Game Theory Problem Set 7

1a	C	D
C	1 1	$\underline{a}$ b
D	b $\underline{a}$	$\underline{0}$ $\underline{0}$

Given: $a>1$, $b<0$, $a+b\leq 1$, $p'+(1-p')b\geq 0$
In what follows, superscripts denote the period, subscripts denote the player and type, and $*$ denotes equilibrium values.

Best responses underlined. By inspection, D is strictly dominant for P1 and P2 $\Rightarrow s_R^{2*}=s_2^{2*}=D$, where R denotes rational P1 and T denotes irrational tit-for-tat P1. By ~~definition of~~ construction of type T, $s_T^{1*}=C$, $s_T^{2*}=s_2^{1*}$.

$\pi_R^{2*}=\pi_R^2(s_R^{2*}=D, s_2^{2*}=D)=0$ for all $s_R^1 \Rightarrow$
~~P1~~ R's payoff $\Pi_R=\pi_R^1+\pi_R^2$ reduces to $\pi_R^1(s_R^1, s_2^1)$.
In words, because rational players defect with certainty in period 2 regardless of period 1 actions, rational P1 maximises total payoff by maximising period 1 payoff. Then, D is strictly dominant $\Rightarrow$ ~~$s_R^{1*}=$~~ $s_R^{1*}=D$

$\Pi_2(s_2^1=C, s_2^2=D; s_R^*, s_T^*)=p'(1+a)+(1-p')(b)$
$\Pi_2(s_2^1=D, s_2^2=D; s_R^*, s_T^*)=p'(a)+(1-p')(0)$
~~$\nexists \Leftrightarrow p$~~
~~$\Pi_2(s_2^1=C, s_2^2=D; s_R^*, s_T^*)>\Pi_2(s_2^1=D, s_2^2=D; s_R^*, s_T^*)\Leftrightarrow$~~
~~$p'(1+a)+(1-p')b > p'a+(1-p')\ldots \Leftrightarrow$~~
~~$p'+(1-p')b\geq 0$~~
$\Pi_2(s_2^1=D, s_2^2=D; s_R^*, s_T^*)>\Pi_2(s_2^1=C, s_2^2=D; s_R^*, s_T^*)\Leftrightarrow$
$p'a > p'(1+a)+(1-p')b \Leftrightarrow p'+(1-p')b<0$
By reductio, given $p'+(1-p')b\geq 0$, $\Pi_2(s_2^1=C)\geq \Pi_2(s_2^1=D)$
Then $s_2^{1*}=C$. In words, P2's best response to s_R^* that ~~always~~ defects with certainty in each period and s_T^* that plays tit for tat is to cooperate then defect since the probability of encountering an irrational P1 is sufficiently high such that ~~the P2~~ the additional payoff of 1 from an additional period of cooperation, ~~against~~ with probability p' outweighs the penalty of b from being exploited by a rational P1 with probability $1-p'$.

$s_R^*=(D,D)$, $s_T^*=(C,C)$, $s_2^*=(C,D)$
P2's beliefs are such that at $t=1$, P2 believes P1 is T with probability p' and at $t=2$, P2 believes P1 is R with certainty if $s_1^1=D$ and P2 believes P1 is T with certainty if $s_1^1=C$.

b Suppose R plays C with certainty in $t=1$, then by Bayes rule, P2's belief in period 2, $p^2 = p^1/p^1+(1-p^1) = p^1$. In words, P1's action in equilibrium in $t=1$ is entirely uninformative.

By the result in a, ~~$s_R^{2*}, s_R^{3*} = D, D$, $s_T^{2*}, s_T^{3*} = C, D$~~ C, C
~~$s_2^{2*}, s_2^{3*} = C, D$.~~ ~~$s_R^* = (C, D, D)$, $s_T^* = (C, C, C)$, &~~
~~$s_2^* = (C, C, D)$.~~ $s_R^{2*}, s_R^{3*} = D, D$, $s_T^{2*}, s_T^{3*} = C, C$,
$s_2^{2*}, s_2^{3*} = C, D. \Rightarrow s_R^* = (C, D, D), s_T^* = (C, C, C), s_2^* = (C, C, D)$

~~$\Pi_2(s_2^1 = C, s_2^{2*}, s_2^{3*}; s_R^*, s_T^*) = p^1(1+1+a) + (1-p$~~
~~$= p^1(1+1+a) + (1-p^1)(1+b+0) = p^1(2+a) + (1-p^1)(1+b)$~~
~~$\Pi_2(s_2^1 = D, s_2^{2*}, s_2^{3*}; s_R^*, s_T^*)$~~
~~$= p^1($~~

$$\Pi_2(s_2^*; s_R^*, s_T^*) = p^1(1+1+a) + (1-p^1)(1+b+0) = p^1(2+a) + (1-p^1)(1+b)$$

Suppose P2 deviates from this candidate eqm by choosing $s_2^1 = s_2^{1\prime} \neq s_2^{1*} = C$. Then $s_2^1 = D \Rightarrow s_T^2 = D$, $s_R^2 = D$ (because if R plays C with non-zero probability, it reveals itself and invites defection in period 3, ~~and also receive~~ receives lower payoff in period 3, and also receives lower payoff in period 2), $s_R^3 = D$ (because D is strictly dominant), $s_2^3 = D$ (for the same reason).

Suppose further that $s_2^2 = C$, then
$\Pi_2 = p^1(a+b+a) + (1-p^1)(a+b+0) = p^1 a + a + b$
Suppose instead that $s_2^2 = D$, then
$\Pi_2 = p^1(a+0+0) + (1-p^1)(a+0+0) = a$

$\Pi_2^* \geq \Pi_2(D, C, D) \Leftrightarrow$
$p^1(2+a) + (1-p^1)(1+b) \geq p^1 a + a + b \Leftrightarrow$
~~$2p^1 + 1 - p^1 - bp^1 \geq a \Leftrightarrow$~~ $p^1 + (1-p^1)b + 1 + ap^1 \geq p^1 a + a + b \Leftrightarrow$
~~$1 + p^1 - bp^1 \geq a \Leftrightarrow$~~ $p^1 + (1-p^1)b + 1 \geq a + b \Leftarrow$
$p^1(1-p^1)b \geq 0$ and $a + b \leq 1$
so $\Pi_2^* \geq \Pi_2(D, C, D)$

$\Pi_2^* \geq \Pi_2(D, D, D) \Leftrightarrow$
$p^1(2+a) + (1-p^1)(1+b) \geq a \Leftrightarrow$
$p^1 + (1-p^1)b - (1-p^1)a + 1 \geq 0 \Leftarrow$
$p^1(1-p^1)b \geq 0$ and $(1-p^1)a \leq 1$
so $\Pi_2^* \geq \Pi_2(D, D, D)$

$s_2^{1*} = C$ is optimal for P2. It is trivial that $s_R^{1*} = C$ is optimal for R since R forgos one period of cooperation payoff if it defects earlier.

Rational P1 preserves its reputation as a tit-for-tat player by playing C in period 1 to extract an additional period of cooperation payoff before defecting to extract greater payoff a and irreversibly revealing itself and terminating cooperation

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2a Given P1, P2, P3 bid truthfully, i.e. bid equal to their respective valuations $v_1=10, v_2=4, v_3=2$, their respective bids are $s_1=10, s_2=4, s_3=2$. Given that the highest bidder pays the second highest bid and so on, the highest bidder, P1 and second highest bidder, P2, have cost per click through $c_1=s_2=4$ and $c_2=s_3=2$ respectively. Given that the highest bidder ~~ha~~ receives 200 click throughs per hour and the second highest bidder receives 100, P1 and P2 have volume of click throughs $q_1=200, q_2=100$. Net profits per hour $\pi_1=(v_1-c_1)q_1=1200$, $\pi_2=(v_2-c_2)q_2=200$, $\pi_3=0$.

b By inspection, π_1 is independent of s_1 for all $s_1 \geqslant s_2 = 4$. Suppose that $2 \nless s_1 \nless 4$ (and for simplicity P1 loses in the event of a tie). Then $c_1=s_3=2, q_1=100$, $\pi_1=(v_1-c_1)q_1=(10-2)(100)=800<1200$. Suppose instead that $s_1 \nless 2$ (and again that P1 loses in the event of a tie. Then $q_1=0, \pi_1=0<1200$. Consequently, P1 has no incentive to deviate from ~~the~~ the truthful strategy profile.

By inspection, π_2 is independent of s_2 for all $s_1=10 \geqslant s_2 \geqslant s_3=2$. Suppose that $s_2 \geqslant 10$ (and for simplicity that P2 wins in the event of a tie). Then $c_2=s_1=10, q_2=200, \pi_2=(4-10)200=-1200<200$. Suppose instead that $s_2<2$, then $q_2=0, \pi_2=0<200$. Consequently, P2 has no incentive to deviate from the truthful strategy profile.

Similarly, π_3 ~~is indepe~~ $=0$ for all $s_3 \overset{<}{\ngtr} 4$ (supposing P3 ~~lose~~ wins in the event of a tie). For all $s_3 \geqslant 4$, given $s_1=10, s_2=4$, we have $c_3=4>v_3=2$, hence $\pi_3=(v_3-c_3)q_3 \nless 0$. Consequently, P3 has no incentive to deviate.

No player has incentive to deviate from the truthful strategy profile, so it is an NE.

c Suppose instead that the second highest bidder receives 200 click throughs per hour. Then, at the truthful strategy profile, i.e. $s_1=10, s_2=4, s_3=2$, we have $c_1=s_2=4$, $c_2=s_3=2$, $q_1=200$, $q_2=200$, $\pi_1=(v_1-c_1)q_1=1200$, $\pi_2=(v_2-c_2)q_2=400$, $\pi_3=0$.

Suppose P1 deviates to $s_1'=3$. Then $c_1=s_3=2$, $c_2=s_1=3$, $q_1=200$, $q_2=200$, $\pi_1=(v_1-c_1)q_1=1600>1200$, $\pi_2=(v_2-c_2)q_2=200$. P1 has incentive to deviate from the truthful strategy profile, the truthful strategy profile is not an NE under the revised example.

d A true VCG mechanism in this environment would involve two auctions (one for each sponsored link spot, where bidders submit their bid in hourly terms (or equivalently in per click ~~the~~ through per hour, if the click through ~~rate~~ volume is equivalent between advertisers) and the winning bidder for each spot pays the second highest bid for that spot.

Google's mechanism differs because it "rolls the two auctions into one". This generates incentives for untruthful reporting because players may prefer to make the second highest bid if the value of the second prize is sufficiently close to the value of the first prize. There is no such incentive in the VCG mechanism since the second highest bidder wins nothing.

3a Given: $0 \leq b \leq a \leq 1$

Suppose for simplicity that in the event of a tie, P2 ~~gains~~. ~~loses~~. wins

P2 wins one unit iff $0 \leq b \leq a \leq X_1$ or ~~$0 \leq b$~~ $0 \leq b \leq X_1 \leq a$ iff $b \leq X_1$, which occurs with probability $1-b^2$.

P2 wins two units iff $0 \leq X_1 \leq b \leq a$ iff $X_1 \leq b$. ~~which occurs with probability~~

If P2 wins one unit, P2 wins one unit of the object which P2 has valuation X_2 for and P2 pays bid b, P2's payoff is $X_2 - b$.

If P2 wins two units, P2 has valuation $X_2 + Y_2$ for those units, and pays $2X_1$ which has distribution given by $F_{X_1}(x) = x^2$

P2's expected payoff is ~~$(1-b^2)(X_2-b) + \int_0^b (X_2+Y_2-2X$~~

$(1-b^2)(X_2-b) + \int_0^b (X_2+Y_2-2x) F'_{X_1}(x)\,dx$

$= (1-b^2)(X_2-b) + \int_0^b (X_2+Y_2-2x) 2x \,dx$

b ~~P2 chooses~~ By inspection, P2's expected payoff π_2 is independent of a. P2's optimisation problem reduces to $\max_b \pi_2$.

FOC: $\partial \pi_2 / \partial b = -2b(X_2-b) + (1-b^2)(-b) + \partial/\partial b \left((X_2+Y_2) \int_0^b 2x\,dx + \int_0^b -4x^2\,dx \right) = 0 \Leftrightarrow$

$-2bX_2 + 2b^2 - b + b^3 + \partial/\partial b \left[(X_2+Y_2)(b^2) - 4b^3/3 \right] = 0 \Leftrightarrow$

$-2bX_2 + 2b^2 - b + b^3 + 2(X_2+Y_2)b - 4b^2 = 0 \Leftrightarrow$

$-2b^2 - b + b^3 + 2bY_2 = 0$ ~~$\Leftarrow b=0$~~ $\Leftrightarrow$

~~$-2b - 1 + b^2 + 2Y_2 = 0$~~ $b=0$ or $2Y_2 = 2b + 1 - b^2 \Leftrightarrow$

~~$b=0 \Rightarrow \pi_2 = X_2$~~ $b=0$ or $Y_2 = -\frac{1}{2}b^2 + b + \frac{1}{2} \Leftrightarrow$

$b=0$ or $Y_2 = \frac{-1 \pm \sqrt{1-1}}{-1} = 1$ (reject since $P(Y_2=1)=0$)

~~SOC: $-2X_2 + 4b - 1 + 3b^2 + 2X_2 + 2Y_2 - 8b$~~

~~$= -4b - 1 + 3b^2 + 2Y_2$~~

Supposing that the FOC is sufficient for a maximum, P2 maximises π_2 given $s_1 = \{0, X_1\}$ by choosing $b=0$ and any a, so $s_2 = \{0, X_2\}$ is a best response to $s_1 = \{0, X_1\}$. By symmetry, $s_1 = \{0, X_1\}$ is a best response to $s_2 = \{0, X_2\}$. ~~This strategy profile is~~ The strategy profile is a BNE.

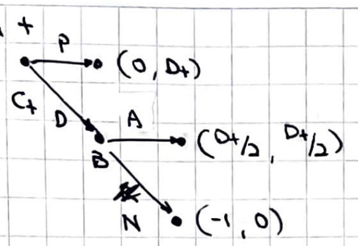
c The outcome is not necessarily efficient. With non-zero probability, $X_1 \geq Y_1 > X_2 \geq Y_2$, i.e. P1 has higher valuation for the second unit than P2 has for the first, so it is optimal for P1 to receive both units. The outcome of the above BNE is such that each player receives one unit regardless of their valuations, so it is not necessarily efficient.

The following mechanism is efficient in this setup.

P1 and P2 first bid for one unit and the highest bidder pays the second highest bid and receives one unit of the good. Then, P1 and P2 bid for the second unit and the highest bidder ~~pays th~~ again pays ~~for~~ the second highest bid and receives the second unit of the good. This is equivalent to the single-stage auction where each player reports two valuations, the first unit is awarded to the player with the highest valuation, who pays an amount equal to the other player's higher valuation. ~~The s~~ The highest valuation is then excluded in what follows. The second unit is awarded to the player with the highest remaining valuation, who pays an amount equal to the ~~p~~ other player's highest remaining valuation.

In a uniform price auction, each bidder submits any number of price-quantity pairs. Then the good is allocated to the highest bidder, in the ~~p~~ corresponding quantity, then to the second highest bidder, and so on until all units of the good have been allocated.

3a



For $t \in \{1, c\}$, where C_1 and C_2 denote country 1 and country 2 respectively, B denotes the bank, P denotes pay, D denotes default, A denotes the bank's absorbing some losses to bail out the borrower, and N denotes the bank refusing to bail out the borrower.

It is given ~~natural to suppose~~ that $D_t > 0$. ~~In the D-st subgame~~. Solving for SPE by backward induction, in the D-subgame, by inspection, it is optimal for B to choose A. Then, ~~in the~~ it is optimal for C_t to choose D, which yields a payoff of $D_t/2$ while P yields a payoff of 0. In the one shot game, C_t defaults knowing that the bank rationally will bail it out.

b The only SPE of the ~~two stage ga~~ two period game is such that each country defaults and the bank bails out each country.

Solving for SPE by backward induction, from (a), the equilibrium strategy of country 2, $s_c^{2*} = D$ and the equilibrium action of the bank in period 2, $s_B^{2*} = A$. This is regardless of C_1 and B's actions in period 1. So C_2's payoff and B's payoff in period 2 are independent of earlier actions. In period 1, B maximises total payoff by maximising period 1 payoff, i.e. B treats each period as an independent one shot game. Then, by the result in (a), the SPE is $s_B^* = AA$, $s_c^{1*} = D$, $s_c^{2*} = D$.

c Let p^t denote the probability that C_t assigns to B's being T (tough) at the stage where C_t is called to act. Let R denote the rational bank. In what follows, superscripts denote the period, ~~and~~ subscripts denote the player and type, and * denotes eqm values.

Given $\forall t: D_t > 0 \Rightarrow D_t/2 > 0 \Rightarrow s_R^{2*} = A$

Given: $s_T^{1*} = s_T^{2*} = N$

$\Rightarrow \pi_c^2(s_c^2 = D \mid p^2) = p^2(-1) + (1-p^2)D_t/2$

Given: $\pi_c^2(s_c^2 = P) = 0$

~~$s_c^2 = D$ if $-p^2 + (1-p^2)D_t/2 > 0$~~

~~$s_c^2 = P$ if $-p^2 + (1-p^2)D_t/2 < 0$~~

~~If $-p^2 + (1-p^2)D_t/2 = 0$, C_2 potentially mixes~~

~~Suppose $\sigma_R^{1*} = (1-q)A + qR$ for $q \in [0,1]$~~

$\pi_C^2(s_C^2 = D \mid \mu^2) = \pi_C^2(s_C^2 = P) \Leftrightarrow$

$\mu^2(-1) + (1-\mu^2)D_2/2 = 0 \Leftrightarrow$

$D_2/2 = \mu^2 (1 + D_2/2) \Leftrightarrow$

$\mu^2 = D_2/(2+D_2) \Rightarrow \mu^2 > 1/2$ (given $D_2 > 2$)

So if R plays N with certainty at $t=1$, then by Baye's rule, $\mu^2 = \mu^1 = 1/2$, then C_2 finds it optimal to play D, then R would be better off playing A in $t=1$. So at ~~the of~~ PBE, R does not play P with certainty in $t=1$.

If R plays A with certainty at $t=1$, then by Baye's rule, $\mu^2 = 1$ if R plays N, then C_2 finds it optimal to play P, then R would be better off playing R in $t=1$. So at PBE, R does not play A with certainty in $t=1$.

Intuitively, R has some incentive to imitate T and thereby deter ~~default~~ D of C_2, but if R perfectly imitates T, then R's playing N is uninformative and fails to deter C_2 from D-ing, perfect imitation fails to build a reputation for being T.

If R plays A in $t=1$, then by Baye's rule, $\mu^2 = 0$, then C_2 finds it optimal to D and R always As. If R plays N in $t=1$, then C_2 mixes. If $a_R^1 = N$ induces D with certainty, R ~~has no incentive~~ would be better off had it chosen $a_R^1 = A$. If $a_R^1 = N$ induces P with certainty, then, supposing $D_2 > D_1$, R has strict incentive to choose $a_R^1 = N$ and does not mix.

Intuitively, if imitating T is entirely effective at detering C_2's D-ing, then R has incentive to perfectly imitate T, then such imitation fails to deter C_2's D-ing. So it cannot be that imitating T is entirely effective at detering C_2's D-ing.

q is such that C_2 is indifferent between P and D, then q is such that $\mu^2 = D_2/(2+D_2)$. By Bayes rule, $\mu^2 = \mu^1/(\mu^1 + (1-\mu^1)q)$. Then

$D_2/(2+D_2) = \mu^1/(\mu^1 + (1-\mu^1)q) \Leftrightarrow$

$D_2(\mu^1 + (1-\mu^1)q) = (2+D_2)\mu^1 \Leftrightarrow$

$D_2(1/2 + q/2) = (2+D_2)1/2 \Leftrightarrow$

$D_2(q+1) = 2 + D_2 \Leftrightarrow D_2 q = 2 \Leftrightarrow q = 2/D_2$

$$
\begin{aligned}
\pi_C^1(a_C^1 = D) &= (\mu^1 + (1-\mu^1)q)(-1) + (1-\mu^1)(1-q)D_1/2 \\
&= -(1/2 + q/2) + (1/2)(1-q)D_1/2 \\
&= -(1/2 + 1/D_2) + (1/2)(D_2 - 2/D_2)(D_1/2) \\
&= -(1/2 + 1/D_2) + (1/2 - 1/D_2)(D_1/2)
\end{aligned}
$$

$\pi_C^1(a_C^1 = P) = 0$

$\pi_C^1(a_C^1 = D) > \pi_C^1(a_C^1 = P) \Leftrightarrow$

$1/2 + 1/D_2 < (1/2 - 1/D_2)(D_1/D_2)$

$D_2/2D_1 + 1/D_1 < (1/2 - 1/D_2)$

C_1 defaults if this condition is satisfied.

Derivation is incomplete. Strategies of C_1, C_2, R to be fully spelled out, and beliefs