : Xx$ is valid but $\exists \alpha \forall x : x \in \alpha$ is not. $\forall X : (Xa \rightarrow Xb)$ implies $a = b$, but $\forall \alpha : (a \in \alpha \rightarrow b \in \alpha)$ does not.
- Boolos argues positively for considering second-order logic to be logic on the ground that second-order logic can be defined as a natural extension of first-order logic. The definition of a model as an ordered pair of a domain and an

interpretation function is unchanged. The interpretation function is extended in a perfectly natural way. Likewise for the valuation function. Definitions of logical validity and semantic consequence are entirely unchanged.

  - Boolos argues that second-order logic does not have high "ontological costs", i.e. it is not committed to the existence of dubious entities. That $\exists X \forall x : (Xx \leftrightarrow \neg x \in x)$ is valid means simply that it is true in all interpretations. But this does not imply that there is a set of all non-self-membered sets unless there is some interpretation whose domain is the set of all sets. But there is no reason to think a set of all sets exists (and apparently plenty of reason from set theory to think that no such set exists). So second-order logic is not committed to the existence of a set of all non-self-membered sets. Second-order logic does not simply obfuscate the "ontological cost", and is properly exempt from such cost. This seems to show that "it is impermissible to use the notation of second-order logic in the formalisation of discourse about certain sorts of objects [...] in case there is no set to which all objects of that sort belong".
- Boolos argues positively for considering second-order logic to be logic on the ground of its greater "expressive capacity".
  - It is possible to form a sentence in second-order logic that is true only in interpretations with infinite domains, and also to form a sentence in second-order logic that is true only in interpretations with countable domains. This is not possible in first-order logic. Similarly, notions of progression, ancestral, and identity cannot be characterised in first-order logic.
  - First-order logic cannot consider as inconsistent some apparently inconsistent infinite sets of sentences, such as $\{$ 'Smith is an ancestor of Jones', 'Smith is not a parent of Jones', 'Smith is not a grandparent of Jones', ... $\}$. But this sort of inconsistency seems to be the same sort of inconsistency as that in the set $\{$ 'There are fewer than three stars', 'Not: there are exactly no stars', 'Not: there is exactly one star', 'Not: there are exactly two stars' $\}$. The reason for inconsistency in both cases seems to be the kind of reason "it has always been the business of logic to give". The sort of inconsistency in both sorts of cases is "palpably logical".
- Boolos rejects Quine's suggestion of the incompleteness of second-order logic as grounds for rejecting that it is logic.
  - First-order logic is not decidable, but some fragments of first-order logic are. This means that there exists an "effective method", like constructing a truth table for determining logical truths for fragments of first-order logic, like propositional logic. Such methods require rigour but not ingenuity, terminate after a finite number of steps, and are in some sense mechanical. Some monadic fragment of second-order logic is also decidable. It is not clear why the "partial" standard of effectiveness, completeness, should be used to delineate between logic and mathematics (rather than the more "thorough" standard of effectiveness, decidability).
- A related reason Quine entertains is a "remarkable concurrence of diverse definitions of logical truth" in the case of first-order logic but not second-order logic. In first-order logic, Quine proves, logical truth under the usual definition of truth under all interpretations coincides with logical truth defined as truth in any reasonably rich object language under all substitutions (of the logical symbols for the sorts of things they stand in place of). Boolos notes that such these only weakly coincide rather than strongly coincide in the sense that there is no such coincidence for definitions of semantic consequence. So it is not clear that alternative definitions of truth, and their coinciding with the usual definition in the case of first-order logic, is relevant.