Game Theory Paper 150608

1a Players : $N = \{1, n=2\}$

Actions : $B_i = [0, a]$ for each player $i \in N$

Strategies : ΔB_i for each player $i \in N$
where ΔB_i denotes the set of probability distributions over B_i

Payoffs : $\pi_i(b_i, b_{-i}) = \begin{cases} a - b_i & \text{if } b_i > b_{-i} \\ -b_i & \text{if } b_i < b_{-i} \\ a/2 - b_i & \text{if } b_i = b_{-i} \end{cases}$

for each player $i \in N$.

b Consider arbitrary pure strategy profile (b_1, b_2).
Suppose $b_1 \neq b_2$ and that, without loss of generality $b_1 < b_2$. ~~Then $\pi_1(b_1, b_2) =$~~ Suppose further that $b_1 > 0$. Then $\pi_1(b_1, b_2) = -b_1 < 0$. P1 has ~~profitable~~ strictly profitable deviation to $b_1' = 0$. $\pi_1(b_1', b_2) = 0$. So there is no such NE. Suppose instead that $b_1 = 0$, then $\pi_2(b_2, b_1) = a - b_2$. P2 has strictly profitable deviation to $b_2' = b_2/2$. $\pi_2(b_2', b_1) = a - b_2/2 > a - b_2$. So there is no such NE. Suppose instead that $b = b_1 = b_2 < a$, then ~~each~~ $\pi_1(b_1, b_2) = \pi_2(b_2, b_1) = a/2 - b$. Each player has strictly profitable deviation to $b' = b + \varepsilon$ for sufficiently small ε. $\pi(b', b) = a - b - \varepsilon > a/2 - b$ for $\varepsilon < a/2$ (and $b + \varepsilon \leq a$), so there is no such NE. Suppose instead that $b = b_1 = b_2 = a$, then each player has strictly profitable deviation to 0 which yields 0 rather than $-a/2$, so there is no such NE.

By cases, generalisation, there is no pure NE.

c Suppose for reductio that there exists some mixed NE $b^* = (b_1^*, b_2^*)$, where firms randomise over a finite number of bids. Let $\bar{b}_1$ and $\bar{b}_2$ denote the highest bid that each firm randomises over.

Suppose that $\bar{b}_1 \neq \bar{b}_2$ and without loss of generality, $\bar{b}_1 < \bar{b}_2$. Then P2 has strictly profitable deviation to b_2' which is identical to b_2^* except in reallocating all probability mass from $\bar{b}_2$ to $\bar{b}_2' = \bar{b}_1 + \bar{b}_2 / 2$. P2 plays $\bar{b}_2'$ on all occasions ~~when~~ where he would originally have played $\bar{b}_2$. Both bids always win, but $\bar{b}_2'$ yields a higher payoff because P2 pays less for winning with $\bar{b}_2'$ than with $\bar{b}_2$. So there is no such NE.

Suppose that $\bar{b}_1 = \bar{b}_2 = a$. Then each player i has strictly profitable deviation ~~by~~ reallocating all probability mass from $\bar{b}_i$ to 0. $\bar{b}_i$ yields zero payoff against ~~$b_{-i} \neq$~~ $b_{-i} \neq a$ and negative payoff against $\bar{b}_{-i} = a$, which is played with non-zero probability and so has negative expected payoff. $b_i = 0$ always has zero payoff, so this deviation is strictly profitable, so there is no such NE.

Suppose that $\bar{b}_1 = \bar{b}_2 < a$, then each player i has strictly profitable deviation to b_i' which is identical to b_i^* except in reallocating all probability mass from $\bar{b}_i$ to $\bar{b}_i' = \bar{b}_i + \varepsilon$ for sufficiently small ε. By so deviating, i pays ε more on the occasions that he plays $\bar{b}_i$ but wins $a/2$ more on the occasions that both players play their highest bids. This is strictly profitable for $\varepsilon < \bar{p}_{-i} \, a/2$, where $\bar{p}_{-i}$ is the probability b_{-i}^* assigns to $\bar{b}_{-i}$. So there is no such NE.

By cases, reductio, generalisation, there is no mixed NE where players mix over a finite number of pure actions.

d ~~Suppose both firms randomise.~~ Consider the strategy profile $b^* = (b_1^*, b_2^*)$ where each b_i^* is a uniform distribution over $B_i = [0, a]$.

$$
\begin{aligned}
\pi_i(b_i, b_{-i}^*) &= P(b_i < b_{-i}^*)(-b_i) + P(b_i > b_{-i}^*)(a - b_i) \\
&= (1 - b_i/a)(-b_i) + (b_i/a)(a - b_i) \\
&= (b_i/a)a - b_i \\
&= 0
\end{aligned}
$$

$b_i = b_{-i}^*$ occurs with zero probability given that b_{-i}^* has a continuous distribution.

From the above, every pure action yields payoff zero against b_{-i}^*, so b_i^* and any (potentially degenerate) mix b_i' yields payoff zero against b_{-i}^*, so neither player has a strictly profitable deviation and b^* is indeed a NE.

Each player has expected payoff zero.

e $\pi(0, b^0 = 1/2) = 0$, $\pi(1/4, b^0) = -1/4$,

$\pi(1/2, b^0) = a/2 - 1/2 = 0$, $\pi(3/4, b^0) = a - 3/4 = 1/4$

$\pi(1, b^0) = a - 1 = 0$

The best response against level 0 player's strategy b^0 is $3/4$ with certainty because

$\frac{3}{4}$ is a strict best response. $b_1 = \frac{3}{4}$

$\pi(0, \cancel{\tfrac{3}{4}}\, b' = \frac{3}{4}) = 0$, $\pi(\frac{1}{4}, b') = -\frac{1}{4}$, $\pi(\frac{1}{2}, b') = -\frac{1}{2}$,
$\pi(\frac{3}{4}, b') = \frac{a}{2} - \frac{3}{4} = -\frac{1}{4}$, $\pi(1, b') = a - 1 = 0$.

0 and 1 are ~~the~~ ~~pure~~ the pure best responses against b'. $b^2 = \frac{1}{2}0 + \frac{1}{2}1$.

$\pi(0, b^2) = 0$, $\pi(\frac{1}{4}, b^2) = \frac{1}{2}(a - \frac{1}{4}) + \frac{1}{2}(-\frac{1}{4}) = \frac{1}{4}$,
$\pi(\frac{1}{2}, b^2) = \frac{1}{2}(a - \frac{1}{2}) + \frac{1}{2}(-\frac{1}{2}) = 0$,
$\pi(\frac{3}{4}, b^2) = \frac{1}{2}(a - \frac{3}{4}) + \frac{1}{2}(-\frac{3}{4}) = -\frac{1}{4}$
$\pi(1, b^2) = \frac{1}{2}(a - 1) + \frac{1}{2}(\frac{a}{2} - 1) = -\frac{1}{4}$

$\frac{1}{4}$ is the only pure best response against b^2, $b^3 = \frac{1}{4}$.

$\pi(0, b^3) = 0$, $\pi(\frac{1}{4}, b^3) = \frac{a}{2} - \frac{1}{4} = \frac{1}{4}$, $\pi(\frac{1}{2}, b^3) = a - \frac{1}{2} = \frac{1}{2}$
$\pi(\frac{3}{4}, b^3) = a - \frac{3}{4} = \frac{1}{4}$, $\pi(1, b^3) = a - 1 = 0$

$\frac{1}{2}$ is the only pure best response against b^3, $b^4 = \frac{1}{2}$

$\pi(0, b^4) = 0$, $\pi(\frac{1}{4}, b^4) = -\frac{1}{4}$, $\pi(\frac{1}{2}, b^4) = \frac{a}{2} - \frac{1}{2} = 0$,
$\pi(\frac{3}{4}, b^4) = a - \frac{3}{4} = \frac{1}{4}$, $\pi(1, b^4) = a - 1 = 0$

$\frac{3}{4}$ is the only pure best response against b^4, $b^5 = \frac{3}{4}$.

So $z = 4$ is the required z.

~~Each~~ ~~Suppose that~~

~~From the above, given that b^n is entirely a~~
~~function of b^{n+1}, by induction, b^n for all n is~~

$b^0 = \frac{1}{2}$, $b^1 = \frac{3}{4}$, $b^2 = \frac{1}{2}0 + \frac{1}{2}1$, $b^3 = \frac{1}{4}$, ~~$b^4 = \frac{1}{2}$~~,
$b^4 = \frac{1}{2}$, $b^5 = \frac{3}{4}$

~~By induction~~ The argument in (e) suffices to
prove that if $b^n = \frac{1}{2}$, then $b^{n+1} = \frac{3}{4}$ and if
$b^n = \frac{3}{4}$, then $b^{n+1} = \frac{1}{2}0 + \frac{1}{2}1$ and so on. Then,
~~by hypothetical~~ if $b^n = \frac{1}{2}$ then $b^{n+1} = \frac{3}{4}$
then $b^{n+2} = \frac{1}{2}0 + \frac{1}{2}1$ then $b^{n+3} = \frac{1}{4}$ then $b^{n+4} =$
$\frac{1}{2} = b^n$. Similarly for $b^n = \frac{3}{4}, \frac{1}{2}0 + \frac{1}{2}1, \frac{1}{4}$.
~~For all~~ By induction, for all n, $b^n = \frac{1}{2}, \frac{3}{4}$,
$\frac{1}{2}0 + \frac{1}{2}1$, or $\frac{1}{4}$. Then, for all n, $b^n = b^{n+4}$. Then,
for all ~~k,~~ ~~k~~ $k \geq 4$, $b^k = b^{k-z}$.

2a Let C and F denote the consumer and the firm respectively. Let σ_C and σ_F denote their respective strategies. $\sigma_C = pS + (1-p)N$, where ~~*~~ S and N denote steal and do not steal respectively. ~~The~~ F's pure actions are $E = [0,1]$

$$\pi_F(e, \sigma_C) = -e^2/2 - p(1-e)a$$
$$= -e^2/2 - ap + ape$$

where payoff in the event of zero effort and where the widget W is stolen with probability 0 is normalised to 0.

Given σ_C, F's payoff maximisation problem is $\max_e \pi_F(e, \sigma_C)$

FOC: $-e + ap = 0 \Rightarrow e = ap$

SOC: $-1 < 0$

$e = ap$ solves F's payoff maximisation problem.

F's optimal level of effort e given σ_C is ap.

b Consider arbitrary NE $\sigma^* = (\sigma_C^*, \sigma_F^*)$. Suppose $\sigma_C^* = S$, i.e. $p^* = 1$. By definition of NE, F plays a best response. ~~$e^* =$~~ $\sigma_F^* = e^* = ap^* = a$. By definition of NE, C plays a best response. This is iff $\pi_C(S, \sigma_F^*) \geq \pi_C(N, \sigma_F^*) \Leftrightarrow (1-e^*)a + e^*(-qa) \geq 0$ ~~$\Leftrightarrow a - ae^* - aqe^* \geq 0 \Leftrightarrow$~~ $\Leftrightarrow (1-a)a + a(-qa) \geq 0 \Leftrightarrow -qa^2 - a^2 + a \geq 0 \Leftrightarrow qa + a \leq 0$. Iff $qa + a \leq 0$, then σ^* is a NE, and a NE ~~for~~ where C ~~always~~ plays S with certainty exists.

c Consider arbitrary ~~SS NE~~ strategy profile $\sigma = (\sigma_C, \sigma_F)$ such that $\sigma_C = N$, i.e. $p = 0$. Suppose for reductio that this is a NE. By definition of NE, F plays a best response, $e = ap = 0$. Then $\pi_C(S, \sigma_F^*) = (1-e)a + e(-qa) = a > 0 = \pi_C(N, \sigma_F^*)$. By definition of NE, C plays a best response, so C plays S with certainty. By reductio, generalisation no strategy profile such that C plays N with certainty is a NE. No such NE exists.

Intuitively, C does not play pure N at NE because if C plays pure N, then $e = 0$ is strictly optimal for F (if C never ~~st~~ steals, effort ~~at~~ exerted to catch stealing is costly and entirely fruitless), but if F plays $e = 0$, then S is strictly optimal for C (if F exerts zero effort to catch stealing, stealing is never caught and is strictly optimal). So there is no equilibrium where C plays S with certainty.

d Consider arbitrary NE ~~&~~ $\sigma^* = (\sigma_C^*, \sigma_F^*)$. Suppose that $p^* \in (0,1)$, i.e. C mixes non-degenerately. By definition of NE, C has no profitable deviation. This is only if C is indifferent between S and N given σ_F^*. $\pi_C(S, \sigma_F^*) = \pi_C(N, \sigma_F^*) \Leftrightarrow (1-e^*)a + e^*(-qa) = 0 \Leftrightarrow a - ae^* - aqe^* = 0 \Leftrightarrow e^* = a/(a+qa) = 1/(1+q)$. By definition of NE, F plays a best response i.e. chooses the optimal ~~e^* given~~ $e = e^*$ given p^*. From (a), $e^* = ap^*$. Then $ap^* = 1/(1+q) \Leftrightarrow p^* = 1/(a+qa)$. $e^* \in [0,1]$ and $p^* \in (0,1)$ iff $q \geq 0$ and $a + qa > 1$. When these conditions are satisfied, $\sigma^* = (\sigma_C^*, \sigma_F^*)$, where $\sigma_C^* = p^*S + (1-p^*)N$, $p^* = 1/(a+qa)$, $\sigma_F^* = e^* = ap^* = 1/(1+q)$ is a NE. From the above, at σ^*, C and F play mutual best responses.

e

Date

3a

	L	M	R
t	4	1	2
	4	0	3
m	0	y=1	-1
	1	x=1	1
b	3	1	2
	2	-1	2

Best responses underlined.

A strategy is rationalisable iff it is a best response to some (potentially correlated) mix of other player's rationalisable strategies. Equivalently, a rationalisable strategy is one that survives iterated elimination of strategies that are not best responses to yet uneliminated strategies. Pearce's Lemma states that a strategy is a best response to some pure or mixed strategy iff it is not strictly dominated by some pure or mixed strategy. Then, all and only rationalisable strategies survive iterated strict dominance.

~~By inspection, each pure action is a unique best response against some pure action of the other player. So no strategy is strictly dominated, and every strategy is rationalisable~~

~~L ≻ R~~, t ≻ b, L ≻ R, then no remaining strategies are strictly dominated. All and only t, m, L, M survive iterated strict dominance. All and only t, m, L, M are rationalisable.

b Only rationalisable strategies are played at NE because at NE players play mutual best responses, so it is sufficient to restrict attention to the reduced game

	L	M
t	4	1
	4	0
m	0	1
	1	1

Best responses underlined. By inspection, there are two pure NE where players play pure mutual best responses.

Best responses are unique, so there are no hybrid NE.

Suppose there exists mixed NE $\sigma^* = (\sigma_1^*, \sigma_2^*)$ where $\sigma_1^* = pt + (1-p)m$, $\sigma_2^* = qL + (1-q)M$. By definition of NE, P1 has no profitable deviation, so P1 is indifferent between t and M. $\pi_1(t, \sigma_2^*) = \pi_1(m, \sigma_2^*) \Leftrightarrow 4q = 1 \Leftrightarrow q = 1/4$. By similar argument, $p = 1/4$. The unique mixed NE is $\sigma^* = (\frac{1}{4}t + \frac{3}{4}m, \frac{1}{4}L + \frac{3}{4}M)$.

The NE are (t, L), (m, M), $(\frac{1}{4}t + \frac{3}{4}m, \frac{1}{4}L + \frac{3}{4}M)$.

c

	L	M	R
t	4	1	2
	4	0	3
m	0	y=0	-1
	1	x=0	1
b	3	1	2
	2	-1	2

Best responses underlined.

t ≻ b, L ≻ R

Only t, m, L, M are rationalisable. It is sufficient to consider only these when checking NE.

	L	M
t	4	1
	4	0
m	0	0
	1	0

Best responses underlined. By inspection there are two pure NE where players play ~~mutual~~ pure mutual best responses. These are (t, L) and (m, M)

Suppose there exists NE $\sigma^* = (\sigma_1^*, \sigma_2^*)$ such that P1 mixes. ~~σ_1^*~~ Then P1 has no profitable deviation and P1 is indifferent. This is iff P2 plays M with certainty. Then ~~by definition~~, P2 has no profitable deviation, which is iff P1 plays m with certainty. By reductio, there is no NE such that P1 mixes. By symmetry, there is no NE such that P2 mixes.

No mixed NE exists because each player only mixes if the other player plays m (or M), which is a best response only against pure M (or m)

d This is a Bayesian game where each player's type is (or corresponds to) that player's observation of x or y. ~~For $\beta = 1$, types are perfectly correlated and P1 is type 1 iff P2 is type 1. For $\beta = 1/4$, types are uncorrelated.~~

~~For $\beta = 1$, the state space is $\Omega = \{\omega\}$. Each player has a perfectly informative~~

types are correlated because for all $\beta \in (1/4, 1)$, ~~$P(x=1, y$~~ $P(x=1 \mid y=0) \neq P(x=1 \mid y=1)$ and likewise for y.

The Bayesian game is formally defined as follows.

States: $\Omega = \{11, 10, 01, 00\}$

Signals: $t_1(\omega) = 1$ iff $\omega = 11, 10$, 0 otherwise

$t_2(\omega) = 1$ iff $\omega = 11, 01$, 0 otherwise.

Beliefs: $P_1(11 \mid t_1=1) = \beta / (\beta + {1-\beta}/3) = 3\beta/1+2\beta$

$P_1(10 \mid t_1 = 1) = {1-\beta}/3 \,/\, (\beta + {1-\beta}/3) = 1-\beta/1+2\beta$

$P_1(01 \mid t_1=1) = P_1(00 \mid t_1 = 1) = 0$

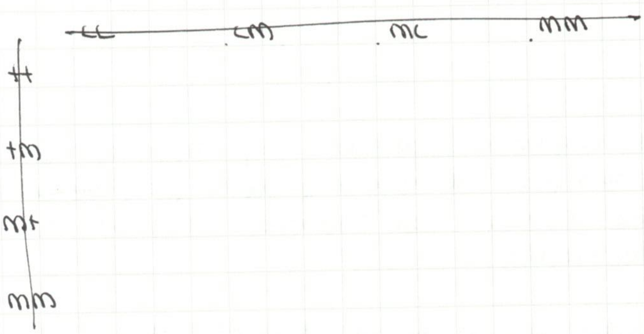
similarly for $t_1 = 0$, P2

Players: $N = \{1, n=2\}$

Actions: $A_1 = \{t, m, b\}$, $A_2 = \{L, M, R\}$

Payoffs ~~are as~~ given in the payoff matrix.

e Each player's strategy is a type-contingent complete plan of action. P1's ~~strate~~ pure strategies are tt, tm, mt, mm and P2's pure strategies are LL, LM, ML, MM, where the first letter denotes that player's action ~~as~~ iff ~~is~~ type 1.



suppose ~~there is~~ that $\sigma^* = (mt, ML)$ is a BNE, then no player-type has a profitable deviation.

$$\pi_1(mt, MT; t_1 = 1) = P(\omega = 11 \mid t_1 = 1)(1) + P(\omega = 10 \mid t_1 = 1)(1) = 1$$

$$\pi_1(tx, MT; t_1 = 1) = P(\omega = 11 \mid t_1 = 1)(0) + P(\omega = 10 \mid t_1 = 1)(4) = 4-4\beta/1+2\beta$$

Type 1 P1 has no profitable deviation iff

$\pi_1(mt, MT; t_1 = 1) \geq \pi_1(tx, MT; t_1 = 1) \Leftrightarrow$

$1 \geq 4-4\beta/1+2\beta \Leftrightarrow$

$1 + 2\beta \geq 4 - 4\beta \Leftrightarrow$

$6\beta \geq 3 \Leftrightarrow$

$\beta \geq 1/2$

$$\pi_1(mt, MT; t_1 = 0) = P(\omega = 01 \mid t_1 = 0)(0) + P(\omega = 00 \mid t_1 = 0)(4) = 1/2(0) + 1/2(4) = 2$$

$$\pi_1(xm, MT; t_1 = 0) = P(\omega = 01 \mid t_1 = 0)(1) + P(\omega = 00 \mid t_1 = 0)(0) = 1/2$$

Type 0 P1 has no profitable deviation iff $\pi_1(mt, MT; t_1 = 0) \geq \pi_1(xm, MT; t_1 = 0) \Leftrightarrow$ $2 \geq 1/2$, which holds for all β

Then neither type of P1 has a profitable deviation iff $\beta \geq 1/2$. By symmetry, neither type of P2 has a profitable deviation iff $\beta \geq 1/2$. Then, the given strategy profile is a BNE iff $\beta \geq 1/2$.

f The four candidate BNE such that each player plays an unconditional pure action are ~~(tt,~~ (tt, LL), (tt, MM), (mm, LL), (mm, MM).

In ex ante expectation,

$\pi_1(mm, LL) = 1$, $\pi_1(tt, LL) = 4$.

P1 has a strictly profitable deviation from the third candidate BNE in ex ante expectation, so this is not a BNE.

Similarly, in ex ante expectation

$\pi_2(MM, tt) = 1$, $\pi_2(LL, tt) = 4$

P2 has a strictly profitable deviation, the second candidate BNE fails to this deviation.

For (mm, MM), in interim expectations, deviation by any player-type yields zero with certainty, equilibrium play yields one or zero. so deviation is not strictly profitable for every player-type, (mm, MM) is a BNE).

Similarly for (tt, LL) in interim expectations, deviation by any player-type yields one, whereas equilibrium play yields 4. There is no profitable deviation, (tt, LL) is a BNE.

4a Solve for the SPE by backward induction.

Firm 2 takes θ and q_1 as given.
Then firm 2's profit maximization problem is
$\max_{q_2} \pi_2(q_2, q_1; \theta)$
FOC: $\partial \pi_2 / \partial q_2 = (\theta - q_1 - q_2) - q_2 = 0 \Rightarrow$
$q_2 = \theta - q_1/2$
SOC: $\partial^2 \pi_2 / \partial q_2^2 = -2 < 0$
$q_2 = \theta - q_1/2$ solves firm 2's profit maximization problem.

Given common knowledge of rationality and incentives, firm 1 takes θ and $q_2 = \theta - q_1/2$ as given.
Then firm 1's profit maximization problem is
$\max_{q_1} \pi_1(q_1, q_2 = \theta - q_1/2; \theta)$
$\pi_1(q_1, q_2 = \theta - q_1/2; \theta)$
$= (\theta - q_1 - \theta - q_1/2) q_1$
$= 1/2 (\theta - q_1) q_1$
FOC: $1/2(\theta - q_1 - q_1) = 0 \Rightarrow$
$q_1 = \theta/2$
SOC: $-2 < 0$
$q_1 = \theta/2$ solves firm 1's profit maximization problem.

The SPE is the strategy profile under which firm 1 plays $q_1 = \theta/2$ and firm 2 plays $\theta - q_1/2$. On the equilibrium path, $q_1 = \theta/2$, $q_2 = \theta/4$.

At the SPE outcome, $\pi_1 = (\theta - \theta/2 - \theta/4)\theta/2 = \theta/8$,
$\pi_2 = (\theta - \theta/2 - \theta/4)\theta/4 = \theta/16$

b ~~From FOC~~
$\pi_2(q_2, q_1; \mu(q_1))$
$= \mu(q_1)[(3 - q_1 - q_2) q_2] + (1 - \mu(q_1))[(4 - q_1 - q_2) q_2]$

~~Firm 2's~~ Supposing that firm 2 is risk-neutral and maximizes expected profit, firm 2's profit maximization problem is
$\max_{q_2} \pi_2(q_2, q_1; \mu(q_1))$
FOC: $\mu(q_1)[3 - q_1 - 2q_2] + (1 - \mu(q_1))[4 - q_1 - 2q_2] = 0 \Rightarrow$
$[3\mu(q_1) + 4(1 - \mu(q_1)) - q_1] / 2 = q_2 \Rightarrow$
$q_2 = [4 - \mu(q_1) - q_1]/2$

c $\pi_1^*(\theta) \cancel{= (\theta - q_1(\theta) - q_2(q_1(\theta), \mu(q_1(\theta))) q_1(\theta)}$
$= (\theta - q_1(\theta) - q_2(q_1(\theta), \mu(q_1(\theta)))) q_1(\theta)$
$\cancel{= (\theta - q_1(\theta) - q_2(q_1(\theta)}$

$= \begin{cases} (3 - q_1(3) - q_2(q_1(3), 1)) q_1(3) & \text{if } \theta = 3 \\ (4 - q_1(4) - q_2(q_1(4), 0)) q_1(4) & \text{if } \theta = 4 \end{cases}$

$= \begin{cases} (3 - q_1(3) - 3 - q_1(3)/2) q_1(3) & \text{if } \theta = 3 \\ (4 - q_1(4) - 4 - q_1(4)/2) q_1(4) & \text{if } \theta = 4 \end{cases}$

$= \begin{cases} 1/2 (3 - q_1(3)) q_1(3) & \text{if } \theta = 3 \\ 1/2 (4 - q_1(4)) q_1(4) & \text{if } \theta = 4 \end{cases}$

$\cancel{\pi_1(q_1(3)}$

Profit from choosing $q_1(3)$ when $\theta = 4$
$= (4 - q_1(3) - 3 - q_1(3)/2) q_1(3)$
$= (5/2 - q_1(3)/2) q_1(3)$
$= 1/2 (5 - q_1(3)) q_1(3)$

~~At PBE, $q_1(4)$ is sequentially rational if $\theta=4$, so~~
$\cancel{1/2(4 - q_1(4)) q_1(4) \geq 1/2(5 - q_1(3)) q_1(3) \Rightarrow}$
$\cancel{4q_1(4) - q_1^2(4) \geq 5q_1(3) - q_1(3)^2 \Rightarrow}$
$\cancel{q_1(4) \geq q_1(3)}$

Profit from choosing $q_1(4)$ when $\theta = 3$
$= (3 - q_1(4) - 4 - q_1(4)/2) q_1(4)$
$= 1/2 (2 - q_1(4)) q_1(4)$

At PBE, $q_1(4)$ is sequentially rational if $\theta = 4$ and $q_1(3)$ is sequentially rational if $\theta = 3$.
$1/2(4 - q_1(4)) q_1(4) \geq 1/2(5 - q_1(3)) q_1(3)$
$1/2(3 - q_1(3)) q_1(3) \geq 1/2(2 - q_1(4)) q_1(4)$
$\Rightarrow$
$1/2(2) q_1(4) \geq 1/2(2) q_1(3)$
$\Rightarrow$
$q_1(4) \geq q_1(3)$

q_1 is weakly increasing in θ

d Suppose for reductio that there exists some PBE such that $q_1(3) = 3/2$, $q_1(4) = 4/2$. ~~Then $\mu(3/2) = 1$, $\mu(4/2) = 0$.~~ By Bayesian beliefs, $\mu(3/2) = 1$, $\mu(4/2) = 0$. Then, by sequential rationality, $q_2(q_1(3), \mu(q_1(3))) = 3 - q_1(3)/2 = 3/4$ and $q_2(q_1(4), \mu(q_1(4))) = 4 - q_1(4)/2 = 1$. Suppose that $\theta = 4$. Then playing $q_1(3)$ yields $\pi_1 = (4 - 3/2 - 3/4) 3/2 = 21/8$, whereas playing $q_1(4)$ yields $\pi_1 = (4 - 2 - 1)(2) = 2$. So $q_1(4)$ is not sequentially rational when $\theta = 4$. By reductio, there is no such PBE.

There is no PBE such that firm 1 plays the Stackelberg quantity found in (a) because if firm 2 formed its belief about the state of demand on the assumption that firm 1 plays the Stackelberg quantities, then firm 1 has incentive to signal that demand is low when it is in fact high so firm 2 chooses lower output that has a large benefit for firm 1. If firm 2 had such beliefs, firm 1 has a strictly profitable deviation from the strategy that such beliefs assume.

e ~~Consider~~ Suppose that at PBE $q_1(4) = 2$, ~~then~~ and that $q_1(3) \neq q_1(4)$ (the eqm is a separating eqm). Then ~~$\mu(2) = 0$ so~~ by Bayesian beliefs $\mu(2) = 0$. Then by sequential rationality, $q_2(q_1(4)) = 4 - q_1(4)/2 = 1$. Then when $\theta = 4$, $\pi_1 = (4 - 2 - 1) 2 = 2$. When $\theta = 3$, playing $q_1 = 2$ ~~yields~~ induces $q_2 = 1$ hence yields $\pi_1 = (3 - q_1 - q_2) q_1 = 0$, ~~so~~ deviation to $q_1(4)$ is not sequentially rational. When $\theta = 4$, playing $q_1(3)$ ~~yields~~ induces $q_2 = \frac{3 - q_1(3)}{2}$ hence yields $\pi_1 = \left(4 - q_1(3) - \frac{3 - q_1(3)}{2}\right) q_1(3) = \frac{1}{2}(5 - q_1(3)) q_1(3)$

Playing $q_1(4)$ is sequentially rational only if $\frac{1}{2}(5 - q_1(3))(q_1(3)) \leq 2 \Leftrightarrow (5 - q_1(3))(q_1(3)) \leq 4 \Leftrightarrow$ $q_1(3) \leq 1$ or $q_1(3) \geq 4$ (reject because $q_1(3) \geq 4$ yields negative profit so is not sequentially rational). If firm 1 has profit less than 0.5 at ~~PBE~~ when $\theta = 3$, then firm 1 has strictly ~~suppose firm 1 chooses $q_1 = 1$ when $\theta = 3$~~ profitable deviation to $q_1 = 1$ when $\theta = 3$. If ~~firm 2 believes $\theta = 4$~~ $\mu(1) = 1$, then $q_2(q_1 = 1) = 1$, $\pi_1 = 1$. If $\mu(1) = 0$, then $q_2(q_1 = 1) = \frac{4-1}{2} = \frac{3}{2}$, $\pi_1 = (3 - 1 - \frac{3}{2}) \cdot 1 = 0.5$. So regardless of $\mu(1)$, deviation to $q_1 = 1$ is strictly profitable. So $(\pi_1 \mid \theta = 3) \geq 0.5$ at PBE.

$(\pi_1 \mid \theta = 3) = \left(3 - q_1(3) - \frac{3 - q_1(3)}{2}\right) q_1(3) = \frac{1}{2}(3 - q_1(3)) q_1(3) \geq 0.5 \Leftrightarrow (3 - q_1(3)) q_1(3) \geq 1 \Leftarrow q_1(3) \geq 0.5$.

?

5a

	L	R
U	3 3	$\underline{5}$ $\underline{1}$
M	$\underline{0}$ 0	$\underline{0}$ 0
D	$\underline{1}$ $\underline{4}$	0 0

Best responses underlined.

By inspection, there are two pure NE (U,R) and (D,L), where players play pure mutual best responses.

U ≻ M. M is played with zero probability at any NE. In the reduced game where M is eliminated, ~~≥~~ best responses are unique, so there are no hybrid NE.

Suppose that P1 mixes at NE, ~~σ_1^*~~ i.e. $\sigma_1^* = pU + (1-p)D$ for $p \in (0,1)$. Then by definition of NE P1 has no profitable deviation, so P1 is indifferent between U and D. $\pi_1(U,\sigma_2^*) = \pi_1(D,\sigma_2^*) \Leftrightarrow 3q + (1-q) = 4q \Leftrightarrow q = 1/2$, where $\sigma_2^* = qL + (1-q)R$. Then P2 is also indifferent. $\pi_2(L,\sigma_1^*) = \pi_2(R,\sigma_1^*) \Leftrightarrow 3p + (1-p) = 5p \Leftrightarrow$ ~~$p=1/3$~~ $1-p = 2p \Leftrightarrow p = 1/3$.
The unique mixed NE is $(1/3 U + 2/3 D,\ 1/2 L + 1/2 R)$

The NE are (U,R), (D,L), $(1/3 U + 2/3 D + 1/2 L + 1/2 R)$.
The mixed NE yields payoffs $(2, 5/3)$

b Consider the strategy profile under which players play (U,L) in the first period and every period not immediately following a deviation, and play (D,R) in every period immediately following ~~a~~ a deviation (including deviations from (D,R)).

$$(U,L) \circlearrowleft (U,L) \underset{(D,R)}{\overset{(U,L)}{\rightleftarrows}} (D,R) \circlearrowleft \neg(D,R)$$

This strategy ~~~~ profile punishes deviation ~~by~~ by (D,R) which yields each player's minimum feasible payoff, so ~~the δ such that this~~ the ~~δ such that this is a SPE is a minimum.~~ minimum δ such that (U,L) forever is sustainable in SPE is the minimum δ such that this strategy profile is a SPE.

In the cooperation phase (any period not immediately following a deviation), ~~the best~~ optimal one shot deviation by P1 is to D, this yields 4 then 0 then 3 indefinitely. Eqm play yields 3 indefinitely. Compare PVs.

$3 + 3\delta + 3\delta^2/(1-\delta) \geq 4 + 0\delta + 3\delta^2/(1-\delta) \Leftrightarrow 3\delta \geq 1 \Leftrightarrow \delta \geq 1/3$. Optimal one shot deviation by P2 is to R, which yields 5 then 0 then 3 indefinitely. Eqm play yields 3 indefinitely. Compare PVs.
$3 + 3\delta + 3\delta^2/(1-\delta) \geq 5 + 0\delta + 3\delta^2/(1-\delta) \Leftrightarrow 3\delta \geq 2 \Leftrightarrow \delta \geq 2/3$.

In the punishment phase (immediately following a deviation), optimal one shot deviation by either player yields 1 then 0 then 3 indefinitely. Eqm play yields 0 then 3 indefinitely. Compare PVs. ~~ϕ~~ $0 + \cancel{3\delta} 3\delta + 3\delta^2/(1-\delta) \geq 1 + 0\delta + 3\delta^2/(1-\delta) \Leftrightarrow 3\delta \geq 1 \Leftrightarrow \delta \geq 0$.

Iff $\delta \geq 2/3$, neither player has a profitable one shot deviation at any history, then the given strategy profile is a SPE. The minimum δ such that (U,L) forever is sustainable in SPE with a one period punishment is $\delta = 2/3$.

c The required SPE is the strategy profile under which players play (U,L) in the first period and every subsequent period ~~provide~~ where there has been no prior deviation and (U,R) in any period where there has been a deviation and P1 deviated (weakly) earlier and (D,L) in any period where there has been no prior deviation and P2 deviated (strictly) earlier.

In the cooperation phase (any period where there has been no prior deviation), optimal one shot deviation by P1 is to D which yields 4 then 1 indefinitely. Eqm play yields 3 indefinitely. Compare PVs. ~~$3 + 3\delta + 3\delta^2/(1-\delta) \geq$~~
$3 + 3\delta/(1-\delta) \geq 4 + 1\delta/(1-\delta) \Leftrightarrow 2\delta/(1-\delta) \geq 1 \Leftrightarrow 2\delta \geq 1-\delta \Leftrightarrow$ ~~$3\delta \geq$~~ $\delta \geq 1/3$. Optimal one shot deviation by P2 is to R, which yields 5 then 1 indefinitely. Eqm play yields 3 indefinitely. Compare PVs.
$3 + 3\delta/(1-\delta) \geq 5 + 1\delta/(1-\delta) \Leftrightarrow 2\delta/(1-\delta) \geq 2 \Leftrightarrow 2\delta \geq 2 - 2\delta \Leftrightarrow \delta \geq 1/2$.

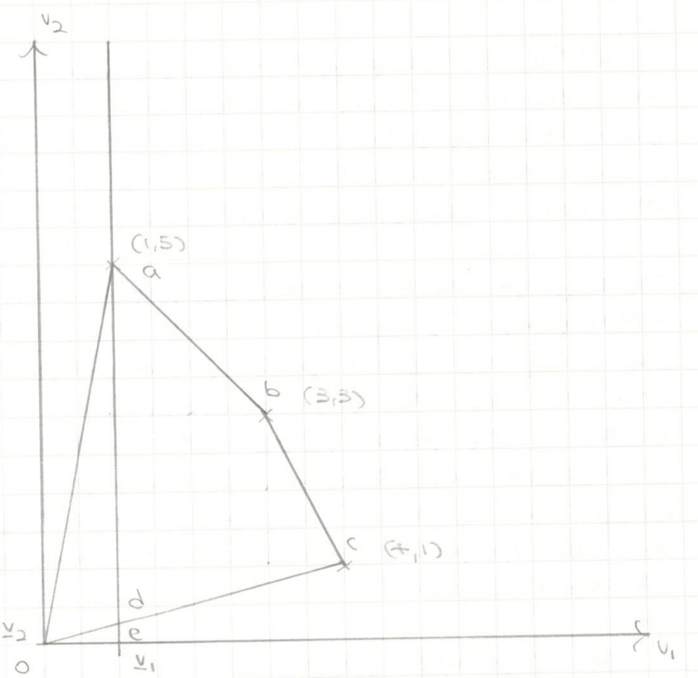
In either punishment phase (any period where there has been no prior deviation), because future prescribed play is independent of present ~~play~~ prescribed play and prescribed play in each period is the stage game NE, neither player has a profitable one shot deviation for all δ.

No player has a profitable one shot deviation in any phase iff $\delta \geq 1/2$

d P1 minmaxes P2 by playing M, then P2's maximum payoff is equal to the minimum feasible payoff 0. P1 holds P2 to payoff 0 by playing M, P2 guarantees payoff 0 by playing any strategy.

P1 is weakly worse off if P2 plays R. So P2 minmaxes P1 by playing R and P1 optimally ~~f~~ responds by playing U. P2 holds P1 to payoff 1 by playing R. P1 guarantees payoff 1 by playing U.

$\underline{v}_1 = 1$, $\underline{v}_2 = 0$



The set of feasible payoff vectors is represented by area abc0. The set of individually rational payoff vectors is represented by the area ~~to the~~ above ~~#~~ the $v_2 = \underline{v}_2 = 0$ line and to the right of the $v_1 = \underline{v}_1 = 1$ line. The set of feasible and individually rational payoff vectors is given by the intersection area abcd.

e At SPE, ~~each player's average discounted value~~ ADV payoff vector is equal to some feasible and individually rational payoff vector. From the diagram in (d), there is no feasible and individually rational payoff vector such that $v_2 < 1/4$. So there is no SPE such that P2 has ADV payoff $< 1/4$.

More rigorously, at any SPE, P1 has ADV ~~AD~~ ~~less~~ payoff no less than 1. Any candidate SPE such that P1 has ADV payoff less than one fails to deviation by P1 to U forever, which yields no less than 1 in every period hence ADV payoff ≥ 1. Such candidate equilibria are not NE let alone SPE. Given

Given that P1 has ADV payoff no less than 1 at SPE, P2 has ADV payoff no less than 1/4 at SPE ~~&~~ because no payoff pair $(v_1 \geq 1, v_2 < 1/4)$ is feasible ~~at SPE~~ (can be achieved by any play) in the dynamic game.

From (e), by the Fudenberg Maskin theorem, there is some SPE ~~st~~ with ADV payoffs $(1, 1/4)$.

~~Consider the strategy~~ Consider some arbitrary SPE. Any continuation play must also be a SPE. ~~At~~ SPE, the minimum ADV payoff to P1 is 1, the minimum feasible payoff to P2 is 1/4 because ~~the~~ at SPE, the payoff vector at SPE is feasible and individually rational. Then, the harshest punishment for ~~of~~ P1 is such that P1 has ADV payoff 1 in the continuation play and ~~P2 has ADV ?~~ the harshest punishment for P2 is such that P2 has ADV payoff 1/4 in the continuation play.

Consider the SPE where deviation is punished with the harshest punishment for the deviating player and (U,C) is played iff there has been no prior deviation.

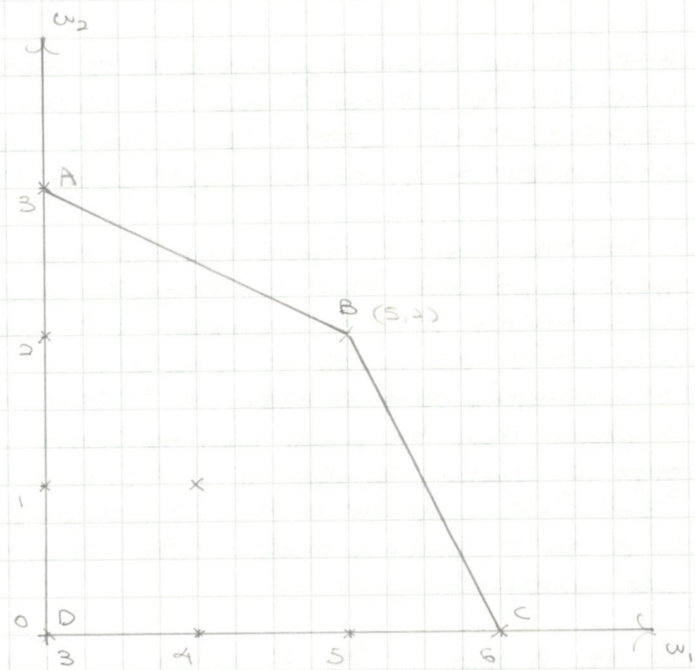
One shot deviation by P1 in the cooperation phase has ADV payoff $4(1-\delta) + 1\delta = 4 - 3\delta$. Eqm play yields ADV payoff ~~3~~ 3. $3 \geq 4 - 3\delta \Leftrightarrow \delta \geq 1/3$.

One shot deviation by P2 in the cooperation phase has ADV payoff $5(1-\delta) + \frac{1}{4}\delta = 5 - \frac{19}{4}\delta$. Eqm play yields ADV payoff 3. $3 \geq 5 - \frac{19}{4}\delta \Leftrightarrow \frac{19}{4}\delta \geq 2 \Leftrightarrow \delta \geq 8/19 \approx 0.42105$

P2 has a profitable one shot deviation if $\delta = 0.41 < 0.42105$. Then (U,C) forever is not sustainable in SPE, even by the harshest possible ~~individual~~ punishments for such δ.

Ta	x_1	y_1	x_2	y_2	u_1	v_1	u_2	v_2	w_1	w_2
	1	1	0	0	4	2	0	0	6	0
	1	0	0	1	4	0	0	1	4	1
	0	1	1	0	3	2	2	0	5	2
	0	0	1	1	3	0	2	1	3	3
	1	0	0	0	4	0	0	0	4	0
	0	0	1	0	3	0	2	0	3	2
	0	1	0	0	3	2	0	0	5	0
	0	0	0	1	3	0	0	1	3	1
	0	0	0	0	3	0	0	0	3	0

Given that each utility function is linear, the bargaining set is the minimal convex hull containing all the above pairs (w_1, w_2).



The bargaining set is represented by area ABCD.

The disagreement point is $\vec{d} = (d_1 = 3, d_2 = 2d)$

The Nash bargaining solution is
$w^* = \arg\max_{\vec{w} \in W} (w_1 - d_1)(w_2 - d_2)$
where W is the bargaining set

~~As~~ The Nash bargaining solution lies on the frontier represented by ABC. Any candidate maximum that does not lie on this frontier fails to deviation by increasing w_1 and w_2 each by a sufficiently small amount ε such ~~that~~ that the deviation remains feasible.

~~Along AB, $(w_1-d_1)(w_2-d_2)$~~
~~$(w_1-d_1)(w_2-d_2)$~~
~~$= (w_1-3)(3-\frac{1}{2}(w_1-3)-2d) = 0$~~

$\max (w_1-3)(w_2-2d)$ s.t.
$w_2 \leq 3 - \frac{1}{2}(w_1-3) \Leftrightarrow \frac{1}{2}w_1 + w_2 \leq \frac{9}{2}$
$w_2 \leq 6 - 2(w_1-3) \Leftrightarrow 2w_1 + w_2 \leq 12$
$\mathcal{L} = (w_1-3)(w_2-2d) - \lambda_1(\frac{1}{2}w_1 + w_2 - \frac{9}{2}) - \lambda_2(2w_1 + w_2 - 12)$
FOC w_1: $w_2 - 2d - \lambda_1/2 - 2\lambda_2 = 0$
FOC w_2: $w_1 - 3 - \lambda_1 - \lambda_2 = 0$
CS_1: $\lambda_1 \geq 0$, $\frac{1}{2}w_1 + w_2 \leq \frac{9}{2}$, $\lambda_1(\frac{1}{2}w_1 + w_2 - \frac{9}{2}) = 0$
CS_2: $\lambda_2 \geq 0$, $2w_1 + w_2 \leq 12$, $\lambda_2(2w_1 + w_2 - 12) = 0$
~~Suppose $\lambda_1 > 0$~~.

~~Suppose $\lambda_1 > 0$, then $\frac{1}{2}w_1 + w_2 - \frac{9}{2} = 0$, so $w_2 = \frac{9}{2} - \frac{w_1}{2}$~~.

Suppose $\lambda_1 > 0$, $\lambda_2 = 0$. From CS_1, $\frac{1}{2}w_1 + w_2 - \frac{9}{2} = 0$, then $w_2 = \frac{9-w_1}{2}$. By substitution into FOC w_1 $\frac{9-w_1}{2} - 2d - \lambda_1/2 = 0$. By substitution into FOC w_2, $w_1 - 3 - \lambda_1 = 0$. Solve simultaneously.
$w_1 - 3 = 9 - w_1 - 4d \Leftrightarrow 2w_1 = 12 - 4d \Leftrightarrow w_1 = 6 - 2d$.
Then $w_2 = \frac{3}{2} + d$. $\lambda_1 = 3 - 2d > 0 \Leftrightarrow d < \frac{3}{2}$.
$\frac{1}{2}w_1 + w_2 = \frac{9}{2}$. $2w_1 + w_2 = \frac{27}{2} - 3d \leq 12 \Leftrightarrow$
$d \geq \frac{1}{2}$. $(6-2d, \frac{3}{2}+d)$ uniquely solves the KT FOC iff $d \geq \frac{1}{2}$.

Suppose $\lambda_1 = 0$, $\lambda_2 > 0$. From CS_2, $2w_1 + w_2 - 12 = 0$, then $w_2 = 12 - 2w_1$. By substitution into FOC w_1 $12 - 2w_1 - 2d - 2\lambda_2 = 0$. By substitution into FOC w_2, $w_1 - 3 - \lambda_2 = 0$. Solve simultaneously,
$12 - 2w_1 - 2d = 2w_1 - 6 \Leftrightarrow 4w_1 = 18 - 2d \Leftrightarrow$ ~~$w_1 = \frac{9}{2} - \frac{d}{2}$~~
$w_1 = \frac{9-d}{2} \Rightarrow w_2 = 3 + d$. $\lambda_2 = w_1 - 3 = \frac{3-d}{2} > 0 \Leftrightarrow$
$d < 3$. $\frac{1}{2}w_1 + w_2 = \frac{9-d}{4} + 3 + d = \frac{21+3d}{4} \leq \frac{9}{2} \Leftrightarrow$
$d \leq -1$. For $d \geq 0$, there is no solution to the KT FOC such that $\lambda_1 = 0$, $\lambda_2 > 0$

Suppose $\lambda_1 = \lambda_2 = 0$. Then from CS_1, CS_2, $\frac{1}{2}w_1 + w_2 = \frac{9}{2}$, $2w_1 + w_2 = 12 \Rightarrow w_1 = 5, w_2 = 2$. By substitution into FOC w_1, $2 - 2d - \lambda_1/2 - \lambda_2 = 0$. By substitution into FOC w_2, $2 - \lambda_1 - \lambda_2 = 0$. Solve simultaneously,
$2 - 2d - \lambda_1/2 - (2 - \lambda_1) = 0 \Leftrightarrow -2d + \lambda_1/2 = 0 \Leftrightarrow \lambda_1 = 4d$.
$4 - 4d - 2\lambda_2 - (2 - \lambda_2) = 0 \Leftrightarrow 2 - 4d - \lambda_2 = 0 \Leftrightarrow \lambda_2 = 2 - 4d$
$\lambda_1, \lambda_2 \geq 0$ iff $0 \leq d \leq \frac{1}{2}$. For such d, $(w_1 = 5, w_2 = 2)$ uniquely solve the KT FOC.

Supposing the KT FOC are necessary and sufficient, the Nash bargaining solution is $(6-2d, \frac{3}{2}+d)$ for $d \geq \frac{1}{2}$, $(5,2)$ otherwise.

b ~~a~~ Solve by backward induction.

Consider the subgame in which P1's offer is rejected. Suppose without loss of generality that P2 intends to induce P1 to choose (x, y). Then ~~P2's payoff m~~ P2's ~~maxim~~ payoff maximisation problem is

$$\max_{x,y} 2(1-x) + (1-y) \quad \text{s.t.}$$

$$x + 3 + 2y \geq (1-x) + 3 + 2(1-y) \iff$$

$$2x + 4y \geq 3 \quad (\text{and } x, y \in [0,1])$$

~~At the any optimum the constraint binds. Any candidate optimum such that the constraint fails does not bind fails to deviation by decreasing x by sufficiently small amount such that the constraint remains satisfied. At any optimum,~~ ~~$2x+4y=3 \iff x = \frac{3-4y}{2}$. By substitution, P2's payoff is $2(1-\frac{3-4y}{2}) + (1-y) = 2-(3-4y)+(1-y)$ $= 3y$. P2 maximises payoff by choosing~~ $y=1$, ~~$x = \frac{3-4y}{2} =$~~ $x=0$

At the optimum $x=0$, then y is such that P1 is indifferent between the two packages.
$x+3+2y = (1-x)+3+2(1-y) \iff 3+2y = 4+2(1-y) \iff$
$2y = 1+2(1-y) \iff 4y = 3 \iff y = 3/4$.

P2 offers $(x=0, y=3/4)$, P1 accepts this then P2 has $(1-x, 1-y) = (1, 1/4)$ and payoff $2(1)+1/4 = 9/4$, and P1 has payoff $3+2(3/4) = 9/2$

In the first stage, P2 rejects any offer ~~such that~~ that yields payoff less than $9/4$. P2's payoff as a function of z if the offer is accepted is $3z$. So P2 rejects any offer where $3z \leq 9/4$ $z \leq 3/4$. P1's payoff as a function of z if the first offer is accepted is $(1-z)+3+2(1-z) =$ $6-3z$. For $z \geq 3/4$ such that the offer is accepted, P1 has payoff $\leq 6-3(3/4) = 15/4 < 9/2$. P1 is strictly better off making an unacceptable offer.

At SPE, P1 offers $z < 3/4$, P2 rejects iff $z < 3/4$ and P2 always offers $x, y = 0, 3/4$, and P1 chooses the (x,y) bundle iff $x+2y \geq (1-x)+2(1-y)$ which is iff $2x+4y \geq 3$.

The outcome is unique. P1 prefers to have his offer rejected because P2 makes a more efficient offer that allocates more of each good to the player that values it more. There is a unique solution to P2's optimisation problem.

c ~~P2's optimisation problem in the second stage is~~ effectively ~~unchanged because the uniform scaling of the pies is equivalent to a change in units given that utility functions are linear.~~

At the optimum, P2 ~~finds at~~ offers $x = \cancel{0.5}\ 0$ to P1 because P1 has ~~higher~~ lower valuations for the first pie. P2 chooses y such that P1 is indifferent. $x+3+2y = (0.5-x)+3+2(0.5-y) \iff$ $2x+4y = 3/2$, $4y = 3/2$, $y = 3/8$.

~~P2's op.~~ The solution to P2's optimisation is simply halved because, given linear utilities, halving both pies is equivalent to a relabelling of variables.

~~If 2~~ In the second stage subgame, P2 offers $(x,y) = (0, 3/8)$, P1 chooses this, P2 has $(0.5-x, 0.5-y) = (1/2, 1/8)$, P1 has payoff $3+0+2(3/8) = 3+3/4$. P2 has payoff $2(1/2)+1/8 = 9/8$

In the first stage, P2 rejects any offer that yields payoff less than $9/8$. ~~P2 re~~ P2's payoff as a function of z is unchanged, so P2 rejects any $z < 3/8$. P1's payoff as a function of z is unchanged, ~~so if P1 offers~~ at $3+3(1-z)$. This is greater than the payoff from ~~the second stage~~ a rejected first offer iff $1-z \geq 1/4$, $z \leq 3/4$. $3/8 \leq z \leq 3/4$ is acceptable to P2 and profitable to P1 so P1 optimally offers $z = 3/8$, which is just acceptable to P2.

At SPE P1 offers $z = 3/8$, P2 accepts all $z \geq 3/8$. If P2 offers, P2 offers $(x,y) = (0, 3/8)$, P1 chooses (x,y) iff $x+2y \geq (0.5-x)+2(0.5-y) \iff$ $2x+4y \geq 3/2$. P1 has payoff $3+3(5/8) = 3+15/8$, P2 has payoff $3(3/8) = 9/8$.

The SPE is unique.