~~Multivaria~~ Microeconomic Analysis Problem Set 2

1a $f(x,y) = xy$

$\cancel{\|(x,y)-(x',y')\| =}$

$\|(x,y)-(0,0)\| = \sqrt{x^2+y^2}$

$\|f(x,y)-f(0,0)\| = \|xy\| = xy$

$x = \sqrt{x^2} \le \sqrt{x^2+y^2}$, $y = \sqrt{y^2} \le \sqrt{x^2+y^2} \Rightarrow xy \le \sqrt{x^2+y^2}^{\,2}$

$\Rightarrow \cancel{\forall \varepsilon \in \mathbb{R}^2}\ \forall \varepsilon > 0 : \exists \delta > 0$, namely $\sqrt{\varepsilon}$: if $\|(x,y)-(0,0)\| < \delta$ then $\|f(x,y)-f(0,0)\| < \varepsilon \Rightarrow$

by definition of continuity, f is continuous at $(0,0)$

b suppose $f : \mathbb{R}^m \to \mathbb{R}$ is continuous at $\vec{x}_0$ and $g : \mathbb{R}^m \to \mathbb{R}$ is continuous at $\vec{x}_0$, then by definition of continuity,

$\forall \varepsilon_f > 0 : \exists \delta_f > 0$: if $\cancel{\|(\vec{x}-\vec{x}_0}$ $\|\vec{x}-\vec{x}_0\|$ ~~then~~ $< \delta_f$ then $\|f(\vec{x})-f(\vec{x}_0)\| < \varepsilon_f$ and $\forall \varepsilon_g\, \cancel{\exists \delta_g} > 0 : \exists \delta_g > 0$: if $\|\vec{x}-\vec{x}_0\| < \delta_g$ then $\|g(\vec{x})-g(\vec{x}_0)\| < \varepsilon_g$. $* \Rightarrow$

$\forall \varepsilon > 0 : \exists \delta_f > 0 : \exists \delta_g > 0$: if $\|\vec{x}-\vec{x}_0\| < \delta_f$ and $\|\vec{x}-\vec{x}_0\| < \delta_g$ then $\|f(\vec{x})-f(\vec{x}_0)\| < \varepsilon$ and $\|g(\vec{x})-g(\vec{x}_0)\| < \varepsilon \Rightarrow$

$\forall \varepsilon > 0 : \exists \delta$, namely $\min\{\delta_f, \delta_g\}$: if $\|\vec{x}-\vec{x}_0\| < \delta$ then $\|f(\vec{x})-f(\vec{x}_0)+g(\vec{x})-g(\vec{x}_0)\| < 2\varepsilon \Rightarrow$

$\forall \varepsilon' > 0$, namely 2ε : $\exists \delta > 0$ namely $\min\{\delta_f, \delta_g\}$: if $\|\vec{x}-\vec{x}_0\| < \delta$ then $\|(f+g)(\vec{x})-(f+g)(\vec{x}_0)\| < \varepsilon' \Rightarrow$

By definition of continuity, $(f+g)$ is continuous at $\vec{x}_0$

2a $\partial_x f(0,0) = \lim_{h\to 0} f(h,0)-f(0,0)/h$

$= \lim_{h\to 0} [h^3/h^2 - 0]/h = 1$

$\partial_y f(0,0) = \lim_{h\to 0} f(0,h)-f(0,0)/h$

$= \lim_{h\to 0} [0-0]/h = 0$

$\partial_x g(0,0) = \lim_{h\to 0} g(h,0)-g(0,0)/h$

$= \lim_{h\to 0} [0-0]/h = 0$

By symmetry, $\partial_y g(0,0) = 0$

b $\cancel{\partial_x f = \begin{cases} 3x^2(-1)(x^2+y^2)^{-2}(2x) = -6x^3/(x^2+y^2)^2 & \text{if } (x,y)\ne(0,0) \\ 1 & \text{otherwise} \end{cases}}$

$\partial/\partial x\ x^3/x^2+y^2 = 3x^2/x^2+y^2 + \cancel{x^3}\, x^3(-1)(x^2+y^2)^{-2}(2x)$

$= 3x^2/x^2+y^2\ \cancel{*} - 2x^4(x^2+y^2)^{-2}$

$= 3x^2(x^2+y^2)-2x^4/(x^2+y^2)^2$

$\partial_x f = \begin{cases} 3x^2(x^2+y^2)-2x^4/(x^2+y^2)^2 & \text{if } (x,y)\ne 0 \\ 1 & \text{otherwise} \end{cases}$

$\cancel{\partial_x f(1/n, 1/n) =}$

$\cancel{\partial_x f(0, 1/n) =}$

$\cancel{\partial_x f(1/n, 0) = n^{-4}/n^{-4} = 1/4\ n^{-4}}$

$\partial_x f(1/n, 2/n) = 13n^{-4}/25n^{-4} = 13/25$

$\lim_{n\to\infty} (1/n, 2/n) = (0,0)$

$\lim_{n\to\infty} \partial_x f(1/n, 2/n) = 13/25 \ne \partial_x f(0,0) = 1$

By definition of continuity, $\partial_x f$ is not continuous at $(0,0)$, $\partial_x f$ is not continuous at all points in the domain of f, f is not C^1

$\partial/\partial x\ x^{1/2}y^{1/2} = \frac{1}{2}x^{-1/2}y^{1/2}$

$\partial_x g(1/n, 1/n) = \frac{1}{2} n^{1/2} n^{-1/2} = 1/2$

$\lim_{n\to\infty} (1/n, 1/n) = (0,0)$

$\lim_{n\to\infty} \partial_x g(1/n, 1/n) = 1/2 \ne \partial_x g(0,0) = 0$

By definition of continuity, $\partial_x g$ is not continuous at $(0,0)$, $\partial_x g$ is not continuous at all points in the domain of g, g is not C^1.

f is differentiable at $\vec{x}$ iff there is some matrix A such that $\lim_{\|\vec{h}\|\to 0} \|f(\vec{x}+\vec{h})-f(\vec{x})-\cancel{Df(\vec{x})}\, A\vec{h}\|/\|\vec{h}\|$ $\cancel{A} = 0$

If such A exists, $A = Df(\vec{x})$

$Df(0,0) = (1,0)$, $f(0,0) = 0$

Let $\vec{h} = \cancel{\vec{h}}\ (x_h, y_h)$

$\|f(\vec{h})-f(0,0)-Df(0,0)\vec{h}\|/\|\vec{h}\| =$

$= \cancel{x_h^3}\,(x_h^3/x_h^2+y_h^2 - x_h)/\sqrt{x_h^2+y_h^2}$

$= -x_h y_h^2/(x_h^2+y_h^2)^{3/2}$

$\cancel{\text{const } \{(1/n, 1/n)\}_{n=1}^{\infty}}$

$\{\|(1/n, 1/n)\|\}_{n=1}^{\infty}$ converges to 0

$\lim_{\|\vec{h}\|\to 0} -x_h y_h^2/(x_h^2+y_h^2)^{3/2}$ along this sequence

$= -n^{-3}/2^{3/2}n^{-3} = -2^{-3/2} \ne 0$

By definition of differentiability, f is not differentiable

$Dg(0,0) = (0,0)$ $g(0,0) = 0$

$\|g(\vec{h})-g(0,0)-Dg(0,0)\vec{h}\|/\|\vec{h}\|$

$= x_h^{1/2}y_h^{1/2}/\sqrt{x_h^2+y_h^2}$

$\lim_{\|\vec{h}\|\to 0} x_h^{1/2}y_h^{1/2}/\sqrt{x_h^2+y_h^2}$ along the sequence $\{(1/n, 1/n)\}_{n=1}^{\infty}$

$= 1/n / \sqrt{2/n^2} = 1/\sqrt{2}$

$\{\|(1/n, 1/n)\|\}_{n=1}^{\infty}$ converges to 0

By definition of differentiability, g is not differentiable

4a Let $f(x,y,z) = z^2+xz+yx^2+y^3$

$\partial_x f = z+2xy$

$\partial_y f = x^2+3y^2$

$\partial_z f = 2z+x$

By inspection, each partial derivative of f is continuous since it is a polynomial, so f is C^1.

$\partial_z f(s^*) = 2(0)+1 = 1 \ne 0$

Since s^* solves $f(x,y,z)=0$, f is C^1, and $\partial_z f(s^*) \ne 0$, by the implicit function theorem, there is a function $g(x,y)$ such that $z^* = g(x^*,y^*)$, $f(x^*,y^*,g(x^*,y^*))=0$, and $g_a(x,y) = -f_a(x,\cancel{g}\,y,g(x,y))/f_z(x,y,g(x,y))$

6 $\cancel{z = g(1,0) =}$ $\cancel{z^*}$ $\cancel{z = g(x,y) =}$

$\cancel{g_x(1,0)}$

$z = g(1,0) = g(x^*,y^*) = z^* = 0$

$g_x(1,0) = -f_x(1,0,0)/f_z(1,0,0) = 0$

$g_y(1,0) = -f_y(1,0,0)/f_z(1,0,0) = -1/1 = -1$

5a Let $f_1(x,y,u,v) = x^2-y^2-u^3+v^2+4$, ~~and~~

$f_2(x,y,u,v) = 2xy+y^2-2u^2+3v^4+8$, and

$\vec{f} = (f_1, f_2)$

$D_{\cancel{xyuv}}\vec{f} = \begin{pmatrix} \partial_x f_1 & \partial_y f_1 & \partial_u f_1 & \partial_v f_1 \\ \partial_x f_2 & \partial_y f_2 & \partial_u f_2 & \partial_v f_2 \end{pmatrix}$

$= \begin{pmatrix} 2x & -2y & -3u^2 & 2v \\ 2y & 2x+2y & -4u & +12v^3 \end{pmatrix}$

By inspection, each partial derivative of $\vec{f}$ is continuous since ~~f~~ each partial derivative is a polynomial, so $\vec{f}$ is C^1.

$D_{uv}\vec{f}(s^*) = \begin{pmatrix} -3u^2 & 2v \\ -4u & 12v^3 \end{pmatrix} = \begin{pmatrix} -12 & 2 \\ -8 & 12 \end{pmatrix}$

$\det D_{uv}\vec{f} = -144 - -16 = -128 \ne 0$ ~~so~~

so $D_{uv}f$ is invertible

By the implicit function theorem, since f is C' ~~and~~ $D_{u,v}f$ is invertible, and $\vec{s}^*$ solves $\vec{f}(x,y,u,v)=\vec{0}$, there is a ~~function~~ C' function $\vec{g}(x,y)$ such that $u^*, v^* = \vec{g}(x^*, y^*)$, $\vec{f}(x^*, y^*, g(x^*, y^*)) = \vec{0}$ and $D_{x,y}g = D_{u,v}f^{-1} D_{x,y}\vec{f}$.

$u,v = \vec{g}(x=2, y=-1) = \vec{g}(x^*, y^*) = u^*, v^* = 2,1$

~~$D_{u,v}f(x=2, y=-1)$~~

$D_{u,v}f(x=2, y=-1) = \begin{pmatrix} -12 & 2 \\ -8 & 12 \end{pmatrix}$

By Cramer's Rule, $D_{u,v}f^{-1} = 1/128 \begin{pmatrix} 12 & -2 \\ 8 & -12 \end{pmatrix}$

$D_x f(x=2, y=-1) = \begin{pmatrix} 2x \\ 2y \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$

$$D_x g(x=2, y=-1) = (D_{u,v}f^{-1} D_x f)(x=2, y=-1)$$
$$= 1/128 \begin{pmatrix} 12 & -2 \\ 8 & -12 \end{pmatrix} \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$
$$= 1/128 \begin{pmatrix} 52 \\ 56 \end{pmatrix}$$
$$= \begin{pmatrix} 13/32 \\ 14/32 \end{pmatrix}$$