



The budget set is represented by area Oabc. P_s is the price, in units of good 2, at which the household can sell good 1. P_b is the price, in units of good 1, at which the household can buy good 1.

The household's optimisation problem is
 $\max_{x_1, x_2} u(x_1, x_2) = x_1^\alpha + x_2^\beta$ s.t.
 $g_1(x_1, x_2) = x_1 \geq 0, g_2(x_1, x_2) = x_2 \geq 0$
 $g_3(x_1, x_2) = P_s x_1 + x_2 \leq P_s M_1 + M_2$
 $g_4(x_1, x_2) = P_b x_1 + x_2 \leq P_b M_1 + M_2$
where $0 < \alpha < 1; M_1, M_2 > 0; 0 < P_s < P_b$

$$Dg = \begin{pmatrix} \frac{\partial g_1}{\partial x_1}, \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1}, \frac{\partial g_2}{\partial x_2} \\ \frac{\partial g_3}{\partial x_1}, \frac{\partial g_3}{\partial x_2} \\ \frac{\partial g_4}{\partial x_1}, \frac{\partial g_4}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ P_s & 1 \\ P_b & 1 \end{pmatrix}$$

Suppose ~~that~~ the constraint qualification holds if the number of binding constraints $e \leq 2$. Suppose that both g_3 and g_4 bind, then, given that $M_1, M_2 > 0$, $x_1 = M_1 > 0$ and $x_2 = M_2 > 0$, so neither g_1 nor g_2 bind. Suppose that both g_1 and g_3 bind, then $x_1 + x_2 = 0$, $P_s x_1 + x_2 = 0 < P_s M_1 + M_2$, and $P_b x_1 + x_2 = 0 < P_b M_1 + M_2$, so neither g_2 nor g_4 bind. So $e \leq 2$, and the constraint qualification holds. The Kuhn-Tucker first-order conditions and complementary slackness conditions are necessary conditions for all optima.

Very good!

Alternatively, given that this is a concave problem it is enough to argue that

the interior of the constraint set is NOT empty.

$$\begin{aligned} L(x_1, x_2; \mu_1, \mu_2; \lambda_S, \lambda_B) \\ = x_1^\alpha + x_2^\beta + \mu_1 x_1 + \mu_2 x_2 \\ - \lambda_S (P_s x_1 + x_2 - (P_s M_1 + M_2)) - \lambda_B (P_b x_1 + x_2 - (P_b M_1 + M_2)) \end{aligned}$$

Try to simplify the problem as much as possible before attacking it!

$x_1 > 0, x_2 > 0$ always?

$$FOC_{x_1}: \beta x_1^{\beta-1} + \mu_1 - \lambda_S p_S - \lambda_B p_B = 0$$

$$FOC_{x_2}: \beta x_2^{\beta-1} + \mu_2 - \lambda_S - \lambda_B = 0$$

$$CSy_1: \mu_1 \geq 0, x_1 \geq 0, \mu_1 x_1 = 0$$

$$CSy_2: \mu_2 \geq 0, x_2 \geq 0, \mu_2 x_2 = 0$$

$$CSS_{\lambda_S}: \lambda_S \geq 0, p_S x_1 + x_2 \leq p_S m_1 + m_2,$$

$$\lambda_S (p_S x_1 + x_2 - (p_S m_1 + m_2)) \geq 0$$

$$CSS_{\lambda_B}: \lambda_B \geq 0, p_B x_1 + x_2 \leq p_B m_1 + m_2,$$

$$\lambda_B (p_B x_1 + x_2 - (p_B m_1 + m_2)) \geq 0$$

Suppose both g_1 and g_2 bind, i.e. $x_1 = x_2 = 0$. Then, there are zero degrees of freedom, there are no (x_1, x_2) around $(x_1=0, x_2=0)$ such that both these constraints bind, and it is not necessary to verify that the second order condition holds at $(x_1=0, x_2=0)$

and not g_2

Suppose ~~that only~~ g_1 binds, i.e. $x_1 = 0$. Then, $x_1 = 0, \mu_1 > 0, x_2 > 0, \mu_2 = 0$. $u(x_1=0, x_2) = x_2^\beta$. Given $\beta > 0 < \beta < 1$, by inspection, $u(x_1=0, x_2)$ is increasing in x_2 , for all x_2 . So $\max_{x_2} u(x_1=0, x_2) = \arg\max_{x_2} x_2^\beta$. Only $(x_1=0, x_2 = p_S m_1 + m_2)$ satisfies the FOCs and CSS.

$$\begin{aligned} \nabla L &= (\frac{\partial L}{\partial x_1}, \frac{\partial L}{\partial x_2}) = (\beta x_1^{\beta-1} + \mu_1 - \lambda_S p_S, \\ &\quad \beta x_2^{\beta-1} + \mu_2 - \lambda_S - \lambda_B p_B) \\ D_x^2 L &= \begin{pmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} \end{pmatrix} \\ &= \begin{pmatrix} \beta(\beta-1)x_1^{\beta-2} & 0 \\ 0 & \beta(\beta-1)x_2^{\beta-2} \end{pmatrix} \end{aligned}$$

The relevant bordered Hessian

$$H = \begin{pmatrix} 0 & 1 & 0 \\ 1 & \beta(\beta-1)x_1^{\beta-2} & 0 \\ 0 & 0 & \beta(\beta-1)x_2^{\beta-2} \end{pmatrix} \quad ? \text{ when are you evaluating if?}$$

$$|H| = -1 \begin{vmatrix} 1 & 0 \\ 0 & \beta(\beta-1)x_2^{\beta-2} \end{vmatrix} = -\beta(\beta-1)x_2^{\beta-2} > 0$$

By the bordered Hessian test, $(x_1=0, x_2 = p_S m_1 + m_2)$ is a strict local maximum. X

Analogously, supposing that g_2 and not g_1 binds, only $(x_1 = p_B m_1 + m_2 / p_B, x_2=0)$ satisfies the FOCs and CSS. X

The relevant bordered Hessian

$$H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \beta(\beta-1)x_1^{\beta-2} & 0 \\ 1 & 0 & \beta(\beta-1)x_2^{\beta-2} \end{pmatrix}$$

$$|H| = \begin{vmatrix} 0 & \beta(\beta-1)x_1^{\beta-2} \\ 1 & 0 \end{vmatrix} = -\beta(\beta-1)x_1^{\beta-2} > 0$$

By the bordered Hessian test, $(x_1 = p_B m_1 + m_2 / p_B, x_2=0)$ is a strict local maximum.

Suppose only g_S binds, then $x_1, x_2 \geq 0, \mu_1 = \mu_2 = \lambda_B = 0, \lambda_S > 0$

$$p_S x_1 + x_2 = p_S m_1 + m_2$$

$$\beta x_1^{\beta-1} - \lambda_S p_S = 0, \beta x_2^{\beta-1} - \lambda_S = 0$$

$$\lambda_S = \beta/p_S x_1^{\beta-1} = \beta x_1^{\beta-1}, x_1^{\beta-1} = p_S x_2^{\beta-1}, x_1 = p_S^{1/\beta-1} x_2$$

$$p_S p_S^{1/\beta-1} x_2 + x_2 = p_S m_1 + m_2$$

$$x_2 = (p_S m_1 + m_2) / (p_S^{1/\beta-1} + 1)$$

$$x_1 = (p_S m_1 + m_2) / \beta (p_S + p_S^{-1/\beta-1})$$

✓ you need to check the CS! when this

Maybe it is worth seeing whether the problem is a concave problem? ..

Then FOC are sufficient..

✓ you need to check the CS! when this

The relevant bordered Hessian

$$H = \begin{pmatrix} 0 & p_s & 1 \\ p_s & \beta(\beta-1)x_1^{\beta-2} & 0 \\ 1 & 0 & \beta(\beta-1)x_2^{\beta-2} \end{pmatrix}$$

$$|H| = -p_s \begin{vmatrix} p_s & 0 \\ 1 & \beta(\beta-1)x_2^{\beta-2} \end{vmatrix} + \begin{vmatrix} p_s & \beta(\beta-1)x_1^{\beta-2} \\ 1 & 0 \end{vmatrix}$$

$$= -p_s^2 \beta(\beta-1)x_2^{\beta-2} - \beta(\beta-1)x_1^{\beta-2} > 0$$

By the bordered Hessian test,
 $(x_1 = (p_s M_1 + M_2) / (p_s + p_s^{-1/\beta-1}), x_2 = (p_s M_2 + M_2) / (p_s^{-\beta/\beta-1} + 1))$

is a strict local maximum.

Suppose only g_b binds, then $x_1, x_2 \geq 0, \mu_1 = \mu_2 = \lambda_S = 0, \lambda_B > 0$

$$p_b x_1 + x_2 = p_b M_1 + M_2$$

$$\beta x_1^{\beta-1} - \lambda_B p_b = 0, \beta x_2^{\beta-1} - \lambda_B = 0$$

$$\lambda_B = 1/p_b, \beta x_1^{\beta-1} = \beta x_2^{\beta-1}, x_1^{\beta-1} = p_b x_2^{\beta-1}, x_1 = p_b^{1/\beta-1} x_2$$

$$p_b^{\beta/\beta-1} x_2 + x_2 = p_b M_1 + M_2$$

$$x_2 = (p_b M_1 + M_2) / (p_b^{\beta/\beta-1} + 1)$$

$$x_1 = (p_b M_1 + M_2) / (p_b + p_b^{-1/\beta-1})$$

check the CS!

By an analogous bordered Hessian test,

$$(x_1 = (p_b M_1 + M_2) / (p_b + p_b^{-1/\beta-1}), x_2 = (p_b M_2 + M_2) / (p_b^{-\beta/\beta-1} + 1))$$

is a strict local maximum.

Suppose that only g_s and g_b bind, then $x_1 = M_1$ and $x_2 = M_2$

and it is not necessary to verify the SOR ~~etc~~ is satisfied.

The candidate optima are

$$(x_1 = 0, x_2 = 0) \quad ①$$

$$(x_1 = 0, x_2 = p_s M_1 + M_2) \quad ②$$

$$(x_1 = p_b M_1 + M_2 / p_b, x_2 = 0) \quad ③$$

$$(x_1 = (p_s M_1 + M_2) / (p_s + p_s^{-1/\beta-1}), x_2 = (p_s M_1 + M_2) / (p_s^{\beta/\beta-1} + 1)) \quad ④$$

$$(x_1 = (p_b M_1 + M_2) / (p_b + p_b^{-1/\beta-1}), x_2 = (p_b M_2 + M_2) / (p_b^{\beta/\beta-1} + 1)) \quad ⑤$$

$$(x_1 = M_1, x_2 = M_2) \quad ⑥$$

By the since U is increasing in x_1, x_2 , $(x_1 = 0, x_2 = 0)$ is not a maximum.

Each of the remaining candidate optima correspond to one of the following points

It can be proven graphically that $(x_1 = 0, x_2 = p_s M_1 + M_2)$ and $(x_1 = p_b M_1 + M_2 / p_b, x_2 = 0)$ are not optima because at these points, $MRS/m = -M\mu_1 / M\mu_2 = -(x_1/x_2)^{\beta-1} + MRT$, which is either equal to p_s at the former and p_b at the latter.

$$MRS/m = -(M_1/M_2)^{\beta-1}$$

the optimum is ④ iff

$$MRS/m < p_s < p_b$$

$$\text{the optimum is } ⑤ \text{ iff } -MRS/m > p_b > p_s$$

$$\text{the optimum is } ⑥ \text{ iff } p_s < MRS/m < p_b$$

You could
have
argued
this from
the start!

Can you state these conditions
in terms of the primitives?