Sider Modal Propositional Logic Semantics Exercises

~~1 Base case~~

~~Consider arbitrary MPL-wff ϕ such that cont complexity, i.e. number of connectives $\sim, \to, \Box$, $C(\phi)=0$. Consider arbitrary Then ϕ is a sentence letter~~.

~~Suppose that~~

Consider arbitrary MPL-wff ϕ. Suppose that $\vDash_{S5}\phi$, then $V_M(\phi, w) = 1$ for all S5-models $M = \langle W, R, I\rangle$ and all $w \in W$. Every O-model is a S5-model, because if R is total on W, then it is also an equivalence relation on W. So $V_M(\phi, w) = 1$ for all O-models $M = \langle W, R, I\rangle$ and all $w \in W$, hence $\vDash_O \phi$.

~~Suppose that $\vDash_O\phi$~~. Suppose ~~for reductio~~ that $\nvDash_{S5}\phi$. Then, by definition, for some S5 model $M = \langle W, R, I\rangle$, and some world $w \in W$, $V_M(\phi, w) = 0$. Let M° be the O model ~~whose~~ $\langle W^\circ, R^\circ, I^\circ\rangle$ such that $W^\circ = \{u \in W : Rwu\}$, $R^\circ = \{\langle u^\circ, v^\circ\rangle : u^\circ, v^\circ \in W^\circ\}$, $I^\circ(\alpha, w^\circ) = I(\alpha, w)$ for all sentence letters α and $w^\circ \in W^\circ$. $V_{M^\circ}(\phi, w) = 0$ (proof below), so $\nvDash_O \phi$.

By biconditional proof, $\vDash_O\phi$ iff $\vDash_{S5}\phi$

~~Base case~~

~~Consider arbitrary MPL-wff ϕ such that complexity, i.e. number of connectives $(\sim, \to, \Box)$ $C(\phi) = 0$.~~

Consider arbitrary S5-model $M = \langle W, R, I\rangle$ and arbitrary $w \in W$. Let M° be the O model $\langle W^\circ, R^\circ, I^\circ\rangle$ such that $W^\circ = \{u \in W : Rwu\}$, $R^\circ = \{\langle u^\circ, v^\circ\rangle : u^\circ, v^\circ \in W^\circ\}$, $I^\circ(\alpha, w^\circ) = I(\alpha, w^\circ)$ for all sentence letters α, worlds $w^\circ \in W^\circ$.

~~Consider arbitrary MPL-wff ϕ such that complexity, i.e. number of connectives $(\sim, \to, \Box)$, $C(\phi) = 0$.~~

Base case.

Consider arbitrary MPL-wff ϕ such that complexity, i.e. number of connectives $(\sim, \to, \Box)$ in ϕ, $C(\phi) = 0$. Then ϕ is a sentence letter α. $V_M(\phi, w) = I(\alpha, w) = I^\circ(\alpha, w^\circ) = V_{M^\circ}(\alpha, w^\circ)$. ~~For all ϕ~~ By generalization, for all MPL-wffs ϕ such that $C(\phi) = 0$, $V_M(\phi, w^\circ) = V_{M^\circ}(\phi, w^\circ)$ for all $w^\circ \in W^\circ$

Inductive Hypothesis

For given n, for all $m < n$, for all MPL-wffs ϕ such that $C(\phi) = m$, $V_M(\phi, w^\circ) = V_{M^\circ}(\phi, w^\circ)$ for all $w^\circ \in W^\circ$

Induction step

Consider arbitrary MPL-wff ϕ such that $C(\phi) = n$. Then $\phi = \sim\psi$, $\psi \to \chi$, or $\Box\psi$.

Suppose $\phi = \sim\psi$, then $C(\psi) = C(\phi) - 1 = n-1 < n$. $V_M(\phi, w^\circ) = 1$ iff by definition of $V_M(\phi, w^\circ)$ $\sim$ clause $V_M(\psi, w^\circ) = 0$ iff by IH $V_{M^\circ}(\psi, w^\circ) = 0$ iff $V_{M^\circ}(\phi, w^\circ) = 1$, so $V_M(\phi, w^\circ) = V_{M^\circ}(\phi, w^\circ)$.

Suppose $\phi = \psi \to \chi$, then $C(\psi) + C(\chi) + 1 = C(\phi)$ so $C(\psi)$, $C(\chi) < n$. $V_M(\phi, w^\circ) = 1$, by $\to$ clause, iff $V_M(\psi, w^\circ) = 0$ or $V_M(\chi, w^\circ) = 1$ iff by IH $V_{M^\circ}(\psi, w^\circ) = 0$ or $V_{M^\circ}(\chi, w^\circ) = 1$ iff $V_{M^\circ}(\phi, w^\circ) = 1$. So $V_M(\phi, w^\circ) = V_{M^\circ}(\phi, w^\circ)$ for all $w^\circ \in W^\circ$.

Suppose $\phi = \Box\psi$, then $C(\psi) = C(\phi) - 1 = n-1 < n$. $V_M(\phi, w^\circ) = 1$ iff $V_M(\psi, u) = 1$ for all u such that $Rw^\circ u$ iff by IH $V_{M^\circ}(\psi, u) = 1$ for all u such that $Rw^\circ u$ iff by construction of M°, $V_{M^\circ}(\psi, u^\circ) = 1$ for all u° such that ~~Rw°~~ $R^\circ w^\circ u^\circ$, iff $V_{M^\circ}(\phi, w^\circ) = 1$. So ~~$V_M(\phi, w^\circ)$~~ $V_{M^\circ}(\phi, w^\circ) = V_M(\phi, w)$

~~By IH~~ By cases, for all ϕ such that $C(\phi) = n$, $V_{M^\circ}(\phi, w^\circ) = V_M(\phi, w)$ for all $w^\circ \in W^\circ$.

By induction, for all ~~MPL~~ MPL-wffs ϕ, $V_{M^\circ}(\phi, w^\circ) = V_M(\phi, w^\circ)$ for all ~~$\phi \in$~~ $w^\circ \in W^\circ$.

So if for some S5-model $M = \langle W, R, I\rangle$ and $w \in W$, if $V_M(\phi, w) = 0$ then $V_{M^\circ}(\phi, w^\circ) = 0$

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2a $\vDash_D [\Box P \wedge \Box(\sim P \vee Q)] \to \Diamond Q$

consider arbitrary D-model $M = \langle W, R, I \rangle$ and arbitrary world $w \in W$. Suppose for reductio that

(1) $V_M([\Box P \wedge \Box(\sim P \vee Q)] \to \Diamond Q, w) = 0$

(1), $\to$ $\Rightarrow$

(2) $V_M(\Box P \wedge \Box(\sim P \vee Q), w) = 1$

(3) $V_M(\Diamond Q, w) = 0$

(2), derived $\wedge$ $\Rightarrow$

(4) $V_M(\Box P, w) = 1$

(5) $V_M(\Box(\sim P \vee Q), w) = 1$

(4), $\Box$ $\Rightarrow$

(6) $\forall u \in W, Rwu: V_M(P, u) = 1$

(5), $\Box$ $\Rightarrow$

(7) $\forall u \in W, Rwu: V_M(\sim P \vee Q, u) = 1$

(6), (7), $\sim$, derived $\vee$ $\Rightarrow$

(8) $\forall u \in W, Rwu: V_M(Q, u) = 1$

(3), derived $\Diamond$ $\Rightarrow$

(9) $\forall u \in W, Rwu: V_M(Q, u) = 0$

serialness of R on W $\Rightarrow$

(10) $\exists u \in W: Rwu$

(8), (9), (10) $\Rightarrow$

(11) $\exists u \in W: Rwu, V_M(Q, u) = 1, V_M(Q, u) = 0$

(11), reductio

(12) $V_M([\Box P \wedge \Box(\sim P \vee Q)] \to \Diamond Q, w) = 1$

(12), generalisation, definition of $\vDash_D$

(13) $\vDash_D [\Box P \wedge \Box(\sim P \vee Q)] \to \Diamond Q$

b $\vDash_{S4} \Diamond\Diamond(P \wedge Q) \to \Diamond Q$

consider arbitrary S4-model $M = \langle W, R, I \rangle$ and ~~for~~ arbitrary world $w \in W$. Suppose for reductio that

(1) $V_M(\Diamond\Diamond(P \wedge Q) \to \Diamond Q, w) = 0$

(1), $\to$ $\Rightarrow$

(2) $V_M(\Diamond\Diamond(P \wedge Q), w) = 1$

(3) $V_M(\Diamond Q, w) = 0$

(2), derived $\Diamond$ $\Rightarrow$

(4) $\exists u \in W, Rwu: V_M(\Diamond(P \wedge Q), u) = 1$

(4), derived $\Diamond$ $\Rightarrow$

(5) $\exists u, v \in W, Rwu, Ruv: V_M(P \wedge Q, v) = 1$

(5), derived $\wedge$ $\Rightarrow$

(6) $\exists u, v \in W, Rwu, Ruv: V_M(Q, v) = 1$

(6), transitivity of R on W $\Rightarrow$

(7) $\exists v \in W, Rwv: V_M(Q, v) = 1$

(3), derived $\Diamond$ $\Rightarrow$

(8) $\nexists v \in W, Rwv: V_M(Q, v) = 1$

(7), (8), reductio

(9) $V_M(\Diamond\Diamond(P \wedge Q) \to \Diamond Q, w) = 1$

(9), generalisation, definition of $\vDash_{S4}$

(10) $\vDash_{S4} \Diamond\Diamond(P \wedge Q) \to \Diamond Q$

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