Sider Axiomatic Proof Exercises 1

64 K: $\Box(\phi\to\psi)\to(\Box\phi\to\Box\psi)$ K◇: $\Box(\phi\to\psi)\to(\Diamond\phi\to\Diamond\psi)$

D: $\Box\phi\to\Diamond\phi$

T: $\Box\phi\to\phi$ T◇: $\phi\to\Diamond\phi$

B: $\Diamond\Box\phi\to\phi$ B◇: $\phi\to\Box\Diamond\phi$

S4: $\Box\phi\to\Box\Box\phi$ S4◇: $\Diamond\Diamond\phi\to\Diamond\phi$

S5: $\Diamond\Box\phi\to\Box\phi$ S5◇: $\Diamond\phi\to\Box\Diamond\phi$

MP: ϕ, $\phi\to\psi \Rightarrow \psi$

NEC: $\phi \Rightarrow \Box\phi$

$\vdash_K \Box((P\to Q)\to(P\to P))$

(1) $(P\to((P\to P)\to P))\to((P\to(P\to P))\to(P\to P))$ (PL 2)

(2) $P\to((P\to P)\to P)$ (PL 1)

(3) $(P\to(P\to P))\to(P\to P)$ (1, 2, MP)

(4) $(P\to(P\to P))$ (PL 1)

(5) $P\to P$ (3, 4, MP)

(6) $(P\to P)\to((P\to Q)\to(P\to P))$ (PL 1)

(7) $(P\to Q)\to(P\to P)$ (5, 6, MP)

(8) $\Box((P\to Q)\to(P\to P))$ (7, NEC)

5 $\vdash_K \sim\Diamond(P\wedge\sim P)$ Dualize/Abbreviate

(1) $\sim(P\wedge\sim P)$ (PL LEM)

(2) $\Box\sim(P\wedge\sim P)$ (1, NEC)

(3) $\sim\sim\Box\sim(P\wedge\sim P)$ (2, PL DNI)

$= \sim\Diamond(P\wedge\sim P)$

6 $\vdash_K \Box(P\wedge Q)\to(\Box P\wedge\Box Q)$ Conjunctive consequent

(1) $(P\wedge Q)\to P$ (PL)

(2) $\Box(P\wedge Q)\to\Box P$ (1, NEC)

(3) $\Box(P\wedge Q)\to\Box Q$ (PL, NEC)

(4) $\Box(P\wedge Q)\to(\Box P\wedge\Box Q)$ (2, 3, PL composition)

7 $\vdash_K (\Box P\vee\Box Q)\to\Box(P\vee Q)$ Disjunctive antecedent

(1) $P\to(P\vee Q)$ (PL)

(2) $\Box P\to\Box(P\vee Q)$ (1, NEC, K, MP)

(3) $\Box Q\to\Box(P\vee Q)$ (PL, NEC, K, MP)

(4) $(\Box P\vee\Box Q)\to\Box(P\vee Q)$ (2, 3, PL dilemma)

8 $\vdash_K (\Box P\wedge\Box Q)\to\Box(P\wedge Q)$ Conjunctive antecedent

~~$= \sim(\Box P\to\sim\Box Q)\to\Box\sim(P\to\sim Q)$~~ import/export

(1) $P\to(Q\to(P\wedge Q))$ (PL)

(2) $\Box P\to\Box(Q\to(P\wedge Q))$ (1, NEC, K, MP)

(3) $\Box(Q\to(P\wedge Q))\to(\Box Q\to\Box(P\wedge Q))$ (K)

(4) $\Box P\to(\Box Q\to\Box(P\wedge Q))$ (2, 3, PL syllogism)

(5) $(\Box P\wedge\Box Q)\to\Box(P\wedge Q)$ (4, PL import/export)

9 $\vdash_K \Box\Box(P\wedge Q)\to\Box\Box P$

(1) $(P\wedge Q)\to P$ (PL)

(2) $\Box(P\wedge Q)\to\Box P$ (1, NEC, K, MP)

(3) $\Box\Box(P\wedge Q)\to\Box\Box P$ (2, NEC, K, MP)

K◇ $\vdash_K \Box(\phi\to\psi)\to(\Diamond\phi\to\Diamond\psi)$

$= \Box(\phi\to\psi)\to(\sim\Box\sim\phi\to\sim\Box\sim\psi)$

(1) $(\phi\to\psi)\to(\sim\psi\to\sim\phi)$ (PL contraposition)

(2) $\Box(\phi\to\psi)\to\Box(\sim\psi\to\sim\phi)$ (1, NEC, K, MP)

(3) $\Box(\sim\psi\to\sim\phi)\to(\Box\sim\psi\to\Box\sim\phi)$ (K)

(4) $(\Box\sim\psi\to\Box\sim\phi)\to(\sim\Box\sim\phi\to\sim\Box\sim\psi)$

(PL contraposition)

(5) $\Box(\phi\to\psi)\to(\sim\Box\sim\phi\to\sim\Box\sim\psi)$ (2, 3, 4, PL syllogism)

contraposition, Becker, K, contraposition

10 $\vdash_K \Box P\to(\Diamond Q\to\Diamond(P\wedge Q))$ K, K◇

(1) $P\to(Q\to(P\wedge Q))$ (PL)

(2) $\Box P\to\Box(Q\to(P\wedge Q))$ (1, NEC, K, MP)

(3) $\Box(Q\to(P\wedge Q))\to(\Diamond Q\to\Diamond(P\wedge Q))$ (K◇)

(4) $\Box P\to(\Diamond Q\to\Diamond(P\wedge Q))$ (2, 3, PL syllogism)

11 $\vdash_K \Diamond P\to(\Box Q\to\Diamond(P\wedge Q))$ K, K◇, import/export

(1) $Q\to(P\to(P\wedge Q))$ (PL)

(2) $\Box Q\to\Box(P\to(P\wedge Q))$ (1, NEC, K, MP)

(3) $\Box(P\to(P\wedge Q))\to(\Diamond P\to\Diamond(P\wedge Q))$ (K◇)

(4) $\Box Q\to(\Diamond P\to\Diamond(P\wedge Q))$ (2, 3, PL syllogism)

(5) $\Diamond P\to(\Box Q\to\Diamond(P\wedge Q))$ (4, PL import/export)

12 $\vdash_K \sim\Box\phi\to\Diamond\sim\phi$

$= \sim\Box\phi\to\sim\Box\sim\sim\phi$

(1) $\sim\sim\phi\to\phi$ (PL DNE)

(2) $\Box\sim\sim\phi\to\Box\phi$ (1, NEC, K, MP)

(3) $\sim\Box\phi\to\sim\Box\sim\sim\phi$ (2, PL contraposition)

MN $\vdash_K \sim\Diamond\phi\to\Box\sim\phi$

(1) $\sim\sim\Box\sim\phi\to\Box\sim\phi$ (PL DNE)

$\vdash_K \Box\sim\phi\to\sim\Diamond\phi$

(1) $\Box\sim\phi\to\sim\sim\Box\sim\phi$ (PL DNI)

$\vdash_K \Diamond\sim\phi\to\sim\Box\phi$

(1) $\phi\to\sim\sim\phi$ (PL DNI)

(2) $\Box\phi\to\Box\sim\sim\phi$ (1, NEC, K, MP)

(3) $\sim\Box\sim\sim\phi\to\sim\Box\phi$ (2, PL contraposition)

13 $\vdash_K \Box\Diamond\Box\sim P\to\sim\Diamond\Box\Diamond P$

(1) $\Box\Diamond\Box\sim P\to\sim\sim\Box\Diamond\Box\sim P$ (PL DNI)

$= \Box\Diamond\Box\sim P\to\sim\Diamond\Box\Diamond P$

5a $\vdash_K \Diamond(P\wedge\exists)\to(\Diamond P\wedge\Diamond Q)$ Conjunctive consequent

(1) $(P\wedge Q)\to P$ (PL) $\Diamond\phi\to\Diamond\psi \Rightarrow K\Diamond$

(2) $\Box((P\wedge Q)\to P)$ (1, NEC)

(3) $\Box((P\wedge Q)\to P)\to(\Diamond(P\wedge Q)\to\Diamond P)$ (K◇)

(4) $\Diamond(P\wedge\exists)\to\Diamond P$ (2, 3, MP)

(5) $\Diamond(P\wedge Q)\to\Diamond\exists$ (PL, NEC, K◇, MP)

(6) $\Diamond(P\wedge Q)\to(\Diamond P\wedge\Diamond Q)$ (PL composition)

b $\vdash_K \Box\sim P\to\Box(P\to Q)$

(1) $\sim P\to(P\to Q)$ (PL)

(2) $\Box\sim P\to\Box(P\to Q)$ (1, NEC, K, MP)

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c $\vdash_K \sim\Diamond(Q\wedge R)\leftrightarrow\Box(Q\rightarrow\sim R)$

(1) $\sim(Q\wedge R)\rightarrow(Q\rightarrow\sim R)$ (PL)

(2) $\Box\sim(Q\wedge R)\rightarrow\Box(Q\rightarrow\sim R)$ (1, NEC, K, MP)

(3) $\sim\sim\Box\sim(Q\wedge R)\rightarrow\Box\sim(Q\wedge R)$ (PL DNE)

(4) $\sim\Diamond(Q\wedge R)\rightarrow\Box(Q\rightarrow\sim R)$ (2, 3, PL syllogism)

(5) $(Q\rightarrow\sim R)\rightarrow\sim(Q\wedge R)$ (PL)

(6) $\Box(Q\rightarrow\sim R)\rightarrow\Box\sim(Q\wedge R)$ (5, NEC)

(7) $\Box\sim(Q\wedge R)\rightarrow\sim\sim\Box\sim(Q\wedge R)$ (PL DNI)

(8) $\Box(Q\rightarrow\sim R)\rightarrow\sim\Diamond(Q\wedge R)$ (6, 7, PL syllogism)

(9) $\sim\Diamond(Q\wedge R)\leftrightarrow\Box(Q\rightarrow\sim R)$ (PL)

d $\vdash_K \Box(P\leftrightarrow Q)\rightarrow(\Box P\leftrightarrow\Box Q)$

(1) $(P\leftrightarrow Q)\rightarrow(P\rightarrow Q)$ (PL)

(2) $\Box(P\leftrightarrow Q)\rightarrow\Box(P\rightarrow Q)$ (1, NEC, K, MP)

(3) $\Box(P\rightarrow Q)\rightarrow(\Box P\rightarrow\Box Q)$ (K)

(4) $\Box(P\leftrightarrow Q)\rightarrow(\Box P\rightarrow\Box Q)$ (2, 3, PL syllogism)

(5) $\Box(P\leftrightarrow Q)\rightarrow(\Box Q\rightarrow\Box P)$ (PL, NEC, K, MP, PL syllogism)

(6) $\Box(P\leftrightarrow Q)\rightarrow(\Box P\leftrightarrow\Box Q)$ (PL)

e $\vdash_K [\Box(P\rightarrow Q)\wedge\Box(P\rightarrow\sim Q)]\rightarrow\sim\Diamond P$

(1) $(P\rightarrow Q)\rightarrow((P\rightarrow\sim Q)\rightarrow\sim P)$ (PL)

(2) $\Box(P\rightarrow Q)\rightarrow\Box((P\rightarrow\sim Q)\rightarrow\sim P)$ (1, NEC, K, MP)

(3) $\Box((P\rightarrow\sim Q)\rightarrow\sim P)\rightarrow(\Box(P\rightarrow\sim Q)\rightarrow\Box\sim P)$ (K)

(4) $\Box(P\rightarrow Q)\rightarrow(\Box(P\rightarrow\sim Q)\rightarrow\Box\sim P)$
(2, 3, PL syllogism)

(5) $[\Box(P\rightarrow Q)\wedge\Box(P\rightarrow\sim Q)]\rightarrow\Box\sim P$ (4, PL import/export)

(6) $\Box\sim P\rightarrow\sim\sim\Box\sim P$ (PL DNI)

(7) $[\Box(P\rightarrow Q)\wedge\Box(P\rightarrow\sim Q)]\rightarrow\sim\Diamond P$
(5, 6, PL syllogism)

f $\vdash_K (\Box P\wedge\Box Q)\rightarrow\Box(P\leftrightarrow Q)$

(1) $P\rightarrow(Q\rightarrow(P\leftrightarrow Q))$ (PL)

(2) $\Box P\rightarrow\Box(Q\rightarrow(P\leftrightarrow Q))$ (1, NEC, K, MP)

(3) $\Box(Q\rightarrow(P\leftrightarrow Q))\rightarrow(\Box Q\rightarrow\Box(P\leftrightarrow Q))$ (K)

(4) $\Box P\rightarrow(\Box Q\rightarrow\Box(P\leftrightarrow Q))$ (2, 3, PL syllogism)

(5) $(\Box P\wedge\Box Q)\rightarrow\Box(P\leftrightarrow Q)$ (PL import/export)

g $\vdash_K \Diamond(P\rightarrow Q)\leftrightarrow(\Box P\rightarrow\Diamond Q)$

(1) $P\rightarrow((P\rightarrow Q)\rightarrow Q)$ (PL)

(2) $\Box P\rightarrow\Box((P\rightarrow Q)\rightarrow Q)$ (1, NEC, K, MP)

(3) $\Box((P\rightarrow Q)\rightarrow Q)\rightarrow(\Diamond(P\rightarrow Q)\rightarrow\Diamond Q)$ (K$\Diamond$)

(4) $\Box P\rightarrow(\Diamond(P\rightarrow Q)\rightarrow\Diamond Q)$ (2, 3, PL syllogism)

(5) $\Diamond(P\rightarrow Q)\rightarrow(\Box P\rightarrow\Diamond Q)$ (4, PL import/export)

(6) $P\rightarrow(Q\rightarrow(P\rightarrow Q))$ (PL)

(7) $\Box P\rightarrow\Box(Q\rightarrow(P\rightarrow Q))$ (6, NEC, K, MP)

(8) $\Box(Q\rightarrow(P\rightarrow Q))\rightarrow(\Diamond Q\rightarrow\Diamond(P\rightarrow Q))$ (K$\Diamond$)

(9) $\Box P\rightarrow(\Diamond Q\rightarrow\Diamond(P\rightarrow Q))$ (7, 8, PL syllogism)

(10) $(\Box P\rightarrow\Diamond Q)\rightarrow\Diamond(P\rightarrow Q)$ (9, PL import/export)

(11) $\Diamond(P\rightarrow Q)\leftrightarrow(\Box P\rightarrow\Diamond Q)$ (5, 10, PL)

h $\vdash_K \Diamond P\rightarrow(\Box Q\rightarrow\Diamond Q)$

(1) $Q\rightarrow(P\rightarrow Q)$ (PL)

(2) $\Box Q\rightarrow\Box(P\rightarrow Q)$ (1, NEC, K, MP)

(3) $\Box(P\rightarrow Q)\rightarrow(\Diamond P\rightarrow\Diamond Q)$ (K$\Diamond$)

(4) $\Box Q\rightarrow(\Diamond P\rightarrow\Diamond Q)$ (2, 3, PL syllogism)

(5) $\Diamond P\rightarrow(\Box Q\rightarrow\Diamond Q)$ (4, PL import/export)

i $\vdash_K \sim\Diamond\Diamond\Box(P\vee Q)\rightarrow\Box\Box\Diamond\sim P$

$=\sim\sim\Box\sim\sim\Box\sim\Box(P\vee Q)\rightarrow\Box\Box\sim\Box\sim\sim P$

(1) $\sim\sim P\rightarrow(P\vee Q)$ (PL)

(2) $\Box\sim\sim P\rightarrow\Box(P\vee Q)$ (1, NEC, K, MP)

(3) $\sim\Box(P\vee Q)\rightarrow\sim\Box\sim\sim P$ (2, PL contraposition)

(4) $\Box\sim\Box(P\vee Q)\rightarrow\Box\sim\Box\sim\sim P$ (3, NEC, K, MP)

(5) $\sim\sim\Box\sim\Box(P\vee Q)\rightarrow\Box\sim\Box(P\vee Q)$ (PL DNE)

(6) $\sim\sim\Box\sim\Box(P\vee Q)\rightarrow\Box\sim\Box\sim\sim P$
(4, 5, PL syllogism)

(7) $\Box\sim\sim\Box\sim\Box(P\vee Q)\rightarrow\Box\Box\sim\Box\sim\sim P$ (6, NEC, K, MP)

(8) $\sim\sim\Box\sim\sim\Box\sim\Box(P\vee Q)\rightarrow\Box\sim\sim\Box\sim\Box(P\vee Q)$
(PL DNE)

(9) $\sim\Diamond\Diamond\Box(P\vee Q)\rightarrow\Box\Box\Diamond\sim P$ (7, 8, PL syllogism)