~~Philoso~~ Studd Philosophical Logic Exercises.

3 Prove by induction that for all PL-wffs ϕ containing at most one occurrence of each sentence letter, there exists some PL-interpretation I_0 and some PL-interpretation I_1 such that $V_{I_0}(\phi)=0$, $V_{I_1}(\phi)=1$.

Base Case

Consider arbitrary PL-wff ϕ such that ~~comp~~ complexity $C(\phi)=0$, then ϕ is some sentence letter α. $V_{I_0}(\phi)=I_0(\alpha)=0$, by construction of I_0. $V_{I_1}(\phi)=I_1(\alpha)=1$, by construction of I_1. So there exists ~~som~~ such PL-interpretations. By generalisation, there exist such PL-~~int~~ interpretations for all such ϕ.

Induction Hypothesis.

Given n, for all $m<n$, for all PL-wff ϕ such that $C(\phi)=m$, if ϕ contains at most one ..., there exists I_0 and I_1 such that $V_{I_0}(\phi)=0$ and $V_{I_1}(\phi)=1$.

Induction Step

Consider arbitrary PL-wff ϕ such that $C(\phi)=n$. Then ϕ ~~is~~ $=\sim\psi$ or $\psi\to\chi$. Suppose that ϕ contains at most one occurrence of each sentence letter. Suppose $\phi=\sim\psi$, then ~~there exists~~ there exists ~~$I_{\psi 0}$~~ $C(\psi)=n-1$, ~~so by IH, there exists $I_{\psi 0}$ such that~~ and ψ contains at most one occurrence of each sentence letter, so by IH, there exists $I_{\psi 0}$ and $I_{\psi 1}$ such that $V_{I_{\psi 0}}(\psi)=0$ and $V_{I_{\psi 1}}(\psi)=1$. By $\sim$ clause, $V_{I_{\psi 0}}(\phi)=1$, $V_{I_{\psi 1}}(\phi)=0$. So ~~ϕ~~ there exists I_0 and I_1, namely $I_{\psi 1}$ and $I_{\psi 0}$ such that $V_{I_0}(\phi)=0$ and $V_{I_1}(\phi)=1$.

Suppose $\phi=\psi\to\chi$. ~~Then~~ Then $C(\psi)+C(\chi)=n-1$, $C(\psi), C(\chi)<n$, and each of ψ and χ contain at most one occurrence of each sentence letter and the sentence letters ~~in~~ occurring in ψ do not occur in χ or vice versa. By IH, there exists $I_{\psi 0}$, $I_{\psi 1}$, $I_{\chi 0}$, $I_{\chi 1}$ that satisfy the obvious. Let $I_{\psi 1 \chi 0}$ be the interpretation that agrees with $I_{\psi 1}$ on sentence letters in ψ and with ~~I~~ $I_{\chi 0}$ on those in χ. Given that ψ and χ contain distinct sentence letters, such I exists. Then $V_{I_{\psi 1 \chi 0}}(\psi)=1$, $V_{I_{\psi 1 \chi 0}}(\chi)=0$ so $V_{I_{\psi 1 \chi 0}}(\phi)=0$. Similarly, there exists $I_{\psi 0 \chi 1}$ such that $V_{I_{\psi 0 \chi 1}}(\phi)=1$.

By cases, conditional proof, generalisation, for all ϕ s.t. $C(\phi)=n$, if ϕ contains at most ..., then there exists $I_{\phi 0}$, $I_{\phi 1}$ s.t. ....

By induction, such is true for all ϕ, ~~such~~ ~~then for all~~ then for all ϕ containing at most one ..., $\exists I_{\phi 0}: V_{I_{\phi 0}}(\phi)=0$, so by definition, $\nvDash_{PL}\phi$.

6a By definition of tV, $\to$ clause,

$$tV_I(\phi\to\psi)=\begin{cases}\cancel{1 \text{ iff } tV_I(\phi)=0 \text{ or } tV_I(\psi)=1}\\ \cancel{0 \text{ iff } tV_I(\phi)=1 \text{ and } tV_I(\psi)=0}\\ \cancel{\# \text{ otherwise}}\end{cases}$$

$$\begin{cases}1 \text{ if } tV_I(\phi)=0 \text{ or } tV_I(\psi)=1\\ 1 \text{ if } tV_I(\phi)=tV_I(\psi)=\#\\ 0 \text{ if } tV_I(\phi)=1 \text{ and } tV_I(\psi)=0\\ \# \text{ otherwise}\end{cases}$$

If $tV_I(\phi)=0$ then $tV_I(\phi)\leq tV_I(\psi)$.
If $tV_I(\psi)=1$ then $tV_I(\phi)\leq tV_I(\psi)$
If $tV_I(\phi)=tV_I(\psi)=\#$, then $tV_I(\phi)\leq tV_I(\psi)$
If $tV_I(\phi)\leq tV_I(\psi)$, then $(tV_I(\phi), tV_I(\psi))=$ ~~(1,0), (1,#), (1,1)~~ (1,1), (#,#), (#,1), (0,0), (0,#), (0,1), so either (1) $tV_I(\phi)=0$ or (2) $tV_I(\psi)=1$ or (3) $tV_I(\phi)=tV_I(\psi)=\#$. So the above reduces to

$$tV_I(\phi\to\psi)=\begin{cases}1 \text{ iff } tV_I(\phi)\geq tV_I(\psi)\\ 0 \text{ iff } tV_I(\phi)=1 \text{ and } tV_I(\psi)=0\\ \# \text{ otherwise}\end{cases}$$

If $tV_I(\phi)=1$ and $tV_I(\psi)=0$, then $tV_I(\phi\to\psi)=0=1-(tV_I(\phi)-tV_I(\psi))$.
For $tV_I(\phi)<tV_I(\psi)$ and ~~tV_I~~ not ($tV_I(\phi)=1$ and $tV_I(\psi)=0$), $tV_I(\phi)-tV_I(\psi)=\#$. ~~So the above further~~ for such ϕ, ψ, $tV_I(\phi\to\psi)=\#=1-(tV_I(\phi)-tV_I(\psi))$. So the above further reduces to

$$tV_I(\phi\to\psi)=\begin{cases}1 \text{ iff } tV_I(\phi)\geq tV_I(\psi)\\ 1-(tV_I(\phi)-tV_I(\psi)) \text{ otherwise.}\end{cases}$$

b Suppose $KV_I(\phi\to\psi)=1$, then $KV_I(\phi)=0$ or $KV_I(\psi)=1$ ~~or $KV_I(\phi)=KV_I(\psi)$~~ then $I(\phi)=0$ or $I(\psi)=1$, then $SV_I(\phi)=0$ or $SV_I(\psi)=1$ then $SV_I(\phi\to\psi)=1$.

Suppose $KV_I(\phi\to\psi)=0$, then $KV_I(\phi)=1$ and $KV_I(\psi)=0$ then $I(\phi)=1$ and $I(\psi)=0$ then $SV_I(\phi)=1$ and $SV_I(\psi)=0$ then $SV_I(\phi\to\psi)=0$.

Suppose $KV_I(\phi\to\psi)=\#$, then either (1) $KV_I(\phi)=1$ and $KV_I(\psi)=\#$, (2) $KV_I(\phi)=\#$ and $KV_I(\psi)=$ ~~#~~ 0, or (3) $KV_I(\phi)=KV_I(\psi)=\#$. Suppose (1), then there exists precisification $I_{\psi 1}$ ~~s.t.~~ such that ~~V_I~~ $I_{\psi 1}(\psi)=1$ and precisification $I_{\psi 0}$ such that $I_{\psi 0}(\psi)=0$, then $V_{I_{\psi 1}}(\phi\to\psi)=1$, $V_{I_{\psi 0}}(\phi\to\psi)=0$, so $SV_I(\phi\to\psi)=\#$. Suppose (2) then $\exists$ prec. $I_{\phi 1}$ s.t. $V_{I_{\phi 1}}(\phi)=1$ and $\exists$ prec. $I_{\phi 0}$ s.t. $V_{I_{\phi 0}}(\phi)=0$ then $V_{I_{\phi 1}}(\phi\to\psi)=0$, $V_{I_{\phi 0}}(\phi\to\psi)=1$ so $SV_I(\phi\to\psi)=\#$. Suppose (3), then $\exists$ prec $I_{\phi 1 \psi 0}$ s.t. $V_{I_{\phi 1 \psi 0}}(\phi)=1$, $V_{I_{\phi 1 \psi 0}}(\psi)=0$, $V_{I_{\phi 1 \psi 0}}(\phi\to\psi)=0$ and $\exists$ prec $I_{\phi 0 \psi 1}$ s.t. $V_{I_{\phi 0 \psi 1}}(\phi)=0$, $V_{I_{\phi 0 \psi 1}}(\psi)=1$, $V_{I_{\phi 0 \psi 1}}(\phi\to\psi)=1$, so $SV_I(\phi\to\psi)=\#$. By cases, $SV_I(\phi\to\psi)=\#$.

By cases, $KV_I(\phi \to \psi) = SV_I(\phi \to \psi)$ for distinct atomic ϕ, ψ, assumed throughout.

Does not hold when non-distinct and atomic. Consider $\phi = \psi = p$, $I(p) = \#$, then ~~$KV_I(p) = \#$, $SV_I(p$. $KV_I(p \to p) = \#$~~, ~~$SV_I(p \to$~~ $KV_I(\phi \to \psi) = \# \neq SV_I(\phi \to \psi) = 1$.

Does not hold when non-atomic and distinct. Consider $\phi = (p \wedge \sim p)$, $\psi = (q \wedge \sim q)$, $I(p) = I(q) = \#$. $KV_I(\phi \to \psi) = \# \neq SV_I(\phi \to \psi) = 1$.

Does not hold when non-atomic and non-distinct. Consider $\phi = (p \wedge \sim p)$, $\psi = (p \wedge \sim p)$, $I(p) = \#$, ~~$KV_I(\phi) = 0 \neq SV_I$~~ $KV_I(\phi \to \psi) = \# \neq SV_I(\phi \to \psi) = 1$.

< Suppose ~~$\gamma \nvDash_{pc}$~~ $T \nvDash_{pc} \phi$, then there exists bivalent interpretation I such that for all $\gamma \in T$, $V_I(\gamma) = 1$ and $V_I(\phi) = 0$, then $SV_I(\gamma) = 1$ and $SV_I(\phi) = 0$ (because for all precisifications of I, namely I itself, $V_I(\gamma) = 1$ and $V_I(\phi) = 0$) so $T \nvDash_{sv} \phi$.

Suppose $T \nvDash_{sv} \phi$. then there exists trivalent interpretation I^3 such that for all $\gamma \in T$, $SV_{I^3}(\gamma) = 1$ and $SV_{I^3}(\phi) = 0$, then ~~for a~~ there exists precisification I^2 of I^3 such that $V_{I^2}(\gamma) = 1$ (for all $\gamma \in T$), and $V_{I^2}(\phi) = 0$, then $T \nvDash_{pc} \phi$.

By biconditional proof, $T \nvDash_{sv} \phi$ iff $T \nvDash_{pc} \phi$, so ~~$T \nvDash$~~ $T \vDash_{pc} \phi$ iff $T \vDash_{sv} \phi$.