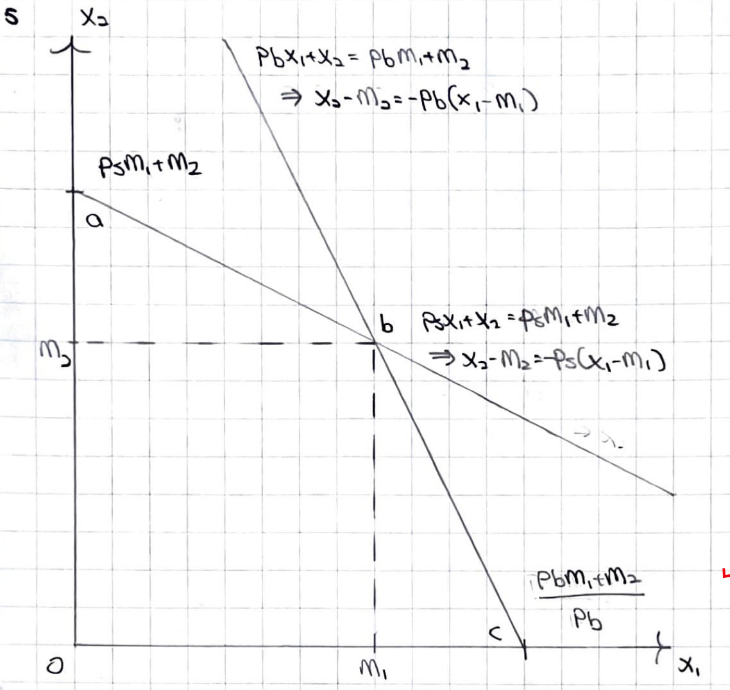
5



The budget set is represented by area 0abc. p_s is the price, in units of good 2, at which the household can sell good 1. p_b is the price, in units of good 2, at which the household can buy good 1.

The household's optimisation problem is

$\max_{x_1,x_2} u(x_1,x_2) = x_1^\beta + x_2^\beta$ st

$g_1(x_1,x_2) = x_1 \geq 0$, $g_2(x_1,x_2) = x_2 \geq 0$

$g_s(x_1,x_2) = p_s x_1 + x_2 \leq p_s m_1 + m_2$

$g_b(x_1,x_2) = p_b x_1 + x_2 \leq p_b m_1 + m_2$

where $0 < \beta < 1$; $m_1, m_2 > 0$; $0 < p_s < p_b$

$$Dg = \begin{pmatrix} \partial g_1/\partial x_1 & \partial g_1/\partial x_2 \\ \partial g_2/\partial x_1 & \partial g_2/\partial x_2 \\ \partial g_s/\partial x_1 & \partial g_s/\partial x_2 \\ \partial g_b/\partial x_1 & \partial g_b/\partial x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ p_s & 1 \\ p_b & 1 \end{pmatrix}$$

Alternatively, given that this is a concave problem it is enough to argue that the interior of the constraint set is NOT empty.

~~Suppose that~~ The constraint qualification holds iff the number of binding constraints $e \leq 2$. Suppose that both g_s and g_b bind, then, given that $m_1, m_2 > 0$, $x_1 = m_1 > 0$ and $x_2 = m_2 > 0$, so neither g_1 nor g_2 bind. Suppose that both g_1 and g_2 bind, then $x_1 = x_2 = 0$, $p_s x_1 + x_2 = 0 < p_s m_1 + m_2$, and $p_b x_1 + x_2 = 0 < p_b m_1 + m_2$, so neither g_s nor g_b bind. So $e \leq 2$, and the constraint qualification holds. The Kuhn-Tucker first-order conditions and complementary slackness conditions are necessary conditions for all optima. Very good!

$L(x_1, x_2; \mu_1, \mu_2; \lambda_s, \lambda_b)$
$= x_1^\beta + x_2^\beta + \mu_1 x_1 + \mu_2 x_2$
$- \lambda_s(p_s x_1 + x_2 - (p_s m_1 + m_2)) - \lambda_b(p_b x_1 + x_2 - (p_b m_1 + m_2))$

Try to simplify the problem as much as possible before attacking it! Can you argue that

$x_1 > 0 \quad x_2 > 0$ always?

$FOC_{x_1}: \beta x_1^{\beta-1} + \mu_1 - \lambda_s p_s - \lambda_b p_b = 0$

$FOC_{x_2}: \beta x_2^{\beta-1} + \mu_2 - \lambda_s - \lambda_b = 0$

$CS\mu_1: \mu_1 \geq 0,\ x_1 \geq 0,\ \mu_1 x_1 = 0$

$CS\mu_2: \mu_2 \geq 0,\ x_2 \geq 0,\ \mu_2 x_2 = 0$

$CS\lambda_s: \lambda_s \geq 0,\ p_s x_1 + x_2 \leq p_s m_1 + m_2,$
$\lambda_s (p_s x_1 + x_2 - (p_s m_1 + m_2)) = 0$

$CS\lambda_b: \lambda_b \geq 0,\ p_b x_1 + x_2 \leq p_b m_1 + m_2,$
$\lambda_b (p_b x_1 + x_2 - (p_b m_1 + m_2)) = 0$

Suppose both g_1 and g_2 bind, i.e. $x_1 = x_2 = 0$. Then, there are zero degrees of freedom, there are no (x_1, x_2) around $(x_1=0, x_2=0)$ such that both these constraints bind, and it is not necessary to verify that the second order condition holds at $(x_1=0, x_2=0)$

Suppose ~~that only~~ g_1 and not g_2 binds, i.e. $x_1 = 0$. Then, $x_1 = 0, \mu_1 > 0$, $x_2 > 0$, $\mu_2 = 0$. $u(x_1=0, x_2) = x_2^\beta$. ~~Since this~~ Given ~~$\neq$~~ $0 < \beta < 1$, by inspection, $u(x_1=0, x_2)$ is increasing in x_2, for all x_2. So $\max_{x_2} = \text{argmax}_{x_2}\ u(x_1=0, x_2)$. Only $(x_1=0, x_2 = \cancel{p_s}\, p_s m_1 + m_2)$ satisfies the FOCs and CSs.

$\cancel{\partial L}\ D_x L = (\partial L/\partial x_1,\ \partial L/\partial x_2) = \cancel{(\beta x_1^{\beta-1} + \mu_1 - \lambda_s p_s}$
$= (\beta x_1^{\beta-1} + \mu_1 - \lambda_s p_s - \lambda_b p_b,\ \beta x_2^{\beta-1} + \mu_2 - \lambda_s - \lambda_b)$

$$D_x^2 L = \begin{pmatrix} \partial^2 L/\partial x_1^2 & \partial^2 L/\partial x_1 \partial x_2 \\ \partial^2 L/\partial x_2 \partial x_1 & \partial^2 L/\partial x_2^2 \end{pmatrix} = \begin{pmatrix} \beta(\beta-1)x_1^{\beta-2} & 0 \\ 0 & \beta(\beta-1)x_2^{\beta-2} \end{pmatrix}$$

The relevant bordered Hessian

$$H = \begin{pmatrix} 0 & 1 & 0 \\ 1 & \beta(\beta-1)x_1^{\beta-2} & 0 \\ 0 & 0 & \beta(\beta-1)x_2^{\beta-2} \end{pmatrix}$$

? when are you evaluating it?

$$|H| = -1 \begin{vmatrix} 1 & 0 \\ 0 & \beta(\beta-1)x_2^{\beta-2} \end{vmatrix} = -\beta(\beta-1)x_2^{\beta-2} > 0$$

By the bordered Hessian test, $(x_1=0, x_2 = p_s m_1 + m_2)$ is a strict local maximum. X

Analogously, supposing that g_2 and not g_1 binds, only $(x_1 = p_b m_1 + m_2 / p_b,\ x_2 = 0)$ satisfies the FOCs and CSs. X

The relevant bordered Hessian

$$H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \beta(\beta-1)x_1^{\beta-2} & 0 \\ 1 & 0 & \beta(\beta-1)x_2^{\beta-2} \end{pmatrix}$$

$$|H| = \begin{vmatrix} 0 & \beta(\beta-1)x_1^{\beta-2} \\ 1 & 0 \end{vmatrix} = -\beta(\beta-1)x_1^{\beta-2} > 0$$

By the bordered Hessian test, $(x_1 = p_b m_1 + m_2 / \cancel{p_b}\, p_b,\ x_2 = 0)$ is a strict local maximum.

Maybe it is worth seeing whether the problem is a concave problem?.. Then FOC are sufficient...

Suppose only g_s binds, then $x_1, x_2 > 0$, $\mu_1 = \mu_2 = \lambda_b = 0$, $\lambda_s > 0$

$p_s x_1 + x_2 = p_s m_1 + m_2$

$\beta x_1^{\beta-1} - \lambda_s p_s = 0,\quad \beta x_2^{\beta-1} - \lambda_s = 0$

$\lambda_s = \beta/p_s\, x_1^{\beta-1} = \beta x_2^{\beta-1},\quad x_1^{\beta-1} = p_s x_2^{\beta-1},\ x_1 = p_s^{1/\beta-1} x_2$

$p_s p_s^{1/\beta-1} x_2 + x_2 = p_s m_1 + m_2$

$x_2 = (p_s m_1 + m_2)/(p_s^{\beta/\beta-1} + 1)$

$x_1 = (p_s m_1 + m_2)/\cancel{p_s}\,(p_s + p_s^{-1/\beta-1})$ ✓

You need to check the CS! when is this

a solution?

The relevant bordered Hessian

$$H = \begin{pmatrix} 0 & p_s & 1 \\ p_s & \beta(\beta-1)x_1^{\beta-2} & 0 \\ 1 & 0 & \beta(\beta-1)x_2^{\beta-2} \end{pmatrix}$$

$$|H| = -p_s \begin{vmatrix} p_s & 0 \\ 1 & \beta(\beta-1)x_2^{\beta-2} \end{vmatrix} + \begin{vmatrix} p_s & \beta(\beta-1)x_1^{\beta-2} \\ 1 & 0 \end{vmatrix}$$

$$= -p_s^2\beta(\beta-1)x_2^{\beta-2} - \beta(\beta-1)x_1^{\beta-2} > 0$$

By the bordered Hessian test,
$(x_1 = (p_s m_1 + m_2)/(p_s + p_s^{-1/\beta-1}),\ x_2 = (p_s m_1 + m_2)/(p_s^{\beta/\beta-1}+1))$
is a strict local maximum.

Suppose only g_b binds, then $x_1, x_2 > 0$, $\mu_1 = \mu_2 = \lambda_s = 0$, $\lambda_b > 0$

$p_b x_1 + x_2 = p_b m_1 + m_2$

$\beta x_1^{\beta-1} - \lambda_b p_b = 0$, $\beta x_2^{\beta-1} - \lambda_b = 0$

$\lambda_b = 1/p_b\ \beta x_1^{\beta-1} = \beta x_2^{\beta-1}$, $x_1^{\beta-1} = p_b x_2^{\beta-1}$, $x_1 = p_b^{1/\beta-1} x_2$

$p_b^{\beta/\beta-1} x_2 + x_2 = p_b m_1 + m_2$

$x_2 = (p_b m_1 + m_2)/(p_b^{\beta/\beta-1}+1)$

$x_1 = (p_b m_1 + m_2)/(p_b + p_b^{-1/\beta-1})$

check the CS!

By an analogous bordered Hessian test,
$(x_1 = (p_b m_1 + m_2)/(p_b + p_b^{-1/\beta-1}),\ x_2 = (p_b m_1 + m_2)/(p_b^{\beta/\beta-1}+1))$ is
a strict local maximum.

Suppose that only g_s and g_b bind, then $x_1 = m_1$ and $x_2 = m_2$
and it is not necessary to verify the SOC ~~are~~ is
satisfied.

The candidate optima are

$(x_1 = 0,\ x_2 = 0)$ ①

$(x_1 = 0,\ x_2 = p_s m_1 + m_2)$ ②

$(x_1 = p_b m_1 + m_2/p_b,\ x_2 = 0)$ ③

$(x_1 = (p_s m_1 + m_2)/(p_s + p_s^{-1/\beta-1}),\ x_2 = (p_s m_1 + m_2)/(p_s^{\beta/\beta-1}+1))$ ④

$(x_1 = (p_b m_1 + m_2)/(p_b + p_b^{-1/\beta-1}),\ x_2 = (p_b m_1 + m_2)/(p_b^{\beta/\beta-1}+1))$ ⑤

$(x_1 = m_1,\ x_2 = m_2)$ ⑥

~~By this~~ Since u is increasing in x_1, x_2, $(x_1 = 0, x_2 = 0)$ is
not a maximum.

~~Each of the remaining candidate optima correspond to~~
~~to one of the following points~~

~~It can be proven graphically that~~ $(x_1 = 0,\ x_2 = p_s m_1 + m_2)$
and $(x_1 = p_b m_1 + m_2/p_b,\ x_2 = 0)$ are not optima because at
these points, $MRS = -MU_1/MU_2 = -(x_1/x_2)^{\beta-1} \neq MRT$, which
is ~~either~~ equal to p_s at the former and p_b at the latter.

You could have argued this from the start!

$MRS|_m = -(m_1/m_2)^{\beta-1}$

~~The optimum is ④ iff~~

The optimum is ④ iff $-MRS|_m < p_s < p_b$

The optimum is ⑤ iff $-MRS|_m > p_b > p_s$

The optimum is ⑥ iff $p_s < MRS|_m < p_b$

Can you state these conditions in terms of the primitives?