Game Theory Problem Set 3

1a

	L	R
T	$\underline{3+\varepsilon_2}$	0
	$\underline{1}$	0
B	0	$\underline{1}$
	0	$\underline{3+\varepsilon_1}$

Best responses underlined.

By inspection, (T,L) and (B,R) are the only pure strategy Nash equilibria where players play mutual best responses.

Suppose that there is some NE σ^* where Pl.1 ~~mixes~~ plays a mixed strategy, and plays action T with probability p and action B with probability $1-p$. Then, $\pi_1(T, \sigma_2^*) = \pi_1(B, \sigma_2^*)$, $q = (3+\varepsilon_1)(1-q)$, where q is the probability that strategy of ~~player~~ Pl. 2 σ_2^* at σ^* assigns to L such that Pl.1 has no profitable deviation $q = \cancel{1/4+\varepsilon_1}$ $3+\varepsilon_1/4+\varepsilon_1$. Then Pl.2 plays a mixed strategy at σ^*, so $\pi_2(L, \sigma_1^*) = \pi_2(R, \sigma_1^*)$, $(3+\varepsilon_2)p = 1-p$, $p = 1/4+\varepsilon_2$.

So ~~##~~ $\sigma^* = (\cancel{1/4}\ p \times T + (1-p) \times B, q \times L + (1-q) \times R)$ is a mixed strategy NE where $p = 1/4+\varepsilon_2$ and $q = \cancel{1/4+\varepsilon_1}$ $3+\varepsilon_1/4+\varepsilon_1$.

From the above argument if Pl.1 mixes, so does Pl.2. By symmetry, if Pl.2 mixes, so does Pl.1. Then, there are no hybrid NE.

2 Let s^* denote the pure BNE where Pl.1 plays T iff $\varepsilon_1 < \varepsilon_1^*$ and B otherwise and Pl.2 plays L iff $\varepsilon_2 > \varepsilon_2^*$ and R otherwise.

$\pi_1(T, s_2^*) = P(\varepsilon_2 > \varepsilon_2^*)(1) + P(\varepsilon_2 < \varepsilon_2^*)(0)$

$= \cancel{\varepsilon_2^*/\bar{\varepsilon}}\ 1 - \varepsilon_2^*/\bar{\varepsilon}$

$\pi_1(B, s_2^*) = P(\varepsilon_2 > \varepsilon_2^*)(0) + P(\varepsilon_2 < \varepsilon_2^*)(3+\varepsilon_1)$

$= \varepsilon_2^*/\bar{\varepsilon}\ \cancel{(3+\varepsilon_1)}\ (3+\varepsilon_1)$

$\pi_1(T, s_2^*) \geq \pi_1(B, s_2^*)$ iff

$1 - \varepsilon_2^*/\bar{\varepsilon} \geq \varepsilon_2^*/\bar{\varepsilon}\ \cancel{(3+\varepsilon_1)}\ (3+\varepsilon_1)$

$1 \geq \varepsilon_2^*/\bar{\varepsilon}(4+\varepsilon_1)$

$4+\varepsilon_1 \leq \bar{\varepsilon}/\varepsilon_2^*$

$\varepsilon_1 \leq \bar{\varepsilon}/\varepsilon_2^* - 4$

Since s^* is a BNE, Pl.1 has no profitable deviation from s_1^*, so under s_1^* Pl.1 plays T iff $\pi_1(T, s_2^*) \geq$ ~~π_1~~ $\pi_1(B, s_2^*)$ iff $\varepsilon_1 \leq \bar{\varepsilon}/\varepsilon_2^* - 4$ iff $\varepsilon_1 < \bar{\varepsilon}/\varepsilon_2^* - 4$, ~~$\varepsilon^*$~~ so $\varepsilon_1^* = \bar{\varepsilon}/\varepsilon_2^* - 4$

$\pi_2(L, s_1^*) = P(\varepsilon_1 < \varepsilon_1^*)(3+\varepsilon_2) + P(\varepsilon_1 > \varepsilon_1^*)(0)$

$= \varepsilon_1^*/\bar{\varepsilon}\ (3+\varepsilon_2)$

$\pi_2(R, s_1^*) = P(\varepsilon_1 < \varepsilon_1^*)(0) + P(\varepsilon_1 > \varepsilon_1^*)(1)$

$= 1 - \varepsilon_1^*/\bar{\varepsilon}$

$\pi_2(L, s_1^*) \geq \pi_2(R, s_1^*)$ iff

$\varepsilon_1^*/\bar{\varepsilon}(3+\varepsilon_2) \geq 1 - \varepsilon_1^*/\bar{\varepsilon}$

$\varepsilon_1^*/\bar{\varepsilon}(\cancel{3}\ 4+\varepsilon_2) \geq 1$

$4+\varepsilon_2 \geq \bar{\varepsilon}/\varepsilon_1^*$

$\varepsilon_2 \geq \bar{\varepsilon}/\varepsilon_1^* - 4$

$\varepsilon_2^* = \bar{\varepsilon}/\varepsilon_1^* - 4$

~~By substitution,~~

$\cancel{\varepsilon_1^* = \bar{\varepsilon}/(\bar{\varepsilon}/\varepsilon_1^* - 4) - 4}$

$\cancel{= \varepsilon_1^* - 4\ \ \bar{\varepsilon}\varepsilon_1^*/\bar{\varepsilon} - 4\varepsilon_1^*}$

Supposing that the BNE is symmetric, i.e. $\varepsilon_1^* = \varepsilon_2^*$,

$\varepsilon_1^* = \bar{\varepsilon}/\varepsilon_1^* - 4$,

$(\varepsilon_1^*+4)\varepsilon_1^* = \bar{\varepsilon}$

$(\varepsilon_1^*+2)^2 = \bar{\varepsilon}+4$

$\varepsilon_1^* = -2+\sqrt{\bar{\varepsilon}+4}$ or $-2-\sqrt{\bar{\varepsilon}+4}$ (reject since $\varepsilon_1^* \in [0,\bar{\varepsilon}]$)

Then $\varepsilon_2^* = \varepsilon_1^* = -2+\sqrt{\bar{\varepsilon}+4}$.

$\varepsilon_2^* = \bar{\varepsilon}/\varepsilon_1^* - 4$, so Pl.2 has no profitable deviation, and s^* is a BNE.

c Let p and q denote the ex ante probability that Pl.1 plays T and the ex ante probability that Pl.2 plays L respectively.

$p = P(\varepsilon_1 < \varepsilon_1^*) = \varepsilon_1^*/\bar{\varepsilon} = -2+\sqrt{\bar{\varepsilon}+4}/\bar{\varepsilon}$

$= (-2+\sqrt{\bar{\varepsilon}+4})(+2+\sqrt{\bar{\varepsilon}+4})/\bar{\varepsilon}(\sqrt{\bar{\varepsilon}+4}+2)$

$= 1/\sqrt{\bar{\varepsilon}+4}+2$

$q = P(\varepsilon_2 > \varepsilon_2^*) = 1-\varepsilon_2^*/\bar{\varepsilon} = 1 - 1/\sqrt{\bar{\varepsilon}+4}+2$

$\lim_{\bar{\varepsilon}\to 0} p = 1/4$

$\lim_{\bar{\varepsilon}\to 0} q = 3/4$

The (ex ante) probability distributions induced by the pure NE of the perturbed game converge to the probability distribution of the mixed NE in the unperturbed game (where $\varepsilon_1=\varepsilon_2=0$), where from the result in (a), $p = 1/4+\varepsilon_2 = 1/4$, $q = 3+\varepsilon_1/4+\varepsilon_1 = 3/4$.

d Mixed NE are difficult to justify because at the mixed NE, ~~each player has equal~~ for each player, each of the actions he mixes yields equal expected payoff, so each player has no incentive to mix over these actions in any particular way.

Harsanyi's purification theorem is that the probability distributions induced by the pure BNE of the perturbed game in which players use threshold strategies converges, as the perturbation vanishes, to the probability distributions of the mixed NE in the unperturbed strategic form game.

Harsanyi's purification theorem ~~can be interprete~~ justifies mixed NE in the sense that the players who mix can be understood as responding to an unobservable, vanishingly small payoff shock, i.e. a whim.

2a Players : $N = \{St, G\}$

Actions : $A_{St} = \{L, R\}$, $A_G = \{l, r\}$

States : $\Omega = \{StL, StR\}$

Signals : $t_{St}(StL) = StL$, $t_{St}(StR) = StR$, $t_G(StL) = t_G(StR)$

Beliefs : ~~$P_{St}(\omega = StL \mid StL\ t_{St}$~~

$P_G(t_{St} = StL \mid t_G) = P_G(t_{St} = StR \mid t_G) = 1/2$

Payoffs :

StL

	l	r
L	$\underline{0.3}$, 0.7	0, $\underline{1}$
R	0.2, $\underline{0.8}$	$\underline{0.4}$, 0.6

StR

	l	r
L	$\underline{0.4}$, 0.6	0.2, $\underline{0.8}$
R	0, $\underline{1}$	$\underline{0.3}$, 0.7

Best responses underlined

In a Bayesian ~~static~~ game, player i's strategy is a type-contingent plan of action. Since G has only one type, its strategy σ_G is some action ~~$a_G \in \{l$~~ $a_G \in A_G = \{l, r\}$ or some probability distribution over A_G. St's strategy σ_{St} is some pair $(\sigma_{StL}, \sigma_{StR})$ where St plays σ_{StL} if $t_{St} = StL$ and σ_{StR} if $t_{St} = StR$, and each of σ_{StL} and σ_{StR} is some action $a_{St} \in A_{St} = \{L, R\}$ or some probability distribution over A_{St}.

By inspection of the payoff tables above, St's best response ~~$B_{St}(a_G) = (R,R)$ if $a_G = l$~~

$B_{St}(\sigma_G) = (R,R)$ if $\sigma_G = l$

(L,L) if $\sigma_G = r$

G's best response

$B_G(\sigma_{St}) =$ ~~l if $\sigma_{St} = (R,R)$~~

l if $\sigma_{St} = (L,L)$

r if $\sigma_{St} = (R,R)$

$\{l, r\}$ if $\sigma_{St} =$ ~~(L,L) or (R,R)~~

(L,R) or (R,L)

~~The payoff table can be rewritten as~~

	~~l~~	~~r~~
~~LL~~	~~0.35~~	~~0.1~~
	~~0.65~~	~~0.9~~
~~LR~~	~~0.15~~	~~0.15~~
	~~0.85~~	~~0.85~~
~~RR~~	~~0.1~~	~~0.35~~
	~~0.9~~	~~0.65~~
~~RL~~	~~0.3~~	~~0.3~~
	~~0.7~~	~~0.7~~

By inspection, there are no mutual best responses in pure strategies, so there is no pure BNE.

b Let σ^* be a hybrid BNE where G ~~St~~ mixes l and r, ~~then σ^*_G is a pr~~ Let ~~p~~ q be the probability σ^*_G assigns to l and $1-q$ be the probability it assigns to r.

~~$\pi_{St}(L, \sigma^*_G; t_{St} = StL) = 0.7q + (1-q) = 1 - 0.3q$~~

~~$\pi_{St}(R, \sigma^*_G; t_{St} = StL) = 0.8q + 0.6(1-q) = 0.6 + 0.2q$~~

~~$\pi_{St}(L, \sigma^*_G; t_{St} = StR) = 0.6q + 0.8(1-q) = 0.8 - 0.2q$~~

~~$\pi_{St}(R, \sigma^*_G; t_{St} = StR) = q + 0.7(1-q) = 0.7 + 0.3q$~~

~~σ^* is a BNE only if G has no profitable deviation which is iff $\pi_G(l, \sigma^*_{St})$ $E(\pi_G(l, \sigma^*_{St}; t_{St})) = E(\pi_G(r, \sigma^*_{St}; t_{St}))$~~

Suppose St plays $\sigma_{St} = (L, L)$

$E_G(\pi_G(l, \sigma_{St}; t_{St})) = 0.35 <$

$E_G(\pi_G(r, \sigma_{St}; t_{St})) = 0.1$

so G can profitably deviate from σ^* by reallocating probability mass from r to l

By symmetry, if St plays $\sigma_{St} = (R, R)$, G can profitably deviate ~~by~~ from σ^* by reallocating probability mass from l to r.

Suppose St plays $\sigma_{St} = (L, R)$

$E_G(\pi_G(l, \sigma_{St}; t_{St})) = \cancel{\tfrac{1}{2}[0.3q + 0(1-q)] + \tfrac{1}{2}[0q + 0.3(1-q)]}$

$\cancel{= 0.15q + 0.15 - 0.15q}$

$\cancel{= 0.15}$

$\cancel{E_G(\pi_G(r, \sigma_{St}; t_{St})) = \tfrac{1}{2}[}$

$= \cancel{\tfrac{1}{2}[0.3} \; \tfrac{1}{2}(0.3) + \tfrac{1}{2}(0) = 0.15$

$E_G(\pi_G(r, \sigma_{St}; t_{St})) = \tfrac{1}{2}(0) + \tfrac{1}{2}(0.3) = 0.15$

Then G has no incentive to deviate for any q.

$\cancel{\pi_{St}(LR, \sigma_2^*; t_{St} = StL) =}$

$\pi_{St}(\sigma_{St}, \sigma_G^*; t_{St} = StL) = 0.7q + 1(1-q) = 1 - 0.3q >$

$\pi_{St}(\sigma_{St}', \sigma_G^*; t_{St} = StL) = 0.8q + 0.6(1-q) = 0.6 + 0.2q$

where σ_{St}' is a pure strategy of St that yields a different action given t_{St}, and

$\pi_{St}(\sigma_{St}, \sigma_G^*; t_{St} = StR) = q + 0.7(1-q) = 0.7 + 0.3q >$

$\pi_{St}(\sigma_{St}', \sigma_G^*; t_{St} = StR) = 0.6q + 0.8(1-q) = 0.8 - 0.2q$

$1 - 0.3q > 0.6 + 0.2q$, $0.4 > \cancel{0.1q}\ 0.5q$, $q < 0.8$

$0.7 + 0.3q > 0.8 - 0.2q$, $0.5q > 0.1$, $q > 0.2$

So ~~all~~ $\sigma^* = (LR, \; q \times l + (1-q) \times r)$ is a hybrid BNE for all $q \in (0.2, 0.8)$

Suppose that St plays $\sigma_{St} = (R, L)$

$E_G(\pi_G(l, \sigma_{St}; t_{St})) = \tfrac{1}{2}(0.2) + \tfrac{1}{2}(0.4) = 0.3$

$E_G(\pi_G(r, \sigma_{St}; t_{St})) = \tfrac{1}{2}(0.4) + \tfrac{1}{2}(0.2) = 0.3$

Then G has no incentive to deviate for any q.

St has no incentive to deviate if

$\pi_{St}(\sigma_{St}, \sigma_G^*; t_{St} = StL) > \pi_{St}(\sigma_{St}', \sigma_G^*; t_{St} = StL)$

$0.6 + 0.2q > 1 - 0.3q$

$0.5q > 0.4$

$q > 0.8$

$\pi_{St}(\sigma_{St}, \sigma_G^*; t_{St} = StR) > \pi_{St}(\sigma_{St}', \sigma_G^*; t_{St} = StR)$

$0.8 - 0.2q > 0.7 + 0.3q$

$0.1 > 0.5q$

$q < 0.2$

so ~~G~~ St always has incentive to deviate, there are no hybrid BNE where ~~G~~ St plays pure strategy RL and G mixes.

c Suppose that some type L St mixes at equilibrium. Then, by definition of NE, this StL has no profitable deviation. Then, the payoff to this StL from ~~StL~~ L and the payoff from R are equal. This is possible only if G mixes l and r and assigns high probability to l. If G so mixes, then ~~an StR~~ a type R St has ~~StR~~ higher payoff from R than from L, and so always chooses R. Then G ~~is very badly off~~ enjoys low payoff against StR and fails to maximise his expected payoff, so it is not an equilibrium.

3a ~~x+y=1~~

All strategy profiles $s^* = (x^*, y^*)$ where $x^* + y^* = 1$ are pure NE. $(1,1)$ is also a pure NE.

b $u_1(x,y) = \begin{cases} x^\alpha & \text{if } x+y \le z \\ 0 & \text{otherwise} \end{cases}$

$u_2(y,x) = \begin{cases} y^\alpha & \text{if } x+y \le z \\ 0 & \text{otherwise} \end{cases}$

$E_1[u_1(x,y)] = \begin{cases} 0 & \text{if } x+y \cancel{\ge 1} > 1 \\ (P(x+y \le z))x^\alpha & \text{otherwise} \end{cases}$

$= \begin{cases} 0 & \text{if } x+y > 1 \\ \cancel{1-(x+y)^n} \\ ([1-(x+y)^n]x^\alpha & \text{otherwise} \end{cases}$

~~By symmetry~~

~~$E_2[u_2(y,x)] = \{0$ if $x+y>1$, $([1-(x+y)^n]x^\alpha$ otherwise~~

~~Suppose~~

~~Supposing that~~

~~Suppose~~ Given that $z \in [0,1]$, ~~$F(z) = (x+y$~~ $F(z=1) = 1$, so ~~$E(z)$~~ the expected utility function simplifies to

$E_1[u_1(x,y)] = [1-(x+y)^n]x^\alpha$

By symmetry,

$E_2[u_2(y,x)] = [1-(x+y)^n]y^\alpha$

c FOCs:

$\partial/\partial x \; E_1[u_1(x,y)] = \partial/\partial x \; [1-(x+y)^n]x^\alpha = 0$

$\partial/\partial y \; E_2[u_2(y,x)] = \partial/\partial y \; [1-(x+y)^n]y^\alpha = 0$

$[1-(x+y)^n]\alpha x^{\alpha-1} + -n(x+y)^{n-1}x^\alpha = 0$

$\alpha[1-(x+y)^n] = nx(x+y)^{n-1}$

$\alpha[1-(x+y)^n] = ny(x+y)^{n-1}$

$2\alpha[1-(x+y)^n] = n(x+y)^n$

$2\alpha = (2\alpha+n)(x+y)^n$

$x+y = (2\alpha/(2\alpha+n))^{1/n}$

~~$x = \alpha[1-$~~

$x = y = \alpha[1-(x+y)^n]/n(x+y)^{n-1}$

$x = y = \frac{1}{2}(2\alpha/(2\alpha+n))^{1/n}$

~~At~~ optimal demands are $x^* = y^* = \frac{1}{2}(2\alpha/(2\alpha+n))^{1/n}$

d Suppose $z \sim U(0,1)$, then $F(z) = z$ for $z \in [0,1]$

$E_1[u_1(x,y)] = [1-(x+y)]x^\alpha$

$E_2[u_2(y,x)] = [1-(x+y)]y^\alpha$

FOCs:

$\partial/\partial x \; E_1[u(x,y)] = 0$

$\partial/\partial y \; E_2[u_2(y,x)] = 0$

$[1-x-y]\alpha x^{\alpha-1} - x^\alpha = 0$

$[1-x-y]\alpha - x = 0$

$(\alpha+1)x = (1-y)\alpha$

$x = \alpha/(\alpha+1)\,(1-y)$

By symmetry,

$y = \alpha/(\alpha+1)\,(1-y)$

$x = y = (1-x-y)\alpha$

~~$x = \alpha/(\alpha+1)\,(1-x)$~~ By substitution $y = x$

$x = \alpha/(\alpha+1) - \alpha x/(\alpha+1)$

$(2\alpha+1)/(\alpha+1)\; x = \alpha/(\alpha+1)$

$x = y = \alpha/(2\alpha+1)$

As players become more risk averse, the marginal utility of money decreases more rapidly, α decreases. Each players ~~demands~~ demand decreases.

Are we expected to know l'Hopital's rule?

e $\cancel{\ln x^* = \tfrac{1}{n}\ln\left(\tfrac{1}{2}\left(\tfrac{2\alpha}{2\alpha+n}\right)^{1/n}\right)}$

$\cancel{= \tfrac{1}{n}\ln\tfrac{1}{2} + \tfrac{1}{n}\ln 2\alpha - \tfrac{1}{n}\ln(2\alpha+n)}$

$\cancel{= \tfrac{1}{n}\left[\ln 2\alpha - \ln 2 - \ln(2\alpha+n)\right]}$

~~Let $f(n) = [\ln 2\alpha - \ln 2 - \ln(2\alpha+n)]$ and $g(n) = n$~~

~~$f'(n) = -\tfrac{1}{2\alpha+n}$, $g'(n) = 1$~~

~~$\lim_{n\to\infty}$~~

$$\ln x^* = \ln\left(\tfrac{1}{2}\left(\tfrac{2\alpha}{2\alpha+n}\right)^{1/n}\right)$$
$$= \ln\tfrac{1}{2} + \tfrac{1}{n}\ln 2\alpha - \tfrac{1}{n}\ln(2\alpha+n)$$

Let $f(n) = \ln 2\alpha - \ln(2\alpha+n)$ and $g(n) = n$

$f'(n) = -\tfrac{1}{2\alpha+n}$, $g'(n) = 1$

$\lim_{n\to\infty} \tfrac{1}{n}\ln 2\alpha - \tfrac{1}{n}\ln(2\alpha+n)$

$= \lim_{n\to\infty} f'(n)/g'(n)$ by l'Hopital's law

$= \lim_{n\to\infty} -\tfrac{1}{2\alpha+n}$

$= 0$

$\lim_{n\to\infty} \ln x^* = \lim_{n\to\infty} \ln\tfrac{1}{2} + \lim_{n\to\infty} \tfrac{1}{n}\ln 2\alpha - \tfrac{1}{n}\ln(2\alpha+n)$

(by sum law)

$= \ln\tfrac{1}{2}$

$\lim_{n\to\infty} x^* = e^{\lim_{n\to\infty}\ln x^*} = \tfrac{1}{2}$

By symmetry, $\lim_{n\to\infty} y^* = \tfrac{1}{2}$.

As n increases, the probability distribution of z narrows, z converges in probability to 1. ~~This can be interpreted as a "perturbance" in z becoming~~ This can be interpreted as a perturbation of the original full information game becoming vanishingly small. This process identifies the risk dominant equilibrium in the original game in (a).

4a ~~State space $\Omega = \{K, C, B\}$~~

State space $\Omega = \{Kcb, Ccb, Bcb, Kcv, Ccv, Bcv\}$

~~#~~ where in each K-state, the sauce is ketchup, in each C-state the sauce is chocolate, in each B-state the sauce is Bechamel, in each cb state Charlie is colourblind, and in each cv state Charlie has colour vision.

b $P_C = (\{Kcb, Ccb\} \{Bcb\}, \{Kcv\}, \{Ccv\}, \{Bcv\})$

$P_A = (\{Kcb, Ccb, Kcv, Ccv\}, \{Bcb, Bcv\})$

c ~~Bcb and Bcv~~ Let the event that the cup contains Bechamel sauce be $B = \{Bcb, Bcv\}$. Player i knows ~~d Bcb and Bcv~~ ~~B iff~~ in state ω iff $P_i(\omega) \subseteq B$.

By inspection, C knows B in states Bcb and Bcv.

$K_C(B) = \{Bcb, Bcv\}$

d By inspection, A knows $K_C(B)$ in states Bcb and Bcv.

e ~~By inspection,~~ Let Choc be the event that the cup contains chocolate sauce.

By inspection, $K_C(Choc) = \{Kcv\}$

f By inspection, $K_A(K_C(Choc)) = \emptyset$, A never knows that C knows that Choc.