Microeconomic Analysis Paper 200604

(a) i $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$

$F_1 = F_2 = 1$

$\vec{u}_n = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}$

$\vec{u}_{n+1} = A\vec{u}_n \iff$

$\begin{pmatrix} F_{n+2} \\ F_{n+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} \iff$

$\begin{pmatrix} F_{n+1}+F_n \\ F_{n+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} \iff$

$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

$\det A = 1\times 0 - 1\times 1 = -1 \neq 0 \Rightarrow A$ is invertible

~~det~~

$A\vec{v} = \lambda\vec{v} \Rightarrow (A-\lambda I)\vec{v} = \vec{0} \Rightarrow \det(A-\lambda I) = 0 \Rightarrow$

$\det\begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = -\lambda(1-\lambda) - (1\times 1) = \lambda^2 - \lambda - 1 = 0 \Rightarrow$

$\lambda = 1 \pm \sqrt{1+4}/2 = \cancel{\pm}\, 1 \pm \sqrt{5}/2$

The eigenvalues are $\lambda_1 = 1+\sqrt{5}/2$, $\lambda_2 = 1-\sqrt{5}/2$

$A\vec{v}_1 = \lambda\vec{v}_1 \iff \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = (1+\sqrt{5}/2)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \iff$

$\begin{pmatrix} v_1+v_2 \\ v_1 \end{pmatrix} = (1+\sqrt{5}/2)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \iff v_1 = 1+\sqrt{5}/2\, v_2 \iff$

$\vec{v}_1 = \begin{pmatrix} 1+\sqrt{5}/2 \\ 1 \end{pmatrix}$

$A\vec{v}_2 = \lambda\vec{v}_2 \iff \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = (1-\sqrt{5}/2)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \iff$

$\begin{pmatrix} v_1+v_2 \\ v_1 \end{pmatrix} = (1-\sqrt{5}/2)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \iff v_1 = (1-\sqrt{5}/2)v_2 \iff$

$\vec{v}_2 = \begin{pmatrix} 1-\sqrt{5}/2 \\ 1 \end{pmatrix}$

ii The eigenbasis of A is $\begin{pmatrix} 1+\sqrt{5}/2 & 1-\sqrt{5}/2 \\ 1 & 1 \end{pmatrix}$

$\cancel{u_{2020} = A^{2018}u_2 = \begin{pmatrix} 1+\sqrt{5}/2 & 1-\sqrt{5}/2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1+\sqrt{5}/2 & 0 \\ 0 & 1-\sqrt{5}/2 \end{pmatrix}}$

denote this eigenbasis V. ~~V^{-1}~~

compute V^{-1} by Gauss-Jordan elimination.

$\left(\begin{array}{cc|cc} 1+\sqrt{5}/2 & 1-\sqrt{5}/2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \xrightarrow{\substack{R_2 \leftarrow R_2 - (2/1+\sqrt{5})R_1 \\ R_1 \leftarrow R_1 - (1-\sqrt{5}/2)R_2}}$

$\left(\begin{array}{cc|cc} \sqrt{5} & 0 & 1 & -1-\sqrt{5}/2 \\ 0 & 1 - 1-\sqrt{5}/1+\sqrt{5} & -2/1+\sqrt{5} & 1 \end{array}\right)$

$= \left(\begin{array}{cc|cc} \sqrt{5} & 0 & 1 & -1-\sqrt{5}/2 \\ 0 & 2\sqrt{5}/1+\sqrt{5} & -2/1+\sqrt{5} & 1 \end{array}\right) \xrightarrow{\substack{R_1 \leftarrow 1/\sqrt{5}R_1 \\ R_2 \leftarrow 1+\sqrt{5}/2\sqrt{5}R_2}}$

$\left(\begin{array}{cc|cc} 1 & 0 & 1/\sqrt{5} & -1-\sqrt{5}/2\sqrt{5} \\ 0 & 1 & -1/\sqrt{5} & 1+\sqrt{5}/2\sqrt{5} \end{array}\right)$

$V^{-1} = \begin{pmatrix} 1/\sqrt{5} & -1-\sqrt{5}/2\sqrt{5} \\ -1/\sqrt{5} & 1+\sqrt{5}/2\sqrt{5} \end{pmatrix}$

$= 1/\sqrt{5}\begin{pmatrix} 1 & -1-\sqrt{5}/2 \\ -1 & 1+\sqrt{5}/2 \end{pmatrix}$

$\cancel{F_{2020} =}$

$\cancel{u_{2020} = A^{2018}u_2}$

$\cancel{= V\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^{2018} V^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

$\cancel{= 1/\sqrt{5}\begin{pmatrix} 1 & -1-\sqrt{5}/2 \\ -1 & 1+\sqrt{5}/2 \end{pmatrix}\begin{pmatrix} 1+\sqrt{5}/2 & 0 \\ 0 & 1-\sqrt{5}/2 \end{pmatrix}^{2018}}$

$\cancel{\begin{pmatrix} 1+\sqrt{5}/2 & 1-\sqrt{5}/2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1+\sqrt{5}/2 & 0 \\ 0 & 1-\sqrt{5}/2 \end{pmatrix}^{2018}(1/\sqrt{5})\begin{pmatrix} 1 & -1-\sqrt{5}/2 \\ -1 & 1+\sqrt{5}/2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

$\cancel{F_{2020} = (1\ 0)}$

$\cancel{= 1/\sqrt{5}(1\ 0)\begin{pmatrix} 1+\sqrt{5}/2 & 1-\sqrt{5}/2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1+\sqrt{5}/2 & 0 \\ 0 & 1-\sqrt{5}/2 \end{pmatrix}^{2018}\begin{pmatrix} 1 & -1-\sqrt{5}/2 \\ -1 & 1+\sqrt{5}/2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

$\cancel{= 1/\sqrt{5}(1\ 0) \ldots \begin{pmatrix} (1+\sqrt{5}/2)^{2018} & 0 \\ 0 & (1-\sqrt{5}/2)^{2018} \end{pmatrix}\begin{pmatrix} 1+\sqrt{5}/2 \\ -1+\sqrt{5}/2 \end{pmatrix}}$

$\cancel{= 1/\sqrt{5}(1\ 0) \ldots \begin{pmatrix} (1+\sqrt{5}/2)^{2018}(1+\sqrt{5}/2) \\ (1-\sqrt{5}/2)^{2018}(-1+\sqrt{5}/2) \end{pmatrix}}$

$\cancel{= 1/\sqrt{5}(1\ 0) \ldots \begin{pmatrix} \lambda_1^{2018}\lambda_2 \\ -\lambda_2^{2019} \end{pmatrix}}$

$\cancel{= 1/\sqrt{5}(1\ 0)\begin{pmatrix} \lambda_1^{2019}\lambda_2 - \lambda_2^{2020} \\ \lambda_1^{2018}\lambda_2 - \lambda_2^{2019} \end{pmatrix}}$

$F_{2020} = (1\ 0)A^{2018}u_2$

$= (1\ 0)VDV^{-1}u_2$

$= (1\ 0)\begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^{2018}(1/\sqrt{5})\begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$= (1\ 0)\begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \lambda_1^{2018} & 0 \\ 0 & \lambda_2^{2018} \end{pmatrix}(1/\sqrt{5})\begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix}$

$= 1/\sqrt{5}\,(1\ 0)\begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \lambda_1^{2019} & 0 \\ 0 & -\lambda_2^{2019} \end{pmatrix}$

$= 1/\sqrt{5}\,(1\ 0)\begin{pmatrix} \lambda_1^{2020} - \lambda_2^{2020} \\ \lambda_1^{2019} - \lambda_2^{2019} \end{pmatrix}$

$= 1/\sqrt{5}\,(\lambda_1^{2020} - \lambda_2^{2020})$

bi The equations are polynomials, so each is C^1 (hence C^1 in an open ball around $(0,1,0,1)$)

$x^5 + sy^2 + s^2 + 2tx\,|_{0,1,0,1} = 0$

$2xy^2 + s^2x + t - 1\,|_{0,1,0,1} = 0$

$(0,1,0,1)$ solves the system of equations.

$D_{x,y}\vec{f} = \begin{pmatrix} 5x^4+2t & 2sy \\ 2y^2+s^2 & 4xy \end{pmatrix}$

$\det D_{x,y}\vec{f}(0,1,0,1) = \det\begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix} = \cancel{0}\ 0-0 = 0 \Rightarrow$

$D_{x,y}\vec{f}^{(0,1,0,1)}$ is not invertible, so the IFT is not applicable to define x, y as a function of s, t in a neighbourhood of $(0,1,0,1)$.

ii $\det D_{x,y}\vec{f}(0,1,-1,1) = \det\begin{pmatrix} 2 & -2 \\ 3 & 0 \end{pmatrix} = 6 \neq 0$

compute $D^{-1}_{x,y}\vec{f}$ by Gauss-Jordan elimination

$\left(\begin{array}{cc|cc} 2 & -2 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\substack{R_1 \leftarrow R_1 - 2/3R_2 \\ R_2 \leftarrow 1/3R_2}} \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1/3 \\ 0 & -2 & 1 & -2/3 \end{array}\right) \xrightarrow{R_2 \leftarrow -1/2R_2}$

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 1/3 \\ 0 & 1 & -1/2 & 1/3 \end{array}\right)$$

$$D^{-1}_{x,y}\vec{f} = \begin{pmatrix} 0 & 1/3 \\ -1/2 & 1/3 \end{pmatrix}$$

$$D_{s,t}\vec{f} = \begin{pmatrix} y^2+2s & 2x \\ 2xs & 1 \end{pmatrix}$$

$$D_{s,t}\vec{f}\,(0,1,-1,1) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{aligned} D_{s,t}\vec{g}\,(-1,1) &= -D^{-1}_{x,y}\vec{f}\,(0,1,-1,1)\, D_{s,t}\vec{f}\,(0,1,-1,1) \\ &= -\begin{pmatrix} 0 & 1/3 \\ -1/2 & 1/3 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= -\begin{pmatrix} 0 & 1/3 \\ 1/2 & 1/3 \end{pmatrix} \end{aligned}$$

$\partial x/\partial s = 0$, $\partial x/\partial t = -1/3$, $\partial y/\partial s = -1/2$, $\partial y/\partial t = -1/3$

2ai $f:\mathbb{R}^n \to \mathbb{R}$ is concave iff for all $\vec{x}, \vec{x}' \in \mathbb{R}^n$, for all $t \in (0,1)$, $f(t\vec{x} + (1-t)\vec{x}') \geq tf(\vec{x}) + (1-t)f(\vec{x}')$.

ii f is concave $\Rightarrow \forall \vec{x}, \vec{x}' \in \mathbb{R}^n, t \in (0,1)$:
$f(t\vec{x} + (1-t)\vec{x}') \geq tf(\vec{x}) + (1-t)f(\vec{x}') \Rightarrow$
$\forall \ldots \; f(t\vec{x} + (1-t)\vec{x}') \geq t\min\{f(\vec{x}), g(\vec{x})\} + (1-t)\min\{f(\vec{x}'), g(\vec{x}')\}$

g is concave $\Rightarrow \forall \ldots \; g(t\vec{x} + (1-t)\vec{x}') \geq t\min\{f(\vec{x}), g(\vec{x})\} + (1-t)\min\{f(\vec{x}'), g(\vec{x}')\}$

$\Rightarrow \forall \ldots \; \min\{f(t\vec{x} + (1-t)\vec{x}'), g(t\vec{x} + (1-t)\vec{x}')\} \geq t\min\{f(\vec{x}), g(\vec{x})\} + (1-t)\min\{f(\vec{x}'), g(\vec{x}')\} \Rightarrow$
$\min\{f, g\}$ is concave.

iii $f(x,y) = \ln x + 3\ln y \quad (x>0, y>0)$
$Df(x,y) = (1/x, 3/y)$
$D^2f(x,y) = \begin{pmatrix} -x^{-2} & 0 \\ 0 & -3y^{-2} \end{pmatrix}$
$\mathrm{tr}\, D^2f(x,y) = -x^{-2} - 3y^{-2} < 0$
$\det D^2f(x,y) = 3x^{-2}y^{-2} > 0$
Both eigenvalues of $D^2f(x,y)$ are strictly negative, $D^2f(x,y)$ is negative definite, f is strictly concave.

$g(x,y) = xy^3 \quad (x,y \in \mathbb{R})$
$Dg(x,y) = (y^3 \quad 3xy^2)$
$D^2g(x,y) = \begin{pmatrix} 0 & 3y^2 \\ 3y^2 & 6xy \end{pmatrix}$
$\mathrm{tr}\, D^2g(x,y) = 6xy$
$\det D^2g(x,y) = -9y^4 < 0$
The eigenvalues have different signs, $D^2g(x,y)$ is indefinite, g is not concave.

$h(x,y) = -x^2 - 3y^2 + 5xy \quad (x,y \in \mathbb{R})$
$Dh(x,y) = (-2x + 5y \quad -6y + 5x)$
$D^2h(x,y) = \begin{pmatrix} -2 & 5 \\ 5 & -6 \end{pmatrix}$
$\mathrm{tr}\, D^2h(x,y) = -8 < 0$
$\det D^2h(x,y) = 12 - 25 = -13 < 0$
The eigenvalues have different signs, $D^2h(x,y)$ is indefinite, h is not concave.

bi $MU_x = \partial u/\partial x = 1/x$, $MU_y = \partial u/\partial y = 3/y$
Marginal utilities approach infinity as the amount of each good approaches zero, so positivity constraints never bind.

ii $\max_{x,y} \ln x + 3\ln y$ s.t.
$KC: x + 4y \leq 20$
$LC: 4x + y \leq 20$
$\mathcal{L} = \ln x + 3\ln y - \lambda_K(x + 4y - 20) - \lambda_L(4x + y - 20)$
$FOC_x: 1/x - \lambda_K - 4\lambda_L = 0$
$FOC_y: 3/y - 4\lambda_K - \lambda_L = 0$
$CS_K: \lambda_K \geq 0,\ x + 4y \leq 20,\ \lambda_K(x + 4y - 20) = 0$
$CS_L: \lambda_L \geq 0,\ 4x + y \leq 20,\ \lambda_L(4x + y - 20) = 0$

λ_K is the marginal utility from an additional unit of capital, λ_L is the marginal utility from an additional unit of labour.

iii Suppose $\lambda_K > 0, \lambda_L = 0$. By FOC_x, FOC_y,
$1/x = \lambda_K,\ 3/y = 4\lambda_K \Rightarrow 3/y = 4/x \Rightarrow x = 4/3\, y$
By CS_K, $x + 4y = 20 \Rightarrow 16/3\, y = 20 \Rightarrow y = 60/16$
$\Rightarrow x = 80/16 = 5 \Rightarrow x + 4y = 400/16 = 25 > 20$,
CS_L is violated. There is no solution where $\lambda_K > 0, \lambda_L = 0$.

Suppose $\lambda_K = 0, \lambda_L > 0$. By FOC_x, FOC_y,
$1/x = 4\lambda_L,\ 3/y = \lambda_L \Rightarrow 1/x = 12/y \Rightarrow x = 1/12\, y$.
By CS_L, $4x + y = 20 \Rightarrow 4/3\, y = 20 \Rightarrow y = 15$
$\Rightarrow x = 5/4 \Rightarrow x + 4y = 60 + 5/4 > 20$, CS_K is violated. There is no solution where $\lambda_K = 0, \lambda_L > 0$.

Suppose $\lambda_K = \lambda_L = 0$. By FOC_x, $1/x = 0$, x is undefined. There is no solution where $\lambda_K = \lambda_L = 0$.

Suppose $\lambda_K, \lambda_L > 0$. By CS_L, CS_K, $x + 4y = 20$, $4x + y = 20 \Rightarrow x = y = 4$. By FOC_x, FOC_y,
$1/4 - \lambda_K - 4\lambda_L = 0,\ 3/4 - 4\lambda_K - \lambda_L = 0 \Rightarrow$
$1/2 - 3\lambda_K + 3\lambda_L = 0 \Rightarrow \lambda_K - \lambda_L = 1/6 \Rightarrow$
$1/4 - (\lambda_L + 1/6) - 4\lambda_L = \cancel{12/24}\ \cancel{5/12 = 5\lambda_L \Rightarrow \lambda_L = 1/12}$
$\cancel{\Rightarrow \lambda_K}\quad 1/12 = 5\lambda_L \Rightarrow \lambda_L = 1/60 \Rightarrow \lambda_K = 11/60$
KT-FOCs are uniquely satisfied at $(x,y) = (4,4)$ with $\lambda_L = 1/60$, $\lambda_K = 11/60$.

iv From (aiii), ~~f is c~~ the objective function is concave. Each constraint is linear hence convex. The optimisation problem is convex. So KT-FOCs are sufficient for a maximum. The constraint set has non-empty interior, so the KT-FOCs are also necessary. The unique solution corresponds to a unique global maximum. The solution is optimal.

v The analysis would be unchanged. $\ln x + 3\ln y = \ln(xy^3)$. $\ln$ is a monotonic transformation, so the maxima of $\ln x + 3\ln y$ and xy^3 coincide.

3a $L_1 \succ L_2 \iff$ (by definition of $\succ$)
$L_1 \succsim L_2$ and $L_2 \not\succsim L_1 \iff$ (by independence)
$pL_1+(1-p)L \succsim pL_2+(1-p)L$
and $pL_2+(1-p)L \not\succsim pL_1+(1-p)L \iff$ (by definition)
$pL_1+(1-p)L \succ pL_2+(1-p)L$

$L_1 \sim L_2 \iff$ (by definition of $\sim$)
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$pL_1+(1-p)L \succsim pL_2+(1-p)L$
and $pL_2+(1-p)L \succsim pL_1+(1-p)L \iff$ (by definition)
$pL_1+(1-p)L \sim pL_2+(1-p)L$

$L_1 \succsim L_2 \Rightarrow$ (by independence)
$pL_1+(1-p)L_3 \succsim pL_2+(1-p)L_3$
$L_3 \succsim L_4 \Rightarrow$ (by independence)
$pL_2+(1-p)L_3 \succsim pL_2+(1-p)L_4$
$\Rightarrow$ (by transitivity)
$pL_1+(1-p)L_3 \succsim pL_2+(1-p)L_4$

b $L_1 = (3/15, 3/15, 3/15, 3/15, 3/15)$
$L_2 = (1/15, 7/15, 0/15, 5/15, 2/15)$
L_1 is a mean-preserving spread of L_2 obtained by reallocating $2/15$ units of probability mass from ~~x~~ outcome -1 to each of outcome -2 and outcome 0 and by reallocating $1/15$ units of probability mass from outcome $+1$ to each of outcome 0 and outcome $+2$.

L_2 SOSDs L_1, any risk averse expected utility maximiser prefers L_2 to L_1.

ci Let L_p and ~~L_p~~ L_f respectively denote the ~~lotteries~~ lottery in poor weather and the lottery in fine weather. $L_p \succ_{FOSD} L_f$ iff ~~L_p~~ L_f allocates lower probabilities to worse outcomes. This is iff the cumulative distribution function associated with L_f is weakly less than that ~~of~~ for L_p, for all ~~x~~ ~~poss values~~ outcome. ~~ie.~~ ~~$\pi_1 \leq$~~

This is iff $p_1 \leq 1/3$, $p_1+p_2 \leq 1/3+1/3$, $p_1+p_2+p_3 \leq 1/3+1/3+1/3$. CR is not monotonically increasing iff it is not the case that $p_1 \leq p_2 \leq p_3$. So, for example $p_1 = 0.2$, $p_2 = 0.41$, $p_3 = 0.39$ satisfies both requirements.

ii Suppose CR is monotonically increasing, i.e. $p_1/\tfrac{1}{3} \leq p_2/\tfrac{1}{3} \leq p_3/\tfrac{1}{3} \iff p_1 \leq p_2 \leq p_3$. Given that $p_1+p_2+p_3=1$, $p_3 \geq 1/3$, otherwise $p_1 \leq p_2 \leq p_3 < 1/3$ so $p_1+p_2+p_3<1$. Then $p_1+p_2 \leq 2/3$. and $p_1 \leq p_2$ so $p_1 \leq 1/3$ otherwise $p_2 \geq p_1 > 1/3$ so $p_1+p_2 > 2/3$. So we have $p_1 \leq 1/3$, $p_1+p_2 \leq 2/3$, $p_1+p_2+p_3=1$. The associated cdf is $F_f(x)$ = ~~1/3 f p_1 for $x \leq \pi_1$, p_1+p_2 for~~ 0 for $x<\pi_1$, p_1 for $\pi_1 \leq x < \pi_2$, p_1+p_2 for $\pi_2 \leq x < \pi_3$, 1 for all other x. This is weakly less than the cdf for L_p for all x, which is $F_p(x) = 0$ for $x < \pi_1$, $1/3$ for $\pi_1 \leq x < \pi_2$, $2/3$ for $\pi_2 \leq x < \pi_3$, 1 for all other x. So L_f FOSDs L_p if CR is monotonically increasing.

Date

4a When effort is observable, principal P offers a contract that just satisfies agent A's participation constraint PC. Such a contract has the form (w, e). Any candidate optimum that strictly satisfies PC fails to deviation to a less generous wage w by sufficiently small amount ε.

Suppose P induces $e=2$. PC binds, $u(w,e)=0 \iff \sqrt{w}-2=0 \iff w=4$. Then P has expected gross (of wage) profit ~~#~~ $E\pi = \frac{1}{4}4 + \frac{3}{4}40 = 31$ and expected net profit $E\pi - w > 31-4 = 27$.

Suppose P induces $e=0$. PC binds. $u(w,e)=0 \iff \sqrt{w}-e=0 \iff w=0$. Then P has $E\pi = \cancel{3/4}$ $\frac{3}{4}4 + \frac{1}{4}40 = 13$ and $E\pi - w = 13$.

So it is optimal to induce $e=2$ and P optimally does so by offering contract $(w,e) = (\cancel{0}\ 4, 2)$.

b When effort is unobservable, a contract to ~~#~~ induce e must satisfy PC and ~~IC, the~~ incentive constraint IC. To induce $e=0$, the optimal contract is unchanged from the case with observable effort because IC does not bind. Under a fixed wage, given costly effort, $e=0$ is strictly incentive compatible, so P induces ~~x~~ $e=0$ with fixed wage schedule $(w_L, w_H) = (0,0)$.

~~To induce~~ In inducing $e=2$, both PC and IC bind. Any candidate optimum such that PC does not bind fails to deviation by reducing w_L by small amount ε such that PC remains satisfied. Such deviation loosens IC ~~because there expec~~ and yields ~~higher~~ lower expected wage hence higher expected profit. Any candidate optimum such that IC ~~fa~~ does not bind fails to deviation consisting in (1) a small mean-preserving (given 1/4, 3/4 probabilities) contraction of w_L, w_H, ~~the~~ which continues to satisfy IC given that it is initially "slack" and loosens PC given concavity of u in w, and (2) a small decrease ~~increase~~ in each of w_L and w_H, which satisfies IC and PC given both are "slack".

PC: $\frac{1}{4}(\sqrt{w_L}-2) + \frac{3}{4}(\sqrt{w_H}-2) \geq 0$

IC: $\frac{1}{4}(\sqrt{w_L}-2) + \frac{3}{4}(\sqrt{w_H}-2) \geq \frac{3}{4}(\sqrt{w_L}) + \frac{1}{4}(\sqrt{w_H})$

Both bind.

PC $\Rightarrow \frac{1}{4}\sqrt{w_L} + \frac{3}{4}\sqrt{w_H} = 2$

IC $\Rightarrow \frac{1}{4}\sqrt{w_L} + \frac{3}{4}\sqrt{w_H} - 2 = \frac{3}{4}\sqrt{w_L} + \frac{1}{4}\sqrt{w_H}$

$\Rightarrow \frac{1}{2}\sqrt{w_H} - 2 = \frac{1}{2}\sqrt{w_L} \Rightarrow \sqrt{w_H} = \sqrt{w_L} + 4$

$\Rightarrow \sqrt{w_L} + 3 = 2$ (reject)

$\Rightarrow \sqrt{w_L} + 3 = 2$ (reject)

So the ~~po~~ positivity constraint on w_L will bind, then IC binding implies $\sqrt{w_H} = 4 \Rightarrow w_H = 16$. The contract will strictly satisfy PC but there is no strictly profitable deviation because the positivity constraint binds.

Under this contract to induce high wage, $E(\pi - w) = \frac{1}{4}(4-0) + \frac{3}{4}(40-16) = 1+18 = 19 > 13$. #

It remains optimal to induce high effort. P offers $(w_L, w_H) = (0, 16)$ and suffers agency cost $27-19=8$. This cost is incurred because a variable wage scheme is necessary to induce high effort, i.e. make it incentive compatible. Then A is made to bear risk, but A is risk averse and must be compensated for risk-bearing with a higher expected wage. This generally holds but in this case expected wage is also pushed up by the positivity constraints on wage.

c If the machine were bad P proposes $w=0$ because profit is independent of effort, so P is better off inducing low effort (because effort is costly to A and A must be compensated), and P optimally induces low effort with the just incentive compatible fixed wage ~~w~~ $w=0$ found in (a).

If the machine were not bad, again to induce low effort, the optimal contract is the fixed wage contract $(w_L, w_H) = (0,0)$. $E(\pi - w) = \frac{2}{3}(4-0) + \frac{1}{3}(40-0) = 16$.

To induce high effort, ~~again PC and IC. IC and positivity~~ constraint binds, so the ~~optimal contract is $(w_L, w_H) =$ again.~~ again IC and positivity constraint bind.

IC: $\sqrt{w_H} - 2 \geq \frac{2}{3}\sqrt{w_L} + \frac{1}{3}\sqrt{w_H}$

$\Rightarrow \sqrt{w_H} - 2 = \frac{1}{3}\sqrt{w_H} \Rightarrow \frac{2}{3}\sqrt{w_H} = 2 \Rightarrow w_H = 9$, $w_L = 0$

$E(\pi - w) = 40 - 9 = \cancel{31}\ 31$

~~Eff cost~~ In ex ante expectation, when the state of the machine is unknown, $E(\pi - w) = \frac{1}{4}(16) + \frac{3}{4}(31) = \frac{109}{4} = \cancel{27}\ 27\frac{1}{4}$

From (b), under unobservable effort, maximum expected net profit without inspection is 19. P is willing to pay up to $27\frac{1}{4} - 19 = 8\frac{1}{4}$ for inspection.

P ~~induces~~ optimally induces $e=2$ when either (1) it is known that the machine is not bad or (2) no inspection has happened. In (1), ~~that~~ there is no agency cost because the outcome is certain and A is not made to bear risk. Though the wage schedule offered is still variable and expected wage is higher than in the observable effort case, it is lower than in (2) because ~~A~~ A faces no risk with $e=2$.