Microeconomic Analysis Problem set 5

1 $f(x,y) = x^2 + 2y^2 + xy + 3x + 19y - 4$

$Df = \begin{pmatrix} \partial f/\partial x & \partial f/\partial y \end{pmatrix} = \begin{pmatrix} 2x+y+3 & 4y+x+19 \end{pmatrix}$

$D^2f = \begin{pmatrix} \partial^2 f/\partial x^2 & \partial^2 f/\partial x \partial y \\ \partial^2 f/\partial y \partial x & \partial^2 f/\partial y^2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$

$|D^2f| = 7$

$tr(D^2f) = 6$

$|D^2f - \lambda I| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix}$; $(2-\lambda)(4-\lambda) - 1 = \lambda^2 - 6\lambda + 7$

~~$\lambda = -6 \pm \sqrt{36-28}/12 =$~~

$\lambda = 6 \pm \sqrt{36-28}/2 = 3 \pm \sqrt{2} > 0$

By the eigenvalue test, D^2f ~~is~~ is positive definite for all (x,y), ~~so D^2f is~~ f is strictly convex.

is a proof of this relationship required? How can this be proven?

FOC: $Df = \vec{0}$

$2x+y+3=0$ ①

$4y+x+19=0$ ②

① - 2②: $2x+y+3 - 2(4y+x+19) = y+3-8y-38 = -7y-35 = 0$

$y = -5$ ③

Sub ③ in ①: $2x - 5 + 3 = 0$, $x = 1$

~~Since D^2f is positive definite, $(x=1, y=-5)$ is a stric~~

Given that f is strictly convex, the stationary point $(x=1, y=-5)$ is the unique global ~~maximum~~. minimum.

Date

2 $\max/\min_{x,y}\ x^2+y^2$ st $x^2+xy+y^2=3$

By inspection, the constraint set $\{(x,y): x^2+xy+y^2=3\}$ is ~~not bounded~~ compact, i.e. closed and bounded. Then, by the Weierstrass extreme value theorem, a global maximum and a global minimum exist.

+ necessary that objective function is continuous for application of Weierstrass theorem

Let $h(x,y) \equiv x^2+xy+y^2$

$Dh = (\partial h/\partial x,\ \partial h/\partial y) = (2x+y \quad 2y+x)$

rank $Dh = 1$ iff $(x,y) \neq (0,0)$

The constraint qualification is satisfied iff $(x,y)\neq(0,0)$. $(0,0) \notin \{(x,y): x^2+xy+y^2=3\}$, so the constraint qualification is satisfied for all (x,y) in the constraint set. Then, by Lagrange's theorem, the Lagrangian first-order ~~FOCs~~ conditions ~~and constraint conditions~~ are necessary for a maximum and for a minimum.

~~$L = x^2+y^2-\lambda(x^2+xy+y^2-3)$~~

$L(x,y;\lambda) = x^2+y^2-\lambda(x^2+xy+y^2-3)$

FOC_x: $\partial L/\partial x = 2x - \lambda(2x+y) = 0$ ①

FOC_y: $\partial L/\partial y = 2y - \lambda(2y+x) = 0$ ②

$C: x^2+xy+y^2=3$ ③

From ① $\lambda = 2x/2x+y$ ④

From ~~④~~ 2 $\lambda = 2y/2y+x$ ⑤

From ④ and ⑤ $2x/2x+y = 2y/2y+x$,

$2x(2y+x) = 2y(2x+y)$,

$4xy+2x^2 = 4xy+2y^2$,

~~$x=\pm y$ ⑥~~ $y=\pm x$ ⑥

From ③ and ⑥

$x^2+x^2+x^2=3$ or $x^2-x^2+x^2=3$

$x=\pm1$ or $x=\pm\sqrt{3}$

$(x,y) = (1,1), (-1,-1), (\sqrt{3},-\sqrt{3})$ or $(-\sqrt{3},\sqrt{3})$

The candidate optima are $(1,1), (-1,-1), (\sqrt{3},-\sqrt{3})$ and $(-\sqrt{3},\sqrt{3})$.

$D_{x,y}L = (\partial L/\partial x \quad \partial L/\partial y) = (2x-\lambda(2x+y),\ 2y-\lambda(2y+x))$

$$D^2_{x,y}L = \begin{pmatrix} \partial^2 L/\partial x\partial x & \partial^2 L/\partial x\partial y \\ \partial^2 L/\partial y\partial x & \partial^2 L/\partial y\partial y \end{pmatrix} = \begin{pmatrix} 2-2\lambda & -\lambda \\ -\lambda & 2-2\lambda \end{pmatrix}$$

Not strictly necessary since by Weierstrass theorem global maximum and global minimum exist, so it is sufficient to evaluate and compare.

Let H denote the bordered Hessian matrix.

$$H = \begin{pmatrix} 0 & 2x+y & 2y+x \\ 2x+y & 2-2\lambda & -\lambda \\ 2y+x & -\lambda & 2-2\lambda \end{pmatrix}$$

Jacobian of constraint

Hessian of Lagrangian

Denote the bordered Hessian with BH

There are two variables and there is one constraint, so there is one degree of freedom, and only one leading principal minor to compute for the bordered Hessian test, namely $|H|$.

Exactly correct

$$|H| = -(2x+y)\left|\begin{pmatrix} 2x+y & -\lambda \\ 2y+x & 2-2\lambda \end{pmatrix}\right| + (2y+x)\left|\begin{pmatrix} 2x+y & 2-2\lambda \\ 2y+x & -\lambda \end{pmatrix}\right|$$

$$= -a\left|\begin{pmatrix} a & -\lambda \\ b & 2-2\lambda \end{pmatrix}\right| + b\left|\begin{pmatrix} a & 2-2\lambda \\ b & -\lambda \end{pmatrix}\right|$$

where $a = 2x+y$, $b = 2y+x$, by substitution

$= -a^2(2-2\lambda) + ab(-\lambda) + ba(-\lambda) - b^2(2-2\lambda)$

$=$ ~~$(a^2+b^2)(2-2\lambda)$~~ $(a^2+b^2)(2\lambda-2) - 2ab\lambda$

Suppose that $(x,y)=(1,1)$, then $\lambda=2/3$, $a=3$, $b=3$,
$|H|=(a^2+b^2)(2\lambda-2)-2ab\lambda=(18)(-2/3)-12<0$
By the bordered Hessian test, $(1,1)$ is a strict local ~~maxima~~. minimum.

$\text{sign}(-1)^m \Rightarrow$ minimum
$\text{sign}(-1)^n \Rightarrow$ maximum

Suppose that $(x,y)=(-1,-1)$, then $\lambda=2/3$, $a=-3$, $b=-3$,
$|H|=(18)(-2/3)-12<0$.
By the bordered Hessian test, $(-1,-1)$ is a strict local minimum.

Suppose that $(x,y)=(\sqrt{3},-\sqrt{3})$, then $\lambda=2$, $a=\sqrt{3}$, $b=-\sqrt{3}$,
$|H|=(6)(2)+12>0$.
By the bordered Hessian test, $(\sqrt{3},-\sqrt{3})$ is a strict local maximum.

Suppose that $(x,y)=(-\sqrt{3},\sqrt{3})$, then $\lambda=2$, $a=-\sqrt{3}$, $b=\sqrt{3}$,
$|H|=(6)(2)+12>0$
By the bordered Hessian test, $(-\sqrt{3},\sqrt{3})$ is a strict local maximum.

Let $f(x,y)=x^2+y^2$.
$f(1,1)=2$, $f(-1,-1)=2$, $f(\sqrt{3},-\sqrt{3})=6$, $f(-\sqrt{3},\sqrt{3})=6$.
$(1,1)$ and $(-1,-1)$ are weak global minima, and
$(\sqrt{3},-\sqrt{3})$ and $(-\sqrt{3},\sqrt{3})$ are weak global maxima.

3: $\max_{x,y} u(x,y) = x^2y$ st $2x+3y \leq 9$, $x,y \geq 0$

Suppose $(x,y)=(1,1)$, then $u(x,y)=u(1,1)=1$, and $2x+3y = 5 \leq 9$, and $x,y \geq 0$, i.e. $(x,y)=(1,1)$ is in the constraint set and yields $u=1$.

Suppose that $x=0$, then $u=0 < 1 = u(1,1)$, so ~~if $x=0$,~~ no $(x=0, y)$ is a constrained maximum. Suppose that $y=0$, then $u=0<1=u(1,1)$, so no $(x, y=0)$ is a constrained maximum. Let (x^*, y^*) denote the solution to the given constrained maximization problem. $x^*, y^* \neq 0$. Given that $x,y \geq 0$, $x^*, y^* > 0$. ~~Supp Suppose that $x,y \neq 0$, then by inspection, u is increasing in x and in y~~

Suppose that $2x^* + 3y^* < 9$, then some (x', y^*) ~~is and in the constrer~~ such that $x' > x^*$ is in the constraint set, namely $x' = 9-3y^*/2$. Given that $y^* > 0$, by inspection, u is increasing in x, so $u(x', y^*) > u(x^*, y^*)$, i.e. (x^*, y^*) is not a constrained maximum. By reductio, $2x^* + 3y^* \geq 9$. Given that $2x^* + 3y^* \leq 9$, $2x^* + 3y^* = 9$.

ii $L(x,y;\lambda) = x^2y - \lambda(2x+3y-9)$

$FOC_x = 2xy - \lambda(2) = 0$ ①

$FOC_y = x^2 - \lambda(3) = 0$ ②

$C: 2x+3y=9$ ③

From ①, $\lambda = xy$ ④

Substituting ④ into ②

$x^2 - 3xy = 0$, $x=0$ ~~or~~ (reject) or $x = 3y$ ⑤

Substituting ⑤ into ③

$6y+3y=9$, $y=1$, $x=3$

The only candidate optimum is $(3,1)$

iii $Du = (\partial u/\partial x \quad \partial u/\partial y) = (2xy \quad x^2)$

$D_{x,y}L = (\partial L/\partial x \quad \partial L/\partial y) = (2xy - \cancel{2}2\lambda \quad x^2 - 3\lambda)$

$$D^2_{x,y}L = \begin{pmatrix} \partial^2 L/\partial x \partial x & \partial^2 L/\partial x \partial y \\ \partial^2 L/\partial y \partial x & \partial^2 L/\partial y \partial y \end{pmatrix} = \begin{pmatrix} 2y & 2x \\ 2x & 0 \end{pmatrix}$$

Let H denote the bordered Hessian

$$H = \begin{pmatrix} 0 & 2xy & x^2 \\ 2xy & 2y & 2x \\ x^2 & 2x & 0 \end{pmatrix}$$

There are two variables and there is one constraint, so there is one degree of freedom and one leading principal minor to compute for the bordered Hessian test, namely $|H|$.

$$|H| = -2xy \left| \begin{pmatrix} 2xy & 2x \\ x^2 & 0 \end{pmatrix} \right| + x^2 \left| \begin{pmatrix} 2xy & 2y \\ x^2 & 2x \end{pmatrix} \right|$$

$= -2xy(-2x^3) + x^2(4x^2y - 2x^2y)$

$= 4x^4y + 2x^4y$

$= 6x^4y$

At $(x,y)=(3,1)$, $|H| = 6(3)^4(1)^{\cancel{4}} > 0$

By the bordered Hessian test, $(x,y)=(3,1)$ is a strict local maximum.

Always attempt to simplify

~~$MU_x(x=0, y=0)$~~ $MU_x \to \infty$ as $x \to 0$, $MU_y \to \infty$ as $y \to 0$

Generally correct approach

Compactness: boundaries are in that set, roughly: weak inequalities. Bounded: ball exists that contains that set
Not required to prove compactness, sufficient to state

+ By Weierstrass, solution exists

constraint set is ~~complex~~ compact, function is continuous, so Weierstrass theorem is applicable

Linear constraints ⇒ each element of the relevant Jacobian (of the constraints) is a constant ⇒ full rank

↳ what if $x+y=1$, $2x+2y=2$
then these are the same constraint
what if $x+y=1$ $y+z=1$ $x+z=1$ $x+y+z=3$

we know it is a max and not a min since $x,y \neq 0$. Granted the solution to the FOC could be a minimum, but we can eliminate this.

Not required to understand ~~mech~~ validity of the BH test.

iv The constraint set is $\{(x,y): 2x+3y\leq 9, x\geq 0, y\geq 0\}$. By inspection, the constraint set is compact (closed and bounded). By the Weierstrass extreme value theorem, a global maximum and a global minimum exist. So the ~~≠~~ unique local maximum is also a global maximum.

4 $\min_{K,L}\ rK + wL$ st $K, L \geq 0$, $Y(K,L) = (K+1)^{1-\alpha}(L+1)^{\alpha} - 1 \geq y$

Let $c(K,L) = rK + wL$.

$Y(K,L)$ is increasing in ~~the constraint set K and L~~ each of K and L in the constraint set, given that $K, L \geq 0$ in the constraint set and $0 < \alpha < 1$. Suppose for reductio that the constraint $Y(K,L) \geq y$ is not binding at the global minimum (K^*, L^*), i.e. $Y(K^*, L^*) > y$. Then, there exists (K', L') such that $Y(K', L') = y$ and $K' \leq K^*$, $L' \leq L^*$, and $K' < K^*$ or $L' < L^*$ so $c(K', L') < c(K^*, L^*)$, then (K^*, L^*) is not a minimum. By reductio, the constraint $Y(K,L) \leq y$ is binding at the global minimum. The minimisation problem reduces to

This argument is correct and necessary

$\min_{K,L}\ rK + wL$ st $K, L \geq 0$, $Y(K,L) = y$

← Linear objective functions are both concave and convex

~~$DY = (\partial Y/\partial K \ \ \partial Y/\partial L) = ((L+1)^{\alpha}(1-\alpha)(K+1)^{-\alpha}$~~
~~$= ((L+1)^{\alpha}$~~

Check concavity / concavity of constraint function by checking definiteness of Hessian, which in turn can be done by checking det and tr of Hessian of constraint.

which is equivalent to
$\min_{K,L}\ rK + wL$ st $K, L \geq 0$, $Y'(K,L) = y'$
where $Y'(K,L) = (1-\alpha)\ln(K+1) + \alpha\ln(L+1)$ and $y' = \ln(y+1)$

Concave / convex problems ⇒ FOCs sufficient

$$DY' = (\partial Y/\partial K \ \ \partial Y/\partial L) = \left(\tfrac{1-\alpha}{K+1} \ \ \tfrac{\alpha}{L+1} \right)$$

$$D(Y', g_1, g_2) = \begin{pmatrix} \partial Y/\partial K & \partial Y/\partial L \\ \partial g_1/\partial K & \partial g_1/\partial L \\ \partial g_2/\partial K & \partial g_2/\partial L \end{pmatrix} = \begin{pmatrix} \tfrac{1-\alpha}{K+1} & \tfrac{\alpha}{L+1} \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

FOCs necessary ~~iff~~ if constraint set has non empty interior, i.e. there is some point that satisfies all constraints with strict inequality

~~By inspecti~~ Given that $0 < \alpha < 1$ and in the constraint set $K, L \geq 0$, in the constraint set $\tfrac{1-\alpha}{K+1}$, $\tfrac{\alpha}{L+1} > 0$. Then, by inspection, ~~the in the constraint set~~, the constraint qualification holds for all points in the constraint set since at most two constraints are binding.

FOCs necessary if CQ holds

Then, the Kuhn-Tucker first order conditions are necessary for an optima.

$\mathcal{L} = rK + wL - \lambda_Y((1-\alpha)\ln(K+1) + \alpha\ln(L+1) - y') - \mu_K K - \mu_L L$

$FOC_K: r - \lambda_Y\left(\tfrac{1-\alpha}{K+1}\right) - \mu_K = 0$

$FOC_L: w - \lambda_Y\left(\tfrac{\alpha}{L+1}\right) - \mu_L = 0$

$C_Y: (1-\alpha)\ln(K+1) + \alpha\ln(L+1) = \ln(y+1)$

equivalently $(K+1)^{1-\alpha}(L+1)^{\alpha} - 1 = y$

$CS_K: \mu_K \geq 0,\ K \geq 0,\ \mu_K K = 0$

$CS_L: \mu_L \geq 0,\ L \geq 0,\ \mu_L L = 0$

Suppose that neither positivity constraint is binding. Then $\mu_K = \mu_L = 0$ (and $K, L > 0$)

By substitution into FOC_K, FOC_L,

$r - \lambda_Y\left(\tfrac{1-\alpha}{K+1}\right) = 0$, $w - \lambda_Y\left(\tfrac{\alpha}{L+1}\right) = 0$

$r = (1-\alpha)\lambda_Y/K+1$, $w = \alpha\lambda_Y/L+1$

$K+1 = (1-\alpha)\lambda_Y/r$, $L+1 = \alpha\lambda_Y/w$

$L+1 = \tfrac{\alpha}{1-\alpha}\tfrac{r}{w}(K+1)$

By substitution into C_Y,
$$(K+1)^{1-\alpha}\left(\frac{\alpha}{1-\alpha}\frac{r}{w}\right)^{\alpha}(K+1)^{\alpha} = y+1$$
$$K+1 = \left(\frac{1-\alpha}{\alpha}\frac{w}{r}\right)^{\alpha}(y+1)$$
$$L+1 = \left(\frac{\alpha}{1-\alpha}\frac{r}{w}\right)\left(\frac{1-\alpha}{\alpha}\frac{w}{r}\right)^{\alpha}(y+1)$$
$$= \left(\frac{\alpha}{1-\alpha}\frac{r}{w}\right)^{1-\alpha}(y+1)$$
$$(K,L) = \left(\left(\frac{1-\alpha}{\alpha}\frac{w}{r}\right)^{\alpha}(y+1),\ \left(\frac{\alpha}{1-\alpha}\frac{r}{w}\right)^{1-\alpha}(y+1)\right)$$

substitute into CS to find conditions for validity
i.e. find conditions on r, w st $L, K > 0$

Check all conditions (incl sub conditions of each CS)

"Only one solution but which it is depends on the parameters"

Similarly for Q5

~~The relevant bordered Hessian is~~

~~$H =$~~

$\cancel{D_{K,L}\mathcal{L} = \left(\partial\mathcal{L}/\partial K \quad \partial\mathcal{L}/\partial L\right) = \left(r - \lambda_Y\left(\frac{1-\alpha}{K+1}\right) - \mu_K,\right.}$

$\cancel{= \left(r - \lambda_Y\left(\frac{1-\alpha}{K+1}\right) - \mu_K,\ w - \lambda_Y\left(\frac{\alpha}{L+1}\right) - \mu_L\right)}$

$\cancel{D^2_{K,L}\mathcal{L} = \begin{pmatrix} \partial^2\mathcal{L}/\partial K^2 & \partial^2\mathcal{L}/\partial K\partial L \\ \partial^2\mathcal{L}/\partial L\partial K & \partial^2\mathcal{L}/\partial L^2 \end{pmatrix}}$

~~$= ($~~

Suppose that only the positivity constraint on K binds,
then $\mu_K > 0$, $K = 0$, $\mu_L = 0$, $L \neq 0$.
By substitution into FOC_K, FOC_L,
$$r - \lambda_Y(1-\alpha) - \mu_K = 0,\quad w - \lambda_Y\left(\frac{\alpha}{L+1}\right) = 0$$
By substitution into C_Y,
$$(L+1)^{\alpha} = y+1,\quad L = (y+1)^{1/\alpha} - 1$$
$$(K,L) = \left(0,\ (y+1)^{1/\alpha} - 1\right)$$

Suppose that only the positivity constraint on L binds,
then $L = 0$.
By substitution into C_Y
$$(K+1)^{1-\alpha} = y+1,\quad K = (y+1)^{1/(1-\alpha)} - 1$$
$$(K,L) = \left((y+1)^{1/(1-\alpha)} - 1,\ 0\right)$$

~~By inspection, the constraint set is closed~~

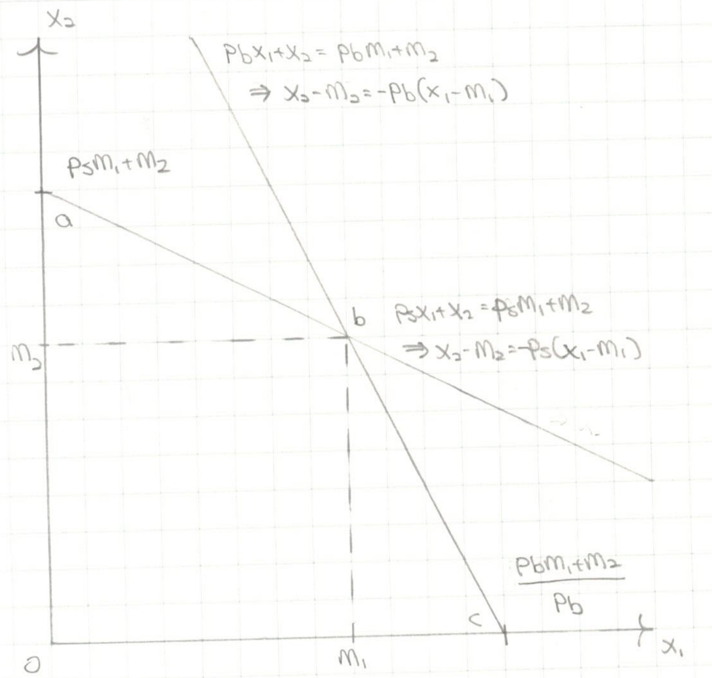
~~Evaluating $c(K,L)$ at the three candidate optima,~~

$\cancel{c(K_1,L_1) = r\left(\frac{1-\alpha}{\alpha}\frac{w}{r}\right)^{\alpha}(y+1) + w\left(\frac{\alpha}{1-\alpha}\frac{r}{w}\right)^{1-\alpha}(y+1)}$

$\cancel{= (y+1)\left[\frac{1-\alpha}{\alpha}\frac{w}{r^{1-1/\alpha}} + \frac{\alpha}{1-\alpha}\frac{r}{w^{1-1/\alpha}}\right]}$

~~$c(K_2,L_2) =$~~

5



The budget set is represented by area 0abc. p_s is the price, in units of good 2, at which the household can sell good 1. p_b is the price, in units of ~~good 2~~, at which the household can buy good 1.

The household's optimisation problem is

$\max_{x_1, x_2} u(x_1, x_2) = x_1^{\beta} + x_2^{\beta}$ st

$g_1(x_1, x_2) = x_1 \geq 0$, $g_2(x_1, x_2) = x_2 \geq 0$

$g_s(x_1, x_2) = p_s x_1 + x_2 \leq p_s m_1 + m_2$

$g_b(x_1, x_2) = p_b x_1 + x_2 \leq p_b m_1 + m_2$

where $0 < \beta < 1$; $m_1, m_2 > 0$; $0 < p_s < p_b$

$$Dg = \begin{pmatrix} \partial g_1/\partial x_1 & \partial g_1/\partial x_2 \\ \partial g_2/\partial x_1 & \partial g_2/\partial x_2 \\ \partial g_s/\partial x_1 & \partial g_s/\partial x_2 \\ \partial g_b/\partial x_1 & \partial g_b/\partial x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ p_s & 1 \\ p_b & 1 \end{pmatrix}$$

~~Suppose that~~ The constraint qualification holds iff the number of binding constraints $e \leq 2$. Suppose that both g_s and g_b bind, then, given that $m_1, m_2 > 0$, $x_1 = m_1 > 0$ and $x_2 = m_2 > 0$, so neither g_1 nor g_2 bind. Suppose that both g_1 and g_2 bind, then $x_1 = x_2 = 0$, $p_s x_1 + x_2 = 0 < p_s m_1 + m_2$, and $p_b x_1 + x_2 = 0 < p_b m_1 + m_2$, so neither g_s nor g_b bind. So $e \leq 2$, and the constraint qualification holds. The Kuhn-Tucker first-order conditions and complementary slackness conditions are necessary conditions for all optima.

$L(x_1, x_2; \mu_1, \mu_2; \lambda_s, \lambda_b)$
$= x_1^{\beta} + x_2^{\beta} + \mu_1 x_1 + \mu_2 x_2$
$- \lambda_s(p_s x_1 + x_2 - (p_s m_1 + m_2)) - \lambda_b(p_b x_1 + x_2 - (p_b m_1 + m_2))$

~~First~~ Argue that positivity constraints do not bind
Argue that constraint set is compact
Argue that problem is concave

Argue that constraint set ~~is not em~~ has non empty interior
so FOCs are necessary and sufficient for a solution
~~Argue that the~~ Function is continuous.

So by Weierstrass: solution exists
FOCs necessary and sufficient for maximum
↔ including CS

concave / convex problem ⇒ FOCs suff for max/min
non empty interior of constraint set (or CQ)
⇒ FOCs nec for max/min

Lagrangian sufficiency is the only other condition

$MU_{x_1} \to \infty$ as $x_1 \to 0$ so $\forall p_s, p_b$, $x_1 > 0$

concave prob: (weakly) concave obj (weakly) convex constraints

~~Lagrangian sufficiency~~

* save the points where CQ does not hold
evaluate obj fn on these points

FOCx_1: $\beta x_1^{\beta-1} + \mu_1 - \lambda_s p_s - \lambda_b p_b = 0$
FOCx_2: $\beta x_2^{\beta-1} + \mu_2 - \lambda_s - \lambda_b = 0$
CSμ_1 : $\mu_1 \geq 0$, $x_1 \geq 0$, $\mu_1 x_1 = 0$
CSμ_2 : $\mu_2 \geq 0$, $x_2 \geq 0$, $\mu_2 x_2 = 0$
CSλ_s: $\lambda_s \geq 0$, $p_s x_1 + x_2 \leq p_s m_1 + m_2$,
$\lambda_s (p_s x_1 + x_2 - (p_s m_1 + m_2)) = 0$
CSλ_b: $\lambda_b \geq 0$, $p_b x_1 + x_2 \leq p_b m_1 + m_2$,
$\lambda_b (p_b x_1 + x_2 - (p_b m_1 + m_2)) = 0$

Suppose both g_1 and g_2 bind, i.e. $x_1 = x_2 = 0$. Then, there are zero degrees of freedom, there are no (x_1, x_2) around $(x_1=0, x_2=0)$ such that both these constraints bind, and it is not necessary to verify that the second order condition holds at $(x_1=0, x_2=0)$

Suppose ~~that~~ ~~only~~ g_1 and not g_2 binds, i.e. $x_1 = 0$. Then, $x_1 = 0$, $\mu_1 > 0$, $x_2 > 0$, $\mu_2 = 0$. $u(x_1=0, x_2) = x_2^\beta$. ~~...~~ Given $0 < \beta < 1$, by inspection, $u(x_1=0, x_2)$ is increasing in x_2, for all x_2. So $\max_{x_2} = \text{argmax}_{x_2}\, u(x_1=0, x_2)$. Only $(x_1=0, x_2 = p_s m_1 + m_2)$ satisfies the FOCs and CSs.

$D_x L = (\partial L/\partial x_1 \;\; \partial L/\partial x_2)$ ~~$= (\beta x_1^{\beta-1} + \mu_1 - \lambda_s p_s$~~
$= (\beta x_1^{\beta-1} + \mu_1 - \lambda_s p_s - \lambda_b p_b \,,\; \beta x_2^{\beta-1} + \mu_2 - \lambda_s - \lambda_b)$

$$D_x^2 L = \begin{pmatrix} \partial^2 L/\partial x_1^2 & \partial^2 L/\partial x_1 \partial x_2 \\ \partial^2 L/\partial x_2 \partial x_1 & \partial^2 L/\partial x_2^2 \end{pmatrix} = \begin{pmatrix} \beta(\beta-1)x_1^{\beta-2} & 0 \\ 0 & \beta(\beta-1)x_2^{\beta-2} \end{pmatrix}$$

The relevant bordered Hessian

$$H = \begin{pmatrix} 0 & 1 & 0 \\ 1 & \beta(\beta-1)x_1^{\beta-2} & 0 \\ 0 & 0 & \beta(\beta-1)x_2^{\beta-2} \end{pmatrix}$$

$$|H| = -1 \begin{vmatrix} 1 & 0 \\ 0 & \beta(\beta-1)x_2^{\beta-2} \end{vmatrix} = -\beta(\beta-1)x_2^{\beta-2} > 0$$

By the bordered Hessian test, $(x_1=0, x_2 = p_s m_1 + m_2)$ is a strict local maximum.

Analogously, supposing that g_2 and not g_1 binds, only $(x_1 = p_b m_1 + m_2 / p_b, x_2 = 0)$ satisfies the FOCs and CSs.
The relevant bordered Hessian

$$H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \beta(\beta-1)x_1^{\beta-2} & 0 \\ 1 & 0 & \beta(\beta-1)x_2^{\beta-2} \end{pmatrix}$$

$$|H| = \begin{vmatrix} 0 & \beta(\beta-1)x_1^{\beta-2} \\ 1 & 0 \end{vmatrix} = -\beta(\beta-1)x_1^{\beta-2} > 0$$

By the bordered Hessian test, $(x_1 = p_b m_1 + m_2 / p_b, x_2 = 0)$ is a strict local maximum.

Suppose only g_s binds, then $x_1, x_2 > 0$, $\mu_1 = \mu_2 = \lambda_b = 0$, $\lambda_s > 0$
$p_s x_1 + x_2 = p_s m_1 + m_2$
$\beta x_1^{\beta-1} - \lambda_s p_s = 0$, $\beta x_2^{\beta-1} - \lambda_s = 0$
$\lambda_s = \beta/p_s\, x_1^{\beta-1} = \beta x_2^{\beta-1}$, $x_1^{\beta-1} = p_s x_2^{\beta-1}$, $x_1 = p_s^{1/\beta-1} x_2$
$p_s\, p_s^{1/\beta-1} x_2 + x_2 = p_s m_1 + m_2$
$x_2 = (p_s m_1 + m_2) / (p_s^{\beta/\beta-1} + 1)$
$x_1 = (p_s m_1 + m_2) / (p_s + p_s^{-1/\beta-1})$

The relevant bordered Hessian

$$H = \begin{pmatrix} 0 & p_s & 1 \\ p_s & \beta(\beta-1)x_1^{\beta-2} & 0 \\ 1 & 0 & \beta(\beta-1)x_2^{\beta-2} \end{pmatrix}$$

$$|H| = -p_s \begin{vmatrix} p_s & 0 \\ 1 & \beta(\beta-1)x_2^{\beta-2} \end{vmatrix} + \begin{vmatrix} p_s & \beta(\beta-1)x_1^{\beta-2} \\ 1 & 0 \end{vmatrix}$$

$$= -p_s^2\beta(\beta-1)x_2^{\beta-2} - \beta(\beta-1)x_1^{\beta-2} > 0$$

By the bordered Hessian test, $(x_1 = (p_s m_1 + m_2)/(p_s + p_s^{-1/\beta-1}),\ x_2 = (p_s m_1 + m_2)/(p_s^{\beta/\beta-1} + 1))$ is a strict local maximum.

Suppose only g_b binds, then $x_1, x_2 > 0$, $\mu_1 = \mu_2 = \lambda_s = 0$, $\lambda_b > 0$

$p_b x_1 + x_2 = p_b m_1 + m_2$

$\beta x_1^{\beta-1} - \lambda_b p_b = 0$, $\beta x_2^{\beta-1} - \lambda_b = 0$

$\lambda_b = 1/p_b\, \beta x_1^{\beta-1} = \beta x_2^{\beta-1}$, $x_1^{\beta-1} = p_b x_2^{\beta-1}$, $x_1 = p_b^{1/\beta-1} x_2$

$p_b^{\beta/\beta-1} x_2 + x_2 = p_b m_1 + m_2$

$x_2 = (p_b m_1 + m_2)/(p_b^{\beta/\beta-1} + 1)$

$x_1 = (p_b m_1 + m_2)/(p_b + p_b^{-1/\beta-1})$

By an analogous bordered Hessian test, $(x_1 = (p_b m_1 + m_2)/(p_b + p_b^{-1/\beta-1}),\ x_2 = (p_b m_1 + m_2)/(p_b^{\beta/\beta-1} + 1))$ is a strict local maximum.

Suppose that only g_s and g_b bind, then $x_1 = m_1$ and $x_2 = m_2$ and it is not necessary to verify the SOC ~~are~~ is satisfied.

The candidate optima are

$(x_1 = 0, x_2 = 0)$ ①

$(x_1 = 0, x_2 = p_s m_1 + m_2)$ ②

$(x_1 = p_b m_1 + m_2/p_b, x_2 = 0)$ ③

$(x_1 = (p_s m_1 + m_2)/(p_s + p_s^{-1/\beta-1}), x_2 = (p_s m_1 + m_2)/(p_s^{\beta/\beta-1} + 1))$ ④

$(x_1 = (p_b m_1 + m_2)/(p_b + p_b^{-1/\beta-1}), x_2 = (p_b m_1 + m_2)/(p_b^{\beta/\beta-1} + 1))$ ⑤

$(x_1 = m_1, x_2 = m_2)$ ⑥

~~By this~~ Since u is increasing in x_1, x_2, $(x_1 = 0, x_2 = 0)$ is not a maximum.

~~Each of the remaining candidate optima correspond to one of the following points.~~

~~It can be proven graphically that~~ $(x_1 = 0, x_2 = p_s m_1 + m_2)$ and $(x_1 = p_b m_1 + m_2/p_b, x_2 = 0)$ are not optima because at these points, $MRS = -MU_1/MU_2 = -(x_1/x_2)^{\beta-1} \neq MRT$, which is ~~either~~ equal to p_s at the former and p_b at the latter.

$MRS|_m = -(m_1/m_2)^{\beta-1}$

~~The optimum is ④ iff~~

The optimum is ④ iff $-MRS|_m < p_s < p_b$

The optimum is ⑤ iff $-MRS|_m > p_b > p_s$

The optimum is ⑥ iff $p_s < MRS|_m < p_b$

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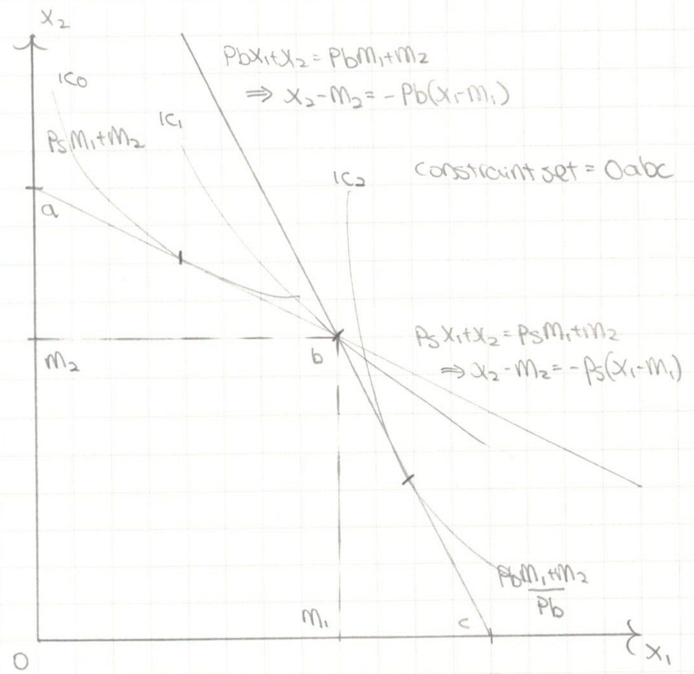

The household's optimisation problem is

$\max_{x_1, x_2} u(x_1, x_2) = x_1^{\beta} + x_2^{\beta}$ where $0 < \beta < 1$

subject to

$g_1(x_1, x_2) = x_1 \geq 0$

$g_2(x_1, x_2) = x_2 \geq 0$

$g_s(x_1, x_2) = p_s x_1 + x_2 \leq p_s m_1 + m_2$, ~~where m_1~~

$g_b(x_1, x_2) = p_b x_1 + x_2 \leq p_b m_1 + m_2$, where $m_1, m_2 > 0$; $0 < p_s < p_b$

p_s is the price, in ~~terms~~ units of good 2, at which the household can sell good 1. p_b is the price, in units of good 2, at which the household can buy good 1.

~~$\partial u / \partial x_1 = \beta x_1^{\beta - 1}$~~

By inspection of u, u is increasing in each of x_1 and x_2 for $x_1, x_2 \geq 0$. So only (x_1, x_2) that maximise some weighted sum ~~of x_1~~ $\alpha x_1 + (1-\alpha) x_2$, i.e. that lie on the frontier abc maximise u.

$MRS = -MU_1/MU_2 = -(x_1/x_2)^{\beta - 1} = -(x_2/x_1)^{1-\beta}$

At a, ~~MRS = 0~~ MRS is undefined. The household would substitute any marginal amount of x_2 for a marginal unit of x_1. MRT $= -p_s$, so the household can substitute p_s marginal units of x_2 for one marginal unit of x_1, and this substitution leaves the household with greater utility. So a is not an optimum.

At c, MRS is zero, MRT $= -p_b$, so the household can substitute one marginal unit of x_1 for p_b marginal units of x_2 and this substitution leaves the household better off. So c is not an optimum.

Suppose that some point x on line ab exclusive is the optimum. Then $MRS|_x = MRT = -p_s$.

$-(x_1/x_2)^{\beta-1} = -p_s$, $x_1^{\beta-1} = p_s x_2^{\beta-1}$, $x_1 = p_s^{1/\beta-1} x_2$

By substitution into $g_s = p_s m_1 + m_2$,

$p_s^{\beta/\beta-1} x_2 + x_2 = p_s m_1 + m_2$, $x_2 = p_s m_1 + m_2 / 1 + p_s^{\beta/\beta-1}$

$x_1 = \cancel{p_s^{1/\beta-1}}\ p_s m_1 + m_2 / p_s^{1/1-\beta} + p_s = 1/p_s\ p_s m_1 + m_2 / 1 + p_s^{\beta/1-\beta}$

Suppose that some point x on line bc exclusive is the optimum. Then $MRS|_x = -p_b$. By an analogous argument,

$x_1 = 1/p_b\ p_b m_1 + m_2 / 1 + p_b^{\beta/1-\beta}$, $x_2 = p_b m_1 + m_2 / 1 + p_b^{\beta/\beta-1}$

b is an optimum iff $p_s \leq -MRS|_b = (m_1/m_2)^{\beta-1} \leq p_b$

if $(m_1/m_2)^{\beta-1} < p_s < p_b$, then the optimum is

$(x_1 = p_s m_1 + m_2 / p_s^{1/1-\beta} + p_s,\ x_2 = p_s m_1 + m_2 / 1 + p_s^{\beta/1-\beta})$

if $p_s < p_b < (m_1/m_2)^{\beta-1}$, then the optimum is

$(x_1 = p_b m_1 + m_2 / p_b^{1/1-\beta} + p_b,\ x_2 = p_b m_1 + m_2 / 1 + p_b^{\beta/1-\beta})$