Microeconomic Analysis Supplementary Exercises 1

1ai (210618 21)

$\vec{x} = (1,0,3)$, $\vec{y} = (-5,3,1)$, $\vec{z} = (-1,1,2)$

Consider system of linear equations

$\cancel{1a_x - 5a_y - 1a_z = a}$ ①

$\cancel{0b_x + 3b_y + 1b_z = b}$ ②

$\cancel{3c_x + 1c_y + 2c_z = c}$ ③

~~From~~ ②,

$\cancel{1\alpha_x - 5\alpha_y - 1\alpha_z = a}$ ①

$\cancel{0\alpha_x + 3\alpha_y + 1\alpha_z = b}$ ②

$\cancel{3\alpha_x + 1\alpha_y + 2\alpha_z = c}$ ③

~~From ②, $\alpha_y = b - \alpha_z$ ④ $(b-\alpha_2)/3$ ④~~

~~Sub ④ into ①, $\alpha_x - 5(b-\alpha_z)/3 - \alpha_z = a$,~~

~~α_x~~

$1\alpha_x - 5\alpha_y - 1\alpha_z = \beta_1$ ①

$0\alpha_x + 3\alpha_y + 1\alpha_z = \beta_2$ ②

$3\alpha_x + 1\alpha_y + 2\alpha_z = \beta_3$ ③

From ②, $\alpha_y = (\beta_2 - \alpha_z)/3$ ④

Sub ④ into ①, $\alpha_x - 5\beta_2/3 + 2\alpha_z/3 = \beta_1$,

$\alpha_x = \beta_1 + 5\beta_2/3 - 2\alpha_z/3$ ⑤

Sub ④, ⑤ into ③, $3\beta_1 + 5\beta_2 - 2\alpha_z + (\beta_2 - \alpha_z)/3 + 2\alpha_z = \beta_3$,

$-\alpha_z/3 + 3\beta_1 + 16\beta_2/3 = \beta_3$, $\cancel{-\alpha_z = 9}$

$-\alpha_z + 9\beta_1 + 16\beta_2 = \beta_3$

$\alpha_z = 9\beta_1 + 16\beta_2 - \beta_3$ ⑥

Sub ⑥ into ⑤

$\alpha_x = \beta_1 + 5\beta_2/3 - 18\beta_1/3 - 32\beta_2/3 + 2\beta_3/3$ ⑦

Sub ⑥ into ④

$\alpha_y = \beta_2/3 - 3\beta_1 - 16\beta_2/3 + \beta_3/3$

$$\vec{\beta} = (\beta_1, \beta_2, \beta_3) = (-15/3\,\beta_1 - 9\beta_2 + 2/3\,\beta_3)\vec{x} + (3\beta_1 - 5\beta_2 + 1/3\,\beta_3)\vec{y} + (9\beta_1 + 16\beta_2 - \beta_3)\vec{z}$$

$\forall \vec{\beta} \in \mathbb{R}^3 : \vec{\beta}$ is a linear combination of $\vec{x}$, $\vec{y}$ and $\vec{z}$.

~~so $\vec{x}$, $\vec{y}$ and $\vec{z}$ form a ba~~

By definition of a basis, $\vec{x}$, $\vec{y}$ and $\vec{z}$ form a basis of $\mathbb{R}^3$

ii $\vec{f} = (1,-1,2)$ in the basis $(\vec{x}, \vec{y}, \vec{z})$

~~$\vec{f} = (4$~~

$\vec{f} = (4,-1,6)$ in the standard basis since

$$\begin{pmatrix} 1 & -5 & -1 \\ 0 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ 6 \end{pmatrix}$$

bi $1, e^x, e^{2x}, \dots, e^{nx}$

$= 1, 1, 1, \dots, 1$ when $x = 0$

$1, e, e^2, \dots, e^n$ when $x = 1$

$1, e^2, e^4, \dots, e^{2n}$ when $x = 2$

$\vdots$

$1, e^n, e^{2n}, \dots, e^{n^2}$ when $x = n$

The linear system of functions is ~~not~~ linearly independent iff $\nexists a_0, a_1, \dots, a_n \in \mathbb{R}$ ~~such~~ : $\exists i\ a_i \neq 0$ and $\forall x \in D : \sum_{i=0}^{n} a_i f_i(x) = 0$ iff

~~$\nexists \vec{a} \in \mathbb{R}^{n+1} : \vec{a} \neq \vec{0}^{n+1}$ and~~

~~only iff $\nexists \vec{a} \in \mathbb{R}^{n+1} : \vec{a} \neq \vec{0}^{n+1}$ and $\nexists$~~

~~$F \cdot \vec{a}$~~

Suppose the system of functions is linearly dependent.

Let F be the $n+1 \times n+1$ matrix

~~$F = 1, 1, 1, \dots,$~~

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & e & e^2 & \dots & e^n \\ 1 & e^2 & e^4 & \dots & e^{2n} \\ \vdots & & & & \vdots \\ 1 & e^n & e^{2n} & \dots & e^{n^2} \end{pmatrix}$$

Then there is no $n+1$ vector $\vec{a} = (a_0, a_1, a_2, \dots a_n)$ such that $\exists i\ a_i \neq 0$ and $F \cdot \vec{a} = \vec{0}^{n+1}$

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$F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$

$F_1 = F_2 = 1$

$\vec{u}_n = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}$, $\vec{u}_{n+1} = \begin{pmatrix} F_{n+2} \\ F_{n+1} \end{pmatrix} = \begin{pmatrix} F_{n+1} + F_n \\ F_{n+1} \end{pmatrix}$

Let A be a 2×2 matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ such that

$\vec{u}_{n+1} = A \cdot \vec{u}_n$

$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} \cdot \begin{pmatrix} a_{11}F_{n+1} + a_{12}F_n \\ a_{21}F_{n+1} + a_{22}F_n \end{pmatrix} = \begin{pmatrix} F_{n+1} + F_n \\ F_{n+1} \end{pmatrix}$

By inspection, $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

$\det A = 1\times 0 - 1\times 1 = -1$

A is invertible iff $\det A \neq 0$, so A is invertible

An eigenvalue and eigenvector of an n-square matrix A are a scalar λ and n-dimensional vector $\vec{x}$ such that $A\cdot\vec{x} = \lambda\vec{x}$.

Then $A\cdot\vec{x} = \lambda(I\cdot\vec{x}) = (\lambda I)\cdot\vec{x}$, $(A - \lambda I)\vec{x} = \vec{0}$

$\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = -\lambda(1-\lambda) - 1\times 1$

$= \lambda^2 - \lambda - 1 = 0$

By the quadratic formula,

$\lambda = \frac{1}{2} + \frac{\sqrt{5}}{2}$ or $\frac{1}{2} - \frac{\sqrt{5}}{2}$, $\lambda_1 = \frac{1}{2} + \frac{\sqrt{5}}{2}$, $\lambda_2 = \frac{1}{2} - \frac{\sqrt{5}}{2}$

$\cancel{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \vec{x}_1 = \left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)\vec{x}_1}$

$\cancel{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 \end{pmatrix} = \left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}$

$\cancel{x_1 = \frac{1}{2} + \frac{\sqrt{5}}{2} x_2, \quad (x_1 + x_2) = \left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)x_1}$

$\cancel{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \vec{x}_1 = \left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)\vec{x}_1}$

$\cancel{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1^T \\ x_1^B \end{pmatrix} = \left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)\begin{pmatrix} x_1^T \\ x_1^B \end{pmatrix}}$,

$\cancel{\begin{pmatrix} x_1^T + x_1^B \\ x_1^T \end{pmatrix} = \left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)\begin{pmatrix} x_1^T \\ x_1^B \end{pmatrix}}$,

$\cancel{\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)x_1^T + x_1^B = 0}$

$\cancel{x_1^T + \left(-\frac{1}{2} - \frac{\sqrt{5}}{2}\right)x_1^B = 0}$

~~Solving simultaneously~~,

$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 \end{pmatrix} = \left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \frac{\sqrt{5}}{2} \\ 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 \end{pmatrix} = \left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} - \frac{\sqrt{5}}{2} \\ 1 \end{pmatrix}$

Eigenvalues and eigenvectors

$\lambda_1 = \frac{1}{2} + \frac{\sqrt{5}}{2}$, $\vec{x}_1 = \left(\frac{1}{2} + \frac{\sqrt{5}}{2}, 1\right)$

$\lambda_2 = \frac{1}{2} - \frac{\sqrt{5}}{2}$, $\vec{x}_2 = \left(\frac{1}{2} - \frac{\sqrt{5}}{2}, 1\right)$

Let $S = \begin{pmatrix} \frac{1}{2} + \frac{\sqrt{5}}{2} & \frac{1}{2} - \frac{\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix}$ Transforms from standard basis to eigenbasis

$SS^{-1} = I$, $\begin{pmatrix} \frac{1}{2} + \frac{\sqrt{5}}{2} & \frac{1}{2} - \frac{\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} s^{-1}_{11} & s^{-1}_{12} \\ s^{-1}_{21} & s^{-1}_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$S^{-1} = \begin{pmatrix} 1 & -\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right) \\ -1 & \left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right) \end{pmatrix}$ "Makes eigenvectors to be the basis"

~~Effient~~

< ~~1 firms mix over a larger interval and are mo-~~
~~invest more on average when the probability of~~
~~facing an efficient opponent increases. Since~~
~~inefficient firms never invest, efficient firms compete~~
~~only with other efficient firms, and so must~~
~~compete more aggressively when competition~~
~~increases.~~

3 (190530 Q1)

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

scalar λ and vector $\vec{v}$ are an eigenvalue and an ~~eigenvector~~ of A iff $A\cdot\vec{v} = \lambda\vec{v}$, ~~iff $A\vec{v} = \lambda(I\vec{v})$~~

~~By def.~~ iff $A\cdot\vec{v} = \lambda(I\cdot\vec{v})$

iff $A\cdot\vec{v} = (\lambda I)\cdot\vec{v}$

iff $(A-\lambda I)\cdot\vec{v} = \vec{0}$

iff $\det(A-\lambda I) = 0$

$\lambda = 1$, $A-\lambda I = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} - 1\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$\det(A-\lambda I) = 1\det\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - 1\det\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + 1\det\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$= 0 - 0 + 0 = 0$

so $\lambda = 1$ is an eigenvalue of A

$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\cdot\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1+v_2+v_3 \\ v_1+v_2+v_3 \\ v_1+v_2+v_3 \end{pmatrix} = \vec{0}$ iff ~~$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} =$~~

$v_1+v_2+v_3 = 0$.

All $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ are eigenvectors of A with eigenvalue

$\lambda = 1$ iff ~~$v_1 = v_2 + v_3$~~ $v_1+v_2+v_3 = 0$

$\lambda = 4$, $A-\lambda I = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$

$\det(A-\lambda I) = -2\det\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} - 1\det\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} + 1\det\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$

$= -2(3) - 1(-3) + 1(3) = 0$

so $\lambda = 4$ is an eigenvalue of A

$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\cdot\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -2v_1+v_2+v_3 \\ v_1-2v_2+v_3 \\ v_1+v_2-2v_3 \end{pmatrix} = \vec{0}$

iff $v_1 = v_2 = v_3$

All $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ are eigenvectors of A with eigenvalue

$\lambda = 4$ iff $v_1 = v_2 = v_3$

∴ ~~$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ are eigenvectors of A that~~

form a basis of $\mathbb{R}^3$

$V = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ is a matrix such that

~~$VAV^{-1} = D$:~~ $VDV^{-1} = A$ for $D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

~~$V^{-1}VDV^{-1}V = D = V^{-1}AV$~~

~~V can be interpreted as a change of basis matrix from the eigen basis formed by $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ to the standard basis.~~

~~$A^6 = VD^6V^{-1}$~~

$V\cdot V^{-1} = I$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} v^{-1}_{11} & v^{-1}_{12} & v^{-1}_{13} \\ v^{-1}_{21} & v^{-1}_{22} & v^{-1}_{23} \\ v^{-1}_{31} & v^{-1}_{32} & v^{-1}_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

~~Solving by calculator,~~

$$V^{-1} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$A^6 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 4096 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} -4096 & 4096 & 4096 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -4094 & 4095 & 4095 \\ -4095 & 4096 & 4095 \\ -4095 & 4095 & 4096 \end{pmatrix}$$

$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ are eigenvectors of A that form a basis of $\mathbb{R}^3$ ~~so~~. Let $V = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$ then

$VDV^{-1} = A$ for $D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, so $D = V^{-1}AV$

Informally, V^{-1} ~~transforms~~ is a transformation from the standard basis to the eigenbasis, D is the analogue of A in the eigenbasis, and V^{-1} is a transformation of the eigenbasis to the standard basis.

Solving by calculator, $V^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$

$$A^6 = VD^6V^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 4^6 & 0 & 0 \\ 0 & 1^6 & 0 \\ 0 & 0 & 1^6 \end{pmatrix}\left(\frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\right)$$

$$= \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 4096 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 4096 & 4096 & 4096 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 4098 & 4095 & 4095 \\ 4095 & 4098 & 4095 \\ 4095 & 4095 & 4098 \end{pmatrix} = \begin{pmatrix} 1366 & 1365 & 1365 \\ 1365 & 1366 & 1365 \\ 1365 & 1365 & 1366 \end{pmatrix}$$