Game Theory Paper 190611

1a At every pure NE, no player has a profitable deviation.

For arbitrary player i,
$u_i(a_i = 1, a_{-i}) = am - p$
$u_i(a_i = 0, a_{-i}) = 0$
If $am - p > 0$, then $a_i = 1$ is a best response for player i. If $am - p < 0$, then $a_i = 0$ is a best response and if $am - p = 0$, both are best responses. ~~By~~ generalisation, if $am - p > 0$, then 1 is a best response for all players, if $am - p < 0$, then 0 is a best response for all players, if $am - p = 0$, both are best responses.

Then, there are three types of NE. In the first which exists if $an - p > 0$, $\Leftrightarrow p < an$, every player plays 1. In the second, which exists for all $p \geq 0$, every player plays 0. In the third, which exists if $p > 0$, some players play 1 while others play 0 such that $am - p = 0 \Leftrightarrow m = P/a \Leftrightarrow N' = P/a + 1$, where N' is the number of players who play 1.

There is one NE of each of the former types, because the strategy profile under which every player plays 1 ~~for all~~ is unique. Likewise for the strategy profile where every player plays 0. There are nCN', where $N' = P/a + 1$ NE of the third type because there are nCN' strategy profiles where n' players play 1.

b The owner's ~~revenue~~ expected revenue is 0.

Suppose that $p = 0$, then for all N, revenue $pN = 0$.

Suppose that $p > 0$, then there exists NE such that no player plays 1, at this NE, each player has payoff 0 and deviation to playing 1 yields payoff $-p < 0$ (so this is in fact a NE). The (pessimistic) expected number of 1 players is $N = 0$, then the (pessimistic) expected revenue is $pN = 0$

c Suppose that $y < P/a$, then ~~for each paying player,~~ the strategy profile under which every paying player plays 0 is an NE. At this strategy profile, each paying player has payoff 0. (Unilateral) deviation to 1 yields payoff $ay - p < 0$, so no player has a profitable deviation. The (pessimistic) expected number of 1 players is $N = 0$

We suppose that the players who are allowed to join for free are in fact compelled to join (rather than simply having zero membership fee), otherwise, there remains a NE at which no players join (because unilateral deviation from not joining to joining, even ~~for the~~ when joining is free is not strictly profitable).

Suppose that $y = P/a$, then ~~universal~~ the strategy profile under which every paying player plays 0 remains a NE because ~~the~~ unilateral deviation to 1 yields $ay - p = 0$ and is not strictly profitable. Then the (pessimistic) expected number of 1 players is $N = 0$

Suppose that $y > P/a$, then the strategy profile under which every paying player plays 1 is the unique pure NE. Suppose for reductio that some paying player plays 0 at NE, then this player has strictly profitable deviation to 1 which yields a minimum $ay - p > 0$, then this is not in fact a NE. By reductio, every paying player plays 1 at NE, then this NE is unique, and the ~~expected~~ (pessimistic) expected number of 1 players is $N = n - y$

$$N(y, p, a) = \begin{cases} 0 & \text{if } y \leq P/a \\ n - y & \text{if } y > P/a \end{cases}$$

d The owners optimisation problem is

$$\max_{y,p} \; pN(y, p, a)$$

which reduces to

$$\max_{y,p} \; p(n - y)$$
subject to $y > P/a$

Consider the similar problem

$$\max_{y,p} \; p(n - y)$$
subject to $y \geq P/a$

The objective function is decreasing in y, so the constraint binds at the optimum, and the above reduces to

$$\max_p \; p(n - P/a)$$

FOC: $\partial/\partial p \; p(n - P/a) = n - 2P/a = 0 \Rightarrow 2P/a = n \Rightarrow p = an/2$
$\Rightarrow y = n/2$ SOC: $-2/a < 0$

~~At~~ Then the "optimum" in the original profit maximisation is such that $p = {}^{an}/_{2}$ and y is arbitrarily greater than $^{n}/_{2}$. (strictly speaking an optimum does not exist because the ~~It is somewhat~~ constraint is a strict inequality)

It is somewhat surprising that the optimal number of "free" members is a fixed proportion of n, and is invariant to a. The intuitive explanation for this is presumably that the effect of a is "absorbed" by p, ~~so~~ variation in p is sufficient to optimally respond to variation in a.

2a ~~One player's~~ Player i's strategy σ_i weakly dominates ~~another~~ strategy σ_i' iff for all ~~strategy~~ profiles of other players' strategies σ_{-i}, σ_i yields weakly greater payoff to player i than σ_i' and for some profile of other players' strategies σ_{-i}', σ_i yields ~~greater~~ strictly greater payoff to player i than σ_i'. A strategy is weakly dominant iff it weakly dominates every other strategy (of the player whose strategy it is).

~~Players~~ Player i's strategy σ_i is strictly dominated ~~iff~~ by player i's strategy σ_i' iff for all profiles of other players' strategies σ_{-i}, σ_i yields strictly greater payoff to player i than σ_{-i}.

A strategy is eliminated by iterated strict dominance only if, against some profile of other player's strategies, some other strategy yields ~~a~~ strictly greater payoff. Then, by definition of weak dominance, a strategy that is so eliminated is not weakly dominant. So ~~only~~ weakly dominant strategies survive iterated strict dominance.

b ~~A strat~~ ~~2~~ A strategy is rationalisable iff it is a best response to some potentially correlated mix of other players' strategies. Equivalently, a strategy is rationalisable iff it survives iterated elimination of strategies that are not best responses to any pure or (potentially correlated) mixed strategies of other players.

Pearce's Lemma: A strategy is a best response against some pure or mixed strategy iff it is not strictly dominated by some pure or mixed strategy.

By Pearce's Lemma, iterated elimination of strategies that are not best responses is equivalent to iterated elimination of strictly dominated strategies. Then all and only strategies that survive iterated strict dominance are rationalisable.

c

	w	x	y	z
a	$\underline{9}$	$\underline{7}$	2	3
	0	2	$\underline{7}$	0
b	4	$\underline{5}$	4	3
	5	$\underline{3}$	5	0
c	2	7	$\underline{9}$	3
	$\underline{7}$	2	0	0
d	$\underline{2}$	0	$\underline{2}$	1
	0	0	0	$\underline{8}$

Best responses underlined.

By iterated strict dominance,
$\frac{1}{2}w + \frac{1}{2}y \succ z$, then
$b \succ d$,
eliminate z and d.

~~Each of~~ By inspection, each of the remaining strategies is a best response to one of the other remaining strategies. All and only a, b, c, w, x, y survive iterated strict dominance and are rationalisable.

Let "... R ..." ~~denote~~ abbreviate "... is rationalisable because it is a best response to ... which ~~is~~ ~~it~~ ~~itself~~ is also rationalisable".

~~aRy~~, bRx, cRw, wRa, xRb, yRc.

d ~~Best~~ Only rationalisable strategies are played at NE (at which players play mutual best responses), so ~~it is~~ ~~reason~~ restrict attention to only rationalisable strategies. The above game reduces to

~~R~~	w	x	y
a	$\underline{9}$	7	2
	0	2	$\underline{7}$
b	4	$\underline{5}$	4
	5	$\underline{3}$	5
c	2	7	$\underline{9}$
	$\underline{7}$	2	0

Best responses underlined.

By inspection, the only pure NE is (b, x) where players play mutual best responses.

e Suppose for reductio that P1 (Row) mixes a and b (only) at NE, then P2 (Column) plays y with zero probability because x yields strictly greater payoff against P1's strategy. Then P1 plays a with zero probability because b yields strictly greater payoff against any (potentially degenerate) mix of w and x. Then P1 does not mix a and b. By reductio, there is no NE where P1 mixes a and b (only)

Suppose for reductio that P1 mixes b and c, then P2 plays w with zero probability because x yields ~~greater payoff against P1's so~~ strictly greater payoff against any mix of b and c, then P1 plays c with zero probability because b yields strictly greater payoff against any ~~mix of~~ (potentially degenerate) mix of x and y. Then P1 does not mix b and c. By reductio, P1 does not mix b and c at NE.

Suppose P1 mixes a and c at NE with probability p_a, $1-p_a$. ~~Let the~~ Let q_w, q_x, $1-q_w-q_x$ be the probabilities with which P2 plays w, x, and y respectively at this NE. At NE, P1 has no profitable deviation, hence

$\pi_1(a, \sigma_2^*) = \pi_1(c, \sigma_2^*) \geq \pi_1(b, \sigma_2^*) \Leftrightarrow$
$2q_x + 7(1-q_w-q_x) = 7q_w + 2q_x \geq 5q_w + 3q_x + 5(1-q_w-q_x) \Leftrightarrow$
$7 - 7q_w - 7q_x = 7q_w, \quad 7q_w + 2q_x \geq 5 - 2q_x \Leftrightarrow$
$\cancel{q_w = 1/3 \quad 1-q_x = 2q_w}$
$q_x = 1-2q_w, \quad 7q_w + 4q_x \geq 5 \Leftrightarrow$
$q_x = 1-2q_w, \quad -q_w \geq 1 \Rightarrow$
$q_w \leq -1$

Given that $q_w \geq 0$ at any NE, by reductio, there is no NE such that P1 mixes a and c.

~~E~~ Then, there is no NE such that P1 mixes exactly two pure actions.

f Given that only rationalisable strategies are played at NE, the required NE, if it exists, is such that P1 mixes a, b, c.

Suppose such NE exists, denote it σ^*. Let q_w, q_x, $1-q_w-q_x$ be the probabilities σ_2^* assigns to w, x, y respectively. At NE, P1 has no profitable deviation. Then,

$\pi_1(a, \sigma_2^*) = \pi_1(b, \sigma_2^*) = \pi_1(c, \sigma_2^*) \Leftrightarrow$
$2q_x + 7(1-q_w-q_x) = 5q_w + 3q_x + 5(1-q_w-q_x) = 7q_w + 2q_x \Leftrightarrow$
$7 - 7q_w - 5q_x = 5 - 2q_x = 7q_w + 2q_x \Leftrightarrow$
$2 - 7q_w - 3q_x = 0, \quad 5 - 7q_w - 4q_x = 0, \quad 7 - 14q_w - 7q_x = 0 \Rightarrow$
$2 = 5 - q_x \Rightarrow q_x = 3$

Given that $q_x \leq 1$ at any NE, by reductio, there is no NE such that P1 mixes exactly three pure actions.

g ~~The unique~~ There is no NE where P1 mixes two pure actions, nor where P1 mixes three pure actions, and P1 does not play unrationalisable d, so at any NE, P1 plays a pure strategy. By inspection, P2 has strict ~~best resp~~ unique best responses to each of P1's pure strategies, so given that P1 plays a pure strategy at NE, so does P2, so no hybrid NE exists. The only pure NE is (b, x), this is the unique NE.

3a consider arbitrary firm i

$\pi_i = (p-c)q_i = (a - \sum_{j=1}^{n} q_j - c)q_i$

$= (a - c - Q_{-i} - q_i)q_i$

where ~~q~~ $Q_{-i} = \sum_{j \neq i} q_j$

$\max_{q_i} \pi_i(q_i)$

FOC: $(a-c-Q_{-i}-q_i) - q_i = 0 \Rightarrow q_i = a - \cancel{Q} - c - Q_{-i}/2$

SOC: $-2 < 0$

Best response function $BR_i(Q_{-i}) = a-c-Q_{-i}/2$

Consider arbitrary firms i and j.

Suppose for reductio that at NE $q_i^* \neq q_j^*$.

~~Suppose without loss of generality that $q_i^* > q_j^*$.~~

By definition of NE, at NE, each firm chooses its best response against the profile of other firms' strategies.

$q_i^* = BR_i(Q_{-i}^*) = BR_i(Q_{-ij}^* + q_j^*) = a-c/2 - Q_{-ij}^*/2 - q_j^*/2$

$q_j^* = \dots = a-c/2 - Q_{-ij}^*/2 - q_i^*/2$

Let $k = a-c/2 - Q_{-ij}^*/2$

$q_i^* = k - q_j^*/2, \quad q_j^* = k - q_i^*/2 \Rightarrow$

~~$2k$ ... k ...~~

$q_i^* = k - \frac{1}{2}(k - q_i^*/2) = k/2 + q_i^*/4 \Rightarrow q_i^* = 3k/2$,

$q_j^* = k - \frac{1}{2}(k - q_j^*/2) = k/2 + q_j^*/4 \Rightarrow q_j^* = 3k/2 \Rightarrow$

$q_i^* = q_j^*$.

By ~~gen~~ reductio, by generalisation, at NE, for all firms i, j, $q_i^* = q_j^*$, i.e. the NE is symmetric.

~~By symmetry,~~

Consider arbitrary firm i,

By definition of NE, $q_i^* = BR_i(Q_{-i}^*)$

By symmetry, ~~$q_i^* = BR_i((n-1)q_i^*)$~~

$q_i^* = BR_i((n-1)q_i^*)$

$= a-c/2 - (n-1)q_i^*/2$

$\Rightarrow q_i^*(1 + n-1/2) = a-c/2$

$\Rightarrow q_i^* = a-c/2 \,/\, n+1/2 = a-c/n+1$

By generalisation, at NE, for all firms i, $q_i^* = a-c/n+1$

The unique pure NE is such that each firm i plays $q_i = a-c/n+1$

$p^* = a - Q^* = a - nq^* = a - n/n+1(a-c) = \cancel{\ldots}$

$= 1/n+1\, a + n/n+1\, c$

$\pi^* = (p^*-c)q^* = (1/n+1\, a - 1/n+1\, c)\, 1/n+1\,(a-c)$

$= (1/n+1)^2 (a-c)^2$

As n approaches ∞, equilibrium price p^* converges to marginal cost c, equilibrium profit of each firm converges to 0. The ~~ot~~ Cournot oligopoly outcome converges to the perfectly competitive outcome as n approaches ∞.

b $N=2$, $a=12$, $c=0$, $q_i \in \{2,3,4,6\}$

From the above, $\pi_i = (a-c-q_{-i}-q_i)q_i \cancel{=} = (12-q_{-i}-q_i)q_i$

	2	3	4	6
2	16 / 16	21 / 14	$\underline{24}$ / 12	$\underline{24}$ / 8
3	14 / 21	18 / 18	$\underline{20}$ / 15	18 / $\underline{9}$
4	12 / $\underline{24}$	15 / $\underline{20}$	$\underline{16}$ / $\underline{16}$	12 / 8
6	8 / $\underline{24}$	$\underline{9}$ / 18	8 / 12	0 / 0

Best responses underlined

Iterated strict dominance

$3 \succ_1 2$ and $3 \succ_2 2$, then

$4 \succ_1 6$ and $4 \succ_2 6$, then

$4 \succ_1 3$ and $4 \succ_2 3$

Only ~~(3~~ (4,4) survives iterated strict dominance

From (a), $q^* = a-c/n+1 = 12-0/2+1 = 4$. The NE is such that each player plays $q^*=4$. Only this NE survives iterated strict dominance.

There is no mixed NE because for each player, only one strategy survives iterated strict dominance (and is rationalisable).

At NE, no player plays 2 with non-zero probability because 3 strictly dominates 2, and yields strictly higher payoff against any strategy of the other player. Then, no player plays 6 with non-zero probability because given that 2 is not played, 4 yields strictly higher payoff than 6 against all strategies. Then no player plays 3 with non-zero probability for similar reasons. Players only play 4 with non-zero probability, so play 4 with certainty.

All and only rationalisable strategies survive iterated strict dominance. A strategy is rationalisable iff it is a best response to some (potentially correlated) mix of other players' strategies. Equivalently, this is iff it survives iterated elimination of non-best responses.

Pearce's Lemma: a strategy is a best response to some pure or mixed strategy iff it is not strictly dominated by some pure or mixed strategy.

By ~~pearf~~ Pearce's Lemma, iterated strict dominance is equivalent to iterated elimination of non-best responses. Then all and only strategies that survive iterated strict dominance are rationalisable. Only ~~the strategies~~ 4 is rationalisable for each player because only 4 survives iterated strict dominance.

c

	2	6
2	16, 16	<u>24</u>, <u>8</u>
6	<u>8</u>, <u>24</u>	0, 0

~~By inspection, the~~ Best responses underlined.

By inspection, there are two pure NE, where players play mutual pure best responses, (2,6) and (6,2).

Suppose that at NE σ^*, P1 mixes, then by definition of NE, P1 has no profitable deviation, and so is indifferent between 2 and 6, i.e.

$\pi_1(2, \sigma_2^*) = \pi_1(6, \sigma_2^*) \Leftrightarrow$

$16q + 8(1-q) = 24q \Leftrightarrow$

$q = \cancel{12}, \frac{1}{2}$

where q is the probability σ_2^* assigns to 2. Then P2 mixes, and by a similar argument, $p = \frac{1}{2}$, where p is the probability ~~P1~~ σ_1^* assigns to 2.

If one player mixes, so does the other, there are no hybrid NE. ~~The~~ if ~~a~~ both players mix at NE, they mix with the above probabilities. the unique mixed NE is $(\frac{1}{2} 2 + \frac{1}{2} 6, \frac{1}{2} 2 + \frac{1}{2} 6)$.

~~Rationality does not select a NE from~~

Under the mixed NE, each player has payoff $\frac{1}{4} 16 + \frac{1}{4} 8 + \frac{1}{4} 24 + \frac{1}{4} 0 = 12$

Rationality does not select a NE because ~~each~~ ~~At~~ at each NE, no player has a strictly profitable deviation, so it is rational to play one's part in each NE supposing the other player does the same. Then, the outcome depends on each player's beliefs and is not determined by rationality.

d ~~If P1 plays~~ Solve by backward induction.

If P1 plays 2, P2 plays 6 then P1 has payoff 8. If P1 plays 6, P2 plays 2 then P1 has payoff 24 so P1 plays 6, then P2 plays 2 (and if P1 plays 2 then P2 plays 6) at the SPE. The SPE is unique.

There is a first-mover advantage in Stackelberg competition because quantities are strategic substitutes, choosing high quantity decreases the rival's marginal profit from quantity, hence the rival chooses lower quantity, which benefits the first-mover. The first-mover is able to exploit this strategic effect because it is able to commit to a quantity.

4a Solve by backward induction.

Let x^Y_Z denote the amount Y proposes to distribute to Z

In the D voting stage, for all $x^D_E \leq 100$, E votes A (against) because for all $x^D_E \leq 99$, E has more coins if the proposal fails and for $x^D_E = 100$, E eliminates D if the proposal fails, both increase E's payoff. Then D has strict incentive to vote F (for), and any proposal passes.

In the D proposal stage, given that any proposal passes, D proposes $(x^D_D, x^D_E) = (100, 0)$.

In the C stage, E votes F iff $x^C_E \geq 1$, D votes F iff $x^C_D > 100$, and C always votes F. Then C optimally proposes $(x^C_C, x^C_D, x^C_E) = (99, 0, 1)$ and this passes.

In the B stage, E votes F iff $x^B_E \geq 2$, D votes F iff $x^B_D \geq 1$, C votes F iff $x^B_C \geq 100$, then B optimally proposes $(x^B_B, x^B_C, x^B_D, x^B_E) = (99, 0, 1, 0)$ and this passes.

In the A stage, E votes F iff $x^A_E \geq 1$, D votes F iff $x^A_D \geq 2$, C votes F iff $x^A_C \geq 1$, B votes F iff $x^A_B \geq 100$, then A optimally offers $(x^A_A, x^A_B, x^A_C, x^A_D, x^A_E) = (98, 0, 1, 0, 1)$.

The SPE is fully described in the above, for clarity, the strategy of each player is as follows.

A: ~~offer~~ propose $(98, 0, 1, 0, 1)$, vote F

B: vote F iff $x^A_B \geq 100$, then offer $(99, 0, 1, 0)$, then vote F

C: vote F iff $x^A_C \geq 1$, vote F iff $x^B_C \geq 100$, then offer $(99, 0, 1)$, then vote F

D: vote F iff $x^A_D \geq 2$, vote F iff $x^B_D \geq 1$, vote F iff $x^C_D \geq 100$, then offer $(100, 0)$, then vote F.

E: vote F iff $x^A_E \geq 1$, $x^B_E \geq 2$, $x^C_E \geq 1$, $x^D_E > 100$

b The "elimination payoff" ~~allows as to~~ serves as a tiebreaker in the event that a player receives as many coins from ~~continuation play~~ subgame perfect continuation play as from some proposal. In that event, the elimination payoff makes voting ~~against~~ A strictly optimal.

With no such tiebreaker, there are complicated cases of indifference and potentially a multiplicity of SPE.

~~In the D voting stage, E votes F iff $x^D_E = 100$, A otherwise. Then D votes And D votes~~

Subgame NE in the D stage is unchanged except in that off the equilibrium path, where $x^D_E = 100$, E votes F and D is indifferent.

In the C stage, E votes F iff $x^C_E \geq 0$, D votes F iff $x^C_D \geq 100$, C optimally offers $(100, 0, 0)$, and this passes.

In the B stage, E votes F iff $x^B_E \geq 0$, D votes F iff $x^B_D \geq 0$, C votes F iff $x^B_C \geq 100$, B optimally offers $(100, 0, 0, 0)$

Similarly in the A stage, the optimal offer is $(100, 0, 0, 0, 0)$ and it passes.

The SPE is such that each player proposes to keep all the coins, in each voting stage, the subsequent proposer votes A iff not offered 100 and every other player votes F.
~~Neglecting cases~~

The proposing player need not "bribe" voters with coins because the proposing player's survival is sufficient incentive for all but the next proposing player to accept any proposal.

c In the final stage, the penultimate ~~final~~ player proposes $(G, 0)$, this player votes F, the final player votes A and the proposal passes.

In the penultimate stage, the third last player bribes the penultimate player with 1, proposes $(G-1, 0, 1)$, and the proposal passes.

~~Generalising, the first player offers $(G - (N-1)/2, 0, 1, \dots)$, each odd player votes F iff offered at least one, each even player votes F iff offered at the second player votes F iff offered~~

Generalising, for odd N, P1 offers $(G - (N-1)/2, 0, 1, \dots 0, 1)$, votes F ~~and the~~ P2 votes F iff offered ~~weakly less~~ more than $G - (N-1)/2 + 1$, each odd player votes F iff offered at least 1, each even player votes F iff offered at least 2.

For even N, P1 offers $(G - (N-2)/2, 0, 1, \dots, 0)$, votes F. P2 votes F iff offered more than ~~G~~ $G - \frac{N-2}{2}$, each odd ~~pte~~ player votes F iff offered at least 1, each even player votes F iff offered at least 2.

d $G = 100$, $N =$ ~~2N+2~~ 202

P1 offers $(0, 0, 1, \dots, \cancel{0, 1}\ 0)$

P1 votes ~~for~~ F

each odd player votes F iff offered at least 1

~~each~~ P2 votes F iff offered at least 1

each other even player votes F iff offered at least 2

$G = 100$, $N > 202$

P1 does not survive.

5a Given some ~~set~~ bargaining problem (U, d), where U is the set of possible agreement payoff vectors and d is the disagreement payoff vector, the Nash bargaining solution is

$u^* = \arg\max_{u \in U} \prod_i (u_i - d_i)$

where i indexes players or bargaining parties.

The Nash bargaining solution uniquely satisfies the four Nash bargaining axioms, which are weak Pareto principle, symmetry, independence of irrelevant alternatives, and invariance to equivalent payoff representations.

Let $F(U, d)$ denote some bargaining solution.

F satisfies the weak Pareto principle iff $F(U, d) = u$ and $\nexists u' \in U : \forall i : u'_i > u_i$, i.e. there is no alternative solution u' that strictly Pareto-dominates u (under which every player i is strictly better off).

F satisfies symmetry iff for all U, d such that for all $(u_1, u_2) \in U$, $(u_2, u_1) \in U$ and $d_1 = d_2$, $F(U, d) = u$ is such that $u_1 = u_2$.

F satisfies independence of irrelevant alternatives iff for all U, U' such that $U' \subseteq U$, $F(U, d) = u \in U'$, then $F(U', d) = u$.

F satisfies invariance to equivalent payoff representations iff for all ~~$f_i(u_i) = d_i + \beta_i u_i$~~ $f_i(u_i) = \alpha_i + \beta_i u_i$, $d' = (f_1(d_1), \dots, f_n(d_n))$, $U' = \{(f_1(u_1), \dots, f_n(u_n)) : (u_1, \dots, u_n) \in U'\}$, $F(U, d) = u$ $F(U', d') = (f_1(u_1), \dots, f_n(u_n))$.

Invariance to equivalent payoff representations is reasonable because a payoff function is a representation of some preferences and its cardinality is not generally meaningful. Symmetry and the weak Pareto principle are intuitively plausible. Independence of irrelevant alternatives is the most questionable. Intuitively, the alternatives available but not selected have some bearing on each party's bargaining position.

b The set of agreement payoff vectors (supposing costless disposal), given e, is

$U(e) = \{(u_1, u_2) : u_1 + u_2 \leq R(e), u_1, u_2 \geq 0\}$
$d = (0, 0)$

By the weak Pareto principle, the Nash bargaining solution lies on the bargaining frontier, where $u_1 + u_2 \leq R(e)$ binds.

$\max_{u_1, u_2} (u_1 - d_1)(u_2 - d_2)$ s.t.
$u_1 + u_2 \leq R(e)$, $u_1, u_2 \geq 0$

This reduces to

$\max_{u_1, u_2} u_1 u_2$ s.t.
$u_1 + u_2 = R(e)$, $u_1, u_2 \geq 0$

Along the bargaining frontier, $u_2 = R(e) - u_1$

$\max_{u_1} u_1(R(e) - u_1)$

FOC: $R(e) - 2u_1 = 0 \Rightarrow u_1 = R(e)/2 \Rightarrow u_2 = R(e)/2$
SOC: $-2 < 0$

By inspection, the positivity constraints are satisfied.

The Nash bargaining solution is $u_1 = u_2 = R(e)/2$, hence $T = R(e)/2$.

c Joint profit maximisation consists in maximising $R(e) - e$.

FOC: $R'(e) - 1 = 0 \Rightarrow R'(e) = 1$
SOC: $R''(e) < 0$

Supposing that the supplier receives $T = R(e)/2$, profit maximisation for the supplier consists in maximising $T - e = R(e)/2 - e$.

FOC: $\frac{1}{2}R'(e) - 1 = 0 \Rightarrow R'(e) = 2$
SOC: $\frac{1}{2}R''(e) < 0$

Given that $R''(e) < 0$, the supplier chooses higher e when maximising joint profit than when maximising individual profit. This is because, when the transfer in the second stage is determined by Nash bargaining, some of the increase in total profit due to an increase in e is retained by the firm, so the supplier does not enjoy the full effect of an increase in e on profit, hence has less incentive to choose high e than if it were to maximise total profit.

di If the firm could make a take it or leave it offer, it has all the bargaining power and offers the supplier a transfer of zero ~~(or arbitrarily above zero)~~ such that the supplier is ~~just~~ indifferent, and the firm captures all the surplus. Then, given that in the second stage, the supplier always receives a transfer of zero, the supplier finds it optimal to produce zero units.

Note that the firm does not offer to cover the supplier's costs because these costs are sunk and the supplier has ~~zero surp.~~ ~~total payoff~~ net payoff of $-e$ if it does not accept the offer, so the firm can offer $T=0$ such that the supplier has payoff $-e$ even if it does accept the offer and is indiffent.

If the supplier can make a take it or leave it offer, the supplier has all the barganig power and offers the firm e units ~~at~~ for transfer $T = R(e)$, such that the firm has zero payoff if it accepts and is indifferent. Then the supplier's net payoff is $R(e)-e$, equal to joint profit, and the supplier chooses e to maximise joint profit. (which it captures entirely).

~~The situation is not symmetrical in the sense~~

The party that can ~~one~~ offer an ultimatum has all the bargaining power and captures all ~~the~~ the profit. If this is the supplier, it has incentive to maximise joint profit. If this is the firm, the supplier ~~the~~ is not paid for its components so produces nothing. The situation is asymmetrical in the sense that the supplier has zero profit without bargaing power ~~but~~ and positive profit with bargaing power but the firm has zero profit at either extreme. The firm has positive profit only if bargaing power is shared. This is somewhat counterintuitive. ~~The explanation is presumably that the firm requires the "cooperation" of the~~

e Take e as given.

~~At equilibrium, the firm is indifferent between~~

Suppose that the equilibrium is stationary, ~~then in each period,~~ ~~each~~ then each player makes the same offer when it is his turn. At equilibrium each player is indifferent between accepting and rejecting. If some player strictly prefers to accept, then the offering player has a strictly profitable deviation to a marginally less generous offer. If some player strictly prefers to reject, ~~then that player does not accept any offer in the stationary equilibrium, and the outcome is either breakdown or~~ ~~so the rejecting player expects greater payoff if he offers~~ then the offering player has a strictly profitable deviation to an offer that is just acceptable ~~(because the other player~~.

Let T_s denote the transfer demanded by the supplier when the supplier makes the offer and T_f denote the transfer offered by the firm when the firm makes the offer.

$$T_f = (1-\alpha) T_s$$
$$R(e) - T_s = (1-\alpha) (R(e) - T_f)$$

By substitution,

$$R(e) - T_s = (1-\alpha)(R(e) - (1-\alpha) T_s)$$
$$= (1-\alpha) R(e) - (1-\alpha)^2 T_s \Leftrightarrow$$
$$\alpha R(e) = T_s - (1 - 2\alpha + \alpha^2) T_s \Leftrightarrow$$
$$\alpha R(e) = (2\alpha - \alpha^2) T_s \Leftrightarrow$$
$$R(e) = (2-\alpha) T_s \Leftrightarrow$$
$$T_s = R(e) / 2-\alpha \Leftrightarrow$$
$$T_f = R(e) \, {}^{1-\alpha}/_{2-\alpha}$$

The firm makes the first offer, $T_f = {}^{1-\alpha}/_{2-\alpha} R(e)$, and this is accepted.

As $\alpha \to 0$, the outcome of this game converges to the Nash bargaining solution. This result holds more generally for infinitely repeated offer-counteroffer games with breakdown.

6a In a symmetric two player game, the strategy α^* is an ESS iff (α^*, α^*) is a NE, and there is no other strategy $\alpha' \neq \alpha^*$ such that α' is a best response against α^* and $u(\alpha', \alpha') \geq u(\alpha^*, \alpha')$, i.e. α' fares better against itself than α^* fares against α'.

This definition captures the idea of resistance to invasion by mutants and is motivated by the phenomenon of evolution in biology. A population of α^* players is evolutionarily stable if no α^* player faces evolutionarily pressure to deviate (which is iff α^* is a best response to α^*) and no mutant α' player could do as well against the α^* population and better than α^* against itself (so no mutant α' players can fare better than α^* on average).

	L	R
L	-1, -1	$\underline{0}$, $\underline{2}$
R	$\underline{2}$, 0	1, 1

Best responses underlined.

By inspection, there are two pure NE, (L,R) and (R,L). Neither is symmetrical, hence neither is a candidate ESS.

Suppose there exists NE σ^* such that P1 mixes, then by definition of NE, P1 has no profitable deviation, then P1 is indifferent between L and R.

$\pi_1(L, \sigma_2^*) = \cancel{\pi_1(R, \sigma_2^*)}\ \pi_1(R, \sigma_2^*) \Leftrightarrow$
$-q + 2(1-q) = 0 + (1-q) \Leftrightarrow$
$1 - 2q = 0 \Leftrightarrow$
$q = \frac{1}{2}$,
where q is the probability σ_2^* assigns to L.

Then P2 mixes, P2 has no profitable deviation, P2 is indifferent.

$\pi_2(L, \sigma_1^*) = \pi_2(R, \sigma_1^*) \Leftrightarrow -p + 2(1-p) = 0 + (1-p) \Leftrightarrow p = \frac{1}{2}$
where p is the probability σ_1^* assigns to L.

There is a mixed NE $(\frac{1}{2}L + \frac{1}{2}R, \frac{1}{2}L + \frac{1}{2}R)$, it is symmetric and corresponds to a candidate ESS.

By the above argument, each of L and R is a best response against $\frac{1}{2}L + \frac{1}{2}R$, then any mix of L and R is also a best response. $\frac{1}{2}L + \frac{1}{2}R$ is an ESS ~~only if~~ iff for all $\sigma' \neq \sigma^*$, $\pi(\sigma^*, \sigma') > \pi(\sigma', \sigma')$. Let p denote the probability σ' assigns to L.

$$\pi(\sigma', \sigma') = p^2(-1) + p(1-p)(0) + p(1-p)(2) + (1-p)^2(1)$$
$$= -p^2 + (2p - 2p^2) + (1 - 2p + p^2)$$
$$= -2p^2 + 1$$
$$\pi(\sigma^*, \sigma') = \tfrac{1}{2}(-p + 2(1-p)) + \tfrac{1}{2}(0p + 1(1-p))$$
$$= \tfrac{1}{2}(2 - 3p) + \tfrac{1}{2}(1-p)$$
$$= \tfrac{3}{2} - 2p$$
$$\pi(\sigma', \sigma') \geq \pi(\sigma^*, \sigma') \Leftrightarrow$$
$$1 - 2p^2 \geq \tfrac{3}{2} - 2p \Leftrightarrow$$
$$-2p^2 + 2p - \tfrac{1}{2} \geq 0 \Leftrightarrow$$
$$4p^2 - 4p + 1 \leq 0 \Leftrightarrow$$
$$p = \tfrac{1}{2} \Leftrightarrow$$
$$\sigma' = \sigma^*$$

~~$\nexists\, \sigma' \neq \sigma^*$~~: $\frac{1}{2}L + \frac{1}{2}R$ is an ESS.

b Under the replicator dynamic, the number of players in a population playing some strategy grows from one period to the next by an amount that is directly proportionate to the number of such players in the former period and the average payoff of such players in the former period.

Then the proportion of players playing some strategies grows from one period to the next iff these players fare better than the average player against the population. This proportion grows at a rate proportionate to the difference between these player's average payoff and the average payoff in the population.

The reverse is true for strategies that fare worse than average. These proportions shrink at a rate proportionate to how ~~poorly~~ much worse than avge they fare.

Let $A = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$ and $p = \begin{pmatrix} p_L \\ p_R \end{pmatrix}$. A is the matrix of Row payoffs and p is the vector ~~of proport~~ denoting the proportion of players of each strategy.

Replicator Equation
$\dot{p}_x = p_x [\, [Ap]_x - p^T A p \,]$
where $[Ap]_x$ is the average payoff of x players and $p^T A p$ is the population average payoff. $x \in \{L, R\}$.

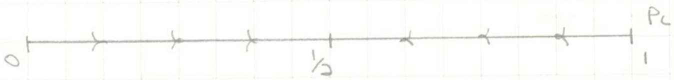
In a game with only two strategies.
$\dot{p}_x = p_x (1 - p_x)(\, [Ap]_x - [Ap]_{\neg x})$

By substitution,
$\dot{p}_L = p_L(1 - p_L)(\, (-p_L + 2p_R) - p_R)$
$= p_L(1 - p_L)(2p_R - 2p_L)$

$= p_c(1-p_c)(2-4p_c)$

$\dot{p}_L = 0 \iff p_c = 0, 1, 1/2$

There are three absorbing states, $p_c=0$, $p_c=1$, $p_c=1/2$. For $p_c \in (0, 1/2)$, $\dot{p}_L > 0$, for $p_c \in (1/2, 1)$, $\dot{p}_L < 0$, all $p_c \in (0, 1/2)$ and $p_c \in (1/2, 1)$ converge to $p_c = 1/2$, which coincides with the ESS. The basin of attraction of $p_c = 1/2$ is $(0,1)$.



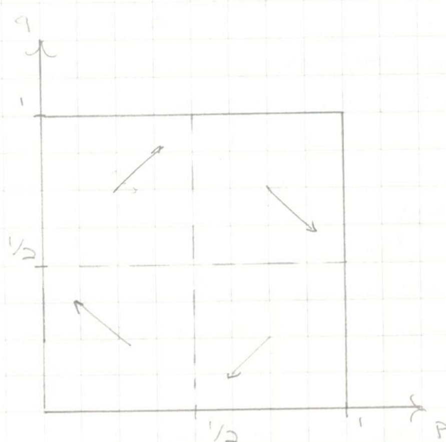
c	h	t
H	$\underline{1}$, 0	0, $\underline{1}$
T	0, $\underline{1}$	$\underline{1}$, 0

$\pi_R(H, \sigma_C) = q$, $\pi_R(T, \sigma_C) = 1-q$
where σ_C denotes the "strategy" of the column population $\sigma_C = (qh + (1-q)t)$.

~~Given~~ Replicator equation for R (Row)
$\dot{p} = p(1-p)[\pi_R(H, \sigma_C) - \pi_R(T, \sigma_C)]$
$= p(1-p)(2q-1)$

$\pi_C(h, \sigma_R) = 1-p$, $\pi_C(t, \sigma_R) = p$
Replicator equation for C (Column)
$\dot{q} = q(1-q)[\pi_C(h, \sigma_R) - \pi_C(t, \sigma_R)]$
$= q(1-q)(1-2p)$

For $p \in (0,1)$, $p(1-p) > 0$, then $\text{sign}(\dot{p}) = \text{sign}(2q-1)$. $q \gtreqless 1/2 \iff \dot{p} \gtreqless 0$. For $q \in (0,1)$, $q(1-p) > 0$, then $\text{sign}(\dot{q}) = \text{sign}(1-2p)$. $p \gtreqless 1/2 \iff \dot{q} \lesseqgtr 0$.



~~$\partial/\partial t \; p(1-p)q(1-q) =$~~

Let $x = \ln p + \ln(1-p) + \ln q + \ln(1-q)$
$\partial/\partial t \, x = \frac{1}{p}\partial p/\partial t + \frac{1}{1-p}\partial(1-p)/\partial t + \frac{1}{q}\partial q/\partial t + \frac{1}{1-q}\partial(1-q)/\partial t$
$= \dot{p}/p + -\dot{p}/(1-p) + \dot{q}/q + -\dot{q}/(1-q)$
$= (1-p)\dot{p} - p\dot{p}/p(1-p) + (1-q)\dot{q} - q\dot{q}/q(1-q)$
$= (1-2p)\dot{p}/p(1-p) + (1-2q)\dot{q}/q(1-q)$
$= (1-2p)(2q-1) + (1-2q)(1-2p)$
$= (1-2p)(2q-1) - (1-2p)(2q-1)$
$= 0$
$\Rightarrow \partial/\partial t \, e^x = \partial/\partial t \; p(1-p)q(1-q) = 0$

At NE of Matching Pennies, $p = q = 1/2$, $p(1-p)q(1-q) = 1/16$. Given that under the replicator dynamic, $p(1-p)q(1-q)$ is time-invariant and does not converge to $1/16$, the state ~~of the game~~ does not converge to NE.

e A plausible candidate ESS is (σ^*, τ^*) where $\sigma^* = (1/2 H + 1/2 T)$ and $\tau^* = (1/2 h + 1/2 t)$. The corresponding strategy profile is a symmetric NE. (It is symmetric by construction). The type R (Row) player has no (strictly) profitable deviation because each pure action yields payoff 1/2. Similarly for the type C player.

Any strategy is a best response to (σ^*, τ^*). ~~Given that Matching Pennies is zero sum with total payoff 1, any player playing any in the symmetric game where players Bayesian game where types are uniformly randomly assigned has ex ante expected payoff 1/2~~

Given that Matching Pennies is zero sum with total payoff 1, in the symmetric Bayesian game where players are randomly assigned with equal probability to R or C, any strategy against itself yields ex ante expected payoff 1/2, ~~which is equal to~~.
which is equal to $\pi((\sigma^*, \tau^*), (\sigma^*, \tau^*))$
~~Then any strategy forms a symmetric NE but is vulnerable to~~

So (σ^*, τ^*) is vulnerable to invasion by any mutant strategy.

~~No other strategy (σ', τ') is a forms a symmetric NE. If σ' assigns greater weight to~~

Any other strategy (σ', τ') forms a symmetric NE only if σ', τ' is a NE in the regular Matching Pennies game. Otherwise, σ'', τ'' such that σ'' best responds to τ' and τ'' best responds to σ' is a strictly profitable deviation. So only (σ^*, τ^*) is a candidate ESS, and it fails, so no ESS exists.

7a

	L	C	R
T	10, 10	$\underline{12}$, $\underline{6}$	8, $\underline{6}$
B	$\underline{6}$, $\underline{12}$	0, 4	4, 4

Best responses underlined.

By inspection, the only pure NE where players play mutual pure best responses ~~is~~ are (B,L), (T,C).

By inspection, each player has a strict pure best response against each pure strategy of the other player, so there is no hybrid NE.

$L \succ_2 R$, so P2 (column) plays R with zero probability at every NE.

Suppose P1 mixes at NE, then P1 has no profitable deviation, and is indifferent between T and B.
$\pi_1(T,\sigma_2^*) = \pi_1(B,\sigma_2^*) \Leftrightarrow$
$10q + 6(1-q) = 12q + 4(1-q) \Leftrightarrow$
$2q = 2(1-q) \Leftrightarrow q = 1-q \Leftrightarrow q = 1/2$
where σ_2^* is P2's strategy at this NE, and q is the probability σ_2^* assigns to L.
Then P2 mixes L and C and is indifferent.
$\pi_2(L,\sigma_1^*) = \pi_2(C,\sigma_1^*) \Leftrightarrow$
$10p + 6(1-p) = 12p \Leftrightarrow$
$6(1-p) = 2p \Leftrightarrow 3-3p = p \Leftrightarrow 4p = 3 \Leftrightarrow p = 3/4$
where σ_1^* is P1's strategy at this NE and p is the probability σ_1^* assigns to T.

The unique mixed NE is $(\frac{3}{4}T + \frac{1}{4}B, \frac{1}{2}L + \frac{1}{2}C)$

b P2's payoff from any pure action, hence also from any potentially mixed strategy is strictly greater if P1 plays T. Then P1 minmaxes P2 by playing B. Then, P2 optimally responds by playing L, realising minmax payoff $\underline{v}_2 = 6$.

P1's payoff from any pure action, hence also from any potentially mixed strategy is strictly greater if P2 plays L and equal if P2 plays either C or R, so P2 minmaxes P1 by playing either C or R (or some mix). Then P1 optimally responds by playing T realising minmax payoff $\underline{v}_1 = 6$.

P1 guarantees payoff no less than $\underline{v}_1 = 6$ by playing T, P2 guarantees u_1 no more than $\underline{v}_1 = 6$ by playing C and/or R. Similarly, P2 guarantees payoff no less than $\underline{v}_2 = 6$ by playing L, and P1 holds P2 with no more than $\underline{v}_2 = 6$ by playing B.

c The maximum ~~attainable~~ total payoff in the stage game is 20, at (T,L).

The mixed NE in (a) yields $\pi_1 = \pi_1(T,\sigma_2^*) = \frac{1}{2}10 + \frac{1}{2}6 = 8$ and $\pi_2 = \pi_2(L,\sigma_1^*) = \frac{3}{4}(10) + \frac{1}{4}(6) = 9$.

In the final stage, at SPE, a stage game Nash is always played. The maximum total NE payoff is 18, achieved by either pure NE.

So ~~the~~ one "theoretical" maximum total payoff is achieved by playing (T,L) in every stage except the last, and one of (T,C) and (B,L) in the last stage. This is not a SPE because, supposing (T,C) is played, P1 has profitable deviation to B in the penultimate stage then T in the final stage. P1 has no incentive to "cooperate".

The second best candidate SPE is such that on the equilibrium path, (T,L) is played in all but the final period. And in the final period, the mixed stage game NE is played. The SPE is such that ~~P1 plays: T in each period if every period except the final period, unless there has been a prior deviation,~~ then players play their part of (T,L) in every period except the final period iff there has been no prior deviation, ~~and their part~~ the mixed stage game NE in the final period iff there has been no prior deviation, and the unfavourable pure stage game NE for the first deviating player if there has been a prior deviation.

This is in fact a NE because no player has a profitable one shot deviation, which yields at most 12 instead of 10 in one period, but 6 instead of 8 or 9 in each subsequent period.

The total payoff is $20(T-1) + 17$.

d The required SPE is as follows

Play T, L in the first period and every subsequent period iff there has been no prior deviation.

If P1 deviates, play T, R for one period then T, C in every subsequent period.

If P2 deviates, play B, L in every subsequent period.

Verify that this is a SPE.

In the indefinite ~~stage~~ subphase of each punishment phase, players play the stage game, so have no profitable one-shot deviation.

In the cooperation phase, one shot deviation yields 12 then 6 indefinitely, with PV $12 + 6\delta/1-\delta$ while eqm play yields 10 indefinitely with PV $10/1-\delta = 10 + 10\delta/1-\delta$
$10 + 10\delta/1-\delta \geq 12 + 6\delta/1-\delta \Leftrightarrow 4\delta/1-\delta \geq 2 \Leftrightarrow$
$\delta/1-\delta \geq 1/2 \Leftrightarrow \delta \geq 1-\delta/2 \Leftrightarrow 2\delta \geq 1-\delta \Leftrightarrow \delta \geq 1/3$
so there is no profitable one shot deviation.

In the T, R ~~pha~~ subphase of the punishment of P1, one-shot deviation and eqm play yield 6 indefinitely for P1. One shot deviation yields 12 then 6 indefinitely for P2. eqm play yields 8 then 12 indefinitely. Compare PVs. $12 + 6\delta/1-\delta \leq 8 + 12\delta/1-\delta \Leftrightarrow 4 \leq 6\delta/1-\delta \Leftrightarrow \delta/1-\delta \geq 2/3 \Leftrightarrow$
$3\delta \geq 2-2\delta \Leftrightarrow \delta \geq 2/5$

In the first period and every subsequent period, ~~if the~~ if there has been no previous deviation, play (T,L). If any player deviates, play (B,R) for one period, then return to (T,L). Failing to play (B,R) after a deviation, itself counts as a deviation.

Verify that this is a SPE.

In the cooperation phase, for either player, one shot deviation (at best) yields 12 then 4 then 10 indefinitely. Eqm play yields 10 indefinitely. Compare PVs. $12 + 4\delta + 10\delta^2/1-\delta \leq 10 + 10\delta + 10\delta^2/1-\delta$
$\Leftrightarrow 2 \leq 6\delta \Leftrightarrow \delta \geq 1/3$. Then there is no profitable one-shot deviation in this phase.

In the punishment phase (any period immediately following a deviation), for ~~each~~ either player, one shot deviation (at best) yields 6 then 4 then 10 indefinitely. Eqm play yields 4 then 10 indefinitely. Compare PVs. $6 + 4\delta + 10\delta^2/1-\delta \leq 4 + 10\delta + 10\delta^2/1-\delta \Leftrightarrow 2 \leq 6\delta \Leftrightarrow \delta \geq 1/3$. Then there is no profitable one-shot deviation in this phase.

This is in fact a SPE.

This SPE is not weakly renegotiation ~~to~~ proof because continuation from (T,L) in the cooperation phase is strictly preferred by each player to (B,R)

e The required SPE is the following.

~~In each period, play~~ In the first period and in every subsequent period, if there has been no ~~pre~~ deviation, play (T,L), otherwise, if P1 was the first to deviate, play (T,C) and if P2 was the first to deviate, play (B,L). (And if the first deviation was by both P1 and P2, play (T,L).).

Verify that this is a SPE.

In the cooperation phase, optimal one shot deviation by either player yields 12 then 6 indefinitely. Eqm play yields 10 indefinitely. Compare PVs $12 + 6\delta/1-\delta \leq 10/1-\delta \Leftrightarrow$
$2 \leq 4\delta/1-\delta \Leftrightarrow 1-\delta \leq 2\delta \Leftrightarrow \delta \geq 1/3$. Then there is no profitable one shot deviation in the cooperation phase.

In the P1 punishment phase (at any history where there has been some deviation and P1 deviated (weakly) earlier, one shot deviation by P1 yields 4 then 6 indefinitely, eqm play yields 6 indefinitely, P1 has no profitable one shot deviation. One shot deviation by P2 (at best) yields 10 then 12 indefinitely, eqm play yields 12 indefinitely, P2 has no profitable one shot deviation.

By analogous argument, there is no profitable one shot deviation in the P2 punishment phase (at any history where there has been some deviation, and P2 deviated first).

There is no profitable one shot deviation at any history, so this is in fact a SPE.

The continuation plays are (T,L), (T,C), (B,L) which have payoffs (10,10), (6,12), (12,6). By inspection, no continuation play is strictly preferred to any other continuation play by every player, so this SPE is weakly renegotiation proof.

f The required SPE is as follows.

In the first stage and every subsequent stage, if there has been no deviation, play (T,L). If there has been some deviation, play the mixed stage game NE $(3/4T + 1/4B, 1/2L + 1/2C)$ indefinitely.

verify that this is a SPE for sufficiently large δ.

~~Ac~~ At any history where there has been a deviation, optimal one shot deviation by P1 yields 8, then 8 indefinitely, because P1 is indifferent between T and B and any mix. Eqm play yields 8 indefinitely, so P1 has no (strictly) profitable one shot deviation. Optimal one shot deviation by P2 yields 9, then 9 indefinitely. eqm play yields 9 indefinitely, so P2 has no profitable one shot deviation.

At any history where there has been no ~~prior~~ deviation, optimal one shot deviation by P1 yields 12 then 8 indefinitely, eqm play yields 10 indefinitely. Optimal one shot deviation by P2 yields 12 then 9 indefinitely, ~~P2 has~~ ~~Given common dis~~ eqm play yields 10 indefinitely. Given common discount factor δ, if P2 has no profitable one shot deviation, neither does P1. This is iff $12 + 9\delta/1-\delta \leq 10/1-\delta \iff 2 \leq \delta/1-\delta \iff 2-2\delta \leq \delta \iff \delta \geq 2/3$.

There is no profitable one shot deviation hence this is a SPE iff $\delta \geq 2/3$.

This SPE is not weakly renegotiation proof because continuation play from ~~(T,B)~~ (T,C) is strictly preferred by every player to continuation play from $(3/4\,T + 1/4\,B,\ 1/2\,L + 1/2\,C)$.

Date

a. Denote the first period column player as P2 and the second period column player as P2'. Denote the 'hard-wired' type as H and the 'normal' type as N.

	L	R
T	$\underline{3}$	2
	3	1
B	0	$\underline{1}$
	$\underline{4}$	$\underline{2}$

Best responses underlined.

By inspection, P1 has a strictly dominant strategy B. Then, in period 2, by sequential rationality, type N P1, simply N, plays B with certainty. It is given that H plays T with certainty. Then ~~to~~ the expected payoffs to P2' from each action are as follows.

$\pi_{2'}(L;\sigma_N,\sigma_H;\mu_2) = 3\mu_2$
$\pi_{2'}(R;\sigma_N,\sigma_H;\mu_2) = 2\mu_2 + (1-\mu_2)$
$\pi_{2'}(L;\dots) \gtreqless \pi_{2'}(R;\dots) \Leftrightarrow \mu_2 \gtreqless 1/2$

By sequential rationality, if $\mu_2 > 1/2$, the PBE in period 2 is such that N plays B, H plays T, P2' plays L, with certainty.

If $\mu_2 = 1/2$, N plays B, H plays T, P2' plays any (potentially degenerate) mix of L and R.

If $\mu_2 < 1/2$, N plays B, H plays T, P2' plays R.

b. Suppose for reductio that N plays T with certainty in t=1, then by Bayesian beliefs, because playing T is entirely uninformative, $\mu_2 = 1/6$, then by the above result, P2' plays R with certainty. Then, N playing T in t=1 is not sequentially rational. ~~Note that~~ N always plays B with certainty in t=2 ~~?~~, so ~~?? N playing T in t=1 yields~~ ~~?~~ the outcome in t=2 is (B, R), regardless of N's ~~?~~ action in t=1. Then N has strictly profitable deviation to B in t=1, which is strictly dominant. By reductio, N does not play T with certainty in t=1.

Suppose for reductio that N plays B with certainty in t=1. Given that N ~~H~~ plays T with certainty in t=1, by ~~Bayesian beliefs, $\mu_2 = 1$. Note again that N plays B with certainty in t=2. Recall from (a) that if $\mu_2 = 1$, then P2' plays L with certainty. Then, N has strictly profitable deviation~~

~~By Bayesian beliefs, $\mu_2 = 1$. If P2' observes~~ By Bayesian beliefs, if P2' observes B in t=1, then $\mu_2 = 0$ and if P2' observes T, then $\mu_2 = 1$. By the result in (a), which follows from sequential rationality, if P2' observes B, then P2' plays R and if P2' observes T, then P2' plays L. Then, ~~?~~ B ~~?~~ in t=1 is not sequentially rational for P1 because deviation to T induces P2' to play L, then ~~?~~ N ~~has total payoff~~ loses 1 util in t=1 but gains 2 utils in t=2, so this deviation is strictly profitable. By reductio, N does not play B with certainty in t=1.

Then, N mixes in t=1. Intuitively, if N perfectly imitates H, by playing T, so playing does not ~~?~~ serve to signal H, and has no effect on future payoffs. If N does not imitate, it has strict incentive to imitate because imitation is perfectly convincing.

c. Suppose for reductio that P2' plays L with certainty upon observing T. Then, N has strict incentive to play T in t=1 because this ~~?~~ costs 1 util in t=1 but induces L which N takes advantage of to gain 2 utils in t=2. Then N does not mix in t=1. By reductio, P2' does not play L with certainty upon observing T.

Suppose P2' plays R with certainty upon observing T. Then mixing is not sequentially rational for N in t=1 because N has strict incentive to play B since P2' plays R regardless and B is strictly dominant in t=1. By reductio, P2' does not play R with certainty upon observing T.

~~?~~ So P2' mixes upon observing T. Intuitively, if T is rewarded with certainty, N has strict incentive to imitate, and if T is never rewarded, N has no incentive to imitate.

Then, by the result in (a), if ~~?~~ P2' observes T, $\mu_2 = 1/2$. By Bayesian beliefs, this is iff N ~~?~~ mixes in t=1, playing T with probability 1/5.

Given that N mixes, N is indifferent between T and B. B yields 1 ~~?~~ more utils in t=1 than T, induces R in t=2, and yields 2 utils in t=2. T yields ~~?~~ 1 fewer util in t=1 than B, and induces some mix ~~?~~ $q'L + (1-q')R$ in t=2 which yields $4q' + 2(1-q') = 2 + 2q'$ in t=2. Then $1 = 2q'$, $q' = 1/2$. P2' mixes with probability $1/3$, $1/2$ in $t=2$ upon observing ~~*~~ T.

Given that N mixes ~~with~~ ~~prob~~ $\frac{1}{5}T + \frac{4}{5}B$, and that N has probability $5/6$ and H $1/6$, the total probability of T in $t=1$ is $1/3$, and of B is $2/3$. Then, $\pi_2(L; \sigma_N, \sigma_H) = 1/3 \times 3 = 1$, $\pi_2(R; \sigma_N, \sigma_H) = 1/3 \times 2 + 2/3 \times 1 = 4/3$. Sequential rationality requires P2 plays R with certainty in $t=1$.

The required PBE is the assessment (σ, μ), where $\sigma = (\sigma_N, \sigma_H, \sigma_2, \sigma_{2'})$ is ~~#~~ the PBE strategy profile and $\mu = \mu_2$ is the PBE beliefs. μ_2 describes ~~P2' belief~~ the probability P2' attributes to facing a H type on observing T.

$\sigma_N = \begin{pmatrix} \frac{1}{5}T + \frac{4}{5}B \text{ in } t=1 \\ B \text{ in } t=2 \text{ regardless} \end{pmatrix}$

$\sigma_H = (T \text{ in } t=1,\ T \text{ in } t=2 \text{ regardless})$

$\sigma_2 = R$

$\sigma_{2'} = \begin{pmatrix} R \text{ upon observing } B \\ \frac{1}{2}L + \frac{1}{2}R \text{ upon observing } T \end{pmatrix}$

$\mu_2 = 1/2$

b This game is a two-stage version of ~~the Milgrom - Roberts reputation model of entry predatory pricing. P1 is an incumbent and each of P2, P2' is a potential entrant in chain store and each of P2, P2' is a potential entrant local store in a different local market. The incumbent is either~~

P1 is some agent, who undertakes projects with determinate outcomes, for example, an individual producer of widgets. P1's actions T and B correspond to high effort and low effort.

P2 is some investor who supplies P1 with capital ~~the ca~~ for the project, actions L and R correspond to high investment and low investment. Investment increases the total profit from the project but also the investor's exposure to the agent's effort.

If investors entertain the possibility of a commitment type agent, who always chooses high effort, then the normal self-interested agent has incentive to partially imitate the diligent type to preserve a reputation for diligence. Though in the final period, the self-interested type always slacks off.