Game Theory Problem Set 6

1a

$$\begin{array}{c|cc} & T & D \\ \hline T & 3/s & 1/s - 1 \\ D & 4 + 1/s & 1/s \end{array}$$

Let G_s denote ~~the sy~~ the symmetric game where two players each ~~choose T or~~ have actions T and D and payoffs given by the above matrix, A_s. Let a^s_{ij} denote the element of A_s in the i^{th} row and j^{th} column. Let $n^s_T(t)$ and $n^s_D(t)$ for $t \in \{1, 2, \ldots\}$ denote the number of players playing T and D respectively in period t. Let $n^s(t) = n^s_T(t) + n^s_D(t)$, $p^s_T(t) = n^s_T(t)/n^s(t)$ and $p^s_D(t) = n^s_D(t)/n^s(t)$. Let $\pi_x(t) = \sum_{x' \in \{T,D\}} a_{xx'} p_x(t) = [Ap(t)]_x$ for $x \in \{T, D\}$. Let λ denote some scale factor.

For $x \in \{T, D\}$,

$n_x(t+1) = n_x(t) + \lambda n_x(t) \pi_x(t) = n_x(t)(1 + \lambda \pi_x(t))$

$n(t+1) = n(t) + \lambda n(t) \sum_{x' \in \{T,D\}} p_{x'}(t) \pi_{x'}(t)$

$= n(t)(1 + \lambda \sum_{x'} p_{x'}(t) \pi_{x'}(t))$

$p_x(t+1) = n_x(t+1)/n(t+1)$

$= \frac{n_x(t)}{n(t)} (1 + \lambda \pi_x(t)) / (1 + \lambda \sum_{x'} p_{x'} \pi_{x'}(t))$

$= p_x(t) (1 + \lambda \pi_x(t)) / (1 + \lambda \sum_{x'} p_{x'} \pi_{x'}(t))$

$p_x(t+1) > p_x(t) \Leftrightarrow 1 + \lambda \pi_x(t) > 1 + \lambda \sum_{x'} p_{x'} \pi_{x'}(t)$

$\Leftrightarrow \pi_x(t) > \sum_{x'} p_{x'} \pi_{x'}(t)$

$\Leftrightarrow \pi_x(t) > p_x \pi_x(t) + (1 - p_x) \pi_{x'}(t)$

$\Leftrightarrow \pi_x(t) > \pi_{x'}(t)$ for $x' \neq x \in \{T, D\}$

The population of $x \in \{T, D\}$ grows relative to the total population iff x has ~~or~~ higher expected payoff than $x' \neq x \in \{T, D\}$.

$\pi_T(t) = 3/s \, p_T(t) + (1/s - 1)(1 - p_T(t))$

$\pi_D(t) = (4 + 1/s) p_T(t) + \cancel{(1/s - 1)} (1/s)(1 - p_T(t))$

$\pi_T(t) > \pi_D(t) \Leftrightarrow 3p + (1-s)(1-p) > (4s+1)p + (1-p)$

where p denotes $p_T(t)$

$\Leftrightarrow 3p + 1 - s - p + sp > 4sp + p + 1 - p$

$\Leftrightarrow 1 + 2p - s + sp > 1 + 4sp$

$\Leftrightarrow \cancel{(2 - 3sp)} (2 - 3s)p > s$

$\Leftrightarrow p > s/(2 - 3s)$

The basin of attraction of $p_T = 1$ is $p_T > s/(2-3s)$. $\pi_T(t) < \pi_D(t) \Leftrightarrow p < s/(2-3s)$, so the basin of attraction of $p_D = 1 \Leftrightarrow p_T = 0$ is $p_T < s/(2-3s) \Leftrightarrow p_D > (2-4s)/(2-3s)$.

b

$$\begin{array}{c|ccc} & C & T & D \\ \hline C & 3/s & 3/s & 0 \\ T & 3/s & 3/s & 1/s - 1 \\ D & 5/s & 4 + 1/s & 1/s \end{array}$$

Under the replicator dynamic

$p_T(t+1) > p_T(t) \Leftrightarrow \pi_T(t) > p_C(t)\pi_C(t) + p_T(t)\pi_T(t) + \varepsilon \pi_D(t)$

$\Leftrightarrow [Ap]_T > p^T A p$, where A is the above payoff matrix and $p^T = (p_C \; p_T \; \varepsilon)$

$\Leftrightarrow \pi_T(t) > \frac{p_C}{p_C + \varepsilon} \pi_C(t) + \frac{\varepsilon}{p_C + \varepsilon} \pi_D(t)$

$\cancel{\Leftrightarrow (3/5)p_C + (2/5)p_T + (1/5 - 1)\varepsilon >}$

$\Leftrightarrow \pi_T(t) > \pi_C(t)$ given vanishingly small ε

$\Leftrightarrow (3/5)p_C + (3/5)p_T + (1/5 - 1)\varepsilon > (3/5)p_C + (3/5)p_T + 0\varepsilon$

$\Leftrightarrow \varepsilon > 0$

The expected proportion of T players increases in each period for all p_C, p_T (supposing $p_T < 1$). This is because, ~~for $\varepsilon = 0$~~, T strictly dominates C and for $\varepsilon > 0$, $\pi_T > \pi_C$, so the population of T players grows faster than the population of C players. Supposing that ε is sufficiently small, the change in the population of D players has negligible effect on the total population, so ~~only~~ the change in proportion of T players is determined entirely by the relative growth of the C and T populations.

2a

	H	D
H	-1 -1	$\underline{0}$ $\underline{2}$
D	$\underline{2}$ $\underline{0}$	1 1

~~B~~ Best responses underlined. By inspection, the only pure NE are (H,D) and (D,H).

Suppose that there is a mixed NE $\sigma^* = (\sigma_1^*, \sigma_2^*)$ where σ_1^* assigns probability p to H and probability $1-p$ to D. Then, by definition of NE, P1 has no profitable deviation, then $\pi_1(H) = \pi_1(D)$, ~~$-q+$~~ $-q + 2(1-q) = 1-q$, where q is the probability σ_2^* assigns to H. $2-3q = 1-q$, $1 = 2q$, $q = 1/2$. Then, by definition of NE, P2 has no profitable deviation, then $\pi_2(H) = \pi_2(D)$, $-p + 2(1-p) = 1-p$, $p = 1/2$. The only mixed NE is (σ_1^*, σ_2^*) where $\sigma_1^* = \frac{1}{2}H + \frac{1}{2}D$ and $\sigma_2^* = \frac{1}{2}H + \frac{1}{2}D$.

From the above argument, if P1 mixes, so does P2. By symmetry, if P2 mixes, so does P1. So there are no hybrid NE.

By definition of ESS, ~~the only cand~~ σ^* is an ESS if (σ^*, σ^*) is an NE. So the only candidate ESS is $\frac{1}{2}H + \frac{1}{2}D$. Denote this strategy σ^*.

$\pi(H, \sigma^*) = \pi(D, \sigma^*) = \frac{1}{2}$, so any ~~pure or mixed~~ ~~stra~~ (potentially degenerate) mixed strategy $\sigma' = p'H + (1-p')D$ is a best response to σ^*.

$$\pi(\sigma', \sigma') = -p'^2 + 2p'(1-p') + (1-p')^2 \cancel{= -p'^2 + 2p' - 2p'^2}$$
$$= -p'^2 + 2p' - 2p'^2 + 1 - 2p' + p'^2$$
$$= -2p'^2 + 1$$
$$\pi(\sigma^*, \sigma') = -\tfrac{1}{2}p' + (1-p') + \tfrac{1}{2}(1-p')$$
$$= \tfrac{3}{2} - 2p'$$
$$\pi(\sigma^*, \sigma') > \pi(\sigma', \sigma') \Leftrightarrow \tfrac{3}{2} - 2p' > -2p'^2 + 1$$
$$\Leftrightarrow 2p'^2 - 2p' + \tfrac{1}{2} > 0$$
$$\Leftrightarrow 2(p' - \tfrac{1}{2})^2 > 0$$
$$\Leftrightarrow p' \neq \tfrac{1}{2}$$

So if $\sigma' \neq \sigma^*$ (i.e. $p' \neq \frac{1}{2}$) is a best response to σ^*, then $\pi(\sigma^*, \sigma') > \pi(\sigma', \sigma')$. By definition of ESS, σ^* is an ESS.

b Let $n_H(t)$, $n_D(t)$, $n(t)$, $\pi_H(t)$, $\pi_D(t)$, $p(t) \equiv p_H(t)$, and $p_D(t)$ be defined in the conventional way. Let λ denote the scale factor

$$n_H(t+1) = n_H(t) + \lambda n_H(t)\pi_H(t) = n_H(t)(1 + \lambda\pi_H(t))$$
$$n(t+1) = n(t) + \lambda n(t) \textstyle\sum_{x \in \{H,D\}} p_x(t)\pi_x(t)$$
$$= n(t)(1 + \lambda(p_H(t)\pi_H(t) + p_D(t)\pi_D(t)))$$
$$= n(t)(1 + \lambda(p(t)\pi_H(t) + (1-p(t))\pi_D(t)))$$
$$\pi_H(t) = \cancel{-p + 2(1-p) = 2-3p} - p(t) + 2(1-p(t)) = 2 - 3p(t)$$
$$\pi_D(t) = 1 - p(t)$$
$$n_H(t+1) = n_H(t)(1 + (2 - 3p(t))\lambda)$$
$$n(t+1) = n(t)(1 + (2p(t) - 3p(t)^2 + (1-p(t))^2 - 2p(t))\lambda)$$
$$= n(t)(1 + (-2p(t)^2 + 1)\lambda)$$

can the replicator equations be given without proof?

Replicator equation

$\dot p = p(1-p)(\pi_H(p) - \pi_D(p))$

$= p(1-p)(2-3p-(1-p))$

$= p(1-p)(1-2p)$

$\dot p = 0$ iff $p=0$, $p=1$, or $p=1/2$

p is stable ~~at p=0~~ over time at $p=0$, $p=1$, or $p=1/2$

$\dot p>0$ for $p\in(0,1/2)$, $\dot p<0$ for $p\in(1/2,1)$

For all initial $p\neq 0,1$, the process evolves to $p=1/2$, $(0,1)$ is the basin of attraction of $p=1/2$.

c ~~Let the superscript of π denote either Row~~

Let π_X^Y denote the expected payoff ~~to~~ from playing action X to a Row player if Y= R and a column player if Y=C.

$\pi_H^R(q)=2-3q$, $\pi_D^R=1-q$, $\pi_H^C(p)=2-3p$, $\pi_D^C(\cancel{q}p)=1-p$

$\dot p = p(1-p)(\pi_H^R(q)-\pi_D^R(q)) = p(1-p)(1-2q)$

$\dot q = q(1-q)(\pi_H^C(p)-\pi_D^C(p)) = q(1-q)(1-2p)$

i $p_0>1/2$, $q_0<1/2$, $\dot p_0>0$, $\dot q_0<0$

Then $p_1>p_0>1/2$, $q_1<q_0<1/2$ so $\dot p_1>0$, $\dot q_1<0$

P evolves to ~~q~~ 1 and q evolves to 0

~~The state evolves to (H,D)~~

ii $p_0=q_0$ $\dot p_0 = p_0(1-p_0)(1-2p_0)$, $\dot q_0 = q_0(1-q_0)(1-2q_0) = p_0(1-p_0)(1-2p_0)$

$\dot p_0 = \dot q_0$, then $p_1=q_1$, $\dot p_1=\dot q_1$, and so on.

p and q evolve to 1/2 and 1/2, ~~the~~

It is as if the two populations were in fact a single population, that evolves as in (b).

iii $q_0<p_0<1/2$, $\dot p_0>0$, $\dot q_0>0$, $\dot p_0>\dot q_0$

Suppose $q_t<p_t=1/2$, then $\dot p_t>0$, $\dot q_t=0$

~~Suppose $q_t<1/2<$~~ $q_{t+1}<1/2<p_{t+1}$, $\dot p_{t+1}>0$, $\dot q_{t+1}<0$

Initially, p ~~and evolves~~ and q evolve upwards, ~~then when p=1/2, q is stationary and~~ p more quickly than q. Then when p=1/2, P continues to evolve upwards while q is stationary. Then when p>1/2, p evolves upwards to 1 while q evolves downwards to 0.

iv In the long run, each population plays ~~one of~~ a pure strategy, and both populations play different strategy (i.e. one plays H with certainty and the other plays D with certainty) The ~~strategy~~ action that each population plays is that to which it ~~is~~ plays with greater initial frequency than the other population. The symmetric ~~evolution~~ equilibrium is unlikely in the long run ~~is~~ unlikely because it is in some sense unstable, ~~and vulnerable to minor~~ Any minor shock to the ~~pop~~ population's strategies such that $p\neq q$ ~~sets the~~ results in the "divergent" evolution path.

3a

	Br	C	Bt
Br	$\underline{1}$	0	b
	$\underline{1+a}$	a	a
C	0	$\underline{1}$	b
	0	$\underline{1}$	0
Bt	0	0	$\underline{1+b}$
	0	0	$\underline{1}$

Best responses underlined. By inspection, the pure NE are (Br, Br), (C, C), and (Bt, Bt) where players play mutual best responses.

b Each pure NE ~~is an absorbing~~ corresponds to an absorbing, i.e. if (a_1, a_2) (denoting the strategy profile where P1 plays ~~a_t~~ a_1 and P2 plays a_2) is a pure NE, then the state ~~st~~
$x = (x_{a_1} = N, x_{a_1'} = 0, x_{a_1''} = 0), y = (y_{a_2} = N, y_{a_2'} = 0, y_{a_2''} = 0)$
is an absorbing state, i.e. remains unchanged in all subsequent periods. ~~Where st~~ Suppose that some such state is reached in period t. At the start of period t+1, some Row players and some Column players update their strategies to best respond to the distribution of strategies in period t ~~t2~~. Some Row players update their strategies to best respond to a_2 and some Column players update their strategies to best respond to a_1. By definition of NE, a_1 ~~is a best response to a_2~~ and a_2 are mutual best ~~respo~~ responses, so the updating players' strategies are unchanged. Then the state is unchanged. By induction, the state remains unchanged in all subsequent periods.

Suppose that some strategy profile is not an NE, then ~~the s~~ either Row is not playing a best response to Column's strategy or Column is not playing a best response to Row's strategy. Suppose without loss of generality that the former is true. Suppose that in period t, the corresponding state is reached. With non-zero probability, at least one Row player updates his strategy at the start of period t+1 to best respond to Column. These Row player's strategies change since they were not previously playing a best response. Then the state changes, so it is not an absorbing state.

Suppose that some strategy profile is a mixed or hybrid NE, and the corresponding state (either where each player plays the ~~###~~ corresponding mixed strategy, or the distribution of players playing pure strategies in each population corresponds to the probability distribution of that population's mixed strategy in NE) is reached in period t. At the start of period t+1, with non-zero probability, ~~one at least~~ one player from the mixing population ~~is selected~~ updates his strategy. Such players are indifferent between the ~~pure~~ actions that their population's corresponding strategy mixes over. So with non-zero probability, ~~one~~ this player changes his strategy. Then the state changes, so it is not an absorbing state.

Consequently a state is an absorbing state iff it corresponds to some pure NE.

c Consider some arbitrary ~~initial~~ state ~~in period~~ in period t. At the start of period $t+1$, with non-zero probability, all and only players from one population update their strategies, and with non zero probability, all such players choose the same pure strategy, which best responds to the other population's strategy distribution in period t. Then, ~~the updating population~~ in period $t+1$, each member of the updating population plays a common ~~strategy. At the start of period~~ pure strategy. At the start of period $t+2$, with non-zero probability, all and only players ~~from~~ from the non-updating population in $t+1$ update their strategies, and each of these players chooses the unique ~~best~~ pure best response to the pure strategy chosen by each player from the ~~other~~ $t+1$-updating-population. So the state in $t+2$ is an absorbing state that corresponds to a pure NE, and remains unchanged in all subsequent periods.

~~In any~~ At any period t, there is a non-zero probability that at period $t+2$ and all subsequent periods, ~~the~~ an absorbing state will have been reached. Once reached, an absorbing state remains unchanged in all subsequent periods. Then, as the number of periods increases, the probability of being in an absorbing state approaches 1. The process necessarily converges to an absorbing state.

d

	C	Bt
Br	0 a	$\underline{b}$ $\underline{a}$
C	$\underline{1}$ $\underline{1}$	b 0

Best responses underlined. By inspection the only pure NE are (C, C) and (Br, Bt). By the ~~argument result given in (b),~~ argument in b, the only absorbing states are the two which correspond to these pure NE, namely
$x = (x_C = 1, x_{Br} = 0)$ $y = (y_C = 1, y_{Bt} = 0)$ and
$x = (x_{Br} = 1, x_C = 0)$, $y = (y_{Bt} = 1, y_C = 0)$
The former state is Pareto optimal.

Date

- Let p^R and p^C denote the proportion of Row players that play C and the proportion of Column players that play C respectively.
  $\pi^R(C, p^C) = p^C$, $\pi^R(Br, p^C) = a$
  $\pi^C(C, p^R) = p^R$, $\pi^C(Bt, p^R) = b$
  For Row, C is a best response iff $p^C \geq a$, and Br is a best response iff $p^C \leq a$.
  For Column, C is a best response iff $p^R \geq b$ and Bt is a best response iff $p^R \leq b$.
  ~~Suppose that~~

  Suppose that best reply dynamics are deterministic, then if ~~$p^C \leq a$ and~~ ~~$p^C \geq$~~ $p^C > a$ and $p^R > b$, $\dot{p}^C > 0$ and $\dot{p}^R > 0$. ~~So the~~ So the basin of attraction of $(p^R = 1, p^C = 1)$ includes $\{(p^R, p^C) : p^R > b, p^C > a\}$ By an analogous argument, the basin of attraction of $(p^R = 0, p^C = 0)$ includes $\{(p^R, p^C) : p^R < b, p^C < a\}$.

  For simplicity, we ~~neglect to consider~~ do not consider best response dynamics where $p^R > b$ and $p^C < a$, or where $p^R < b$ and $p^C > a$.

  The ~~minimum volume~~ number of uncorrected errors required to ~~exit the basin of~~ leave $(p^R = 1, p^C = 1)$ and enter the basin of $(p^R = 0, p^C = 0)$ is $N(1-a) + N(1-b)$. The ~~minimum~~ number of uncorrected errors required to leave $(p^R = 0, p^C = 0)$ and enter the basin of $(p^R = 1, p^C = 1)$ is $Na + Nb$. ~~The latter is~~ $(p^R = 0, p^C = 0)$ is more likely in the long run than $(p^R = 1, p^C = 1)$ iff $N(2-a-b) < N(a+b)$, $\Leftrightarrow a+b > 1$