Microeconomic Analysis Paper 220613

1a Let $\vec{u}_1 = \begin{pmatrix} c \\ 1 \\ 1 \end{pmatrix}$, $\vec{u}_2 = \begin{pmatrix} 1 \\ c \\ 1 \end{pmatrix}$, $\vec{u}_3 = \begin{pmatrix} 1 \\ 1 \\ c \end{pmatrix}$. Let $A = \begin{pmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{pmatrix}$ be the matrix whose columns are $\vec{u}_1, \vec{u}_2, \vec{u}_3$.

exactly correct approach

$\vec{u}_1, \vec{u}_2, \vec{u}_3$, ~~given~~ are ~~be~~ a basis of $\mathbb{R}^3$ iff they span $\mathbb{R}^3$ given that they would then ~~be~~ form a minimal collection of vectors that spans $\mathbb{R}^3$. $\vec{u}_1, \vec{u}_2, \vec{u}_3$ span $\mathbb{R}^3$ iff they are linearly independent which is iff A is full rank which is iff A is invertible which is iff $\det A \neq 0$.

$$\det A = c \det \begin{pmatrix} c & 1 \\ 1 & c \end{pmatrix} - 1 \det \begin{pmatrix} 1 & 1 \\ 1 & c \end{pmatrix} + 1 \det \begin{pmatrix} 1 & c \\ 1 & 1 \end{pmatrix}$$
$$= c(c^2-1) - (c-1) + (1-c)$$
exactly correct
$$= c^3 - c - c + 1 + 1 - c$$
$$= c^3 - 3c + 2$$
by intuition from the form of the matrix
$$= (c-1)(c^2+c-2)$$
$$= (c-1)(c-1)(c+2)$$

$\det A = 0 \iff c = 1$ or $c = -2$. For $c=1$ ~~or~~ and $c=-2$, $\vec{u}_1, \vec{u}_2, \vec{u}_3$ fail to form a basis of $\mathbb{R}^3$. For $c=1$, $\vec{u}_1 = \vec{u}_2 = \vec{u}_3$, then ~~span they span~~ no two of these are linearly independent, they span 1 dimension (A has rank 1). For $c=-2$, any two of these are linearly independent, they span 2 dimensions (A has rank 2).

b Let λ and $\vec{v}$ be an eigenvalue and the associated eigenvector of A. Then, by definition, $A\vec{v} = \lambda\vec{v}$.

This is the definition

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ ~~Let~~

exactly correct appr

Let $\lambda_1 = 1$, $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\lambda_2 = -1$, $\vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$A\vec{v}_1 = \lambda_1 \vec{v}_1 \iff \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\iff a_{11} + 2a_{12} = 1$, $a_{21} + 2a_{22} = 2$

$A\vec{v}_2 = \lambda_2 \vec{v}_2 \iff \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = -1\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\iff 2a_{11} + a_{12} = -2$, $2a_{21} + a_{22} = -1$

Solve simultaneously by elimination,

$a_{11} + 2a_{12} - 2(2a_{11} + a_{12}) = 1 - 2(-2) \Rightarrow -3a_{11} = 5$

$\Rightarrow a_{11} = -5/3 \Rightarrow a_{12} = \cancel{13/6}\ 4/3$

$a_{21} + 2a_{22} - 2(2a_{21} + a_{22}) = 2 - 2(-1) \Rightarrow -3a_{21} = 4$

$\Rightarrow a_{21} = -4/3 \Rightarrow a_{22} = 5/3$

$$\Rightarrow A = \begin{pmatrix} -5/3 & 4/3 \\ -4/3 & 5/3 \end{pmatrix}$$

This part ~~here onward~~ out of syllabus

$\dot{\vec{y}} = A\vec{y}$

$\vec{y} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow \dot{\vec{y}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow \vec{y}' = \vec{y} + \dot{\vec{y}} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

$\Rightarrow \dot{\vec{y}}' = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \Rightarrow \vec{y}'' = \vec{y}' + \dot{\vec{y}}' = \begin{pmatrix} 4 \\ 8 \end{pmatrix} \Rightarrow \cdots$

$\vec{y} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow \dot{\vec{y}} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} \Rightarrow \vec{y}' = \vec{y} + \dot{\vec{y}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$

$\dot{\vec{y}}' = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{y}'' = \vec{y}' + \dot{\vec{y}}' = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \cdots$

$\vec{y}' = \vec{y} + \dot{\vec{y}} = \vec{y} + A\vec{y} = I\vec{y} + A\vec{y} = (A+I)\vec{y}$
$= \begin{pmatrix} -2/3 & 4/3 \\ -4/3 & 8/3 \end{pmatrix}\vec{y}$

Let $A' = A + I = \begin{pmatrix} -2/3 & 4/3 \\ -4/3 & 8/3 \end{pmatrix} = 2/3\begin{pmatrix} -1 & 2 \\ -2 & 4 \end{pmatrix}$

$\det A' - \lambda I = \det \begin{pmatrix} -2/3 - \lambda & 4/3 \\ -4/3 & 8/3 - \lambda \end{pmatrix}$
$= (-2/3 - \lambda)(8/3 - \lambda) - (4/3)(-4/3)$
$= (-16/9 - 6/3\lambda + \lambda^2) - (-16/9)$
$= -2\lambda + \lambda^2$

$\det A' - \lambda I = 0 \iff \lambda = 0$ or $\lambda = 2$

One eigenvalue of A' is 0, the other is 2

$A'\vec{v}_1 = 0\vec{v}_2 \iff 2/3\begin{pmatrix} -1 & 2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Leftarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$A'\vec{v} = 2\vec{v} \iff 2/3\begin{pmatrix} -1 & 2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2v_1 \\ 2v_2 \end{pmatrix} \iff$

$-v_1 + 2v_2 = 3v_1$, $-2v_1 + 4v_2 = 3v_2 \iff$

$v_2 = 2v_1 \Leftarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

The eigenvalues and eigenvectors of ~~A+I~~ $A' = A + I$ are 0 and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, 2 and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Any $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ such that $y_1 = 2y_2$ converges immediately to $\vec{0}$ and remains at $\vec{0}$ indefinitely. Any $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ such that $y_2 = 2y_1$ increases exponentially by a factor of 2 in each period and "explodes" to $\begin{pmatrix} \infty \\ 2\infty \end{pmatrix}$

c i $f(x,y) = \begin{cases} 3xy / x^2+2y^2 & \text{if } x \neq 0 \\ 0 & \text{if } x=0 \end{cases}$

$f_x(0,0) = \lim_{h\to 0} f(h,0) - f(0,0)/h$
$= \lim_{h\to 0} 0-0/h$
$= 0$

$f_y(0,0) = \lim_{h\to 0} f(0,h) - f(0,0)/h$
$= \lim_{h\to 0} 0-0/h$
$= 0$

$Df(0,0) = (0 \;\; 0)$

$\nabla f(0,0) = (Df(0,0))^T = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Directional derivative of $\vec{v}$ ~~$\nabla f(0$~~ at $(0,0)$ is
$\nabla f(0,0) \cdot \vec{v} = 0$

ii Consider the sequence $\{(1/n, 1/n)\}_{n=1}^{\infty}$. As n approaches ∞, $(1/n, 1/n)$ converges to $(0,0)$. As n approaches ∞, $f(1/n, 1/n) = 3 \cdot 1/n \cdot 1/n \,/\, 1/n^2 + 2/n^2 = 3n^{-2}/3n^{-2} = 1$ converges to $1 \neq f(0,0)$. Then, f is not continuous at $(0,0)$ hence not differentiable at $(0,0)$.

d i $g(t,-t,1) = 1(t+-t) - -t \ln 1 = 1(0) + t(0) = 0$

$g_x = z$, $g_y = z - \ln z$, $g_z = (x+y) - y/z$

The implicit function theorem applies ~~to~~ defines z such that $g(x,y,z)=0$ as a function of x,y around $(t,-t)$ iff ~~$(t,-$~~ $g(t,-t,z)=0$, g is C^1 in an open ball around $(t,-t,z)$ and $g_z(t,-t,z) \neq 0$.

Given $x=t$, $y=-t$, $g(x,y,z)=0$, then $z=1$. Then by inspection of the partial derivatives, g is C^1 in an open ball around all such points. ~~and~~
$g(t,-t,z)=0$ is satisfied by $z=1$ and $g_z \neq 0$ iff $y=-t \neq 0 \iff t \neq 0$.

$g(x,y,z)=0$ defines z as an implicit function of (x,y) around $(t,-t)$ iff $t \neq 0$.

ii ~~$z_x = g_x/g_z$~~
$z_x(t,-t) = -g_x(t,-t,1)/g_z(t,-t,1) = 1/t$
$z_y(t,-t) = -g_y(t,-t,1)/g_z(t,-t,1) = 1/t$

negative sign here

$Df(0,0,v) =$ ~~$f(tv_1, tv_2) -$~~

$= \lim_{t\to 0} \frac{f(tv_1, tv_2) - f(0,0)}{t}$

(This is in the slides)

directional derivative $= \nabla f(\vec{x}) \cdot \vec{v}$ only if f is C^1.

Be careful ~~that~~ to note which positivity constraints are made redundant.

2ai (P): $\max_{x,y} f(x,y)$ s.t. False, add^{e} requires $\lambda \geq 0$

$x + 2y \leq 9$

$x \geq 0,\ y \geq 0$

Bonus points for counterexample

By Lagrangian sufficiency, the statement is true.

Lagrangian sufficiency:
If there exists $\vec{\lambda}^*, \vec{x}^*$ such that $\mathcal{L}(\vec{x}^*, \vec{\lambda}^*) = f(\vec{x}^*) - \vec{\lambda}^*(\vec{g}(\vec{x}^*) - \vec{b})$ is maximised, and $\vec{\lambda}^*(\vec{g}(\vec{x}^*) - \vec{b}) = \vec{0}$, and $g(\vec{x}^*) \leq \vec{b}$, i.e. $\vec{x}^*$ is feasible, then $\vec{x}^*$ ~~maximise~~ solves the maximisation problem $\max_{\vec{x}} f(\vec{x})$ ~~st~~ s.t. $\vec{g}(\vec{x}) \leq \vec{b}$.

ii By the Weierstrass extreme value theorem, if f is continuous, and the constraint set is convex then a maximum exists. By inspection, the constraint set is convex, so indeed a maximum exists.

If the constraint qualification is satisfied, the KT FOCs are necessary for an optimum. The CQ is the condition that the Jacobian matrix of the binding constraints is full rank. By inspection, at any point, at most two constraints bind. Then, the Jacobian of the binding constraints is one of $(1\ \ 2)$, $(1\ \ 0)$, $(0\ \ 1)$, $\begin{pmatrix}1 & 2\\ 1 & 0\end{pmatrix}$, $\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$, $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$. By inspection, each of these is full rank, so CQ is satisfied at every point and KT FOCs are necessary for an optimum at every point.

If (x^*, y^*) uniquely satisfies the KT FOCs, it is the only candidate maximum. The Weierstrass extreme value theorem guarantees the existence of a maximum, so (x^*, y^*) then ~~is the~~ must be the maximum. all exactly correct

iii ? False, counterexample quasi concave utility fn can have corner sol^{n}

Are there conditions on the constraint set for this? Is MU or MRS the impt condition?

bi $u_x = 3(1/2)x^{-1/2} = 3/2\, x^{-1/2}$

As x approaches 0, u_x explodes to ∞, (and u_y remains constant at 1), so the consumer never chooses $x = 0$. The implicit positivity constraint on x never binds.

ii $\mathcal{L} = 3\sqrt{x} + y - \lambda(x + 2y - m)$ $+ \gamma y$ should include +vity constraint for y

$FOC_x: 3(1/2)x^{-1/2} - \lambda = 3/2\, x^{-1/2} - \lambda = 0$

$FOC_y: 1 - 2\lambda = 0$

$CS_\lambda: \lambda \geq 0,\ x + 2y \leq m,\ \lambda(x + 2y - m) = 0$

$CS_\gamma: \gamma \geq 0,\ y \geq 0,\ \gamma y = 0$

~~From FOCy,~~

$FOC_y \Rightarrow \lambda = 1/2$

$CS_\lambda,\ \lambda = 1/2 \Rightarrow x + 2y = m$

$FOC_x,\ \lambda = 1/2 \Rightarrow 3/2\, x^{-1/2} = 1/2$

$\Rightarrow x^{-1/2} = 1/3 \Rightarrow x^{1/2} = 3 \Rightarrow x = 9$

$x = 9,\ x + 2y = m \Rightarrow y = m - 9/2$

The solution to the KT FOCs is $(x, y) = (9, m - 9/2)$

By inspection, the objective function u is concave, and the constraint is linear hence (weakly) convex. Then the ~~KT FOCs~~ optimisation problem is concave and the KT FOCs are sufficient for an optimum. By inspection, the constraint set has non empty interior ($(m/10, m/10)$ satisfies the constraints with strict equality) so the KT FOCs are also necessary for an optimum. ~~Then,~~ exactly correct explanation

So $(9, m - 9/2)$ is in fact an optimum.

iii The time constraint is binding for m such that the optimal bundle under only the budget constraint is not feasible, which is iff this bundle violates the time constraint.

~~3x + 4y >~~

$3x + 4y > 48 \Leftrightarrow 3(9) + 4(m - 9/2) > 48 \Leftrightarrow$
$27 + 2m - 18 > 48 \Leftrightarrow 2m > 39,\ m > 39/2$.

iv $\max_{x,y} 3\sqrt{x} + y$ s.t.

BC: $x + 2y \leq m$

TC: $3x + 4y \leq 48$

Two approaches:
One Lagrangian many cases
Multiple Lagrangian, fewer cases each

$\mathcal{L} = 3\sqrt{x} + y - \lambda_B(x + 2y - m) - \lambda_T(3x + 4y - 48)$

$FOC_x: 3/2\, x^{-1/2} - \lambda_B - 3\lambda_T = 0$

$FOC_y: 1 - 2\lambda_B - 4\lambda_T = 0$

$CS_B: \lambda_B \geq 0,\ x + 2y \leq m,\ \lambda_B(x + 2y - m) = 0$

$CS_T: \lambda_T \geq 0,\ 3x + 4y \leq 48,\ \lambda_T(3x + 4y - 48) = 0$

Suppose that $\lambda_T = 0$, then FOC_x, FOC_y, CS_B are identical to the problem above and have solution $(9, m - 9/2)$, which also solves CS_T iff $m \leq 39/2$.

Suppose that $\lambda_B = \lambda_T = 0$, then FOC$_y$ reduces to $1=0$. ~~By~~ By reductio, there is no solution such that $\lambda_B = \lambda_T = 0$.

Suppose that $\lambda_B = 0$, then $\lambda_T \neq 0$ and FOC$_x$, FOC$_y$, ~~CS$_T$~~ reduce to ~~$\frac{3}{5}$~~

$\frac{3}{2}x^{-1/2} - 3\lambda_T = 0$

$1 - 4\lambda_T = 0$

$\Rightarrow \lambda_T = \frac{1}{4}$, $\frac{1}{2}x^{-1/2} = \frac{1}{4} \Rightarrow x^{-1/2} = \frac{1}{2} \Rightarrow x^{1/2} = 2 \Rightarrow x = 4$

$\Rightarrow 3(4) + 4y = 48 \Rightarrow y = 9$

$(x,y) = (4,9)$ solve FOC$_x$, FOC$_y$ and satisfy CS$_T$.
CS$_B$ is satisfied iff $x + 2y \leq m \Leftrightarrow m \geq 22$.

Suppose that $\lambda_B, \lambda_T \neq 0$, then CS$_B$, CS$_T$ $\Rightarrow$
$x + 2y = m$, $3x + 4y = 48 \Rightarrow y = \frac{m-x}{2} \Rightarrow 3x + 2(m-x) = 48$
$\Rightarrow x = 48 - 2m$, $y = \frac{3m-48}{2}$. ~~For suitable λ_B~~
~~For~~ It can be verified that ~~this implies $\lambda_B, \lambda_T > 0$ so CS$_B$, CS$_T$ are satisfied~~ suitable $\lambda_B, \lambda_T > 0$ satisfy FOC$_x$, FOC$_y$, for $\frac{39}{2} \leq m \leq 22$.

The solution to the KT-FOCs is as follows
For $m \leq \frac{39}{2}$, $x = 9$, $y = \frac{m-9}{2}$
For $\frac{39}{2} \leq m \leq 22$, $x = 48 - 2m$, $y = \frac{3m-48}{2}$
For $m \geq 22$, $x = 4$, $y = 9$

Given that the additional constraint is linear hence weakly convex, this optimisation problem remains concave and by the earlier argument, the KT-FOCs are necessary and sufficient for a maximum. The above are the solutions to the optimisation problem.

Possible to draw diagram

3a ~~For all menus A ⊆ A'~~, choice function c satisfies α iff for all menus $A \subseteq A'$, if $c(A') = x$ and ~~$x \in A'$~~ $x \in A$, then $c(A) = x$. In other words, choice function c satisfies α iff the removal of unselected options from some menu does not affect the option that c selects. Good to give definition.

α cannot be applied because neither $A_1 \subseteq A_2$ nor $A_2 \subseteq A_1$. For example, $(20,0) \in A_1$ but $(20,0) \notin A_2$ and $(0,30) \in A_2$ but $(0,30) \notin A_1$.
exactly correct.

ii ~~$(16,0) \in A_1$ (because $16\times1 + 0\times1 \leq 20$), then $c(A_1) = (10,10)$ reveals $(10,10) \succsim (16,0)$, where $\succsim$ denotes D's weak preference relation.~~

Suppose that D chooses by choice function $c(\cdot, \succsim)$ where $\succsim$ is D's weak preference relation. ~~This mechanis~~ Not nec to introduce these mechanics.

$(16,0) \in A_1$ because $16\times1 + 0\times1 \leq 20$, then $c(A_1, \succsim) = (10,10) \Rightarrow ((10,10) \succsim (16,0))$. By strict monotonicity, $(16,0) \succ (15,0)$, i.e. $(16,0) \succsim (15,0)$ but not $(15,0) \succsim (16,0)$. Suppose for reductio that D has rational preferences, then $\succsim$ is complete and transitive. ~~By transitivity of $\succsim$, $(10,10) \succsim (15,0)$~~ $(10,10) \in A_2$ because $10\times2 + 10\times1 \leq 30$, then $c(A_2, \succsim) = (15,0) \Rightarrow (15,0) \succsim (10,0)$. By transitivity of $\succsim$, $(15,0) \succsim (16,0)$. By reductio, if D chooses by choice function $c(\cdot, \succsim)$, then D's weak preference relation $\succsim$ is irrational.
Exactly correct to find contradiction

iii Weak axiom of revealed preference is violated. Choice function c satisfies the weak axiom of revealed preference iff for all menus A, A' for all options $x \neq x' \in A, A'$, it is not the case that $c(A) = x$ and $c(A') = x'$. exactly correct

In this case, $(10,10) \neq (15,0) \in A_1, A_2$, but $c(A_1) = (10,10)$ and $c(A_2) = (15,0)$, so the weak axiom of revealed preference is violated.

very important

There is no strictly monotone utility function that represents the preferences which yield these choices. Suppose for reductio that there is such utility function u. Then from the argument in (aii) $u(10,10) \geq u(16,0) > u(15,0) \geq u(10,10) \Rightarrow u(10,10) > u(10,10)$. By reductio, there is no such utility function.

bi D is transitive.
Possible to graph these preferences

Consider arbitrary $\vec{x}, \vec{y}, \vec{z}$. Suppose $\vec{x} D \vec{y}$ and $\vec{y} D \vec{z}$. Then by construction of D, $3x_1 + x_2 \geq 3y_1 + y_2$, $x_1x_2 \geq y_1y_2$, $3y_1 + y_2 \geq 3z_1 + z_2$, $y_1y_2 \geq z_1z_2$. Then by transitivity of $\geq$ on the real numbers, $3x_1 + x_2 \geq 3z_1 + z_2$ and $x_1x_2 \geq z_1z_2$, then by construction of D, $\vec{x} D \vec{z}$. By conditional proof, generalisation, for all $\vec{x}, \vec{y}, \vec{z}$, if $\vec{x} D \vec{y}$ and $\vec{y} D \vec{z}$ then $\vec{x} D \vec{z}$. By definition of transitivity, D is transitive (on ~~bundles~~ $X = \mathbb{R}^2_+$)

ii D is reflexive. Consider arbitrary $\vec{x}$. $3x_1 + x_2 \geq 3x_1 + x_2$, $x_1x_2 \geq x_1x_2$, then by construction of D, $\vec{x} D \vec{x}$. By generalisation, for all $\vec{x}$, $\vec{x} D \vec{x}$, then by definition of reflexivity, D is reflexive.

D is not (weakly) connected.
Consider $\vec{x} = (10,0)$ and $\vec{y} = (5,5)$.
$3y_1 + y_2 = 20 \not\geq 3x_1 + x_2 = 30 \Rightarrow \neg \vec{y} D \vec{x}$
$x_1x_2 = 0 \not\geq y_1y_2 = 25 \Rightarrow \neg \vec{x} D \vec{y}$
$\Rightarrow \neg \vec{x} D \vec{y}$ and $\neg \vec{y} D \vec{x} \Rightarrow$ D is not (weakly) connected $\Rightarrow$ D is not complete (i.e. strictly connected).

iii ~~Consider arbitrary $\vec{x}$. Suppose for reductio that D is antisymmetric. Consider arbitrary $\vec{x}, \vec{x}'$. Suppose that $\vec{x} D \vec{x}'$ and~~

~~Consider arbitrary $\vec{x}, \vec{x}'$. Suppose $\vec{x} D \vec{x}'$ and $\vec{x}' D \vec{x}$. Then $3x_1 + x_2 \geq 3x_1' + x_2'$, $x_1x_2 \geq x_1'x_2'$, $3x_1' + x_2' \geq 3x_1 + x_2$ and $x_1'x_2' \geq x_1x_2$, then $3x_1 + x_2 = 3x_1' + x_2'$ and $x_1x_2 = x_1'x_2'$. Let $x_1x_2 = x_1'x_2' = k$, then $x_2 = k/x_1$, $x_2' = k/x_1'$. Let $3x_1 + x_2 = 3x_1' + x_2' = c$. Then $3x_1 + k/x_1 = 3x_1' + k/x_1' = c$, $3x_1^2 - cx_1 + k = 0$ and $3x_1'^2 - cx_1' + k = 0$~~

~~Let $\vec{x} = (1,0)$, $\vec{x}' = (0,3)$, then $3x_1 + x_2 = 3 = 3x_1' + x_2'$ and x_1~~
Easier with $(1,0)$, $(0,3)$ "turn off second ineq"
Let $\vec{x} = (1,6)$, $\vec{x}' = (2,3)$, then $3x_1 + x_2 = 9 = 3x_1' + x_2'$ and $x_1x_2 = 6 = x_1'x_2'$, then $\vec{x} D \vec{x}'$ and $\vec{x}' D \vec{x}$, so D is not antisymmetric.

ci Consider some arbitrary bundle $\vec{x}$ not in the frontier of the budget constraint, i.e. $p_1x_1 + p_2x_2 < m$, then there exists bundle $\vec{x}'$ such that $x_1' = x_1 + \varepsilon$, $x_2' = x_2 + \varepsilon$ such that $p_1x_1' + p_2x_2' = (p_1x_1 + p_2x_2) + (p_1 + p_2)\varepsilon \leq m$ for sufficiently small ε. Then, $3x_1' + x_2' > 3x_1 + x_2$, $x_1'x_2' > x_1x_2$, hence $\vec{x}' D \vec{x}$ and not $\vec{x} D \vec{x}'$, i.e. $\vec{x}$ is strictly worse than $\vec{x}'$, and $\vec{x}$ is eliminated. By generalisation, all $\vec{x}$ not in the frontier are eliminated. exactly correct

ii In A_1, ~~(20,0) maximises~~ for all $\vec{x}$ in the frontier, ~~$x_1 = 20$~~ $x_2 = 20 - x_1$, then $3x_1 + x_2 = 2x_1 + 20$ and $x_1x_2 = x_1(20 - x_1)$. Then $(20,0)$ maximises $3x_1 + x_2$ and $(10,10)$ maximises x_1x_2. From $(10,10)$ to $(20,0)$ along the frontier, $3x_1 + x_2$

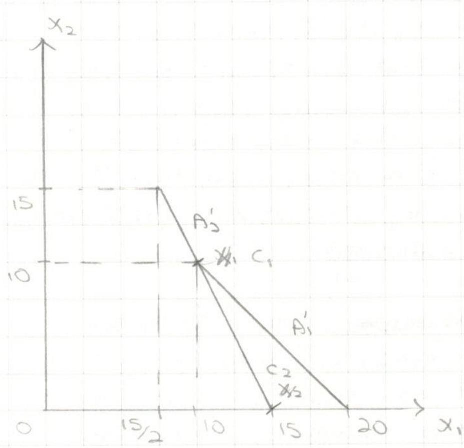
increases and x_1x_2 decreases, so within this range, no bundle is strictly preferred to any other. Outside this range, on the frontier, from $(0,20)$ to $(10,10)$, $3x_1+x_2$ and x_1x_2 are both lower than at $(10,10)$. All such bundles are eliminated. No bundles are strictly ~~preffe pref~~ preferred over those along $(10,10)$ $(20,0)$, so all such bundles survive and D froze.

In A_2, along the frontier, $x_2 = 30-2x_1$, $3x_1+x_2 = 30+x_1$, $x_1x_2 = x_1(30-2x_1)$. $3x_1+x_2$ is maximised at ~~(x_1,x_2)~~ $(15,0)$. x_1x_2 is maximised at $(^{15}/_2, 15)$. From $(^{15}/_2,15)$ to $(15,0)$ along the frontier, $3x_1+x_2$ increases and x_1x_2 decreases, so none of these bundles is strictly preferred to another. From $(0,30)$ to $(^{15}/_2,15)$, $3x_1+x_2$ and x_1x_2 are ~~lower~~ both lower than at $(^{15}/_2,15)$, so $(^{15}/_2,15)$ is strictly preferred to these bundles and these bundles are eliminated. All and only bundles between $(^{15}/_2,15)$ and $(\cancel{20}\,15,0)$ on the frontier survive and D froze. exactly correct

iii ~~Such a menu is suc~~ The required menu is such that the maximum of $3x_1+x_2$ and x_1x_2 coincide. On the frontier, $x_2 = {}^{m-p_1x_1}/_{p_2}$. ~~set~~ Without loss of generality, let ~~*~~ $p_2=1$. Then $x_2 = m-px_1$. Then $3x_1+x_2 = m+(3-p)x_1$. $x_1x_2 = x_1(m-px_1)$. $\partial/\partial x_1\ x_1x_2 = m-2p_1x_1$. A maximum obtains when $m-2p_1x_1=0 \Rightarrow x_1 = {}^m/_{2p_1} \Rightarrow x_2 = {}^m/_2$. This maximises $3x_1+x_2$ iff $\partial/\partial x_1\ 3x_1+x_2 = 3-p_1=0 \Leftrightarrow p_1=3$. exactly correct

~~The the~~ The menu that satisfies the requirements is $\{(x_1,x_2): 3x_1+x_2 \leq 12\}$, then $(x_1,x_2)=(2,6)$ maximises both $3x_1+x_2$ and x_1x_2, ~~so i~~ and strictly maximises x_1x_2, so is strictly preferred to every other bundle.

iv ~~The~~ Let the submenu of uneliminated bundles be A_1' A_2'. $A_1' = \{(x_1,x_2): x_1\in[10,20], x_1+x_2=20\}$, $A_2' = \{(x_1,x_2): x_1\in[^{15}/_2, 15], \cancel{x_1+x_2}\ 2x_1+x_2=30\}$.



It could not have been the same rational friend. ~~$(10,10) \in A_2'$ and A_1' and $(15,5)$~~ Suppose for reductio that it was. $(10,10)\in A_1'$, $(15,5)\in A_1'$. ~~that~~ $c'(A_1') = (10,10)$ $\Rightarrow (10,10) \succ' (15,5)$. By strict monotonicity and transitivity, $(10,10) \succ' (15,0)$. ~~c'f~~ $(10,10), (15,0) \in A_2'$. $c'(A_2') = (15,0)$ $\Rightarrow (15,0) \succsim (10,10) \Rightarrow \neg\,(10,10) \succ' (15,0)$. By reductio the choices do not correspond to a common rational (strictly monotone) preference $\succsim$.

4a $\int_{-10}^{10} f(\varepsilon)\,d\varepsilon = 1 \Leftrightarrow$

$\int_{-10}^{10} c(10+\varepsilon)^2\,d\varepsilon = \cancel{8}\,[c(1/3)(10+\varepsilon)^3]_{-10}^{10}$

$= c/3(10+10)^3 - c/3(10-10)^3$

$= c/3(20)^3$

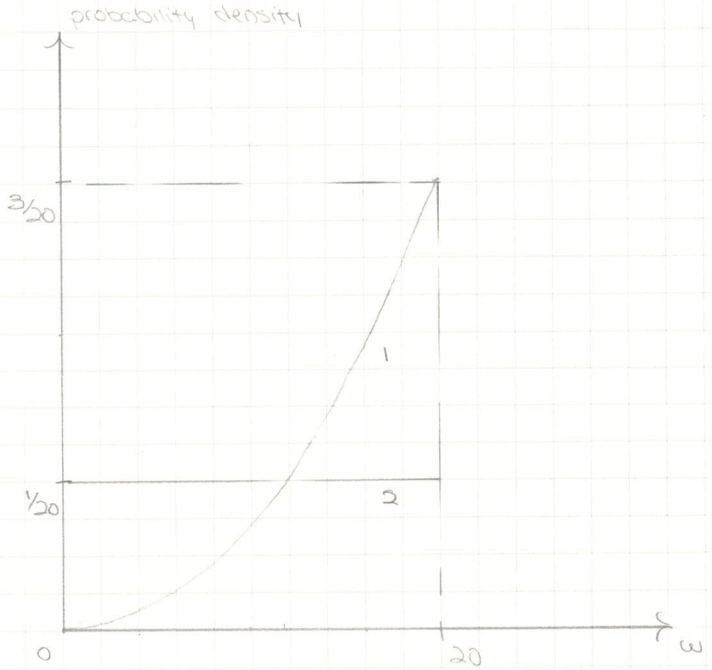
$= 1 \Leftrightarrow$

$8000c/3 = 1 \Leftrightarrow$

$c = 3/8000$

$f(-10) = c(10-10)^2 = 0$

$f(10) = c(10+10)^2 = 3/20$ exactly correct



ii $P(\varepsilon < 20a^{1/3}-10) = \int_{-10}^{20a^{1/3}-10} f(\varepsilon)\,d\varepsilon$

$= \int_{-10}^{20a^{1/3}-10} c(10+\varepsilon)^2\,d\varepsilon$

$= c\,[1/3(10+\varepsilon)^3]_{-10}^{20a^{1/3}-10}$

$= c\,[1/3(20a^{1/3})^3 - 1/3(0)^3]$

$= c\,[1/3(8000a)]$

$= 3/8000 \cdot 8000a/3$

$= a$ exactly correct

iii $P(\varepsilon > 5) = 1 - P(\varepsilon < 5)$

$20a^{1/3} - 10 = 5 \Leftrightarrow 20a^{1/3} = 15 \Leftrightarrow a^{1/3} = 3/4 \Leftrightarrow a = (3/4)^3 = 27/64$

$P(\varepsilon > 5) = 1 - P(\varepsilon < 5)$

$= 1 - P(\cancel{20a^{1/3}}\ \varepsilon < 20a^{1/3} - 10)$

where $a = 27/64$

$= 1 - 27/64$

by the result in (aii)

$= 37/64$

bi Let the lottery in final wealth values associated with ~~Job~~ job 1 ~~and job 2~~ be L_1 and that associated with job 2 be L_2.

~~$L_1 \succ_{FOSD} L_2$~~ Some lottery with pdf h FOSDs another with pdf h' iff for all $a \in \mathbb{R}$, $\int_{-\infty}^{a} h(x)\,dx \le \int_{-\infty}^{a} h'(x)\,dx$.

$\int_{-10}^{a} f(x)\,dx = [c/3(10+\varepsilon)^3]_{-10}^{a}$

$= c/3(10+a)^3$

$\int_{-10}^{a} g(x)\,dx = 10+a/20$

$\int_{-10}^{a} f(x)\,dx \le \int_{-10}^{a} g(x)\,dx \Leftrightarrow$

$c/3(10+a)^3 \le 10+a/20 \Leftrightarrow$

$1/400\,(10+a)^2 \le 1 \Leftrightarrow$

$(10+a)^2 \le 400 \Leftrightarrow$

$-30 \le a \le 10$

For all $a \in [-10, 10]$,

$\int_{-\infty}^{a} f(x)\,dx = \int_{-10}^{a} f(x)\,dx \le \int_{-10}^{a} g(x)\,dx = \int_{-\infty}^{a} g(x)\,dx$

For all $a \ge 10$,

$\int_{-\infty}^{a} f(x)\,dx = 1 \le \int_{-\infty}^{a} g(x)\,dx = 1$

Then, the lottery of shocks associated with job 1 with pdf f FOSDs the lottery of shocks associated with job 2 with pdf g. Then, the lottery of final wealth values associated with job 1 FOSDs that associated with job 2. This is because the pdfs in final wealth are obtained from the pdfs in shocks by an identical translation of 10 units in the positive w direction. So the inequalities contince to hold.

L_2 does not FOSD L_1 ~~because the lottery of μ does not FOSD that of ε.~~ By inspection of the graph in (ai), for small w, the area under the pdf for job 2 exceeds that for job 1, so the cdf for job 2 exceeds that for job 1.

~~The cd~~ L_2 attaches higher probabilities to less favourable outcomes, L_1 FOSDs L_2, any expected utility maximiser prefers L_1 to L_2.

Then $CE_1 > CE_2$. By definition, given an ~~agent~~ expected utility maximiser with Bernoulli utility u, $L_1 \sim [1; CE_1]$ and $L_2 \sim [1; CE_2]$, then $U(L_1) = u(CE_1)$ and $U(L_2) = u(CE_1)$. Given that $L_1 \succ_{FOSD} L_2$, it is also the case that $L_1 \succ L_2$ hence $U(L_1) > U(L_2)$ and $u(CE_1) > u(CE_2)$. By monotonicity of u, $CE_1 > CE_2$.

ii $EV_1 = \int_{-10}^{10} (10+\varepsilon) f(\varepsilon)\,d\varepsilon$

$= \int_{-10}^{10} (10+\varepsilon)\,c\,(10+\varepsilon)^2\,d\varepsilon$

$= [c/4(10+\varepsilon)^4]_{-10}^{10}$

$= [c/4(20)^4 - c/4(0)^4]$

$= 3/2^5 10^3 \quad 2^4 10^4$

$= 30/2$

$= 15$

ii Given Bernoulli utility, $u(w) = w^{1-r}$,
$u'(w) = (1-r)w^{-r}$, $u''(w) = (1-r)(-r)w^{-r-1} < 0$
(given $r<1$)

A is risk averse and has positive risk premium RP on any ~~less than cer~~ non degenerate lottery (~~where there is more than or~~ where the outcome is not certain).

Then A has positive risk premium on L_1, hence $CE_1 = EV_1 - RP_1 < EV_1 = 15$.

By concavity of u, the expected utility of L_1 is less than the utility of the expected value, then by monotonicity of u and definition of CE, the certainty equivalent of L_1 is less than the expected value of L_1.

iii It is not always true that $CE_1 < 10$. For ~~sufficiently~~ ~~r sufficiently close to~~ consider r arbitrarily close to 1, close to zero then $u(w)$ is only marginally concave, then the expected utility of L_1 is only marginally below the utility of ~~the exp~~ EV_1, then CE_1 is only marginally below $EV_1 = 15$. So $CE_1 > 10$.

iii
$$u(L_1) = \int_{-10}^{10} (10+\varepsilon)^{1-r}\, c(10+\varepsilon)^2 \, d\varepsilon$$
$$= c\left[\tfrac{1}{4-r}(10+\varepsilon)^{4-r}\right]_{-10}^{10}$$
$$= c\left[\tfrac{1}{4-r}(20)^{4-r}\right]$$
$$= \tfrac{1}{4-r}(c)(20)^4 \tfrac{1}{20^r}$$
$$= \tfrac{1}{4-r}(60)\left(\tfrac{1}{20^r}\right)$$
$$= \tfrac{60}{4-r}(20^{-r})$$

iii $u(CE_1) = u(L_1) \iff$
$$CE_1^{1-r} = \tfrac{60}{4-r}(20^{-r})$$

ci True.

~~be~~ consider arbitrary wealth distribution over $[0,20]$ described by pdf h. Denote the lottery associated with this distribution L_h.

Let $u(L_h) = \int_{\underline{w}}^{\bar{w}} h(x)\,dx$. $u(L_h)$ is the fraction of wealth levels belonging to the interval $[\underline{w}, \bar{w}]$ in the distribution described by h. Then ~~by~~ given the description of B's preferences, B ~~prefers~~ $h \succsim_B h'$ iff $u(L_h) \geq u(L_{h'})$. By the completeness and transitivity of $\geq$ over the real numbers (and the fact that for all L_h, $u(L_h)$ exists and is a real number), $\succsim$ is complete and transitive on all such ~~distribut~~ lotteries.

ii Whether B prefers job 1 or job 2 is contingent on the values of $\underline{w}$ and $\bar{w}$. By inspection of the graph in (ai), there exist $[\underline{w}, \bar{w}]$, for example $[19,20]$ such that ~~the~~ the distribution associated with job 1 has greater probability mass within this band, ~~and there ex~~ so B prefers job 1. There also exist intervals, for example $[0,20]$, such that job 2 has greater probability mass in this interval and B prefers job 2.

5a $E(u(w,e) \mid e=1) = \frac{3}{5}(\sqrt{100}-1) + \frac{2}{5}(\sqrt{0}-1) = 5$

$E(u(w,e) \mid e=0) = \frac{2}{5}(\sqrt{100}-0) + \frac{3}{5}(\sqrt{0}-0) = 4$

$E(u(w,e) \mid e=1) > E(u(w,e) \mid e=0)$.

Agent (A) ~~option~~ maximises expected utility, so optimally chooses $e=1$ which yields utility 5. This is A's reservation utility $\bar{u}$.

b Given a contract that ~~requires that~~ requires e, the principal P chooses f, ~~&~~ c to maximize expected payoff $P(S|e)f - P(F|e)c$ subject to the participation constraint (PC) ~~P(S|e)~~
$P(S|e)[u(100-f, e)] + P(F|e)[u(0+c, e)] \geq \bar{u} = 5$, where S and F denote the events success and failure respectively.

$\max_{f,c} \; P(S|e)f - P(F|e)c$ s.t.

PC: $P(S|e)u(100-f, e) + P(F|e)u(c,e) \geq \bar{u} = 5$

At any optimum PC binds. Every candidate optimum such that PC does not bind fails to deviation by reducing c or increasing f by sufficiently small ε such that PC remains satisfied.

At any optimum, $100-f = c$. Every candidate optimum such that $100-f \neq c$ fails to the following deviation. Let $\bar{w} = P(S|e)(100-f) + P(F|e)c$. By strict concavity of u in w, $u(\bar{w}, e) > P(S|e)u(100-f, e) + P(F|e)u(c,e) \geq \bar{u}$. Then set f, c such that $100-f = c = \bar{w} - \varepsilon$ for sufficiently small ε such that PC remains satisfied. This increases expected payoff to P by $\varepsilon > 0$.

From the above, the unique optimum is such that PC binds and $100-f = c$.

Suppose $e=1$. ~~Denote~~ Let $100-f = c = w_1$. Then
$\frac{3}{5}(\sqrt{w_1}-1) + \frac{2}{5}(\sqrt{w_1}-1) = \sqrt{w_1}-1 = \bar{u} = 5 \Rightarrow w_1 = 36$
$\Rightarrow f = 64, c = 36. \Rightarrow \pi_1 = \frac{3}{5}64 - \frac{2}{5}36 = \cancel{50.3}\ 24$

Suppose $e=0$. Let $100-f = c = w_0$. Then
$\frac{2}{5}\sqrt{w_0} + \frac{3}{5}\sqrt{w_0} = \sqrt{w_0} = \bar{u} = 5 \Rightarrow w_0 = 25 \Rightarrow f = 75, c = 25$
$\Rightarrow \pi_0 = \frac{2}{5}75 - \frac{3}{5}25 = 15$

P's optimal contract is $(e^*, f^*, c^*) = (1, 64, 36)$, which yields profit $\pi^* = 24$.

Intuitively, P finds it optimal to bear all the risk because A is risk averse and P is risk neutral so A must be compensated for risk bearing with a higher expected wage, which ~~the~~ decreases P's payoff. Then P offers a fixed wage which is just acceptable to A. P finds it optimal to require high effort because this maximises total payoff of which A captures a fixed amount and P captures any excess.

c P's optimisation problem in inducing high effort is

$\max_{f,c} \; \frac{3}{5}f - \frac{2}{5}c$ s.t.

PC: $\frac{3}{5}u(100-f, 1) + \frac{2}{5}u(c, 1) \geq \bar{u} = 5 \Leftrightarrow$
$\frac{3}{5}\sqrt{100-f} + \frac{2}{5}\sqrt{c} - 1 \geq 5 \Leftrightarrow$
$\frac{3}{5}\sqrt{100-f} + \frac{2}{5}\sqrt{c} \geq 6$

IC: $\frac{3}{5}u(100-f, 1) + \frac{2}{5}u(c, 1) \geq \frac{2}{5}u(100-f, 0) + \frac{3}{5}u(c, 0)$
$\Leftrightarrow \frac{3}{5}\sqrt{100-f} + \frac{2}{5}\sqrt{c} - 1 \geq \frac{2}{5}\sqrt{100-f} + \frac{3}{5}\sqrt{c} \Leftrightarrow$
$\frac{1}{5}\sqrt{100-f} - 1 \geq \frac{1}{5}\sqrt{c} \Leftrightarrow$
$\sqrt{100-f} \geq \sqrt{c} + 5$

At any optimum PC binds. Every candidate optimum such that PC does not bind fails to deviation by reducing c by sufficiently small amount ε such that PC remains satisfied. ~~PC~~ IC remains satisfied and is satisfied more strictly. Payoff increases by $\frac{2}{5}\varepsilon$.

At any optimum IC binds. Every candidate optimum such that IC does not bind fails to the following deviation. Let $\sqrt{100-f} = v_f$ and $\sqrt{c} = v_c$. Increase v_f by ~~sufficient~~ ε and decrease v_c by $\frac{3}{2}\varepsilon$ for sufficiently small ε such that IC remains satisfied. PC remains satisfied. f increases and c decreases so payoff increases.

Then at the optimum, $\frac{3}{5}v_f + \frac{2}{5}v_c = 6$, $v_f = v_c + 5$
$\Rightarrow \frac{3}{5}v_c + \frac{2}{5}v_c + 3 = 6 \Rightarrow v_c = 3 \Rightarrow v_f = 8$
$\Rightarrow 100-f = 64$, $c = 9$, $f = 36 \Rightarrow \pi = \frac{3}{5}(36) - \frac{2}{5}(9) = 18$

To induce low effort, P offers a fixed wage as in (b). ~~A~~ The optimum in (b) satisfies the PC (for low effort) because under a fixed wage A has strict incentive to choose $e=0$. So the optimal contract is $(f=75, c=25)$ which yields payoff to P of $\pi = 15$.

By comparison, the optimal contract ~~is~~ is $(f=36, c=9)$ which induces $e=1$.

Intuitively, PC and IC bind in inducing high effort because incentivising high effort is costly to P because this requires A to bear risk for which A must be compensated so that participation remains optimal, so P finds it optimally to have high effort only just incentive compatible.

d The optimal contract in (c) compared to (b) is a variable wage and not a fixed wage, and yields lower profit. A variable wage is necessary to induce high effort because under a fixed wage, A has strict incentive

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to choose low effort (effort only has a negative direct effect on utility). Then, under a variable wage, because A is risk averse, participation is optimal only if A has higher expected wage than under the fixed wage. So P must offer A higher expected wage and has lower expected profit. The difference is equal to the agency cost which is the cost incurred in incentivising high effort when effort is unobservable. Because incentivising high effort is costly, P optimally "only just" incentivises high effort, so IC binds.

Under different parameters, if the agency cost is sufficiently high, it might be that it is optimal to induce low effort when it is unobservable, but that is not the case here.