

Microeconomic Analysis Problem Set 2

a By the (ε, δ) definition of continuity, function $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is continuous iff $\forall \varepsilon > 0 \in \mathbb{R}, \exists \delta > 0 \in \mathbb{R}$:
 at \vec{x}_0 iff $\forall \varepsilon > 0 \in \mathbb{R}: \exists \delta > 0 \in \mathbb{R}$: if $\|\vec{x}_1 - \vec{x}_0\| < \delta$ then
 $\|f(\vec{x}_1) - f(\vec{x}_0)\| < \varepsilon$.

$$f(x, y) = xy$$

$$\|(x, y) - (0, 0)\| = \|(x, y)\| = \sqrt{x^2 + y^2}$$

$$\|f(x, y) - f(0, 0)\| = \|xy\| = xy$$

$$x, y \leq \sqrt{x^2 + y^2}$$

$$xy \leq (\sqrt{x^2 + y^2})^2$$

If $\|(x, y) - (0, 0)\| < \delta$, then $\sqrt{x^2 + y^2} < \delta$, then
 $xy < \delta^2$, then $\|f(x, y) - f(0, 0)\| < \delta^2$.

So if $\forall \varepsilon > 0: \exists \delta > 0$ namely $\delta = \sqrt{\varepsilon}$ such that if
 $\|(x, y) - (0, 0)\| < \delta$ then $\|f(x, y) - f(0, 0)\| < \varepsilon$.

b By the (ε, δ) definition of continuity, $f(x, y) = xy$
 is continuous at $0, 0$.

b $f: \mathbb{R}^m \rightarrow \mathbb{R}$ is continuous at \vec{x}_0 . ①

$g: \mathbb{R}^m \rightarrow \mathbb{R}$ is continuous at \vec{x}_0 . ②

From ① & by the (ε, δ) definition of continuity,

$\forall \varepsilon > 0: \exists \delta > 0: \text{if } \|\vec{x}_1 - \vec{x}_0\| < \delta \text{ then } \|f(\vec{x}_1) - f(\vec{x}_0)\| < \varepsilon$ ③

$\forall \varepsilon' > 0: \exists \delta' > 0: \text{if } \|\vec{x}_1 - \vec{x}_0\| < \delta' \text{ then } \|g(\vec{x}_1) - g(\vec{x}_0)\| < \varepsilon'$ ④

From ③ and ④

$\forall \varepsilon, \varepsilon' > 0: \exists \delta, \delta' > 0: \text{if } \|\vec{x}_1 - \vec{x}_0\| < \delta \text{ then } \|f(\vec{x}_1) - f(\vec{x}_0)\| < \varepsilon$

and if $\|\vec{x}_1 - \vec{x}_0\| < \delta'$ then $\|g(\vec{x}_1) - g(\vec{x}_0)\| < \varepsilon'$ ⑤

From ⑤

$\forall \varepsilon, \varepsilon' > 0: \exists \delta^* > 0$, namely $\exists \delta, \delta' > 0: \text{if } \|\vec{x}_1 - \vec{x}_0\| <$

$\min\{\delta, \delta'\} = \delta^*$ then $\|f(\vec{x}_1) - f(\vec{x}_0)\| < \varepsilon$ and

$\|g(\vec{x}_1) - g(\vec{x}_0)\| < \varepsilon'$ ⑥

From ⑥

$\forall \varepsilon, \varepsilon' > 0: \exists \delta^* > 0: \text{if } \|\vec{x}_1 - \vec{x}_0\| < \delta^*$ then

$\|(f+g)(\vec{x}_1) - (f+g)(\vec{x}_0)\| < \varepsilon + \varepsilon' = \varepsilon^*$ ⑦

From ⑦

$\forall \varepsilon^* > 0: \exists \delta^* > 0: \text{if } \|\vec{x}_1 - \vec{x}_0\| < \delta^*$ then

$\|(f+g)(\vec{x}_1) - (f+g)(\vec{x}_0)\| < \varepsilon^*$ ⑧

From ⑧, by the (ε, δ) definition of continuity,

$f+g$ is continuous at \vec{x}_0 .

a) the k^{th} partial derivative of the function f at $\vec{x} \in \mathbb{R}^n$

$\frac{\partial f}{\partial x_k}(\vec{x})$ is formally defined as $\lim_{h \rightarrow 0} \frac{f(x_1, \dots, x_{k-1}, x_k + h, x_{k+1}, \dots, x_n) - f(\vec{x})}{h}$

$$f(x, y) = \begin{cases} x^3 / (x^2 + y^2) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\partial f}{\partial x} = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h}$$

$$(0, 0) = \lim_{h \rightarrow 0} \frac{f(h, 0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{1}{h}$$

$$= 1$$

$$\frac{\partial f}{\partial y} = \lim_{h \rightarrow 0} \frac{f(0, h) - f(0, 0)}{h}$$

$$(0, 0) = \lim_{h \rightarrow 0} \frac{f(0, h)}{h}$$

$$= 0$$

$$g(x, y) = \begin{cases} x^{1/2} y^{1/2} & \text{if } x \geq 0 \text{ and } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\partial g}{\partial x}(0, 0) = \lim_{h \rightarrow 0} \frac{g(h, 0) - g(0, 0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{0}{h}$$

$$= 0$$

By symmetry, $\frac{\partial g}{\partial y} = 0$

b) $f_x = x^3 (-1)(x^2 + y^2)^{-2} (2x) + 3x^2 (x^2 + y^2)^{-1}$ if $(x, y) \neq (0, 0)$

$$= -2x^4 (x^2 + y^2)^{-2} + 3x^2 (x^2 + y^2)^{-1}$$

$$= x^2 (x^2 + y^2)^{-2} [-2x^2 + 3(x^2 + y^2)]$$

$$= x^2 (x^2 + y^2)^{-2} [x^2 + 3y^2] \quad \text{if } (x, y) \neq (0, 0)$$

$$f_y = x^2 (x^2 + y^2)^{-2} x^3 (-1)(x^2 + y^2)^{-2} (2y) \quad \text{if } (x, y) \neq (0, 0)$$

$$= -2x^3 y (x^2 + y^2)^{-2} \quad \text{if } (x, y) \neq (0, 0)$$

$$g_x = (\frac{1}{2}) x^{-1/2} y^{1/2} \quad \text{if } x \geq 0 \text{ and } y \geq 0$$

$$g_y = (\frac{1}{2}) y^{-1/2} x^{1/2} \quad \text{if } x \geq 0 \text{ and } y \geq 0$$

$$\exists u \quad u = \sum_{i=1}^L d_i \ln x_i, \quad \forall i: d_i > 0, \quad \sum_{i=1}^L d_i = 1$$

$$\partial u = (\partial_1 u \quad \partial_2 u \cdots \partial_L u)$$

$$= (d_1/x_1 \quad d_2/x_2 \cdots d_L/x_L)$$

$$\begin{aligned} b D^2 u &= \begin{pmatrix} \partial_1 \partial_1 u & \partial_1 \partial_2 u & \cdots & \partial_1 \partial_L u \\ \partial_2 \partial_1 u & \partial_2 \partial_2 u & \cdots & \partial_2 \partial_L u \\ \vdots & \vdots & \ddots & \vdots \\ \partial_L \partial_1 u & \partial_L \partial_2 u & \cdots & \partial_L \partial_L u \end{pmatrix} \\ &= \begin{pmatrix} -d_1 x_1^{-2} & 0 & \cdots & 0 \\ 0 & -d_2 x_2^{-2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -d_L x_L^{-2} \end{pmatrix} \end{aligned}$$

c Let $D^2 u_{ij}$ denote the element in the i^{th} row and j^{th} column of $D^2 u$. By definition of $D^2 u$, $\forall i, j \in \{1, \dots, L\}$:

$$D^2 u_{ij} = \partial_i \partial_j u = \partial_i (d_j/x_j) = 0 \text{ for all } i \neq j.$$

$D^2 u$ is a diagonal matrix so $D^2 u$ is a symmetric matrix. This is confirmed by inspection.

Let $D^2 u^k$ denote the square submatrix of A with only the first k rows and columns retained. Then $D^2 u^k = (-d_i x_i^{-2})$

$$D^2 u^2 = \begin{pmatrix} -d_1 x_1^{-2} & 0 \\ 0 & -d_2 x_2^{-2} \end{pmatrix} \text{ and so on.}$$

$$\det(-1)^1 \det D^2 u^1 = d_1 x_1^{-2}$$

$$(-1)^2 \det D^2 u^2 = (d_1 x_1^{-2})(d_2 x_2^{-2})$$

$$(-1)^k \det D^2 u^k = (-1)^k \prod_{i=1}^k (-d_i x_i^{-2})$$

since the determinant of a diagonal matrix is the product of its diagonal elements

$$= \prod_{i=1}^k (d_i x_i^{-2})$$

> 0 for all k

Given $\forall i: d_i > 0$, assuming that u is well-defined then

$\forall i: \ln x_i$ is well-defined then $\forall i: x_i \neq 0$ then $\forall i: x_i^{-2} > 0$.

By the determinant test, $D^2 u$ is negative definite.

d $\vec{y}^T D^2 u \vec{y}$ is well-defined only if \vec{y} is a vector of length L . Let $\vec{y} = (y_1, \dots, y_L)^T$

$$\begin{aligned} D^2 u \vec{y} &= \begin{pmatrix} -d_1 x_1^{-2} & 0 & \cdots & 0 \\ 0 & -d_2 x_2^{-2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -d_L x_L^{-2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_L \end{pmatrix} \\ &= \begin{pmatrix} -y_1 d_1 x_1^{-2} \\ -y_2 d_2 x_2^{-2} \\ \vdots \\ -y_L d_L x_L^{-2} \end{pmatrix} \end{aligned}$$

$$\vec{y}^T D^2 u \vec{y} = (y_1, y_2, \dots, y_L) \begin{pmatrix} -y_1 d_1 x_1^{-2} \\ -y_2 d_2 x_2^{-2} \\ \vdots \\ -y_L d_L x_L^{-2} \end{pmatrix}$$

$$= -y_1^2 d_1 x_1^{-2} - y_2^2 d_2 x_2^{-2} - \cdots - y_L^2 d_L x_L^{-2}$$

$$= -\sum_{i=1}^L y_i^2 \cdot d_i x_i^{-2}$$

< 0 for all $\vec{y} \neq 0 \in \mathbb{R}^L$

Given $\forall i: d_i > 0$, assuming that u is well-defined hence

$$\forall i: x_i^{-2} > 0$$

By checking $\vec{y}^T D^2 u \vec{y}$, $D^2 u$ is negative definite.

$$\begin{aligned} e \quad u(\vec{x}') &\approx u(\vec{x}) + D^2 u(\vec{x})(\vec{x}' - \vec{x}) + \frac{1}{2} (\vec{x}' - \vec{x})^T D^2 u(\vec{x})(\vec{x}' - \vec{x}) \\ &= 0 + (d_1 x_1^{-2} \cdots d_L x_L^{-2})(x_1' - 1 \ x_2' - 1 \ \cdots x_L' - 1) \\ &\quad + \frac{1}{2} (x_1' - 1 \ x_2' - 1 \ \cdots x_L' - 1)^T D^2 u(x_1' - 1 \ x_2' - 1 \ \cdots x_L' - 1) \end{aligned}$$

$$\begin{aligned} &= \alpha_1(x'_1 - 1) + \alpha_2(x'_2 - 1) + \dots + \alpha_L(x'_L - 1) \\ &+ \frac{1}{2} [-\alpha_1(x'_1 - 1)^2 - \alpha_2(x'_2 - 1)^2 - \dots - \alpha_L(x'_L - 1)^2] \\ &= \sum_{i=1}^L \alpha_i (x'_i - 1) - \frac{1}{2} (x'_i - 1)^2 \end{aligned}$$

$$4a \text{ Let } f(x, y, z) = z^2 + xz + 4x^2 + y^3$$

$$f(x, y, z) = 0$$

$$z^2 + xz + 4x^2 + y^3 = 0$$

$$z^2 + xz = -4x^2 - y^3$$

$$(z + x/2)^2 - x^2/4 = -4x^2 - y^3$$

$$(z + x/2) = \sqrt{-4x^2 - y^3 + x^2/4} - x/2$$

$$z = g(x, y) = \sqrt{-4x^2 - y^3 + x^2/4} - x/2$$

$$f_x = z + 2xy$$

$$f_y = x^2 + 3y^2$$

$$f_z = 2z + x$$

By inspection, f_x , f_y , and f_z are continuous, so f is C^1

$f_z(x^*) = f_z(z^*) = 1 \neq 0$. Then, by the implicit function

theorem, $f(x, y, z) = 0$ defines a $z = g(x, y)$ in the neighbourhood of $(x^*, y^*) = (1, 0)$ such that $z^* = g(x^*, y^*)$,

$$f(x^*, y^*, g(x^*, y^*)) = 0 \text{ and } g_x(x, y, z) =$$

$$g_x(x, y, z) = -\frac{\partial f}{\partial z} f_x(x, y, z) / f_z(x, y, z) \text{ and}$$

$$g_y(x, y, z) = -\frac{\partial f}{\partial z} f_y(x, y, z) / f_z(x, y, z)$$

b ~~$z = g(x=1, y=0) =$~~

$$z = g(x=1, y=0) = 0$$

$$g_x(x=1, y=0)$$

$$g_x(1, 0, 0) = -\frac{\partial f}{\partial z} f_x(1, 0, 0) = 0$$

$$g_y(1, 0, 0) = -\frac{\partial f}{\partial z} f_y(1, 0, 0) = -1/1 = -1$$

5a Let $f_1(x, y, u, v) = x^2 - y^2 - u^3 + v^2 + 4$,
 $f_2(x, y, u, v) = 2xy + y^2 - 2u^2 + 3v^4 + 8$,

and $\mathbf{f} = (f_1, f_2)$

Given $\mathbf{f}(s^*) = (0, 0)$

$$\mathbf{D}\mathbf{f} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{pmatrix}$$

$$= \begin{pmatrix} 2x & -2y & -3u^2 & 2v \\ 2y & 2x & -4u & 12v^3 \end{pmatrix}$$

By inspection, each element of $D\mathbf{f}$ is continuous, so \mathbf{f} is.

∴

$$D_{u,v}\mathbf{f} = \begin{pmatrix} -3u^2 & 2v \\ -4u & 12v^3 \end{pmatrix}$$

$$\det D_{u,v}\mathbf{f} = -36u^2v^3 + 8uv$$

$$\text{At } s^*, \det D_{u,v}\mathbf{f} = -36(2^2)(1^3) + 8(2)(1)$$

$$= -144 + 16$$

$$= -128 \neq 0$$

so $D_{u,v}\mathbf{f}$ is invertible at s^*

Then, by the implicit function theorem, $\mathbf{f}(x, y, u, v) = (0, 0)$

defines a function $(u, v) = g(x, y)$ such that (x^*, y^*)

$= g(x^*, y^*)$, $\mathbf{f}(x^*, y^*, g(x^*, y^*)) = (0, 0)$.

$D_x g = -[D_{u,v}\mathbf{f}]^{-1} D_x \mathbf{f}$ and $D_y g = -[D_{u,v}\mathbf{f}]^{-1} D_y \mathbf{f}$

b) $(u, v) = g(2, -1) = (2, 1)$

At $(2, -1, 2, 1)$, $D_{u,v}\mathbf{f} = \begin{pmatrix} -12 & 2 \\ -8 & 12 \end{pmatrix}$, $\det D_{u,v}\mathbf{f} = -128$

By Cramer's rule, $[D_{u,v}\mathbf{f}]^{-1} = -\frac{1}{128} \begin{pmatrix} 12 & -2 \\ 8 & -12 \end{pmatrix}$

$$D_x \mathbf{f} = \begin{pmatrix} \frac{\partial f_1}{\partial x} \\ \frac{\partial f_2}{\partial x} \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

At $(2, -1, 2, 1)$, $D_x \mathbf{f} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$

$$D_x g = -[D_{u,v}\mathbf{f}]^{-1} D_x \mathbf{f} = \frac{1}{128} \begin{pmatrix} 12 & -2 \\ 8 & -12 \end{pmatrix} \begin{pmatrix} 4 \\ -2 \end{pmatrix} = \frac{1}{128} (52, 56)$$

$$= \frac{1}{32} (13, 14)$$

$$\frac{\partial u}{\partial x} = \frac{13}{32}, \quad \frac{\partial v}{\partial x} = \frac{14}{32}$$

6 $f(x,y) = x^4 + x^2 - 6xy + 3y^2$
 $Df(x,y) = (-4x^3 + 2x - 6y \quad -6x + 6y)$

FOCs:

$$4x^3 + 2x - 6y = 0 \quad (1)$$

$$-6x + 6y = 0 \quad (2)$$

$$\text{From (2), } x=y \quad (3)$$

Sub (3) into (1)

$$4x^3 - 4x = 4x(x+1)(x-1) = 0, x = -1, 0, \text{ or } 1$$

Critical points: $(-1, -1), (0, 0), (1, 1)$

$$D^2f(x,y) = \begin{pmatrix} 12x^2 + 2 & -6 \\ -6 & 6 \end{pmatrix}$$

SOCs:

for $(-1, -1)$

$$\det D^2f(-1, -1) = \det \begin{pmatrix} 14 & -6 \\ -6 & 6 \end{pmatrix} = 48 > 0$$

$\det [D^2f(-1, -1)]_{11} = \det (14) = 14 > 0$
 $D^2f(-1, -1)$ is positive definite, $(-1, -1)$ is a strict local minimum

$$\det D^2f(0, 0) = \det \begin{pmatrix} 2 & -6 \\ -6 & 6 \end{pmatrix} = -24 < 0$$

$D^2f(0, 0)$ is indefinite, $(0, 0)$ is a saddle point.

$$\det D^2f(1, 1) = \det \begin{pmatrix} 14 & -6 \\ -6 & 6 \end{pmatrix} = 48 > 0$$

$$\det [D^2f(1, 1)]_{11} = \det (14) = 14 > 0$$

$D^2f(1, 1)$ is positive definite, $(1, 1)$ is a strict local minimum.