Game Theory Paper 180605

1a

	A	B	C
A	5	-1	$\underline{6}$
	5	-1	$\underline{2}$
B	~~+~~-1	$\underline{0}$	-1
	-1	$\underline{0}$	1
C	$\underline{2}$	1	-1
	$\underline{6}$	-1	-1

A strategy is rationalisable iff it is a best response to some potentially correlated mix of other players rationalisable strategies. This is iff it survives iterated elimination of non-best responses

Pearce's Lemma: a strategy is not a best ~~r~~ response to some pure or mixed strategy iff it is ~~is~~ strictly dominated by ~~p~~ some pure or mixed strategy.

By Pearce's Lemma, iterated elimination of non-best-responses is equivalent to iterated strict dominance. So all and only rationalisable strategies survive iterated strict dominance.

By inspection, each pure action is a ~~strict~~ best response to some pure action of the other player, so all pure actions in this game are rationalisable.

b By inspection of the payoff matrix (best responses underlined), there is one ~~pure~~ symmetric pure NE, (B, B).

Consider candidate symmetric mixed NE such that P1 and P2 mix non-degenerately over A and B only, with probability p_A, $p_B = 1 - p_A$, ~~st~~ $p_A \in (0,1)$. This candidate NE fails to deviation by either player ~~to C~~. ~~# by $\pi(C, \sigma^*_2) = 6p_A - p_B > 5p \; \pi_1(A, \sigma^*_2)$~~ by reallocating probability mass from A to C.
$\pi_2(C, \sigma^*_1) = 6p_A - p_B > \pi_2(A, \sigma^*_1) = 5p_A - p_B$.

Consider candidate symm mixed NE s.t. P1, P2 mix non-degenerately over only B and C. This fails to deviation by reallocating all probability mass from C to B which always yields higher payoff against B or C.

Consider candidate symm mixed NE s.t. P1, P2 mix non-degenerately over A and C only. Then each is indifferent and has no profitable deviation.
$\pi_1(A, \sigma^*_2) = \pi_1(C, \sigma^*_2) > \pi_1(B, \sigma^*_2) \iff$
$5p_A + 2(1-p_A) = 6p_A - 1(1-p_A) > -1p_A + (1-p_A) \iff$
$3(1-p_A) = p_A$, $7p_A > 2(1-p_A) \iff$
$3 = 4p_A$, $7p_A > 2(1-p_A) \iff$
$p_A = 3/4$, $21/4 > 2/4 \Rightarrow$
$p_C = 1/4$.
$\sigma^* = (3/4A + 1/4C, \; 3/4A + 1/4C)$ is a symmetric mixed NE.

Consider candidate symm mixed NE s.t. each mixes non-degenerately over ~~P1, P2,~~ A, B, C. Then each is indifferent.

$\pi_1(A, \sigma^*_2) = \pi_1(B, \sigma^*_2), \pi_1(C, \sigma^*_2) \iff$
$5p_A - p_B + 2(1-p_A-p_B) = -p_A + (1-p_A-p_B) = 6p_A - p_B - (1-p_A-p_B)$
$\iff$
$3p_A - 3p_B + 2 = -2p_A - p_B + 1 = 7p_A - 1 \Rightarrow$
$5p_A = \cancel{4p_B}\; 2p_B - 1$, $\quad 9p_A = -p_B + 2 \iff$
$p_A = 2/5 p_B - 1/5 = \quad -p_B/9 + 2/9 \quad \Rightarrow$
$18/45 p_B + 5/45 p_B = 10/45 + 9/45 \Rightarrow$
$p_B = 19/23$ ~~p_A~~, $p_A = 3/23$, $p_C = 1/23$
~~p_A~~
$\sigma^* = (3/23 A + 19/23 B + 1/23 C, \; 3/23 A + 19/23 B + 1/23 C)$ is a symmetric mixed NE.

c The required correlated eqm is as follows.
$\Omega = \{x, y\}$
$P\{\omega = x\} = P\{\omega = y\} = 1/2$
$\mathcal{P}_1 = \{\{x\}, \{y\}\}$
$\mathcal{P}_2 = \{\{x\}, \{y\}\}$

$$\sigma_1(\omega) = \begin{cases} A & \text{if } \omega = x \\ C & \text{if } \omega = y \end{cases}$$

$$\sigma_2(\omega) = \begin{cases} A & \text{if } \omega = y \\ C & \text{if } \omega = x \end{cases}$$

These strategies are consistent with the above information ~~partition~~ partitions, and yield expected payoff $1/2 \, 2 + 1/2 \, 6 = 4$ for each player. In ~~a~~ each state, neither player has incentive to deviate because prescribed play is a NE.

d The required ~~#~~ correlated eqm is as follows.
$\Omega = \{x, y, z\}$
$P\{\omega = x\} = P\{\omega = y\} = P\{\omega = z\} = 1/3$
$\mathcal{P}_1 = \{\cancel{x}\{x\}, \{y, z\}\}$.
$\mathcal{P}_2 = \{\{x, y\}, \{z\}\}$.

$$\sigma_1(\omega) = \begin{cases} A & \text{if } \omega = y \text{ or } z \\ C & \text{if } \omega = x \end{cases}$$

$$\sigma_2(\omega) = \begin{cases} A & \text{if } \omega = x \text{ or } y \\ C & \text{if } \omega = z \end{cases}$$

These strategies are consistent with the above information partitions, and yield expected payoff $1/3 \, \cancel{6}\, 2 + 1/3 \, 5 + 1/3 \, 6 = 4\,1/3$ for each player.

Consider P1. In the $\omega = x$ state, P1 knows P2 plays A, so P1's ~~best r~~ prescribed play is a BR, C. In the other information ~~point~~ set, P1 knows P2 plays A and C with equal probability. A yields $\frac{1}{2}8 + \frac{1}{2}2 = \frac{7}{2}$, B yields $\frac{1}{2}\cdot -1 + \frac{1}{2}1 = 0$, C yields $\frac{1}{2}6 + \frac{1}{2}\cdot -1 = \frac{5}{2}$, so A is a BR in interim expectations. By symmetry, P2's prescribed play in this correlated eqm is also a BR in each information set, so this is indeed a correlated eqm.

e Correlated eqm expand the set of possible payoff vectors but not necessarily in a Pareto-improving direction.

Some degenerate correlated eqm where P1 and P2 play (B,B) most of the time and occassionally are instructed to both play C could yield worse payoff than (0,0).

2a Players : $N = \{1, \dots, n = 60\}$

Actions : $A_i = \{HW, R\}$ for all $i \in N$

Payoffs : $u_i(a_i, a_{-i}) = \begin{cases} -[(1+n_{HW}) + (51 + n_{HW}/10)] & \text{if } a_i = HW \\ -[(51 + n_R/10) + (1 + n_R)] & \text{if } a_i = R \end{cases}$

$= \begin{cases} -(52 + {}^{11}/_{10}\, n_{HW}) & \text{if } a_i = HW \\ -(52 + {}^{11}/_{10}\, n_R) & \text{if } a_i = R \end{cases}$

for all $i \in N$.

where $n_{HW} = \sum_{i \in N} [A_i = HW]$, $n_R = \sum_{i \in N} [A_i = R]$

b At every pure NE, no player has strict incentive to deviate. Let $*$ denote NE values.

HW - players have no strict incentive to deviate iff $-(52 + {}^{11}/_{10} n^*_{HW}) \geq -(52 + {}^{11}/_{10}(n^*_R + 1))$ similarly R - players have no strict incentive to deviate iff $-(52 + {}^{11}/_{10} n^*_R) \geq -(52 + {}^{11}/_{10}(n^*_{HW} + 1))$.

Both conditions are satisfied iff

$-{}^{11}/_{10}\, n^*_{HW} \geq -{}^{11}/_{10} n^*_R - {}^{11}/_{10}$,

$-{}^{11}/_{10}\, n^*_R \geq -{}^{11}/_{10} n^*_{HW} - {}^{11}/_{10}$ $\iff$

$n^*_{HW} \geq n^*_R - 1$, $n^*_R \geq n^*_{HW} - 1$ $\iff$

$n^*_{HW} = n^*_R$ $\Rightarrow$

$n^*_{HW} = n^*_R = 30$.

At every pure NE, $n^*_{HW} = n^*_R = 30$. There is a large number of such NE because there is a large number of possible combinations such that 30 players play each pure action.

At each NE, travel time $t^*_{HW} = 52 + {}^{11}/_{10} n^*_{HW} = 85$,
$t^*_R = 52 + {}^{11}/_{10} n^*_R = 85$

c We are interested only in NE, so it is sufficient to consider the game in strategic rather than extensive form, with three pure strategies : HW, R, and HWR, where the latter denotes taking the route Ox → HW → R → C.

At NE, payoffs to each pure action are equal (neglecting integer constraints), so time taken on each route is equal.

$t^*_{HW} = (1 + n_{HW} + n_{HWR}) + (51 + n_{HW}/10)$
$= 52 + {}^{11}/_{10} n_{HW} + n_{HWR}$

$t^*_R = (51 + n_R/10) + (1 + n_R + n_{HWR})$
$= 52 + {}^{11}/_{10} n_R + n_{HWR}$

$t^*_{HWR} = (1 + n_{HW} + n_{HWR}) + (10 + n_{HWR}/10) + (1 + n_R + n_{HWR})$
$= 12 + {}^{21}/_{10} n_{HWR} + n_{HW} + n_{HR}$

$t^*_{HW} = t^*_R = t^*_{HWR}$ $\iff$

$52 + {}^{11}/_{10} n_{HW} + n_{HWR} = 52 + {}^{11}/_{10} n_R + n_{HWR} = 12 + {}^{21}/_{10} n_{HWR} + n_{HW} + n_R$

$\iff$

$40 + {}^{11}/_{10} n_{HW} = 40 + {}^{11}/_{10} n_R = {}^{11}/_{10} n_{HWR} + n_{HW} + n_R$ $\Rightarrow$

$n_{HW} = n_R$, $n_{HW} + n_R = 40$ $\Rightarrow$

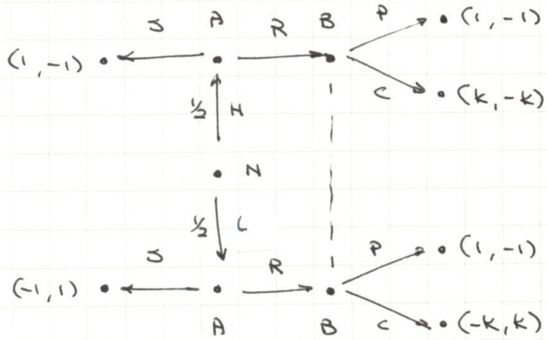
$n_{HW} = n_R = n_{HWR} = 20$ $\Rightarrow$

$t^*_{HW} = t^*_R = t^*_{HWR} = 52 + {}^{11}/_{10}(20) + 20 = 94$

Eqm travel time on each route, somewhat counterintuitively, has increased as a result of the construction of the additional road.

The above NE is also an SPE because in the HW - subgame, the HW → R → C route has equal travel time as the ~~R~~ HW → C route, so both yield ~~of~~ equal payoff hence ~~there is no~~ play in this subgame is NE.

3a



	P	C
RR	-1; 1; $\underline{1},\underline{1}$;	0; $\underline{0}$; $\underline{k},-k$;
RS	0; $\underline{0}$; $\underline{1},-1$;	$k-1/2$; $1-k/2$; $\underline{k},\underline{-1}$;
SR	-1; 1; $\underline{1},\underline{1}$;	$1-k/2$; $\underline{k-1/2}$; $1,-k$
SS	0; $\underline{0}$; $\underline{1},-1$;	0; $\underline{0}$; $1,\underline{-1}$;

Payoffs in interim expectation, payoff for high H type A given first. Best responses underlined.

b By inspection, there is no pure BNE where players play mutual pure best responses in interim expectation.

c ~~By inspection, each pure strategy for A, in interim expectation, A is a best response against some B strategy for some type of A A. No pure strategy is strictly dominated in interim expectation.~~

In ~~ex a~~ ex ante expectation, a uniform mix between RR and RS strictly dominates SS. The former strategy yields payoff $1+0/2$ against P in ex ante expectation ~~while the latter yie~~ and payoff $k-1/4 > 0$ against C in ex ante expectation, whereas the latter strategy yields payoff 0 against both P and C in ex ante expectation. No other strategies are strictly dominated in ex ante expectation.

No strategies are strictly dominated in interim expectation because each strategy yields equal payoff of 1 against P for H type A.

d At BNE, the strictly dominated strategy SS is played with zero probability.

At BNE B mixes non-degenerately. Suppose that B plays pure P at BNE. ~~Then P play~~ Then A plays RS (and SS) up 0 because L type H strictly prefers to R. Then B strictly prefers to play C with certainty which yields $0 > -1$ against RR and $k-1/2 > -1$ against SR, and so does better ~~against~~ than P against any rational A strategy. Suppose that B plays pure C at BNE, then A plays RR and SR up 0 because L type A strictly prefers S, so A plays RS with certainty, then P is strictly rational for B. By reduction, B must mix at ~~BNE~~ BNE.

Given that B mixes, A plays SR up 0 because RR yields strictly higher payoff in interim expectation for H types.

So at BNE, A mixes RR and RS and B mixes P and C. Let p denote the probability that H type A plays RR at BNE and q denote the probability that B plays P. At BNE, players have no strictly profitable deviation, so they are indifferent.

$\pi_B(P, \sigma_A^*) = \pi_B(C, \sigma_A^*) \Longleftrightarrow$

$-1p + 0(1-p) = 0p + (1-k/2)(1-p) \Longleftrightarrow$

$-p = (1-k/2)(1-p) \Longleftrightarrow$

$k-1/2 = k+1/2\, p \Longleftrightarrow$

$p = k-1/k+1$

~~Itbf~~ $\pi_A(RR, \sigma_B^*; t_A = L) = \pi_A(RS, \sigma_B^*; t_A = L) \Longleftrightarrow$

$q(1) + (1-q)(-k) = -1 \Longleftrightarrow$

$q - k + kq = -1 \Longleftrightarrow$

~~kf~~ $q(k+1) = k-1 \Longleftrightarrow$

$q = k-1/k+1$

The unique BNE = $(p\,RR + (1-p)RS,\ qP + (1-q)C)$ where $p = q = k-1/k+1$.

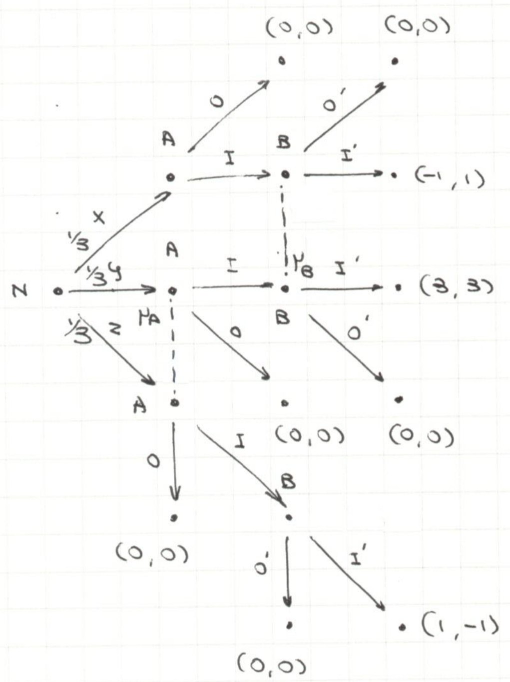
At BNE, ~~A has ex ante expected payoff $\pi_A(RR, \sigma_B^*)$ $= 1/2(q + (1-q)k) + 1/2($~~ type H A has expected payoff $q + (1-q)k = k-1/k+1 + 2k/k+1 = 3k-1/k+1$ type L A has expected payoff $q - (1-q)k = k-1/k+1 - 2k/k+1 = -k-1/k+1 = -1$ A has ex ante expected payoff $1/2(-1) + 1/2(3k-1/k+1) = k-1/k+1$. B has expected payoff $-1p + 0(1-p) = -p = -k-1/k+1$.

e The probability of bluffing is $p = k-1/k+1$, which increases with increasing k, from $p = 1/3$ when $k = 2$, ~~this~~ p converges to 1 as k becomes large. Intuitively, ~~so the~~ this is because P yields payoff -1 with certainty while C yields negative payoff that becomes ~~ever~~ more negative with increasing k, so as k increases, B has less incentive to play C. A can get away with bluffing more frequently so A bluffs more.

4a Event E is known to player i in state ω ~~*~~ iff $P_i(\omega) \subseteq E$, where $P_i(\omega)$ is player i's information set that contains ω. The event that E is known to player i is $K_i(E) = \{\omega : P_i(\omega) \subseteq E\}$. The event that E is known by all players is $K(E) = \bigcap_i K_i(E)$. The event that all players know that all players know that event E is $K^2(E) = K(K(E))$. The event that E is commonly known is $K^\infty(E)$.

State x is known to A in state x but never known to B in any state, so the event that A and B both know x is ∅, ~~x is A~~ so x is never commonly known. By symmetry, the same is true for z with the roles reversed. State y is not known to either in any state, so the event that both know y is ∅, so y is never commonly known. Every state is never commonly known.

b



The required NE is (OO, O'O'), where OO for A means Out in the x information set and Out in the yz information set and O'O' for B means Out in the xy information set and Out in the ~~x~~ z information set.

The only proper subgame of this game ~~is at~~ has its root at B's z ~~=~~ information set. ~~The~~ There, O' is strictly optimal for B because it ~~induces~~ yields payoff 0 rather than −1.

(OO, O'O') is a SPE because unilateral deviation by either player is not strictly profitable. It yields payoff 0 because the other player plays Out anyway which is equal to eqm payoff.

So (OO, O'O') is a NE, and induces a NE in every (one) proper subgame, so it is a SPE.

There is no PBE such that each player (at every information set) plays Out. This is because In is strictly sequentially rational for B at B's xy information set. I' yields ~~$\mu_B 3$~~ $3\mu_B + 1(1-\mu_B)$ ~~>0~~ here whereas O' yields 0.

~~There is a PBE such that each player at on (any) eqm path plays Out. This is ((OO, I'O'), ($\mu_A = 1/2$, μ_B~~

d Conjecture that the required ~~strategy~~ PBE ~~p~~ includes the strategy profile (OI, I'O'). This yields payoff 3 to each player in the y state ~~but~~ ~~sp~~ and 0 to each player in every other state hence expected payoff 1 to each player.

O' in z info set is strictly SR for B. I' in xy info set is strictly SR for B. So SR uniquely pins down B's ~~each~~ strategy at PBE. Then, at A's x info set, given B's strategy, O is strictly SR, yielding 0 instead of −1. So by BB, $\mu_B = 1$. By BB, $\mu_A = 1/2$. Then at A's yz infoset, I yields $\frac{1}{2}3 + \frac{1}{2}0 = \frac{3}{2}$ whereas O yields 0, so I is strictly SR. ~~This pi~~ The PBE is entirely pinned down by SR and BB, i.e. by definition of PBE, so it is unique. The PBE consists in strategy profile $\sigma^* = (OI, I'O')$ and beliefs $\mu^* = (\mu_A^* = 1/2, \mu_B^* = 1)$.

Date

5a Given bargaining problem (U, d), where U is a set of possible agreement payoff vectors and d is the disagreement payoff vector, the Nash bargaining solution is $F^N(U,d) = \arg\max_{u \in U} \prod_i (u_i - d_i)$, where i indexes bargaining parties. The Nash bargaining ~~axio~~ solution is characterised by the four Nash bargaining axioms.

Weak Pareto efficiency (WPAR): bargaining solution $F(U,d) = u^*$ satisfies WPAR iff $\nexists u' \in U$ such that $\forall i : u_i' > u_i^*$.

Symmetry (SYM): $F(U,d) = u^*$ satisfies SYM iff if (U,d) is symmetric ~~(the~~ in the two-party case, this is iff $\cancel{\forall u \in U}$ $(u_1, u_2) \in U : (u_2, u_1) \in U$ and $d_1 = d_2$) then u^* is symmetric, i.e. $u_1^* = u_2^*$.

Invariance to equivalent payoff representations (INV): $F(U,d) = u^*$ satisfies ~~SYM iff~~ INV iff for all affine transformations $f(x) = ax + b$ with $a > 0$, $F(U', d') = u'$ where $U' = \{ (f(u_1), \ldots f(u_n)) : (u_1, \ldots, u_n) \in U \}$, $d' = (f(d_1), \ldots, f(d_n))$, and $u' = (f(u_1), \ldots, f(u_n))$

Independence of irrelevant alternatives (IIA): $F(U,d) = u^*$ satisfies IIA iff for all $U' \subseteq U$ s.t. $u^* \in U'$, then $F(U', d) = u^*$.

Of these, IIA is generally considered least plausible.

b By WPAR, the Nash bargaining solution exhausts the endowment, i.e. the constraint $x_1 + x_2 \leq 1$ binds at the optimum. The Nash bargaining solution reduces to

$\cancel{x_1^* = a}$

$x_1^N = \arg\max_{x_1} (x_1^\rho - 0) \cancel{(x_2 - d_2)} (1 - x_1 - d_2)$

FOC: $x_1^\rho(-1) + \rho x_1^{\rho - 1}(1 - x_1 - d_2) = 0 \Rightarrow$

$\cancel{-x_1^\rho + \rho x_1^{\rho-1} - \rho x_1^\rho - d_2 \rho x_1^{\rho-1} = 0 \Rightarrow}$

$\cancel{(x_1^{\rho-1})(-x_1 + \rho - \rho x_1 - d_2 \rho x_1) = 0 \Rightarrow}$

$\cancel{(x_1^{\rho-1})(\rho - (1 + \rho + d_2 \rho) x_1) = 0 \Rightarrow}$

$\cancel{x_1 = 0 \text{ or } x_1 = \rho / 1 + \rho + d_2 \rho}$

$\cancel{SOC: -\rho x_1^{\rho-1} + \rho(\rho-1)x_1^{\rho-2} - \rho^2 x_1^{\rho-1} - d_2 \rho(\rho-1) x_1^{\rho}}$

$x_1^{\rho-1} \left(-x_1 + \rho(1 - x_1 - d_2) \right) = 0 \Rightarrow$

$x_1^{\rho-1} \left(\rho - \rho d_2 - x_1(1+\rho) \right) = 0 \Rightarrow$

$x_1 = 0$ or $x_1 = (1 - d_2)\rho / 1 + \rho$

SOC: $(\rho - 1) x_1^{\rho-2}(-x_1 + \rho(1 - x_1 - d_2))$
$\quad + \rho x_1^{\rho-1}(1+\rho)$

~~SOC~~ < 0 for $x_1 = (1-d_2)\rho / 1 + \rho$, $\quad 0$ for $x_1 = 0$

$x_1^N = (1 - d_2)\rho / 1 + \rho$, $\quad x_2^N = 1 - x_1^N = 1 - (1-d_2)\rho / 1 + \rho$

This allocation uniquely solves the ~~Nas~~ condition for the Nash bargaining solution.

$\cancel{\partial / \partial \rho}$ $\partial / \partial \rho \, x_1^N = (1 - d_2)[- - \rho^{-2}] > 0$,

$\partial / \partial d_2 \, x_1^N = -\rho / 1 + \rho < 0$

P1's share under the Nash bargaining solution is increasing in ρ and decreasing in d_2. Intuitively, the more risk-averse P1, ~~i.e.~~ and the less utility P1 gets from the pie, i.e. the lower ρ, the ~~less~~ weaker the demands that P1 will make in bargaining hence the smaller P1's share. The better the disagreement point for P2, the stronger P2's bargaining position, hence the more P2 receives from Nash bargaining.

In the ~~sat~~ symmetric case where $\rho = 1$, $d_2 = 0$, the Nash bargaining solution is symmetric, so $x_1^N = x_2^N = 1/2$.

As ρ approaches zero, P1's bargaining position weakens, and x_1^N converges to 0. As d_2 approaches one, P2's bargaining position strengthens, and x_1^N converges to 0.

c In the stationary SPE, each player makes the same offer in each offering period, and each player is exactly indifferent between accepting and rejecting each offer. If ~~A stri~~ Pi strictly prefers accepting, then $P_{\neg i}$ has strictly profitable deviation to a less generous, still strictly acceptable offer. If Pi strictly prefers to reject, then $P_{\neg i}$'s offer is never ~~accep~~ accepted, and this is because ~~Pi w~~ Pi ~~s~~ makes a counteroffer that is less generous to $P_{\neg i}$ than it expects $P_{\neg i}$ to accept, so $P_{\neg i}$ has strictly profitable deviation to a just acceptable offer. ~~tet~~

Let superscripts denote the offering player.

$\cancel{x_1^1}$ $x_2^1 = \delta x_2^2$

$x_1^2 = \delta x_1^1$

$x_1^1 = (1 - x_2^1)$, $x_1^2 = (1 - x_2^2)$ (from above argument)

$\Rightarrow$

$x_2^1 = \delta(1 - x_1^2) = \delta(1 - \delta x_1^1) = \delta(1 - \delta(1 - x_2^1)) \Rightarrow$

$x_2^1 = \delta - \delta^2(1 - x_2^1) = \delta - \delta^2 + \delta^2 x_2^1 \Rightarrow$

$(1 - \delta^2) x_2^1 = \delta - \delta^2 \Rightarrow$

$x_2^1 = \delta - \delta^2 / 1 - \delta^2 = \delta(1-\delta) / (1+\delta)(1-\delta) = \delta / 1 + \delta \Rightarrow$

$\cancel{x_2^2}$ $x_1^1 = 1 / 1 + \delta \Rightarrow$

$\cancel{x_2^2}$ $x_1^2 = \delta / 1 + \delta$, $x_2^2 = 1 / 1 + \delta$

~~P1~~ At SPE, P1 makes the first offer $x_1^1 = 1/1+\delta$, $x_2^1 = \delta / 1 + \delta$ and this is immediately accepted. P1 receives $x_1^* = 1/1+\delta$ and P2 receives $\cancel{\delta/1+\delta}$ $x_2^* = \delta / 1 + \delta$

d P2 exercises the ~~option~~ disagreement ~~opti~~ option iff P2's payoff from continuation of play is less than d_2, which is iff (at the stationary SPE), P2's SPE payoff $x_2^* = \delta/_{1+\delta}$ is less than d_2.

If $d_2 \cancel{\neq} < \cancel{\delta/_{1+\delta}}$ $x_2^* = \delta/_{1+\delta}$, P2 never exercises the disagreement option, and the SPE is unchanged. If $d_2 \cancel{\neq} \geq x_2^* = \delta/_{1+\delta}$, then P2 receives d_2 at SPE either because P1 offers this or because P2 immediately exercises the disagreement option.

So at SPE, if $d_2 < x_2^* = \delta/_{1+\delta}$, P1 receives $x_1^* = 1/_{1+\delta}$ otherwise P1 receives $1-d_2$.

The result of this ~~offer~~ infinitely repeated offer-counter offer game with "breakdown" fails to coincide with the Nash bargaining solution. This ~~fa~~ ~~consi~~ consists in the disagreement payoff having a ~~non~~ ~~discont~~ discontinuous effect on payoffs, ~~~~ ~~~~ only serving to guarantee a minimum payoff. ~~but~~ This is because "breakdown" in this game is endogenous, i.e. it is an option exercised by P2 rather than ~~simply~~ an exogenous fixed probability.

6a. In a two player symmetric game, strategy α^* is an ESS iff (α^*, α^*) is a NE and for all strategies $\alpha' \neq \alpha^*$, it is not the case that both (1) α' is a BR to α^*, and (2) $\pi(\alpha', \alpha') \geq \pi(\alpha^*, \alpha')$.

This captures the idea of stability against invasion by mutants. A population of α^* players is stable against α' mutants iff α' mutants do worse against the α^* population than α^* players do or α' players do worse than α^* players against α', and so do worse than α^* on average within a partially invaded population.

b

	C	R
C	$\underline{4}$, $\underline{4}$	3, 0
R	0, 3	$\underline{3}$, $\underline{3}$

Best responses underlined.

By inspection, there are two pure NE where players play pure mutual best responses. These NE are symmetric and strict (so no other strategy is a best response), so these correspond to ESS. These are (C, C) and (R, R).

~~Consider mixed NE σ^*. By definition, neither player has profitable deviation, so both are~~

There is no mixed ESS. At mixed NE σ^*, by definition of NE, both players have no profitable deviation and are indifferent. Then any strategy is a best response against the mixed strategy at NE, including C. (C, C) strictly maximises payoffs for each player, so the mixed strategy at NE fares worse against C than C does against itself. So C is a successful mutant invader against any candidate mixed ESS. The only ESS are C and R

c Under the replicator dynamic, the number of X-players in a population grows in each period by some amount directly proportionate to the initial number of X-players and the average payoff to X players. Then, the proportion of X-players in a population grows in each period by an amount that is in direct proportion to the initial proportion of X players and the difference between the average payoff to X players and the average payoff in the population.

Let p_C denote the proportion of C players. Let A denote the Row payoff matrix and p denote the vector of player proportions.

$$A = \begin{pmatrix} 4 & 0 \\ 3 & 3 \end{pmatrix} \quad p = \begin{pmatrix} p_C \\ p_R \end{pmatrix}$$

Then $[Ap]_X$ is the average payoff to X players and p^TAp is the population average payoff.

Replicator equation.

$$\dot{p}_C = p_C ([Ap]_C - p^TAp)$$
$$= p_C ([Ap]_C - (p_C[Ap]_C + (1-p_C)[Ap]_R))$$
$$= p_C (1-p_C)([Ap]_C - [Ap]_R)$$
$$= p_C (1-p_C)(4p_C - 3)$$

$\dot{p}_C = 0 \iff p_C = 0, p_C = 1$, or $p_C = 3/4$

$p_C \in (0, 3/4) \Rightarrow \dot{p}_C < 0$

$p_C \in (3/4, 1) \Rightarrow \dot{p}_C > 0$

Under the replicator dynamic, there are three absorbing states, $p_C = 0$, $p_C = 1$, $p_C = 3/4$. $p_C = 3/4$ is an unstable absorbing state. $p_C = 0$ has basin of attraction $[0, 3/4)$, $p_C = 1$ has basin of attraction $(3/4, 1]$. This means that if the initial state of the system is $p_C \in [0, 3/4)$, the system converges to $p_C = 0$ and if the initial state is $p_C \in (3/4, 1]$, it converges to $p_C = 1$. The state of the system converges to one that corresponds to an ESS. The absorbing states correspond to NE.

d Let $\sigma = pC + (1-p)R$ denote the strategy corresponding to a population with proportion p of C players and proportion $1-p$ of R players.

$\pi(C, \sigma) \gtreqless \pi(R, \sigma) \iff$

$4p \gtreqless 3 \iff$

$p \gtreqless 3/4$

C is a BR against the population iff $p \geq 3/4$

R is a BR against the population iff $p \leq 3/4$

Then, beginning from the $p=0$ absorbing state, ~~3/4 of the population~~ the minimum number of non-best-response "mistakes" required for the population to evolve to the $p=1$ state is $3/4 N$, where N is the size of the population.

Beginning from the p=1 absorbing state, ~~+~~ the ~~total~~ minimum number of such errors required to reach the p=0 state is $N = \frac{c}{a}$.

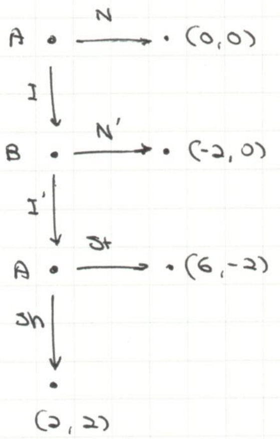
For ~~sufficiently~~ small ε, the probability of such movements is dominated by the ~~effect of the~~ $(\frac{\varepsilon}{2})^m$ term, where m is the minimum number of mistakes required. So the probability of evolution from the p=1 state to the p=0 state is much higher than the probability of evolution in the other direction.

In the long run, the population spends most of the time at or around one of the two states p=0, p=1 and spends ~~sign~~ much more time ~~aro~~ at or around the p=0 state than at or around the p=1 state.

The population ~~oscillates between = states that corres~~ remains mostly at or around an ESS and remains at or around the risk-dominant (R,R) ~~ESS~~ more than the other (L,L), ~~ES ESS~~ state.

e Evolutionary models are of limited use because there are few ~~games wh~~ ~~ec~~ economic interactions that involve few agents (two players) interacting repeatedly. For ~~ec +~~ economic interactions with many players, such theories as consumer theory ~~are~~ are generally more useful. It is also generally the case that economic interactions have unique eqm, so an evolutionary theory is not necessary.

7a



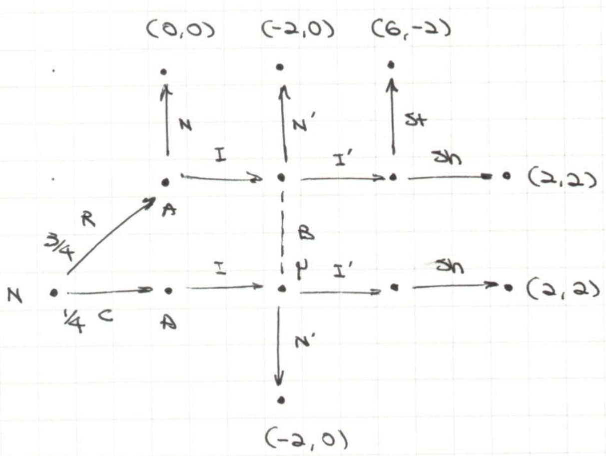
A's payoff given first.

Solve for SPE by backward induction. In the II' subgame, ~~Sh~~ St is strictly rational for A yielding payoff 6 to A rather than 2 from Sh. In the I subgame, given that the NE outcome of I' yields ~~-6,~~ -2 to B, N' is strictly rational for P2, yielding 0. Then, in the first stage, knowing that SPE play in the I subgame yields ~~-2~~ -2 to A, N is strictly rational for A which instead yields 0.

The unique SPE is the strategy profile where A plays N St and B plays N'. (where N St means Not invest and Steal) It is necessary to specify off eqm path actions to fully specify a player's strategy.

At SPE, no investment happens because A cannot be trusted to Sh and so B cannot be trusted to I'.

b



c From (a), rational A always plays St in the RII' subgame.

$\pi_B(I', \sigma_R, \sigma_C; \mu) = 2\mu + (-2)(1-\mu) = 4\mu - 2$

$\pi_B(N', \sigma_R, \sigma_C; \mu) = 0$

$\pi_B(I', \sigma_R, \sigma_C; \mu) \geq \pi_B(N', \sigma_R, \sigma_C; \mu) \iff \cancel{\mu}$

$4\mu - 2 \geq 0 \iff$

$\mu \geq 1/2$

B weakly prefers I' to N' iff $\mu \geq 1/2$.

d Suppose for reductio that at PBE, R ~~(rational~~ (rational, non technically) type A plays pure N. Then by BB, $\mu = 1$, then B strictly prefers I' to N', and by SR, B plays I' at PBE. Then N is not SR for A because ~~∃~~ ∃ deviation to I is profitable and yields payoff 6 rather than 0 (because B then plays I' and A then plays St). By reductio, ~~At~~ at PBE, R type A does not play pure N.

Suppose for reductio that at PBE R type A plays pure I. Then by BB, $\mu = 1/4$, i.e. ~~playing I is cost~~ B's observing I is completely uninformative for B. From (c), ~~B then~~ by SR, B plays N'. Then I is not SR for A because deviation to N is strictly profitable, yielding 0 rather than -2. By reductio, R type A does not play pure I at PBE.

So R type A mixes at PBE.

e At PBE, B mixes. Suppose for reductio that B plays pure N' at PBE. Then N is SR for R type A, so R type A plays N at PBE. From (d), R type A mixes at PBE, so by reductio, B does not play pure N' at PBE. Suppose for reductio that B plays pure I' at PBE, then pure I is strictly SR for ~~R~~ R type A so R type A plays pure I. From (d), by reductio, ~~R type~~ B does not play pure I' at PBE.

From (c), mixing is ~~only~~ SR for B ~~iff~~ ~~only~~ iff $\mu = 1/2$. Then, by BB, R type A plays I wp 1/3. Mixing is SR for R ~~type~~ type A iff I and N yield equal expected payoff. This is iff $6p + (-2)(1-p) = 0$, where p is the prob that B plays I'. $6p - 2(1-p) = 0 \iff 8p - 2 = 0 \iff p = 1/4$. Then, the unique PBE is s.t. ~~plays~~ $\sigma_R = 1/3\, I + 2/3\, N$, St, $\sigma_C = I\ Sh$, $\sigma_B = 1/4\, I' + 3/4\, N'$, $\mu = 1/2$.

At PBE, R type A does not imitate C types perfectly because perfect imitation renders I a non-credible signal. B does not reward I with certainty because if B so rewards, then ~~A has~~ R type A has strict incentive to I, which breaks the eqm. So at eqm both mix to keep each other indifferent.

~~A C +~~ C type A is taken advantage of or not trusted most of the time because of the possibility of being an imitating R type. R types occasionally imitate and are occasionally rewarded

3a

L	a	b
A	4, 4, 4	10, 0, 0
B	0, 0, 10	1, 2, 3

R	a	b
A	−1, 1, −1	0, 1, −1
B	1, 1, 0	3, 2, 1

Best responses underlined.

By inspection, there are two pure NE, ~~(A,b,L)~~ ~~and~~ (B,b,L) and (B,b,R) where players play pure mutual ~~B~~ BRs.

B is strictly dominant for P1, b is SD for P2.

b ~~P1~~ B is SD for P1 so P1 always plays B even when minmax punished. So P2 ~~and~~ and P3 minmax P1 by playing a and R respectively. This holds P1 to payoff 0 while P1 guarantees payoff 0 by playing B. Then P1's minmax payoff is $\underline{v}_1 = 0$.

b is SD for P2 so P2 always plays b incl when minmax punished. P1 and P3 minmax P2 by ~~play~~ playing A and R respectively which holds P2 to minmax payoff $\underline{v}_2 = 0$. P2 guarantees $\underline{v}_2 = 0$ by playing b.

P1 and P2 minmax P3 by correlated mixing with ~~even~~ equal probabilities between B,a and Ab. This holds ~~P2~~ P3 to minmax payoff $\underline{v}_3 = \frac{1}{2}$. P3 guarantees this payoff by mixing between L and R with equal probabilities.

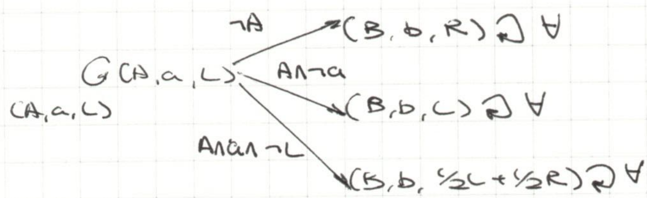
c Consider the following ~~fo~~ strategy profile ~~st~~. Play (A,a,L) in every period except the last 6 iff there has been no prior deviation. If there has been a prior deviation and P1 ~~plan~~ deviated first, play (B,b,R) forever. If P2 deviated first, play (B,b,L) forever. If P3 deviated first play (B,b,R) forever. In the last six periods, if there has been no prior deviation, play $(B, b, \frac{1}{2}L + \frac{1}{2}R)$ in every period.

In any punishment phase and in the last six periods, prescribed play is the stage game NE in every period, so no player has a profitable one shot deviation.

In the cooperation phase, ~~and~~ P3 has no profitable one shot deviation because eqm play yields 4 in each period till the last six, then 2 for the last six whereas ~~of~~ one shot deviation yields 1 then two in every subsequent period.

P1 has no profitable one shot deviation because eqm play ~~yields~~ in the ~~co~~ cooperation phase yields 4 in any period then 2 in ~~the~~ each of the last six whereas optimal one shot deviation yields 10 then 1 ~~~~ for the remaining periods (and there are always at least 6 remaining periods in the cooperation phase). Likewise for P2.

d The required SPE is represented by the following automaton.



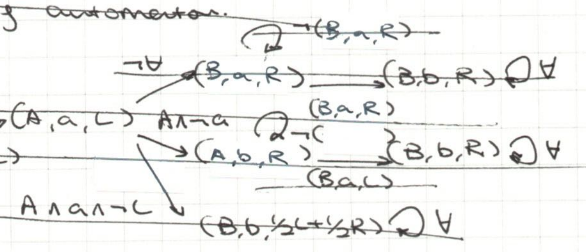
No player has profitable deviation in each punishment phase because each punishment phase consists in playing the stage game NE indefinitely.

In the cooperation phase, optimal OS dev by P1 is to B, which yields 10 then 1 indef., eqm play yields 4 indef. P1 ∄ π-able OS dev iff ~~$10 + 1\delta/1-\delta \geq 4 + 4\delta/1-\delta$~~ ⟺ $4 + 4\delta/(1-\delta) \geq 10 + 1\delta/(1-\delta)$ ⟺ $3\delta/(1-\delta) \geq 6$ ⟺ $3\delta \geq 6 - 6\delta$ ⟺ $9\delta \geq 6$ ⟺ $\delta \geq 2/3$. The analysis is identical for P2.

OS devn by P3 yields 1 then 2 indef, eqm play yields 4 indef, it is trivial that P3 has no π-able devn.

So for $\delta \geq 2/3$, no player has a π-able devn and the above is a SPE.

e ~~The required SPE~~ is represented by the ~~following automaton.~~



~~In each indef punishment phase, no player has π able OS dev because prescribed play is stage game NE forms~~

~~In the cooperation phase, by the argument from (d), it is trivial that P3 has no π able OS dev.~~

~~In the cooperation phase, for P1, optimal OS dev yields 10 then 0 then 1 indef. equm play yields 4 indef. ∄ πable OS dev for P1 iff~~ $\cancel{4 + 4\delta + 4\delta^2/1-\delta \geq 10 + 0\delta + 1\delta^2/1-\delta \iff 4\delta + 3\delta^2/1-\delta \geq 6}$ ~~which holds for some sufficiently large~~ $\cancel{\delta < 2/3}$. ~~Similarly for P2.~~

~~In the transitory P1 punishment phase, equm play by P1 yields~~