Propositional Logic Rough Notes

Exam Technique

- Answers to parts of a question often draw on earlier results.
- It is necessary to argue that counterexamples succeed.
- In general, a useful tactic for proving a conditional is to prove that the negation of the consequent implies the negation of the antecedent. For example, prove $\vDash_{SV} \phi \Rightarrow \vDash_{PL} \phi$ by proving $\not\vDash_{PL} \phi \Rightarrow \not\vDash_{SV} \phi$.
- When given some abbreviation, it is sometimes expedient to construct truth tables or write a derived rule which is then applied or referred to in semantic arguments, rather than disabbreviating within semantic arguments.
  - For example, it is expedient to construct the truth tables for $\Delta\phi = \neg(\phi \to \neg\phi)$ and $\nabla\phi = \neg\Delta\neg\phi$ and refer to those in a semantic argument.

Bivalent Propositional Logic

Syntax

- Definition (PL-wff)
  - Each sentence letter $\alpha \in \{P, Q, R, P_1, Q_1, R_1, P_2, \ldots\}$ is a PL-wff.
  - If each of ϕ and ψ is a PL-wff, then each of $\neg\phi$ and $(\phi \to \psi)$ is a PL-wff.
  - Only strings that can be shown to be PL-wffs by the above rules are PL-wffs.
- Abbreviations
  - "$\phi \wedge \psi$" abbreviates "$\neg(\phi \to \neg\psi)$".
  - "$\phi \vee \psi$" abbreviates "$(\neg\phi \to \psi)$".
  - "$\phi \leftrightarrow \psi$" abbreviates "$((\phi \to \psi) \wedge (\psi \to \phi))$".
  - The familiar bracketing conventions apply.

Semantics

- Definition (PL-Interpretation)
  - A PL-interpretation (bivalent interpretation) $\mathcal{I}$ is a function from the set of sentence letters $\{P, Q, R, P_1, Q_1, R_1, P_2, \ldots\}$ to the set of truth values $\{0, 1\}$.
- Definition (PL-Valuation)
  - The PL-valuation $V_{\mathcal{I}}$ for a given PL-interpretation $\mathcal{I}$ is the unique function from the set of PL-wffs to the set of truth values $\{0, 1\}$ such that:
    - for all sentence letters α, $V_{\mathcal{I}}(\alpha) = \mathcal{I}(\alpha)$,
    - $V_{\mathcal{I}}(\neg\phi) = 1 \text{ iff } V_{\mathcal{I}}(\phi) = 0,$
    - and $V_{\mathcal{I}}(\phi \to \psi) = 0 \text{ iff } V_{\mathcal{I}}(\phi) = 1 \text{ and } V_{\mathcal{I}}(\psi) = 0.$
- Definition (PL-Validity)
  - PL-wff ϕ is PL-valid iff for all PL-interpretations $\mathcal{I}$, $V_{\mathcal{I}}(\phi) = 1$.
  - "$\vDash_{PL} \phi$" is equivalent to "ϕ is PL-valid".
- Definition (PL-Semantic Consequence)
  - PL-wff ϕ is a PL-semantic consequence of set of PL-wffs $\Gamma = \{\gamma_1, \gamma_2, \ldots\}$ iff for all PL-interpretations $\mathcal{I}$, if for all $\gamma_i \in \Gamma$, $V_{\mathcal{I}}(\gamma_i) = 1$, then $V_{\mathcal{I}}(\phi) = 1$.
  - "$\Gamma \vDash_{PL} \phi$" is equivalent to "ϕ is a PL-semantic consequence of Γ".

Trivalent Propositional Logic

- Motivation (Bivalence Failure)
  - Presupposition failure, vagueness, and indeterminacy yield prima facie bivalence failure in natural language. [See Studd, 2020, Notes on Propositional Logic, p. 4 for examples.]

Syntax

- Definition (Trivalent wff)
  - Each sentence letter $\alpha \in \{P, Q, R, P_1, Q_1, R_1, P_2, \ldots\}$ is a trivalent wff.
  - If each of ϕ and ψ is a trivalent wff, then each of $\neg\phi$, $(\phi \to \psi)$, $(\phi \wedge \psi)$, and $(\phi \vee \psi)$ is a trivalent wff.
  - Only strings that can be shown to be trivalent wffs by the above rules are trivalent wffs.

Semantics

- Definition (Trivalent Interpretation)
  - A trivalent interpretation $\mathcal{I}$ is a function from the set of sentence letters $\{P, Q, R, P_1, Q_1, R_1, P_2, \ldots\}$ to the set of truth values $\{0, 1, \#\}$.
- Definition (Weak Kleene-Valuation)
  - The weak Kleene-valuation $WV_{\mathcal{I}}$ for a given trivalent interpretation $\mathcal{I}$ is the unique function from the set of trivalent wffs to the set of truth values $\{0, 1, \#\}$ such that:
    - for all sentence letters α, $WV_{\mathcal{I}}(\alpha) = \mathcal{I}(\alpha)$,
    - $WV_{\mathcal{I}}(\neg\phi) = \# \text{ iff } WV_{\mathcal{I}}(\phi) = \#$,
    - for all $\circ \in \{\wedge, \vee, \to\}$, $WV_{\mathcal{I}}(\phi \circ \psi) = \# \text{ iff } WV_{\mathcal{I}}(\phi) = \# \text{ or } WV_{\mathcal{I}}(\psi) = \#$,
    - and $WV_{\mathcal{I}}$ coincides with $V_{\mathcal{I}}$ for all other $WV_{\mathcal{I}}(\phi)$ and $WV_{\mathcal{I}}(\psi)$.
- Discussion (Weak Kleene-Valuation)
  - Weak Kleene-valuation is plausible if we take $\#$ to indicate "meaningless" and suppose that any sentence with a meaningless component is thereby also meaningless.
- Definition (Kleene-Valuation)
  - The (strong) Kleene-valuation $KV_{\mathcal{I}}$ for a given trivalent interpretation $\mathcal{I}$ is the unique function from the set of trivalent wffs to the set of truth values $\{0, 1, \#\}$ such that:
    - for all sentence letters α, $KV_{\mathcal{I}}(\alpha) = \mathcal{I}(\alpha)$,
    - and for all trivalent wffs ϕ, ψ and $\circ \in \{\wedge, \vee, \to\}$, $KV_{\mathcal{I}}(\phi \circ \psi) = 1$ iff (with some violence to notation) $KV^*_{\mathcal{I}}(\phi) \circ KV^*_{\mathcal{I}}(\psi) = 1$ for all $KV^*_{\mathcal{I}}$ that differ from $KV_{\mathcal{I}}$ only in assigning 0 or 1 instead of $\#$, $KV_{\mathcal{I}}(\phi \circ \psi) = 0$ iff $KV^*_{\mathcal{I}}(\phi) \circ KV^*_{\mathcal{I}}(\psi) = 0$ for all $KV^*_{\mathcal{I}}$ that differ from $KV_{\mathcal{I}}$ only in assigning 0 or 1 instead of $\#$, and $KV_{\mathcal{I}}(\phi \circ \psi) = \#$ otherwise.
- Definition (Lukasiewicz-Valuation)
  - The Lukasiewicz-valuation $LV_{\mathcal{I}}$ for a given trivalent interpretation $\mathcal{I}$ is the unique function from the set of trivalent wffs to the set of truth values $\{0, 1, \#\}$ that differs from $KV_{\mathcal{I}}$ only in that $LV_{\mathcal{I}}(\phi \to \psi) = 1$ if $LV_{\mathcal{I}}(\phi) = \#$ and $LV_{\mathcal{I}}(\psi) = \#$.
- Definition (Kleene-Validity)
  - Trivalent wff ϕ is Kleene-valid iff for all trivalent interpretations $\mathcal{I}$, $KV_{\mathcal{I}}(\phi) = 1$.
  - "$\vDash_K \phi$" is equivalent to "ϕ is Kleene-valid".
- Definition (Kleene-Semantic Consequence)
  - Trivalent wff ϕ is a Kleene-semantic consequence of set of trivalent wffs $\Gamma = \{\gamma_1, \gamma_2, \ldots\}$ iff for all trivalent interpretations $\mathcal{I}$, if for all $\gamma_i \in \Gamma$, $KV_{\mathcal{I}}(\gamma_i) = 1$, then $KV_{\mathcal{I}}(\phi) = 1$.
  - "$\Gamma \vDash_K \phi$" is equivalent to "ϕ is a Kleene-semantic consequence of Γ".
- Definition (Lukasiewicz-Validity)

- Trivalent wff ϕ is Lukasiewicz-valid iff for all trivalent interpretations $\mathcal{I}$, $LV_{\mathcal{I}}(\phi) = 1$.
- "$\vDash_L \phi$" is equivalent to "ϕ is Lukasiewicz-valid".
- Definition (Lukasiewicz-Semantic Consequence)
  - Trivalent wff ϕ is a Lukasiewicz-semantic consequence of set of trivalent wffs $\Gamma = \{\gamma_1, \gamma_2, \ldots\}$ iff for all trivalent interpretations $\mathcal{I}$, if for all $\gamma_i \in \Gamma$, $LV_{\mathcal{I}}(\gamma_i) = 1$, then $LV_{\mathcal{I}}(\phi) = 1$.
  - "$\Gamma \vDash_L \phi$" is equivalent to "ϕ is a Lukasiewicz-semantic consequence of Γ".
- Definition (LP-Validity)
  - Trivalent wff ϕ is LP-valid iff for all trivalent interpretations $\mathcal{I}$, $KV_{\mathcal{I}}(\phi) \in \{1, \#\}$.
  - "$\vDash_{LP} \phi$" is equivalent to "ϕ is LP-valid".
- Definition (LP-Semantic Consequence)
  - Trivalent wff ϕ is a LP-semantic consequence of set of trivalent wffs $\Gamma = \{\gamma_1, \gamma_2, \ldots\}$ iff for all trivalent interpretations $\mathcal{I}$, if for all $\gamma_i \in \Gamma$, $KV_{\mathcal{I}}(\gamma_i) \in \{1, \#\}$, then $KV_{\mathcal{I}}(\phi) \in \{1, \#\}$.
  - "$\Gamma \vDash_L P\phi$" is equivalent to "ϕ is a LP-semantic consequence of Γ".
- The following table summarises the above definitions.

System	Valuation	Designated Values
Kleene	Kleene	1
Lukasiewicz	Lukasiewicz	1
LP (Logic of Paradox)	Kleene	1,#

- Motivation (Supervaluation)
  - Penumbral (i.e. indefinite, marginal, "shadowy") connections (of sentences) yield prima facie counterexamples to Kleene-valuation. A penumbral connection is a logical connection between indefinite (neither definitely true nor definitely false) sentences. $\mathcal{I}(P) = \# \Rightarrow KV_{\mathcal{I}}(P \wedge \neg P) = \#, KV_{\mathcal{I}}(P \vee \neg P) = \#$. Suppose P represents the sentence "Henry is tall" and Henry is a borderline case, so Henry is not (unqualifiedly) tall but also not (unqualifiedly) not tall, hence we construct $\mathcal{I}$ such that $\mathcal{I}(P) \neq 0, 1$. Intuitively, we think that $P \wedge \neg P$ should evaluate as (simply) false, and $P \vee \neg P$ should evaluate as (simply) true.
- Definition (Supervaluation)
  - The supervaluation $SV_{\mathcal{I}}$ for a given trivalent interpretation $\mathcal{I}$ is the unique function from the set of trivalent wffs to the set of truth values $\{0, 1, \#\}$ such that:
    - $SV_{\mathcal{I}}(\phi) = 1$ iff for all precisifications $\mathcal{I}^+$ of $\mathcal{I}$, $V_{\mathcal{I}^+}(\phi) = 1$,
    - $SV_{\mathcal{I}}(\phi) = 0$ iff for all precisifications $\mathcal{I}^+$ of $\mathcal{I}$, $V_{\mathcal{I}^+}(\phi) = 0$,
    - $SV_{\mathcal{I}}(\phi) = \#$ otherwise.
- Definition (Refinement)
  - A trivalent interpretation $\mathcal{I}^+$ is a refinement of trivalent interpretation $\mathcal{I}$ iff for all sentence letters α, if $\mathcal{I}(\alpha) = 1$ then $\mathcal{I}^+(\alpha) = 1$, and if $\mathcal{I}(\alpha) = 0$ then $\mathcal{I}^+(\alpha) = 0$.
- Definition (Precisification)
  - A bivalent interpretation $\mathcal{I}^+$ is a precisification of trivalent interpretation $\mathcal{I}$ iff for all sentence letters α, if $\mathcal{I}(\alpha) = 1$ then $\mathcal{I}^+(\alpha) = 1$, and if $\mathcal{I}(\alpha) = 0$ then $\mathcal{I}^+(\alpha) = 0$.
- Definition (SV-Validity)
  - Trivalent wff ϕ is SV-valid iff for all trivalent interpretations $\mathcal{I}$, $SV_{\mathcal{I}}(\phi) = 1$.
  - "$\vDash_{SV} \phi$" is equivalent to "ϕ is SV-valid".
- Definition (SV-Semantic Consequence)
  - Trivalent wff ϕ is a SV-semantic consequence of set of trivalent wffs $\Gamma = \{\gamma_1, \gamma_2, \ldots\}$ iff for all trivalent interpretations $\mathcal{I}$, if for all $\gamma_i \in \Gamma$, $SV_{\mathcal{I}}(\gamma_i) = 1$, then $SV_{\mathcal{I}}(\phi) = 1$.
  - "$\Gamma \vDash_{SV} \phi$" is equivalent to "ϕ is a SV-semantic consequence of Γ".
- Relationship (SV-Semantic Consequence and PL-Semantic Consequence)
  - $\Gamma \vDash_{SV} \phi$ iff $\Gamma \vDash_{PL} \phi$.

- Discussion (Truth-Functionality)
  - Supervaluationist connectives are not truth-functional (whereas PL, Kleene, and Lukasiewicz connectives are) in the sense that $SV_{\mathcal{I}}(\phi \circ \psi)$ is not (simply) a function of $SV_{\mathcal{I}}(\phi)$ and $SV_{\mathcal{I}}(\psi)$ for all connectives $\circ \in \{\wedge, \vee, \rightarrow\}$.
- Discussion (Non-Classical Logics)
  - In general non-classical logics "scale back classical logic's set of logical truths and logical consequences". Intuitionists want to reject the law of excluded middle (as a logical truth), paraconsistent logicians want to reject ex falso quodlibet (as a logical consequence). (See 210611 Philosophical Logic Paper Q1.)