Infinity in Predicate Logic

- Some set Γ of sentences is finitely satisfiable iff each finite subset of Γ is satisfiable.

Natural Numbers

- In predicate logic with identity, the sentence "there is at least one F thing" can be formalised as "$\exists x Fx$". The sentence "there are at least two F things" can be formalised as "$\exists x \exists x' : (Fx \land Fx' \land x \neq x')$" (where "$x \neq x'$" abbreviates "$\neg x = x'$"). Similarly, the sentence "there are at least three F things" can be formalised as "$\exists x \exists x' \exists x'' : (Fx \land Fx' \land Fx'' \land x \neq x' \land x \neq x'' \land x' \neq x'')$". This strategy generalises. Let "$\exists_n F$" abbreviate the formalisation of "there are at least n F things".
- $\{\exists_n F : n \in \mathbb{N}\}$ is the set of sentences that formalise "there is at least one F thing", "there are at least two F things", and so on.

Inequality

- The sentence "there exists a one-to-one mapping from the F things to the G things" can be formalised in second-order logic as
  - "$\begin{aligned} &\exists R[\forall x \forall y(Rxy \to Fx \land Gy) \\ &\quad \land \neg\exists x \exists y_1 \exists y_2(Rxy_1 \land Rxy_2 \land y_1 \neq y_2 \land Fx \land Gy_1 \land Gy_2) \\ &\quad \land \neg\exists x_1 \exists x_2 \exists y(Rx_1y \land Rx_2y \land x_1 \neq x_2 \land Fx_1 \land Fx_2 \land Gy)] \end{aligned}$".
  - "$\begin{aligned} &\exists R[\forall x \forall y(Rxy \to Fx \land Gy) \land \forall x(Fx \to \exists y Rxy) \\ &\quad \land \neg\exists x \exists y_1 \exists y_2(Rxy_1 \land Rxy_2 \land y_1 \neq y_2) \\ &\quad \land \neg\exists x_1 \exists x_2 \exists y(Rx_1y \land Rx_2y \land x_1 \neq x_2)] \end{aligned}$" seems more accurate.
  - This reads as "there exists binary relation R, from F things to G things, whose domain is all the F things, no F thing is mapped to two distinct G things (this relation is functional), and no two distinct F things map to the same G thing (this relation is one-to-one). Let "$F \leq G$" abbreviate the formalisation of "there exists a one-to-one mapping from the F things to the G things".
- The sentence "there are strictly more F things than G things" can be formalised as "$(G \leq F) \land \neg(F \leq G)$". Let "$F < G$" abbreviate this formalisation.

Subsets

- The sentence "the F things are a subset of the G things" can be formalised as "$\forall x(Fx \to Gx)$". Let "$F \subseteq G$" abbreviate this formalisation.
- The sentence "the F things are a strict subset of the G things" can be formalised as "$(F \subseteq G) \land \neg(G \subseteq F)$". Let $F \subset G$ abbreviate this formalisation.

Infinity

- The sentence "there are (Dedekind) infinitely many F things" can be formalised as "$\exists F'[F' \subset F \land (F \leq F') \land (F' \leq F)]$". Let "$\infty F$" abbreviate this sentence.
  - Some set S is Dedekind infinite iff there is some proper subset $S' \subset S$ such that $|S| = |S'|$, i.e. S and S' have the same number of elements.
- So, for example, the argument "there are finitely many earthlings, there are infinitely many aliens, so there are more aliens than earthlings" can be formalised as "$\neg\infty E; \infty A; E < A$".

Existence in First-Order Logic

- In first-order logic (with identity) the set $\{\exists_n F : n \in \mathbb{N}\} \cup \{\neg\infty F\}$ is not satisfiable, but is finitely satisfiable. So if "$\neg\infty F$" exists in first-order logic, then first-order logic is not compact. Given that first-order logic is compact (finite satisfiability implies satisfiability), by reductio, "$\neg\infty F$" does not exist in first-order logic. Because "$\neg\infty F$" exists iff "∞F" exists, "∞F" does not exist in first-order logic either.
- In first-order logic (with identity) the set $\{F > G\} \cup \{\exists_n F : n \in \mathbb{N}\} \cup \{\exists_n G : n \in \mathbb{N}\}$ is satisfiable in a model with an uncountably infinite domain, but not in a model with a countably infinite domain, or a model with a finite domain. Given

that in first-order logic satisfiability in an uncountably infinite domain implies satisfiability in a countably infinite domain, by reductio, "$F > G$" does not exist in first-order logic.