Microeconomic Analysis Problem set 2

1a By the (ε, δ) definition of continuity, function $f: \mathbb{R}^n \to \mathbb{R}^m$ is continuous ~~iff $\forall \varepsilon>0 \in \mathbb{R}: \exists \delta>0 \in \mathbb{R}:$~~ at $\vec{x}_0$ iff $\forall \varepsilon>0 \in \mathbb{R}: \exists \delta>0 \in \mathbb{R}:$ if $\|\vec{x}_1-\vec{x}_0\|<\delta$ then $\|f(\vec{x}_1)-f(\vec{x}_0)\|<\varepsilon$.

$f(x,y)=xy$

$\|(x,y)-(0,0)\| = \|(x,y)\| = \sqrt{x^2+y^2}$

$\|f(x,y)-f(0,0)\| = \|xy\| = xy$

$x, y \leq \sqrt{x^2+y^2}$

$xy \leq (\sqrt{x^2+y^2})^2$

If $\|(x,y)-(0,0)\|<\delta$, then $\sqrt{x^2+y^2}<\delta$, then $xy<\delta^2$, then $\|f(x,y)-f(0,0)\|<\delta^2$.

So if ~~iff~~ $\forall \varepsilon>0: \exists \delta>0$ namely $\delta=\sqrt{\varepsilon}$ such that if $\|(x,y)-(0,0)\|<\delta$ then $\|f(x,y)-f(0,0)\|<\varepsilon$.

∴ By ~~defn~~ the (ε,δ) definition of continuity, $f(x,y)=xy$ is continuous at $0,0$.

b $f: \mathbb{R}^m \to \mathbb{R}$ is continuous at $\vec{x}_0$ ①

$g: \mathbb{R}^m \to \mathbb{R}$ is continuous at $\vec{x}_1$ ②

From ① ~~&~~, by the (ε,δ) definition of continuity,

$\forall \varepsilon>0: \exists \delta>0:$ if $\|\vec{x}_1-\vec{x}_0\|<\delta$ then $\|f(\vec{x}_1)-f(\vec{x}_0)\|<\varepsilon$ ③

$\forall \varepsilon'>0: \exists \delta'>0:$ if $\|\vec{x}_1-\vec{x}_0\|<\delta'$ then $\|g(\vec{x}_1)-g(\vec{x}_0)\|<\varepsilon'$ ④

From ③ and ④

$\forall \varepsilon, \varepsilon'>0: \exists \delta, \delta'>0:$ if $\|\vec{x}_1-\vec{x}_0\|<\delta$ then $\|f(\vec{x}_1)-f(\vec{x}_0)\|<\varepsilon$ and if $\|\vec{x}_1-\vec{x}_0\|<\delta'$ then $\|g(\vec{x}_1)-g(\vec{x}_0)\|<\varepsilon'$ ⑤

From ⑤

$\forall \varepsilon, \varepsilon'>0:$ ~~$\exists \delta^*>0$, namely~~ $\exists \delta, \delta'>0:$ if $\|\vec{x}_1-\vec{x}_0\|<\min\{\delta,\delta'\}\equiv\delta^*$ then $\|f(\vec{x}_1)-f(\vec{x}_0)\|<\varepsilon$ and $\|g(\vec{x}_1)-g(\vec{x}_0)\|<\varepsilon'$ ⑥

From ⑥

$\forall \varepsilon, \varepsilon'>0: \exists \delta^*>0:$ if $\|\vec{x}_1-\vec{x}_0\|<\delta^*$ then $\|(f+g)(\vec{x}_1)-(f+g)(\vec{x}_0)\|<\varepsilon+\varepsilon'\equiv\varepsilon^*$ ⑦

From ⑦

$\forall \varepsilon^*>0: \exists \delta^*>0:$ if $\|\vec{x}_1-\vec{x}_0\|<\delta^*$ then $\|(f+g)(\vec{x}_1)-(f+g)(\vec{x}_0)\|<\varepsilon^*$ ⑧

From ⑧, by the (ε,δ) definition of continuity, $f+g$ is continuous at $\vec{x}_0$.

2a The k^{th} partial derivative of the function f at $\vec{x} \in \mathbb{R}$ $\partial f / \partial x_k (\vec{x})$ is formally defined as ~~$\lim_{h\to 0}(1/h$~~

$\lim_{h\to 0} (1/h)[f(x_1, \dots, x_k+h, \dots x_n) - f(\vec{x})]$

$f(x,y) = \begin{cases} x^3/x^2+y^2 & \text{if } (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases}$

$\partial f/\partial x = \lim_{h\to 0} (1/h)[f(h,0) - f(0,0)]$

$(0,0) = \lim_{h\to 0} (1/h)[h-0]$

$= \lim_{h\to 0} 1$

$= 1$

$\partial f/\partial y = \lim_{h\to 0} (1/h)[f(0,h) - f(0,0)]$

$(0,0) = \lim_{h\to 0} (1/h)[0]$

$= 0$

$g(x,y) = \begin{cases} x^{1/2} y^{1/2} & \text{if } x \geq 0 \text{ and } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$

$\partial g/\partial x (0,0) = \lim_{h\to 0} (1/h)[g(h,0) - g(0,0)]$

$= \lim_{h\to 0} (1/h)[0]$

$= 0$

By symmetry, $\partial g/\partial y = 0$

b $f_x = x^3(-1)(x^2+y^2)^{-2}(2x) + 3x^2(x^2+y^2)^{-1}$ if $(x,y) \neq (0,0)$

$= -2x^4(x^2+y^2)^{-2} + 3x^2(x^2+y^2)^{-1}$

$= x^2(x^2+y^2)^{-2}[-2x^2 + 3(x^2+y^2)]$

$= x^2(x^2+y^2)^{-2}[x^2+3y^2]$ if $(x,y) \neq (0,0)$

$f_y = \cancel{x^2(x^2+y^2}\ x^3(-1)(x^2+y^2)^{-2}(2y)$ if $(x,y) \neq (0,0)$

$= -2x^3y(x^2+y^2)^{-2}$ if $(x,y) \neq (0,0)$

$g_x = (1/2)x^{-1/2}y^{1/2}$ if $x \geq 0$ and $y \geq 0$

$g_y = (1/2)y^{-1/2}x^{1/2}$ if $x \geq 0$ and $y \geq 0$

3a $u=\sum_{i=1}^{L} a_i \ln x_i$, $\forall i: a_i>0$, $\sum_{i=1}^{L} a_i=1$

$Du=(\partial_1 u \ \partial_2 u \cdots \partial_L u)$

$=(a_1/x_1 \ a_2/x_2 \cdots a_L/x_L)$

b $$D^2u=\begin{pmatrix}\partial_1\partial_1 u & \partial_1\partial_2 u & \cdots & \partial_1\partial_L u\\ \partial_2\partial_1 u & \partial_2\partial_2 u & \cdots & \partial_2\partial_L u\\ \vdots & \vdots & \ddots & \vdots\\ \partial_L\partial_1 u & \partial_L\partial_2 u & \cdots & \partial_L\partial_L u\end{pmatrix}$$

$$=\begin{pmatrix}-a_1x_1^{-2} & 0 & \cdots & 0\\ 0 & -a_2x_2^{-2} & & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & -a_Lx_L^{-2}\end{pmatrix}$$

c Let D^2u_{ij} denote the element in the i^{th} row and j^{th} column of D^2u. By definition of D^2u, $\forall i,j\in\{1,\dots L\}$: $D^2u_{ij}=\partial_i\partial_j u=\partial_i(a_j/x_j)=0$ for all $i\neq j$. Then D^2u is a diagonal matrix so D^2u is a symmetric matrix. This is confirmed by inspection.

Let D^2u^k denote the square submatrix of A with only the first k rows and columns retained. Then $D_2u^1=(-a_1x_1^{-2})$

$D_2u^2=\begin{pmatrix}-a_1x_1^{-2} & 0\\ 0 & -a_2x_2^{-2}\end{pmatrix}$ and so on.

~~det~~ $(-1)^1 \det D_2u^1 = a_1x_1^{-2}$

$(-1)^2 \det D_2u^2 = (a_1x_1^{-2})(a_2x_2^{-2})$

$(-1)^k \det D_2u^k = (-1)^k \prod_{i=1}^{k}(-a_ix_i^{-2})$

since the determinant of a diagonal matrix is the product of its diagonal elements

$=\prod_{i=1}^{k}(a_ix_i^{-2})$

>0 for all k

Given $\forall i\, a_i>0$, assuming that u is well-defined then $\forall i: \ln x_i$ is well-defined then $\forall i: x_i\neq 0$ then $\forall i\, x_i^{-2}>0$.

By the determinant test, D^2u is negative definite.

d $\vec{y}^T D^2u\, \vec{y}$ is well-defined only if $\vec{y}$ is a vector of length L. Let $\vec{y}=(y_1,\dots,y_L)^T$

$$D^2u\,\vec{y}=\begin{pmatrix}-a_1x_1^{-2} & 0 & \cdots & 0\\ 0 & -a_2x_2^{-2} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & -a_Lx_L^{-2}\end{pmatrix}\begin{pmatrix}y_1\\ y_2\\ \vdots\\ y_L\end{pmatrix}$$

$=\cancel{(y_1a_1x_1^{-2} \ y_2a_2x_2^{-2} \cdots y_La_Lx_L^{-2})}$

$\cancel{\vec{y}^TD^2u\vec{y}}$ $\begin{pmatrix}-y_1a_1x_1^{-2}\\ -y_2a_2x_2^{-2}\\ \\ -y_La_Lx_L^{-2}\end{pmatrix}$

$$\vec{y}^TD^2u\vec{y}=(y_1\ y_2\cdots y_L)\begin{pmatrix}-y_1a_1x_1^{-2}\\ -y_2a_2x_2^{-2}\\ \\ -y_La_Lx_L^{-2}\end{pmatrix}$$

$=-y_1^2a_1x_1^{-2}-y_2^2a_2x_2^{-2}-\dots-y_L^2a_Lx_L^{-2}$

$=\cancel{\sum a} -\sum_{i=1}^{L} \cancel{a_i y_i^2 x} \ a_i y_i^2 x_i^{-2}$

<0 for all $\vec{y}\neq 0\in\mathbb{R}^n$

Given $\forall i: a_i>0$, assuming that u is well-defined hence $\forall i: x_i^{-2}>0$

By checking $\vec{y}^TD^2u\vec{y}$, D^2u is negative definite.

e $u(\vec{x}')\approx u(\vec{x})+Du(\vec{x})(\vec{x}'-\vec{x})+\frac{1}{2}(\vec{x}'-\vec{x})^TD^2u(\vec{x})(\vec{x}'-\vec{x})$

$=0+(a_1/\cancel{x_1}\ a_2/\cancel{x_2}\cdots a_L/\cancel{x_L})(x_1'-1\ x_2'-1\cdots x_L'-1)$

$+\frac{1}{2}(x_1'-1\ x_2'-1\cdots x_L'-1)D^2u(x_1'-1\ x_2'-1\cdots x_L'-1)^T$

$$= \alpha_1(x_1'-1) + \alpha_2(x_2'-1) + \cdots + \alpha_L(x_L'-1)$$
$$+ \frac{1}{2}\left[-\alpha_1(x_1'-1)^2 - \alpha_2(x_2'-1)^2 + \cdots + \alpha_L(x_L'-1)^2\right]$$
$$= \sum_{i=1}^{L} \alpha_i \ (x_i'-1) - \frac{1}{2}(x_i'-1)^2$$

4a Let $f(x,y,z) = z^2 + xz + yx^2 + y^3$

$f(x,y,z) = 0$

$z^2 + xz + yx^2 + y^3 = 0$

$z^2 + xz = -yx^2 - y^3$

$(z + x/2)^2 - x^2/4 = -yx^2 - y^3$

$(z + x/2) = \sqrt{-yx^2 - y^3 + x^2/4}$

$z = g(x,y) \equiv \sqrt{-yx^2 - y^3 + x^2/4} - x/2$

$f_x = z + 2xy$

$f_y = x^2 + 3y^2$

$f_z = 2z + x$

By inspection, f_x, f_y, and f_z are continuous, so f is C'

$\cancel{f_y(s^*)}$ $f_z(s^*) = 1 \neq 0$. Then, by the implicit function theorem, $f(x,y,z) = 0$ defines a $z = g(x,y)$ in the neighbourhood of $(x^*, y^*) = (1,0)$ such that $z^* = g(x^*, y^*)$, $f(x^*, y^*, g(x^*, y^*)) = 0$ and $\cancel{g_x(x,y,z) =}$

$g_x(x,y,z) = -f_x(x,y,z)/f_z(x,y,z)$ and

$g_y(x,y,z) = -f_y(x,y,z)/f_z(x,y,z)$

b $\cancel{g_x}$ $\cancel{z = g(x=1, y=0) =}$

$z = g(x=1, y=0) = 0$

$\cancel{g_x(x=1, y=0}$

$g_x(1,0,0) = -0/0 = 0$

$g_y(1,0,0) = -1/1 = -1$

5a Let $f_1(x,y,u,v) = x^2-y^2-u^3+v^2+4$,

$f_2(x,y,u,v) = 2xy+y^2-2u^2+3v^4+8$,

and $f = (f_1, f_2)$

Given $f(s^*) = (0,0)$

$$Df = \begin{pmatrix} \partial f_1/\partial x & \partial f_1/\partial y & \partial f_1/\partial u & \partial f_1/\partial v \\ \partial f_2/\partial x & \partial f_2/\partial y & \partial f_2/\partial u & \partial f_2/\partial v \end{pmatrix}$$

$$= \begin{pmatrix} 2x & -2y & -3u^2 & 2v \\ 2y & 2y & -4u & 12v^3 \end{pmatrix}$$

By inspection, each element of Df is continuous, so f is C_1

$$D_{u,v}f = \begin{pmatrix} -3u^2 & 2v \\ -4u & 12v^3 \end{pmatrix}$$

$\det D_{u,v}f = -36u^2v^3 + 8uv$

At s^*, $\det D_{u,v}f = -36(2^2)(1^3) + 8(2)(1)$

$= -144 + 16$

$= -128 \neq 0$

so $D_{u,v}f$ is invertible at s^*

Then, by the implicit function theorem, $f(x,y,u,v) = (0,0)$ defines a function $(u,v) = g(x,y)$ such that $(u^*,v^*) = g(x^*,y^*)$, $f(x^*,y^*,g(x^*,y^*)) = (0,0)$,

$D_x g = -[D_{u,v}f]^{-1} D_x f$ and $D_y g = -[D_{u,v}f]^{-1} D_y f$

b $(u,v) = g(2,-1) = (2,1)$

At $(2,-1,2,1)$, $D_{u,v}f = \begin{pmatrix} -12 & 2 \\ -8 & 12 \end{pmatrix}$, $\det D_{u,v}f = -128$

By Cramer's rule, $[D_{u,v}f]^{-1} = -\frac{1}{128}\begin{pmatrix} 12 & -2 \\ 8 & -12 \end{pmatrix}$

$$D_x f = \begin{pmatrix} \partial f_1/\partial x \\ \partial f_2/\partial x \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

At $(2,-1,2,1)$, $D_x f = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$

$$D_x g = -[D_{u,v}f]^{-1} D_x f = 1/128 \begin{pmatrix} 12 & -2 \\ 8 & -12 \end{pmatrix}\begin{pmatrix} 4 \\ -2 \end{pmatrix} = 1/128\,(52, 56)$$

$= 1/32\,(13, 14)$

$\partial u/\partial x = 13/32$, $\partial v/\partial x = 14/32$

Date

6 $f(x,y) = x^4 + x^2 - 6xy + 3y^2$

$Df(x,y) = (4x^3+2x-6y \quad -6x+6y)$

FOCs:

$4x^3+2x-6y=0$ ①

$-6x+6y=0$ ②

From ②, $x=y$ ③

Sub ③ into ①

$4x^3-4x = 4x(x+1)(x-1)=0$, $x=-1, 0,$ or 1

critical points: $(-1,-1)$, $(0,0)$, $(1,1)$

$$D^2f(x,y) = \begin{pmatrix} 12x^2+2 & -6 \\ -6 & 6 \end{pmatrix}$$

SOCs:

~~For (-1,-1)~~

$$\det D^2f(-1,-1) = \det\begin{pmatrix} 14 & -6 \\ -6 & 6 \end{pmatrix} = 48 > 0$$

$\det [D^2f(-1,-1)]_1 = \det(14) = 14 > 0$

$D^2f(-1,-1)$ is positive definite, $(-1,-1)$ is a strict local minimum

$$\det D^2f(0,0) = \det\begin{pmatrix} 2 & -6 \\ -6 & 6 \end{pmatrix} = -24 < 0$$

$D^2f(0,0)$ is indefinite, $(0,0)$ is a saddle point.

$$\det D^2f(1,1) = \det\begin{pmatrix} 14 & -6 \\ -6 & 6 \end{pmatrix} = 48 > 0$$

$\det [D^2f(1,1)]_1 = \det(14) = 14 > 0$

$D^2f(1,1)$ is positive definite, $(1,1)$ is a strict local minimum.