Quantitative Economics Problem Set 4

1 Given that $u_i \perp\!\!\!\perp x_{1i}, \dots, x_{ki}$, $E[u_i|x_{1i},\dots,x_{ki}] \overset{E(u_i)=}{=} 0$ and $E[u_i^2|x_{1i},\dots,x_{ki}] = E(u_i^2)$, then by definition of homoskedasticity u_i is homoskedastic.

2a $Y=\beta_0+\beta_1 X+u$ where $Eu = EXu = 0$

$H_0: \beta_1 = 0$, $H_1: \cancel{\beta_1 \neq 0}\ \beta_1 > 0$

Compute the OLS estimator $\hat\beta_1$ for β_1 by ~~$\hat\beta_1 = \widehat{cov}$~~ $\hat\beta_1 = \widehat{cov}(Y,X)/\widehat{var}(X)$

Compute the OLS residual $\hat u = Y - \hat\beta_0 - \hat\beta_1 X$

If homoskedasticity is plausible, compute the sample variance s_X^2 ~~$\widehat{var}(X)$~~ and ~~$\widehat{var}(\hat u)$~~ $s_{\hat u}^2$ of X and $\hat u$, then compute the ~~homoskedasticity t-stat~~ standard error $se(\hat\beta_1) = n^{-1/2}\sqrt{\cancel{\widehat{var}(\hat u)/\widehat{var}(X)}\ s_{\hat u}^2/s_X^2}$. ~~Then, the~~

If homoskedasticity is not plausible, compute the sample variance ~~$\widehat{var}(X)$ and~~ s_X^2 ~~$\widehat{var}(X)$~~ and ~~$\widehat{var}(X\hat u)$~~ $s_{X\hat u}^2$ of X and $X\hat u$, then compute the standard error $se(\hat\beta_1) = n^{-1/2}\sqrt{\cancel{\widehat{var}(X\hat u)/[\widehat{var}(X)]^2}\ s_{X\hat u}^2/(s_X^2)^2}$.

Then compute the t-statistic $t = (\hat\beta_1 - 0)/se(\hat\beta_1)$.

Under the null, ~~given~~ by CLT, given a sufficiently large iid random sample, $t \overset{a}{\sim} N(0,1)$

Reject H_0 if ~~$|t| > c_\alpha$~~ $t > c_\alpha$ where $\alpha = 0.10 = \cancel{2}\Phi(-c_\alpha) \Rightarrow \cancel{c_\alpha = 1.64}\ c_\alpha = 1.28$

b i The type II error rate is the probability of failing to reject a false null.

When $\beta_1 = 0.01$, the ~~distr~~ ~~$(\hat\beta_1-\beta_1)/se(\hat\beta_1) \overset{a}{\sim} N(0,1)$~~, the sampling distribution of the t-statistic lies close to the sampling distribution of the t-statistic under the null, large ~~to~~ positive values of the t-statistic are relatively unlikely, hence the probability that the false null is rejected is low and the above test has low power.

For larger β_1, the sampling distribution of the t-statistic lies further from that under the null, large positive values of the t-statistic are more likely, hence the false null is more likely to be rejected and the above test has higher power.

ii When $\beta_1 = -1$, the sampling distribution of the t-statistic is ~~far lower~~ lies lower on the real number line than that under the null. Large positive values of the t-statistic are even less likely than under the null. The probability that the false null is rejected is very small. The above test has no power to detect such departures from the null.

c A two-sided test would have ~~be~~ less power to detect deviations in the positive direction. Analytically, this is because the critical values ~~as of~~ for a two-sided test ~~have great~~ are larger than those for a one-sided test at the same level of significance. Intuitively, and informally, this is because a positive t-statistic constitutes stronger evidence that the ~~para~~ parameter of interest is higher than hypothesised than it constitutes evidence that the parameter of interest is different from the hypothesised value.

A two-sided test has equal power when $\beta = 1$ as when $\beta = -1$.

3a $Y = \beta_0 + \beta_1 X + u$ where $Eu = EXu = 0$

$H_0: \beta_0 = 0$, $H_1: \beta_1 \neq 0$

Restricted model

~~$Y = \beta_0 + u$~~ $Y = \beta_{0,rs} + u_{rs}$

$TSS = \sum_{i=1}^n (Y_i - \bar Y)^2$

$$\begin{aligned} SSR_{rs} &= \sum_{i=1}^n \hat u_{rs}^2 \\ &= \sum_{i=1}^n (Y_i - \hat\beta_{0,rs})^2 \\ &= \sum_{i=1}^n (Y_i - \bar Y)^2 \\ &= TSS \end{aligned}$$

where the penultimate equality follows by construction of the linear regression model of Y (on no dependent variables).

Intuitively, the restricted model is the constant $\bar Y$, which is the best constant predictor of Y in the sample, then ~~SSR_{rs} is simply TSS~~. The residual in the restricted model is the displacement from the mean, so $SSR_{rs} = TSS$.

b $SSR_{un} = \sum_{i=1}^n \hat u_{un}^2$
$= \sum_{i=1}^n (Y_i - \hat\beta_{0,un} - \hat\beta_{1,un} X_i)^2$

$SSR_{rs} - SSR_{un} = \cancel{\sum_{i=1}^n (-\bar Y + \hat\beta_{0,un} - \hat\beta_{1,un} X_i)}$

$= \sum_{i=1}^n [(Y_i - \bar Y)^2 - (Y_i - \hat\beta_{0,un} - \hat\beta_{1,un} X_i)^2]$

$\hat\beta_1^2 = [\widehat{cov}(Y,X)/\widehat{var}(X)]^2$
$= [\sum_{i=1}^n (Y_i - \bar Y)(X_i - \bar X) / \sum_{i=1}^n (X_i - \bar X)^2]^2$

$SSR_{un} = \sum_{i=1}^n (Y_i - \hat\beta_{0,un} - \hat\beta_{1,un} X_i)^2$
$= \sum_{i=1}^n (Y_i - \bar Y + \hat\beta_{1,un}\bar X - \hat\beta_{1,un} X_i)^2$
$= \sum_{i=1}^n (Y_i - \bar Y)^2 + \hat\beta_1^2 \sum_{i=1}^n (X_i - \bar X)^2 - 2\hat\beta_1 \sum_{i=1}^n (Y_i - \bar Y)(X_i - \bar X)$

$SSR_{un} = \sum_{i=1}^{n} (Y_i - \hat{\beta}_{0,un} - \hat{\beta}_1 X_i)^2$
$= \sum_{i=1}^{n} (Y_i - (\bar{Y} - \hat{\beta}_1 \bar{X}) - \hat{\beta}_1 X_i)^2$
$= \sum_{i=1}^{n} (Y_i - \bar{Y} - \hat{\beta}_1 (X_i - \bar{X}))^2$
$= \sum_{i=1}^{n} (Y_i - \bar{Y})^2 + \sum_{i=1}^{n} \hat{\beta}_1^2 (X_i - \bar{X})^2$
$- 2\sum_{i=1}^{n} \hat{\beta}_1 (Y_i - \bar{Y})(X_i - \bar{X})$

$SSR_{rs} = TSS = \sum_{i=1}^{n} (Y_i - \bar{Y})^2$

$SSR_{rs} - SSR_{un} = 2\hat{\beta}_1 \sum_{i=1}^{n} (Y_i - \bar{Y})(X_i - \bar{X}) - \sum_{i=1}^{n} \hat{\beta}_1^2 (X_i - \bar{X})^2$

$\sum_{i=1}^{n} (Y_i - \bar{Y})(X_i - \bar{X}) = n\widehat{cov}(Y, X)$
$= n\hat{\beta}_1 \widehat{var}(Y, X)$
$= \hat{\beta}_1 \sum_{i=1}^{n} (X_i - \bar{X})^2$

$SSR_{rs} - SSR_{un} = 2\hat{\beta}_1^2 \sum_{i=1}^{n} (X_i - \bar{X})^2 - \hat{\beta}_1^2 \sum_{i=1}^{n} (X_i - \bar{X})^2$
$= \hat{\beta}_1^2 \sum_{i=1}^{n} (X_i - \bar{X})^2$

c $F = {}^{LR}/_q = {}^1/_q \; SSR_{rs} - SSR_{un} / SSR_{un} / n-k-1$
$= {}^{n-k-1}/_q \; SSR_{rs} - SSR_{un} / SSR_{un}$

$t = (\hat{\beta}_1 - 0) / se(\hat{\beta}_1)$
$= \hat{\beta}_1 / se(\hat{\beta}_1)$
$= \hat{\beta}_1 [\sum_{i=1}^{n} (X_i - \bar{X})^2]^{1/2} / \sqrt{1/n-k-1 \sum_{i=1}^{n} \hat{u}_i^2}$

$t^2 = \hat{\beta}_1^2 \sum_{i=1}^{n} (X_i - \bar{X})^2 / {}^1/_{n-k-1} \sum_{i=1}^{n} \hat{u}_i^2$
$= {}^{n-k-1}/_q \; \hat{\beta}_1^2 \sum_{i=1}^{n} (X_i - \bar{X})^2 / \sum_{i=1}^{n} \hat{u}_i^2$
$= {}^{n-k-1}/_q \; SSR_{rs} - SSR_{un} / SSR_{un} = F$

Noting that the number of restrictions, $q=1$ and that $\hat{u}_i$ in the equation for t denotes the residual of the unrestricted model.

d $F_{1,\infty} = {}^1/_1 \; \chi_1^2 = \sum_{i=1}^{1} Z^2 = Z^2$

For two-sided t-test, reject Ho iff $|t| > c_\alpha$

For F-test, reject Ho iff $F > c'_\alpha \Leftrightarrow t^2 > c'_\alpha$
$\Leftrightarrow |t| = c_\alpha$ for appropriate c_α, c'_α namely $c'_\alpha = c_\alpha^2$

4a $R^2 = 1 - {}^{SSR}/_{TSS} = 1 - {}^{TSS-ESS}/_{TSS} = 0.13394$
$\bar{R}^2 = 1 - {}^{n-1}/_{n-k-1} \; {}^{SSR}/_{TSS} = 1 - {}^{n-1}/_{n-k-1} \; {}^{TSS-ESS}/_{TSS}$
$= 0.13264$

Ho: coefficients on every regressor is zero
H1: coefficient on at least one regressor is non-zero

$F = {}^1/_q \; SSR_{rs} - SSR_{un} / SSR_{un}/n-k-1$
$= {}^{n-k-1}/_q \; SSR_{rs} - SSR_{un} / SSR_{un}$
$= {}^{n-k-1}/_q \; TSS - (TSS - ESS) / TSS - ESS$
$= {}^{6028-9-1}/_9 \; {}^{12851}/_{95948 - 12851}$
$= 103.41$

Under the null, $F \overset{a}{\sim} F_{9,\infty}$

Reject the null if $F > c_\alpha$

From the statistical table, $c_\alpha = 1.94$

Reject the null.

b $H_0: \beta = 0$
$H_1: \beta \neq 0$
where β is the population linear regression coefficient on teacher_exp.

t-Statistic
~~$t = (13.93 - 0)/$~~ $t = {}^{0.04 - 0}/_{0.02} = 0$

Under the null, $t \overset{a}{\sim} N(0,1)$

Reject Ho if $|t| > c_\alpha$
$\alpha = 0.05 = 2\Phi(t - c_\alpha) \Rightarrow c_\alpha = 1.96$

Reject Ho

c $H_0: \beta = 0$, $H_1: \beta > 0$, where β is the coefficient on small_class in the population linear regression of maths_test on the given regressors.

$t = {}^{0.66}/_{0.30} = 2.2$
Under the null, $t \overset{a}{\sim} N(0,1)$

~~$p = P(|t| > c_\alpha) = \Phi(t < -c_\alpha)$~~

~~$p = P(|N(0,1)| > t) = 2P(Z < t)$~~

~~$p = P(|N(0,1)| > |t|) = 2\Phi(-|t|) = 2(0.0139) = 0.0278$~~

~~$H_0: \beta = 0, H_1: \beta > 0$~~
$p = P(N(0,1) > t) = \Phi(-t) = 0.0139$

Under the null, the probability of observing a test statistic at least as unfavourable to the null as that actually observed is 0.0139. Reject the null at any level of significance greater than 0.0139.

d The required confidence interval is
$C = [-0.57 - c_\alpha(0.11), -0.57 + c_\alpha(0.11)]$
where $\alpha = 0.01 = 2\Phi(-c_\alpha) \Rightarrow c_\alpha = 2.575$
$C = [-0.85325, -0.28675]$

The interval C contains the population regression coefficient on summer_baby with 99% probability.

e The required elasticity is $\beta \; \widehat{teacher\_exp} / \widehat{maths\_test}$ where β is the population linear regression coefficient of teacher_exp. ~~Neglecting~~ This is estimated by $\hat{\beta} \; \widehat{teacher\_exp} / \widehat{maths\_test}$. ~~Neglecti~~
The estimate is $0.04 \; {}^{13.93}/_{61.81} = 0.0090147$. Neglecting sampling variation of sample means, this has

standard error $\frac{13.93}{61.81}$ 0.02 = 0.0045074.

The required confidence interval is
$C = [\hat{\varepsilon} - c_\alpha se(\hat{\varepsilon}), \hat{\varepsilon} + c_\alpha se(\hat{\varepsilon})]$
where $\alpha = 0.10 = 2\Phi(-c_\alpha) \Rightarrow c_\alpha = 1.645$
$C = [0.0090147 - 1.645(0.0045074),$
$\quad 0.0090147 + 1.645(0.0045074)]$
$= [0.0016000, 0.016429]$

f On average black students had scores 1.53 points lower than white students, holding the remaining observed determinants constant. On average, students who were neither black nor white had scores 0.90 points higher than white students, holding the remaining observed determinants constant.

Let β denote the coefficient on black in the population regression model
$H_0: \beta = 0$, $H_1: \beta \neq 0$
$t_n(\beta^*) = (\hat{\beta} - \beta^*)/se(\hat{\beta}) = -1.53/0.16 = -9.5625$
Under the null, $t_n(\beta^*) \overset{a}{\sim} N(0,1)$
$p = P(|t_n(\beta^*)| > |t^{act}|)$
$= P(|N(0,1)| > 9.5625)$
$= 2\Phi(-9.5625)$
≈ 0

Under the null, the probability of observing a test statistic as unfavourable to the null as that actually observed is ~~almost neg~~ vanishingly low. Reject the null at any reasonable level of significance. There is strong evidence that an ethnic difference in math test scores between black students and white students exists in the population.

Repeat the above for the coefficient on other_non_~~white~~ in the population regression model β.
$H_0: \beta = \beta^* = 0$, $H_1: \beta \neq \beta^*$
$t_n(\beta^*) = (\hat{\beta} - \beta^*)/se(\hat{\beta}) = 0.90/0.59 = 1.5254$
Under the null, $t_n(\beta^*) \overset{a}{\sim} N(0,1)$
$p = P(|t_n(\beta^*)| > |t^{act}|)$
$= P(|N(0,1)| > 1.5254)$
$= 2\Phi(-1.5254)$
$= 2(0.0637)$
$= 0.1274$

Under the null, the probability of observing a test statistic at least as unfavourable to the null as that actually observed is 0.1274. Reject the null at any level of significance greater than 0.1274. Fail to reject the null at the conventional 10%, 5%, 1% levels of significance. There is limited evidence that an ethnic difference in math test scores ~~exists~~ between other non white and non white students exists in the population.

g HM education secretary's proposal is premised on a causal interpretation of the negative coefficient on free-lunch in the given regression. Such an interpretation is appropriate only if the OLS regression coefficient is a consistent estimator of the ~~causal~~ corresponding coefficient in the causal model of math test scores, which in turn is only if free_lunch is uncorrelated with other unobserved causal determinants of math test. This is implausible. One such determinant is access to educational resources and opportunities outside of school. This is plausibly highly correlated with household income, which in turn is highly correlated with free lunch, supposing that students who receive free lunch are beneficiaries of financial assistance programmes intended to benefit underprivileged students. Then the coefficients in the population linear regression model do not coincide with those of the causal model. OLS regression is consistent for the former but not the latter. The free lunch variable is intended to serve as a proxy for household income and ~~externa~~ access to external opportunities and resources, it is not a direct causal determinant of math test scores, hence a causal interpretation is inappropriate.

5a $\forall i: C_i + T_i + R_i = 1$

bi Each of β_X and γ_X is equal to the average difference in wage associated with a ~~larg~~ higher (by one unit) number of years of experience, holding the type of area of residence constant.

ii β_C is equal to the average difference in hourly wage between an ~~indi~~ individual in a city and an individual in a rural area, holding years of experience (and trivially implicitly, whether an individual ~~lives~~ lives in a town) constant).

γ_C is equal to the average difference in hourly wage between an individual in a city and an individual in a town, holding years of experience (and whether an individual lives in a rural area constant).

c $W_i = \beta_0 + \beta_X X_i + \beta_C C_i + \beta_T T_i + u_i$
$= \beta_0 + \beta_X X_i + \beta_C C_i + \beta_T(1 - C_i - R_i) + u_i$
$= \beta_0 + \beta_T + \beta_X X_i + (\beta_C - \beta_T)C_i - \beta_T R_i + u_i$
where $Eu_i = 0$, $EX_iu_i = EC_iu_i = ER_iu_i = 0$ (given that $EC_iu_i = ET_iu_i = 0$ and R_i is a linear function of C_i, T_i). Then, the above is a linear regression model of W_i on X_i, C_i, R_i, and its coefficients necessarily

coincide with those of
$W_i = \gamma_0 + \gamma_x x_i + \gamma_C C_i + \gamma_R R_i + v_i$,
hence $-\beta_T = \gamma_R$, $\beta_T = -\gamma_R$.

~~In~~ β_T is the average difference in hourly wage between ~~an~~ ~~individual~~ a town resident and a rural resident. This is equal ~~to~~ in magnitude and of opposite sign to the average difference in hourly wage between a rural resident and a town resident, given by γ_R.

d $H_0: \beta_C = \beta_T = 0$
$H_1: \beta_C \neq 0$ or $\beta_T \neq 0$

Fit the unrestricted model
$W_i = \beta_{0,un} + \beta_{x,un} x_i + \beta_C C_i + \beta_T T_i + u_{un,i}$

Fit the restricted model
$W_i = \beta_{0,rs} + \beta_{x,rs} x_i + u_{rs,i}$

Compute the residuals of each model
$\hat{u}_{un,i} = W_i - \hat{\beta}_{0,un} - \hat{\beta}_{x,un} x_i - \hat{\beta}_C C_i - \hat{\beta}_T T_i$
$\hat{u}_{rs,i} = W_i - \hat{\beta}_{0,rs} - \hat{\beta}_{x,rs} x_i$

~~Supposing that residuals are homoskedastic,~~

Compute the F statistic
$F = {}^{CR}/_{q} = {}^{1}/_{q}\,(SSR_{rs} - SSR_{un})/(SSR_{un}/n-k-1)$
$= {}^{n-k-1}/_{q}\,(SSR_{rs} - SSR_{un}/SSR_{un})$
where n is the sample size, $k=3$ is the number of regressors, $q=2$ is the number of restrictions, $SSR_{rs} = \sum_{i=1}^{n} \hat{u}_{rs,i}^2$, $SSR_{un} = \sum_{i=1}^{n} \hat{u}_{rs,i}^2$.

Reject the null if $F > c_\alpha$, where c_α is some appropriate critical value, drawn from the $F_{q,\infty}$ distribution at the level of significance α.

e Fit the model
$W_i = \beta_0 + \beta_x X_i + \beta_{Cx} C_i X_i + \beta_{Tx} T_i X_i + \beta_C C_i + \beta_T T_i + u_i$

~~Supposing that orthogonality is satisfied, β_x, β_{Cx}, and β_{Tx} respectively consistently estimate the returns to experience for an a rural resident, for a city resident, and for a town resident.~~

Perform an F test at level of significance α of $H_0: \beta_{Cx} = \beta_{Tx} = 0$ against $H_1: \beta_{Cx} = 0$ or $\beta_{Tx} = 0$.

β_{Cx} is equal to the average difference between a city resident and a rural resident of the average difference in hourly wage associated with a higher number of years of experience. Supposing orthogonality holds, this is the difference in return to experience between a city resident and a rural resident.

β_{Tx} can be given an analogous interpretation.

6 $Y_i = A_i L_i^{\alpha} K_i^{\beta} e^{\varepsilon_i}$
$\Rightarrow \ln Y_i = \ln A_i + \alpha \ln L_i + \beta \ln K_i + \varepsilon_i$
$= \ln A + \alpha \ln L_i + \beta \ln K_i + \varepsilon_i$
~~where $E\varepsilon_i = EL_i\varepsilon_i = EK_i\varepsilon_i = 0$~~
~~Then, $\ln A$, α, and β are the population linear regression parameters of the regression of $\ln Y_i$ on $\ln L_i$ and $\ln K_i$. So these can be consistently estimated by OLS regression of $\ln Y_i$ on $\ln L_i$ and $\ln K_i$.~~
$= \ln A + k + \alpha \ln L_i + \beta \ln K_i + \varepsilon_i - k$
where $k = E\varepsilon_i$, then given $\varepsilon_i \perp\!\!\!\perp K_i, L_i$, and supposing iid random sampling, $\forall i \neq j$ $\varepsilon_i \perp\!\!\!\perp K_j, L_j$, hence $E\varepsilon_i \perp\!\!\!\perp K_i, L_i$ and $\varepsilon_i - k \perp\!\!\!\perp K_i, L_i$. Then, the above satisfies orthogonality, and coincides with a population linear regression model of $\ln Y_i$ on $\ln L_i$ and $\ln K_i$. Then, the coefficients α and β are consistently estimated by the OLS regression coefficients on $\ln L_i$ and $\ln K_i$.

6 The production function exhibits constant returns to scale iff $\alpha + \beta = 1$. Conduct the following hypothesis test.

$H_0: \alpha + \beta = 1$, $H_0: \alpha + \beta \neq 1$

Under the null,
$\ln Y_i = \ln A_i + k + \alpha \ln L_i + (1-\alpha)\ln K_i + \varepsilon_i - k$
$= \ln A_i + k + \alpha(\ln L_i - \ln K_i) + \ln K_i + \varepsilon_i - k$
$\Rightarrow \ln Y_i - \ln K_i = \ln A_i + k + \alpha(\ln L_i - \ln K_i) + \varepsilon_i - k$

Fit the restricted model
$\ln Y_i - \ln K_i = \beta_{0,rs} + \beta_{1,rs}(\ln L_i - \ln K_i) + u_{rs,i}$
(Regress $(\ln Y_i - \ln K_i)$ on $(\ln L_i - \ln K_i)$)

Fit the unrestricted model
$\ln Y_i = \beta_{0,un} + \beta_{1,un}(\ln L_i) + \beta_{2,un}(\ln K_i) + u_{un,i}$

Compute the residuals
$\hat{u}_{rs} = \ln Y - \ln K - \hat{\beta}_{0,rs} - \hat{\beta}_{1,rs}(\ln L - \ln K)$
$\hat{u}_{un} = \ln Y - \hat{\beta}_{0,un} - \hat{\beta}_{1,un} \ln L - \hat{\beta}_2 \ln K$

Compute the F statistic
$F = {}^{CR}/_{q} = {}^{1}/_{q}\,(SSR_{rs} - SSR_{un})/(SSR_{un}/n-k-1)$
$= {}^{n-k-1}/_{q}\,(SSR_{rs} - SSR_{un})/SSR_{un}$
where n is the sample size, $k=2$ is the number of regressors, $q=1$ is the number of restrictions, $SSR_{rs} = \sum_{i=1}^{n} \hat{u}_{rs,i}^2$, $SSR_{un} = \sum_{i=1}^{n} \hat{u}_{un,i}^2$

Under the null, $F \overset{a}{\sim} F_{1,\infty}$ (supposing homoskedastic errors)

Reject H_0 if $F > c_\alpha$, where c_α is some critical value drawn from $F_{1,\infty}$ at level of significance α.

~~Alternatively, a t-test of~~

$$\ln Y = \ln A + k + \alpha \ln L + \beta \ln K + \varepsilon - k$$
$$\cancel{= \ln A + k + \alpha \ln L + \varepsilon + \alpha \ln}$$
$$= \ln A + k + \alpha \ln L + (1-\alpha)\ln K + (\beta - (1-\alpha))\ln K + \varepsilon - k$$
$$= \ln A + k + \alpha(\ln L - \ln K) + \ln K + (\alpha + \beta - 1)\ln K + \varepsilon - k$$
$$\Rightarrow \ln Y - \ln K = \ln A + k + \alpha(\ln L - \ln K) + (\alpha + \beta - 1)\ln K + \varepsilon - k$$

Alternatively a t-test of $H_0: \gamma_2 = 0$ against $H_1: \gamma_2 \neq 0$ in the regression

$$\ln Y - \ln K = \gamma_0 + \gamma_1(\ln L - \ln K) + \gamma_2 \ln K + u$$

constitutes a heteroskedasticity robust test of the hypothesis that there are constant returns to scale.

c Yes. Orthogonality fails, i.e. there is an unobserved determinant of Y that is correlated with L and K, hence the population regression model does not coincide with the causal model. OLS regression is consistent for the former but not the latter.

d For arbitrary i, ~~$A_{i,2015}/A_{i,2014}$~~

$$Y_{i,2015}/Y_{i,2014} = A_{i,2015} L_{i,2015}^{\alpha} K_{i,2015}^{\beta} e^{\varepsilon_{i,2015}} / A_{i,2014} L_{i,2014}^{\alpha} K_{i,2014}^{\beta} e^{\varepsilon_{i,2014}}$$
$$= (L_{i,2015}/L_{i,2014})^{\alpha} (K_{i,2015}/K_{i,2014})^{\beta} e^{\varepsilon_{i,2015} - \varepsilon_{i,2014}}$$

$$\ln(Y_i/Y_i') = \alpha \ln L_i/L_i' + \beta \ln K_i/K_i' + (\varepsilon_i - \varepsilon_i')$$

where Y_i and Y_i' abbreviate $Y_{i,2015}$, $Y_{i,2014}$, and similarly for L, K, ε.

Given $\varepsilon_i \perp\!\!\!\perp L_i, K_i$, $\varepsilon_i' \perp\!\!\!\perp L_i', K_i'$, supposing further that $\varepsilon_i \perp\!\!\!\perp L_i', K_i'$, $\varepsilon_i' \perp\!\!\!\perp L_i, K_i$, we have that $(\varepsilon_i - \varepsilon_i') \perp\!\!\!\perp \ln L_i/L_i', \ln K_i/K_i'$, then orthogonality holds, and α and β are consistently estimated by OLS regression of $\ln Y_i/Y_i'$ on $\ln L_i/L_i'$ and $\ln K_i/K_i'$.

7ai Let $W_1 = X_1$, $W_2 = X_1^2$, $W_3 = X_2$, $W_4 = X_1X_2$, then the given causal model can be written as

$$Y = \beta_0 + \beta_1 W_1 + \beta_2 W_2 + \beta_3 W_3 + \beta_4 W_4 + u,$$

hence the given causal model is linear in the ~~parameters~~ parameters.

ii Let $W_1 = \ln X_1$, $W_2 = X_2$, $W_3 = X_2 \ln X_1$, then the given causal model can be written as

$$Y = \beta_0 + \beta_1 W_1 + \beta_2 W_2 + \beta_3 W_3 + u,$$

hence the given causal model is linear in the parameters.

bi
$$Eu = EX_1u = EX_1^2u = EX_2u = EX_1X_2u = 0$$
$$E[u \mid X_1, X_2] = 0$$

ii
$$Eu = Eu\ln X_1 = EX_2u = EX_2u\ln X_1 = 0$$
$$E[u \mid X_1, X_2] = 0$$

Date