Microeconomic Analysis Problem Set 2

1a By the (ε, δ) definition of continuity, function $f: \mathbb{R}^n \to \mathbb{R}^m$ is continuous ~~iff $\forall \varepsilon > 0 \in \mathbb{R}: \exists \delta > 0 \in \mathbb{R}:$~~ at $\vec{x}_0$ iff $\forall \varepsilon > 0 \in \mathbb{R}: \exists \delta > 0 \in \mathbb{R}:$ if $\|\vec{x}_1 - \vec{x}_0\| < \delta$ then $\|f(\vec{x}_1) - f(\vec{x}_0)\| < \varepsilon$.

$f(x,y) = xy$

$\|(x,y) - (0,0)\| = \|(x,y)\| = \sqrt{x^2+y^2}$

$\|f(x,y) - f(0,0)\| = \|xy\| = xy$

$x, y \leq \sqrt{x^2+y^2}$

$xy \leq (\sqrt{x^2+y^2})^2$

If $\|(x,y) - (0,0)\| < \delta$, then $\sqrt{x^2+y^2} < \delta$, then $xy < \delta^2$, then $\|f(x,y) - f(0,0)\| < \delta^2$.

So if ~~iff~~ $\forall \varepsilon > 0: \exists \delta > 0$ namely $\delta = \sqrt{\varepsilon}$ such that if $\|(x,y) - (0,0)\| < \delta$ then $\|f(x,y) - f(0,0)\| < \varepsilon$.

∴ By ~~defin~~ the (ε, δ) definition of continuity, $f(x,y) = xy$ is continuous at $0,0$.

b $f: \mathbb{R}^m \to \mathbb{R}$ is continuous at $\vec{x}_0$ ①

$g: \mathbb{R}^m \to \mathbb{R}$ is continuous at $\vec{x}_1$ ②

From ① ~~&~~, by the (ε, δ) definition of continuity,

$\forall \varepsilon > 0: \exists \delta > 0:$ if $\|\vec{x}_1 - \vec{x}_0\| < \delta$ then $\|f(\vec{x}_1) - f(\vec{x}_0)\| < \varepsilon$ ③

$\forall \varepsilon' > 0: \exists \delta' > 0:$ if $\|\vec{x}_1 - \vec{x}_0\| < \delta'$ then $\|g(\vec{x}_1) - g(\vec{x}_0)\| < \varepsilon'$ ④

From ③ and ④

$\forall \varepsilon, \varepsilon' > 0: \exists \delta, \delta' > 0:$ if $\|\vec{x}_1 - \vec{x}_0\| < \delta$ then $\|f(\vec{x}_1) - f(\vec{x}_0)\| < \varepsilon$ and if $\|\vec{x}_1 - \vec{x}_0\| < \delta'$ then $\|g(\vec{x}_1) - g(\vec{x}_0)\| < \varepsilon'$ ⑤

From ⑤

$\forall \varepsilon, \varepsilon' > 0:$ ~~$\exists \delta^* > 0$, namely~~ $\exists \delta, \delta' > 0:$ if $\|\vec{x}_1 - \vec{x}_0\| < \min\{\delta, \delta'\} \equiv \delta^*$ then $\|f(\vec{x}_1) - f(\vec{x}_0)\| < \varepsilon$ and $\|g(\vec{x}_1) - g(\vec{x}_0)\| < \varepsilon'$ ⑥

From ⑥

$\forall \varepsilon, \varepsilon' > 0: \exists \delta^* > 0:$ if $\|\vec{x}_1 - \vec{x}_0\| < \delta^*$ then $\|(f+g)(\vec{x}_1) - (f+g)(\vec{x}_0)\| < \varepsilon + \varepsilon' \equiv \varepsilon^*$ ⑦

From ⑦

$\forall \varepsilon^* > 0: \exists \delta^* > 0:$ if $\|\vec{x}_1 - \vec{x}_0\| < \delta^*$ then $\|(f+g)(\vec{x}_1) - (f+g)(\vec{x}_0)\| < \varepsilon^*$ ⑧

From ⑧, by the (ε, δ) definition of continuity, $f+g$ is continuous at $\vec{x}_0$.

2a The k^{th} partial derivative of the function f at $\vec{x} \in \mathbb{R}$

$\partial f/\partial x_k(\vec{x})$ is formally defined as ~~$\lim_{h\to 0}(1/h$~~

$\lim_{h\to 0}(1/h)[f(x_1,\dots,x_k+h,\dots x_n)-f(\vec{x})]$

$f(x,y)=\begin{cases} x^3/x^2+y^2 & \text{if } (x,y)\neq(0,0) \\ 0 & \text{otherwise}\end{cases}$

$\partial f/\partial x(0,0) = \lim_{h\to 0}(1/h)[f(h,0)-f(0,0)]$

$= \lim_{h\to 0}(1/h)[h-0]$

$= \lim_{h\to 0} 1$

$= 1$

$\partial f/\partial y(0,0) = \lim_{h\to 0}(1/h)[f(0,h)-f(0,0)]$

$= \lim_{h\to 0}(1/h)[0]$

$= 0$

$g(x,y)=\begin{cases} x^{1/2}y^{1/2} & \text{if } x\geq 0 \text{ and } y\geq 0 \\ 0 & \text{otherwise}\end{cases}$

$\partial g/\partial x(0,0) = \lim_{h\to 0}(1/h)[g(h,0)-g(0,0)]$

$= \lim_{h\to 0}(1/h)[0]$

$= 0$

By symmetry, $\partial g/\partial y = 0$

b $f_x = x^3(-1)(x^2+y^2)^{-2}(2x)+3x^2(x^2+y^2)^{-1}$ if $(x,y)\neq(0,0)$

$= -2x^4(x^2+y^2)^{-2}+3x^2(x^2+y^2)^{-1}$

$= x^2(x^2+y^2)^{-2}[-2x^2+3(x^2+y^2)]$

$= x^2(x^2+y^2)^{-2}[x^2+3y^2]$ if $(x,y)\neq(0,0)$

$f_y = \cancel{x^2(x^2+y^2}\ x^3(-1)(x^2+y^2)^{-2}(2y)$ if $(x,y)\neq(0,0)$

$= -2x^3y(x^2+y^2)^{-2}$ if $(x,y)\neq(0,0)$

$g_x = (1/2)x^{-1/2}y^{1/2}$ if $x\geq 0$ and $y\geq 0$

$g_y = (1/2)y^{-1/2}x^{1/2}$ if $x\geq 0$ and $y\geq 0$

3a $u = \sum_{i=1}^{L} \alpha_i \ln x_i$, $\forall i: \alpha_i > 0$, $\sum_{i=1}^{L} \alpha_i = 1$

$Du = (\partial_1 u \;\; \partial_2 u \cdots \partial_L u)$

$= (\alpha_1/x_1 \;\; \alpha_2/x_2 \cdots \alpha_L/x_L)$

b $$D^2u = \begin{pmatrix} \partial_1\partial_1 u & \partial_1\partial_2 u & \cdots & \partial_1\partial_L u \\ \partial_2\partial_1 u & \partial_2\partial_2 u & \cdots & \partial_2\partial_L u \\ \vdots & \vdots & \ddots & \vdots \\ \partial_L\partial_1 u & \partial_L\partial_2 u & \cdots & \partial_L\partial_L u \end{pmatrix}$$

$$= \begin{pmatrix} -\alpha_1 x_1^{-2} & 0 & \cdots & 0 \\ 0 & -\alpha_2 x_2^{-2} & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\alpha_L x_L^{2} \end{pmatrix}$$

c Let D^2u_{ij} denote the element in the i^{th} row and j^{th} column of D^2u. By definition of D^2u, $\forall i,j \in \{1, \dots L\}$: $D^2u_{ij} = \partial_i\partial_j u = \partial_i(\alpha_j/x_j) = 0$ for all $i \neq j$. Then D^2u is a diagonal matrix so D^2u is a symmetric matrix. This is confirmed by inspection.

Let D^2u^k denote the square submatrix of A with only the first k rows and columns retained. Then $D_2u^1 = (-\alpha_1 x_1^{-2})$

$D_2u^2 = \begin{pmatrix} -\alpha_1 x_1^{-2} & 0 \\ 0 & -\alpha_2 x_2^{-2} \end{pmatrix}$ and so on.

~~det~~ $(-1)^1 \det D_2u^1 = \alpha_1 x_1^{-2}$

$(-1)^2 \det D_2u^2 = (\alpha_1 x_1^{-2})(\alpha_2 x_2^{-2})$

$(-1)^k \det D_2u^k = (-1)^k \prod_{i=1}^{k} (-\alpha_i x_i^{-2})$

since the determinant of a diagonal matrix is the product of its diagonal elements

$= \prod_{i=1}^{k} (\alpha_i x_i^{-2})$

> 0 for all k

Given $\forall i\, \alpha_i > 0$, assuming that u is well-defined then $\forall i: \ln x_i$ is well-defined then $\forall i: x_i \neq 0$ then $\forall i\, x_i^{-2} > 0$.

By the determinant test, D^2u is negative definite.

d $\vec{y}^T D^2u\, \vec{y}$ is well-defined only if $\vec{y}$ is a vector of length L. Let $\vec{y} = (y_1, \dots, y_L)^T$

$$D^2u\, \vec{y} = \begin{pmatrix} -\alpha_1 x_1^{-2} & 0 & \cdots & 0 \\ 0 & -\alpha_2 x_2^{-2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\alpha_L x_L^{-2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_L \end{pmatrix}$$

$= \cancel{(-y_1\alpha_1 x_1^{-2} \quad -y_2\alpha_2 x_2^{-2} \cdots -y_L\alpha_L x_L^{-2})}$

$\cancel{\vec{y}^T D^2u\, \vec{y}} \begin{pmatrix} -y_1\alpha_1 x_1^{-2} \\ -y_2\alpha_2 x_2^{-2} \\ \\ -y_L\alpha_L x_L^{-2} \end{pmatrix}$

$$\vec{y}^T D^2u\, \vec{y} = (y_1 \; y_2 \cdots y_L) \begin{pmatrix} -y_1\alpha_1 x_1^{-2} \\ -y_2\alpha_2 x_2^{-2} \\ \\ -y_L\alpha_L x_L^{-2} \end{pmatrix}$$

$= -y_1^2\alpha_1 x_1^{-2} - y_2^2\alpha_2 x_2^{-2} - \cdots - y_L^2\alpha_L x_L^{-2}$

$= \cancel{\sum \alpha} - \sum_{i=1}^{L} \cancel{\alpha_i y_i^2 x} \; \alpha_i y_i^2 x_i^{-2}$

< 0 for all $\vec{y} \neq 0 \in \mathbb{R}^n$

Given $\forall i: \alpha_i > 0$, assuming that u is well-defined hence $\forall i: x_i^{-2} > 0$

By checking $\vec{y}^T D^2u\, \vec{y}$, D^2u is negative definite.

e $u(\vec{x}') \approx u(\vec{x}) + Du(\vec{x})(\vec{x}' - \vec{x}) + \frac{1}{2}(\vec{x}' - \vec{x})^T D^2u(\vec{x})(\vec{x}' - \vec{x})$

$= 0 + (\alpha_1/\cancel{x} \;\; \alpha_2/\cancel{x} \cdots \alpha_L/\cancel{x})(x_1' - 1 \;\; x_2' - 1 \cdots x_L' - 1)$

$+ \frac{1}{2}(x_1' - 1 \;\; x_2' - 1 \cdots x_L' - 1) D^2u (x_1' - 1 \;\; x_2' - 1 \cdots x_L' - 1)^T$

$$= \alpha_1(x_1'-1) + \alpha_2(x_2'-1) + \cdots + \alpha_L(x_L'-1)$$
$$+ \tfrac{1}{2}\left[-\alpha_1(x_1'-1)^2 - \alpha_2(x_2'-1)^2 + \cdots + \alpha_L(x_L'-1)^2\right]$$
$$= \Sigma_{i=1}^{L} \alpha_i \ (x_i'-1) - \tfrac{1}{2}(x_i'-1)^2$$

4a Let $f(x,y,z) = z^2 + xz + yx^2 + y^3$

$f(x,y,z) = 0$

$z^2 + xz + yx^2 + y^3 = 0$

$z^2 + xz = -yx^2 - y^3$

$(z + x/2)^2 - x^2/4 = -yx^2 - y^3$

$(z + x/2) = \sqrt{-yx^2 - y^3 + x^2/4}$

$z = g(x,y) \equiv \sqrt{-yx^2 - y^3 + x^2/4} - x/2$

$f_x = z + 2xy$

$f_y = x^2 + 3y^2$

$f_z = 2z + x$

By inspection, f_x, f_y, and f_z are continuous, so f is C'

~~$f_y(s^*) =$~~ $f_z(s^*) = 1 \neq 0$. Then, by the implicit function theorem, $f(x,y,z) = 0$ defines a $z = g(x,y)$ in the neighbourhood of $(x^*, y^*) = (1,0)$ such that $z^* = g(x^*, y^*)$, $f(x^*, y^*, g(x^*, y^*)) = 0$ and ~~$g_x(x,y,z) =$~~

$g_x(x,y,z) = -$ ~~f_z~~ $f_x(x,y,z)/f_z(x,y,z)$ and

$g_y(x,y,z) = - f_y(x,y,z)/f_z(x,y,z)$

b ~~g_x~~ ~~$z = g(x=1, y=0) =$~~

$z = g(x=1, y=0) = 0$

~~$g_x(x=1, y=0$~~

$g_x(1,0,0) = -0/0 = 0$

$g_y(1,0,0) = -1/1 = -1$

a Let $f_1(x,y,u,v) = x^2 - y^2 - u^3 + v^2 + 4$,
$f_2(x,y,u,v) = 2xy + y^2 - 2u^2 + 3v^4 + 8$,
and $f = (f_1, f_2)$
Given $f(s^*) = (0,0)$

$$Df = \begin{pmatrix} \partial f_1/\partial x & \partial f_1/\partial y & \partial f_1/\partial u & \partial f_1/\partial v \\ \partial f_2/\partial x & \partial f_2/\partial y & \partial f_2/\partial u & \partial f_2/\partial v \end{pmatrix}$$

$$= \begin{pmatrix} 2x & -2y & -3u^2 & 2v \\ 2y & 2y & -4u & 12v^3 \end{pmatrix}$$

By inspection, each element of Df is continuous, so f is C_1

$$D_{u,v}f = \begin{pmatrix} -3u^2 & 2v \\ -4u & 12v^3 \end{pmatrix}$$

$\det D_{u,v}f = -36u^2v^3 + 8uv$

At s^*, $\det D_{u,v}f = -36(2^2)(1^3) + 8(2)(1)$
$= -144 + 16$
$= -128 \neq 0$

so $D_{u,v}f$ is invertible at s^*

Then, by the implicit function theorem, $f(x,y,u,v) = (0,0)$ defines a function $(u,v) = g(x,y)$ such that $(u^*, v^*) = g(x^*, y^*)$, $f(x^*, y^*, g(x^*, y^*)) = (0,0)$,
$D_x g = -[D_{u,v}f]^{-1} D_x f$ and $D_y g = -[D_{u,v}f]^{-1} D_y f$

b $(u,v) = g(2,-1) = (2,1)$

At $(2,-1,2,1)$, $D_{u,v}f = \begin{pmatrix} -12 & 2 \\ -8 & 12 \end{pmatrix}$, $\det D_{u,v}f = -128$

By Cramer's rule, $[D_{u,v}f]^{-1} = -\frac{1}{128}\begin{pmatrix} 12 & -2 \\ 8 & -12 \end{pmatrix}$

$$D_x f = \begin{pmatrix} \partial f_1/\partial x \\ \partial f_2/\partial x \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

At $(2,-1,2,1)$, $D_x f = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$

$$D_x g = -[D_{u,v}f]^{-1} D_x f = \frac{1}{128}\begin{pmatrix} 12 & -2 \\ 8 & -12 \end{pmatrix}\begin{pmatrix} 4 \\ -2 \end{pmatrix} = \frac{1}{128}(52, 56)$$

$= \frac{1}{32}(13, 14)$

$\partial u/\partial x = 13/32$, $\partial v/\partial x = 14/32$

6 $f(x,y) = x^4 + x^2 - 6xy + 3y^2$

$Df(x,y) = (4x^3+2x-6y \quad -6x+6y)$

FOCs:

$4x^3+2x-6y=0$ ①

$-6x+6y=0$ ②

From ②, $x=y$ ③

Sub ③ into ①

$4x^3-4x = 4x(x+1)(x-1)=0$, $x=-1, 0,$ or 1

critical points: $(-1,-1), (0,0), (1,1)$

$$D^2f(x,y) = \begin{pmatrix} 12x^2+2 & -6 \\ -6 & 6 \end{pmatrix}$$

SOCs:

~~For (-1,-1)~~

$\det D^2f(-1,-1) = \det\begin{pmatrix} 14 & -6 \\ -6 & 6 \end{pmatrix} = 48 > 0$

$\det [D^2f(-1,-1)]_1 = \det(14) = 14 > 0$

$D^2f(-1,-1)$ is positive definite, $(-1,-1)$ is a strict local minimum

$\det D^2f(0,0) = \det\begin{pmatrix} 2 & -6 \\ -6 & 6 \end{pmatrix} = -24 < 0$

$D^2f(0,0)$ is indefinite, $(0,0)$ is a saddle point.

$\det D^2f(1,1) = \det\begin{pmatrix} 14 & -6 \\ -6 & 6 \end{pmatrix} = 48 > 0$

$\det [D^2f(1,1)]_1 = \det(14) = 14 > 0$

$D^2f(1,1)$ is positive definite, $(1,1)$ is a strict local minimum.

Date

2i $\forall, \exists$ notation is good
Make explicit the intermediate step that
$\sqrt{x^2+y^2} < \delta \Rightarrow \sqrt{x^2} < \delta$ and $\sqrt{y^2} < \delta \Rightarrow$ ~~x^2~~ $xy < \delta^2$
so if ... namely ... notation is correct

b Note: $|f(\vec{x}_0) - f(\vec{x}_1)| + |g(\vec{x}_0) - g(\vec{x}_1)| \geq |f+g(\vec{x}_0) - f+g(\vec{x}_1)|$
Not $=$
Easy to construct problems to show continuity

Do not need to prove continuity where obvious

2a ~~$\frac{\partial}{\partial x} f$~~ $\partial_x f$ at $(x,y) \neq (0,0)$ is found simply by differentiating x^3 / x^2+y^2

Can simply claim that f_x is continuous by inspection

Arithmetic error in f_x
Partial derivative can be written as $= \begin{cases} 1 & \text{if } (x,y)=(0,0) \\ \cdots \end{cases}$

can show ~~lim f_x~~ f_x is non-continuous by finding some sequence ~~to show f~~ such that $\lim f_x$ is not ~~$\neq$~~ 1 from some direction
Sequences are useful to find a counterexample
$\{0, \frac{1}{n}\}_{n=1}^{\infty} \to (0,0)$, $\{f(0,\frac{1}{n})\}_{n=1}^{\infty} \not\to 1$

C^1 implies differentiable, but not the reverse

Differentiable iff $\lim_{\|\vec{h}\| \to 0} \frac{f(h_1,h_2) - f(0,0) - Df(0,0)\binom{h_1}{h_2}}{\|\vec{h}\|} = 0$

simplify then check diagonal $\{(\frac{1}{n}, \frac{1}{n})\}_{n=1}^{\infty}$, use $1/n$, $2/n$, etc.

Find partial derivatives at $(0,0)$ by limit formula
~~calc~~ Compute pds elsewhere
Find partial derivatives at $(0,y)$ $(x,0)$ by limit method

Partial derivatives dont exist on the axes
can show partial derivatives not continuous to $(0,0)$ from diagonals

* Extra continuity weaker requirement than differentiability
can be not C^1 but still differentiable
f^n not continuous $\Rightarrow f^n$ not differentiable
differentiable: "everywhere you can approximate w a line"
continuity nec for diffability
C^1 not nec for diffability

4 Correct to first check if f is C^1
can be ~~given~~ "by inspection"
Correct to check $f_2(s^*) \neq 0$
Then by implicit fn theorem
$z = g(x,y)$
$\cancel{0 = g(1,0)}$ $z^* = g(x^* y^*)$
g has cont pd at (x^*, y^*)
Not necessary to find the function

5 All polynomials are C^1
(a) is all correct
(b) is all correct

6 Saddle: incr in some directions, decr in others
(informally)
Correct

How to check global?
- Find counterexample
- Prove function is bounded

3 When providing Taylor approx, evaluate ~~the~~ Jacobian & Hessian at the point that you are approximating.
(c) can check definiteness in many ways
∴ notation is correct

Taylor approx gives intuition behind definiteness check.