# Bargaining Notes

## Cooperative Game Theory of Bargaining

- Definition (Bargaining Problem)
  - A bargaining problem is a pair $(U, d)$, where $U$ is the set of possible agreement payoff vectors, and $d$ is the disagreement payoff vector. $U = \{(u_1(\overrightarrow{\mathbf{x}}), \ldots, u_n(\overrightarrow{\mathbf{x}}) : \overrightarrow{\mathbf{x}} \in X\}$ where $X$ is the set of possible agreements. $\overrightarrow{\mathbf{d}} = (d_1 \equiv u_1(\overrightarrow{\mathbf{D}}), \ldots, d_n \equiv u_n(\overrightarrow{\mathbf{D}}))$ where $\overrightarrow{\mathbf{D}}$ is the disagreement.
  - It is common to suppose that $\overrightarrow{\mathbf{D}} \in X \Leftrightarrow \overrightarrow{\mathbf{d}} \in U$, i.e. it is possible to "agree to disagree", $\exists \overrightarrow{\mathbf{v}} \in U : \forall i : v_i > d_i$, i.e. some agreement Pareto-dominates disagreement, and $U$ is convex, closed, and bounded, i.e. $\forall \overrightarrow{\mathbf{v_1}}, \overrightarrow{\mathbf{v_2}} \in U : \forall \alpha \in [0,1] : \alpha \overrightarrow{\mathbf{v_1}} + (1-\alpha)\overrightarrow{\mathbf{v_2}} \in U$ and $\forall \{\overrightarrow{\mathbf{v_i}}\}_{i=1}^{\infty} : (\forall i : \overrightarrow{\mathbf{v_i}} \in U) \Rightarrow \overrightarrow{\mathbf{v_\infty}} \in U$ and $\nexists \overrightarrow{\mathbf{u}} \in U : \exists i : u_i = \infty \vee u_i = -\infty$ , i.e. if $\overrightarrow{\mathbf{v_1}}, \overrightarrow{\mathbf{v_2}} \in U$ then any weighted average of the two elements is also an element of $U$, and if all members of a sequence are in $U$ then the limit of that sequence is also in $U$, and there is a finite upper bound and a finite lower bound on each dimension of $U$.
- Definition (Bargaining Solution)
  - A bargaining solution is a function $F(U, \overrightarrow{\mathbf{d}})$ from bargaining problems $(U, \overrightarrow{\mathbf{d}})$ to agreements $\overrightarrow{\mathbf{u}} \in U$.

## Nash Bargaining Theorem

- Definition (Weak Pareto Efficiency)
  - Suppose that $F(U, \overrightarrow{\mathbf{d}}) = \overrightarrow{\mathbf{u}}$, then $\nexists \overrightarrow{\mathbf{v}} \in U : \forall i : v_i > u_i$, i.e. there is no agreement that strictly Pareto-dominates $\overrightarrow{\mathbf{u}}$.
- Definition (Symmetry)
  - Consider $n = 2$. Bargaining problem $(U, \overrightarrow{\mathbf{d}} \equiv (d_1, d_2))$ is symmetric iff $d_1 = d_2$ and $\forall (u_1, u_2) \in U : (u_2, u_1) \in U$. Suppose that $F(U, \overrightarrow{\mathbf{d}}) = \overrightarrow{\mathbf{u}} \equiv (u_1, u_2)$, then $u_1 = u_2$, i.e. if a bargaining problem is symmetric, then its solution is also symmetric.
- Definition (Invariance to Equivalent Payoff Representations)
  - Let $f_i(u_i) = \alpha_i u_i + \beta_i$ where $\alpha_i > 0$ for all $i \in \{1, \ldots, n\}$. Let $U' = \{(f_1(u_1), \ldots, f_n(u_n)) : (u_1, \ldots, u_n) \in U\}$, $\overrightarrow{\mathbf{d'}} = (f_1(d_1), \ldots, f_n(d_n))$, and $\overrightarrow{\mathbf{u'}} = (f_1(u_1), \ldots, f_n(u_n))$. Suppose that $F(U, \overrightarrow{\mathbf{d}}) = \overrightarrow{\mathbf{u}}$ then $F(U', \overrightarrow{\mathbf{d}'}) = \overrightarrow{\mathbf{u}'}$, i.e. if one bargaining problem is a linear transformation of another, then the solution of the former is obtained by applying the same transformation to the solution of the latter.
- Definition (Independence of Irrelevant Alternatives)
  - Suppose that $U' \subseteq U$ and $\overrightarrow{\mathbf{d'}} = \overrightarrow{\mathbf{d}}$ and $F(U, \overrightarrow{\mathbf{d}}) \subseteq U'$, then $F(U', \overrightarrow{\mathbf{d}'}) = F(U, \overrightarrow{\mathbf{d}})$, i.e. removal of non-solution possible agreements does not affect the solution.
- Nash Bargaining Theorem
  - $F(U, \overrightarrow{\mathbf{d}}) = \overrightarrow{\mathbf{u^*}} \equiv (u_1^*, \ldots, u_n^*) \equiv \arg\max_{(u_1, \ldots, u_n)} \Pi_{i=1}^n (u_i - d_i)$ subject to $(u_1, \ldots, u_n) \in U$ and $\forall i : u_i \geq d_i$ uniquely satisfies the Nash bargaining axioms.
- Definition (Kalai Smorodinsky Bargainign Solution)
  - The Kalai-Smorodinsky bargaining solution is the point on the Pareto frontier $(u_1^*, u_2^*)$ such that $\frac{u_1^* - d_1}{u_2^* - d_2} = \frac{\max u_1 - d_1}{\max u_2 - d_2}$.

## Non-Cooperative Game Theory of Bargaining

- Relationship (Cooperative and Non-Cooperative Game Theory of Bargaining)
  - In the infinite repetition offer-counteroffer game, as the probability $\alpha$ of breakdown converges to zero, the subgame perfect equilibrium allocation converges to the Nash bargaining solution.
- At the stationary equilibrium of the infinite repetition offer-counteroffer game, each player makes the same offer every time, and each player is indifferent between accepting and rejecting each offer.
  - If some player strictly prefers accepting, then the offering player has a strictly profitable deviation to a less generous but still acceptable offer.
  - If some player $A$ strictly prefers rejecting, then this is only because $A$ has greater payoff from rejecting and making an acceptable counteroffer. This counteroffer is less generous to the initial offering player $B$ than $B$'s initial offer. So $B$ has strictly profitable deviation to a more generous, acceptable offer.

- The general algebraic solution to such a game is as follows.
  - $x_2^1 = (1-\alpha)x_2^2$,
  - $x_1^2 = (1-\alpha)x_1^1 \Rightarrow (1-x_2^2) = (1-\alpha)(1-x_2^1)$.
- The result of this Rubinstein model converges to the Nash bargaining solution as the probability of breakdown converges to zero.