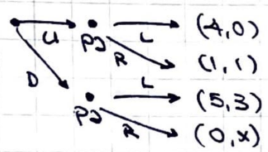
Game Theory Problem Set 4

1a) P1



	LL	LR	RL	RR
U	4, 0	4, 0	1, 1	1, 1
D	5, 3	0, x	5, 3	0, x

where P2's strategy XY denotes action X if P1 chooses U, action Y if P1 chooses D.

x = 2
Best responses underlined ~~##~~ in the strategic form representation.

By inspection, there are three pure strategy NE, (U,RR), (D,LL), and (D, RL).

~~Suppose that there is a mixed NE $\sigma^* = (\sigma_1^*, \sigma_2^*)$ where $\sigma_1^* = pU + (1-p)D$, and $\sigma_2^* = (q_1 L + (1-q_1)R, q_2 L + (1-q_2)R)$, i.e. P1 plays U with probability p, P2 plays L in response to U with probability q_1 and P2 plays L in response to D with probability q_2. σ^* is a mixed NE iff neither player has a profitable deviation, which is only if ~~all a~~ for each player, each action assigned non zero probability by σ^* yields equal payoff. Suppose $p \in (0,1)$.~~

$\cancel{\pi_1(U, \sigma_2^*) = \pi_1(D, \sigma_2^*)}$

$\cancel{4q_1 + 1(1-q_1) = 5q_2 + 0(1-q_2)}$

$\cancel{3q_1 + 1 = 5q_2}$

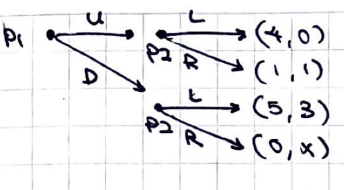
$\cancel{\pi_2(}$

~~By inspection, R is a stric~~ RR and ~~RL are weak best responses to~~

The hybrid ~~equilibria~~ NE are $(D, q_1 LL + (1-q_1) RL)$ and $(U, q_2 RL + (1-q_2) RR)$ $q_1 \in (0,1)$ and $q_2 \in (0, 1/5)$.

Finding the ~~SS~~ SPE by backward induction. ~~Suppose~~ Suppose that P1 chooses U, then ~~P2~~ it is optimal for P2 to choose R, yielding payoffs (1,1). Suppose that P2 chooses D, then it is optimal for P2 to choose L, yielding payoffs (5,3). So P1 finds it optimal to choose U. The unique pure SPE is (D,L).

By inspection of the extensive form, P2 has a strictly dominant ~~str~~ pure strategy in each subgame, so P2 never mixes in equilibrium. By inspection, P1 has a strictly dominant pure strategy in ~~each~~ the reduced form game, so P1 never mixes in equilibrium. Then there are no hybrid or mixed SPE.

ii 

	LL	LR	RL	RR
U	4, 0	4, 0	1, 1	1, 1
D	5, 3	0, x	5, 3	0, x

x=4. Best responses underlined.
By inspection, the unique pure NE ~~are~~ is (U, RR), ~~and~~
By inspection, RL and RR are weakly domint against U and LR and RR are weakly ~~dominant~~ optimal against D, and P1's best responses are ~~always~~ all strict best responses, so P1 never mixes, and P2 may mix RL and RR against U ~~and~~ or mix LL and ~~L~~ RL against D. The hybrid NE are (U, q_1RL + (1-q_1)RR) for $q_1 \in (0, 1/5)$. ~~and~~ ~~(D,~~

By backward induction, R is optimal for P2 against U, and R is ~~dominant against~~ optimal for P2 against D.* so ~~the~~ U is optimal for P1. ~~In the reduced for~~ So the unique pure SPE is (U, R). ~~By an arg~~

~~SPE~~ By inspection, P2 has a strictly dominant pure strategy in each subgame, so P2 never mixes in each subgame ~~as~~ NE. By inspection of the reduced form game, P1 has a strictly dominant pure strategy. So P1 never mixes in SPE. So there are no hybrid or mixed SPE.

iii When x=2, ~~P2~~ it is optimal for P2 to play action L against D. When x=4, it is optimal for P2 to play action R against D. If P2 plays L against D, P1 has no incentive to deviate from D, so (D, LL) and (D, RL) are NE ~~T~~ when x=2. If P2 plays R against D, P1 has incentive to deviate, so (D, ~~RR~~) and (D, ~~RE~~ LR) are not NE when x=4. When x=2, RL is weakly dominant for P2. When x=4, RR is weakly dominant for P2.

When x=2, L is optimal for P2 in the subgame where P1 plays D. When x=4, R is optimal. So the reduced form game is different, ~~as~~ and P1's optimal choice in the reduced form game is different, and the SPE is different.

b $u_L = x/(x+y) - x$, $u_C = y/(x+y) - y$

C ~~maxs the~~ chooses y to max u_C.

~~Taking FOCs:~~ Taking FOCs:

$\partial u_C/\partial y = y(-1)(x+y)^{-2} + (x+y)^{-1} - 1 = 0$,

$-y + (x+y) - (x+y)^2 = 0$,

$x = (x+y)^2$,

$y = \sqrt{x} - x$

Substituting into u_L,

$u_L = x/\sqrt{x} - x = \sqrt{x} - x$

Taking FOCs:

$\partial u_L / \partial x = \frac{1}{2}x^{-1/2} - 1 = 0,$

$x^{-1/2} = 2, \; x^{1/2} = 1/2, \; x = 1/4$

By substitution,

$y = \sqrt{1/4} - 1/4 = 1/4$

$u_L = u_C = 1/4$

The SPE is identical to the NE of the simultaneous game. The SPE yields symmetric payoffs, so there is no first mover advantage (or disadvantage). The SPE is $(x = 1/4, y = 1/4)$ and the typical strategy for C is to choose $y = \sqrt{x} - x$.

Does this make sense? It seems that at the simultaneous NE, x and y are strategic substitutes, i.e. $\uparrow x \Rightarrow \downarrow \partial u_C / \partial y$ and $\downarrow y \Rightarrow \uparrow u_L$, so there is a positive strategic effect.

2a Each firm i chooses p_i to maximise $\pi_i(p_i, p_j) = (p_i - c_i)D_i(p_i, p_j) = p_i(1 + p_j - p_i)$.

Taking FOCs:
$\partial \pi_i / \partial p_i = (1 + p_j - p_i) - p_i = 1 + p_j - 2p_i = 0$,
$p_i = 1+p_j/2$
Best response of firm i to price of firm j $BR_i(p_j) = 1+p_j/2$
At NE, ~~each firm i choose~~ firms play mutual best responses.
$p_i^* = BR_i(p_j^*)$, $p_j^* = BR_i(p_i^*)$, $\cancel{1/2 + 1/2(1}$
$p_i^* = 1/2 + 1/2(1/2 + 1/2 p_i^*) = 3/4 + 1/4 p_i^*$, $3/4 p_i^* = 3/4$, $p_i^* = 1$
$p_j^* = 1+p_i^*/2 = 1$
$\pi_i^* = p_i^*((1 + p_j^* - p_i^*) = 1$, $\pi_j^* = p_j^*(1 + p_i^* - p_j^*) = 1$

At NE, $p_1^* = p_2^* = 1$, $\pi_1^* = \pi_2^* = 1$
The ~~Nash~~ NE is symmetric.
Each firm's ~~BR~~ is increasing with the other firm's price since prices are strategic complements, i.e. an increase in one firm's price increases the other firm's marginal profit due to an increase in price, i.e. $\partial \pi_i / \partial p_i = 1 + p_j - 2p_i$ is increasing in p_j, so an increase in one firm's price causes an increase in the other firm's incentive to increase price.

b Firm 2's best response $BR_2(p_1) = 1+p_1/2$ (from a). In the second stage subgame, firm 2 finds it optimal to choose $p_2 = 1+p_1/2$. Given CKR, firm 1's maximisation problem in the first stage reduces to ~~max p_1~~
$\max_{p_1} \; p_1(1 + 1+p_1/2 - p_1) = p_1(3/2 - p_1/2) = p_1/2(3 - p_1)$.

Taking FOCs:
~~$\partial \pi_1 / \partial p_1 =$~~ $d\pi_1/dp_1 = 1/2(3 - p_1) + p_1/2(-1) = 3/2 - p_1 = 0$, $p_1 = 3/2$
At SPE, $p_1^* = 3/2$, $p_2^* = 1+p_1^*/2 = 5/4$, $\pi_1^* = p_1^*(1 + p_2^* - p_1^*) = 9/8$
$\pi_2^* = p_2^*(1 + p_1^* - p_2^*) = 25/16$

The SPE is unique because firm 2 has a unique strict best response to any p_1, so there is a unique reduced form maximisation problem for firm 1, and this problem has a unique solution. ~~The first uniq~~ The solutions are unique because the corresponding ~~maximisation problems are~~ functions are quadratic.

c Firm 1 ~~can~~'s profits increase compared to the simultaneous game because at the ~~S~~NE of the simultaneous game, an increase in price has a positive strategic effect ~~that~~ in the sequential game. ~~An increase in p_1 incre~~ Since prices are strategic complements, an increase in p_1 increases firm 2's incentive to choose a higher price. Firm 2's choosing a higher price in turn increases firm 1's profit.

There is a second-mover advantage because firm 2 ~~can profitab~~ finds it optimal ~~to deviate from~~ to undercut firm 1 ~~since~~ when firm 1's price is fixed and this hurts firm 1 but benefits firm 2.

Firm 1's choosing a higher price has a direct, positive, first-order effect on firm 2's profit that is larger than the ~~indirect,~~ strategic, ~~second-order~~ positive second-order effect ~~that~~ on its own profit.

d In Cournot, each firm i ~~chooses p~~ $\in N=\{1,2\}$ chooses ~~price p_i~~ output q_i given ~~the~~ constant marginal cost $c=0$ to maximise profit $\pi_i(q_i, q_j) = q_i[P(q_i+q_j)-c]$ $= q_i P(q_i+q_j) = q_i(1-q_i-q_j)$ ~~for~~ if $1-q_i-q_j \geq 0$.

Firm i's best response $BR_i(q_j)$,
Taking FOCs:
$\partial\pi_i/\partial q_i = (1-q_i-q_j)-q_i = 0$, $q_i = {}^{1-q_j}/_2$ for $1-q_i-q_j \geq 0$
$BR_i(q_j) = {}^{1-q_j}/_2$ for $1-q_i-q_j \geq 0$

At NE, firms play mutual best responses,
$q_i^* = BR_i(q_j^*)$, $q_j^* = BR_j(q_i^*)$,
$q_i^* = 1/2 - 1/2(1/2 - 1/2 q_i^*) = 1/4 + 1/4 q_i^*$, $3/4 q_i^* = 1/4$, $q_i^* = 1/3$
$q_j^* = 1/2 - 1/2 q_i^* = 1/3$
$\pi_i^* = q_i^*(1-q_i^*-q_j^*) = 1/9$. By symmetry, $\pi_j^* = 1/9$
~~q_i^*~~ At NE, $q_1^* = q_2^* = 1/3$, $\pi_1^* = \pi_2^* = 1/9$

~~Best~~ The NE is symmetric and unique because each firm's profit function is quadratic, so each firm has a ~~strict~~ unique strict best response. Best responses are downward sloping because quantities are strategic substitutes, i.e. $\partial/\partial q_j(\partial\pi_i/\partial q_i) < 0$, i.e. an increase in one firm's quantity decreases the other firm's incentive to increase quantity.

In Stackelberg, ~~firm~~ in the second stage subgame, firm 2 chooses ~~p~~ $q_2 = BR_2(q_1) = 1/2 - q_1/2$. In the first stage, given CKR, firm 1's maximisation problem reduces to $\max_{q_1} \pi_1 = q_1(1-(1/2 - q_1/2) - q_1) = q_1(1/2 - q_1/2) = q_1/2(1-q_1)$.

Taking FOCs:
~~$\partial\pi$~~ $\partial\pi_1/\partial q_1 = q_1/2(-1) + 1/2(1-q_1) = 1/2 - q_1 = 0$, $q_1 = 1/2$
At NE, $q_1^* = 1/2$, $q_2^* = 1/2 - q_1^*/2 = 1/4$, $\pi_1^* = 1/8$, $\pi_2^* = 1/16$

Firm 1 finds it optimal in Stackelberg to choose a higher quantity because at the Cournot NE in the Stackelberg game, ~~the~~ an increase in quantity has a positive second-order strategic effect on firm 1's profit. Since quantities are strategic substitutes, an increase in firm 1's quantity decreases firm 2's incentive to increase quantity. A decrease in firm 2's quantity, in turn causes an increase in (price hence ~~π~~ in) firm 1's profit.

There is a first mover advantage since quantities are strategic substitutes. Firm 1's increasing its profit ~~π~~ through the positive strategic effect has a ~~str~~ negative first order effect on firm 2's profit. Whether there is a first mover advantage depends on whether the strategic effect directs first movers to be more aggressive or less aggressive.

3a

	A	B
A	$\underline{1}$ $\underline{3}$	0 0
B	0 0	$\underline{3}$ $\underline{1}$

Best responses underlined.
By inspection, the only pure NE are (A,A) and (B,B) where players play mutual best responses.

Suppose there is some NE σ^* ~~such that~~ $=(\sigma_1^*, \sigma_2^*)$ such that P1 mixes, i.e. σ_1^* assigns probability p to A and $1-p$ to B for $p \in (0,1)$. σ^* is a NE iff P1 has no profitable deviation from σ_1^*, which is only if each ~~action~~ of A and B yield equal expected payoff. Let q denote the probability that σ_2^* assigns to A.

$\pi_1(A, \sigma_2^*) = \pi_1(B, \sigma_2^*)$

$3q + 0(1-q) = 0q + 1(1-q)$, $4q = 1$, $q = 1/4$

~~$4q=1$~~ so P2 mixes A and B. Then since σ^* is a NE, P2 has no profitable deviation which is only if each of A and B yield equal expected payoff.

$\pi_2(A, \sigma_1^*) = \pi_2(B, \sigma_1^*)$

$1p + 0(1-p) = 0(p) + 3(1-p)$, $4p = 3$, $p = 3/4$.

~~Since~~

Since if P1 mixes then P2 mixes and by symmetry if P2 mixes ~~P1 m~~ then P1 mixes, there are no hybrid NE. The unique mixed NE is $\sigma^* = (pA + (1-p)B, qA + (1-q)B)$ for $p = 3/4$, $q = 1/4$.

b • P1

$\downarrow X$ $\xrightarrow{Y}$

	A	B
A	$\underline{1}$ $\underline{3}$	0 0
B	0 0	$\underline{3}$ $\underline{1}$

	A	B
A	$\underline{1}$ $\underline{2}$	0 -1
B	0 -1	$\underline{3}$ $\underline{0}$

	AA	AB	BA	BB
XA	$\underline{1}$ $\underline{3}$	$\underline{1}$ $\underline{3}$	0 0	0 0
XB	0 0	0 0	$\underline{3}$ 1	$\underline{3}$ $\underline{1}$
YA	$\underline{1}$ 2	0 -1	$\underline{1}$ $\underline{2}$	0 -1
YB	0 -1	$\underline{3}$ 0	0 -1	$\underline{3}$ 0

Aren't there 8 pure strategies for P1 of the form ~~ZC_1~~ ZC_1C_2 where Z is played in the first stage, C_1 if previously X and C_2 if previously Y?

Best responses underlined.
By inspection, the only pure NE are (XA, AA), (~~AA~~ XA, AB), (XB, BB), and (YA, BA). ~~Stage~~ P1's strategy ZC is to be understood as play Z in the first stage and C in the second stage. P2's strategy C_1C_2 is to be understood as play C_1 if P1 ~~has~~ played X and play C_2 if P1 played Y.

By inspection, (XA, AA) induces a NE in the X subgame and in the Y subgame, so (XA, AA) is an SPE.
By inspection, (XA, AB) does not induce a NE in the Y subgame, so (XA, AB) is not a SPE.
By inspection, (XB, BB) induces a NE in the X subgame and in the Y subgame, so (XB, BB) is an SPE.
By inspection, (YA, BA) does not induce a NE in the X subgame, so (YA, BA) is not a SPE.
The SPE are (XA, AA) and (XB, BB)

This result seems entirely uninteresting, did the question mean for the Y subgame to be

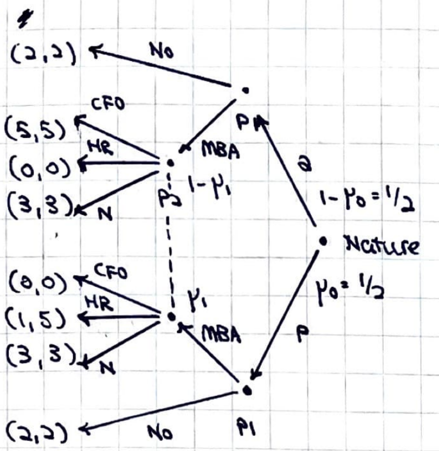
	A		B	?
A		1		0
	2		0	
B		0		3
	0		0	

d $XA \geq_1 YB$, so first eliminate YB.
Then $AA \geq_2 AB$, $BA \geq_2 BB$, so eliminate AB and BB.
Then $YA \geq_1 XB$, so eliminate XB.
Then $AA \geq_2 BA$, so eliminate BA.
Then $XA \geq_1 YA$, so eliminate YA.
The only strategy profile that survives IMEWDS is (XA, AA). This is one of the SPE found in c.

e Suppose P1 plays X in the first stage. Then, conceivably, P1 expects greater payoff from the X subgame than is possible in the Y subgame~~s~~, namely payoff 3 from XAA greater than payoff 2 from YAA. ~~E This expectation is on~~ The only pure strategy of P1 ~~consistent with this expectation is~~ in the X subgame consistent with this expectation is A. Then if P2 anticipates P1 plays A in the X subgame, P2 also plays A, so ~~the (XA, AA) is~~ the outcome of (XA, AA) is realised.

This argument is not reasonable because X is strictly dominant for P1, and so is implied by rationality. P2 need not ~~anticipate~~ think A or B more likely in virtue of P1 choosing X, so P2 has no ~~reason~~ strict incentive to play either A or B in the X subgame.

4



b ~~A PBE is an assessment (s,~~

A pure PBE is an assessment (s, μ) consisting ~~of~~ a pure strategy profile s and beliefs μ such that each player chooses optimally given his beliefs and the other players' equilibrium strategies, and beliefs are computed based on equilibrium strategies via Bayes' rule where possible.

Suppose $s_2 =$ CFO ~~then~~ By definition of PBE, CFO is optimal given s_1 and μ, i.e. $\pi_2(CFO, s_1; \mu) \geq \pi_2(HR, s_1; \mu)$ and $\pi_2(CFO, s_1; \mu) \geq \pi_2(N, s_1; \mu)$.
$5(1-\mu_1) \geq 5\mu_1$, $\mu_1 \leq 1/2$
$5(1-\mu_1) \geq 3$, $\mu_1 \leq 2/5$.
By definition of PBE, s_1 is optimal against CFO (given μ).
s_1 is some pair (a_1^P, a_1^Q) denoting P1's action if type P and ~~P1's~~ P1's action if type Q.
By inspection, $s_1 = (No, MBA)$ is optimal against CFO, i.e. only type Q P1 chooses MBA.
By Bayes' rule, $\mu_1 = 0$
So (s, μ) is a PBE where $s_1 = (No, MBA)$, $s_2 =$ CFO and $\mu_1 = 0$.

Suppose $s_2 = HR$. Then HR is optimal given s_1 and μ.
$\pi_2(HR, s_1; \mu) \geq \pi_2(CFO, s_1; \mu)$, $5\mu_1 \geq 5(1-\mu_1)$, $\mu_1 \geq 1/2$
$\pi_2(HR, s_1; \mu) \geq \pi_2(N, s_1; \mu)$, $5\mu_1 \geq 3$, $\mu_1 \geq 3/5$
By definition of PBE, s_1 is optimal against $s_2 = HR$ (given μ)
By inspection, $s_1 =$ (~~MBA~~ No, No) is optimal against $s_2 = HR$
~~By Bayes' rule,~~ So there are no restrictions on μ_1
So (s, μ) is a PBE where $s_1 =$ (~~MBA~~ No, No), $s_2 = HR$, and $\mu_1 \geq 3/5$

Suppose $s_2 = N$. Then N is optimal given s_1 and μ.
$\pi_2(N, s_1; \mu) \geq \pi_2(CFO, s_1; \mu)$, $3 \geq 5(1-\mu_1)$, $\mu_1 \geq 2/5$
$\pi_2(N, s_1; \mu) \geq \pi_2(HR, s_1; \mu)$, $3 \geq 5\mu_1$, $\mu_1 \leq 3/5$
By definition of PBE, s_1 is optimal against $s_2 = N$
By inspection, (MBA, MBA) is optimal against $s_2 = N$
By Bayes rule, $\mu_1 = 1/2$
So (s, μ) is a PBE where $s_1 =$ (MBA, MBA), $s_2 = N$, and $\mu_1 = 1/2$.

c Intuitive criterion: at unreached information sets which constitute a deviation from the equilibrium strategy profile, the player who is choosing at that information set should assign zero probability to the other player's being any type whose equilibrium payoff is greater than any possible payoff from deviation.

The intuitive criterion does not eliminate the PBE (s, μ) where $s_1 = (No, No)$ ~~and~~ $s_2 = HR$, and $\mu_1 \geq 3/5$ because it does not restrict the beliefs that P2 could have at the non-singleton information set. At this information set, the greatest possible payoff for a type P P1 is 3, ~~and for a type Q P1 also~~ which is greater than the equilibrium payoff for this type P P1, of 2. Similarly for type Q P1. So the intuitive criterion does not eliminate the possibility of either type, hence does not restrict μ_1.

d Yes. Then the greatest possible payoff from deviation to type P P1 is 1, which is less than the equilibrium payoff 2. So the intuitive criterion ~~eliminates~~ requires P2 to assign zero probability to type ~~Q~~ P. Intuitively, type P would never ~~dev~~ deviate from the equilibrium since it is strictly worse off if it does so.

Then, supposing that $s_2 = HR$, $s_1 = (No, No)$ is optimal, so the non-singleton information set is unreachable, the Intuitive Criterion necessitates $\mu_1 = 0$, then ~~s~~ $s_2 = CFO$ is optimal given μ, so the ~~positi~~ strategy profile (s, μ) where $s_1 = (No, No)$, $s_2 = HR$ ~~and~~ for ~~any μ is not a PBE.~~ $\mu_1 = 0$ is not a PBE. Any other μ_1 is inconsistent with the Intuitive Criterion, so the Intuitive Criterion eliminates pooling on No.

But it seems it is not (strictly) optimal against "s_1 given μ", so is this sufficient to reject $s_2 = HR$?