Quantitative Economics Problem Set 8

1a ~~$\Delta X_t = \beta_0 + \alpha t + \delta X_{t-1} + \sum_{i=1}^{p\Delta} \gamma_i Y_{t-i} + u_t$~~

$\Delta X_t = \beta_0 + \alpha t + \delta X_{t-1} + \sum_{i=1}^{p\Delta} \gamma_i \Delta X_{t-i} + u_t$

~~where $\Delta X_t = X_t - X_{t-1}$, $\delta = u_t$~~

where $E[u_t | y_{t-1}] = 0$ by construction

$H_0: \delta = 0$

$H_1: \delta < 0$

$t = (\hat{\delta} - 0) / se(\hat{\delta})$

under the null, $t \xrightarrow{d} DF_{tr}$

Reject the null if $t < c_\alpha$, where c_α is the critical value at the level of significance α from the DF_{tr} distribution.

If the null is rejected, then $\{X_t\}$ is a (trend) stationary AR(p) process. If the null is not rejected, then $\{X_t\}$ is a unit root AR(p) process.

If $\{X_t\}$ is (trend) stationary, then $\{X_t\}$ has order of integration 0, $X_t \sim I(0)$.

If $\{X_t\}$ has a unit root, then the first difference of X_t, ΔX_t is a AR($p\Delta$) process. Then, $\{X_t\}$ has order of integration 1, (equivalently) $X_t \sim I(1)$ iff $\{\Delta X_t\}$ is (trend) stationary and does not have a unit root. Whether $\{\Delta X_t\}$ is stationary or has a unit root can be tested by the augmented Dickey Fuller test.

In general, the choice of $p\Delta$ can be guided by information criteria or by a stepwise testing down procedure, and the test statistic is reported for a range of $p\Delta$ to verify robustness of the result to the choice of $p\Delta$.

b In the constant only ADF test, under the null, X_t is a unit root autoregressive process, and under the alternative, X_t is a stationary autoregressive process. The null and alternative are not collectively exhaustive. A trend stationary process, for example, does not have a unit root, but is also not a stationary autoregressive process. A unit root autoregressive process better accommodates ~~time series with a determinist~~ trend stationary process because a unit root autoregressive process can have a deterministic trend whereas a stationary autoregressive process exhibits mean reversion. Then, under a constant only ADF test, it is possible that we fail to reject the null because of the presence of a deterministic trend in an otherwise stationary process. Where it is possible that a time series has a deterministic trend, the constant and trend ADF test is more appropriate.

In some cases, it is not ~~possible that~~ plausible that there is a deterministic trend, for example, where the time series is a difference of ~~autoregressive~~ a unit root autoregressive process, because differencing eliminates deterministic trends. Then, the constant and trend model overfits the data, and the corresponding ADF test is inappropriate.

c The lag order $p\Delta$ of the ADF regression can be determined by using an information criterion or by a stepwise testing down procedure.

Two common information criteria are the Akaike information criterion (AIC) and the Bayesian information criterion (BIC).

AIC: $\ln \frac{SSR_m}{T} + m(2/T)$

BIC: $\ln \frac{SSR_m}{T} + m(\ln/T)$

The application of each information criterion involves estimating the ADF regression for each $p\Delta$, computing the SSR hence the value of the information criterion, and selecting the $p\Delta$ whose associated regression minimises the information criterion.

The stepwise testing down procedure essentially involves estimating a model for ~~large~~ some large initial $p\Delta$, testing the hypothesis that the coefficient on the longest lag is zero, eliminating that lag if the null is not rejected, and proceeding to test for the next longest lag, until the null of a zero coefficient is rejected.

The information criteria and stepwise testing down procedures are applicable because for large samples, the distribution of the estimates is approximately normal

d For the ADF test for Y, the AIC selects the $p\Delta = 2$ ADF regression. The ADF test statistic is -1.697. The 10%, 5%, 1% critical values from the DF_{tr} distribution are -3.12, -3.41, -3.96 respectively. Fail to reject the null of a unit root at any conventional level of significance. By inspection, the same result obtains for the other reported values of $p\Delta$.

For the ADF test ~~of~~ for ΔY, the AIC selects the $p\Delta = 1$ ADF regression. The ADF test statistic is -7.119. At every conventional level of significance, reject the null of a unit root. ~~Then, $\{\Delta Y_t\}$ is stationary, and $\{Y_t\}$ has order of integration 1, $Y_t \sim I(1)$.~~ By inspection, this result obtains for the other reported values of $p\Delta$. Then, $\{\Delta Y_t\}$ is stationary and $\{Y_t\}$ has order of integration 1.

For the ADF test for C, the AIC selects the model with $p\Delta = 3$. The ADF test statistic is -1.859. Fail to reject the null of a unit root at every conventional level of significance. By inspection, the same result obtains for the other reported values of $p\Delta$. Then, conclude that C has a unit root.

For the ADF test for ΔC, the AIC selects the ADF regression with $p\Delta = 2$. The ADF test statistic for this regression is -5.075. Reject the null of a unit root at every conventional level of significance. By inspection, the result obtains for the other reported ADF regressions. Conclude that $\{\Delta C\}$ is stationary hence that $\{C\}$ has order of integration 1, $C_t \sim I(1)$.

ei From the above, each of $\{Y_t\}, \{C_t\}$ has order of integration 1, so findings of a relationship between the two are potentially spurious. Hence, a test of cointegration is useful to distinguish between genuine and spurious relationships.

Compute $\varepsilon_t = Y_t - C_t$, perform an ADF test for a unit root in ε_t. If the null of a unit root is rejected, then Y_t is not cointegrated with C_t with cointegrating coefficient $\theta = 1$. If the null is not rejected, then Y_t and C_t are cointegrated with cointegrating coefficient $\theta = 1$.

ii Estimate θ by OLS regression of Y_t on C_t. ~~Perform an ADF test~~ Compute $\hat{\varepsilon}_t = Y_t - \hat{\theta} C_t$. Perform an ADF test for a unit root in $\hat{\varepsilon}_t$. If the null of a unit root is rejected, conclude that Y_t and C_t are not cointegrated. Otherwise, conclude that Y_t and C_t are cointegrated. In this case, compare the ADF test statistic against adjusted Engle-Granger critical values.

2a Given the AR(1) model

$\pi_t = \beta_0 + \beta_1 \pi_{t-1} + u_t$

where $E[u_t \mid \pi_{t-1}, \pi_{t-2}, \ldots] = 0$

$H_0: \beta_1 = 0$

$H_1: \beta_1 \neq 0$

Test statistic

$t = (\hat{\beta}_1 - 0)/se(\hat{\beta}_1) = 0.84/0.04 = 21$

Under the null, given a sufficiently large sample, supposing that asymptotic normality of $\hat{\beta}_1$ holds,

$t \xrightarrow{d} N(0,1)$

Reject the null iff $|t| > c_\alpha$.

Reject the null at any conventional level of significance α.

Conclude that lagged inflation is useful in forecasting future inflation.

It is appropriate to take critical values from the $N(0,1)$ distribution only if the test statistic is asymptotically normal under the null, which is if inflation is stationary.

b $\pi_{2017Q2 \mid 2017Q1} = 0.68 + 0.84\,\pi_{2017Q1}$
$= 2.36$

In the vicinity of a unit root, when the coefficient on π_{t-1} is close to 1, non-trivial biases of the OLS estimator become particularly pronounced. Then, forecasts on the basis of OLS estimates are not entirely reliable.

c Regression (2) is an ADF regression.

Let δ denote the coefficient on π_{t-1} in this model.

$H_0: \delta = 0$

$H_1: \delta < 0$

Test statistic

$t = (\hat{\delta} - \delta)/se(\hat{\delta}) = -0.107/0.041 = -2.6098$

Under the null, $t \xrightarrow{d} DF_{cn}$

Reject the null if $t < -c_\alpha$, where c_α is the critical value drawn from the DF_{cn} distribution at level of significance α.

Reject the null at the 10% level of significance.

Conclude that π_t has a unit root.

Then, π_t is not stationary and the OLS estimators for the coefficients on the lags of π_t are biased. So forecasts on the basis of (1) are less ~~reliable~~ accurate than those on the basis of (2).

a ~~Sol~~

~~Let γ_3 denote the coefficient on $\Delta\pi_{t-3}$ in the AR(3) model~~

Given AR(3) model

$\Delta\pi_t = \beta_0 + \beta_1\Delta\pi_{t-1} + \beta_2\Delta\pi_{t-2} + \beta_3\Delta\pi_{t-3} + u_t$

where $E[u_t \mid \Delta\pi_{t-1}, \Delta\pi_{t-2}, \dots] = 0$

$H_0: \beta_3 = 0$

$H_1: \beta_3 \neq 0$

Test statistic

$t = (\hat\beta_3 - 0)/se(\hat\beta_3) = 0.196/0.075 = 2.6133$

Under the null

$t \xrightarrow{d} N(0,1)$

Reject the null if $|t| > c_\alpha$, ~~under~~ where c_α is the critical value drawn from the $N(0,1)$ distribution at level of significance α.

For $\alpha = 0.01$, $c_\alpha = 2.576$

Reject the null at the 1% level of significance. Conclude that the coefficient on $\Delta\pi_{t-3}$ in the above model is non-zero.

$\Delta\pi_t$ follow an AR(3) (or higher order) model.

c The claim suggests two breakpoints in the time series, one at the beginning of Volcker's tenure, and one at the end of Volcker's tenure.

Perform a Quandt likelihood ratio test for a break at the beginning of Volcker's tenure as follows. Identify some interval of time in which a break ~~sth~~ is plausible. For each ~~period at~~ time in this interval, compute the test statistic of the associated Chow breakpoint test. Compute the QLR test statistic as the maximum Chow test statistic in this interval. Compare the QLR test statistic against a suitable critical value.

The Chow ~~test~~ breakpoint test is a F test of

$H_0: \gamma_0 = \gamma_1 = \dots = 0$, against

$H_1: \gamma_0 \neq 0$, or $\gamma_1 \neq 0$, or ...

Given the model

$Y_t = \beta_0 + \sum_{i=1}^{p} \beta_i Y_{t-i} + \gamma_0 D_t(\tau) + \sum_{i=1}^{p} D_t(\tau)\gamma_i Y_{t-i} + u_t$

where $D_t(\tau)$ is a dummy variable that has value 1 iff $t \geq \tau$

The test statistic is the standard F statistic.

So $\beta_1 + \beta_2 = 1$

b $Y_t = \beta_0 + \beta_1 Y_{t-1} + \beta_2 Y_{t-2} + u_t$, where $\beta_1 + \beta_2 = 1$ and $E[u_t \mid Y_t] = 0$

$\Delta Y_t = Y_t - Y_{t-1}$

$= \beta_0 + (\beta_1 - 1)Y_{t-1} + \beta_2 Y_{t-2} + u_t$

$= \beta_0 + (-\beta_2)Y_{t-1} + \beta_2 Y_{t-2} + u_t$

$= \beta_0 - \beta_2 \Delta Y_{t-1} + u_t$

$\gamma_0 = \beta_0$, $\gamma_1 = -\beta_2$

c The order of integration of $\{Y_t\}$ is the minimum d such that $\{\Delta^d Y_t\}$ is stationary. Given that Y_t has a unit root, $\{Y_t\}$ is not stationary. Given further that $\{\Delta Y_t\}$ is stationary, the order of integration of $\{Y_t\}$ is 1.

d ~~test~~ Given that $\{\Delta Y_t\}$ is stationary, for all t, t', $E\Delta Y_t = E\Delta Y_{t'}$. ~~Let μ denote the common~~ Let $\mu = E\Delta Y_t$. Let $v_t = \Delta Y_t - \mu$. ~~$E\mu =$~~

$Ev_t = E\Delta Y_t - E\mu$.

$Y_t = Y_{t-1} + \Delta Y_t$

$= Y_{t-2} + \Delta Y_t + \Delta Y_{t-1}$

$\vdots$

$= Y_0 + \sum_{s=1}^{t} \Delta Y_s$

$= Y_0 + \sum_{s=1}^{t} \mu + v_s$

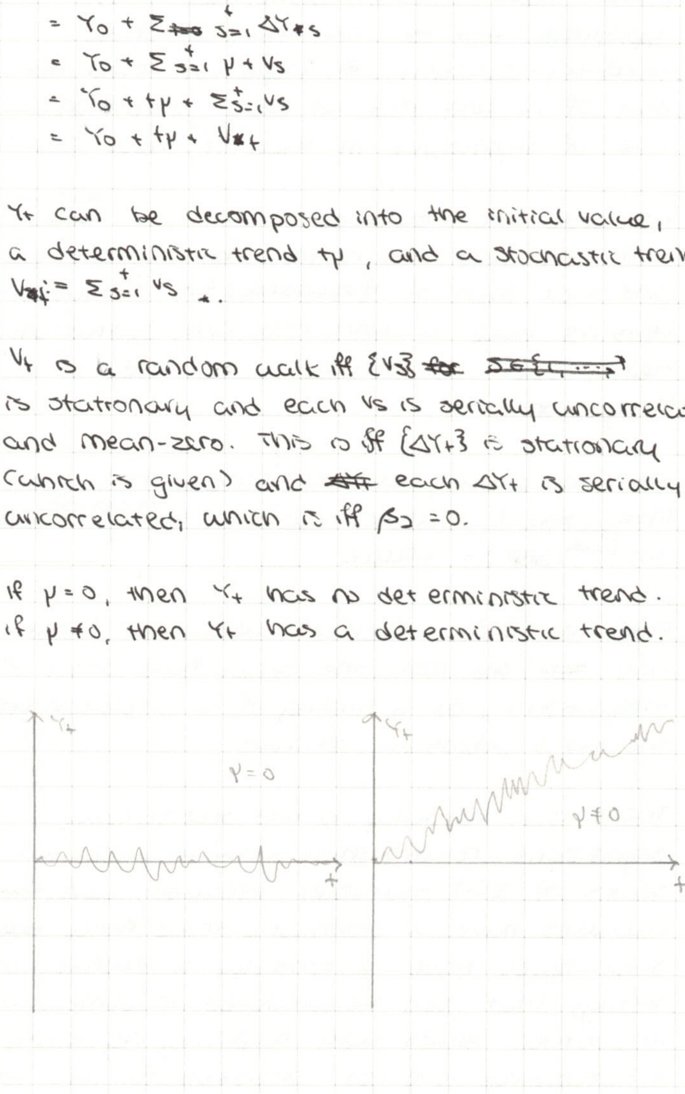
$= Y_0 + t\mu + \sum_{s=1}^{t} v_s$

$= Y_0 + t\mu + V_t$

Y_t can be decomposed into the initial value, a deterministic trend $t\mu$, and a stochastic trend $V_t := \sum_{s=1}^{t} v_s$.

e V_t is a random walk iff $\{v_s\}$ ~~for $s \in \{1, \dots\}$~~ is stationary and each v_s is serially uncorrelated and mean-zero. This is iff $\{\Delta Y_t\}$ is stationary (which is given) and ~~sth~~ each ΔY_t is serially uncorrelated, which is iff $\beta_2 = 0$.

f If $\mu = 0$, then Y_t has no deterministic trend. If $\mu \neq 0$, then Y_t has a deterministic trend.



4a critical values

	10%	5%	1%
DFcn	-2.57	-2.86	-3.43
DFtr	-3.12	-3.41	-3.96

The ADF test is a test of
$H_0: \delta = 0$, against
H_1: ~~$\delta \neq 0$~~ $\delta < 0$, given AR(p) model
$\Delta Y_t =$ ~~$\beta_0 + \delta Y_{t-1} +$~~
$\beta_0 + (\alpha t +) \delta \Delta Y_{t-1} + \sum_{i=1}^{p_0} \gamma_i \Delta Y_{t-i} + u_t$
The test statistic is
$t = (\hat{\delta} - 0) / se(\hat{\delta})$
For the constant only test, this is compared against critical values from the DFcn distribution. For the constant and trend test, this is compared against critical values from the DFtr distribution.

By inspection, fail to reject the null ~~f~~ of a unit root ~~&~~ in both the constant only and the constant and trend test ~~for each of~~ at ~~the~~ every conventional level of significance for each of luSUK$_t$, luSEU$_t$, luSUK$_t$ - luSEU$_t$.

~~By inspection~~ First differences of an AR(p) process have no deterministic trend. Then, the appropriate test for ΔluSUK$_t$, ΔluSEU$_t$, ΔluSUK$_t$ - ΔluSEU$_t$. By inspection, reject the null of a unit root at every conventional level of significance in each of these cases.

Conclude that each of luSUK$_t$, ... has a ~~~~ unit root and ~~the~~ is thus non-stationary and that each of ~~{ΔluSUK$_t$}~~ ΔluSUK$_t$, ... does not have a unit root and is stationary. Hence, each of luSUK$_t$ ... has order of integration 1.

b From the above, luSUK$_t$ - luSEU$_t$ has order of integration 1. luSUK$_t$ - luSEU$_t$ = ln(USDGBP/USDEUR) = ln(EUR/GBP) = LEUUK

c From the above, each of luSEU$_t$ and luSUK$_t$ has ~~~~ one root, the series have order of integration 1, so a finding of a correlation between the two is potentially spurious.

There is a tendency to find statistically significant relationships between unrelated series of I(1) variables because such ~~variable~~ variables have a stochastic trend hence ~~~~ such series have a tendency to exhibit large swings that can be matched to some swing in another series with surprising regularity. t-statistics are not consistent for zero and diverge in magnitude as the sample size grows, without settling on a fixed distribution. R^2 does not converge to zero and remains high with non-negligible probability. Then, because of the possibility of spurious correlation, ~~high~~ large t statistics and high R^2 ~~do not~~ cannot be unambiguously interpreted as evidence of a relationship between time series with order of integration 1.

e Test for cointegration between the two variables. Estimate a cointegrating coefficient by ~~a~~ OLS regression of one variable on the other. Compute the estimate of the equilibrium error $\hat{\varepsilon}_t = Y_t - \hat{\theta} X_t$. Perform an ADF test for a unit root in $\hat{\varepsilon}_t$. If the null of a unit root is rejected, ~~$\hat{\varepsilon}_t$ is~~ conclude that the estimate of the equilibrium error is stationary and that Y_t and X_t are cointegrated. Otherwise, the estimate of the equilibrium error is non stationary and the two variables are not cointegrated. If the variables are cointegrated, then $\hat{\theta}$ consistently estimates the cointegrating coefficient, and the finding of a relationship is genuine, not spurious.

5a The order of integration of each process is 1.

b

5a $Y_t = \sum_{s=1}^{t} v_{ys}$

$v_{yt} = \varepsilon_t + \gamma \varepsilon_{t-1}$

$X_t = \sum_{s=1}^{t} v_{xs}$

$v_{xt} = \varepsilon_t$

$\{\varepsilon_t\}$ is iid, $\varepsilon_0 = 0$

~~$E(Y_t) =$~~

$EY_t = \sum_{s=1}^{t} E(\varepsilon_s + \gamma \varepsilon_{s-1})$

$= \sum_{s=1}^{t} E(\varepsilon_s) + \gamma \sum_{s=1}^{t} E \varepsilon_{s-1}$

$= \sum_{s=1}^{t} \mu + \gamma \sum_{s=1}^{t} \mu$

$= (1+\gamma) t\mu$

~~$cov(Y_t, Y_{t-1}) = cov(\varepsilon_t + \gamma\varepsilon_{t-1}$~~

$var(Y_t) = var(\varepsilon_t) + \gamma^2 var(\varepsilon_{t-1})$

$= (1+\gamma^2)\sigma^2$

$cov(Y_t, Y_{t-1}) = cov(\varepsilon_t + \gamma\varepsilon_{t-1}, \varepsilon_{t-1} + \gamma\varepsilon_{t-2})$

$= \gamma\sigma^2$

$cov(Y_t, Y_{t-h}) = 0$ for $h \geq 2$

Then, Y_t is stationary iff $\gamma = -1$

Given that $\{\varepsilon_t\}$ is ~~iid, it is~~ a ~~station~~ iid process, it is a stationary process, then ~~it is trivial~~ $\{v_{xs}\}$ is stationary and so is $\{v_{ys}\}$ because each is a linear function of ~~the variables of~~ stationary series. So if $\gamma \neq -1$, Y_t is not stationary but $\Delta Y_t = v_{yt}$ is, so $Y_t \sim I(1)$, and if $\gamma = -1$, $Y_t \sim I(0)$.

$EX_t = \sum_{s=1}^{t} E v_{xs}$

$= \sum_{s=1}^{t} \mu$

$= t\mu$

$cov(X_t, X_{t-h}) =$ ~~$cov(\sum_{s=1}^{t} v_{xs} \sum$~~

$= cov(\sum_{s=1}^{t-h} v_{xs} + \sum_{s=t-h+1}^{t} v_{xs}, \sum_{s=1}^{t-h} v_{xs})$

$= cov(\sum_{s=1}^{t-h} v_{xs}, \sum_{s=1}^{t-h} v_{xs})$

$= var(\sum_{s=1}^{t-h} v_{xs})$

$= \sum_{s=1}^{t-h} var\, v_{xs}$

$= (t-h)\sigma^2$

Then X_t is non-stationary (supposing that ε_t is a non-degenerate random variable with $\sigma^2 > 0$. From the above, $\Delta X_t = v_{xt}$ is stationary, so $X_t \sim I(1)$.

b $Y_t = \sum_{s=1}^{t} (\varepsilon_s + \gamma\varepsilon_{s-1})$

$= \varepsilon_t + \sum_{s=1}^{t-1} (\varepsilon_s + \gamma\varepsilon_s) + \gamma\varepsilon_0$

$= \varepsilon_t + (1+\gamma)\sum_{s=1}^{t-1} \varepsilon_s$

$= (1+\gamma)\sum_{s=1}^{t} \varepsilon_s - \gamma\varepsilon_t$

$= (1+\gamma) X_t - \gamma\varepsilon_t$

Given that $\{\varepsilon_t\}$ is iid, ε_t hence $\gamma\varepsilon_t$ is stationary, so $\varepsilon_t = Y_t - \cancel{(1+\gamma)}\, \theta X_t = -\gamma\varepsilon_t$ is stationary, and Y_t and X_t are cointegrated with cointegrating coefficient $\theta = (1+\gamma)$

c $v_{xt} = \varepsilon_t + \delta\eta_t$

$X_t = \sum_{s=1}^{t} v_{xs}$

$= \sum_{s=1}^{t} \varepsilon_s + \delta\eta_s$

$= \sum_{s=1}^{t} \varepsilon_s + \delta\sum_{s=1}^{t} \eta_s$

$Y_t = (1+\gamma)\sum_{s=1}^{t} \varepsilon_s - \gamma\varepsilon_t$

Y_t and X_t are no longer cointegrated.

$Y_t - \theta X_t = -\gamma\varepsilon_t + (1+\gamma-\theta)\sum_{s=1}^{t} \varepsilon_s - \theta\delta\sum_{s=1}^{t} \eta_s$

~~For $\theta = 0$, the third term~~

No θ eliminates both the latter two ~~deterministic~~ stochastic trend terms, and any variable with a stochastic trend term forms a non-stationary series. So there is no candidate cointegrating efficient such that the equilibrium error is stationary.

Date