Game Theory Paper 160606

1a

	a	r
A	$V-c_2$ $V-c_1$	$\underline{V}$ $\underline{V-c_1}$
R	$\underline{V-c_2}$ $\underline{V}$	0 0

Given: $\forall i: 0 < c_i < V$. Best responses underlined. The game is a coordination game. By inspection, the two pure NE are (A, r) and (R, a).

Suppose there exists ~~an~~ NE $\sigma^* = (\sigma_1^*, \sigma_2^*)$ such that P1 mixes. Let p and q denote the probability that σ_1^* assigns to A and the probability that σ_2^* assigns to a respectively. By definition of NE, P1 has no profitable deviation, then $\pi_1(A, \sigma_2^*) = \pi_1(R, \sigma_2^*) \Leftrightarrow$ ~~$V-(1-q)$~~ $V - c_1 = qV \Leftrightarrow q = {}^{V-c_1}/_V$ ~~$\notin$~~. By definition of NE, P2 has no profitable deviation, then $\pi_2(a, \sigma_1^*) = \pi_2(r, \sigma_1^*) \Leftrightarrow V - c_2 = pV \Leftrightarrow p = {}^{V-c_2}/_V$. The unique mixed NE is $(pA + (1-p)R,\ qa + (1-q)r)$ where $p = {}^{V-c_2}/_V$, $q = {}^{V-c_1}/_V$.

If P1 mixes at NE so does P2. By symmetry, if P2 mixes at NE so does P1. Then, there are no hybrid NE.

p increases with decreasing c_2 relative to V. Intuitively this is because P1 mixes ~~such~~ at NE such that P2 is indifferent between a and r. The ~~greater~~ smaller c_2, the ~~lower~~ greater P2's incentive to a, so, informally, ~~the more~~ P1 must be more likely to ~~R~~ A, to reward P2's r-ing. If P2 is not indifferent between a and r, then P1 has a strict pure best response, and players do not mix. ~~When $c_2 = 0$;~~ As $c_2 \to 0$, $p \to 1$, ~~that~~ as $c_2 \to V$, $p \to 0$, by the above intuition.

Symmetrical ~~op~~ intuition holds for q and c_1.

b $V = 2$, $c_1 = 1/4$, $c_2 \in \{1/4, 3\}$, P1 has prior beliefs that $c_2 = 1/4$ with probability $1/2$ and $c_2 = 3$ with probability $1/2$. This game can be modelled as follows.

Players: $N = \{1, 2\}$ (indexed by i)

Actions: $A_1 = \{A, R\}$, $A_2 = \{a, r\}$

States: $\Omega = \{L, H\}$ (where L corresponds to the ~~low-co~~ realisation of a low-cost P2 and H corresponds to the realisation of a high-cost P2)

Signals: ~~$\tau_i(\omega) =$~~ $\tau_1(L) = \tau_1(H) = 0$

$\tau_2(L) = L$, $\tau_2(H) = H$

(P1 receives a perfectly uninformative signal, P2 receives a perfectly informative signal)

Beliefs: $P_1(\omega = L) = P_1(\omega = H) = 1/2$

$P_1(\omega = L \mid t_1) = P_1(\omega = H \mid t_1) = 1/2$

$P_2(\omega = L \mid t_2 = L) = 1$, $P_2(\omega = H \mid t_2 = H) = 1$

Payoffs: Given in the above payoff table.

~~In ex-ante payoffs~~ P1's strategies are not type contingent, since there is only one type of P1. ~~P~~ P2's strategies are type contingent. Denote P2 pure strategies $a_2^L a_2^H$, where a_2^L denotes ~~P2~~ low-cost P2's pure action and a_2^H denotes high-cost P2's pure action.

In ex-ante payoffs

	aa	ar	ra	rr
A	$3/8$ $7/4$	$5/8$ $\underline{7/4}$	$1/2$ $\underline{7/4}$	$\underline{2}$ $\underline{7/4}$
R	$3/8$ $\underline{2}$	$\underline{7/8}$ 1	$-1/2$ 1	0 0

By inspection, the only pure BNE is (A, rr) where players play mutual best responses. ~~That~~ ~~It~~ Intuitively, the ~~NE~~ ~~strategy~~ pure strategy profile under which P1 always plays R and P2 always plays a is not a BNE because type H P2 finds it optimal to play r, even if P1 plays R.

r is strictly dominant for type H P2.

Suppose there exists BNE σ^* such that P1 mixes. Let p and q denote the probability that P1 plays A at this BNE and the probability that type L P2 plays a at this BNE. By definition of BNE, P1 has no profitable deviation, then ~~$\pi_1(A, \sigma^*)$~~ $E_1(u_1(A, \sigma_2^{L*}, \sigma_2^{H*}; \omega)) = E_1(u_1(R, \sigma_2^{L*}, \sigma_2^{H*})) \Leftrightarrow 7/4 = 1/2(2q) + 1/2(0) \Leftrightarrow q = 7/4$. ~~By reductio~~ We have $q \in$ ~~$[0,1]$~~ $[0, 1]$. By reductio, there is no BNE such that P1 mixes.

Given that P1 does not mix at BNE and that type L P2 has strict best responses, type L P2 does not mix at BNE.

There are no mixed or hybrid BNE. The unique BNE is (A, rr). Intuitively, P1, ~~facing the~~ given the risk of facing a type ~~L~~ H P2 that never plays a, ~~prefers~~ always plays A. Then type L P2 free-rides on P1's ~~A~~ playing A.

$S_L =$ ~~$\{LL, LH, HL, HH\}$~~ $\{Ll, Lh, Hl, Hh\}$

~~τ_1~~ $\tau_1(Ll) = \tau_1(Lh) = L$, $\tau_1(Hl) = \tau_1(Hh) = H$

$\tau_2(Ll) = \tau_2(Hl) = l$, $\tau_2(Lh) = \tau_2(Hh) = h$

$P_1(t_2 = L \mid t_1) = P_1(t_2 = H \mid t_1) = 1/2$

$P_2(t_1 = L \mid t_2) = P_2(t_1 = H \mid t_2) = 1/2$

R is strictly dominant for type H P1. r is strictly dominant for type h P2. This fact is used to reduce the following payoff table

In ex ante payoffs

	ar	rr
AR	$\underline{15/8}$ $\underline{15/8}$	1 $\underline{7/8}$
RR	$\underline{7/8}$ 1	0 0

By inspection, the only pure BNE is (AR, ar), where each player plays his strictly dominant strategy.

The game is dominance solvable, so the above BNE is unique.

d Let s^* denote the required BNE under which each player plays the given threshold strategy.

$\pi_1(A, s_2^*) = v - c_1 = 2 - c_1$

$\pi_1(R, s_2^*) = P(c_2 > \bar{c}) \cdot 0 + P(c_2 < \bar{c}) \cdot v = 2\bar{c}/3$

$\pi_1(A, s_2^*) \leq \pi_1(R, s_2^*) \Leftrightarrow 2 - c_1 \leq 2\bar{c}/3 \Leftrightarrow c_1 \geq 2 - 2\bar{c}/3$

$\cancel{c_1 = 2 - 2c_2/3}$ $\bar{c} = 2 - 2\bar{c}/3 \Leftrightarrow 5\bar{c}/3 = 2 \Leftrightarrow \bar{c} = 6/5$

2a $u_i(e_i, e_{-i}) = \frac{1}{N}\sum_{j=1}^{N} ke_j - \frac{1}{2}ce_i^2$

$= k/N \sum_{j\neq i} e_j + k/N\, e_i - 1/2\, ce_i^2$

~~Most~~ Each worker's maximisation problem is

$\max_{e_i} u_i$

FOC: $\partial u_i/\partial e_i = k/N - ce_i = 0 \Leftrightarrow e_i = k/Nc$

SOC: $\partial^2 u_i/\partial e_i^2 = -c < 0$

$\Rightarrow e_i = k/Nc$ is in fact a maximum

Player i's best response function $BR_i(e_{-i}) = k/Nc$

At NE, ~~each player plays~~ players play mutual best responses. For each player i, $e_i^* = BR(e_{-i}^*) = k/Nc$, where * denotes ~~eq~~ NE values. Aggregate effort $E^* = Ne_i^* = k/c$.

Each player's best response function is a constant, so ~~there~~ each ~~the~~ strategy profile such that ~~ea~~ players play mutual best responses is such that each player plays this constant: such a strategy profile is unique.

Eqm (individual) effort e_i^* is increasing in productivity of effort k. The greater k, the greater the ~~out~~ marginal ~~output~~ product of effort, hence the greater the marginal increase in consumption due to effort, hence the greater the incentive for effort.

e_i^* is decreasing with increasing N. The greater N, the lower the marginal increase in consumption due to effort (since output is shared between a larger number of workers), hence the lower the incentive for effort.

e_i^* is decreasing with increasing c. The greater c, the greater the marginal disutility of effort hence the lower the incentive for effort.

b $W(e) = \sum_{i=1}^{N} u_i(e) = \sum_{i=1}^{N}\left(\frac{1}{N}\sum_{j=1}^{N} ke_j - \frac{1}{2}ce_j^2\right)$

$= \sum_{i=1}^{N}\left(\frac{1}{N}\sum_{j=1}^{N} ke_j\right) - \sum_{j=1}^{N}\left(\frac{1}{2}ce_j^2\right)$

$= \sum_{j=1}^{N} ke_j \sum_{i=1}^{N}\frac{1}{N} - \frac{c}{2}\sum_{j=1}^{N} e_j^2$

$= k\sum_{j=1}^{N} e_j - \frac{c}{2}\sum_{j=1}^{N} e_j^2$

$= \sum_{j=1}^{N}\left(ke_j - \frac{c}{2}e_j^2\right)$

The social optimisation problem is $\max_e W(e)$

FOC_j: $\partial W/\partial e_j = k - ce_j = 0 \Rightarrow e_j = k/c$

SOC_j: $\partial^2 W/\partial e_j^2 = -c < 0$

$e = (k/c, \ldots, k/c)$ is a maximum

At the social optimum $e^\dagger = (k/c, \ldots, k/c)$, each player chooses common effort, namely k/c.

$e_i^\dagger > e_i^*$. Precisely, $e_i^\dagger = Ne_i^*$.

The NE outcome is not socially optimal because each player, in choosing his ~~optima~~ (privately) optimal level of effort, does not account for the positive externality of effort, which consists in the increased utility from ~~consu~~ consumption by other workers of the output produced by each worker's effort.

The marginal social benefit of effort is N times as large as the marginal private benefit of effort, ~~The~~ since each worker consumes $1/N$ of the output produced by his effort. The marginal social cost of effort is equal to the marginal private cost of effort, ce_i. ~~Since~~ That $e_i^\dagger = Ne_i^*$ follows from the functional form of marginal cost and marginal social benefit being N times marginal private benefit.

c g is the marginal ~~disutility~~ (also average) disutility of each unit of "unearned" consumption.

$G_i = g\max\{Q/N - q_i, 0\}$

$= g\max\{k(e_i + \sum_{j\neq i}\hat{e})/N - ke_i, 0\}$

$= g\max\{k/N\, e_i + k(N-1)/N\, \hat{e} - ke_i, 0\}$

$= g\max\{k(N-1)/N\,(\hat{e} - e_i), 0\}$

$=_1 g\, k(N-1)/N\,(\hat{e} - e_i) >_2 0$

where $=_1$ ~~follows given $e_i < \hat{e}$~~ and $>_2$ follow given $e_i < \hat{e}$.

$\partial G_i/\partial e_i = -g\, k(N-1)/N < 0$

given that $g>0, k>0, N>1$

$e_i < \hat{e}$ induces positive guilt which decreases as e_i increases.

$G_i = g\max\{k(N-1)/N\,(\hat{e} - e_i), 0\}$

$=_1 0$

where $=_1$ follows from the supposition that $e_i \geq \hat{e}$

$e_i \geq \hat{e}$ induces zero guilt.

d Let $e^* = (k/c, \ldots, k/c)$

$u_i(e^*) = k(k/c) - 1/2\, c(k/c)^2 = k^2/c - k^2/2c = k^2/2c$

~~unilateral~~

~~Consider deviation from e^* to $e_i > e^*$.~~

At e^*, marginal private ~~consump~~ benefit (from consumption) due to effort, ~~is g~~ ~~$\partial G_i/\partial e_i =$~~ $\partial Q/N/\partial e_i = k/N$, marginal disutility of effort $\partial C_i/\partial e_i =^* ce_i =^* k$

Consider unilateral deviation from e^* to $e_i > e^*$. $\partial Q/N/\partial e_i|_{e^*} = k/N$, $\partial C_i/\partial e_i|_{e^*} = k$, $\partial G_i/\partial e_i|_{e^*} = 0$. An increase in effort has no effect on disutility due to guilt and ~~but~~ causes a large increase in total disutility of effort that exceeds the private benefit from consumption due to effort. So ~~an~~ a unilateral increase in effort from e^* is not ~~optimal~~ profitable.

Consider unilateral deviation from e^* to $e_i < e^*$.

$\partial a/N / \partial e_i |_{e^*} = k/N$, $\partial c_i / \partial e_i |_{e^*} = \cancel{ce} - k$,

$\partial G_i / \partial e_i |_{e^{*}-} = -k(N-1)/N$

$\partial u / \partial e_i |_{e^*-} = \cancel{k/N - k}\ k/N - k - -k(N-1)/N = 0$

A decrease in effort causes an increase in guilt and ~~an~~ a decrease in consumption that exactly offsets the decrease in disutility of effort. A unilateral decrease in effort from e^* is not strictly profitable.

There is no strictly profitable deviation from e^*, it is an NE.

Guilt yields NE outcome "closer" to the social optimum because ~~it~~ agents with guilt experience disutility from choosing ~~effort~~ ~~low effort~~, ~~which is equivalent~~ ~~equivalently, have higher than~~ lower effort than others and ~~are dist~~ have less incentive to free ride on the cooperative outcome.

3a Player i's strategy a_i ~~strictly~~ weakly dominates strategy a_i' iff for all strategy profiles of other players, a_i yields a weakly greater payoff than a_i', and for some strategy profile of other players, a_i yields a strictly greater payoff, i.e.
~~$\forall a_i \in$~~ $\forall \alpha_{-i} \in \Delta(A_{-i}): u_i(a_i, \alpha_{-i}) \geq u_i(a_i', \alpha_{-i})$ and
$\exists \alpha_{-i} \in \Delta(A_{-i}): u_i(a_i, \alpha_{-i}) > u_i(a_i', \alpha_{-i})$.

Player i's strategy a_i is ~~strictly~~ weakly dominant iff it ~~strictly~~ weakly dominates all other strategies a_i'.

b ~~Let b_i^+ denote some arbitrary $b_i > v_i$ and b_i^- denote some arbitrary $b_i < v_i$.~~

Consider $b_i^+ > b_i = v_i$. Suppose ~~$v_i < \max_{j \neq i} b_j$, then~~ Suppose $\max_{j\neq i} b_j < b_i < b_i^+$, then $u_i(b_i, b_{-i}) = u_i(b_i^+, b_{-i}) = v_i - \max_{j\neq i} b_j$. Suppose $b_i < \max_{j\neq i} b_j < b_i^+$, then $u_i(b_i, b_{-i}) = 0 > u_i(b_i^+, b_{-i}) = v_i - \max_{j\neq i} b_j$, which follows from $v_i = b_i < \max_{j\neq i} b_j$. Suppose $b_i < \max_{j\neq i} b_j = b_i^+$, then $u_i(b_i, b_{-i}) = 0 > u_i(b_i^+, b_{-i}) = \frac{1}{n}(v_i - \max_{j\neq i} b_j)$ where n is the number of tied highest bids. Suppose $b_i < b_i^+ < \max_{j\neq i} b_j$, then $u_i(b_i, b_{-i}) = u_i(b_i^+, b_{-i}) = 0$. ~~Cons~~ So b_i ~~weakly dom~~ $= v_i$ weakly dominates all $b_i^+ > b_i = v_i$.

Consider $b_i^- < b_i = v_i$. Suppose $\max_{j\neq i} b_j < b_i^- < b_i$, then $u_i(b_i, b_{-i}) = u_i(b_i^-, b_{-i}) = v_i - \max_{j\neq i} b_j > 0$. Suppose $\max_{j\neq i} b_j = b_i^- < b_i$, then $u_i(b_i, b_{-i}) = v_i - \max_{j\neq i} b_j > u_i(b_i^-, b_{-i}) = \frac{1}{n}(v_i - \max_{j\neq i} b_j)$. Suppose $b_i^- < \max_{j\neq i} b_j < b_i$, then $u_i(b_i, b_{-i}) = v_i - \max_{j\neq i} b_j > u_i(b_i^-, b_{-i}) = 0$. Suppose $b_i^- < \max_{j\neq i} b_j = b_i$, then $u_i(b_i, b_{-i}) = \frac{1}{n}(v_i - \max_{j\neq i} b_j) > u_i(b_i^-, b_{-i}) = 0$. Suppose $b_i^- < b_i < \max_{j\neq i} b_j$, then $u_i(b_i, b_{-i}) = u_i(b_i^-, b_{-i}) = 0$. So $b_i = v_i$ weakly dominates all $b_i^- < b_i = v_i$.

Consequently, $b_i = v_i$ is weakly dominant.

~~By definition of weak dominance, each player i has no (strictly) profitable deviation from $b_i = v_i$. Then, the strategy profile $b^* = (v_1, \ldots, v_N)$ is such that no player has a profitable deviation. By definition of NE, b^* is an NE.~~

By definition of a weakly dominant strategy, given that $b_i = v_i$ is a dominant strategy,
$\forall \beta_{-i} \in \Delta(B_{-i}): u_i(b_i = v_i, \beta_{-i}) \geq u_i(b_i' \neq v_i, \beta_{-i})$
$\Rightarrow \forall v_{-i}: u_i(b_i = v_i, b_{-i} = v_{-i}) \geq u_i(b_i' \neq v_i, b_{-i} = v_{-i})$
$\Rightarrow E(u_i(b_i = v_i, b_{-i} = v_{-i})) \geq E(u_i(b_i' \neq v_i, b_{-i} = v_{-i}))$
$\Rightarrow \forall b_i': E(u_i(b_i = v_i, b_{-i} = v_{-i})) \geq E(u_i(b_i'(v_i), b_{-i} = v_{-i}))$
In words, ~~for the strategy $b_i = v_i$~~ at the strategy profile such that each player i plays $b_i = v_i$, the strategy $b_i = v_i$ weakly maximises ex-ante (and interim) payoff for each player i. Consequently, that strategy profile is a BNE.

c No.

~~Suppose that~~ Denote the last bidder N. Suppose that $v_N > \max_{j\neq N} b_j$, then $u_N(b_N = v_N, b_{-N}) = v_N - \max_{j\neq N} b_j$, $u_N(b_N^+ > v_N, b_{-N}) = v_N - \max_{j\neq N} b_j$, $u_N(\max_{j\neq N} b_j < b_N^- < v_N, b_{-N}) = v_N - \max_{j\neq N} b_j$, $u_N(\max_{j\neq N} = b_N^{--} < v_N, b_{-N}) = \frac{1}{n}(v_N - \max_{j\neq N} b_j) < v_N - \max_{j\neq i} b_j$, $u_N(b_N^{---} < \max_{j\neq N} b_j, b_{-i}) = 0 < v_N - \max_{j\neq i} b_j$. So $b_N = v_N$ ~~weakly do~~ is weakly dominant if $v_N > \max_{j\neq N} b_j$.

Suppose that $v_N < \max_{j\neq N} b_j$, then $u_N(b_N = v_N, b_{-N}) = 0$
$u_N(b_N^- < v_N, b_{-N}) = 0$
$u_N(v_N < b_N^+ < \max_{j\neq N} b_j, b_{-N}) = 0$
$u_N(v_N < b_N^{++} = \max_{j\neq N} b_j, b_{-N}) = \frac{1}{n}(v_N - \max_{j\neq N} b_j) < 0$
$u_N(b_N^{+++} > \max_{j\neq N} b_j, b_{-N}) = v_N - \max_{j\neq N} b_j < 0$.
So $b_N = v_N$ is weakly dominant if $v_N < \max_{j\neq N} b_j$.

Suppose that $v_N = \max_{j\neq N} b_j$, then $u_N(b_N = v_N, b_{-N}) = \frac{1}{n}(v_N - \max_{j\neq N} b_j) = 0$, $u_N(b_N < v_N, b_{-N}) = 0$, $u_N(b_N^+ > v_N, b_{-N}) = v_N - \max_{j\neq N} b_j = 0$. So $b_N = v_N$ is weakly dominant if $v_N = \max_{j\neq N} b_j$.

$b_N = v_N$ is weakly dominant for all v_N, so N can do no better than by playing $b_N = v_N$.

d From (c), for each level 1 player, $b_i = v_i$ is weakly dominant. Given that level 0 players randomise, $b_i = v_i$ ~~maximises payoff for 1 expe~~ strictly maximises expected payoff for level 1 players. All level 1 players play $b_i = v_i$.
~~From~~ Then, given that level 1 players' valuations are drawn from the distribution $U(0, \bar{v})$, and that $b_i = v_i$ is weakly dominant, $b_i = v_i$ strictly maximises expected payoff for level 2 players. All level 2 players play $b_i = v_i$.

By induction, all level $k \in \{1, 2, \ldots\}$ players play $b_i = v_i$.

e $\pi_i'(b_i, b_{-i}) = P(b_i > b_{-i})(v_i - b_i) + P(b_i < b_{-i})(0)$
(~~Supposing for simplicity~~ Noting that $P(b_i = b_{-i}) = 0$)
$= F(b_i)(v_i - b_i)$
$= \cancel{(\bar{v} - b_i)(v_i - b_i)}\ b_i/\bar{v}\,(v_i - b_i)$
(Supposing that the maximum "possible" bid is $\bar{v}$)

Level 1 player's maximisation problem is
~~$\max_{b_i}\ \pi_i' = (\bar{v} - b_i)(v_i - b_i) = \bar{v}v_i + b_i^2 - (\bar{v} + v_i)b_i$~~
$\max_{b_i}\ \pi_i' = b_i/\bar{v}\,(v_i - b_i)$
~~FOC: $\partial\pi_i'/\partial b_i = 2b_i - (\bar{v} + v_i) = 0$~~
FOC: $\partial\pi_i'/\partial b_i = 1/\bar{v}\,(v_i - 2b_i) = 0 \Rightarrow b_i = v_i/2$
SOC: $\partial^2\pi_i'/\partial b_i = -2/\bar{v} < 0$
Level 1 player maximises expected payoff by

playing $b_i = v_i/2$

$\pi_i^2(b_i, b_{-i}) = P(b_i > b_{-i})(v_i - b_i)$
$= F_i(b_i)(v_i - b_i)$
$= b_i/\bar{v}/2 \; (v_i - b_i)$

Level 2 player's maximisation problem is
$\max_{b_i} \pi_i^2 = 2b_i/\bar{v} \, (v_i - b_i)$

FOC: $\partial \pi_i^2 / \partial b_i = 2/\bar{v} \, (v_i - 2b_i) = 0 \Rightarrow b_i = v_i/2$
SOC: $\partial^2 \pi_i^2 / \partial b_i^2 = -4/\bar{v} < 0$

Level 2 players each maximise expected payoff by playing $b_i = v_i/2$

By induction ~~for~~ all level $k \geq 1$ players play $b_i = v_i/2$.

f ~~Conjecture that $b^* = (b_1^*, b_2^*)$ such that $b_1^*(v_1) = v_1/2$ and $b_2^*(v_2) = v_2/2$~~ is a BNE.

~~Ex-ante interim payoffs~~
$E($~~$u_i(b_i^*(v_i), b_{-i}^*(v_{-i}); v_i, v_{-i})$~~$|v_i)$
$=$ ~~$E(u_i(v_i/2, v_{-i}/2; v_i, v_{-i})|v_i$~~$)$
$=$ ~~$P(v_i/2 > v_{-i}/2)(v_i - v_i/2)$~~
~~Noting that $P(v_i/2 = v_{-i}/2) = 0$ since valuations are drawn from a uniform distribution~~
~~$= v_i/\bar{v} \, (v_i/2) = v_i^2/2\bar{v}$~~

Conjecture that at BNE $b^* = (b_1^*, b_2^*)$, each player i plays $b_i^* = \alpha v_i$ for some constant α.

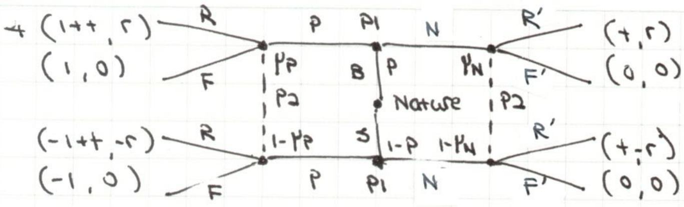
Interim payoffs
$E(u_i(b_i^*(v_i), b_{-i}^*(v_{-i}); v_i, v_{-i})|v_i)$
~~$= E(u_i(\alpha v_i, \alpha v_{-i}; v_i, v_{-i})|v_i)$~~
~~$= P(\alpha v_i > \alpha v_{-i})(v_i - \alpha v_i)$~~
~~$= E(u_i(\alpha v_i, b_{-i}^*; v_i, v_{-i})|v_i)$~~
~~$= E(u_i(\alpha_i v_i, b_{-i}^*; v_i, v_{-i})|v_i)$~~
~~$= P(\alpha v_i > b_{-i}^*)(v_i - \alpha v_i)$~~
~~$= E(u_i(\alpha_i v_i, \alpha_{-i} v_{-i}; v_i, v_{-i})|v_i)$~~
~~$= P(\alpha_i v_i > \alpha_{-i} v_{-i})(v_i - \alpha v_i)$~~
~~$= P(v_{-i} < \alpha_i v_i/\alpha_{-i})(1-\alpha) v_i$~~
~~$= \alpha_i v_i/\alpha_{-i}\bar{v} \, (1-\alpha) v_i$~~

~~FOC: $\partial/\partial \alpha_i$~~
$= E(u_i(b_i, \alpha_{-i} v_{-i}; v_i, v_{-i})|v_i)$
$= P(b_i > \alpha_{-i} v_{-i})(v_i - b_i)$
$= P(v_{-i} < b_i/\alpha_{-i})(v_i - b_i)$
$= b_i/\alpha_{-i}\bar{v} \, (v_i - b_i)$

FOC: $1/\alpha_{-i}\bar{v} \, (v_i - 2b_i) = 0 \Rightarrow b_i = v_i/2 \Rightarrow \alpha_i = 1/2$
SOC: $1/\alpha_{-i}\bar{v} \, (-2) < 0$

$b_1^* = v_1/2$, $b_2^* = v_2/2$ are mutual interim best responses, b^* is a BNE.

Intuitively, bidders have incentive to shade their bids below their ~~tr~~ valuations because, ~~at the truth~~ bidding truthfully, each player receives zero payoff from winning the auction (and from losing). Each bidder trades off a lower probability of winning for a higher payoff from winning.

4 

Given: ~~∅~~ $p \in [0,1]$, $r>0$, $t>1$

b Given: $p > 1/2$

Interim payoffs

	RR'	RF'	FR'	FF'
PP	$(2p-1)r$	$(2p-1)r$	0	0
	$\underline{1+t}, \underline{-1+t}$	$\underline{1+t}, \underline{-1+t}$	$1,-1$	$\underline{1},-1$
PN	$(2p-1)r$	$\underline{pr}$	$(p-1)r$	0
	$\underline{1+t}, \underline{t}$	$\underline{1+t}, 0$	$1, \underline{t}$	$\underline{1}, \underline{0}$
NP	$(2p-1)r$	$(p-1)r$	$\underline{pr}$	0
	$t, -\underline{1+t}$	$0, \underline{-1+t}$	$\underline{t}, -1$	$0,-1$
NN	$\underline{(2p-1)r}$	0	$(2p-1)r$	0
	$t, \underline{t}$	$0,0$	$\underline{t}, \underline{t}$	$0, \underline{0}$

where P1's strategy $a_1^B a_1^S$ denotes P1~~B~~ ~~as~~ plays a_1^B if P1 is bright and P1 plays a_1^S if P1 is feeble-minded, and P2's strategy $a_2^P a_2^N$ denotes P2 plays a_2^P if P1 played P and P2 plays a_2^N if P1 played N.

Best responses underlined.

By ~~##~~ inspection, the only pure BNE are (PP, RF) and (~~PP~~ NN, FR) where ~~players~~ P1B, P1S, and P2 play mutual interim best responses. ~~BNE~~ An assessment (σ, μ) is a PBE only if ~~σ is a B~~ strategy profile σ is a BNE.

Consider $\sigma = (PP, RF)$. By Baye's rule, $\mu_P = p > 1/2$. Bayesian beliefs does not place any constraints on μ_N. ~~RF (specifically~~ $a_2^N = F$ ~~⇒~~ is sequentially rational ~~iff~~ iff $0 \geq \mu_N r + (1-\mu_N)(-r) \Leftrightarrow \mu_N \leq 1/2$. ~~⇔~~ $a_2^P = R$ is sequentially rational iff $\mu_P r + (1-\mu_P)(-r) \geq 0 \Leftrightarrow \mu_P \geq 1/2$. Given that $\mu_P = p > 1/2$, $a_2^P = R$ is sequentially rational. Then $(\sigma, \mu) = ((PP, RF), (\mu_P = p, \mu_N \leq 1/2))$ is a PBE.

Consider $\sigma = (NN, FR)$. By Baye's rule, $\mu_N = p > 1/2$. Bayesian beliefs does not place any constraints on μ_P. $a_2^N = R$ is sequentially rational iff $\mu_N r + (1-\mu_N)(-r) \geq 0 \Leftrightarrow \mu_N \geq 1/2$. Given that $\mu_N = p > 1/2$, $a_2^N = R$ is sequentially rational. $a_2^P = F$ is sequentially rational iff $0 \geq \mu_P r + (1-\mu_P)(-r) \Leftrightarrow \mu_P \leq 1/2$. Then $(\sigma, \mu) = ((NN, FR), (\mu_P \leq 1/2, \mu_N = p))$ is a PBE.

Intuitive criterion: in assigning probabilities to off-equilibrium path histories, assign zero probability to types whose equilibrium payoffs exceeds the maximum deviation payoff for that type.

In the former PBE, IC implies $\mu_N = 0$. This is ~~consist~~ consistent with SR and BB which imply $\mu_N \leq 1/2$, so the former PBE satisfies the IC.

In the latter PBE, IC implies $1-\mu_P = 0 \Leftrightarrow \mu_P = 1$. This is not consistent with SR and BB which imply $\mu_P \leq 1/2$, so the latter PBE does not satisfy the IC.

c Given: $p < 1/2$, $\sigma_1^S = \frac{p}{1-p} P + \frac{1-2p}{p} N$

~~Interi~~ By ~~definit~~ sequential rationality,
~~$\pi_1^S(\sigma_1^S, \sigma_1^B, \sigma_2) = \pi_1^S(\sigma$~~
$\pi_1^S(P, \sigma_2) = \pi_1^S(N, \sigma_2) \Leftrightarrow q_2^P t - 1 = q_2^N t$
$\Leftrightarrow (q_2^P - q_2^N)t = 1$, where $q_2^X = P(a_2^X = R) \Rightarrow q_2^P > q_2^N$
$\pi_1^B(P, \sigma_2) = 1 + q_2^P t$, $\pi_1^B(N, \sigma_2) = q_2^N t$
Then $\pi_1^B(P, \sigma_2) > \pi_1^B(N, \sigma_2) \Rightarrow \sigma_1^B = P$
By Baye's rule, $\mu_P = \frac{p}{p + (1-p)(p/(1-p))} = 1/2$, $\mu_N = 0$

At P2's P information set, expected payoff to R is $\frac{1}{2}r + \frac{1}{2}(-r) = 0$, expected payoff to F is 0, so any $q_2^P \in [0,1]$ is SR.

At P2's N information set, expected payoff to R is $-r$, expected payoff to F is 0, so $q_2^N = 0$ is SR.

~~The candidate PBE is ((P, p/1-p P + 1-2p/1-p N, RF~~

By substitution, $q_2^P t = 1$, $q_2^P = 1/t$

The candidate PBE is $((P, \frac{p}{1-p}P + \frac{1-2p}{1-p}N, \frac{1}{t}R + \frac{t-1}{t}F, N), (\mu_P = 1/2, \mu_N = 0))$. This candidate PBE is derived by application of Bayesian beliefs for μ_P, μ_N, and sequential rationality for P1S, P1B, P2P, and P2N, so Bayesian beliefs and sequential rationality are satisfied, and it is in fact a PBE.

P1S (feeble minded academic) imitates P1B with probability $p/(1-p)$. There is incentive to imitate P1B because P2 rewards P by R. P1S cannot perfectly imitate P1B because P then fails to signal B, and P2 then does not reward P. P2 does not reward P by R with certainty because if it does, P1S has strict incentive to P, and again P fails to signal B.

d There are no off ~~path~~ equilibrium path beliefs, so the above PBE satisfies IC.

An increase in t (analytically) causes a ~~de~~ ~~in~~crease in the probability that P2 plays R on observing P at PBE. Intuitively, the greater the reward for being retained, the greater the incentive for P1S to imitate P1B. Since P1S is indifferent between P and N at PBE, the increase in magnitude of the reward must be offset by a decrease in its probability.

e By inspection of the payoff matrix above, when $p<1/2 \Leftrightarrow (2p-1)r<0$, there are no pure BNE, hence no pure PBE. If P2 plays a pure strategy (RR', RF', FR', or FF'), each of P1B and P1S has a strict best response in pure strategies, so each P1 plays a pure strategy. ~~and PBE is pure~~ By reductio, P2 does not play a pure strategy. Then by sequent- sequential rationality, $\mu_P = 1/2$ or $\mu_N = 1/2$.

~~if P1~~

Suppose P1S mixes with different probabilities, then by a similar argument to that in (c), $q_2^P = q_2^N \Rightarrow \sigma_1^B = P$ $\Rightarrow \mu_N = 0$, $\mu_P \neq 1/2$. By reductio, if P1S mixes, P1S mixes with the given probability.

If P1S does not mix then either $\sigma_1^S = P \Rightarrow \sigma_2^P = F^* \Rightarrow \sigma_1^S \neq P$ where the ~~second~~ implications follows by sequentially rationality or $\sigma_1^S = N \Rightarrow \sigma_2^P = R \Rightarrow \sigma_1^S = P$. ~~where the~~ by sequential rationality. By reductio, P1S mixes.

Consequently, the only PBE is that found in (c).

5a Consider the bargaining problem (U, d), where U is the set of possible agreement payoff vectors, and d is the disagreement payoff vector. Let $F(U,d)=u$ denote the bargaining solution.

$F(U,d)=u$ satisfies WPAR iff ~~$\nexists v \in U: \forall i$~~ $\nexists v \neq u \in U: \forall i \in \{1,2\}: v_i > u_i$, ~~which implies~~. In words, there is no payoff vector v in U such that all players i are strictly ~~weakly~~ better off at ~~still~~ v than at u, ~~and some player is strictly better off~~.

$F(U,d)=u$ satisfies SYM iff if $\forall (u_1,u_2) \in U: (u_2,u_1) \in U$ and $d_1 = d_2$, then $u_1 = u_2$. In words, if the set of possible agreement payoff vectors is symmetric and the disagreement payoff vector is symmetric, then the ~~bargain payoff vec~~ solution is symmetric.

Let $f_i(u_i) = \alpha_i + \beta_i u_i$ for all i, where α_i and β_i are arbitrary constants. $F(U,d)=u$ satisfies INV iff for all such f_i, $F(U',d')=u'$ where $U' = \{(f_1(u_1), \ldots, f_n(u_n)) : (u_1, \ldots, u_n) \in U\}$, $d' = (f_1(d_1), \ldots, f_n(d_n))$, and $u' = (f_1(u_1), \ldots, f_n(u_n))$. In words, if one bargaining problem is a linear transformation of another, the bargaining solution of the former is given by the same linear transformation of the bargaining solution of the latter.

$F(U,d)=u$ satisfies IIA iff if $U' \subseteq U$ and $u \in U'$, then $F(U',d)=u$. In words, the removal of non-solution possible agreements does not affect the solution.

b WPAR: Consider $F(U,d) = (1-\varepsilon)\operatorname{argmax}_u \prod_{i=1}^{n}(u_i - d_i)$. Given that the Nash bargaining solution satisfies SYM, INV, and IIA, it is trivial that if the above solution exists, it satisfies SYM, INV, and IIA, and does not satisfy WPAR since the Nash solution strictly dominates it. For sufficiently small ε, given that U is convex, the above solution exists.

Let superscript N denote the Nash solution.

SYM: Consider $F(U,d) = \operatorname{argmax}_u \prod_{i=1}^{n}(1+\varepsilon_i)(u_i - d_i)$ for small ε_i ~~such that~~ and ~~$\forall i,j: \varepsilon$~~ $\forall i \neq j: \varepsilon_i \neq \varepsilon_j$. It is trivial that such a solution satisfies WPAR, INV, IIA, but not SYM.

INV: Consider $F(U,d) = \operatorname{argmax}_u \sum_{i=1}^{n}(u_i - d_i)$. Such a solution is not invariant to equivalent payoff representations that asymmetrically scale players' utility. It is trivial that this solution satisfies SYM, WPAR, IIA.

IIA: Consider $F(U,d) = \operatorname{argmax}_u \prod_{i=1}^{n}(u_i - \min u_i)$. Such a solution is not IIA ~~since it depends on minimal u such that $\exists i: u_i = \min u_i$~~. It satisfies WPAR, SYM, INV.

c ~~Suppose that P2 makes the first offer and offers P1 $20-x$, where $x \geq 0.22$. If P1 rejects and offers P2 $0.20 \leq x < 0.22$, it is optimal for P1 to accept $x = 20 - 0.22 + \varepsilon$, for sufficiently small it is optimal for P1 to accept since P1 can receive no more than $20 - 0.22$ otherwise. Then P1 receives $0.22 - \varepsilon$ $- 0.20$. If P1 accepts, P1 receives x~~

Suppose that players' strategies are stationary. ~~Let x_j^i denote the offer that player i makes to player j.~~

~~Let $\hat{x}_2^1 = \hat{x}_2^2 - 0.22$, $\hat{x}_1^2 = \hat{x}_1^1 - 0.20$~~

~~Suppose for reductio that P2 offers P1 $20-x$, where $x \geq 0.22$. Then it is optimal for P1 to reject and offer P2 $x - 0.22 + \varepsilon$ because it is optimal for P2 to accept this because if P2 rejects this, P2's continuation payoff is no greater than x and $x - 0.22$. Then, P1 has payoff $20 - (x - 0.22 + \varepsilon) - 0.2$ from $= 20 - x + 0.22 - 0.20 - \varepsilon > 20$ for sufficiently small ε from rejecting P2's initial offer. Then P2 has payoff no greater than~~ No ~~initial $x \geq 0.22$ is accepted. By reductio, $x < 0.22$.~~

Suppose for reductio that the maximum payoff to P2 at SPE is $\bar{u}_2^* \geq 0.22$. Suppose further that the SPE that realises $u_2^* = \bar{u}_2^*$ is such that P2 makes the last offer and P1 accepts. Then the last offer is such that P2 offers P1 ~~$20 - x_2^2$~~ $x_1^2 = 20 - x_2^2$ and receives $x_2^2 \geq \bar{u}_2^* \geq 0.22$ where x_j^i denotes player i's offer to player j. Suppose that P1 rejects this and offers $x_2^1 = x_2^2 - 0.22 + \varepsilon$. It is rational for P2 to accept this counteroffer since P2's maximum feasible payoff at SPE of the continuation, by supposition, is $\bar{u}_2^* - 0.22 \leq$ ~~$x_2^2 - 0.22$~~ $x_2^2 - 0.22 < x_2^1$. ~~It is rational for P1~~ Then, by rejecting $x_1^2 = 20 - x_2^2$, P1 receives at least $20 - 0.2 - (x_2^2 - 0.22 + \varepsilon)$ $= 20 - x_2^2 + 0.02 - \varepsilon > x_1^2$ (for sufficiently small ε), so it is rational for P1 to reject x_1^2. By definition of SPE, there is no SPE such that the last offer is by P2, ~~at $x_2^2 \geq \bar{u}_2^*$~~ P1 accepts, and $u_2^* \geq 0.22$

Suppose instead that the SPE (that realizes $u_2^* = \bar{u}_2^* \geq 0.22$) is such that P1 makes the last offer and P2 accepts. Then $x_2^1 \geq \bar{u}_2^* \geq 0.22$. This is not rational for ~~P2~~ P1 because P2 would accept any $x_2^1 \geq \bar{u}_2^* - 0.22$, which is the maximum ~~payoff~~ possible payoff from rejection, and ~~such~~ $\bar{u}_2^* - 0.22 \leq x_2^1 < \bar{u}_2^*$ yields a greater payoff for P1. By reductio, there is no SPE such that the last offer is by P1, P2 accepts, and $u_2^* \geq 0.22$

~~By reductio~~ Then if $\bar{u}_2^* \geq 0.22$, there is no SPE such that $u_2^* \geq 0.22$. By reductio, $\bar{u}_2^* < 0.22$.

d From (c), at SPE if P2 rejects P1's offer, P2's ~~payoff~~ ~~is neg~~ continuation payoff is less than 0.22. Given that rejection costs P2 0.22, P2's payoff from rejecting any offer is strictly negative. Then, at SPE, P2 accepts any $x_2^1 \geq 0$, P1 maximises payoff by offering $x_1^1 = 20$, $x_2^1 = 0$ in all periods, and P2 always accepts $x_2^1 \geq 0$.

6a Each player's pure strategy is some history-contingent plan of stage-game actions $A=\{C, D\}$. Given that the repeated game ends if any player plays D, the only histories at which players are called upon to act are those consisting of action profiles under which each player plays C, i.e. histories of the form $((C,C), (C,C), \dots, (C,C))$. Then each player's pure strategy is some mapping from such histories to some $a \in A$. ~~So each player's pure strategies are strategy~~ ~~so the set of possible pure strategies for eac~~ so for each player i, every pure strategy is fully characterised by some ~~indefinite~~ ~~string~~ infinite sequence of $a_t \in C, D$ such as $(C, C, C, D, C, \dots)$, where the t^{th} stage game action t in the string is played at the history $(a_1=(C,C), \dots, a_{t-1}=(C,C))$ ~~at the t^{th} period)~~ (at period t). Note that a strategy does not "terminate" after an occurrence of D because a strategy must ~~specify also~~ ~~as~~ specify actions off the equilibrium path.

b ~~Each pure strategy~~ Every two pure strategies a_i and a_i' of player i such that the first occurrence of D in each is at the common period t fare equally well against any ~~a_j~~ strategy a_j of player j because a_i and a_i' are identical up to t, and the game terminates immediately after t. So it is without loss of generality to consider pure strategies that contain at most one occurrence of D. Let a^t for $t = 1, 2, \dots$ denote the pure strategy "C in every period except t" where a^∞ is the pure strategy "C in every period". ~~and a~~

~~$u_1(a^1, a^2) = 4$, $u_2(a^1, a^2) = 0$~~
~~$u_1(a^1, a^1) = 1$, $u_1(a^2, a^2) = 7/2$~~
~~Then a^1 is evolutionarily stable because~~

Consider arbitrary $t > 1$, $t \neq \infty$. Then a^t is not an ESS because $u(a^{t-1}, a^t) > u(a^t, a^t)$ ~~and $u(a^{t-1}, a^{t-1}) > u(a^t, a^{t-1})$, i.e. a^{t-1} is~~ then (a^t, a^t) is not a NE. In words, defecting one period earlier yields an increase in payoff in period $t-1$ of 1 (from 3 to 4), and a decrease in expected payoff in period t of ½ (from ½ to 0), then (a^t, a^t) fails to deviation by either player to a^{t-1}.

(a^∞, a^∞) is a strict NE. Deviation from a^∞ by ~~deviation~~ defection in period t yields an increase in payoff of 1 (from 4 to 3) in period t but a decrease in payoff of ~~€~~ 3 (from $\frac{1}{2}3 + (\frac{1}{2})^2 3 + \dots$ to 0) ~~as~~ from subsequent periods. a^∞ is a strict best response to a^∞, then a^∞ is an ESS.

It is trivial that (a^1, a^1) is a strict NE, then a^1 is an ESS.

The ~~ESS are th~~ pure ESS ~~are the following~~ are the strategy "C in every period" and any strategy that involves "D in the first period"

c Suppose that each player i plays a stationary mixed strategy at NE ~~at~~ under which in ~~ea~~ each period t, there is a constant probability ~~*~~ p that i plays C. Then, by definition of NE, i has no profitable deviation in any period t. This is ~~iff * payoff~~ iff payoff to i from C is equal to that from D. If i plays C at t, i's expected payoff is $3p$ in t and $\frac{1}{2}c$ ~~where th~~ from continuation play, where c is the continuation payoff. If i plays D, i's payoff ~~at t~~ is $4p + 1(1-p) = 3p+1$ in t and 0 from subsequent periods. By stationarity, $c = 3p + \frac{1}{2}c = 3p+1 \Rightarrow c = 2, p = \frac{1}{3}$. The strategy profile under which each player mixes with probability $p = \frac{1}{3}$ (of C) in every period is a NE.

~~This~~ The strategy of mixing with $p = \frac{1}{3}$ is not an ESS because a^∞ is a weak best response against this strategy, and $u(a^\infty, a^\infty) > u(a^{1/3}, a^\infty)$ where a^p for $p \in (0,1)$ is the stationary mixed strategy of playing C with probability p in every period.

d

A	a^1	a^∞
a^1	1	4
a^∞	0	6

Row player's payoffs

Let p_1 denote the proportion of a^1 players and p_∞ denote the proportion of a^2 players. Let $p = (p_1, p_\infty)^{T}$. Then the 2x1 column matrix Ap gives the average payoff of each type and the 1x1 matrix $p^T A p$ gives the population average payoff.

$$Ap = \begin{pmatrix} p_1 + 4p_\infty \\ 6p_\infty \end{pmatrix}$$

Replicator equation

$$\dot p_i = p_i(1-p_i)([Ap]_i - [Ap]_j)$$

$$\dot p_1 = p_1(1-p_1)(p_1 + 4p_\infty - 6p_\infty) = p_1(1-p_1)(p_1 - 2(1-p_1)) = p_1(1-p_1)(3p_1 - 2)$$

$\dot p_1 = 0 \Leftrightarrow p_1 = 0, 1, \frac{2}{3}$. $p_1 \in (\frac{2}{3}, 1) \Leftrightarrow \dot p_1 > 0$. $p_1 \in (0, \frac{2}{3}) \Leftrightarrow \dot p_1 <$

For $1-\alpha = 0, 1, \frac{2}{3}$ ($\alpha = 1, 0, \frac{1}{3}$), the population state does not evolve under the replicator dynamic, and "converges" to the initial state where a proportion $1-\alpha$ play a' and ~~a~~ the remaining α play a^∞.

For $1-\alpha \in (2/3, 1)$ $(\alpha \in (0, 1/3))$, the proportion $1-\alpha$ of a' players grows in the first period and in each subsequent period because given a ~~su~~ sufficiently large proportion of a' players, a' players fare better against the population than a^∞ players. The population state converges to the state where all players play a'

For ~~(t~~ $1-\alpha \in (0, 2/3)$ $(\alpha \in (1/3, 1))$, the proportion $1-\alpha$ of a' players shrinks in the first period and in each subsequent period because given a sufficiently large proportion of a^∞ players, a^∞ players fare better against the population than a' players. The population state converges to the state where all players play a^∞.

e Denote the two populations A and B. Denote the proportion of a^∞ players in population $X \in \{A, B\}$ as $p^X_{a^\infty}$.

From (d), $\pi_X(a^\infty, p^Y_{a^\infty} a^\infty + p^Y_{a'} a') \gtreqless \pi_X(a', p^Y_{a^\infty} a^\infty + p^Y_{a'} a')$, (where $p^Y_{a'} = 1 - p^Y_{a^\infty}$, $X, Y \in \{A, B\}$, $X \neq Y$) $\Leftrightarrow p^Y_{a^\infty} \gtreqless 1/3$.

Consider the deterministic best response dynamics to determine the absorbing states. ~~b~~ From the above $\dot{p}^X_{a^\infty} \gtreqless 0 \Leftrightarrow p^Y_{a^\infty} \gtreqless 1/3$. Then, the absorbing states are $(p^A_{a^\infty}, p^B_{a^\infty}) = (0,0), (1/3, 1/3), (1,1)$.

The $(1/3, 1/3)$ absorbing state is stochastically unstable because each player in each population is indifferent between a' and a^∞. So with non-zero probability $(1/2)$, the selected agent ~~ch~~ changes his strategy to the other weak best response. Then each agent in the other population has a strict best response. With non-zero probability, even with no agent choosing a non-best response, the population state evolves to one of the other absorbing states.

~~From (d), there are the~~ Let p_∞ denote the proportion of a^∞ players in the population. Let p_1 denote the proportion of a' players in the population. Then $p' = 1 - p^\infty$.

$\pi_1(a^\infty, p_\infty a^\infty + p_1 a') = \cancel{p_1 + 4p_\infty}\ 6p_\infty$

$\pi_1(a', p_\infty a^\infty + p_1 a') = p_1 + 4p_\infty$

From (d), $\pi_1(a^\infty, p_\infty a^\infty + p_1 a') \gtreqless \pi_1(a', p_\infty a^\infty + p_1 a') \Leftrightarrow$ $p_\infty \gtreqless 1/3$

Then the absorbing states under deterministic best ~~resp~~ reply dynamics are ~~$(p_\infty = 1/3)\ p_\infty$~~ $p_\infty = 0, 1/3, 1$.

The $p_\infty = 1/3$ absorbing state is stochastically unstable. ~~In each~~ Each player has two best responses to the distribution of strategies in the population. Then in each period, with probability $1/2$, the selected player updates his strategy to a different strategy, then each player has a strict best response to the distribution in the population. Then, even with no ~~a~~ player selecting a~~n~~ non-best response, the population state converges to one of the other two absorbing states.

The minimum number of agents who must select a non-best response ~~to~~ such that the population state evolves from $p_\infty = 0$ to $p_\infty = 1$ is $\frac{1}{3}(2N)$. For the reverse, it is $\frac{2}{3}(2N)$.

Given small ε, the number of non-best responses which occurs with probability $\varepsilon/2$ is the dominant factor in the probability of such evolutions. Then, ~~the stab~~ the $p_\infty = 1$ state is the stochastically stable state.

In the long run, under stochastic best reply dynamics, the population remains at or near the $p_\infty = 1$ state ~~for a large~~ ~~for more periods~~ more often than it remains at or near the other absorbing states. This state has a larger basin of attraction.

TG	L	R	$x \in (0,3)$
U	$\underline{x}$, $\underline{x}$	0, $\underline{6}$	
D	$\underline{6}$, 0	0, 0	

Best responses underlined. By inspection, U is strictly dominant for P1 and L is strictly dominant for P2. Then at NE, neither player mixes (because each player has strict incentive to play his dominant strategy with certainty). The unique pure NE is (U,L), where players play mutual best responses. (U,L) is the unique NE.

b Let $\underline{v}_i$ denote player i's minmax payoff.

$\underline{v}_1 = \min_{\alpha_2 \in \Delta(A_2 = \{L,R\})} \max_{\alpha_1 \in \Delta(A_1 = \{U,D\})} u_1(\alpha_1, \alpha_2)$

By definition of a strictly dominated strategy, P1's strictly dominant strategy U ~~sol~~, for any given α_2, solves $\max_{\alpha_1 \in \Delta(A_1)} u_1(\alpha_1, \alpha_2)$. Then the above equation reduces to

$\underline{v}_1 = \min_{\alpha_2 \in \Delta(A_2)} u_1(U, \alpha_2)$.

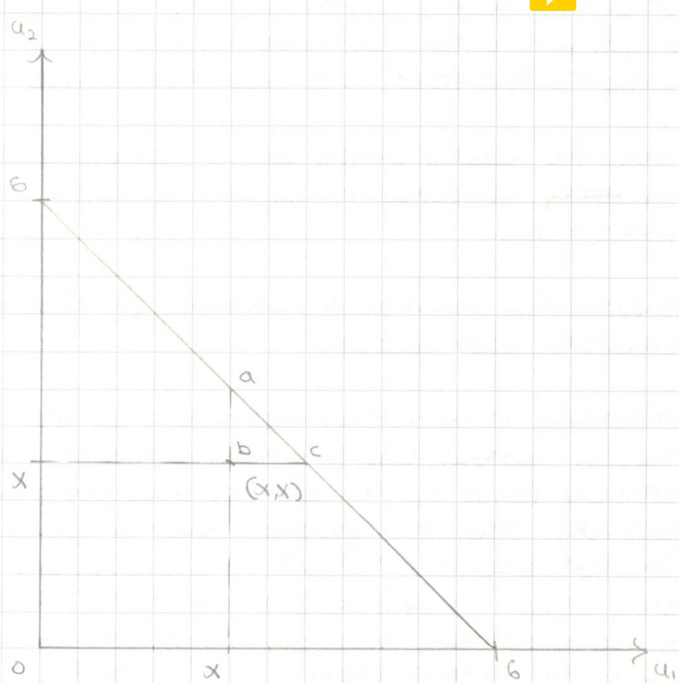
~~By inspection,~~

$= \min_{\alpha_2 \in \Delta(A_2)} p_2 x + (1-p_2)6$

where p_2 is the probability α_2 assigns to L

$= x$

P2 minmaxes P1 by playing L and P1 optimally responds by playing U. P1 then has minmax payoff x. By symmetry, $\underline{v}_2 = x$.



The set of feasible and individually rational payoff pairs is represented by area abc (inclusive of boundaries).

c The given strategy profile is such that play alternates between (D,L) and (U,R) iff no player ~~deviates, and~~ previously deviated and players play the unique NE (U,L) ~~at every period~~ otherwise, in each period.

Neglecting alternation, i.e. ~~treating alternation as~~ the initial "cooperation" phase as homogenous, in which at each period, each player receives average payoff ~~between the alternat~~ from the two alternating strategy profiles) in what follows.

By the one-shot deviation principle, the given strategy profile is a SPE iff at ~~no that~~ every history, each player has no profitable ~~deviat~~ one-shot deviation.

In the initial "cooperation" phase (i.e. at any history such that no player previously deviated), optimal one-shot deviation yields 6 in the period of deviation and x in every subsequent period. Present value of this payoff stream is $6 + \delta/(1-\delta)\, x$. ~~Playing~~ Non-deviation yields 3 in every period. PV of this payoff stream is $3/(1-\delta)$. No player has incentive to deviate only if $3/(1-\delta) \geq 6 + \delta/(1-\delta)\, x \iff 3 \geq 6(1-\delta) + \delta x \iff 3 - 6 \geq (x-6)\delta \iff \delta \geq 3/(6-x)$.

In the "punishment" phase (i.e. at any history such that ~~one~~ some player previously deviated), players play strictly dominant stage game strategies (and continue to do so in every subsequent period), there is no profitable one-shot deviation.

The minimum discount factor δ such that the given strategy profile is an SPE is $\underline{\delta} = 3/(6-x)$.

~~Treating~~ Considering alternation, at the on-equilibrium path prescribing (U,R) ~~((U,D) ((D,L))~~, P1's one-shot deviation yields PV $0 + \cancel{\delta x}\, \delta x + \delta^2 x + \dots = \delta/(1-\delta)\, x$, P1's ~~equili (P2's)~~ equilibrium play yields $\cancel{x+\delta}\ 6 + \delta 0 + \delta^2 6 + \dots = 6/(1-\delta^2)$, P2's ~~(P1's)~~ one-shot deviation yields PV $x + \delta x + \delta^2 x + \dots = x/(1-\delta)$, P2's equilibrium play yields PV $0 + \delta 6 + \delta^2 0 + \delta^3 6 = \delta/(1-\delta^2)\, 6$. ~~By inspection, P1 P2 (P1) has no profitable one shot deviation iff P1 (P2) does not which is iff P1 has no profitable one shot deviation iff $\delta/(1-\delta^2) \geq$~~ By inspection, if P2 has no profitable one-shot deviation, neither does P1. P2 has no such deviation iff $\cancel{x}\ 6\delta/(1-\delta^2) \geq x/(1-\delta) \iff 6\delta(1-\delta) \geq x - x\delta^2$

$\cancel{(x-6)\delta^2 + 6\delta - x \geq 0 \iff}$ $\iff \delta \geq x/(6-x)$

$\cancel{-6+\sqrt{36+4(x-6)}/2(x-6) \leq \delta \leq -6-\sqrt{36+4(x-6)}/2(x-6)}$

$\cancel{\iff -6+\sqrt{12+4x}/2x-12 \leq \delta \leq -6-\sqrt{12+4x}/2x-12}$

$\cancel{\iff 6-\sqrt{12+4x}/12-2x \leq \delta \leq 6+\sqrt{12+4x}/12-2x}$

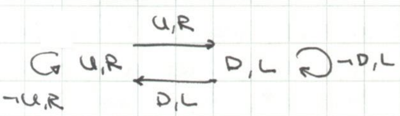
$\cancel{\underline{\delta} = 6-\sqrt{12+4x}/12-2x}$

d Consider the ~~following~~ strategy profile. such that play alternates between ~~(U,L) and (D~~ (U,R) and (D,L) in each period iff no player previously deviated. If player i deviates, begin the phase Punishment^i

Punishment^i: alternate between (U,R) and (D,L) to minmax i.

Neglecting alternation, in the initial cooperation phase, each player i has no profitable deviation iff PV from deviation $6 + \delta x + \delta^2 x + \ldots = 6 + \delta/(1-\delta)\, x$ is less than PV from equilibrium play $3 + \delta 3 + \delta^2 3 + \ldots = 3/(1-\delta)$. From (c), this is iff $\delta \geq 3/(6-x)$. Intuitively, this is because the punishment above is identical, for the deviating player, to the punishment in (c).

Consider the strategy profile represented by the following automaton.



Suppose that in period t, the prescribed (by the above strategy profile) play is U,R. Then, P1's one-shot deviation yields $0 + \delta 6 + \delta^2 0 + \delta^3 6 + \ldots = \delta/(1-\delta^2)\, 6$. P1's equilibrium play yields $6 + \delta 0 + \delta^2 6 + \ldots = 1/(1-\delta^2)\, 6$. P1 has no profitable ~~dev~~ one-shot deviation for all $\delta \leq 1$. P2's one-shot deviation yields $x + \delta 0 + \delta^2 6 + \delta^3 0 + \ldots = x + \delta^2 6/(1-\delta^2)$. P2's equilibrium play yields $0 + \delta 6 + \delta^2 0 + \delta^3 6 + \ldots = \delta 6/(1-\delta^2)$. P2 has no profitable one-shot deviation iff $\delta 6/(1-\delta^2) \geq x + \delta^2 6/(1-\delta^2)$ $\Leftrightarrow \delta 6 \geq x(1-\delta^2) + \delta^2 6 \Leftrightarrow \delta(1-\delta)6 \geq x(1-\delta)(1+\delta)$ $\Leftrightarrow 6\delta \geq x + x\delta \Leftrightarrow \delta \geq x/(6-x)$

By symmetry, at any period where the prescribed play is D,L, there is no profitable one-shot deviation iff $\delta \geq x/(6-x)$

The ~~given~~ above strategy profile is an SPE for all $\delta \geq \underline{\delta} = x/(6-x)$.

This coincides with the critical discount factor found in (c). The critical discount factor found in (c) is the minimum discount factor such that alternation between (U,R) and (D,L) is sustainable in equilibrium since deviation is maximally punished with minmax payoffs in all subsequent periods. Then, ~~the~~ this critical discount factor is the minimum such that alternation is sustainable with only (U,R) and (D,L) played in equilibrium.

e The critical discount factors are equal.

~~Deviation~~ In (c), the most attractive deviation yields x in perpetuity with ~~PV~~ ~~while equilibrium play instead yields~~ PV $x/(1-\delta)$ while equilibrium play instead yields PV $6\delta/(1-\delta^2)$. In (d), the most attractive deviation yields PV $x + 6\delta^2/(1-\delta^2)$ while equilibrium play insted yields $6\delta/(1-\delta^2)$. It is by coincidence that the critical discount values are equal.

f The highest feasible symmetric expected average discounted payoff is 3.

For $x=1$, $\underline{\delta} = 1/5 < \delta = 4/7$. For $x=2$, $\underline{\delta} = 1/2 < \delta = 4/7$.

Each of the above (in (c) and (d)) x-independent strategy profiles is an SPE.

The required payoff pair is achieved only by alternation between (U,R) and (D,L). The required strategy profile is such that on the equilibrium path, players so alternate. Then, by the one shot deviation principle, for any x, each player has no profitable one-shot deviation at (U,R) and (D,L)

The required payoff pair is achievable only if players play only (U,R) and (D,L). The required payoff pair is not achievable by alternating since the "first-mover" will have higher payoff.

Then, the required payoff pair is achievable only by ~~using~~ correlated mixing with x as the public signal.

Suppose that deviation is maximally punished by irreversible Nash reversion.

It remains to be verified that this is an SPE. Suppose ~~without loss of generality~~ that at some arbitrary period t, $x=2$ and the prescribed play is (U,R). Then P1's one-shot deviation yields PV $0 + \delta E(x) + \delta^2 E(x) + \ldots = \delta/(1-\delta)\, 3/2$. P1's equilibrium play yields $6 + \delta(\frac{1}{2}6 + \frac{1}{2}0) + \delta^2(\frac{1}{2}6 + \frac{1}{2}0) = 6 + \delta/(1-\delta)\, 3/2$. P1 has no profitable deviation. P2's one shot deviation yields PV $2 + \delta E(x) + \delta^2 E(x) + \ldots = 2 + \delta/(1-\delta)\, 3/2$. P2's equilibrium play yields PV $0 + \delta(3) + \delta^2(3) + \ldots = \delta/(1-\delta)\, 3$. P2 has no profitable deviation iff $\delta/(1-\delta)\, 3 \geq 2 + \delta/(1-\delta)\, 3/2 \Leftrightarrow$ ~~$3/2 \geq 2(1-\delta) \Leftrightarrow 2\delta \geq \ldots 1/2$~~ $3\delta/2 \geq 2(1-\delta) \Leftrightarrow 2 \leq 7\delta/2 \Leftrightarrow \delta \geq 4/7$. Then, there is no profitable deviation on the equilibrium path. It is trivial that there is no profitable deviation off the equilibrium path. This is in fact an SPE. ~~It is the unique such SPE because the required payoff~~