Game Theory Problem Set 2

1a $u_L(x,y) = x/(x+y) - x$

~~$B_L(y)$ is ed define~~

~~By definition of best response, $B_L(y) = \arg\max_x u_L(x,y)$~~

Taking FOCs:

$\partial u_L/\partial x = x(-1)(x+y)^{-2} + (x+y)^{-1} - 1 = 0$,

$-x + (x+y) - (x+y)^2 = 0$,

$y = (x+y)^2$,

$x = \sqrt{y} - y$

Checking SOCs:

$\partial^2 u_L/\partial^2 x = x(-1)(-2)(x+y)^{-3} + (-1)(x+y)^{-2} - (x+y)^{-2}$

$= 2x(x+y)^{-3} - 2(x+y)^{-2}$

$\partial^2 u_L/\partial^2 x < 0$ iff

$2x(x+y)^{-1} - 2 < 0$ since $x,y>0$ hence $(x+y)^2>0$

$2x < 2(x+y)$ since $x,y>0$ hence $(x+y)>0$

which holds for all $x,y>0$

So $x=\sqrt{y}-y$ maximises $u_L(x,y)$

$B_L(y) = \sqrt{y} - y$

By symmetry, $B_C(x) = \sqrt{x} - x$

b Suppose ∃ ~~pure NE~~ symmetric pure NE $s^* = (x^*, y^*)$

Then by definition of NE and BR,

$x^* = B_L(y^*) = \sqrt{y^*} - y^*$.

~~By our~~ By supposition

$y^* = x^* = \sqrt{y^*} - y^*$,

$2y^* = \sqrt{y^*}$,

$4y^{*2} = y^*$,

$4y^* = 1$,

$y^* = 1/4$.

By supposition

$x^* = y^* = 1/4$

Then

$B_C(x^*) = \sqrt{x^*} - x^* = \sqrt{1/4} - 1/4 = 1/4 = y^*$

$B_L(y^*) = \sqrt{y^*} - y^* = \sqrt{1/4} - 1/4 = 1/4 = x^*$

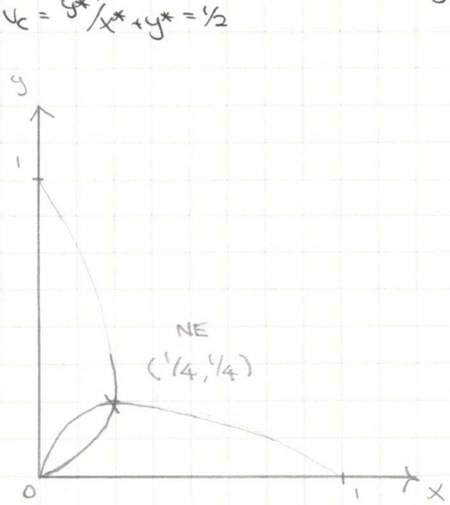
By definition of ~~BR and~~ NE and BR, $s^*=(x^*,y^*)$ is indeed a ~~pur~~ NE.

Let v_L and v_C denote the respective parties' vote share

At s^*, ~~$v_L = v_C = 1/2$~~ $v_L = x^*/(x^*+y^*) = 1/2$,

$v_C = y^*/(x^*+y^*) = 1/2$

c



d At $x=0$, $y=0$, it is reasonable to assume that both parties have equal vote share, ~~stace~~ given that advertising is the sole determinant of vote share, so $v_L = v_C = 1/2$. Then, given that payoffs are equal to vote share minus advertising costs, $u_L = v_L - x = 1/2$, $u_C = v_C - y = 1/2$. There is no equilibrium at (0,0) given these payoffs since this candidate equilibrium fails to the deviation $(\varepsilon, 0)$ for sufficiently small $\varepsilon > 0$.

$u_L(\varepsilon,0) = \frac{\varepsilon}{\varepsilon+0} - \varepsilon = 1-\varepsilon > 1/2$ for $\varepsilon < 1/2$

$u_L(0,0) = 1/2$.

e (0,0) is a NE iff ~~only if~~ $u_L(0,0) \geq$ ~~$u_L(\varepsilon,0)$~~ $u_L(\alpha,0) = 1-\alpha$ for all $\alpha \in \mathbb{R}_{>0}$ and $u_C(0,0) \geq u_C(0,\beta) = 1-\beta$ for all $\beta \in \mathbb{R}_{>0}$, iff $u_L(0,0) \geq 1$ and $u_C(0,0) \geq 1$

2a Let $BR_A(y_B)$ denote A's best response function and $BR_B(y_A)$ denote B's best response function.

~~By definition of best response fn, $BR_A(y$~~

Suppose that ~~$y_B = X$, then~~ $y_B \geq X$, then

~~then $\pi_A(0,y_B) = 0$~~ $\pi_A(y_A=0, y_B) = 0$

$\pi_A(y_A \in (0,y_B), y_B) = -y_A < 0$

$\pi_A(y_A = y_B, y_B) = X/2 - y_A = X/2 - y_B \leq -X/2 < 0$

$\pi_A(y_A > y_B, y_B) = X - y_A <$ ~~$X - y_B$~~ 0

so $BR_A(y_B \geq X) = 0$

~~Suppose that $y_B < X$, then~~

~~$\pi_A(y_A \in (y_B,X), y_B) = X - y_A > 0$~~

~~$BR_A(y_B<X) \neq \emptyset$. Suppose that $y_B < X$~~

~~$BR_A(y_B<X) \not< y_B$ since $u_A(y_A<y_B, y_B) = -y_A$~~

~~$< u_A(y_A$~~

Suppose that $y_B < X$, then

$BR_A(y_B<X) \not< y_B$ since $\pi_A(y_A<y_B, y_B) = -y_A < \pi_A(y_A \in (y_B,X), y_B) = X - y_A > 0$

$BR_A(y_B<X) \neq y_B$ since $\pi_A(y_A=y_B, y_B) = X/2 - y_A = X/2 - y_B < \pi_A(y_A = y_B+\varepsilon, y_B) = X - y_B - \varepsilon$ for $\varepsilon < X/2$

$BR_A(y_B<X) \not> y_B$ since $\pi_A(y_A>y_B, y_B) = X - y_A < \pi_A(y_A' = \frac{y_A+y_B}{2}, y_B) = X - \frac{y_A}{2} - \frac{y_B}{2}$

so $BR_A(y_B<X) = \emptyset$

b ~~By definition of NE~~, Suppose that ∃ pure NE $s^* = (y_A^*, y_B^*)$. Then, by definition of NE and BR, $y_A^* \in BR_A(y_B^*)$ and $y_B^* \in BR_B(y_A^*)$

Then $BR_A(y_B^*) \neq \emptyset$, so $y_A^* = BR_A(y_B^*) = 0$

Then $BR_B(y_A^*=0) = \emptyset$ so $y_B^* \notin BR_B(y_A^*)$

By reductio, there is no pure strategy NE s^*

c Assume that ~~players their at equilibrium~~ the mixed NE σ^* is symmetric. ~~and eac~~

Suppose that each player i ~~plo~~ plays the mixed strategy σ_i^* which assigns positive probability p to the action y. This candidate equilibrium fails to the deviation (σ_A', σ_B^*), where σ_A' is a mixed strategy identical to σ_A^* except in assigning zero probability to y and probability p to $y+\varepsilon$ for sufficiently small $\varepsilon > 0$ ~~such that $y < y+\varepsilon < y'$ where y' is the smallest action in the support of $\sigma_A^* = \sigma_B^*$ such that $y' > y$~~. Such a reallocation of probability mass ~~that~~ decreases expected payoff by ε ~~with probabil on~~ with probability p due to the increased bidding cost, but increases payoff by $X/2$ ~~or~~ with probability p^2 (where A plays action $y+\varepsilon$ and B plays action B), so increases expected payoff on net iff $p^2 X/2 > p\varepsilon$, $\varepsilon < pX/2$.

d Suppose that the symmetric ~~NE~~ mixed NE is $\sigma^* = (\sigma_i^*, \sigma_i^*)$ where σ_i^* is a probability distribution ~~over~~ with cdf $F(y)$.
Suppose that there is a gap in the support of $F(y)$, ~~$[y_1, y_2]$~~ (y_1, y_2). ~~Then~~
Any action $y' \in (y_1, y_2)$ yields a higher payoff than y_2 given that the other player plays σ_i^* since ~~$y' > y$~~ $y' > y_{-i}$ iff $y_2 > y_{-i}$ where y_{-i} is the other player's action and y_2 is played with zero probability so $\pi_i(y', \sigma_i^*) = F(y')X - y' > \pi_i(y_2, \sigma_i^*) = F(y_2)X - y_2$ ~~where W(y) is the probability that~~ then $\pi_i(\sigma_i', \sigma_i^*) > \pi_i(\sigma_i^*, \sigma_i^*)$ since σ_i' and σ_i^* are otherwise identical. So there is a profitable deviation from σ_i^* if there is a gap in the support of $F(y)$. By reductio, since σ_i^* is a NE, there is no gap in the support of $F(y)$.

e Suppose that $F(y)$ has support $[\underline{y}, \bar{y}]$ and $\underline{y} > 0$, then either firm can ~~devia~~ profitably deviate from the strategy profile (σ_i^*, σ_i^*) by playing some alternative mixed strategy σ_i' which assigns ~~the~~ positive probability to y_0 since $\pi_i(\sigma_i^*, \sigma_i^*) = \pi_i(\underline{y}, \sigma_i^*)$ since at NE all actions in the support of the mixed strategy have equal payoff (otherwise there is profitable deviation by reallocating probability mass between actions in the support) $\pi_i(\underline{y}, \sigma_i^*) = -\underline{y}$ since given a continuous distribution with no atoms, there are no ties and $-\underline{y}$ always loses
$\pi_i(\underline{y}, \sigma_i^*) < \pi_i(0, \sigma_i^*) = 0$, then σ^* is not a NE.
By reductio, since σ^* is an NE, $\underline{y} = 0$.

f σ^* is a mixed NE ~~iff~~ only if all actions in the support of $F(y)$ have equal payoff such that there is no profitable deviation by reallocating ~~the~~ probability mass between these actions.
$\pi_i(0, \sigma_i^*) = \pi_i(y, \sigma_i^*)$ for all y in the support of $F(y)$
$0 = F(y)X - y$ since $y > y_{-i}$ with probability $F(y)$
$F(y) = y/X$. This corresponds to a uniform distribution over $[0, X]$.

At the symmetric mixed NE, each player plays a mixed strategy which is a uniform distribution over $[0, X]$.

3a

	L	R
T	6, 6	$\underline{2}$, $\underline{8}$
B	$\underline{8}$, $\underline{2}$	0, 0

Best responses underlined.
By inspection, (T,R) and (B,L) are the only pure NE where players play mutual BRs.
Suppose $\exists$ mixed NE σ^* where Pl.1 mixes T and B, then $\pi_1(T, \sigma_2^*) = \pi_1(B, \sigma_2^*)$, $6q + 2(1-q) = 8q$, where q is the probability assigned to L by σ_2^*, $q = 1/2$ so Pl.2 mixes L and R, then $\pi_2(L, \sigma_1^*) = \pi_2(R, \sigma_1^*)$, $6p + 2(1-p) = 8p$, where p is the probability assigned to T by σ_1^*.
So if Pl.1 mixes so does Pl.2. By symmetry, if Pl.2 mixes so does Pl.1. So there are no hybrid NE.
The unique mixed NE is $(\tfrac{1}{2}T + \tfrac{1}{2}B, \tfrac{1}{2}L + \tfrac{1}{2}R)$
The expected payoffs are
(T,R) (2,8)
(B,L) (8,2)
$(\tfrac{1}{2}T + \tfrac{1}{2}B, \tfrac{1}{2}L + \tfrac{1}{2}R)$ (4,4)