Microeconomic Analysis Paper 220613

1a The vectors form a basis of $\mathbb{R}^3$ iff they are linearly independent. $(c,1,1)^T$ is linearly independent of the other two vectors iff

$\nexists \alpha, \beta : \begin{pmatrix} c \\ 1 \\ 1 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ c \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \\ c \end{pmatrix} \Leftrightarrow$

$\nexists \alpha, \beta : c = \alpha + \beta,\ 1 = \alpha c + \beta,\ 1 = \alpha + c\beta \Leftrightarrow$

$\nexists \alpha, \beta : c = \alpha + \beta,\ 1 = \alpha c + \beta,\ \alpha(c-1) - \beta(c-1) = 0 \Leftrightarrow$

$\nexists \alpha, \beta : c = \alpha + \beta,\ 1 = \alpha c + \beta,\ \alpha = \beta \Leftrightarrow$

$\nexists \alpha : c = 2\alpha,\ c = {}^{1-\alpha}/_{\alpha} \Leftrightarrow$

~~$\nexists \alpha : 2\alpha$ $\nexists \alpha : c = 2\alpha = {}^{1-\alpha}/_{2\alpha}$~~

$c \neq 2\alpha$ where $2\alpha = {}^{1-\alpha}/_{\alpha}$, $2\alpha^2 = 1-\alpha$, $2\alpha^2 + \alpha - 1 = 0$

$\alpha = {}^{-1 \pm \sqrt{1+8}}/_{4} = -1$ or $\frac{1}{2}$ $\Leftrightarrow$

$c \neq -2$ or 1

By symmetry, $(1,c,1)^T$ is linearly independent of the other given vectors iff $c \neq -2$ or 1 and likewise for $(1,1,c)^T$. Then, the three vectors fail to form a basis of $\mathbb{R}^3$ iff $c = -2$ or 1.

For $c = -2$, the ~~number~~ $(c,1,1)^T$ and $(1,c,1)^T$ are linearly independent (by inspection), likewise for any other pair of the three, then the matrix formed by the three vectors is rank 2, the vectors span two dimensions.

For $c = 1$, the three vectors are identical, no two are linearly independent, the matrix formed by the vectors is rank 1, the vectors span one dimension.

b $\lambda_1 = 1,\ \vec{v}_1 = (1,2)^T,\ \lambda_2 = -1,\ \vec{v}_2 = (2,1)^T$

By definition of an eigenvalue and eigenvector, $A\vec{v}_1 = \lambda_1 \vec{v}_1,\ A\vec{v}_2 = \lambda_2 \vec{v}_2 \Rightarrow$

$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} v_1^1 \\ v_1^2 \end{pmatrix} = \lambda_1 \begin{pmatrix} v_1^1 \\ v_1^2 \end{pmatrix},\ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} v_2^2 \\ v_2^3 \end{pmatrix} = \lambda_2 \begin{pmatrix} v_2^2 \\ v_2^3 \end{pmatrix}$

$\Rightarrow$

$\begin{pmatrix} a_{11} + 2a_{12} \\ a_{21} + 2a_{22} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},\ \begin{pmatrix} 2a_{11} + a_{12} \\ 2a_{21} + a_{22} \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$

$\Rightarrow$ ~~By elimination~~ solve simultaneously by elimination

$(a_{11} + 2a_{12}) - 2(2a_{11} + a_{12}) = 1 - 2(-2)$,

$(a_{21} + 2a_{22}) - 2(2a_{21} + a_{22}) = 2 - 2(-1)$

$\Rightarrow$

$-3a_{11} = 5$

$-3a_{21} = 4$

$\Rightarrow$

$a_{11} = -5/3,\ a_{21} = -4/3$

$\Rightarrow$

$a_{12} = 4/3,\ a_{22} = 5/3$

$\Rightarrow$

$A = \begin{pmatrix} -5/3 & 4/3 \\ -4/3 & 5/3 \end{pmatrix}$

For $\vec{y} = (1,2)^T$, $\dot{\vec{y}} = (1,2)^T$, then any $\vec{y} = \alpha(1,2)^T$ for $\alpha \in \mathbb{R}$ "doubles" in each unit interval of time.

For $\vec{y} = (2,1)^T$, $\dot{\vec{y}} = (-2,-1)^T$, then any $\vec{y} = \alpha(2,1)^T$ for $\alpha \in \mathbb{R}$ converges to $(0,0)^T$. ~~in each unit interval of~~

ci $f(x,y) = \begin{cases} 3xy/x^2+2y^2 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

$f_x(0,0) = \lim_{h \to 0} f(h,0) - f(0,0)/h$
$= \lim_{h \to 0} 0 - 0/h$
$= 0$

$f_y(0,0) = \lim_{h \to 0} f(0,h) - f(0,0)/h$
$= \lim_{h \to 0} 0 - 0/h$
$= 0$

$Df(0,0) = (0,0)$

$\nabla f(0,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\forall \vec{v} \in \mathbb{R}^2 : \nabla f \cdot \vec{v} = 0$

The directional derivative of f in any direction $\vec{v} \neq \vec{0} \in \mathbb{R}^2$ is 0.

ii f is differentiable at $(0,0)$ if all partial derivatives of f exist at $(0,0)$, and are continuous at $(0,0)$.

From the above, the partial derivatives of f exist at $(0,0)$.

~~Consider the sequence~~

~~For arbitrar~~

Consider the sequence $\{(1/n, 1/n)\}_{n=1}^{\infty}$. It is trivial that this sequence converges to $(0,0)$. $\{f(1/n,1/n)\}_{n=1}^{\infty} = \{3n^{-2}/n^{-2}+2n^{-2}\}_{n=1}^{\infty} = \{1\}_{n=1}^{\infty}$. The sequence $\{f(1/n, 1/n)\}_{n=1}^{\infty}$ does not ~~converge to~~ converges to $1 \neq f(0,0)$. f is not continuous at $(0,0)$ hence not differentiable at $(0,0)$.

di $g(x,y,z) = z(x+y) - y\ln z$

$g(t,-t,1) = z(t+-t) - -t\ln 1 = 0$

$g(x,y,z)$ defines z as an implicit function of (x,y) around $(t,-t)$ iff $g(t,-t,z) = 0$, g is C^1 in an open ball around $(t,-t,z)$, and $g_z(t,-t,z) \neq 0$.

$g(t,-t,z) = t\ln z = 0$ iff $t = 0$ or $z = 1$

$g_z(t,-t,z) = t/z \neq 0$ iff $t \neq 0$

By inspection, $g(x,y,z)$ is C^1 ~~in an open~~ for all

$(x, y, z) = (t, -t, 1)$

Then for all $t \neq 0$, the implicit function theorem is applicable to define z as an implicit function of x, y in a neighbourhood of $(t, -t, 1)$

~~$g_x(t, -t$~~

$g_x(x,y,z) = z$, $g_y(x,y,z) = z - \ln z$, $g_z(x,y,z) = x + y \cdot y/z$

$g_x(t,-t,1) = 1$, $g_y(t,-t,1) = 1$, $g_z(t,-t,1) = t$

$z_x(t,-t) = g_x(t,-t,1)/g_z(t,-t,1) = 1/t$

$z_y(t,-t) = g_y(t,-t,1)/g_z(t,-t,1) = 1/t$

2ai By Lagrangian sufficiency, if there exists $\vec{x}^*$ and $\vec{\lambda}^*$ such that $\mathcal{L}(\vec{x}^*, \vec{\lambda}^*) \geq \mathcal{L}(\vec{x}, \vec{\lambda}^*)$ for all $\vec{x} \in \mathbb{R}^n$ ~~and~~, $\vec{\lambda}^*(\vec{g}(\vec{x}^*) - \vec{b}) = 0$, and $\vec{x}^*$ is feasible, $\vec{x}^*$ is optimal for P: $\max_{\vec{x}} f(\vec{x})$ s.t. $\vec{g}(\vec{x}) \leq \vec{b}$.

If there exists ~~(x*, y*) and λ* that~~ λ^* and (x^*, y^*) such that (x^*, y^*) maximises $\mathcal{L}(x, y, \lambda^*) = f(x,y) - \lambda^*(x+2y-9)$ where $\lambda^*(x+2y-9) = 0$ and (x^*, y^*) is feasible, then there exists $\lambda^*, \lambda^x, \lambda^y$ and (x^*, y^*) such that (x^*, y^*) maximises $\mathcal{L}(x, y, \lambda^*, \lambda^x, \lambda^y) = f(x,y) - \lambda^*(x+2y-9) + \lambda^x(x-0) + \lambda^y(y-0)$, where $\lambda^*(x+2y-9) - \lambda^x(x-0) - \lambda^y(y-0) = 0$ and (x^*, y^*) is feasible, namely x^*, y^*, λ^* from above and $\lambda^x = \lambda^y = 0$, then by Lagrangian sufficiency, x^*, y^* is optimal for the given P.

ii By inspection, the constraint set is compact (closed and bounded, ~~in that it is define~~ because the constraints have weak inequalities and some upper bound and lower bound exists for each variable). Then, by the Weierstrass extreme value theorem, given that f is continuous, a global optimum of P exists.

By inspection of the constraints, at most two constraints bind at any point, and the Jacobian of the binding constraints, given that there are at most two, is full rank, then the constraint qualification is satisfied at all ~~points~~ feasible points, and the KT FOC are necessary for an optimum.

So if ~~only~~ there is a unique solution to the KT FOC, given that the KT FOC are necessary and an optimum exists, that solution is in fact the global optimum.

iii False. Consider $f(x,y) = 100x + y$. Then, the ~~optim~~ optimum is $(9, 0)$. This f is quasi-concave and increasing in each of x and y.

bi $u_x(x,y) = 3(1/2)x^{-1/2}$

$\lim_{x \to 0} u_x(x,y) = \infty$

Marginal utility of x tends to infinity as x tends to 0, then at any optimum, $x \neq 0$.

ii $\mathcal{L} = 3\sqrt{x} + y - \lambda(x + 2y - m)$

$FOC_x: 3(1/2)x^{-1/2} - \lambda = 0$

$FOC_y: 1 - 2\lambda = 0$

$CS: \lambda \geq 0, \; x + 2y \geq m, \; \lambda(x + 2y - m) = 0$

$FOC_x, FOC_y \Rightarrow 3/2\, x^{-1/2} = 1/2 \Rightarrow x^{-1/2} = 1/3 \Rightarrow x = 9$
$\Rightarrow \lambda = 1/2 \Rightarrow x + 2y - m = 0 \Rightarrow y = m - x/2 = m - 9/2$

Neglecting the positivity constraint on y, the unique solution to the KT FOCs is $(x, y) = (9, m - 9/2)$

$Du = (3/2\, x^{-1/2} \quad 1)$

$D^2u = \begin{pmatrix} -3/4\, x^{-3/2} & 0 \\ 0 & 0 \end{pmatrix}$

$\det D^2u = 0$, $\operatorname{tr} D^2u = -3/4\, x^{-3/2} < 0$, one eigenvalue of D^2u is negative, the other is zero, D^2u is negative semi-definite, u is weakly concave.

The budget constraint is linear hence weakly convex. Then the optimisation problem is concave and the KT FOCs are sufficient for an optimum.

The constraint set has non-empty interior, then the KT FOCs are also necessary for an optimum.

So the unique solution to the KT FOCs found above is the unique global optimum.

iii The time constraint binds iff the optimum under only the budget constraint is not feasible when the time constraint is added. This is iff $3(9) + 4(m - 9/2) > 48 \Leftrightarrow 27 + 2m - 18 > 48 \Leftrightarrow 2m > 39 \Leftrightarrow m > 39/2$

iv $\max_{x,y} u(x,y) = 3\sqrt{x} + y$ s.t.

BC: $x + 2y \leq m$

TC: $3x + 4y \leq 48$

$\mathcal{L} = 3\sqrt{x} + y - \lambda_B(x + 2y - m) - \lambda_T(3x + 4y - 48)$

$FOC_x: 3/2\, x^{-1/2} - \lambda_B - 3\lambda_T = 0$

$FOC_y: 1 - 2\lambda_B - 4\lambda_T = 0$

$CS_B: \lambda_B \geq 0, \; x + 2y \leq m, \; \lambda_B(x + 2y - m) = 0$

$CS_T: \lambda_T \geq 0, \; 3x + 4y \leq 48, \; \lambda_T(3x + 4y - 48) = 0$

Suppose only BC binds, then by CS_T, $\lambda_T = 0$, then FOC_x, FOC_y reduce to those in (ii), and the unique solution to FOC_x, FOC_y, CS_B is $(x, y) = (9, m - 9/2)$. From (iii), this satisfies CS_T iff $m \leq 39/2$. For $m \leq 39/2$, the ~~optimum~~ unique solution is $(9, m - 9/2)$.

Suppose only TC binds, then by CS_T, $3x + 4y = 48$, $x = 16 - 4/3\, y$, and by CS_B, $\lambda_B = 0$. By substitution into FOC_x, FOC_y, $3/2\, x^{-1/2} - 3\lambda_T = 0$, $1 - 4\lambda_T = 0 \Rightarrow \lambda_T = 1/4$, $3/2\, x^{-1/2} - 3(1/4) = 0 \Rightarrow x^{-1/2} = 1/2 \Rightarrow x = 4$, $y = 9$. By substitution into CS_B, $4 + 2(9) \leq m \Rightarrow m \geq 22$. For $m \geq 22$, the unique solution is $(4, 9)$.

Suppose both BC and TC bind, then $x + 2y = m$, $3x + 4y = 48 \Rightarrow y = 12 - \frac{3}{4}x$, $x + 24 - \frac{3}{2}x = m \Rightarrow -\frac{1}{2}x = m - 24 \Rightarrow x = \cancel{2m - 48}\ 48 - 2m \Rightarrow y = 12 - \frac{3}{4}(48 - 2m) = \frac{3}{2}m - 24$. By substitution into FOC_x, FOC_y, $\frac{3}{2}(48-2m)^{-1/2} - \lambda_B - 3\lambda_T = 0$, $1 - 2\lambda_B - 4\lambda_T = 0 \Rightarrow \frac{3}{2}(48-2m)^{-1/2} - \lambda_B - 3\lambda_T = 0$, $\lambda_B = \frac{1-4\lambda_T}{2} \Rightarrow \frac{3}{2}(48-2m)^{-1/2} - \frac{1-4\lambda_T}{2} - 3\lambda_T = 0$
$\Rightarrow \frac{3}{2}(48-2m)^{-1/2} - \frac{1}{2} = \cancel{3\lambda_T}\ \lambda_T \Rightarrow$
$\lambda_B = \frac{1}{2} - \left(3(48-2m)^{-1/2} - 1\right) = \frac{3}{2} - 3(48-2m)^{-1/2}$.

By substitution into CS_B, CS_T,
$\frac{3}{2}(48-2m)^{-1/2} - \frac{1}{2} \geq 0$, $\frac{3}{2} - 3(48-2m)^{-1/2} \geq 0 \Leftrightarrow$
$3(48-2m)^{-1/2} \geq 1$, $3(48-2m)^{-1/2} \leq \frac{3}{2} \Leftrightarrow$
$\frac{1}{3} \leq (48-2m)^{-1/2} \leq \frac{1}{2} \Leftrightarrow 4 \leq 48-2m \leq 9 \Leftrightarrow$
$\frac{39}{2} \leq m \leq 22$. For $\frac{39}{2} \leq m \leq 22$, the unique solution is $(48-2m, \frac{3}{2}m - 24)$.

Suppose neither constraint binds, then by CS_B, CS_T, $\lambda_B = \lambda_T = 0$, then by FOC_y, $1 = 0$. By reductio, at least one constraint binds at any solution to the KT FOCs.

By the argument in (ii) which applies because TC is linear hence weakly ~~concave~~ convex, the KT FOC are necessary and sufficient. Then the optimum is as follows

for $m \leq \frac{39}{2}$, $(9, \frac{m-9}{2})$
for $\frac{39}{2} < m \leq 22$, $(48-2m, \frac{3}{2}m - 24)$
for $m > 22$, $(4, 9)$

3ai ~~choices~~ Choice function c on menus satisfies α iff for all A_1, A_2, a_1 such that $A_2 \subseteq A_1$, $c(A_1) = a_1$, and $a_1 \in A_2$, it is also the case that $c(A_2) = a_1$. Strict preference relation $\succ$ is rational only if the induced choice function $c(A, \succ)$ which selects the maximal element of A according to $\succ$ satisfies α.

α cannot be applied to A_1 and A_2 because neither $A_1 \subseteq A_2$ nor $A_2 \subseteq A_1$. $(20,0) \in A_1$ ~~as~~ but ~~but~~ $(20,0) \notin A_2$, and $(0,30) \in A_2$ but ~~not~~ $(0,30) \notin A_1$.

ii ~~Let $\succ$ denote D's~~ Let c denote D's choice function.

$c(A_1) = (10,10)$ reveals $(10,10)$ as at least weakly preferred to $(15,0)$ given that $(15,0) \in A_1$.

~~By strict monotonicity, (~~

$c(A_2) = (15,0)$ reveals $(15,0)$ as at least weakly preferred to $(10,10)$ given that $(10,10) \in A_2$.

By strict monotonicity, $(16,0)$ is strictly preferred to $(15,0)$.

$c(A_1) = (10,10)$ reveals $(10,10)$ as at least weakly preferred to $(16,0)$ given that $(16,0) \in A_1$.

Then, if D has rational preferences revealed by c, by transitivity, $(15,0) \succsim (10,10) \succsim (16,0) \succ (15,0) \Rightarrow (15,0) \succ (15,0)$. By reductio, D does not have rational preferences.

iii The weak axiom of revealed preference, which is a property of rational choice (choice on the basis of a rational preference relation) is violated. This property is satisfied by choice function c iff for all A_1, A_2, ~~and~~ x_1, x_2, if $x_1, x_2 \in A_1, A_2$, then it is not the case that $c(A_1) = x_1$ and $c(A_2) = x_2$.

There is no strictly monotone utility function that represents D's preference.

$c(A_2) = (15,0) \Rightarrow u(15,0) \geq u(10,10)$
$c(A_1) = (10,10) \Rightarrow u(10,10) \geq u(16,0)$
By strict monotonicity, $u(16,0) > u(15,0)$
Then ~~u(15,0~~ by transitivity of $>, \geq$ on real numbers,
$u(15,0) > u(15,0)$
By irreflexivity of $>$ on real numbers,
$u(15,0) \not> u(15,0)$,

By reductio, there is no such u.

b Yes

Consider arbitrary $(x_1, x_2), (y_1, y_2), (z_1, z_2)$. Suppose $(x_1,x_2) D (y_1,y_2)$ and $(y_1,y_2) D (z_1,z_2)$, then given the ~~rele~~ necessary and sufficient condition for D, $3x_1 + x_2 \geq 3y_1 + y_2$, $x_1x_2 \geq y_1y_2$, $3y_1 + y_2 \geq 3z_1 + z_2$, $y_1y_2 \geq z_1z_2$. Then by transitivity of $\geq$, $3x_1 + x_2 \geq 3z_1 + z_2$ and $x_1x_2 \geq z_1z_2$. Then $(x_1,x_2) D (z_1,z_2)$. By conditional proof, generalisation, ~~D is if $(x_1,x_2) D (y_1,y_2)$ and~~ D is transitive.

ii ~~#~~ D is reflexive.

$3x_1 + x_2 \geq 3x_1 + x_2$, $x_1x_2 \geq x_1x_2$, then $(x_1,x_2) D (x_1,x_2)$ for arbitrary (x_1, x_2), so D is reflexive.

Consider $(x_1,x_2) = (1,1)$ and $(y_1,y_2) = (5, 1/10)$
$3x_1 + x_2 \cancel{\geq} = 4 < 3y_1 + y_2 = 15\frac{1}{10}$,
$x_1x_2 = 1 > y_1y_2 = 1/2$, then neither $(x_1,x_2) D (y_1,y_2)$ nor $(y_1,y_2) D (x_1,x_2)$, D is not connected, hence not complete (strongly connected).

iii ~~D is antisymmetric~~

~~Suppose that there exists $(x_1,x_2) \neq (y_1,y_2)$ such that $(x_1,x_2) D (y_1,y_2)$ but and $(y_1,y_2) D (x_1,x_2)$ then $3x_1 + x_2 = 3y_1 + y_2$ and $x_1x_2 = y_1y_2$. Let $k = x_1x_2 = y_1y_2$, then $3x_1 + k/x_1 = 3y_1 + k/y_1$ $\Rightarrow 3x_1^2 + k = 3y_1^2$ then Let $k = 3x_1 + x_2 = 3y_1 + y_2$, then $x_2 = k - 3x_1$, $y_2 = k - 3y_1$, $x_1(k - 3x_1) = y_1(k - 3y_1)$ $= -3x_1^2 + kx_1 = -3y_1^2 + ky_1$. Let $c = 3x_1^2 - kx_1 = 3y_1^2 - ky_1$~~

$c = 6$ $k = -3$

D is not antisymmetric.

Consider $(x_1 = 1, x_2 = -6)$, $(y_1 = -2, y_2 = 3)$, then $3x_1 + x_2 = -3 = 3y_1 + y_2$, $x_1x_2 = -6 = y_1y_2$, then $(x_1,x_2) D (y_1,y_2)$, $(y_1,y_2) D (x_1,x_2)$, $(x_1,x_2) \neq (y_1,y_2)$.

ci For every $\vec{x}$ ~~Each~~ bundle not at the frontier of the budget constraint ~~is~~, there exists an alternative $\vec{x}'$ which is obtained by increasing each quantity by sufficiently small amount ε such that it remains feasible, ~~then~~ namely $\vec{x}' = (x_1 + \varepsilon, x_2 + \varepsilon)$. Then $3x_1' + x_2' > 3x_1 + x_2$, $x_1'x_2' > x_1x_2$, $\vec{x}' D \vec{x}$ and not $\vec{x} D \vec{x}'$. $\vec{x}$ is strictly worse than $\vec{x}'$ and $\vec{x}$ is eliminated.

ii ~~The frontier of A_1 is $A_1^f = \{(x_1, x_2) : x_1 + x_2 = 10\}$.~~
~~The bundles that cannot be eliminated are those $\vec{x}$ such that $\nexists \vec{x}' : \vec{x}' D \vec{x}$ and not $\vec{x} D \vec{x}'$,~~
~~which are those $\vec{x}$ such that $\forall \vec{x}' \in A_1^f$:~~
~~(not $\vec{x}' D \vec{x}$) or $\vec{x} D \vec{x}'$ which are those~~

~~such that $3x_1'+x_2' < 3x_1+x_2$ or $x_1'x_2' < x_1x_2$~~ or ~~$(3x_1x_2 \geq 3x_1'+x_2'$ and $x_1x_2 \geq x_1'x_2')$ which are those such that $\forall x'$:~~

In A_1^0, $(10,10)$ maximises x_1x_2, and so is ~~weakly preferred to all other but~~ is not strictly worse than any other bundle, $(20,0)$ maximises $3x_1+x_2$ and is not worse than any other bundle. The bundles $(x_1<10, x_2>10)$ are strictly worse than $(10,10)$ because they yield lower x_1x_2 and $3x_1+x_2$, and are eliminated. Along the frontier from $(10,10)$ to $(20,0)$, $3x_1+x_2$ strictly increases, x_1x_2 strictly decreases. None of these bundles are strictly worse than any other. ~~get uneliminated~~ The uneliminated bundles are those in $\{(x_1,x_2)\ x_1 \in [10,20],\ x_1+x_2=20\}$.

By an exactly analogous argument the set of uneliminated bundles from A_2 is $\{(x_1,x_2):$ $x_1 \in [10,15],\ 2x_1+x_2=30\}$.

D freezes in each frontier.

iii ~~The required~~ menu ~~is such that some bundle maximises both $3x_1+x_2$ and x_1x_2. The bundle menu defined by $p_1=3, p_2=1, m=20$ is such that the bundle $(5,5)$~~

Any bundle on the frontier of ~~the~~ any menu with $p_1=3$, $p_2=1$ weakly maximises $3x_1+x_2$. Only one such bundle for a given menu maximises x_1x_2 and ~~does so~~ strictly maximises x_1x_2. This bundle is D-better than every other and D-worse than no other. Every other bundle is D-worse than this and not D-better, and so is eliminated. D does not freeze.

The ~~m~~ menu described by $p_1=3$, $p_2=1$, $m=20$ satisfies the above. Along the frontier, $x_1x_2 = x_1(20-3x_1) = 20x_1-3x_1^2$, ~~this~~ and is maximised when FOC: $20-6x_1$ is satisfied at $x_1=10/3$, $x_2=10$.

iv ~~a~~ Let A_1^0 and A_2^0 denote the ~~~~ reduced menus following elimination of D-strictly worse bundles from A_1 and A_2.

$A_1^0 = \{(x_1,x_2): x_1 \in [10,20],\ x_1+x_2=20\}$
$A_2^0 = \{(x_1,x_2): x_1 \in [10,15]\ 2x_1+x_2=30\}$

~~$(15,5) \in$~~ Suppose D sought advice from the same rational friend ~~~~ with ~~~~ strictly monotone preferences.

$c'(A_1^0) = (10,10) \Rightarrow (10,10) \succsim' (15,5)$
$c'(A_2^0) = (15,0) \Rightarrow (15,0) \succsim' (10,10)$

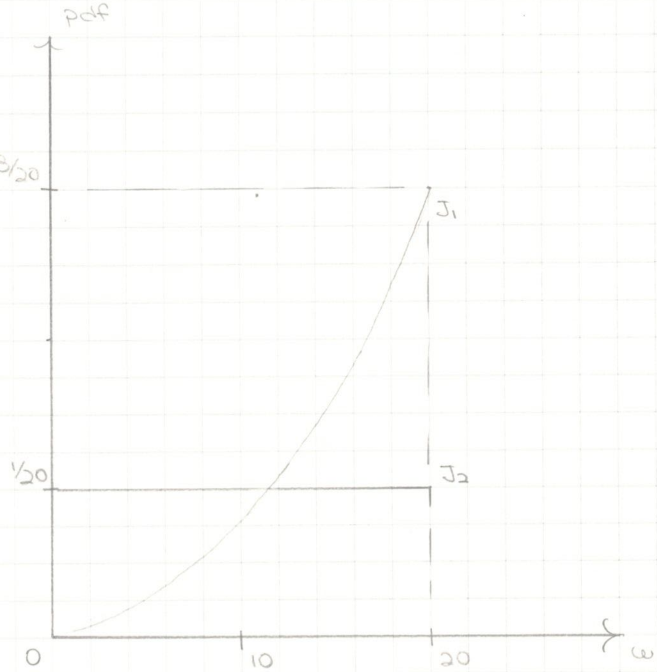
Strict monotonicity $\Rightarrow (15,5) \succ' (15,0)$
$\Rightarrow (10,10) \succ' (10,10)$.

By reductio, D did not seek advice from the same rational friend with strictly monotone preferences.

4a By definition of a probability density function,

$$\int_{-10}^{10} f(\varepsilon)\,d\varepsilon = \int_{-10}^{10} c(10+\varepsilon)^2\,d\varepsilon$$
$$= \left[c\,\tfrac{1}{3}(10+\varepsilon)^3\right]_{-10}^{10}$$
$$= c\left[\tfrac{1}{3}(10+10)^3 - \tfrac{1}{3}(10-10)^3\right]$$
$$= c\left[\tfrac{1}{3}(8000)\right] = 1$$

$\Rightarrow c = 3/8000$



b $P(\varepsilon < 20\alpha^{1/3} - 10) = \int_{-10}^{20\alpha^{1/3}-10} c(10+\varepsilon)^2\,d\varepsilon$
$$= c\left[\tfrac{1}{3}(10+\varepsilon)^3\right]_{-10}^{20\alpha^{1/3}-10}$$
$$= c\left[\tfrac{1}{3}(20\alpha^{1/3})^3 - \tfrac{1}{3}(0)^3\right]$$
$$= \cancel{c[\tfrac{1}{3}(8\alpha)]}\; c\left[\tfrac{1}{3}(8000\alpha)\right]$$
$$= \alpha$$

$P(\varepsilon > 5) = 1 - P(\varepsilon < 5)$
$= 1 - P(\varepsilon < 20\alpha^{1/3} - 10)$
where $20\alpha^{1/3} - 10 = 5 \Rightarrow \alpha^{1/3} = 15/20 = 3/4 \rightarrow \alpha = (3/4)^3$
$= 1 - \alpha$
$= 1 - (3/4)^3$
$= 37/64$

bi $F(\varepsilon) = \int_{-10}^{\varepsilon} f(e)\,de$
$$= c\int_{-10}^{\varepsilon} (10+e)^2\,de$$
$$= c\left[\tfrac{1}{3}(10+e)^3\right]_{-10}^{\varepsilon}$$
$$= c\left[\tfrac{1}{3}(10+\varepsilon)^3\right]$$

$G(\mu) = \int_{-10}^{\mu} g(m)\,dm$
$$= \int_{-10}^{\mu} \tfrac{1}{20}\,dm$$
$$= \left[\tfrac{1}{20}m\right]_{-10}^{\mu}$$
$$= \mu + 10/20$$

$\cancel{F(\varepsilon) \geq G(\mu)}\; F(\varepsilon) \leq G(\varepsilon) \Leftrightarrow c\left[\tfrac{1}{3}(10+\varepsilon)^3\right] \leq \varepsilon + 10/20$
$\Leftrightarrow \tfrac{1}{8000}(10+\varepsilon)^3 \leq \varepsilon + 10/20$
$\Leftrightarrow \cancel{\tfrac{1}{400}(10+\varepsilon)^3 \leq \varepsilon + 10}\; \tfrac{1}{400}(10+\varepsilon)^2 \leq 1$
$\Leftrightarrow \varepsilon = 10$

For all $\varepsilon \in \cancel{[0,10]}\, [-10,10]$, $F(\varepsilon) \leq G(\varepsilon)$, then the distribution F of ε FOSDs the distribution G of $\cancel{\varepsilon}\ \mu$. Then the distribution of w under J_1 FOSDs that of w under J_2.

J_1 FOSDs J_2, then ~~to~~ any expected utility maximiser strictly prefers J_1 to J_2, then $CE_1 > CE_2$ given that A is an expected utility maximiser with monotonic Bernoulli utility.

In symbols, $J_1 \succ_{FOSD} J_2 \Rightarrow J_1 \succ_A J_2 \Rightarrow [1; CE_1] \sim_A J_1 \succ_A J_2 \sim_A [1; CE_2] \Rightarrow [1; CE_1] \succ_A [1; CE_2] \Rightarrow u_A(CE_1) > u_A(CE_2) \Rightarrow CE_1 > CE_2$.

ii $EV_1 = \int_{-10}^{10} 10 + \varepsilon f(\varepsilon)\,d\varepsilon$
$$= 10 + \int_{-10}^{10} \varepsilon c(10+\varepsilon)^2\,d\varepsilon$$
$$= 10 + c\int_{-10}^{10} 100\varepsilon + 20\varepsilon^2 + \varepsilon^3\,d\varepsilon$$
$$= 10 + c\left[50\varepsilon^2 + \tfrac{20}{3}\varepsilon^3 + \tfrac{1}{4}\varepsilon^4\right]_{-10}^{10}$$
$$= 10 + c\left[\tfrac{20}{3}\varepsilon^3\right]_{-10}^{10}$$
$$= 10 + c\left[2(\tfrac{20}{3})(10)^3\right]$$
$$= 10 + \tfrac{3}{8000}\cdot\tfrac{40000}{3}$$
$$= 15$$

$CE_1 < EV_1$ iff A is risk averse because, by definition of risk aversion, risk averse A has concave Bernoulli utility u, and by concavity of u,
~~$u(CE_1) = \int_{-10}^{10} f(\varepsilon)\, 10+\varepsilon\, u(10+\varepsilon)\,d\varepsilon$~~
$u(CE_1) = U(J_1) < EV_1$.

$CE_1 < 10$ only if A is sufficiently risk averse such that ~~RP~~ risk premium $RP_1 > 5$, and prefers to receive final wealth of 10 with certainty than J_1.

iii $U(J_1) = \int_{-10}^{10} f(\varepsilon)u(10+\varepsilon)\,d\varepsilon$
$$= \int_{-10}^{10} c(10+\varepsilon)^2(10+\varepsilon)^{1-r}\,d\varepsilon$$
$$= c\int_{-10}^{10}(10+\varepsilon)^{3-r}\,d\varepsilon$$
$$= c\left[\tfrac{1}{4-r}(10+\varepsilon)^{4-r}\right]_{-10}^{10}$$
$$= c\left[\tfrac{1}{4-r}(20)^{4-r}\right]$$
$$= \tfrac{3}{8000}\cdot\tfrac{1}{4-r}\cdot\tfrac{160{,}000}{20^r}$$
$$= \frac{60}{20^r(4-r)}$$
$$\cancel{= \tfrac{3}{4-r}\,20^{-r}}$$

$u(CE_1) = U(J_1)$
$\Leftrightarrow CE_1^{1-r} = \dfrac{60}{20^r(4-r)}$

ci Let %(J) denote ~~the~~ $P(\underline{w} < J < \bar{w})$ for ~~lotteries in final wealth J~~ random variables J corresponding to some wealth distribution over [0, 20].

For all J, %(J) is defined, and rational. It is given that for all J, J′, $J \succsim_B J'$ iff $\%(J) \geq \%(J')$

consider arbitrary J, J', J''. Suppose that $J \succsim_B J'$ and $J'' ' \succsim_B J''$, then $\%(J) \geq \%(J')$ and $\%(J') \geq \%(J'')$, by the transitivity of $\geq$ on $\mathbb{R}$, $\%(J) \geq \%(J'')$, then $J \succsim_B J''$. For all J, J', J'', if $J \succsim_B J'$ and $J' \succsim_B J''$, then $J \succsim_B J''$, $\succsim_B$ is transitive.

consider arbitrary J, J'. ~~A~~ Given that $\%(J), \%(J')$ are well defined and rational, by the ~~connected~~ connectedness of $\geq$ on $\mathbb{R}$, $\%(J) \geq \%(J')$ or $\%(J') \geq \%(J)$, then $J \succsim_B J'$ or $J' \succsim_B J$. For all J, J' $J \succsim_B J'$ or $J' \succsim_B J$, $\succsim_B$ is complete.

ii whether $J_1 \succsim_B J_2$ or $J_2 \succsim_B J_1$ is contingent on the values of $\underline{w}$ and $\bar{w}$. ~~It is not~~

suppose $\underline{w} = 0$, $\bar{w} = 1$, then $P(\underline{w} < J_1 < \bar{w}) = F(-9) = \frac{3}{8000}[\frac{1}{3}(1)^3] = \frac{1}{8000}$ and $P(\underline{w} < J_2 < \bar{w}) = \frac{1}{20}$, then $J_2 \succsim_B J_1$ and not $J_1 \succsim_B J_2$.

suppose $\underline{w} = 19$, $\bar{w} = 20$, then $P(\underline{w} < J_1 < \bar{w}) = 1 - F(19) = \cancel{3/2}\ 1 - \frac{3}{8000}[\frac{1}{3}(19)^3] = 1 - \frac{19^3}{8000} = \cancel{1141/8}$ $\frac{1141}{8000}$, and $P(\underline{w} < J_2 < \bar{w}) = \frac{1}{20}$, then $J_1 \succsim_B J_2$ and not $J_2 \succsim_B J_1$.

B prefers J_2 to J_1 iff $G(\bar{w}) - G(\underline{w}) \geq F(\bar{w}) - F(\underline{w})$
$\Leftrightarrow \frac{\bar{w} - \underline{w}}{20} \geq \frac{3}{8000} \frac{1}{3} [\bar{w}^3 - \underline{w}^3] = \frac{1}{8000}[\bar{w}^3 - \underline{w}^3]$

5a $E(u(w,e)|e=1) = \frac{3}{5}[\sqrt{100}-1] + \frac{2}{5}[\sqrt{0}-1] = 5$

$E(u(w,e)|e=0) = \frac{2}{5}[\sqrt{100}-0] + \frac{3}{5}[\sqrt{0}-0] = 4$

$E(u(w,e)|e=1) > E(u(w,e)|e=0)$,

$e=1$ solves A's optimisation problem

$\max_{e \in \{0,1\}} E(u(w,e)|e)$

A optimally chooses $e=1$ which yields expected utility 5. This is A's outside option, hence A effectively has reservation utility $\bar{u}=5$.

b Informally, P maximises expected payoff by offering A full insurance, given some e^*. Then, the optimal effort to require is such as to maximise total expected wealth less the disutility of effort in wealth terms. Full insurance is optimal because A is risk averse and P is risk neutral so A will accept a lower expected payoff for lower risk and P benefits from higher expected payoff and has no risk premium.

Formally, given some e^*, P's optimisation problem is

$\max_{f,c} P(S|e^*)f - P(F|e^*)c$ s.t.

PC: $P(S|e^*)(\sqrt{100-f} - e^*) + P(F|e^*)(\sqrt{0+c} - e^*) \geq \bar{u} = 5$

At any optimum PC binds. ~~At~~ Any candidate optimum such that PC does not bind fails to deviation by increasing f by sufficiently small amount ε such that PC remains satisfied.
~~Any candi~~

Any candidate optimum such that $100-f \neq c$ fails to deviation to ~~f' $c'=P(S|e^*)f+P(F|e^*)$~~ f', c' such that $100-f'' = c'' = P(S|e^*)(100-f) + P(F|e^*)c$ and $f' = f'' + \varepsilon$, $c' = c''$ for sufficiently small amount ε. Noting that u is concave in w, $u(P(S|e^*)(100-f) + P(F|e^*)(c), e^*) > P(S|e^*)\,u(100-f, e^*) + P(F|e^*)\,u(c, e^*)$, so PC holds with strict equality at f'', c'' which is equally profitable as f, c. Then f', c' is more profitable than f, c, and for sufficiently small ε, PC remains satisfied.

Suppose $e^*=0$, then by the above,
$\frac{2}{5}(\sqrt{100-f}) + \frac{3}{5}(\sqrt{100-f}) = 5 \Rightarrow 100-f = 25 \Rightarrow f = 75$,
$c = 100-f = 25$. Payoff $\pi := P(S|e^*)f - P(F|e^*)c = \frac{2}{5}75 - \frac{3}{5}25 = 15$.

Suppose $e^*=1$, then by the above,
$\frac{3}{5}(\sqrt{100-f}-1) + \frac{2}{5}(\sqrt{100-f}-1) = 5 \Rightarrow \sqrt{100-f} = 6 \Rightarrow f = 64$,
$c = 100-f = 36$, $\pi = \frac{3}{5}64 - \frac{2}{5}36 = 24$.

P finds it optimal to require $e^* = 1$ and fully insures A with $f=64$, $c=36$.

c Suppose P induces high effort under unobservable effort, then P's optimisation problem is

$\max_{f,c} \frac{3}{5}f - \frac{2}{5}c$ s.t.

PC: $\frac{3}{5}(\sqrt{100-f}-1) + \frac{2}{5}(\sqrt{c}-1) \geq \bar{u} = 5$

IC: $\frac{3}{5}(\sqrt{100-f}-1) + \frac{2}{5}(\sqrt{c}-1) \geq \frac{2}{5}(\sqrt{100-f}) + \frac{3}{5}(\sqrt{c})$

At the optimum IC binds. The optimum of the maximisation problem without IC is that found in (b). That optimum does not satisfy IC because it ~~offers a fixed~~ offers A fixed final wealth, then $e=0$ is optimal for A. In symbols, if $\sqrt{100-f} = \sqrt{c} = \sqrt{k}$, then $\frac{3}{5}(\sqrt{100-f}-1) + \frac{2}{5}(\sqrt{c}-1) = \sqrt{k}-1 < \frac{2}{5}\sqrt{100-f} + \frac{3}{5}\sqrt{c} = \sqrt{k}$. Given that that optimum is a strict optimum, IC binds at the optimum with IC.

At the optimum PC binds. Any candidate optimum such that PC does not bind fails to deviation by reducing c by sufficiently small amount ε such that PC remains satisfied. LHS of IC decreases by more than RHS of IC, so IC remains satisfied. Payoff increases.

~~Then~~ From the above, at the optimum,

$\frac{3}{5}(\sqrt{100-f}-1) + \frac{2}{5}(\sqrt{c}-1) = 5$

$\cancel{\frac{3}{5}\sqrt{100}} = \frac{2}{5}\sqrt{100-f} + \frac{3}{5}\sqrt{c}$

Let $g = \sqrt{100-f}$ and $d = \sqrt{c}$ for notational convenience

$\frac{3}{5}g + \frac{2}{5}d = 6$, $\cancel{\frac{3}{5}g + \frac{2}{5}d =}$

$\frac{2}{5}g + \frac{3}{5}d = 5$,

$\Rightarrow g+d = 11,\ g-d = 5 \Rightarrow g = 8,\ d = 3$

$\Rightarrow \sqrt{100-f} = 8,\ \sqrt{c} = 3$

$\Rightarrow 100-f = 64,\ c = 9$

$\Rightarrow f = 36,\ c = 9$

P induces $e=1$ by offering $f=36$, $c=9$. This yields $\pi = \frac{3}{5}36 - \frac{2}{5}9 = 18$

P optimally induces $e=0$ by offering the fixed final wealth contract that ~~is just P~~ just satisfies PC, found in (b), $f=75$, $c=25$, which yields $\pi = 15$. This remains optimal because given fixed final wealth, A has strict incentive to choose $e=0$, so IC is satisfied.

P optimally induces $e=1$ and offers $f=36$, $c=9$. Then A has (weak) incentive to choose $e=1$, and does so at equilibrium.

d The optimal contract remains such as to induce high effort. Payoff to P decreased under unobservable effort. This is because P must offer A a variable final wealth contract to induce high effort where effort is unobservable, then A takes on some risk, and given A is risk averse, must be compensated ~~by~~ with a higher expected ~~fi~~ final wealth such that participation remains rational. Then, P has lower expected payoff because P incurs an agency cost in incentivising high effort and compensating A for the increased risk under such incentives.