Game theory supplementary Exercises 2

1a

	L	R
T	4, 4	0, 1
B	1, 0	x, y

$P(x=y=1)=\beta\in(1/4,1)$

$P(x=1,y=0)=P(x=0,y=1)=P(x=0,y=0)=1-\beta/3$

This is a Bayesian game because it can be represented as a set of players {Row (Pl.1), Column (Pl.2)}, a set of states $\{w,x,y,z\}$ corresponding to $(x=1,y=1)$, $(x=1,y=0)$, $(x=0,y=1)$ and $(x=0,y=0)$ respectively, a set of actions for each player, $A_1=\{T,B\}$, $A_2=\{L,R\}$, a set of signals for each player $t_1=\{x_1,x_0\}$ and $t_2=\{y_1,y_0\}$ with a signal function ~~$\tau_1(w,x)$~~ $\tau_1(w)=\tau_1(x)=x_1$, $\tau_1(y)=\tau_1(z)=x_0$, $\tau_2(w)=\tau_2(y)=y_1$, $\tau_2(x)=\tau_2(z)=y_0$, a set of posterior beliefs for each player ~~type, x_1 believes $P(t_2=y_1)$ with pro $=3\beta$, $P(t_2=y_0)=1-3\beta$ $= 3\beta/1+2\beta$ and $P(t_2=y_0)=$~~

$P(t_2=y_1|t_1=x_1)=3\beta/1+2\beta$, $P(t_2=y_0|t_1=x_1)=1-\beta/1+2\beta$

~~$P(t_1=x_1|t_2=y_1)=3\beta/1+2\beta$, $P(t_1=x_0|t_2=y_1$~~

$P(t_2=y_1|t_1=x_0)=1/2$, $P(t_2=y_0|t_1=x_0)=1/2$

$P(t_1=x_1|t_2=y_1)=3\beta/1+2\beta$, $P(t_1=x_0|t_2=y_1)=1-\beta/1+2\beta$

$P(t_1=x_1|t_2=y_0)=1/2$, $P(t_1=x_0|t_2=y_0)=1/2$, and a set of payoffs given by the payoff table above.

Types are correlated because since each player's type is drawn from a joint distribution, and the realisation of each player's type is not independent, i.e. $t_1 \not\perp t_2$, i.e. $P(t_1=x_1|t_2)\neq P(t_1=x_1)$.

Pl.1's pure strategies are {TT, TB, BT, BB}, where the first action is played when $t_1=x_1$ and the second action is played when $t_1=x_0$. ~~Likew~~ Similarly, Pl.2's pure strategies are {LL, LR, RL, RR}

~~$E[u_1(TT,LL;t_1,t_2)|t_1]$~~

~~$E[u_1(TT,LL;t_1,t_2)|t_1=x_1]=3\beta/1+2\beta(4)+1-\beta/1+2\beta(4)=4$~~

~~$E[u_1(TT,LL;t_1,t_2)|t_1=x_0]=$~~

~~$E[u_1(TT,LL;t_1,t_2)|t_1]=4$~~

~~$E[u_1(TT,RR;t_1,t_2)|t_1]=0$~~

~~$E[u_1(TT,BB,LL;t_1,t_2)|t_1]=1$~~

~~$E[u_1(BB,RR;t_1,t_2)|t_1]=x$~~

Suppose $\exists$ BNE ~~st Pl.1 plays~~ s^* st $s_1^*=(T,B)$ and $s_2^*=(L,R)$, where the first action is played when player i is type 0 and the second action is played when player i is type 1. ~~then~~

Then $E[u_1(s_1^*,s_2^*;t_1,t_2)|t_1]>E[u_1(s_1',s_2^*;t_1,t_2)|t_1]$ for all $s_1'\neq s_1^*$ and t_1, and ~~the~~ similarly for Pl.2

$E[u_2(s_2^*,s_1^*;t_1,t_2)|t_2]>E[u_2(s_2',s_1^*;t_1,t_2)|t_2]$ for all t_2, $s_2'\neq s_2^*$.

~~$t_1=x_0$~~

$E[u_1(s_1^*,s_2^*;t_1,t_2)|t_1=x_0]=$ ~~$1/2u_1(s_1^*,s_2^*,t_1$~~

$=1/2u_1(s_1^*,s_2^*;t_1=0,t_2=0)+1/2u_1(s_1^*,s_2^*;t_1=0,t_2=1)$

$=1/2(4)+1/2(0)$

$=2$

$E[u_1(s_1^*,s_2^*;t_1,t_2)|t_1=x_1]$

$=3\beta/1+2\beta(1)+1-\beta/1+2\beta(1)=1$

$E[u_1(s_1',s_2^*;t_1,t_2)|t_1=x_0]$

$=1/2(1)+1/2(x=0)=1/2$

$E[u_1(s_1',s_2^*;t_1,t_2)|t_1=x_1]$

$=3\beta/1+2\beta(0)+1-\beta/1+2\beta(4)=4-4\beta/1+2\beta$

If s^* is a BNE, $1>4-4\beta/1+2\beta$, $1+2\beta>4-4\beta$, $6\beta>3$, $\beta>1/2$. By inspection, the conditions for Pl.2 are symmetrical. So s^* is a BNE iff $\beta>1/2$

c There are ~~for~~ four candidate BNE s^* such that each player plays an action, not conditioned on his type (TT, LL), (TT, RR), (BB, LL) and (BB, RR).

(TT, LL) is a BNE iff

$E[u_1(TT,LL;t_1,t_2)|t_1=x_0]\geq E[u_1(s_1',LL;t_1,t_2)|t_1=x_0]$

$4>1$

$E[u_1(TT,LL;t_1,t_2)|t_1=x_1]\geq E[u_1(s_1',LL;t_1,t_2)|t_1=x_1]$

$4>1$,

Similarly for Pl.2

So (TT, LL) is a BNE

(TT, RR) is a BNE iff

$E[u_1(TT,RR;t_1,t_2)|t_1=x_0]\geq E[u_1(s_1',RR;t_1,t_2)|t_1=x_0]$

$0\geq$ ~~0~~ 0

$E[u_1(TT,RR;t_1,t_2)|t_1=x_1]\geq E[u_1(s_1',RR;t_1,t_2)|t_1=x_1]$

$0\geq x=1$

So (TT, RR) is not a BNE

By symmetry (BB, LL) is not a BNE

(BB, RR) is a BNE iff

$E[u_1(BB,RR;t_1,t_2)|t_1=x_0]\geq E[u_1(\cancel{BB}\ s_1',RR;t_1,t_2)|t_1=x_0]$

$0\geq 0$

$E[u_1(BB,RR;t_1,t_2)|t_1=x_1]\geq E[u_1(s',RR;t_1,t_2)|t_1=x_1]$

$1\geq 0$

Similarly for Pl.2

So (BB, RR) is a BNE

2a Players: $N = \{1, 2\}$

States: $\{e|e2, e|i2, i|e2, i|i2\}$

Actions: $A_i = [0, 1]$

Signals: $\tau_1(e|e2) = \tau_1(e|i2) = e$, $\tau_1(i|e2) = \tau_1(i|i2) = i$

$\tau_2(e|e2) = \tau_2(i|e2) = e$, $\tau_2(e|i2) = \tau_2(i|i2) = i$

Beliefs: $P(t_j = e | t_i) = 1 - \gamma$, $p(t_j = i | t_i) = \gamma$

where type $t_i = \tau_i(\omega)$

where ω is the realised state

Payoffs: $u_i(x_i, x_{-i}; t_i, t_{-i})$

$$= \begin{cases} 1 - (x_i + F_i) & \text{if } x_i > x_{-i} \\ 1/2 - (x_i + F_i) & \text{if } x_i = x_{-i} \\ -(x_i + F_i) & \text{if } 0 < x_i < x_{-i} \\ 0 & \text{if } x_i = 0 \end{cases}$$

where $F_i = 1$ if $t_i = i$ and $F_i = 0$ if $t_i = e$

Suppose that there is some symmetric pure BNE $s^* = (s_1^*, s_2^*)$ where each s_i^* is some pair ~~(s_{ie}^*, s_{ii}^*)~~ $(\hat{s}_i^*, \check{s}_i^*)$ such that i plays $\hat{s}_i^*$ if $t_i = e$ and $\check{s}_i^*$ if $t_i = i$, and $s_1^* = s_2^*$

~~s^* is a BNE iff if $\hat{s}_1^* = \hat{s}_2^*$~~

~~$\check{s}_i^* = 0$ since~~

~~$\check{s}_i^* = 0$ since then $E[u_i(s_i^*, s_j^*; t_i, t_j) | t_i = i] = \gamma$~~

~~and $E[u_i(s_i', s_j^*; t_i, t_j) | t_i = i] = -\check{s}_i^{*\prime} + 0 < \gamma$~~

~~Let Consider $s_i' = [\check{s}_i > 0, \hat{s}_i^*]$ $(\hat{s}_i, \check{s}_i)$, i.e. alternative strategy s_i'. Let s_i' be an alternative of strategy of firm i that differs from firm i's strategy at the candidate equilibrium in only the action if inefficient type action.~~

~~$E[u_i(s_i^*, s_j^*; t_i, t_j) | t_i = e] = E[u_i(s_i^{*\prime}, s_j^*; t_i, t_j) | t_i = e]$~~

~~$E[u_i(s_i^* = (\hat{s}_i^*, 0), s_j^*; t_i, t_j) | t_i = i] = \gamma > 0$~~

~~$E[u_i(s_i' = (\hat{s}_i^*, \check{s}_i' \neq 0), s_j^*; t_i, t_j) | t_i = i] =$~~

Suppose $\check{s}_i^* > 0$, then the candidate equilibrium fails to the deviation where some player i plays $s_i' = (\hat{s}_i^*, \check{s}_i' = 0)$ since this strategy does as well as s_i^* when $t_i = e$ and yields greater payoff ~~(0)~~ $(0 > (1-\gamma)(-1-\check{s}_i^*) + \gamma(1/2 - 1 - \check{s}_i^*))$ when $t_i = i$, so if ~~s^*~~ s^* is a BNE, $\check{s}_i^* = 0$

"does as well as ..." not required

Suppose $\hat{s}_i^* = 1$, then the candidate equilibrium fails to the deviation ~~where~~ $s_i = (\hat{s}_i' = \varepsilon, \check{s}_i^*)$ for sufficiently small ε since the strategy does just as well as s_i^* when $t_i = i$ and yields greater payoff

~~$(0 > (1-\gamma)(1/2 - \hat{s}_i^*) + \gamma(1-s$~~

~~$((1-\gamma)0 + \gamma(1-\varepsilon) > (1-\gamma)(1/2 - \hat{s}_i^*) + \gamma(1 - \hat{s}_i^*))$~~

$((1-\gamma)0 + \gamma(1-\varepsilon) = \gamma(1-\varepsilon) > (1-\gamma)(1/2 - 1) + \gamma(1-1))$ when $t_i = e$, so if s^* is a BNE, $\hat{s}_i^* < 1$

Suppose $\hat{s}_i^* \geq 1/2$, then the candidate equilibrium fails to deviation ~~$\hat{s}_i$~~ $s_i = (\hat{s}_i' = \varepsilon, \check{s}_i^*)$ for sufficiently small ε since this strategy does equally well as s_i^* when $t_i = i$ and does better when $t_i = e$

$(1-\gamma)0 + \gamma(1-\varepsilon) = \gamma(1-\varepsilon) > (1-\gamma)(1/2 - \hat{s}_i^*) + \gamma(1 - \hat{s}_i^*)$

Suppose $\hat{s}_i^* < 1/2$, then the candidate equilibrium fails to deviation $s_i' = (\hat{s}_i' = \hat{s}_i^* + \varepsilon, \check{s}_i^*)$ for sufficiently small since this strategy ~~does equally well for t_i~~ outperforms s^* when $t_i = e$

~~$(1-\gamma)(1 - \hat{s}_i^*) + \gamma(1 - \hat{s}_i^*) = 1 - \hat{s}_i^* > (1-\gamma)($~~

$(1-\gamma)(1 - \hat{s}_i^{*\prime}) + \gamma(1 - \hat{s}_i') = 1 - \hat{s}_i' = 1 - \hat{s}_i^* - \varepsilon$

$> (1-\gamma)(1/2 - \hat{s}_i^*) + \gamma(1 - \hat{s}_i^*) = 1 - \hat{s}_i^* - 1/2(1-\gamma)$

~~So there the symmetric pure BNE s^* is~~

~~By reduct if~~ By reductio, there is no symmetric pure BNE s^*.

6 Let σ^* be a symmetric mixed BNE. Then $\sigma^* =$ ~~(σ_i^*, σ_i^*), where~~ $(\sigma_1^* = \sigma_i^*, \sigma_2^* = \sigma_i^*)$, i.e. each player plays strategy σ_i^* at σ^*, and $\sigma_i^* = (\hat{\sigma}_i^*, \check{\sigma}_i^*)$, i.e. the strategy σ_i^* is such that the player whose strategy it is plays $\hat{\sigma}_i^*$ if he is efficient ~~and~~ (type e) and $\check{\sigma}_i^*$ if he is inefficient (type i), and each of $\hat{\sigma}_i^*$ and $\check{\sigma}_i^*$ is a probability distribution over A_i.

Suppose $\check{\sigma}_i^*$ is some probability distribution that ~~assigns non-zero probability~~ does not assign probability 1 to $x_i = 0$. Then the candidate equilibrium fails to deviation $\sigma_i' = (\check{\sigma}_i', \hat{\sigma}_i^*)$ where $\check{\sigma}_i'$ assigns probability 1 to $x_i = 0$ since, when $t_i = i$, the deviation yields higher expected payoff.

Suppose $\hat{\sigma}_i^*$ is some probability distribution characterised by ~~cdf G~~ continuous cdf G on some interval $[0, \bar{x}]$, then when $t_i = e$, firm i's expected payoff from each $x_i \in [0, \bar{x}]$ is equal.

~~$E[u_i(0, \sigma_j^*; t_i, t_j) | t_i = e]$~~ mistake, $G(0) = 0$, so

~~$= (1-\gamma)0 + \gamma(1/2) = \gamma/2$~~ ← never share the market

~~since $x_i = 0$ "loses" to any efficient opponent and "splits" with any inefficient opponent.~~

~~$E[u_i(x_i \in [0, \bar{x}]$~~

~~$E[u_i(x_i = \bar{x}, \sigma_j^*; t_i, t_j) | t_i = e]$~~

~~$= (1-\gamma)(1-\bar{x}) + \gamma(1-\bar{x}) = 1 - \bar{x} = \gamma/2$, $\bar{x} = 1 - \gamma/2$~~

~~since $x_i = \bar{x}$ "wins" against any opponent~~

$E[u_i(x_i \in (0, \bar{x}), \sigma_j^*; t_i, t_j | t_i = e]$

$= (1-\gamma)G(x_i)(1 - x_i) + (1-\gamma)(1 - G(x_i))(-x_i) + \gamma(1 - x_i)$

$= (1-\gamma)(-x_i) + (1-\gamma)G(x_i) + \gamma(1 - x_i)$

$= -x_i + \gamma x_i + \gamma - \gamma x_i + (1-\gamma)G(x_i)$

$= -x_i + \gamma + (1-\gamma)G(x_i) =$ ~~$\gamma/2$~~ π

~~$(1-\gamma)G(x_i) = x_i - \gamma/2$~~

~~$G(x_i) = (x_i - \gamma/2)/1-\gamma$~~

~~Since~~ Given G is a continuous cdf, $G(0) = 0$

$\pi = \gamma$

$(1-\gamma)G(x_i) = x_i$, $G(x_i) = x_i / 1-\gamma$

$E[u_i(x_i = \bar{x}, \sigma_j^*; t_i, t_j) | t_i = e]$

$= (1-\gamma)(1-\bar{x}) + \gamma(1-\bar{x}) = 1 - \bar{x} = \pi = \gamma$, $\bar{x} = 1 - \gamma$

3 Players : $N=\{1,2,3\}$

Actions (of each player ~~i~~i) : $A_i=\{A,B\}$

Payoffs (of each player i given actions of other players j and k) : $u_i(a_i,a_j,a_k)$

$$=\begin{cases} 1 \text{ iff } a_i=A \text{ and } a_j=a_k=B \\ 2 \text{ iff } a_i=B \text{ and } a_j=a_k=A \\ 0 \text{ otherwise} \end{cases}$$

	A	B
A	0,0,0	$\underline{0,2,0}$
B	$\underline{2,0,0}$	$\underline{0,0,1}$

A

	A	B
A	$\underline{0,0,2}$	$\underline{1,0,0}$
B	$\underline{0,1,0}$	0,0,0

B

where Pl.1, Pl.2, and Pl.3 are the ~~Row, Column~~ row, column, and matrix player, and the payoff of Pl.1 is listed first, then that of Pl.2, then that of Pl.3.

Best responses underlined.

No strategy is strictly dominated since each player i has higher payoff from A than B if both other players play B and higher payoff from B than A if both other players play A, i.e. $u_i(A,B,B) > u_i(B,B,B)$ and $u_i(B,A,A) > u_i(A,A,A)$

By inspection, (A,B,A), (A,A,B), (A,B,B), (B,A,A), (B,B,A), (B,A,B) are the only pure strategy NE, i.e. all and only pure strategy profiles where at least one player chooses each action are pure NE.

Let Pl.3 be the player who plays a ~~fixed str~~ pure strategy. This is wlog since the players are identical. Let Pl.1 ~~be~~ be a player who plays a mixed strategy. This is again wlog.

~~Suppose that Pl.3 plays A~~

Let σ^* denote any such hybrid NE

Suppose that ~~σ~~ $\sigma_3^*=A$

Then, by definition of NE,

$\pi_1(A,\sigma_2^*,A)=\pi_1(B,\sigma_2^*,A)$

such that Pl.1 has no profitable deviation

~~$2(1-q)=$~~ $0=2q$, where q is the probability assigned to A by σ_2^*.

$q=0$, Pl.2 plays ~~fixed s~~ pure strategy B

Then, by definition of NE,

$\pi_2(B,\sigma_1^*,A) \geq \pi_2(A,\sigma_1^*,A)$

$2p \geq 0$, where p is the probability assigned to A by σ_1^*.

$\pi_3(A,\sigma_1^*,\sigma_2^*) \geq \pi_3(B,\sigma_1^*,\sigma_2^*)$

$1(1-p) \geq 0$

σ^* is a NE iff $0 \leq p \leq 1$

σ^* is a hybrid NE iff $0<p<1$.

All σ^* such that one player plays pure strategy A, ~~the~~ another plays pure strategy B, and a third plays ~~any~~ ~~$pA+(1-p)B$~~ mixed strategy $pA+(1-p)B$ for $p \in (0,1)$ is a hybrid NE.

If one player plays pure strategy A, at most one other player mixes at equilibrium

Suppose $\sigma_3^*=B$

Then $\pi_1(A,\sigma_2^*,B)=\pi_1(B,\sigma_2^*,B)$, $1-q=0$, $q=1$

So Pl.2 plays pure strategy A.

Then, there are no hybrid NE such that two players mix.

All and only strategy profiles where one player plays A, another player plays B, and another plays mixed strategy $(pA+(1-p)B)$ for $p \in (0,1)$ are hybrid NE.

The remaining NE are completely mixed. Let σ^* ~~be~~ denote any completely mixed NE.

$\pi_1(A,\sigma_2^*,\sigma_3^*)=\pi_1(B,\sigma_2^*,\sigma_3^*)$

~~$0qr+0(1-q)r+$~~ $(1-q)(1-r)=qr$, where r is the probability σ_3^* assigns to A

$1+qr-q-r=qr$, $q+r=1$

By symmetry, $p+q=1$ and $p+r=1$

Solving simultaneously, $p=q=r=1/2$

The only remaining NE is the only mixed NE $(\tfrac{1}{2}A+\tfrac{1}{2}B,\ \tfrac{1}{2}A+\tfrac{1}{2}B,\ \tfrac{1}{2}A+\tfrac{1}{2}B)$

	A	B
A	0	$2+\alpha_2$
	0	1
B	1	0
	$2+\alpha_1$	0

Players : $N=\{1,2\}$

Actions (of each player i) : $A_i=\{A,B\}$

States : $\{(\alpha_1,\alpha_2) : \alpha_1,\alpha_2 \in [0,x]\}$

Signals : $\tau_1((\alpha_1,\alpha_2))=\alpha_1$

$\tau_2((\alpha_1,\alpha_2))=\alpha_2$

Beliefs : $p_1(t_2 \leq x' \mid t_1) =$ ~~$\neq$~~ x'/x

$p_2(t_1 \leq x' \mid t_2) = x'/x$

~~Payoff~~ where $t_i=\tau_i((\alpha_1,\alpha_2))$

Payoffs given in the ~~te~~ matrix above.

Consider the ~~strategy~~ pure strategy profile where each player i plays action A iff α_i ~~$<1/3$~~ ~~$<\alpha_i$~~ $<\hat{\alpha}_i$ and action B otherwise. Denote this strategy profile as s^*. s^* is a BNE iff

$\forall i : \forall t_i :$ ~~$\forall s_{-i}$~~ $\forall s_i' : E[u(s_i^*,s_{-i}^*;t_i,t_{-i}) \mid t_i=t]$

$\geq E[u(s_i',s_{-i}^* \mid t_i,t_{-i}) \mid t_i=t]$, i.e. iff for ~~all~~ each player~~s~~ i, for all signals ~~that~~ he receives t_i, it is optimal to play s_i^* (specifically the action $s_i^*(t_i)$). ~~ed~~

~~$E[u(s_i^*,s_{-i}^*;t_i,t_{-i}) \mid t_i<1/3]$~~

~~$=\tfrac{1}{3}u(A,A;t_i<1/3,t_{-i}<1/3)+\tfrac{2}{3}u(A,B;t_i<1/3,t_{-i}>1/3)$~~

~~$=2/3$~~

~~$E[u(s_i^{*\prime},s_{-i}^*;t_i,t_{-i}) \mid t_i<1/3]$~~

~~$=\tfrac{1}{3}u(B,A;t_i<1/3,t_{-i}<1/3)+\tfrac{2}{3}u(B,B;t_i<1/3,t_{-i}>1/3)$~~

~~$E[u(s_i^*,s_{-i}^*;t_i,t_{-i}) \mid t_i<\hat{\alpha}_i]$~~

~~$=\hat{\alpha}_i u(A,A;t_i<\hat{\alpha}_i,t_{-i}<\hat{\alpha}_i)+(1-\hat{\alpha}_i)u(A,B;t_i<\hat{\alpha}_i,t_{-i}>\hat{\alpha}_i)$~~

~~$E[u(s_i^*,s_{-i}^*;t_i,t_{-i}) \mid t_i<\hat{\alpha}]$~~

~~$=\hat{\alpha}u(A,A;t_i<\hat{\alpha},t_{-i}<\hat{\alpha})+(1-\hat{\alpha})u(A,B;t_i<\hat{\alpha},t_{-i}>\hat{\alpha})$~~

~~$=1-\hat{\alpha}$~~

~~$E[u(s_i',s_{-i}^*;t_i,t_{-i}) \mid t_i<\hat{\alpha}]$~~

~~$=\hat{\alpha}u(B,A;t_i<\hat{\alpha},t_{-i}<\hat{\alpha})+(1-\hat{\alpha})u(B,B;t_i<\hat{\alpha},t_{-i}>\hat{\alpha})$~~

~~$=\hat{\alpha}(2+t_i)$~~

~~$E[u(s_i^*,s_{-i}^*;t_i,t_{-i}) \mid t_i>\hat{\alpha}]$~~

~~$=\hat{\alpha}u(B,A;t_i>\hat{\alpha},t_{-i}<\hat{\alpha})+(1-\hat{\alpha})u(B,B;t_i>\hat{\alpha},t_{-i}>\hat{\alpha})$~~ $=\hat{\alpha}(2+t_i)$

~~$E[u(s_i',s_{-i}^*;t_i,t_{-i}) \mid t_i>\hat{\alpha}]$~~

~~$=\hat{\alpha}u(A,A;t_i>\hat{\alpha},t_{-i}<\hat{\alpha})+(1-\hat{\alpha})u(A,B;t_i>\hat{\alpha},t_{-i}>\hat{\alpha})$~~ $=(1-\hat{\alpha})$

s^* is a BNE iff

$1-\hat{\alpha} \geq \cancel{2x}\,\hat{\alpha}(2+t_i)$ for $t_i < \hat{\alpha}$ and

$\hat{\alpha}(2+t_i) \geq 1-\hat{\alpha}$ for $t_i > \hat{\alpha}$

Since $[1-\hat{\alpha}]-[\hat{\alpha}(2+t_i)]$ is a continuous function of t_i,

~~If~~ s^* is a BNE iff

$\cancel{\hat{\alpha}(2}$ $1-\hat{\alpha} = \hat{\alpha}(2+t_i)$ for $t_i = \hat{\alpha}$

$1-\hat{\alpha} = 2\hat{\alpha} + \hat{\alpha}^2$,

$\hat{\alpha}^2 + 3\hat{\alpha} - 1 = 0$

~~s^* is a BNE iff~~

~~$E[u_1(s_1^*, s_2^*; t_1, t_2) | t_1 < \hat{\alpha}_1]$~~

~~$= \hat{\alpha}_2/x\, u_1(A, A; t_1 < \hat{\alpha}_1, t_2 < \hat{\alpha}_2) + (1-\hat{\alpha}_2/x) u_1(A, B; t_1 < \hat{\alpha}_1, t_2 > \hat{\alpha}_2)$~~

~~$= 0 + (1-\hat{\alpha}_2/x) = 1-\hat{\alpha}_2/x$~~

~~$E[u_1(s_1', s_2^*; t_1, t_2) | t_1 < \hat{\alpha}_1]$~~

~~$= \hat{\alpha}_2/x$~~

~~$BR_1(s_2^*)$~~ :

$\pi_1(A, s_2^*) = \hat{\alpha}_2/x\,(0) + (1-\hat{\alpha}_2/x) = 1-\hat{\alpha}_2/x$

$\pi_1(B, s_2^*) = \hat{\alpha}_2/x\,(2+t_1)$

$\pi_1(A, s_2^*) \geq \pi_1(B, s_2^*)$ iff

$1-\hat{\alpha}_2/x \geq \hat{\alpha}_2/x\,(2+t_1)$

$1-3\hat{\alpha}_2/x \geq \hat{\alpha}_2/x\; t_1$

$t_1 \leq x/\hat{\alpha}_2 - 3$

So $BR_1(s_2^*) = A$ iff $t_1 \leq x/\hat{\alpha}_2 - 3$, B otherwise

By symmetry,

$BR_2(s_1^*) = A$ iff $t_2 \leq x/\hat{\alpha}_1 - 3$, B otherwise

~~If~~ s^* is a BNE iff

$s_1^* = BR_1(s_2^*)$, $\cancel{s_2^*}$ and $s_2^* = BR_2(s_1^*)$

$\hat{\alpha}_1 = x/\hat{\alpha}_2 - 3$, and $\hat{\alpha}_2 = x/\hat{\alpha}_1 - 3$

$\hat{\alpha}_1 = x/(x/\hat{\alpha}_1 - 3) - 3$

$= x\hat{\alpha}_1/x - 3\hat{\alpha}_1 \quad -3$

$1 = x/x - 3\hat{\alpha}_1 \quad -3$

$4 = x/x - 3\hat{\alpha}_1$

$1/4 = x - 3\hat{\alpha}_1/x$

$x/4 = x - 3\hat{\alpha}_1$

$3\hat{\alpha}_1 = 3x/4$

$\hat{\alpha}_1 = x/4$

3 Game 1

	L	C	R
T	$\underline{2}$	0	0
	0	$\underline{2}$	0
M	0	$\underline{2}$	0
	$\underline{2}$	0	0
B	0	0	$\underline{1}$
	0	0	$\underline{1}$

~~B~~

Best responses underlined.

By inspection, the unique pure strategy Nash equilibrium is (B, R)

Since each player has a strict and unique best response to each pure strategy of the opponent, there are no hybrid equilibria.

"unique" does not seem important here.

~~The The unique~~ one mixed strategy Nash equilibrium is $(0.5 \times T + 0.5 \times M,\ 0.5 \times L + 0.5 \times C)$. If player 1 deviates by allocating positive probability mass to B, player 1's expected payoff decreases since the expected payoff from action B is 0 while the expected payoff from the Nash equilibrium strategy is $\frac{1}{4}(2) + \frac{1}{4}(2) = 1$. The same is true of player 2 and R. If player 1 deviates by allocating probability mass asymmetrically between T and M, his expected payoff is unchanged ~~since h~~ since his expected payoff from each action is equal. The same is true of player 2.

If player 1 mixes only M and B, player 2 never plays L since it is strictly dominated by some mix of C and R, then player 1 never plays M, since it is strictly dominated by some mix of T and B. By reductio player 1 never mixes only M and B. The same is true for T and B, C and R, L and R.

At equilibrium if player 1 mixes T, M, and B, player 1's expected payoff from each action is constant.
$2p_L = 2p_C = 1p_R$. $p_L + p_C + p_R = 1$, $p_L = p_C = \frac{1}{4}$ $p_R = \frac{1}{2}$
Then player 2 mixes L, C, and R, so player 2's expected payoff from each action is constant.
$2p_T = 2p_M = p_B$, $p_T + p_M + p_B = 1$, $p_T = p_M = \frac{1}{4}$, $p_B = \frac{1}{2}$
So $(0.25 \times p_T + 0.25 \times p_M + 0.5 \times p_B,$
$0.25 \times p_L + 0.25 \times p_C + 0.5 \times p_R)$ is a mixed strategy Nash equilibrium

Game Theory Exercises 1

1	L	C	R
T	$\underline{4}$, $\underline{4}$	0, 1	$\underline{3}$, 2
M	1, 0	$\underline{1}$, $\underline{1}$	1, -1
B	2, $\underline{3}$	-1, 1	2, 2

Best responses underlined

By inspection, each ~~strategy~~ action is a best response to some action of the other player. So all pure strategies are rationalisable.

By inspection, there are two pure strategy Nash equilibria, ~~(4,4) &~~ (T, L) and (M, C) where players play mutual best responses.

Suppose that there is some mixed strategy Nash equilibrium where player 1 mixes T, M, and B, then the expected payoff to player 1 from each of these actions is equal,

~~$4p_L + p_C + 2p_R = p_C - p_R = 3p_C + p_C$~~

$4p_L + 3p_R = p_L + p_C + p_R = 2p_L - p_C + 2p_R = 1$

$4p_L + 3p_R = 1$ ①

$p_L + p_C + p_R = 1$ ②

$2p_L - p_C + 2p_R = 1$ ③

② + ③: $3p_L + 3p_R = 2$ ④

① − ④: $p_L = -1$ ⑤

By reductio, there is no such mixed strategy Nash equilibrium. By symmetry, there is no mixed strategy Nash equilibrium where player 2 mixes L, C, and R.

Suppose ∃ MNE s.t. player 1 mixes T and M, then

$4p_L + 3p_R = p_L + p_C + p_R = 1$

Solving simultaneously,

~~$p_L = (1-3p_R)/4$, $p_C =$~~

$p_L = 1/4 - 3/4\,p_R$, $p_C = 3/4 - 1/4\,p_R$, $p_R \in [0, 1/3]$

~~Since player 2 never mixes L, C, and R, $p_R \notin (0,1)$, $p_R = 0$ or~~.

~~So~~ Since player 2 never mixes L, C, and R, $p_R = 0$ or $1/3$.

If $p_R = 0$, then $p_L = 1/4$, $p_C = 3/4$, player 2 mixes L and C, then ~~$4p_T = p_T + p_M$~~ by symmetry, $p_T = 1/4$, $p_M = 3/4$.

If $p_R = 1/3$, then $p_C = 2/3$, $p_R = 1/3$, player 2 mixes C and R, then $p_T + p_M = 2p_T - p_M = 1$. Solving simultaneously, $p_T = 2/3$, $p_M = 1/3$. So ~~$(2/3 \times T + 1/3 \times M,$~~

So there are the following ~~NE~~ MNE where player ~~2~~ 1 mixes T and M:

$(1/4 \times T + 3/4 \times M,\ 1/4 \times L + 3/4 \times C)$

$(2/3 \times T + 1/3 \times M,\ 2/3 \times C + 1/3 \times R)$

By symmetry, there are the following additional MNE where player 2 mixes L and C

~~$(2/3 \times L + 1/3 \times C,\ 2/3 \times M + 1/3 \times B)$~~

$(2/3 \times M + 1/3 \times B,\ 2/3 \times L + 1/3 \times C)$

Suppose ∃ MNE st player 1 mixes M and B only, then

$p_L + p_C + p_R = 2p_L - p_C + 2p_R = 1$

Solving simultaneously,

$p_L = 2/3 - p_R$, $p_C = 1/3$, $p_R \in [0, 2/3]$

Since ∄ MNE st player 2 mixes L, C, and R, $p_R = 0$ or $2/3$

By inspection, the two such MNE are among those identified above. By symmetry, there are no additional MNE st player 2 mixes C and R

Suppose ∃ MNE st player 1 mixes T and B only, then

$4p_L + 3p_R = 2p_L - p_C + 2p_R$, $p_L + p_C + p_R = 1$

Solving simultaneously, $p_L = -1$

By reductio, ∄ such MNE

By symmetry, ∄ MNE st player 2 mixes L & R only

The pure and mixed NE are

(T, L)

(M, C)

$(1/4 \times T + 3/4 \times M,\ 1/4 \times L + 3/4 \times C)$

$(2/3 \times T + 1/3 \times M,\ 2/3 \times C + 1/3 \times R)$

$(2/3 \times M + 1/3 \times B,\ 2/3 \times L + 1/3 \times C)$