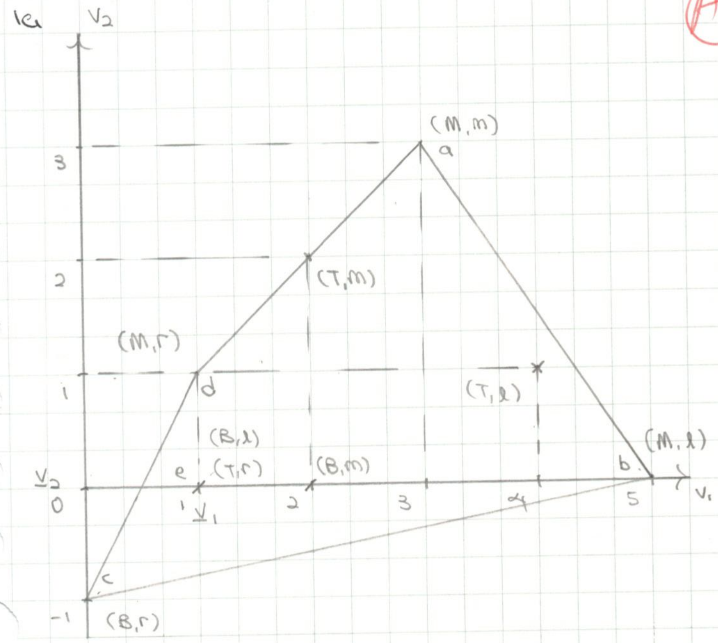
Game Theory Problem Set 8

1a



The set of feasible payoffs is represented by abcd.

b

	l	m	r
T	1	$\underline{2}$	0
	4	2	$\underline{1}$
M	0	$\underline{3}$	1
	$\underline{5}$	$\underline{3}$	$\underline{1}$
B	$\underline{0}$	$\underline{0}$	-1
	1	2	0

Best responses underlined. By inspection, P1 minmaxes P2, i.e. chooses a_1 to minimise ~~the~~ $\max_{a_2} u_2(a_2, a_1)$, ~~by choosing $a_1 = B$~~, i.e. minimise u_2 given P2 best responds to a_1 by playing B, then ~~≠~~ P2's minmax payoff, $\underline{v}_2 = 0$. By inspection, P2 minmaxes P1 by playing r, then $\underline{v}_1 = 1$.

(v_1, v_2) is individually rational iff $v_1 \geq \underline{v}_1$ and $v_2 \geq \underline{v}_2$

The set of feasible and individually rational payoffs is ~~given by~~ represented by abed.

c By inspection, the only pure NE is M,m, where players play mutual best responses.

Suppose there is some hybrid or mixed NE $\sigma^* = (\sigma_1^*, \sigma_2^*)$ such that P1 mixes only T and M. By definition of NE, P1 has no profitable deviation. Then $\pi_1(T, \sigma_2^*) =$ ~~$\pi_2(T, \sigma$~~ $\pi_1(M, \sigma_2^*) \geq \pi_1(B, \sigma_2^*)$. ~~Denote~~ Let $\sigma_1^* = p_T T + p_M M + (1 - p_T - p_M) B$ and $\sigma_2^* = p_l l + p_m m + (1 - p_l - p_m) r$.
$4p_l + 2p_m + 1(p_l - p_m) = 5p_l + 3p_m + 1(p_l - p_m) \geq p_l + 2p_m \Leftrightarrow p_l = p_m = 0$
$\Leftrightarrow \sigma_2^* = r$. Then, by definition of NE, $\pi_2(r, \sigma_1^*) \geq \pi_2(l, \sigma_1^*), \pi_2(m, \sigma_1^*)$. By inspection, if $p_T + p_M = 1$, m is strictly dominant for P2, then $\pi_2(m, \sigma_1^*) > \pi_2(r, \sigma_1^*)$.
By reductio, there is no NE where P1 mixes only T and M.

A

~~Generally~~, look for easiest SPE to construct. This is usually by punishment by Nash reversion

Convex hull implies not only mixing but also correlated mixing, assumes "access to common public signal"

Theorem: 2 P zero-sum games → minmax ≡ maxmin

Generally minmaxing is insufficient punisher must be rewarded for punishing

It domina: It dominates

would have been faster to observe M >₁ B and m >₂ r then M >₁ T and m >₂ l

M strictly dominates B, so ~~P1~~ at NE P1 never mixes M and B (with or without T), so there is no fully mixed equilibrium, and P1 ~~doe~~ never plays pure B at equilibrium.

P2 has strict best responses to each of pure T and pure M, so P2 does not mix in equilibrium against pure T or against pure M. So there are no hybrid NE.

The only NE is the pure NE (M,m)

d ~~By inspection of the diagram in a, some~~ ~~mixed~~ ~~hybrid strategy profile $(M, p_\ell \ell + (1-p_\ell)m)$~~

Let $v=(4,1)$, $w=(2,1/3)$, $w^1=(\cancel{2\tfrac{1}{3}},\ 2, 2/3)$, $w^2=(2\tfrac{1}{3}, 1/3)$

~~(T,L) yields~~

$v = (\pi_1(T,L), \pi_2(T,L))$

$w = (\pi_1(T, \tfrac{2}{3}r + \tfrac{1}{3}\ell), \pi_2(T, \tfrac{2}{3}r + \tfrac{1}{3}\ell))$

$w^1 = (\pi_1(T, \tfrac{8}{15}r + \tfrac{4}{15}\ell + \tfrac{1}{5}m), \pi_2(T, \tfrac{8}{15}r + \tfrac{4}{15}\ell + \tfrac{1}{5}m))$

$w^2 = (\pi_1(\tfrac{1}{3}T + \tfrac{2}{3}M, \tfrac{1}{3}\ell, \tfrac{2}{3}r), \pi_2(\tfrac{1}{3}T + \tfrac{2}{3}M, \tfrac{1}{3}\ell + \tfrac{2}{3}r))$.

∃ simpler punishment

: exists simpler punishment

Possible to punish P2 at (B,r) for one period with $\delta \geq 1/2$

Usually "spot punishment"

~~The SPE is the following strategy profile~~

~~Each player plays his part of (T, ℓ) in the first stage and each subsequent stage iff no player deviates. If player 1 deviates, in period,~~ then in the next T periods, each ~~player plays his part of (M,r).~~ If no ~~player deviates~~ in ~~these T periods~~

$m^1 = (1,1) = (\pi_1(M,r), \pi_2(M,r))$

$m^2 = (2,0) = (\pi_1(B,m), \pi_2(B,m))$

Necessary to give delta?

Need to check there is a δ that works

What are the strict requirements for SPE?

The SPE is the following strategy profile

Phase I

In period 1 and each subsequent period, play the strategy profile corresponding to v iff no player previously deviated. If player i deviates in period t, begin phase II^i in period $t+1$

Phase II^i

~~In the f For T periods~~ In each of T periods, play the strategy profile corresponding to m^i iff no player deviated within this phase. If player j deviates in period t, begin phase II^j in period $t+1$. At the end of this phase, if no player deviated within this phase, begin ~~period~~ phase III^i

Phase III^i

In each period, play the strategy profile corresponding to w^i iff no player deviated within this phase. If player k deviates in period t, begin phase II^k in period $t+1$.

One gain other loses

These payoffs are no greater than ~~that~~ those of the one shot game. The payoffs of the one shot game are Pareto optimal.

e The NE payoff pair of the one shot game is sustainable in an SPE of the repeated game (by replacing v with $v' = (3,3) = (\pi_1(M,m), \pi_2(m,M))$. The SPE payoff pair of the repeated game in (d), (4,L) is not achievable in NE of the one shot game. Repetition makes it possible to achieve any feasible and strictly individually rational payoffs in SPE (for sufficiently patient players, in an infinite time horizon, by Fudenberg and Maskin (1986)'s theorem). So in cases where "cooperation" yields ~~a Pareto op~~ payoffs that Pareto dominate the one shot NE payoffs, repetition makes it "easier" to achieve such "cooperation" payoffs. In this case, the one shot NE payoffs are Pareto optimal, so repetition is "redundant".

Date

2a

	L	R
T	$\underline{5}$, $\underline{2}$	-5, 0
M	-2, 0	$\underline{2}$, $\underline{2}$
B	-1, $\underline{6}$	-1, $\underline{6}$

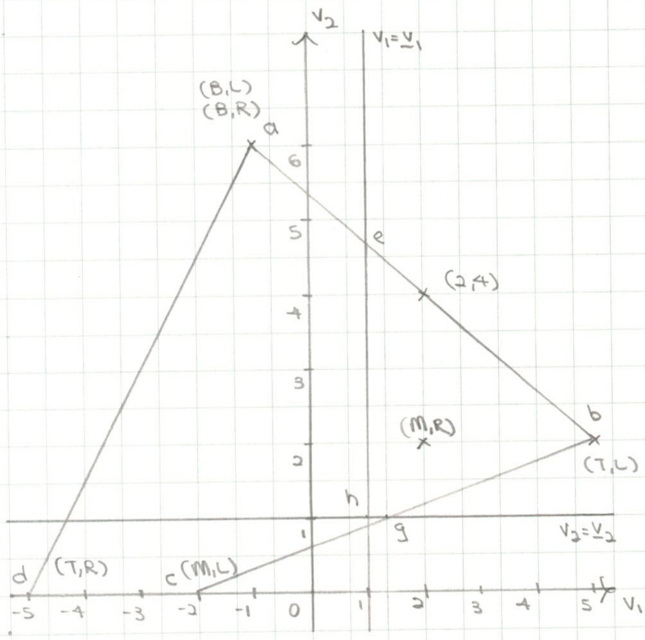
$\longrightarrow$ $\tfrac{1}{2}T+\tfrac{1}{2}M >_1 B$

	L	R
T	$\underline{5}$, $\underline{2}$	-5, 0
M	-2, 0	$\underline{2}$, $\underline{2}$

Best responses underlined. By inspection, P1 minmaxes P2 by playing $\sigma_1 = \tfrac{1}{2}T+\tfrac{1}{2}M$, P2 best responds by playing $\sigma_2 = \tfrac{1}{2}L+\tfrac{1}{2}R$ and has minmax payoff $\underline{v}_2 = 1$. By inspection, P2 minmaxes P1 by playing $\sigma_2 = \tfrac{1}{2}L+\tfrac{1}{2}R$ ~~and~~, P1 best responds by playing $\sigma_1 = \tfrac{1}{2}T+\tfrac{1}{2}M$ and has minmax payoff $\underline{v}_1 = 0$.

is this sufficient? what sort of "steps" need to be shown for finding minmax payoffs?

Spot and explain is fine
Otherwise, solve max and min with parameters
p for ~~p~~ mixing



The set of feasible payoff pairs is represented by abcd. The set of feasible and individually rational payoff pairs is represented by ebgh.

b By inspection of the payoff table above, $\tfrac{1}{2}T+\tfrac{1}{2}M >_1 B$, so in the one shot game, ~~rational P1~~ at NE, ~~the~~ rational P1 plays B with zero probability. The one shot game payoff table reduces to ~~the~~ the table on the right.

By inspection, the pure NE are (T, L) and (M, R).

Suppose P1 mixes at NE, then by definition of NE, P1 has no profitable deviation, then $\pi_1(T,\sigma_2^*) = \pi_1(M,\sigma_2^*) \geq \pi_1(B,\sigma_2^*) = -1$. Let P_T, P_M, and P_L denote the ~~respective~~ probabilities assigned by NE strategies in the obvious way. Given P1 plays B with zero probability, $P_M = 1-P_T$. $5P_L - 5(1-P_L) = -2P_L + 2(1-P_L) \geq -1 \Rightarrow P_L = \tfrac{1}{2}$. P2 mixes, then by definition of NE, $\pi_2(L,\sigma_1^*) = \pi_2(R,\sigma_1^*) \Rightarrow 2P_T = 2(1-P_T) \Rightarrow P_T = \tfrac{1}{2}$. If P1 mixes, so does P2 and vice versa, so there are no hybrid NE. The mixed NE is $(\tfrac{1}{2}T+\tfrac{1}{2}M,\ \tfrac{1}{2}L+\tfrac{1}{2}R)$.

minmax payoffs is NE

The payoffs at each NE of the one shot game are as follows

$\sigma^* = (T, L) \Rightarrow \pi_1(\sigma^*), \pi_2(\sigma^*) = 5, 2$
$\sigma^* = (M, R) \Rightarrow \pi_1(\sigma^*), \pi_2(\sigma^*) = 2, 2$
$\sigma^* = (\frac{1}{2}T + \frac{1}{2}M, \frac{1}{2}L + \frac{1}{2}R) \Rightarrow \pi_1(\sigma^*), \pi_2(\sigma^*) = 0, 1$

c ~~The requi~~
~~Let v_i~~

The required SPE is as follows.

Collaboration
~~In the first period, each player plays his part of (T,L), then in the second period, each player plays his part of (B,R), and players~~ In the first period and in each subsequent odd period, iff no player previously deviated from the SPE, each player plays his part of (T,L). In each even period, iff no player previously deviated from the SPE, each player plays his part of (B,R). If player i deviates in period t, start punishment^i in period t+1

Is there any diffence between T,L B,L alternation and T,L B,R alternation? Yes: P1 has greater incentive to deviate from (B,L)

The solution: T,L ⇔ B,R then Nash reversion to M,R ~~if any deviation~~ forever.

Thought process: Nash reversion → which Nash?

Punishment^i
~~In period~~ In each of the first T periods of this phase, each player plays his part of the strategy profile that minimaxes player i, namely $(\frac{1}{2}T + \frac{1}{2}M, \frac{1}{2}L + \frac{1}{2}R)$ (for all i). If player j deviates ~~st~~ in period t, ~~begin~~ start punishment j. If no player deviated within the first T punishment ~~Reconciliation~~ periods, start reconciliation^i in ~~period~~ the ~~T+1 th~~ the next period.

This seems difficult / imprecise for mixed strategies, how would a deviation be recognised?

Can just use Nash

Reconciliation^i
In each ~~of~~ period of reconciliation i, ~~play the~~ each player plays his part of the strategy profile that yields payoffs w^i. If player ~~j~~ k deviates ~~start~~ in ~~pe~~ period t, start punishment^k in period t+1.

$w^1 = (4/3, 5/3)$, $w^2 = (5/3, 4/3)$

Is it necessary to find the mix that yields these payoffs? That seems like it would be very tedious.

Is the $w^1 = (w_1, w_2+\varepsilon)$, $w^2 = (w_1+\varepsilon, w_2)$ formulation necessary? Or is it sufficient to have $\underline{v}_1 < w_1 < w_1' < v_1$, $\underline{v}_2 < w_2 < w_2' < v_2$ and $w^1 = (w_1, w_2')$, $w^2 = (w_1', w_2)$

d In what follows, superscripts denote players, subscripts denote periods, and * denotes equilibrium values. By inspection, $\frac{1}{2}T + \frac{1}{2}M > B$, ~~so~~ since total payoff Π is simply the sum of individual period payoff, P1 in t=2 maximises Π^1 by choosing σ_2^1 to maximise π_2^1, then σ_2^{1*} assigns zero probability to B. Then, by inspection of the ~~payoff~~ one shot ~~payo~~ game payoff table, $\pi_2^2 \leq 2$. Then, given $\Pi^{2*} = 8$, $\pi_1^{2*} \geq 6$, so $\sigma_1^{1*} = B$. and $\pi_1^{1*} = -1$. Then, given $\Pi^{1*} = 2.5$, we have $\pi_2^{1*} = \cancel{2.5}$ 3.5

Last period: NE "because there is no future"

By inspection, $\pi_2^{1*} = 2.5$ is achievable in NE of the period 2 stage game in a correlated mixed equilibrium where (T,L) has probability ½ and (M,R) has probability ½. We conjecture that correlated mixing is accomplished by correlating on the basis of P2's action in period 1. Then, ~~the~~ one strategy profile that achieves the given expected payoffs is as follows:

~~$\sigma^{2*}_2 = \frac{1}{2}L + \frac{1}{2}R$, σ^{*}~~

$\sigma^{2*} = \sigma^{2*}_1, \sigma^{2*}_2$

$\sigma^{2*}_1 = \frac{1}{2}L + \frac{1}{2}R$

~~$\sigma^{2*}_2 = L$ if $a^2_1 = L$, R if $a^2_1 = R$~~

~~$\sigma^{1*} = \sigma^{1*}_1, \sigma^{1*}_2$~~

~~$\sigma^{1*}_1 = B$~~

~~$\sigma^{1*}_2 = T$ if $a^2_1 = L$, M if $a^2_1 = R$~~

$\sigma^{2*}_2 = L$ if $a^2_1 = L$ and $a^1_1 = B$
R if $a^2_1 = R$ and $a^1_1 = B$
$\frac{1}{2}L + \frac{1}{2}R$ if $a^1_1 \neq B$

$\sigma^{1*} = \sigma^{1*}_1, \sigma^{1*}_2$

$\sigma^{1*}_1 = B$

$\sigma^{1*}_2 = T$ if $a^2_1 = L$ and $a^1_1 = B$
M if $a^2_1 = R$ and $a^1_1 = B$
$\frac{1}{2}T + \frac{1}{2}M$ if $a^1_1 \neq B$

On equilibrium path payoffs are $\Pi^* = (2.5, 8)$

If P1 optimally deviates only in $t=1$, $\pi^1_1 = 0$, $\pi^1_2 = 0$, $\Pi^1 = 0$

~~If P1 optimally deviates in $t=2$, $\pi^1_1 = -1$, $\pi^1_2 =$~~

P1 has no incentive to deviate in only period 2 since given non-deviation in period 1, P1's eqm strategy profile is a best response to P2's.

P2 has no incentive to deviate in ~~per~~ only period 1 since ~~it receives the maximum feasible~~ this has no effect on its total payoff.
P2 has no incentive to deviate in only period 2 for the same reason as P1

By the one shot deviation principle, this strategy profile is indeed an SPE.

Backward induction does not yield "unraveling"

Proof of one shot deviation principle

If no one shot deviation is profitable, no finite deviation is profitable, by induction.

Because of discounting, any deviation must pay off within finite periods.

Does not need to be proven

Date

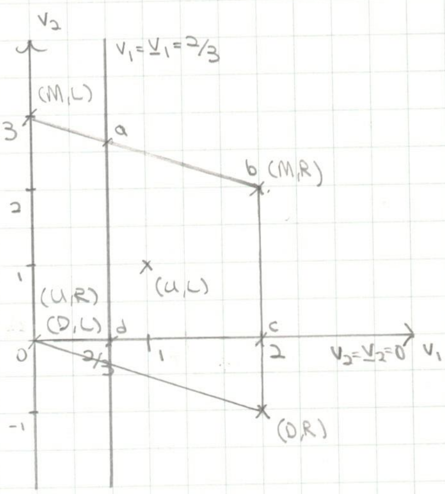
3a

	L	R
U	$\underline{1}$, $\underline{1}$	0, 0
M	0, $\underline{3}$	$\underline{2}$, 2
D	0, $\underline{0}$	$\underline{2}$, -1

Best responses underlined.

By inspection, $\forall a_2 \in \{L, R\}: \pi_2(a_2, D) < \pi_2(a_2, \overset{U}{~~M~~}) < \pi_2(a_2, U)$, i.e. P2 ~~is~~ receives the lowest payoff, regardless of P2's action if P1 plays D, then P1 minmaxes P2 by playing D with certainty, P2 best responds by playing L, so P2's minmax payoff $\underline{v}_2 = 0$. By inspection, P2 minmaxes P1 by playing $\frac{2}{3}L + \frac{1}{3}R$, P1 best responds by ~~mixing over~~ playing any ~~(poten~~ (potentially degenerate) mixed strategy, so P1's minmax payoff $\underline{v}_1 = 2/3$.

R is dominated, P2 never plays R, P1 minmaxes P2 by choosing ~~a~~ st $u_2(a_1, L)$ is a minimum.



The set of feasible and individually rational payoffs is represented by abcd.

b ~~a~~ $L >_2 R$, so rational P2 plays L with certainty in the stage game. P1 best responds by playing U. The only NE is the pure NE (U, L).

c The required SPE is the strategy profile such that each player plays the grim trigger strategy under which each player plays his part of (M, R) iff there has been no deviation, and plays his part of (U, L) ~~thee~~ otherwise.

Check of conds

On equilibrium path, $\pi_1 = \pi_2 = 2$ (in ADV)

P1's one shot deviation yields $\pi_1' \leq (1-\delta)2 + \delta(1) = 2 - \delta$ ~~P1's P2's one shot deviation yields~~ $< \pi_1$ (for all $\delta > 0$)

P2's one shot deviation yields $\pi_2' \leq (1-\delta)3 + \delta(1) = 3 - 2\delta < \pi_2$

By the one shot deviation principle, the above strategy profile is an SPE.

d The required SPE is the strategy profile such that

In the first period and in each subsequent period, ~~if~~ if no player previously deviated, each player plays his part of (M,R), ~~then~~ if P1 deviated then ~~the~~ ~~still~~ each player still plays his part of (M,R), if P2 deviated, then start the punishment phase. In the punishment phase, each player plays his part of (D,R) for T periods. If no player deviates within these T periods, return to the collaboration phase (i.e. play as if ~~the game h~~ no player previously deviated). If ~~one~~ any player deviates ~~from~~ ~~state~~ within the T periods of the punishment phase, the punishment phase begins anew.

Gez δ

In either phase, $\pi_1 = 2$. It is trivial that P1 has no profitable one shot deviation.

T is such that P2 has no profitable one shot deviation in the collaboration phase, i.e.

This is "check that it works as a punishment"

$3 + \delta(-1) + \delta^2(-1) + \dots + \delta^T(-) \leq 2 + \delta(2) + \delta^2(2) + \dots + \delta^T(2)$

$\Leftrightarrow 3 + \delta(-1)(1-\delta^T)/1-\delta \leq 2 + \delta 2(1-\delta^T)/1-\delta$

$\Leftrightarrow 1 - 1/3(1-1/3^T)/2/3 \leq \cancel{2} 2/3(1-1/3^T)/2/3$

$\Leftrightarrow \cancel{8} 1 - (1-1/3^T)/2 \leq (1-1/3^T)$

$\Leftrightarrow (1-1/3^T) \geq 2/3$

$\Leftrightarrow T \geq 1$

~~T is such that P2 has no profitable deviation in the punishment phase~~

~~$-1 + \delta(-1) + \dots + \delta^{T-1}(-1)$~~

Suppose that T=1, then in the punishment phase, the ~~δ SPE~~ SPE ~~yields~~ ~~prescribes~~ yields

$\pi_2 = -1 + \delta(2) + \delta^2(2) + \dots = -1 + 1/1-\delta(2) = 1$

~~optimal~~ one shot deviation yields

$\pi_2' = 0 + \delta(-1) + \delta^2(2) + \dots = \delta\pi_2 < \pi_2$

This seems to imply T = 100 is also an SPE, but surely P2 simply plays L forever after deviation in that case?

No, if π_2 is negative, then it is preferred to defer, so compare $-1+2\delta$ vs $0-\delta$

So P2 has no profitable one shot deviation

By the one shot deviation principle, this is an SPE.

Look at game matrix rather than apply the general recipe

ADV vs non-ADV notation is a matter of preference

Will not be forced to use general recipe