Game Theory Paper 210616

1a

	I	N
W	$\underline{\omega - c}$, $v-\omega-\gamma$	$\omega - c$, $\underline{v-\omega}$
S	0, $\underline{-\gamma}$	$\underline{\omega}$, $-\omega$

b $\omega > c > \gamma$

By inspection of the payoff matrix, there are no pure NE where players play pure mutual best responses. By inspection, each player has a unique best response to each pure strategy of the other player, so there are no hybrid NE.

P1 mixes at NE σ^*. Then P1 has no profitable deviation from σ_1^*, so P1 is indifferent between pure actions W and S.

$\pi_1(W, \sigma_2^*) = \pi_1(S, \sigma_2^*) \Leftrightarrow$
$\omega - c = \omega(1-q)$
where $\sigma_2^* = qI + (1-q)N$

P2 mixes at NE, so P2 is indifferent

$\pi_2(I, \sigma_1^*) = \pi_2(N, \sigma_1^*) \Leftrightarrow$
$(v-\omega)p - \gamma = vp - \omega$
where $\sigma_1^* = pW + (1-p)S$

$\omega - c = \omega(1-q) \Rightarrow q = 1 - \frac{\omega - c}{\omega} = \frac{c}{\omega}$
$(v-\omega)p - \gamma = vp - \omega \Rightarrow -\omega p - \gamma = -\omega \Rightarrow p = \frac{\omega - \gamma}{\omega}$

Intuitively, the probability of inspection is increasing in the cost of effort and decreasing in wage because the higher the cost of effort and the lower the wage, ceteris paribus, the greater the worker's incentive to shirk, hence the more probable inspection must be such that the worker remains indifferent.

Intuitively, the probability of working is increasing in wage and decreasing in cost of inspection because the higher the wage and the lower the cost of inspection, the greater the principal's incentive to inspect, hence the more probable working must be such that the principal remains indifferent.

c $\pi_2(\sigma_2^*, \sigma_1^*)$
$= pq(v-\omega-\gamma) + p(1-q)(v-\omega) + (1-p)q(-\gamma) + (1-p)(1-q)(-\omega)$
$= pv - q\gamma - \omega + (1-p)(q)\omega$
$= \frac{\omega-\gamma}{\omega}v - \frac{c}{\omega}\gamma - \omega + (\frac{\gamma}{\omega})(\frac{c}{\omega})\omega$
$= \frac{\omega v - \gamma v - c\gamma - \omega^2 + \gamma c}{\omega}$
$= \frac{\omega v - \gamma v - \omega^2}{\omega}$
$= v - \omega - \frac{\gamma v}{\omega}$

d $\max_\omega \pi_2(\sigma_2^*(\omega), \sigma_1^*(\omega))$

FOC: $-1 - \gamma v(-\omega^{-2}) = 0 \Rightarrow$
$\gamma v \omega^{-2} = 1 \Rightarrow$
$\omega^{-2} = 1/\gamma v \Rightarrow$
$\omega = \sqrt{\gamma v}$

Given that $\gamma v > c^2$, $\omega > c$, the earlier assumption used in the derivation of the NE holds.

SOC: $-2\gamma v \omega^{-3} < 0 \Rightarrow$
$\omega = \sqrt{\gamma v}$ is a maximum

Intuitively, the greater the value of the product to the principal, the greater the incentive for the principal to ~~offer~~ incentivise working, by offering a high wage. The greater the cost of inspection, the greater the principal's incentive to incentivise working by offering a higher wage rather than by frequent inspection. Where the output is worthless, the principal will pay nothing for it. Where inspection is costless, the principal will offer wage equal to cost of effort ~~(which is zero or negat~~ (zero) and incentivise effort by inspecting with certainty.

e P1's payoff given q is

$\pi_1(\sigma_1, qI + (1-q)N)$
$= p(\omega - c) + (1-p)(1-q)\omega$
~~where $\sigma_1 = pW + (1-p)S$~~

~~$\max_p \pi_1(\sigma_1, qI + (1-q)N)$~~
~~FOC: $(\omega - c) - (1-q)\omega = 0 \Rightarrow$~~
~~$\omega - c = (1-q)\omega \Rightarrow$~~

For $q > c/\omega$, $p = 1$, i.e. W with certainty is optimal. For $q < c/\omega$, $p = 0$, i.e. S with certainty is optimal. For $q = c/\omega$, any mix is optimal.

At SPE, $q \leq c/\omega$. Any SPE such that $q > c/\omega$ fails to deviation by P2 to $\sigma_2' = q'I + (1-q')N$ such that $q > q' > c/\omega$. ~~Such deviation~~ P1 continues to play W with certainty while P2 has ~~lower~~ higher payoff due to lower expected cost of inspection.

At SPE, $q \notin (0, c/\omega)$. Any SPE such that $q \in (0, c/\omega)$ fails to deviation by P2 to $\sigma_2' = N$. P1 continues to play S. This deviation reduces expected cost of inspection.

~~At candidate SPE (S, N), P2 has payoff $-\omega$~~

At SPE such that $q = c/w$, $p=1$, otherwise P2 has strictly profitable deviation to $\sigma' = q'I + (1-q')N$ where $q' = q+\epsilon$ for sufficiently small ϵ. Then P2 has expected payoff $v - w - (c/w)\gamma$. If $v \geq c$, this payoff is greater than the payoff of the simultaneous move game.
~~It is natural to suppose $v>c$~~

~~if $v<c$, then $v<w$, so $v-w<0$, P2 finds it optimal to induce S, and chooses $q=0$, and has payoff~~

It is natural to suppose $v>c$. P2 benefits from commitment. Intuitively, ~~P2~~ the principal commits to inspect such that ~~P1~~ the worker has marginally greater payoff from working. The increase in expected inspection cost is marginal but the increase in expected output value is not, so this is strictly profitable.

2a Suppose firm i plays $s_i(\theta_i)=1$, then it has benefit θ_i^2 and cost $-c$, so its payoff is θ_i^2-c. This ~~is~~ trivially follows from the given description.

Suppose firm i plays $s_i(\theta_i)=0$, then it has benefit θ_i^2 iff firm j plays $s_j(\theta_j)=1$, which has probability p_j and benefit 0 otherwise, so it has expected benefit $p_j\theta_i^2$, and it has zero cost, so its payoff is $p_j\theta_i^2$.

b $\pi_i(1\quad, s_j(\theta_j)) \gtreqless \pi_i(0, s_j(\theta_j)) \Leftrightarrow$
$\theta_i^2 - c \gtreqless p_j\theta_i^2 \Leftrightarrow$
$p_j \lesseqgtr \theta_i^2 - c/\theta_i^2$

Firm i's best response

$$b_i(p_j) = \begin{cases} 1 & \text{if } p_j < \theta_i^2-c/\theta_i^2 \\ 0 & \text{if } p_j > \theta_i^2-c/\theta_i^2 \\ [0,1] & \text{for } p_j = \theta_i^2-c/\theta_i^2 \end{cases}$$

For $p_j = \theta_i^2-c/\theta_i^2$, any potentially degenerate mix of 1 and 0 is optimal, so firm i best responds (though not uniquely), by playing 1 iff $p_j \leq \theta_i^2-c/\theta_i^2$
$p_j\theta_i^2 \leq \theta_i^2 - c \Leftrightarrow (1-p_j)\theta_i^2 \geq c \Leftrightarrow \theta_i^2 \geq c/1-p_j \Leftrightarrow$
$\theta_i \geq \sqrt{c/1-p_j}$. Informally, this "says" that firm i's expected net benefit $(1-p_j)\theta_i^2$ is greater than the expected net cost c. Intuitively, an increase in c increases the cost of investment, so ~~the~~ investment is optimal only for ~~firm~~ "higher" type firms who ~~have~~ enjoy greater benefit from investment. An increase in p_j decreases the expected net (compared to the case of non investment) benefit because it is less likely that investment by firm i will be pivotal, so decreases the incentive to invest, which will only be optimal for high types. Hence $\uparrow c \Rightarrow \uparrow v$ and $\uparrow p_j \Rightarrow \uparrow v$.

c If firm i were the only firm that can invest, its payoff from investment is unchanged but its payoff from non-investment decreases because it enjoys zero benefit with certainty, its payoff from non-investment is zero. It invests iff
$\theta_i^2 - c > 0 \Leftrightarrow$
$\theta_i > \sqrt{c}$

This threshold is weakly lower than the earlier threshold for all p_j ~~and strictly~~ $\in [0,1]$ and strictly lower for all $p_j \in (0,1]$. Firm i has a lower threshold here because it can no longer free ride on the investment of other firms if it does not invest.

If there are more than two firms, let $p_{\neg i}$ denote the total probability that at least one firm other than firm i plays 1. Firm i's best response function is obtained by substituting $p_{\neg i}$ for p_j, so firm i's threshold is obtained by substituting $p_{\neg i}$ for p_j. At equilibrium, $p_{\neg i} > p_j$ because each "additional" firm has incentive to invest with non-zero probability, ~~and the~~ (and the decrease in p_j is less than this), so firm i's threshold increases.

Intuitively, the more firms there are, the more likely it is that some other firm will invest, so the greater the probability that if firm i does not invest, it will still be able to free ride, so the less incentive it has to invest, and the higher its threshold.

d At BNE, firms play mutual best responses in interim expectations.

$p_j = P\{\theta_j > \sqrt{c/1-p_i}\} = 1-\sqrt{c/1-p_i} \Rightarrow 1-p_j = \sqrt{c/1-p_i}$
~~$p_i = P\{\theta_i > \sqrt{c/1-p_j}\} = 1-\sqrt{c/1-p_j}$~~
By symmetry, ~~p_i~~ $1-p_i = \sqrt{c/1-p_j}$
By substitution, $1-p_i = \sqrt{c/\sqrt{c/1-p_i}} \Rightarrow$
$(1-p_i)^2 = c/\sqrt{c/1-p_i} \Rightarrow$
$(1-p_i)^4 = c^2/c/1-p_i = (1-p_i)c \Rightarrow$
$(1-p_i)^3 = c \Rightarrow$
$p_i = 1-c^{1/3} \Rightarrow$
$p_j = 1-\sqrt{c/c^{1/3}} = 1-\sqrt{c^{2/3}} = 1-c^{1/2}$

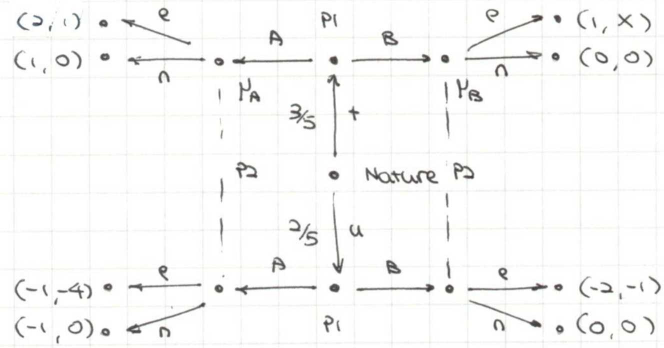
$v_i = \sqrt{c/1-p_j} = \sqrt{c/c^{1/3}} = c^{1/3}$
$v_j = \sqrt{c/1-p_i} = \sqrt{c/c^{1/3}} = c^{1/3}$

At BNE each firm i plays the cutoff strategy: 1 iff $\theta_i \geq c^{1/2}$, 0 otherwise.

Each firm ~~p~~ invests ~~or~~ iff its type, which determines its benefit from the product development is sufficiently high relative to the cost of investment. This is intuitive. The threshold type required to induce investment is lower than if there were only one firm because each firm has some incentive to free ride on the investment of other firms. ~~the~~

Date

1a



b $X=1$

Consider candidate ~~pooling~~ PBE ~~of~~ with pooling on A. By Bayesian beliefs, $\mu_A = 3/5$. ~~Payoff~~ ~~If~~ At the A information set, e yields $3/5(1)+2/5(-4) = -1$ to P2, n yields 0, so n is strictly sequentially rational. ~~For $\mu_B \geq 1/2$, n at the B information set is~~ A is ~~strictly~~ ~~ratio~~ sequentially rational for t ~~if~~ for all P2's strategies at the B information set. A is sequentially rational for u ~~to~~ iff P2 plays e with probability ~~less~~ $\geq 1/2$ at the B information set. e wp $\geq 1/2$ at the B information set is sequentially rational for P2 iff $\mu_B \geq 1/2$. ~~The following~~ ~~Pooling on A is a PBE.~~ Pooling on A is sustained in PBE iff P2 plays n if A, e if B, and $\mu_A = 3/5$, $\mu_B \geq 1/2$.

Consider candidate ~~pooling~~ PBE w pooling on B. By Bayesian beliefs, $\mu_B = 3/5$. At ~~the~~ B information set, e yields $3/5(1)+2/5(-1) = 1/5$ to P2, n yields 0, so e is strictly sequentially rational. B is ~~sequent~~ sequentially rational for t iff P2 plays n with ~~not~~ certainty at A information set. B is sequentially rational for u ~~iff for all P2 strategies iff~~ for all P2 strategies at A. Pooling on B cannot be sustained at PBE.

Consider candidate PBE w separating $t \to A$, $u \to B$. ~~By SR, at B~~, By BB, $\mu_A = 1$, $\mu_B = 0$. By SR, P2 at A plays e, P2 at B plays n. A is SR for t ~~iff for~~, and B is SR for u. ~~This~~ ~~such separation~~ This is a PBE.

Consider candidate ~~PBE~~ w separating $t \to B$, $u \to A$. By BB, $\mu_A = 0$, $\mu_B = 1$. By SR, P2 at A plays n, P2 at B plays e. $t \to B$ and $u \to A$ are SR. This is a PBE.

Three types of PBE: ~~sep~~ pooling on A, separating ~~on~~ $t \to A$ $u \to B$, separating $t \to B$ $u \to A$.

Pooling PBE: P1 always produces an ambitious piece and is never exhibited. Talented P1 has no profitable deviation because the payoff simply from producing an ambitious piece is greater than that from exhibiting a boring piece. Untalented P1 has no profitable deviation because the lousy boring piece will be exhibited with sufficiently high probability, embarassing P1. The boring piece is exhibited with high probability because P2 believes a boring piece more likely to be produced by a talented type.

Separating $t \to A$, $u \to B$. Ambitious piece signals talent and is exhibited, boring piece signals no talent and is not exhibited. Talented artist has no incentive to pass as mediocre. Mediocre artist has no incentive to pass as talented (producing an ambitious piece is effortful).

Separating $t \to B$, $u \to A$. Opposite signals. Talented artist indifferent between exhibiting boring and not exhibiting ambitious. Mediocre artist prefers not exhibiting ambitious to exhibiting boring, embarassment is more costly than effort.

c Intuitive criterion is silent on ~~poo~~ separating eqm where there are no off eqm path beliefs.

Intuitive criterion: off eqm path, assign zero probability to types ~~for~~ whose eqm payoff exceeds maximum attainable deviation payoff. u type eqm payoff is ~~0~~ -1 but could achieve 0 by deviation, t type eqm payoff is 1 but could achieve 1 by deviation. IC is silent on the pooling eqm.

~~Forward induction argument against pooling PBE.~~ ~~t has no strict~~ For all P2 strategies at B, ~~P~~ t has no ~~incent~~ strict incentive to deviate, and has weak incentive only if P2 plays e with certainty at B. ~~P P1 has~~ u has strict incentive to deviate if P2 plays n wp ~~if~~ $> 1/2$ at B and weak incentive if $\geq 1/2$. There is a larger set of P2 strategies such that deviation by ~~P~~ u is profitable than such that deviation by t is possible. By divinity, $\mu_B = 0$, then pooling ~~P~~ on A cannot be sustained.

d Pooling on A fails to deviation by u to B because n is strictly SR for P2 at B.

~~Pooling on B fails to deviation~~ Pooling on B fails to deviation by ~~A to~~ t to A because ~~A yields strictly gree~~ n is strictly SR for P2 at B and any A outcome is strictly preferred to B,n by t. Separating with $t \to B$, $u \to A$ fails for the same reason.

Separating with $t \to A$, $u \to B$ induces $\mu_A = 1$, $\mu_B = 0$ by BB, then e at A and n at B is SR (uniquely), then $t \to A$ and $u \to B$ is SR, so this is a PBE. It is the unique pure PBE.

IC is silent because there are no off eqm path beliefs.

n at B strictly dominates e at B. Then B is strictly dominant for u, and A is strictly dominant for t, then e at A strictly dominates n at A. The game is dominance solvable.

4a $u_A(x,y) = \begin{cases} V-x & \text{if } x+y \geq 100 \\ \alpha(x+y)-x & \text{otherwise} \end{cases}$

$u_B(x,y) = \begin{cases} V-y & \text{if } x+y \geq 100 \\ \beta(x+y)-y & \text{otherwise} \end{cases}$

$u_A(x,y) = \begin{cases} 80-x & \text{if } x+y \geq 100 \\ 0.8(x+y)-x = 0.8y-0.2x & \text{otherwise} \end{cases}$

$u_B(x,y) = \begin{cases} 80-y & \text{if } x+y \geq 100 \\ -y & \text{otherwise} \end{cases}$

$x=0$ is strictly dominant for A. Consider arbitrary y. Suppose $y \geq 100$, then $u_A(x,y) = 80-x$, which is strictly decreasing in x, so payoff maximization consists in minimising x, so $x=0$ is ~~opt~~ a strict best response. Suppose $y<100$, then $u_A(x=0,y) = 0.8y$. For $x \in (0, 100-y)$, $u_A(x,y) = 0.8y-0.2x < 0.8y$. ~~For $x=100-y$~~, $u_A(x=100-y, y) = 80-(100-y) = y-20 < 0.8y$ for all $y<100$. for $x>100-y$, $u_A(x,y) = 80-x < y-20 < 0.8y$. So for all y, $x=0$ is a strict best response, $x=0$ is strictly dominant for A.

Then, $y=0$ is strictly dominant for B against A's strictly dominant strategy. $u_B(x=0,y=0)=0$. For $y \in (0,100)$, $u_B(x=0,y) = -y < 0$. $u_B(x=0,y=100) = 80-100 = -20 < 0$. For $y>100$, $u_B(x=0,y) = 80-y < -20 < 0$. So $y=0$ is a strict best response for B against $x=0$.

The game is dominance solvable, only one pure strategy for each player survives iterated strict dominance. The corresponding strategy profile $(x=0, y=0)$ is the unique ~~PS~~ NE.

The fundraising drive will not be successful. A ~~contributes zero~~ strictly prefers to contribute nothing regardless of B's contribution because if the drive fails A has a large payoff that is decreasing in A's contribution, and A has no incentive to be pivotal in the drive's ~~succ~~ success.

b $u_A(x,y) = \begin{cases} 60-x & \text{if } x+y \geq 100 \\ 0.3(x+y)-x = 0.3y-0.7x & \text{otherwise} \end{cases}$

$u_B(x,y) = \begin{cases} 60-y & \text{if } x+y \geq 100 \\ 0.3(x+y)-x = 0.3y-0.7x & \text{otherwise.} \end{cases}$

Suppose $y \geq 100$, then $u_A(x, y \geq 100) = 60-x$. Payoff maximization consists in minimising x, so $x=0$ is the strict best response. A finds it optimal to induce success given $y<100$ iff the minimum x such that $x+y \geq 100$, i.e. $x = 100-y$, is such that $60-x \geq 0.3y \Leftrightarrow -40+y \geq 0.3y \Leftrightarrow 0.7y \geq 40 \Leftrightarrow y \geq 400/7$. So for $y \geq 400/7$, A's best response is $100-y$, and for $y \leq 400/7$, A's best response is 0. So $x > 100 - 400/7 = 300/7$ is never a best response for A, ~~hence by Pearce's lemma~~, against any pure or mixed strategy by B. By Pearce's lemma, $x > 300/7$ is strictly dominated by some pure or mixed strategy of A. By symmetry, $y > 300/7$ is strictly dominated for B. Then 0 is strictly dominant for both players, so only $(x=0, y=0)$ survives iterated strict dominance. This is the unique NE. The fundraising drive again fails.

c $V=80$, $\alpha=0.8$, $\beta=0$

Consider the subgame in which $x_1 = 20+\varepsilon$ for arbitrarily small ε, arbitrary y_1. Payoffs reduce to:

$$u_A(x_2,y_2) = \begin{cases} 80-(20+\varepsilon)-x_2 = 60-x_2-\varepsilon \\ \quad \text{if } x_2+y_2 \geq 100-(20+\varepsilon)-y_1 = 80-y_1-\varepsilon \\ 0.8(x_1+y_1+x_2+y_2) - (x_1+x_2) \\ = 0.8(y_1+y_2) - 0.2(x_1+x_2) \\ = 0.8y_1+y_2 - 0.2x_2 - 4 - 0.2\varepsilon \\ \quad \text{otherwise} \end{cases}$$

$$u_B(x_2,y_2) = \begin{cases} 80-y_1-y_2 \\ \quad \text{if } x_2+y_2 \geq 80-y_1-\varepsilon \Leftrightarrow y_1+y_2 \geq 80-x_2-\varepsilon \\ \quad \Leftrightarrow 80-y_1-y_2 \leq x_2+\varepsilon \\ 0 \text{ otherwise} \end{cases}$$

B's best response is to induce success by ~~y_2~~ $y_2 = 80-y_1-\varepsilon-x_2$, which yields $x_2+\varepsilon > 0$. Given that at subgame NE, the drive always succeeds, i.e. $w \geq 100$, ~~P2~~ A's payoff at NE is $60-x_2-\varepsilon$, which is decreasing in x_2, so A ~~optimally~~ maximises payoff by playing $x_2=0$. The unique subgame NE is thus $(x_2=0, y_2=80-y_1-\varepsilon)$.

~~In the first stage, if A plays $x_1<20$, by analysis~~

In the first stage, if A plays $x_1 = 20+\varepsilon$, ~~A has payoff~~ this induces $(x_2=0, y_2=80-y_1-\varepsilon)$ at the subgame NE. Then A has payoff $80-(20+\varepsilon) = 60-\varepsilon$, and B has payoff ε. If $x_1+y_1<20$, then by arguments analogous to those in (a), the subgame NE is $(x_2,y_2)=0$, so both players have weakly negative payoff, and deviation to $x_1 = 20+\varepsilon$ is strictly profitable for A. If $x_1+y_1>20$ but $x_1<20$, ~~B has~~ $(x_2=0, y_2=100-x_1-y_1)$ is induced, then P2 has strictly negative payoff, so strictly profitable deviation to $y_1=0$, $y_2=0$.

The unique SPE is $(x_1=20, x_2=0, y_1=0, y_2=80)$. Note that any candidate SPE where $x_1 = 20+\varepsilon$ for $\varepsilon>0$ fails to deviation by ~~B~~ A to $x_1 = 20+\varepsilon/2$, $x_2=0$. So at SPE B is indifferent between $y_2=0$, $y_2=80$. $y_2=0$ is not ~~poss~~ sustainable at SPE because A then has strictly profitable deviation to $x_1=20+\varepsilon$, $x_2=0$.

d In the single round simultaneous game, P1 optimally induces success by playing $x = 100 - y - z$. This yields payoff $40 - (100 - y - z) = y + z - 60$. P1 optimally induces failure by playing $x = 0$ which yields $0.5(y+z)$. $0.5(y+z) > y+z-60 \iff 0.5(y+z) > 60 \iff y+z > 120$. So for $y+z<100$, P1 ~~optimally~~ best responds with $x=0$ to induce failure. For $y+z \geq 100$, P1 has payoff $40-x$ and best responds with $x=0$. So $x=0$ is strictly dominant for P1. Then the game reduces to a two player game ~~with~~ as earlier, with $V=40$, $\alpha_2 = 0.3$, $\alpha_3 = 0$. By analysis similar to that in (b), at NE, $x, y, z = 0$. The fundraising drive cannot be successful in a single round.

The required SPE is such that $z_3 = 40$, P2 is indifferent between contributing y_2 and contributing 0, i.e. ~~$y_2 =$~~ $40 - y_2 = 0.3x_1$, ~~and~~ P1 chooses the minimum x_1 such that $x_1 + y_2 + z_3 = 100$. $z_3 = 40$ is (weakly) optimal for P3 if $x_1 + y_2 = 60$. So the required SPE solves

$x_1 + y_2 = 60$,

$40 - y_2 = 0.3x_1$, $\Rightarrow$

$x_1 + (40 - 0.3x_1) = 60$, $y_2 = 60 - x_1 \Rightarrow$

$0.7x_1 = 20$, $y_2 = 60 - x_1 \Rightarrow$

$x_1 = 200/7$, $y_2 = 220/7$

5a In a two player ~~st~~ symmetric game, the strategy σ^* is an ESS iff (σ^*, σ^*) is a symmetric NE and for all strategies $\sigma' \neq \sigma^*$, it is not the case that both σ' is a best response to σ^*, i.e. $\pi(\sigma', \sigma^*) = \pi(\sigma^*, \sigma^*)$, and σ' fares better than σ^* against σ', i.e. $\pi(\sigma', \sigma') > \pi(\sigma^*, \sigma')$.

This captures the idea of resistance to invasion by mutants. A population of σ^* players is resistant to invasion by any σ' mutants iff σ' fares worse against σ^* than σ^* (i.e. σ' is not a best response), or if it fares equally well, then σ' fares ~~wo~~ worse than σ^* against σ', so σ^* players fare better than σ' players on average (and no σ' fares better than σ^* against σ^*).

The candidate ESS are the symmetric NE.

	H	L
H	$\underline{5}$, $\underline{5}$	-1, 0
L	0, -1	$\underline{1}$, $\underline{1}$

By inspection, there are two pure NE where players play pure mutual best responses.

Consider mixed NE σ^*. P1 has no profitable deviation, so P1 is indifferent between H and L

$\pi_1(H, \sigma^*_2) = \pi_1(L, \sigma^*_2) \Leftrightarrow$

$5q - 1(1-q) = 0q + 1(1-q) \Leftrightarrow$

$7q - 2 = 0 \Leftrightarrow$

$q = 2/7$

where $\sigma^*_2 = qH + (1-q)L$.

P2 has no profitable deviation and is indifferent.

$\pi_2(H, \sigma^*_1) = \pi_2(L, \sigma^*_1) \Leftrightarrow$

$p = 2/7$

where $\sigma^*_1 = pH + (1-p)L$

The unique mixed NE is $(2/7 H + 5/7 L, 2/7 H + 5/7 L)$

Each NE found is symmetric. Each is a candidate ESS.

Mixed candidate ESS fails because H is a best response and H fares better against H (payoff 5) than $2/7 H + 5/7 L$ against H (payoff 5/7).

Both pure candidate ESS are ~~str~~ ESS because the associated NE is a strict NE, so there are no other best responses.

In general, it is necessary to check if there exists a best response against each candidate ESS that would be a successful mutant.

The ESS are ~~{~~ H and L.

b Replicator Equation

$\dot{p}_i = p_i[[Ap]_i - p^T Ap]$

This reduces to

$\dot{p}_i = p_i(1-p_i)([Ap]_i - [Ap]_j)$

for a game with only two strategies, where ~~p_i is the~~ p is the vector of the proportion of players playing each strategy in the population, ~~i.e.~~ $i \neq j \in \{H, L\}$, A is the row payoff matrix $A = \begin{pmatrix} 5 & -1 \\ 0 & 1 \end{pmatrix}$, so $[Ap]_i$ is the average payoff of i players ~~a~~ against the population, and $p^T Ap$ is the average payoff of all players in the population.

$[Ap]_H = 5p_H - 1p_L = 5p_H - (1-p_H) = 6p_H - 1$

$[Ap]_L = 0p_H + 1p_L = p_L = 1 - p_H$

$[Ap]_H - [Ap]_L = 6p_H - 1 - (1-p_H) = 7p_H - 2$.

$\dot{p}_H = p_H(1-p_H)(7p_H - 2)$

There are three absorbing states, $p_H = 0$, $p_H = 2/7$, $p_H = 1$. For $p_H \in (0, 2/7)$, $7p_H - 2 < 0$, $\dot{p}_H < 0$. For $p_H \in (2/7, 1)$, $7p_H - 2 > 0$, $\dot{p}_H > 0$. So $p_H = 2/7$ is unstable. All states $p_H \in (0, 2/7)$ converge to $p_H = 0$ and all states $p_H \in (2/7, 1)$ converge to $p_H = 1$. $p_H = 0$ has basin of attraction $[0, 2/7)$. $p_H = 1$ has basin of attraction $(2/7, 1]$

d The basin of attraction of the $p_H = 1$ state is larger than the basin of attraction of the $p_H = 0$ state, so the system ~~spends the~~ is at or around one of these two absorbing states ~~its~~ the vast majority of the time, and ~~around~~ at or around the $p_H = 1$ state much more ~~often~~ frequently than around the $p_H = 0$ state.

~~Less at the~~ A greater number of "mistakes" is required to leave the ~~basin of attr~~ $p_H = 1$ state and enter the basin of attraction of the $p_H = 0$ state than is required to leave the $p_H = 0$ state and enter the basin of attraction of the $p_H = 1$ state. For sufficiently small probability of error ε, the number of errors required dominates in the probability of "movement" from one state to the other, so it is far more likely to "move" from the $p_H=0$ state to the $p_H=1$ state than the reverse.

The absorbing states under stochastic best reply dynamics ~~and~~ best reply dynamics, and replicator dynamics correspond to the symmetric NE, and of these states, the stochastically stable states are ~~the ESS.~~ ESS and the unique risk dominant NE of the strategic form game.

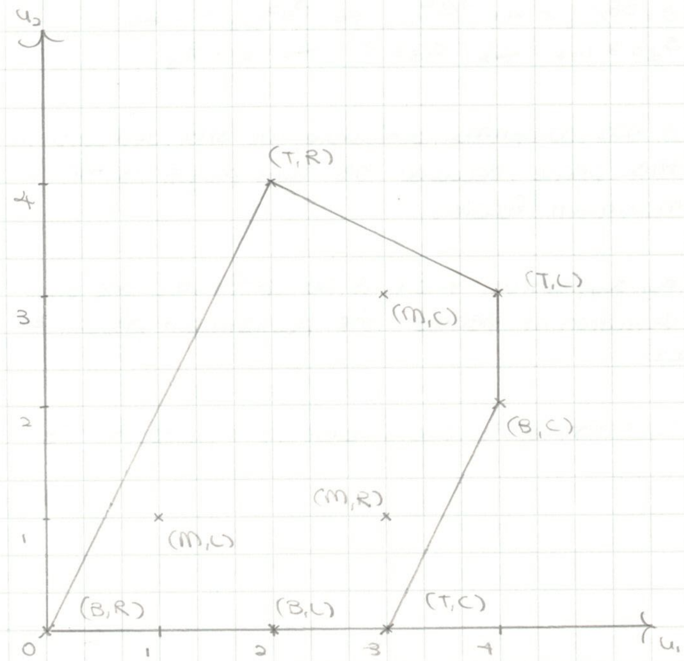
c The long run equilibrium depends on the initial state. For ~~p~~ initial $p_H^0 \in [0, 2/7)$, the long run eqm is $p_H=0$ because C is a strict best response against the population, and remains so as p_H decreases. For initial $p_H^0 \in (2/7, 1]$, the long run eqm is $p_H=1$ because H is a best response against the population and remains so as p_H increases.

For initial $p_H^0 = 2/7$, supposing that the updating player chooses randomly between best responses where there is multiplicity, with ~~non~~ non-zero probability, the updating player changes his strategy, and with certainty some updating player changes his strategy in the long run. This causes $p_H^0 \neq 2/7$, then ~~at~~ either $p_H=0$ or $p_H=1$ is reached in the long run, depending on which direction the updating player changed his strategy in.

Date

6a ~~Supposing that players do not mix, the Nash bargaining solution is the pure strategy profile with payoff (u_1, u_2) that maximises $(u_1-d_1)(u_2-d_2)$, where (d_1, d_2) is the disagreement payoff~~

The set of feasible payoffs is given by the minimal convex hull containing all potential payoff vectors. This is plotted below.



Given that some (u_1, u_2) such that $u_1 > d_1 = 3$, ~~and~~ $u_2 > d_2 = 3$ is feasible, the Nash bargaining solution ~~is some~~ (u_1^*, u_2^*) ~~such that~~ that maximises $(u_1-d_1)(u_2-d_2)$ is ~~some~~ such that $u_1^* > d_1$, $u_2^* > d_2$. The Nash bargaining solution lies on the utility possibility frontier because any point ~~within the~~ below the frontier fails to maximise $(u_1-d_1)(u_2-d_2)$. So (u_1^*, u_2^*) lies on the line segment

$$u_2 = 5 - u_1/2, \quad u_1 \in (3,4).$$

The ~~max~~ maximisation problem reduces to

$$\max_{u_1} (u_1-3)(5-u_1/2-3) \quad \text{s.t. } u_1 \in (3,4)$$

$$= (u_1-3)(2-u_1/2) = -\tfrac{1}{2}u_1^2 + \tfrac{7}{2}u - 6$$

FOC: $-u_1 + 7/2 = 0 \Rightarrow u_1 = 7/2$

SOC: $-1 < 0 \Rightarrow u_1 = 7/2$ is a maximum

The Nash bargaining solution is $(7/2, 13/4)$, which is achieved by the strategy profile $(T, \tfrac{3}{4}L + \tfrac{1}{4}R)$

b

	L	C	R
T	3	0	4
	4	3	2
M	1	3	1
	1	3	3
B	0	2	0
	2	4	0

Best responses underlined.

L seems weak because it is not a best response to any pure strategy. Indeed, it is dominated by a mix of C and R.

$\tfrac{1}{10}C + \tfrac{9}{10}R \succ_2 L$. Then $\tfrac{9}{10}M + \tfrac{1}{10}B \succ_1 T$. Then $C \succ_2 R$. Then $B \succ_1 M$. Only (B,C) survives iterated strict dominance, so (B,C) is the unique NE. The game is dominance solvable.

c ~~At the~~ In the cooperation phase, i.e. at any history at which (T,L) is prescribed, P1 has his maximum feasible stage game payoff, ~~so P1~~ and eqm play yields this payoff indefinitely, so P1 has no profitable one shot deviation. Optimal one shot deviation by P2 is to R which yields 4 then 0 then 3 indefinitely. Eqm play yields 3 indefinitely. Compare PVs, P2 has no profitable one shot deviation iff $3 + 3\delta + 3\delta^2/(1-\delta) \geq 4 + 0\delta + 3\delta^2/(1-\delta) \Leftrightarrow 3\delta \geq 1 \Leftrightarrow \delta \geq 1/3$.

In the punishment phase, i.e. at any history where (B,R) is prescribed. Optimal one shot ~~deviat~~ deviation by P1 is to M which yields 3 then 0 then 4 indefinitely. Eqm play yields 0 then 4 indefinitely. Compare PVs. P1 has no profitable one shot deviation iff $0 + 4\delta + 4\delta^2/(1-\delta) \geq 3 + 0\delta + 4\delta^2/(1-\delta) \Leftrightarrow 4\delta \geq 3 \Leftrightarrow \delta \geq 3/4$. Optimal one shot deviation by P2 is to C which yields 2 then 0 then 3 indefinitely. Eqm play yields 0 then 3 indefinitely. Compare PVs. P2 has no profitable ~~de~~ one shot deviation iff $0 + 3\delta + 3\delta^2/(1-\delta) \geq 2 + 0\delta + 3\delta^2/(1-\delta) \Leftrightarrow 3\delta \geq 2 \Leftrightarrow \delta \geq 2/3$.

No player has a profitable one shot deviation iff $\delta \geq 3/4$. The given strategy profile is a SPE for such δ.

d Consider the strategy profile under which players play (T,L) iff no player previously deviated and (B,C) otherwise.

At any history, P1 is ~~stage~~ receives maximum feasible stage game payoff indefinitely from eqm play, so P1 has no profitable ~~deviation~~ one shot deviation at any history for any δ.

In the punishment phase (i.e. any history where there has been some prior deviation), ~~eqm eqm~~ prescribed play is the stage game Nash in every period, ~~independent of past play~~, so neither player has a profitable one shot deviation.

In the cooperation phase (no prior deviation),

~~P2 optimally deviat~~ optimal one shot deviation is to R, which yields 4 then 2 indefinitely. Eqm play yields 3 indefinitely. Compare PVs. P2 has no profitable one shot deviation iff ~~3+3~~
$3+\frac{3\delta}{1-\delta} \geq 4+\frac{2\delta}{1-\delta} \iff \frac{\delta}{1-\delta} \geq 1 \iff \delta \geq 1-\delta \iff \delta \geq 1/2$.

~~Consider the strategy profile under wh represented by the following automaton.~~

~~(T,C) ¬(B,R)~~

~~¬(T,C) (B,R)~~

~~(T,C) (B,R) (B,C)~~

~~In the (T,C) phase, eqm play yields the maximum feasible stage game payoff to P1 indefinitely, so P1 has no profitable one shot deviation for any δ in this phase.~~

~~In the (B,R) phase, optimal one shot deviation by P1 is to M which yields 3 then 0 then 4 indefinitely. Eqm play yields 0 then 4 indefinitely. Compare PVs.~~ $\cancel{0+4\delta+\frac{4\delta^2}{1-\delta} \geq}$ ~~P1 has no profitable one shot deviation iff~~ $\cancel{0+4\delta+\frac{4\delta^2}{1-\delta} \geq 3+0+\frac{4\delta^2}{1-\delta} \iff 4\delta \geq 3}$

For $\delta = 0.6$, no player has a profitable one shot deviation, the above is a SPE.

c P1 minmaxes P2 by playing $\frac{1}{3}T + \frac{2}{3}B$. P2 best responds with either C or R, which yields minmax payoff $4/3$. This is lower than ~~any~~ the maximum ~~minimum~~ P2 payoff against any pure P1 strategy, so it is verified that P1 minmaxes P2 by mixing. This mix holds P2 to payoff $4/3$, P2 playing C or R (or any mix of the two guarantees payoff $4/3$), ~~so~~ so this is P2's minmax payoff.

~~d~~ P2 minmaxes P1 by playing $\frac{1}{4}C + \frac{3}{4}R$. P2 best responds with T, M, or any mix of the two, and has minmax payoff $5/2$. P2 holds P1 to this payoff by playing $\frac{1}{4}C+\frac{3}{4}R$. P1 guarantees this payoff by playing T, M, or any mix of the two, so this is P1's minmax payoff.

At SPE, each player has ADV no lower than his minmax payoff. Otherwise the SPE fails to deviation by some player to the stage game ~~strategy~~ strategy that guarantees his minmax payoff, in every period. So such a candidate SPE is not a ~~SP~~ NE, let alone a PBE.

In any subgame at SPE, each player has continuation ADV payoff ~~with~~ no lower than his stage game minmax payoff.

Consider some SPE that sustains (T,C) indefinitely by threatening to continue play following a P2-deviation with the continuation play that yields to P2 a payoff with ADV equal to minmax payoff $4/3$.

P2's optimal one shot deviation in the cooperation phase is to R which yields 4 then (effectively) $4/3$ indefinitely. This has PV $4+\frac{(4/3)\delta}{1-\delta}$. Eqm play yields 3 indefinitely. This has PV $3+\frac{3\delta}{1-\delta}$. ~~3≥~~ P2 has no profitable one shot deviation iff
$3+\frac{3\delta}{1-\delta} \geq 4+\frac{(4/3)\delta}{1-\delta} \iff \frac{(5/3)\delta}{1-\delta} \geq 1 \iff \frac{5}{3}\delta \geq 1-\delta \iff \frac{8}{3}\delta \geq 1 \iff \delta \geq 3/8$.

P1 has no ~~optimal~~ profitable one shot deviation in this phase because his eqm payoff is the maximum feasible.

By supposition, there is no profitable one shot deviation in ~~period~~ the punishment phase. ~~So ET~~

(T,C) forever is sustainable in eqm for $\delta \geq 3/8$