Philosophical Logic Paper 160610

1(a) i: A triple $\langle W, R, I\rangle$ is a MPL model iff W is some non-empty set, the set of possible worlds, R is a binary relation over W, the accessibility relation, and I is a function from each sentence letter-world pair (α, w) to some truth value 0 or 1.

A ~~MPL~~ triple $\langle W, R, I\rangle$ is a S4 model iff it is a MPL model and R is reflexive and transitive on W, i.e. $\forall w \in W: Rww$ and $\forall w_1, w_2, w_3 \in W: Rw_1w_2 \wedge Rw_2Rw_3 \rightarrow Rw_1w_3$.

A triple $\langle W, R, I\rangle$ is a S5-model iff it is a MPL model and R is an equivalence relation, i.e. reflexive, symmetric, and transitive on W, i.e. $\forall w \in W: Rww$, $\forall w_1, w_2 \in W: Rw_1w_2 \rightarrow Rw_2w_1$ and $\forall w_1 w_2 w_3: Rw_1w_2 \wedge Rw_2Rw_3 \rightarrow Rw_1w_3$.

ii consider some arbitrary S5 model $M = \langle W, R, I\rangle$ and some arbitrary world $w \in W$.

Suppose for reductio that

(1) $V_M(\Diamond(\Diamond P \wedge \Diamond \sim P) \rightarrow \Box(\Diamond P \wedge \Diamond \sim P), w) = 0$

From (1), by $\rightarrow$

(2) $V_M(\Diamond(\Diamond P \wedge \Diamond \sim P), w) = 1$

(3) $V_M(\Box(\Diamond P \wedge \Diamond \sim P), w) = 0$

From (2), by derived $\Diamond$

(4) $\exists u \in W, Rwu: V_M(\Diamond P \wedge \Diamond \sim P, u) = 1$

From (3), by $\Box$

(5) $\exists v \in W, Rwv: V_M(\Diamond P \wedge \Diamond \sim P, v) = 0$

From (4), by derived $\wedge$

(6) $\exists u \in W, Rwu: V_M(\Diamond P, u) = 1$

(7) $\exists u \in W, Rwu: V_M(\Diamond \sim P, u) = 1$

~~From (6), by the symmetry and transitivity of R on W.~~ (5), (7)

~~(8) $\exists v \in W, Rwv: V_M(\Diamond P \wedge \Diamond \sim P, v) = 0$, $V_M(\Diamond P, v) = 1$, $V_M(\Diamond \sim P, v) = 1$~~

~~From (8), by $\wedge$~~

~~(9) $\exists v \in W, Rwv: V_M(\Diamond P \wedge \Diamond \sim P, v) = 0$. $V_M(\Diamond P \wedge \sim \Diamond P, v)$~~

~~From (9), by reductio~~

From (6), by derived $\Diamond$

(8) $\exists u, u' \in W, Rwu, Ruu': V_M(P, u') = 1$

From (7), by derived $\Diamond$

(9) $\exists u, u' \in W, Rwu, Ruu': V_M(\sim P, u') = 1$

From (5), (8), (9), by symmetry and transitivity of R on W

(10) $\exists v' \in W, Rvv': V_M(P, v') = 1$

(11) $\exists v' \in W, Rvv': V_M(\sim P, v') = 1$

where v is that in (5)

From (10), (11), by derived $\Diamond$, derived $\wedge$

(12) $V_M(\Diamond P \wedge \sim \Diamond P, v) = 1$

where v is that in (5)

From (5), (12), by reductio

(13) $V_M(\Diamond(\Diamond P \wedge \sim \Diamond P) \rightarrow \Box(\Diamond P \wedge \sim \Diamond P), w) = 0$

From (13), by generalisation over w and M, by definition of S5-validity.

(14) $\models_{S5} \Diamond(\Diamond P \wedge \Diamond \sim P) \rightarrow \Box(\Diamond P \rightarrow \Diamond \sim P)$ ∎

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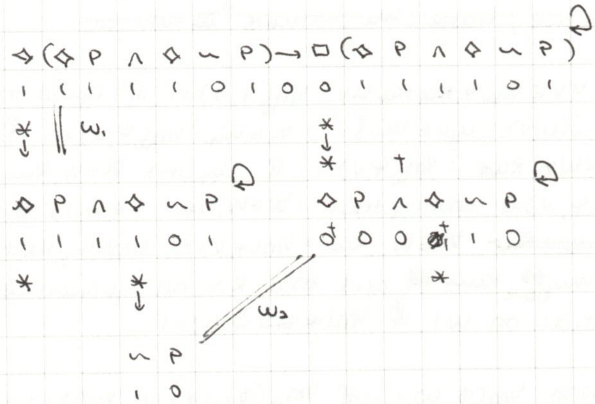

Consider the S4-countermodel

$M = \langle W, R, I\rangle$

$W = \{w_1, w_2\}$

$R = \{\langle w_1, w_1\rangle, \langle w_2, w_2\rangle, \langle w_1, w_2\rangle\}$

$I(P, w_1) = 1$, $I(\alpha, w) = 0$ for all other sentence letter-world pairs α, w

b) i) Base case

Consider arbitrary MPL-wff ϕ such that ~~$C(\phi)$~~ complexity $C(\phi) = 0$. Consider arbitrary $u \in W_w$. From $C(\phi) = 0$, $\phi = \alpha$, where α is some sentence letter. Then $V_{M_w}(\phi, u) = I_w(\alpha, u) = I(\alpha, u) = V_M(\phi, u)$. By generalisation, for all ~~$u \in W_w$~~ ϕ such that $C(\phi) = 0$, for all $u \in W_w$, $V_{M_w}(\phi, u) = V_M(\phi, u)$.

~~Induction Hypothesis~~

~~Given n, for all $m < n$, for all~~

Induction Hypothesis

Given n, for all $m < n$, for all ϕ such that $C(\phi) = m$, for all $u \in W_w$, $V_{M_w}(\phi, u) = V_M(\phi, u)$.

Induction Step

Consider arbitrary MPL-wff ϕ such that $C(\phi) = n$. Consider arbitrary $u \in W_w$. $\phi = \sim\psi$, $\phi = \psi \rightarrow \chi$, or $\phi = \Box\psi$ for some ψ, χ.

Suppose $\phi = \sim\psi$, then $V_{M_w}(\phi, u) = 1$ iff $V_{M_w}(\psi, u) = 0$ iff (by IH) $V_M(\psi, u) = 0$ iff $V_M(\phi, u) = 1$, and $V_{M_w}(\phi, u) = 0$ iff $V_{M_w}(\phi, u) \neq 1$ iff $V_M(\phi, u) \neq 1$ iff $V_M(\phi, u) = 0$, then $V_{M_w}(\phi, u) = V_M(\phi, u)$.

Suppose $\phi = \psi \rightarrow \chi$, then $V_{M_w}(\phi, u) = 1$ iff $V_{M_w}(\psi, u) = 0$ or $V_{M_w}(\chi, u) = 1$ iff (by IH) $V_M(\psi, u) = 0$ or $V_M(\chi, u) = 1$ iff $V_M(\phi, u) = 1$, and $V_M(\phi, u) = 0$ iff $V_{M_w}(\phi, u) \neq 1$ iff $V_M(\phi, u) \neq 1$ iff $V_M(\phi, u) = 0$, then $V_{M_w}(\phi, u) = V_M(\phi, u)$

Suppose ~~that~~ $\phi = \Box\psi$, then $V_{M_u}(\phi, u) = 1$ iff ~~for all $v \in W_u$ & $V_{M_u}(\psi, v) = 1$ iff by IH for all $v \in W_u$. $V_M(\psi, v) = 1$ $\forall v \in W_u, R_u uv : V_{M_u}(\psi, v) = 1$ iff $\forall v \in W_u, R_{uv} : V_M(\psi, v) = 1$ (by IH, given R is reflexive, hence W_u contains u and all~~

iff $\forall v \in W_u, R_u uv : V_{M_u}(\psi, v) = 1$ iff (given that $R_u = \{\langle u, v \rangle : u, v \in W_u\}$) $\forall v \in W_u : V_{M_u}(\psi, v) = 1$ iff $\forall v \in W, R_{uv} : V_{M_u}(\psi, v) = 1$ iff (by IH) $\forall v \in W, R_{uv} : V_M(\psi, v) = 1$ (noting that $\forall v \in W, R_{uv} : v \in W_u$) iff ~~$\forall v \in W : R_{uv}$~~ $\forall v \in W, R_{uv} : V_M(\psi, v) = 1$ (noting that $u \in W_u$, here R_{uu} and that R is an equivalence relation on W) iff $V_M(\phi, u) = 1$.

And $V_{M_u}(\phi, u) = 0$ iff $V_{M_u}(\phi, u) \neq 1$ iff $V_M(\phi, u) \neq 1$ iff $V_M(\phi, u) = 0$. Then $V_{M_u}(\phi, u) = V_M(\phi, u)$.

By cases, by generalisation over u, ϕ, for all ϕ such that $c(\phi) = n$, for all $u \in W_u$, $V_{M_u}(\phi, u) = V_M(\phi, u)$.

By induction over $c(\phi)$, for all ϕ, for all $u \in W_u$, $V_{M_u}(\phi, u) = V_M(\phi, u)$.

ii Suppose for biconditional proof that ϕ is S5-valid, suppose further for reductio that ϕ is not valid in every MPL model whose accessibility relation is the universal relation on its set of possible worlds. Then in some such world of some such model, ϕ evaluates to false. Such a model is an S5-model then ϕ is not S5-valid. By reductio, ϕ is valid in every "universal" model.

Suppose for biconditional proof that ϕ is valid in every "universal" model. Suppose further for reductio that ϕ is not S5-valid. Then in some world w of some S5-model M, ϕ evaluates to false, then in the universal model whose set of possible worlds is the set of worlds accessible from w in M, by the result in (bi), at w, ϕ evaluates to false, then ϕ is not valid in every "universal" model. By reductio, ϕ is S5-valid.

By biconditional proof ϕ is S5-valid iff it is valid in every "universal" model.

iii No. The $\phi = \Box\psi$ case in the induction step of the inductive proof fails. ~~It cannot be inferred~~ $\forall v \in W, R_{uv} : V_M(\psi, v) = 1$ is no longer necessary and sufficient for $\forall v \in W, R_{uv} : V_M(\psi, v) = 1$ when symmetry no longer holds. The proof in (bii) relied on the result of (bi), so that proof fails too.

c Intuitively, it seems reasonable to infer from "it is (metaphysically) possible that (it is possible that P and it is possible that not P)" that "it is (metaphysically) necessary that (it is possible that P and it is possible that not P)" in other words if some thing is ~~metaphy~~ possibly metaphysically contingent, then it is necessarily metaphysically contingent. In other words, metaphysical contingency is not itself metaphysically contingent.

For example if it is metaphysically contingent that there exists a Philosophical Logic exam, this contingency is not itself contingent.

This constitutes reason for favouring S5 over S4.

Plausibly, the universal relation is appropriate for the relation of metaphysical possibility / contingency / necessity. Plausibly, every ~~metaphysically accessible world is accessib~~ all possible worlds are metaphysically accessible from each other. For instance, we think the fact that P is true in some possible has metaphysical implications for all possible worlds, and no worlds are insulated from this influence by asymmetry. All "upstream" and "downstream" worlds ~~are equally~~ "feel" the metaphysical ~~implication~~ effect equally.

2ai A triple $\langle W, \precsim, I\rangle$ is a SC-model iff W is some non empty set, the set of possible worlds, $\precsim$ is some ternary relation over W that encodes a binary relation $\precsim_w$ over W for each $w \in W$ such that $\precsim_w$ is a linear order on W and satisfies the base assumption that $w \precsim_w u$ for all $u \in W$ (where $\precsim_w = \{\langle u,v\rangle : \langle u,v,w\rangle \in \precsim\}$), $\precsim$ is the nearness relation and I is some function from each sentence letter α for each world $w \in W$ to the truth values 0, 1, the interpretation function, that for all wff ϕ, if there exist some world $u \in W$ such that $V_M(\phi, u) = 1$, then for all $v \in W$, there exists some $w \in W$ such that $V_M(\phi, w) = 1$ and for all $w' \in W$ such that $V_M(\phi, w') = 1$, it is also the case that $w \precsim_v w'$ (in other words, for all wff ϕ, if some ϕ world exists, then for all worlds $v \in W$, some v-closest ϕ-world exists).

A triple $\langle W, \precsim, I\rangle$ is an CC model iff W is some non-empty set, the set of possible worlds, $\precsim$ is some ternary relation that ~~encodes~~ for each world $w \in W$, encodes a binary relation $\precsim_w = \{\langle u,v\rangle : \langle u,v,w\rangle \in \precsim\}$, that is a linear preorder on W and satisfies the modified base assumption that if $u \precsim_w w$ then $u = w$ for all $u \in W$. $\precsim$ is the nearness relation, and I is some function from each sentence letter α for each world to some truth value 0, or 1, the interpretation function (that does not necessarily satisfy the limit assumption).

A linear order on W is a binary relation that is reflexive, transitive, weakly connected, and antisymmetric on W. A linear preorder is a binary relation that is reflexive, transitive, and weakly connected on W.

ii ~~Given SC model~~ The SC valuation function for ~~given~~ some given SC model $M = \langle W, \precsim, I\rangle$ is the unique function V_M from wff ϕ for each world $w \in W$ such that

$V_M(\alpha, w) = I(\alpha, w)$, for all sentence letters α

$V_M(\sim\phi, w) = \begin{cases} 1 & \text{iff } V_M(\phi, w) = 0 \\ 0 & \text{otherwise} \end{cases}$

for all wff ϕ

$V_M(\phi \to \psi, w) = \begin{cases} 1 & \text{iff } V_M(\phi, w) = 0 \text{ or } V_M(\psi, w) = 1 \\ 0 & \text{otherwise} \end{cases}$

for all wff ϕ, ψ

$V_M(\Box\phi, w) = \begin{cases} 1 & \text{iff } V_M(\phi, u) = 1 \text{ for all } u \in W \\ 0 & \text{otherwise} \end{cases}$

for all wff ϕ

$V_M(\phi \Box\!\!\rightarrow \psi, w) = \begin{cases} 1 & \text{iff no } \phi\text{-world exists or } V_M(\psi, u) = 1 \text{ in the } w\text{-closest } \phi\text{-world } u \\ 0 & \text{otherwise} \end{cases}$

where a ϕ-world is some world $u \in W$ such that $V_M(\phi, u) = 1$, and the w-closest ϕ-world is the unique world $v \in W$ such that $V_M(\phi, v) = 1$ and for all ϕ-worlds v', $v \precsim_w v'$, if it exists.

The CC valuation function for some given CC model $M = \langle W, \precsim, I\rangle$ is defined identically except in the clause for $\Box\!\!\rightarrow$, which is instead.

$V_M(\phi \Box\!\!\rightarrow \psi, w) = \begin{cases} 1 & \text{iff no } \phi\text{-world exists or there exists some } \phi\text{-world } u \text{ such that in every world } u' \text{ such that } u' \precsim_w u,\ V_M(\phi \to \psi, u') = 1 \\ 0 & \text{otherwise} \end{cases}$

iii Base case

~~Consider arbitrary wff ϕ such that complexity $c(\phi) = 0$. Then ϕ is some sentence letter α. Then under any CC-model $M = \langle W, \precsim, I\rangle$ where $I(\alpha, w) = 0$ for some $w \in W$, $V_M(\phi, w) = 0$, hence ϕ is not CC-valid. Then, if ϕ is CC-valid, it is SC-valid. By generalisation over ϕ, for all wff ϕ such that $c(\phi) = 0$, if ϕ is CC-valid, then it is also SC-valid.~~

Consider some arbitrary wff ϕ. Suppose ~~that~~ for conditional proof that ϕ is CC-valid. Suppose further for reductio that ϕ is not SC-valid. Then by definition of SC-validity, there exists SC-model $M = \langle W, \precsim, I\rangle$ and world $w \in W$ such that $V_M(\phi, w) = 0$. If each $\precsim_w$ encoded by $\precsim$ is a linear order on W, then it is also a linear preorder on W. By antisymmetry of each $\precsim_w$ on W, if each $\precsim_w$ satisfies the base assumption (in SC), it also satisfies the ~~base assumption~~ modified base assumption (in CC), then M is also a CC-model. By the lemma that SC-valuation and CC-valuation ~~for~~ coincide for SC-models, $V_M(\phi, w) = 0$ in CC, then ϕ is not CC-valid. By reductio and conditional proof, if ϕ is CC-valid, it is also SC-valid.

Lemma: CC-valuation and SC-valuation coincide for SC models. It is sufficient ~~for~~ argument for this lemma to prove that for SC-models, the clause for $\Box\!\!\rightarrow$ in SC and in CC coincide.

~~In an SC-model, either no ϕ-wor~~ Consider some arbitrary SC-model M, wff ϕ, ψ, world w. In an SC-model, ~~eit~~ by limit, either no ϕ world exists or some w-closest ϕ-world, u, exists. If no ϕ world exists, $\phi \Box\!\!\rightarrow \psi$ at w evaluates as true under both CC and SC. If some w-closest ϕ-world u exists, then if

ψ evaluates as 1 in u, then SC and LC agree in evaluating $\phi \Box\!\!\rightarrow \psi$ as 1 in w. If ψ evaluates as 0 in u, then SC and LC agree in evaluating $\phi \Box\!\!\rightarrow \psi$ as 0 in w. So the clause for $\Box\!\!\rightarrow$ in SC and LC coincide for SC models.

bi SC-valid

Consider arbitrary SC model $M = \langle W, \precsim, I \rangle$, and arbitrary world $w \in W$.
Suppose for reductio that
(1) $V_M((P \Box\!\!\rightarrow Q) \vee (P \Box\!\!\rightarrow \sim Q), w) = 0$
From (1), by derived $\vee$
(2) $V_M(P \Box\!\!\rightarrow Q) = 0$
(3) $V_M(P \Box\!\!\rightarrow \sim Q) = 0$
From (2), by $\Box\!\!\rightarrow$
(4) $V_M(Q, u) = 0$
where u is the w-closest P-world
which exists given limit and (2), which implies that some P-world exists
From (3), by $\Box\!\!\rightarrow$
(5) $V_M(\sim Q, u) = 0$
where u is the w-closest P-world
From (5), by $\sim$
(6) $V_M(Q, u) = 1$
From (4), (6), by reductio
(7) $V_M((P \Box\!\!\rightarrow Q) \vee (P \Box\!\!\rightarrow \sim Q), w) = 1$
From (7), by generalisation over w, M, by definition of SC-validity.
(8) $\vDash_{SC} (P \Box\!\!\rightarrow Q) \vee (P \Box\!\!\rightarrow \sim Q)$ ■

Not LC-valid

Consider the following countermodel

$M = \langle W, \precsim, I \rangle$
$W = \{w_0, w_1, w_2\}$
$\precsim_{w_0} = \{\langle w_1, w_2 \rangle, \langle w_2, w_1 \rangle, \langle w_0, w_1 \rangle, \langle w_0, w_2 \rangle\}$
$I(P, w_1) = I(P, w_2) = I(Q, w_1) = 1$, $I(\alpha, \omega) = 0$ for all other sentence letter-world pairs α, ω.

For completeness,
$\precsim_{w_1} = \{\langle w_1, w_0 \rangle, \langle w_1, w_2 \rangle, \langle w_2, w_0 \rangle\}$
$\precsim_{w_2} = \{\langle w_2, w_0 \rangle, \langle w_2, w_1 \rangle, \langle w_1, w_0 \rangle\}$

$V_M((P \Box\!\!\rightarrow Q) \vee (P \Box\!\!\rightarrow \sim Q), w) = 0$
$\nvDash_{LC} (P \Box\!\!\rightarrow Q) \vee (P \Box\!\!\rightarrow \sim Q)$ ■

ii SC-valid

Consider arbitrary SC-model $M = \langle W, \precsim, I \rangle$ and arbitrary world $w \in W$.
Suppose for reductio that
(1) $V_M(\Diamond P \rightarrow (\sim(P \Box\!\!\rightarrow \sim Q) \leftrightarrow (P \Box\!\!\rightarrow Q)), w) = 0$
From (1), by $\rightarrow$,
(2) $V_M(\Diamond P, w) = 1$
(3) $V_M(\sim(P \Box\!\!\rightarrow \sim Q) \leftrightarrow (P \Box\!\!\rightarrow Q), w) = 0$
From (2), by derived $\Diamond$
(4) $\exists u \in W : V_M(P, u) = 1$
From (3), by derived $\leftrightarrow$
(5): (6) or (7)
(6): (8) and (9)
(8): $V_M(\sim(P \Box\!\!\rightarrow \sim Q), w) = 1$
(9): $V_M(P \Box\!\!\rightarrow Q, w) = 0$
(7): (10) and (11)
(10): $V_M(\sim(P \Box\!\!\rightarrow \sim Q), w) = 0$
(11): $V_M(P \Box\!\!\rightarrow Q, w) = 1$
From (8), by $\sim$
(12) $V_M(P \Box\!\!\rightarrow \sim Q, w) = 0$
~~From (10), by ~~~
(13) $\cancel{V_M(P \Box\!\!\rightarrow \sim Q, w) = 1}$
Let v denote the w-closest P-world, which exists and is unique given (4) and limit.
From (9), (12)
(14) $V_M(Q, v) = 0$
(15) $V_M(\sim Q, v) = 0$
From (15), by $\sim$
(16) $V_M(Q, v) = 1$
From (14), (16), by reductio
(17) $V_M(\Diamond P \rightarrow (\sim(P \Box\!\!\rightarrow \sim Q) \leftrightarrow (P \Box\!\!\rightarrow Q)), w) = 1$
From the above, by conditional proof
(18) If (6) then (17)
By analogous argument, from the above
(19) If (7) then (17)
From the above, by cases
(20) $V_M(\Diamond P \rightarrow (\sim(P \Box\!\!\rightarrow \sim Q) \leftrightarrow (P \Box\!\!\rightarrow Q)), w) = 1$
From (15), by $\sim$
(16) $V_M(Q, v) = 1$
By ~~analogous arg~~ conditional proof
(17) If (6), then (14) and (16)
By an analogous argument
(18) If (7), then (14) and (16)
By cases
(19) (14) and (16)
From (19) by reductio
(20) $V_M(\Diamond P \rightarrow (\sim(P \Box\!\!\rightarrow \sim Q) \leftrightarrow (P \Box\!\!\rightarrow Q)), w) = 1$
From (20), by generalisation over w, M, by definition of SC-validity.
(21) $\vDash_{SC} \Diamond P \rightarrow [\sim(P \Box\!\!\rightarrow \sim Q) \leftrightarrow (P \Box\!\!\rightarrow Q)]$ ■

Not LC-valid.

The earlier countermodel is again a countermodel for this wff.

~~3a Base Case~~

~~Consider some arbitrary PL-wff ϕ such that complexity $c(\phi) = 0$.~~

PL-wff ϕ is a supervaluationist semantic consequence of PL-wffs Γ iff for all trivalent interpretations I, if for all $\gamma \in \Gamma$, $SV_I(\gamma)=1$ then $SV_I(\phi)=1$, which is iff for all trivalent I, for all precisifications I^+, ~~$V_{I^+}(\Gamma)$~~ if for all $\gamma \in \Gamma$, $V_{I^+}(\gamma)=1$, then $V_{I^+}(\phi)=1$, which is iff, noting that the set of all precisifications of all trivalent interpretations ~~I^+~~ is simply the set of all bivalent interpretations, for all bivalent interpretations I' if for all ~~γ~~ $\gamma \in \Gamma$, $V_{I'}(\gamma)=1$ then $V_{I'}(\phi)=1$, which is iff, by definition of PL-semantic consequence, ϕ is a PL-semantic consequence of Γ.

b: Base Case

Consider some arbitrary MPL-wff ϕ which contains no occurrences of ϕ such that complexity $c(\phi)=0$. Consider some arbitrary trivalent interpretation I. ~~By definition~~, ϕ is some sentence letter α. By definition, $I(\alpha) = 1$, 0 or $\#$.

Suppose that $I(\alpha)=1$, then by definition, every precisification of I, c is such that $c(\alpha)=1$, then in I's induced Kripke model, for every $c \in W$, by definition of MPL-valuation, sentence letter clause, $V_{M_I}(\phi,c) = V_{M_I}(\alpha,c) = H(\alpha,c) = c(\alpha) = 1$. Then by definition of MPL-validity, ϕ is valid in M_I, then by definition of $SV^*_I(\phi)$, $SV^*_I(\phi)=1=SV_I(\phi)$

By an analogous argument, supposing that $I(\alpha)=0$, ~~so~~ $SV^*_I(\phi) = 0 = SV_I(\phi)$.

Suppose that $I(\alpha)=\#$, then by definition, there exists some precisification ~~I^0 such~~ c^0 such that $c^0(\alpha)=0$ and some precisification c' such that $c'(\alpha)=1$. Then $V_{M_I}(\phi,c^0) = V_{M_I}(\alpha,c^0) = H(\alpha,c^0) = c^0(\alpha) = 0$ and similarly $V_{M_I}(\phi,c')=1$. ~~By definition~~ hence $V_{M_I}(\sim\phi,c')=0$. By definition of MPL validity, neither ϕ nor $\sim\phi$ is MPL-valid in M_I. Then by definition of $SV^*_I(\phi)$, $SV^*_I(\phi) = \# = SV_I(\phi)$.

By cases, by generalisation over I, ϕ, for all ϕ containing no occurrences of $\Box$ such that $c(\phi)=0$, for all trivalent interpretations I, $SV^*_I(\phi) = SV_I(\phi)$.

Induction Hypothesis

Given n, for all $m<n$, for all ϕ containing no occurrences of $\Box$ such that $c(\phi)=m$, for all trivalent interpretations I, $SV^*_I(\phi)=SV_I(\phi)$.

Induction Step

Consider some arbitrary MPL-wff ϕ such that ϕ contains no occurrences of $\Box$ and $c(\phi)=n$. Consider some arbitrary trivalent interpretation I. $\phi = \sim\psi$ or $\psi \to \chi$ for some MPL-wff ψ, χ, each containing no occurrences of $\Box$ and with ~~$c(\phi)$~~ $c(\psi), c(\chi) < n$.

Suppose $\phi = \sim\psi$. $SV^*_I(\phi) = 1$ iff $SV^*_I(\sim\psi) = 1$ iff ~~$SV^*_I(\sim\sim$~~ $\sim\psi$ ~~iff~~ is valid in M_I iff ~~$V_{M_I}(\sim\psi, c)=1$ for all $c \in W$ iff $V_{M_I}(\psi,c)=0$ for all $c \in W$ iff $V_{M_I}(\psi,c)=0$ for all $c \in W$ iff $V_{M_I}(\sim\sim\psi,c)=0$ for~~ ~~for all $c \in W$~~ $SV^*_I(\psi)=0$ iff by IH $SV_I(\psi)=0$ iff for all precisifications I^+ of I $V_{I^+}(\psi)=0$ iff for all I^+ $V_{I^+}(\sim\psi)=V_{I^+}(\phi)=1$ iff $SV_I(\phi)=1$. $SV^*_I(\phi)=0$ iff $SV^*_I(\sim\psi)=0$ iff $\sim\sim\psi$ is valid in M_I iff for all precisifications c of I, $V_{M_I}(\sim\sim\psi, c)=1$ iff ~~$V_{M_I}(\psi,c)=1$~~ for all c, $V_{M_I}(\psi,c)=1$ iff $SV^*_I(\psi)=1$ iff by IH $SV^*_I(\psi)=1$ iff for all c $V_c(\psi)=1$ iff for all c $V_c(\sim\psi)=V_c(\phi)=0$ iff $SV_I(\phi)=0$. By mutual exclusivity and collective exhaustiveness, $SV^*_I(\phi) = \#$ iff $SV_I(\phi)=\#$.

Suppose $\phi = \psi \to \chi$. $SV^*_I(\phi)=1$ iff $SV^*_I(\psi\to\psi)=1$ iff $\psi\to\chi$ is valid in M_I iff for all precisifications c of I $V_{M_I}(\psi\to\chi,c)=1$ iff for all c $V_{M_I}(\psi,c)=0$ or $V_{M_I}(\chi,c)=1$

Prove by induction that for all ~~precisificat~~ ~~trival~~ MPL-wffs ϕ containing no occurrences of $\Box$ for all trivalent interpretations I, for all ~~precisI~~ precisifications c of I, $V_{M_I}(\phi,c) = V_c(\phi)$.

Base case such that $c(\phi)=0$

Consider arbitrary MPL wff ϕ with no $\Box$. ~~Consider~~ arbitrary I^3, ~~consider~~ arbitrary prec. c of I^3 ~~V_{M_I}~~ ϕ is some sentence letter α. $V_{M_I}(\phi,c) = V_{M_I}(\alpha,c) = H(\alpha,c) = c(\alpha) = V_c(\alpha) = V_c(\phi)$. By generalisation, for all ϕ with no $\Box$ of $c(\phi)=0$, for all I^3, for all c of I^3, $V_{M_I}(\phi,c) = V_c(\phi)$.

Induction Hypothesis

Given n, for all $m<n$, for all ϕ with no $\Box$ of $c(\phi)=m$, for all I^3, for all c of I^3, $V_{M_I}(\phi,c) = V_c(\phi)$.

Induction Step

Consider arbitrary MPL-wff ϕ with no $\Box$ such that $c(\phi)=n$, arbitrary I^3, arbitrary c for I^3. $\phi = \sim\psi$ or $\psi \to \chi$ for some MPL-wffs ψ, χ each with no $\Box$ and $c(\psi), c(\chi) < n$.

Suppose $\phi = \sim\psi$. $V_{M_I}(\phi,c) = 1$ iff $V_{M_I}(\sim\psi, c)=1$ iff $V_{M_I}(\psi,c)=0$ iff by IH ~~$V_c(c)$~~ $V_c(\psi)=0$ iff $V_c(\phi)=$

By analogous argument, $v_{m_I}(\phi, c) = 0$ iff $v_c(\phi) = 0$ and $v_{m_I}(\phi, c) = \#$ iff $v_c(\phi) = \#$. Then $v_{m_I}(\phi, c) = v_c(\phi)$.

Suppose $\phi = \psi \to \kappa$. $v_{m_I}(\phi, c) = 1$ iff $v_{m_I}(\psi, c) = 0$ or $v_{m_I}(\kappa, c) = 1$ iff by IH $v_c(\psi) = 0$ or $v_c(\kappa) = 1$ iff $v_c(\phi) = 1$. By analogous argument, $v_{m_I}(\phi, c) = 0$ iff $v_c(\phi) = 0$ and $v_{m_I}(\phi, c) = \#$ iff $v_c(\phi) = \#$, then $v_{m_I}(\phi, c) = v_c(\phi)$.

By cases, by generalisation, for all MPL wff ϕ with no $\Box$ such that $c(\phi) = n$, for all I^3, for all c of I^3, $v_{m_I}(\phi, c) = v_c(\phi)$

By induction, for all MPL wff ϕ with no $\Box$, for all I^3, for all c of I^3, $v_{m_I}(\phi, c) = v_c(\phi)$.

~~$SV^*_I(\phi) = 1$ iff for all c of I~~

Consider arbitrary MPL wff ϕ with no $\Box$, arbitrary I^3. $SV^*_I(\phi) = 1$ iff for all c of I^3, $v_{m_I}(\phi, c) = 1$ iff (by above inductive proof) for all such c $v_c(\phi) = 1$ iff $SV_I(\phi) = 1$. ~~By analogous argument,~~ $SV^*_I(\phi) = 0$ iff for all c of I^3, $v_{m_I}(\sim\phi, c) = 1$ iff for all such c $v_c(\sim\phi) = 1$ iff for all such c $v_c(\phi) = 0$ iff $SV_I(\phi) = 0$. $SV^*_I(\phi) = \#$ iff $SV^*_I(\phi) \neq 0, 1$ iff $SV_I(\phi) \neq 0, 1$ iff $SV_I(\phi) = \#$. Then $SV^*_I(\phi) = SV_I(\phi)$. By generalisation, this holds for all ϕ with no $\Box$ and all I^3.

ii ~~Suppose~~ Consider arb. I^3, ~~prec. c of I^3~~, MPL-wff ϕ. Suppose for conditional proof that $SV^*_I(\phi) = 1$, then for all prec. c of I^3, $v_{m_I}(\phi, c) = 1$. Then for all c' of I^3, $v_{m_I}(\Box\phi, c') = 1$ given that for every c', every accessible c'' is such that ϕ evaluates as 1, then $SV^*_I(\Box\phi) = 1$.

iii True

Suppose for conditional proof that $\Gamma \cup \{\phi\} \vDash_{S*} \psi$. Suppose further for reductio that $\Gamma \nvDash_{S*} \phi \to \psi$. Then there exists I^3 such that $SV^*_I(\gamma) = 1$ for all $\gamma \in \Gamma$, $SV^*_I(\phi \to \psi) \neq 1$, then there exists prec. c for I^3 such that $v_{m_I}(\gamma, c) = 1$ for all $\gamma \in \Gamma$, ~~$v_{m_I}(\gamma, c) = 0$~~ $v_{m_I}(\phi, c) = 1$, $v_{m_I}(\psi, c) = 0$. Then $SV^*_c(\gamma) = 1$ for all $\gamma \in \Gamma$, $SV^*_c(\phi) = 1$ and $SV^*_c(\psi) = 0$, and $\Gamma \cup \{\phi\} \nvDash_{S*} \psi$. By reductio, conditional proof, if $\Gamma \cup \{\phi\} \vDash_{S*} \psi$, then $\Gamma \vDash_{S*} \phi \to \psi$.

iv ~~True~~

~~Suppose for conditional proof that $\Gamma \cup \{\phi\} \vDash_{S*} \kappa$ and that $\Gamma \cup \{\psi\} \vDash_{S*} \kappa$. Suppose further for reductio that $\Gamma \cup \{\phi \vee \psi\} \nvDash_{S*} \kappa$. Then for all I^3, if $SV^*_I(\gamma) = 1$ for all $\gamma \in \Gamma$ and $SV^*_I(\phi) = 1$ then $SV^*_I(\kappa) = 1$. Likewise for ψ. There exists I^3 such that $SV^*_I(\gamma) = 1$ for all $\gamma \in \Gamma$, $SV^*_I(\phi \vee \psi) = 1$, $SV^*_I(\kappa) = 0$. Consider some arbitrary such I~~

False.

Consider the following counterexample

$\Gamma = \{\gamma_1\}$
$\gamma_1 = (\Box P \vee \Box \sim P) \to Q$
$\phi = P$
$\psi = \sim P$
$\kappa = Q$

~~I^3 assigns # to P,~~