3a $u = \sum_{i=1}^{L} a_i \ln x_i$, $\forall i: a_i > 0$, $\sum_{i=1}^{L} a_i = 1$

$Du = (\partial_1 u \;\; \partial_2 u \; \cdots \; \partial_L u)$

$= (a_1/x_1 \;\; a_2/x_2 \cdots a_L/x_L)$

b $D^2u = \begin{pmatrix} \partial_1\partial_1 u & \partial_1\partial_2 u & \cdots & \partial_1\partial_L u \\ \partial_2\partial_1 u & \partial_2\partial_2 u & \cdots & \partial_2\partial_L u \\ \vdots & \vdots & \ddots & \vdots \\ \partial_L\partial_1 u & \partial_L\partial_2 u & \cdots & \partial_L\partial_L u \end{pmatrix}$

$= \begin{pmatrix} -a_1x_1^{-2} & 0 & \cdots & 0 \\ 0 & -a_2x_2^{-2} & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -a_Lx_L^{-2} \end{pmatrix}$

c Let D^2u_{ij} denote the element in the i^{th} row and j^{th} column of D^2u. By definition of D^2u, $\forall i,j \in \{1,\dots L\}$: $D^2u_{ij} = \partial_i\partial_j u = \partial_i(a_j/x_j) = 0$ for all $i \neq j$. Then D^2u is a diagonal matrix so D^2u is a symmetric matrix. This is confirmed by inspection.

Let D^2u^k denote the square submatrix of A with only the first k rows and columns retained. Then $D_2u^1 = (-a_1x_1^{-2})$

$D_2u^2 = \begin{pmatrix} -a_1x_1^{-2} & 0 \\ 0 & -a_2x_2^{-2} \end{pmatrix}$ and so on.

~~det~~ $(-1)^1 \det D_2u^1 = a_1x_1^{-2}$

$(-1)^2 \det D_2u^2 = (a_1x_1^{-2})(a_2x_2^{-2})$

$(-1)^k \det D_2u^k = (-1)^k \prod_{i=1}^{k} (-a_ix_i^{-2})$

since the determinant of a diagonal matrix is the product of its diagonal elements

$= \prod_{i=1}^{k} (a_ix_i^{-2})$

> 0 for all k

Given $\forall i\, a_i > 0$, assuming that u is well-defined then $\forall i: \ln x_i$ is well-defined then $\forall i: x_i \neq 0$ then $\forall i\, x_i^{-2} > 0$.

By the determinant test, D^2u is negative definite.

d $\vec{y}^T D^2u \vec{y}$ is well-defined only if $\vec{y}$ is a vector of length L. Let $\vec{y} = (y_1, \dots, y_L)^T$

$D^2u\vec{y} = \begin{pmatrix} -a_1x_1^{-2} & 0 & \cdots & 0 \\ 0 & -a_2x_2^{-2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -a_Lx_L^{-2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_L \end{pmatrix}$

$= \cancel{(y_1a_1x_1^{-2} \;\; y_2a_2x_2^{-2} \cdots y_La_Lx_L^{-2})}$

$\cancel{\vec{y}^T D^2u \vec{y}}$ $\begin{pmatrix} -y_1a_1x_1^{-2} \\ -y_2a_2x_2^{-2} \\ \\ -y_La_Lx_L^{-2} \end{pmatrix}$

$\vec{y}^T D^2u \vec{y} = (y_1 \; y_2 \cdots y_L) \begin{pmatrix} -y_1a_1x_1^{-2} \\ -y_2a_2x_2^{-2} \\ \\ -y_La_Lx_L^{-2} \end{pmatrix}$

$= -y_1^2a_1x_1^{-2} - y_2^2a_2x_2^{-2} - \cdots - y_L^2a_Lx_L^{-2}$

$= \cancel{\Sigma\, a} - \sum_{i=1}^{L} \cancel{a_i y_i^2 x} \; a_i y_i^2 x_i^{-2}$

< 0 for all $\vec{y} \neq 0 \in \mathbb{R}^n$

Given $\forall i: a_i > 0$, assuming that u is well-defined hence $\forall i: x_i^{-2} > 0$

By checking $\vec{y}^T D^2u \vec{y}$, D^2u is negative definite.

e $u(\vec{x}') \approx u(\vec{x}) + Du(\vec{x})(\vec{x}' - \vec{x}) + \frac{1}{2}(\vec{x}' - \vec{x})^T D^2u(\vec{x})(\vec{x}' - \vec{x})$

$= 0 + (a_1/\cancel{x_1} \;\; a_2/\cancel{x_2} \cdots a_L/\cancel{x_L})(x_1'-1 \;\; x_2'-1 \cdots x_L'-1)$

$+ \frac{1}{2}(x_1'-1 \;\; x_2'-1 \cdots x_L'-1) D^2u (x_1'-1 \;\; x_2'-1 \cdots x_L'-1)^T$

$$= \alpha_1(x_1'-1) + \alpha_2(x_2'-1) + \cdots + \alpha_L(x_L'-1)$$

$$+ \frac{1}{2}\left[-\alpha_1(x_1'-1)^2 - \alpha_2(x_2'-1)^2 + \cdots + \alpha_L(x_L'-1)^2\right]$$

$$= \sum_{i=1}^{L} \alpha_i \left[(x_i'-1) - \frac{1}{2}(x_i'-1)^2\right]$$