Microeconomic Analysis Paper 210618

2a. A concave optimization problem is one that has the following form.

$$\max_{\vec{x}} f(\vec{x}) \quad \text{s.t. } \vec{g}(\vec{x}) \leq \vec{b}$$

where the objective function f is concave, and each constraint g (~~#~~ $\vec{g} = (g_1, \dots, g_m)$ and each ~~for~~ $g_i(\vec{x}) \leq b_i$ is one constraint) is convex.

A convex optimization problem is one that has the following form.

$$\min_{\vec{x}} f(\vec{x}) \quad \text{s.t. } \vec{g}(\vec{x}) \geq \vec{b}$$

where f is convex and each g is concave.

If an optimization problem is concave (convex) then the KT-FOCs are sufficient for a maximum (minimum). If, in addition, the constraint set is non-empty, ~~then the KT-FOCs~~ or the constraint qualification cannot fail, then the KT-FOCs are also necessary.

b $f(x,y) = 2x^2 + 2y^2 - 2xy - 9y$

$Df(x,y) = \begin{pmatrix} 4x - 2y & 4y - 2x - 9 \end{pmatrix}$

$D^2f(x,y) = \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}$

$\det D^2f(x,y) = 16 - 4 = 12 > 0$

$\operatorname{tr} D^2f(x,y) = 8 > 0$

Both eigenvalues of $D^2f(x,y)$ are positive ($\det > 0 \Rightarrow$ product is positive, $\operatorname{tr} > 0 \Rightarrow$ sum is positive), so $D^2f(x,y)$ is positive definite, and f is convex.

$g(x,y) = y - 4x^2 + 2$

$Dg(x,y) = \begin{pmatrix} -8x & 1 \end{pmatrix}$

$D^2g(x,y) = \begin{pmatrix} -8 & 0 \\ 0 & 0 \end{pmatrix}$

$\det D^2f(x,y) = 0$

$\operatorname{tr} D^2f(x,y) = -8$

One eigenvalue of $D^2g(x,y)$ is $\cancel{\neq}\, 0$, the other is strictly negative, so $D^2g(x,y)$ is negative semi-definite and g is (weakly) concave.

c ~~Dfx~~ FOC: $Df(x,y) = \vec{0} \Rightarrow$

$4x - 2y = 0, \quad 4y - 2x - 9 = 0 \Rightarrow$

$$\begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 9 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 4 & -2 & 0 \\ -2 & 4 & 9 \end{array}\right) \xrightarrow[R_1 \leftarrow R_1 + \frac{1}{2}R_2]{R_2 \leftarrow R_2 + \frac{1}{2}R_1} \left(\begin{array}{cc|c} 3 & 0 & 9/2 \\ 0 & 3 & 9 \end{array}\right) \rightarrow$$

$$\left(\begin{array}{cc|c} 1 & 0 & 3/2 \\ 0 & 1 & 3 \end{array}\right)$$

$x = 3/2, \ y = 3$

f is strictly convex (at all points in its domain) so the SOC is satisfied for a minimum. The unique global minimum is $(x,y) = (3/2, 3)$.

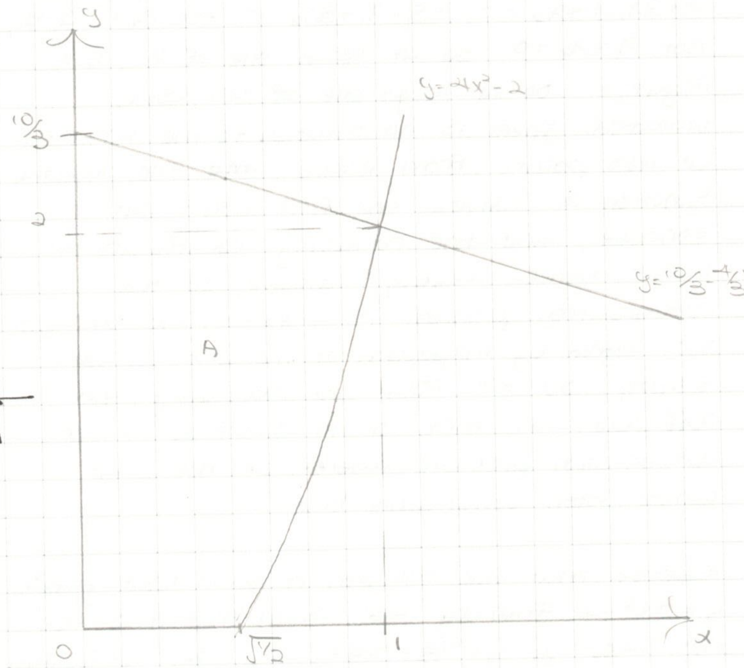
d i $\min\ 2x^2 + 2y^2 - 2xy - 9y$ s.t.

$y - 4x^2 \geq -2 \Leftrightarrow y \geq 4x^2 - 2$

$4x + 3y \leq 10 \Leftrightarrow y \leq 10/3 - 4/3 x$

$x \geq 0$

$y \geq 0$



* The feasible region is area A bounded by the lines $x = 0$, $y = 0$, $y = 10/3 - 4/3 x$ and the curve $y = 4x^2 - 2$.

ii $\mathcal{L} = 2x^2 + 2y^2 - 2xy - 9y - \lambda_1(y - 4x^2 - -2) + \lambda_2(4x + 3y - 10) - \mu_x(x) - \mu_y(y)$

FOCx: $4x - 2y + 8\lambda_1 x + 4\lambda_2 - \mu_x = 0$

FOCy: $4y - 2x - 9 - \lambda_1 + 3\lambda_2 - \mu_y = 0$

CS1: $\lambda_1 \geq 0$, $y - 4x^2 \geq -2$, $\lambda_1(y - 4x^2 - -2) = 0$

CS2: $\lambda_2 \geq 0$, $4x + 3y \leq 10$, $\lambda_2(4x + 3y - 10) = 0$

CSx: $\mu_x \geq 0$, $x \geq 0$, $\mu_x x = 0$

CSy: $\mu_y \geq 0$, $y \geq 0$, $\mu_y y = 0$

iii Given that the positivity constraints do not bind ~~the above FOC~~ $\mu_x = 0$, $\mu_y = 0$, $x, y > 0$.

Suppose neither remaining constraint binds. Then by CS1, CS2, $\lambda_1 = \lambda_2 = 0$. Then by FOCx, FOCy, $4x - 2y = 0$ and $4y - 2x - 9 = 0$. From earlier, this implies $(x,y) = (3/2, 3)$. $4x + 3y = 15 > 10$. CS2 is violated. There is no solution to the KT-FO ~~is not a solution~~ where none of the constraints bind. At such a point, the constraint qualification is vacuously satisfied so the KT-FOCs are necessary for an optimum. So there is no optimum where none of the constraints bind.

The point at which the two (remaining) constraints bind solves $y = 4x^2 - 2$ and $y = {}^{10}/_{3} - {}^{4}/_{3}x$. $4x^2 + {}^{4}/_{3}x - {}^{16}/_{3} = 0 \iff 3x^2 + x - 4 = 0 \iff x = {}^{-1 \pm \sqrt{1+48}}/_{6}$
$x = -{}^{4}/_{3}$ (reject) or 1. $y = 4 - 2 = 2$. The point at which the two remaining constraints bind is $(x,y) = (1,2)$. By FOC_x, FOC_y, $0 + 8\lambda_1 + 4\lambda_2 = 0$, $-3 - \lambda_1 + 3\lambda_2 = 0 \Rightarrow \lambda_1 = -{}^{1}/_{2}\lambda_2$, not $\lambda_1 = \lambda_2 = 0$, so at least one of λ_1, λ_2 is negative, so at least one of CS_1, CS_2 is violated. There is no solution to the KT-FOCs at this point. From earlier, the objective function is convex, the first constraint is concave, and the remaining constraints are linear hence weakly concave, so the optimisation problem is convex. The constraint set (by inspection of the plot) is non-empty, so KT-FOCs are necessary and sufficient. So there is no solution to the optimisation problem ~~where~~ at the point where both constraints bind.

Suppose that the solution is such that $y = 4x^2 - 2$. ~~Then by~~ Given that it is then not also on $y = {}^{10}/_{3} - {}^{4}/_{3}x$, by CS_2, $\lambda_2 = 0$. Then by FOC_x, FOC_y, $4x - 2(4x^2 - 2) + 8\lambda_1 x = 0$ and $4(4x^2 - 2) - 2x - 9 - \lambda_1 = 0 \iff -8x^2 + (4 + 8\lambda_1)x + 4 = 0$
$16x^2 - 2x - 17 - \lambda_1 = 0$. It can presumably (with tedious algebra) be proven that this results in a violation of $\lambda_1 \geq 0$ (or $x, y \geq 0$).

The only remaining case is where $y = {}^{10}/_{3} - {}^{4}/_{3}x$. (and the other constraints do not bind). By CS_1, $\lambda_1 = 0$. Then by FOC_x, FOC_y, $4x - 2y + 4\lambda_2 = 0$,
$4y - 2x - 9 + 3\lambda_2 = 0 \iff 4x - 2({}^{10}/_{3} - {}^{4}/_{3}x) + 4\lambda_2 = 0$,
$4({}^{10}/_{3} - {}^{4}/_{3}x) - 2x - 9 + 3\lambda_2 = 0 \iff {}^{20}/_{3}x - {}^{20}/_{3} + 4\lambda_2 = 0$,
$-{}^{22}/_{3}x + {}^{13}/_{3} + 3\lambda_2 = 0 \iff \lambda_2 = -{}^{5}/_{3}x + {}^{5}/_{3} = {}^{22}/_{9}x - {}^{13}/_{9}$
$\Rightarrow {}^{37}/_{9}x = {}^{28}/_{9} \Rightarrow x = {}^{28}/_{37} \Rightarrow \lambda_2 = {}^{5}/_{3}({}^{9}/_{37}) = {}^{15}/_{37}$
> 0. ~~All KT-FOCs are satis~~ $\Rightarrow y = {}^{86}/_{37}$
$y - 4x^2 \geq -2$. All KT-FOCs are satisfied. The unique solution to the KT-FOCs is $({}^{28}/_{37}, {}^{86}/_{37})$

ii From earlier argument, the KT-FOCs are necessary and sufficient for a minimum, so we can conclude that the solution found is the global minimum.

3a Supposing that all and only finite subsets of [0, 10] are part of the data, ~~denote~~ M's choice function c from arbitrary menu A in the data is $c(A) = \arg\min_{x\in A} (x-\bar{x})^2$ where $\bar{x} = \frac{1}{|A|}\sum_{x\in A} x$. The choice function selects the proposal ~~option the~~ closest to the mean proposal. The formal definition above neglects ties.

bi Sen's α (Nash's IIA): for all menus A, A', if $A' \subseteq A$ and $c(A) \in A'$, then $c(A') = c(A)$.

Consider the following example.

$A = \{4, 5, 6, 10\}$
$A' = \{4, 5, 6\}$
$\bar{x} = \frac{1}{|A|}\sum_{x\in A} x = 25/4 = 6\frac{1}{4}$, so $c(A) = 6$
$\bar{x}' = \frac{1}{|A'|}\sum_{x\in A'} x = 15/3 = 5$, so $c(A) = 5$
$A' \subseteq A$ and $c(A) \in A'$ but $c(A') \neq c(A)$, so choice function c violates α.

ii WARP: for all menus A, A', for all $x \neq x' \in A, A'$, it is not the case that $c(A) = x$ and $c(A') = x'$.

Consider the following example

$A = \{1, 4, 6\}$
$A' = \{4, 6, 10\}$
$\bar{x} = 11/3 = 3\frac{2}{3} \Rightarrow c(A) = 4$
$\bar{x}' = 20/3 = 6\frac{2}{3} \Rightarrow c(A') = 6$
$4 \neq 6 \in A, A'$ but $c(A) = 4$, $c(A') = 6$, so choice function c violates WARP.

iii Consider the menus in ii. Choice function c on menu A ~~reveals 4~~ directly reveals 4 as (strictly) preferred to 6 because both 4 and 6 are in A and c selects 4 over 6.

More generally, for arbitrary menu A and arbitrary $x \neq x' \in A$, ~~choice c(A) = x directly~~ arbitrary choice function c directly reveals x preferred to x' iff $c(A) = x$.

Returning to the menus in ii, $c(A) = 4$ directly reveals 4 (strictly) preferred to 6 and $c(A) = 6$ directly reveals ~~ff~~ 6 (strictly) preferred to 4. So choice function c cannot be rationalised by some rational strict preference relation ≻. The former choice reveals $4 \succ 6$ whereas the latter choice reveals $6 \succ 4$, but a rational strict preference relation is asymmetric, so this is a contradiction. So M's choices are not rationalisable.

~~#~~ Suppose that M's choices are guided by the maximisation of some utility function u, then $c(A) = 4$ implies ~~4 = argmin~~ $4 = \arg\max_{x\in A} u(x)$, which in particular implies $u(4) > u(6)$. Similarly, $c(A') = 6$ implies $u(6) > u(4)$. This is a contradiction so M's choices cannot be guided by the maximisation of ~~s~~ some utility function.

ci Outcome x is indirectly ~~ref~~ revealed preferred to outcome x' by choice function c over set of menus Δ iff ~~*~~ it is possible to infer that x is preferred to x' from directly revealed preferences and transitivity.

ii Consider the following example.

$A = \{3, 4, 5\}$
$A' = \{3, 5, 6\}$
$g = 4$, $g' = 6$
$\bar{x} = 4 \Rightarrow c(A) = 4$
$\bar{x}' = 14/3 = 4\frac{2}{3} \Rightarrow c(A') = 5$

$c(A) = 4$ directly reveals 4 preferred to 5. $c(A') = 5$ directly reveals 5 preferred to 6. So 4 can be claimed to be indirectly revealed preferred to 6.

iii It is not only in the case of M, but also more generally that a choice function with data on two menus indirectly reveals ~~pre~~ any preferences only if the two menus have a common element such that transitivity of preferences ~~of~~ can be applied.

di P has rational preferences over elements of X because P's preferences are complete and transitive. For any two outcomes $x, x' \in X$, either $|x-3|$ (the distance between x and g^*) $\leq |x'-3|$ or $|x'-3| \leq |x-3|$, so either $x \succsim_P x'$ or $x' \succsim_P x$, so P's weak preference relation is complete. For any three outcomes $x, x', x'' \in X$ if $|x-3| \leq |x'-3|$ and $|x'-3| \leq |x''-3|$, then by the transitivity of $\leq$ on $\mathbb{R}$, $|x-3| \leq |x''-3|$, so if $x \succsim_P x'$ and $x' \succsim_P x''$, then $x \succsim_P x''$, so P's weak preference relation is also transitive.

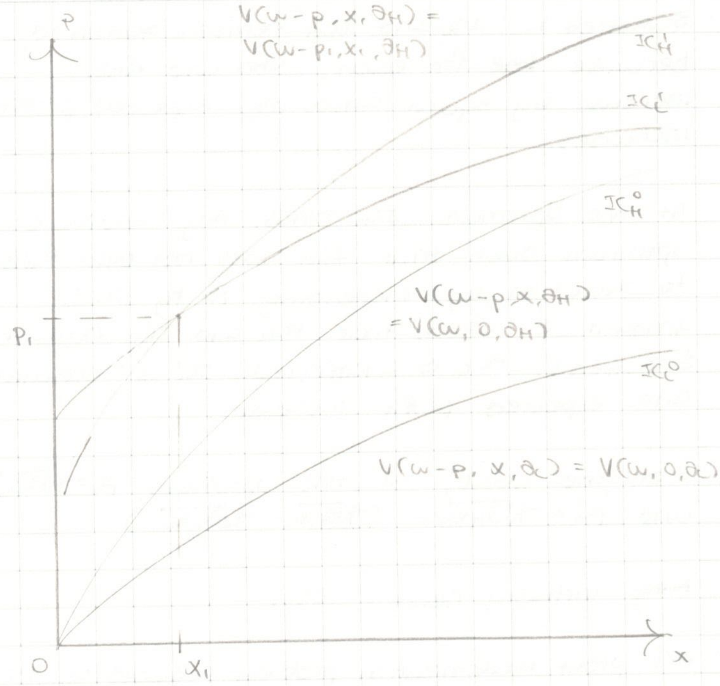
P's preferences ~~satisfy~~ over X satisfy single-peakedness because it is possible to construct some linear order of the elements in X, namely a line from 0 to 10 such that there exists a bliss point $x^* = 3$, such that for all outcomes ~~in~~ ~~either~~ ~~one direction~~ ~~from~~ to the "left" of x^*, those closer to x^* are preferred to those further from x^* and likewise for ~~those to~~ outcomes to the "right".

ii $\succsim_p = \{\langle x_1, x_2 \rangle : |x_1 - 3| \lesssim |x_2 - 3|, \; x_1, x_2 \in X\}$
$u(x) = -|x - 3|$

iii ~~M has ra~~ P has rational preferences that satisfy IIA and WARP while M does not. M's ~~pre~~ choice depends on the alternatives not ~~offer~~ ~~chosen and~~ offered but not chosen whereas P's does not.

We might ~~think~~ like to think that we are like P and have rational preferences. but it is difficult to believe that our choices are unaffected by the ~~option~~ ~~op~~ options that are offered but not chosen.

5a observable θ



Given θ, monopolist M maximises profit subject to the constraint that consumer C is willing and able to buy. Formally, M has the following profit maximisation problem.

$\max_{x,p} \; p - kx$ s.t.

PC: $V(w-p, x, \theta) \geq V(w, 0, \theta) \iff$

$w - p + 4\sqrt{\theta x} \geq w \iff$

$4\sqrt{\theta x} - p \geq 0$

where w is C's endowment or gross income.

At the optimum PC binds. Any candidate optimum such that PC does not bind fails to deviation by M consisting in a small increase (ε) in p such that PC remains satisfied and profit increases.

M's profit maximisation problem reduces to the following.

$\max_x \; 4\sqrt{\theta x} - kx$

FOC: $2\theta^{1/2}x^{-1/2} - k = 0 \Rightarrow$

$2\theta^{1/2}x^{-1/2} = k \Rightarrow$

$x^{-1/2} = \frac{1}{2}\theta^{-1/2}k \Rightarrow$

$x = 4\theta k^{-2}$

SOC: $-1\theta^{1/2}x^{-3/2} < 0$

$x = 4\theta k^{-2}$ is the unique maximum.

$x = 4\theta k^{-2} \Rightarrow p = 4\sqrt{\theta x} = 8\theta k^{-1}$

The optimal contracts are as follows.

$(x_L^*, p_L^*) = (4\theta_L^2 k^{-2}, 8\theta_L k^{-1})$

$(x_H^*, p_H^*) = (4\theta_H^2 k^{-2}, 8\theta_H k^{-1})$

b $V_x(m, x, \theta_H) = \cancel{4\theta_H}\ 2\theta_H x^{-1/2}$

$V_x(m, x, \theta_L) = 2\theta_L x^{-1/2}$

$V_x(m, x, \theta_H) > V_x(m, x, \theta_L)$

The single-crossing property is satisfied. ~~Margin~~ Marginal utility of H types, at any given m, x, is greater than marginal utility of L types.

Because utility is quasilinear in ~~mon~~ net income hence in payment p, indifference curves are vertical shifts of one another. H types have steeper ICs because of higher MU. Any IC_H crosses each IC_L at exactly one point. This is the graphical result of the single crossing property.

ci The monopolist has the following profit maximization problem.

$\max_{x_L, p_L, x_H, p_H} \; \lambda(p_L - kx_L) + (1-\lambda)(p_H - kx_H)$

s.t.

PC_L: $V(w-p_L, x_L, \theta_L) \geq V(w, 0, \theta_L) \iff$

$4\sqrt{\theta_L x_L} - p_L \geq 0$

PC_H: $4\sqrt{\theta_H x_H} - p_H \geq 0$

IC_L: $4\sqrt{\theta_L x_L} - p_L \geq 4\sqrt{\theta_L x_H} - p_H$

IC_H: $4\sqrt{\theta_H x_H} - p_H \geq 4\sqrt{\theta_H x_L} - p_L$

By the revelation principle, it is ~~sde~~ sufficient to restrict attention to contracts of this form (such that "truth-telling" is optimal for C).

ii ~~At the optimum~~, ~~θ PCH satis~~ PC_H is redundant and satisfied with strict equality when PC_L and IC_H are satisfied.

$0 \leq_1 4\sqrt{\theta_L x_L} - p_L <_2 4\sqrt{\theta_H x_L} - p_L \leq_3 4\sqrt{\theta_H x_H} - p_H$,

where $\leq_1$ ~~is~~ follows from PC_L, $<_2$ ~~is~~ follows from $\theta_H > \theta_L$, and $\leq_3$ follows from PC_H.

At the optimum, PC_L binds, i.e. L types are indifferent between buying and not buying. Any candidate optimum such that PC_L does not bind fails to deviation by ~~decreasing~~ increasing each of p_L and p_H by small amount ε such that PC_L and PC_H remain satisfied). IC_L and IC_H remain satisfied because ~~the~~ for each, LHS and RHS increase by equal amounts. Expected profit increases.

At the optimum IC_H binds. Any candidate optimum such that IC_H does not bind fails to deviation by increasing p_H by small amount ε (such that PC_H and IC_H remain satisfied). PC_L is unaffected, IC_L is "loosened" and expected profit increases.

iii From the above, at the optimum, $p_L = 4\sqrt{\theta_L x_L}$ and $p_H = 4\sqrt{\theta_H x_H} - (4\sqrt{\theta_H x_L} - 4\sqrt{\theta_L x_L})$

~~Neg~~ Initially neglect IC_L.

M's profit maximisation problem reduces to the following.

$$\cancel{\max_{x_L, x_H} \lambda(4\sqrt{\theta_H x_H} - (4\sqrt{\theta_H x_L} - 4\sqrt{\theta_L x_L}) - kx_H)}$$
$$\cancel{+ (1-\lambda)(4\sqrt{\theta_L x_L} - kx_L)}$$

~~Differentiate~~

~~Evaluate~~ $\cancel{\partial\pi/\partial x_L \mid_{x_L = x_L^*}}$

$$\cancel{\partial\pi/\partial x_L = \lambda[-4\theta_H^{1/2}(\tfrac{1}{2}x_L^{-1/2}) + 4\theta_L^{1/2}(\tfrac{1}{2}x_L^{-1/2})]}$$
$$\cancel{+ (1-\lambda)[4\theta_L^{1/2}(\tfrac{1}{2}x_L^{-1/2})]}$$
$$\cancel{= (\lambda[-4\theta_H^{1/2} + 4\theta_L^{1/2}] + (1-\lambda)[4\theta_L^{1/2}])(\tfrac{1}{2}x_L^{-1/2})}$$
$$\cancel{= (4\theta_L^{1/2} - 4\lambda\theta_H^{1/2})(\tfrac{1}{2}x_L^{-1/2}) = 0 \iff}$$

$$\cancel{\max_{x_L, x_H} \lambda(4\sqrt{\theta_L x_L} - kx_L)}$$
$$\cancel{+ (1-\lambda)(4\sqrt{\theta_H x_H} - (4\sqrt{\theta_H x_L} - 4\sqrt{\theta_L x_L}) - kx_H)}$$

~~FOC x_L:~~

$$\cancel{\partial\pi/\partial x_L = \lambda[2\theta_L^{1/2}x_L^{-1/2} - k] + (1-\lambda)[-2\theta_H^{1/2}x_L^{-1/2} + 2\theta_L^{1/2}x_L^{-1/2}] = 0}$$
$$\iff$$
$$\cancel{-\lambda k \quad \lambda[2\theta_L^{1/2}x_L^{-1/2}] + (1-\lambda)[2\theta_L^{1/2}x_L^{-1/2} - 2\theta_H^{1/2}x_L^{-1/2}] = 0}$$
$$\iff$$
$$\cancel{x_L^{-1/2}[2\theta_L^{1/2} - (1-\lambda)2\theta_H^{1/2}] = \lambda k}$$
$$\iff$$
$$\cancel{x_L = \lambda^{-2}k^{-2}[2\theta_L^{1/2} - (1-\lambda)2\theta_H^{1/2}]^2}$$

$$\max_{x_L, x_H} \lambda(4\sqrt{\theta_L x_L} - kx_L)$$
$$(1-\lambda)(4\sqrt{\theta_H x_H} - (4\sqrt{\theta_H x_L} - 4\sqrt{\theta_L x_L}) - kx_H)$$

FOC x_H: $(1-\lambda)(2\theta_H^{1/2}x_H^{-1/2} - k) = 0 \iff$
$2\theta_H^{1/2}x_H^{-1/2} = k \iff$
$x_H = 4\theta_H k^{-2} = x_H^*$

FOC x_L: $\lambda(2\theta_L^{1/2}x_L^{-1/2} - k)$
$+ (1-\lambda)(-2\theta_H^{1/2}x_L^{-1/2} + 2\theta_L^{1/2}x_L^{-1/2}) = 0 \iff$
$\lambda k = \cancel{2\theta_L} \; 2\lambda\theta_L^{1/2}x_L^{-1/2} + (1-\lambda)(2\theta_L^{1/2} - 2\theta_H^{1/2})x_L^{-1/2}$
$k = 2\theta_L^{1/2}x_L^{-1/2} + \frac{1-\lambda}{\lambda}(2\theta_L^{1/2} - 2\theta_H^{1/2})x_L^{-1/2}$
$x_L = (2\theta_L^{1/2} + \frac{1-\lambda}{\lambda}(2\theta_L^{1/2} - 2\theta_H^{1/2}))^2 k^{-2} < x_L^*$