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Microeconomic Analysis Problem set 5

1 $f(x,y) = x^2 + 2y^2 + xy + 3x + 19y - 4$

$Df = \begin{pmatrix} \partial f/\partial x & \partial f/\partial y \end{pmatrix} = \begin{pmatrix} 2x+y+3 & 4y+x+19 \end{pmatrix}$

$D^2f = \begin{pmatrix} \partial^2 f/\partial x^2 & \partial^2 f/\partial x \partial y \\ \partial^2 f/\partial y \partial x & \partial^2 f/\partial y^2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$

$|D^2f| = 7$

$tr(D^2f) = 6$

$|D^2f - \lambda I| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda) - 1 = \lambda^2 - 6\lambda + 7$

$\cancel{\lambda = -6 \pm \sqrt{36-28}/12 =}$

$\lambda = 6 \pm \sqrt{86-28}/2 = 3 \pm \sqrt{2} > 0$

By the eigenvalue test, D^2f is positive definite for all (x,y), ~~so D^2f is~~ f is strictly convex.

Is a proof of this relationship required? How can this be proven?

FOC: $Df = \vec{0}$

$2x+y+3=0$ ①

$4y+x+19=0$ ②

① - 2②: $2x+y+3 - 2(4y+x+19) = y+3-8y-38 = -7y-35=0$

$y=-5$ ③

Sub ③ in ①: $2x-5+3=0$, $x=1$

~~Since D^2f is positive definite, $(x=1, y=-5)$ is a stat~~

Given that f is strictly convex, the stationary point $(x=1, y=-5)$ is the unique global ~~maximum~~. minimum.

2 max/min$_{x,y}$ x^2+y^2 st $x^2+xy+y^2=3$

By inspection, the constraint set $\{(x,y): x^2+xy+y^2=3\}$ is ~~not bounded~~ compact, i.e. closed and bounded. Then, by the Weierstrass extreme value theorem, a global maximum and a global minimum exist.

Let $h(x,y) = x^2+xy+y^2$
$Dh = (\partial h/\partial x, \partial h/\partial y) = (2x+y \quad 2y+x)$
rank $Dh = 1$ iff $(x,y) \neq (0,0)$
The constraint qualification is satisfied iff $(x,y)\neq(0,0)$.
$(0,0) \notin \{(x,y): x^2+xy+y^2=3\}$, so the constraint qualification is satisfied for all (x,y) in the constraint set. Then, by Lagrange's theorem, the Lagrangian first-order ~~FOCs~~ conditions ~~and constraint conditions~~ are necessary for a maximum and for a minimum.

~~$L = x^2+y^2 - \lambda(x^2+xy+y^2-3)$~~
$L(x,y;\lambda) = x^2+y^2-\lambda(x^2+xy+y^2 - 3)$
$FOC_x: \partial L/\partial x = 2x - \lambda(2x+y) = 0$ ①
$FOC_y: \partial L/\partial y = 2y - \lambda(2y+x) = 0$ ②
$C: x^2+xy+y^2=3$ ③
From ① $\lambda = 2x/(2x+y)$ ④
From ② $\lambda = 2y/(2y+x)$ ⑤
From ④ and ⑤ $2x/(2x+y) = 2y/(2y+x)$,
$2x(2y+x) = 2y(2x+y)$,
$4xy+2x^2 = 4xy+2y^2$,
~~$x=\pm y$ ⑥~~ $y=\pm x$ ⑥
From ③ and ⑥
$x^2+x^2+x^2=3$ or $x^2-x^2+x^2=3$
$x=\pm1$ or $x=\pm\sqrt{3}$
$(x,y) = (1,1), (-1,-1), (\sqrt{3},-\sqrt{3})$ or $(-\sqrt{3},\sqrt{3})$
The candidate optima are $(1,1), (-1,-1), (\sqrt{3},-\sqrt{3})$ and $(-\sqrt{3},\sqrt{3})$.

~~DL~~ $D_{x,y}L = (\partial L/\partial x \quad \partial L/\partial y) = (2x-\lambda(2x+y),\ 2y-\lambda(2y+x))$

$$D^2_{x,y}L = \begin{pmatrix} \partial^2 L/\partial x\partial x & \partial^2 L/\partial x\partial y \\ \partial^2 L/\partial y\partial x & \partial^2 L/\partial y\partial y \end{pmatrix} = \begin{pmatrix} 2-2\lambda & -\lambda \\ -\lambda & 2-2\lambda \end{pmatrix}$$

Let H denote the bordered Hessian matrix.

$$H = \begin{pmatrix} 0 & 2x+y & 2y+x \\ 2x+y & 2-2\lambda & -\lambda \\ 2y+x & -\lambda & 2-2\lambda \end{pmatrix}$$

There are two variables and there is one constraint, so there is one degree of freedom, and only one leading principal minor to compute for the bordered Hessian test, namely $|H|$.

$$|H| = -(2x+y)\left|\begin{pmatrix} 2x+y & -\lambda \\ 2y+x & 2-2\lambda \end{pmatrix}\right| + (2y+x)\left|\begin{pmatrix} 2x+y & 2-2\lambda \\ 2y+x & -\lambda \end{pmatrix}\right|$$

$$= -a\left|\begin{pmatrix} a & -\lambda \\ b & 2-2\lambda \end{pmatrix}\right| + b\left|\begin{pmatrix} a & 2-2\lambda \\ b & -\lambda \end{pmatrix}\right|$$

where $a = 2x+y$, $b = 2y+x$, by substitution

$= -a^2(2-2\lambda) + ab(-\lambda) + ba(-\lambda) - b^2(2-2\lambda)$
$= \cancel{(a^2+b^2)(2-2\lambda)}\ (a^2+b^2)(2\lambda-2) - 2ab\lambda$

Suppose that $(x,y)=(1,1)$, then $\lambda = 2/3$, $a=3$, $b=3$,
$|H| = (a^2+b^2)(2\lambda-2)-2ab\lambda = (18)(-2/3)-12<0$
By the bordered Hessian test, $(1,1)$ is a strict local ~~maximum~~. minimum.

Suppose that $(x,y)=(-1,-1)$, then $\lambda = 2/3$, $a=-3$, $b=-3$,
$|H| = (18)(-2/3)-12<0$.
By the bordered Hessian test, $(-1,-1)$ is a strict local minimum.

Suppose that $(x,y)=(\sqrt{3},-\sqrt{3})$, then $\lambda=2$, $a=\sqrt{3}$, $b=-\sqrt{3}$,
$|H| = (6)(2)+12>0$.
By the bordered Hessian test, $(\sqrt{3},-\sqrt{3})$ is a strict local maximum.

Suppose that $(x,y)=(-\sqrt{3},\sqrt{3})$, then $\lambda=2$, $a=-\sqrt{3}$, $b=\sqrt{3}$,
$|H| = (6)(2)+12>0$
By the bordered Hessian test, $(-\sqrt{3},\sqrt{3})$ is a strict local maximum.

Let $f(x,y)=x^2+y^2$.
$f(1,1)=2$, $f(-1,-1)=2$, $f(\sqrt{3},-\sqrt{3})=6$, $f(-\sqrt{3},\sqrt{3})=6$.
$(1,1)$ and $(-1,-1)$ are weak global minima, and $(\sqrt{3},-\sqrt{3})$ and $(-\sqrt{3},\sqrt{3})$ are weak global maxima.

3: $\max_{x,y} u(x,y) = x^2y$ st $2x+3y \leq 9$, $x,y \geq 0$

Suppose $(x,y)=(1,1)$, then $u(x,y)=u(1,1)=1$, and $2x+3y = 5 \leq 9$, and $x,y \geq 0$, i.e. $(x,y)=(1,1)$ is in the constraint set and yields $u=1$.

Suppose that $x=0$, then $u=0 < 1 = u(1,1)$, so ~~if x=0,~~ no $(x=0,y)$ is a constrained maximum. Suppose that $y=0$, then $u=0<1=u(1,1)$, so no $(x,y=0)$ is a constrained maximum. Let (x^*,y^*) denote the solution to the given constrained maximisation problem. $x^*,y^* \neq 0$. Given that $x,y \geq 0$, $x^*,y^*>0$. ~~Supp Suppose that x,y≠0, then by inspection, u is increasing in x and in y~~

Suppose that $2x^*+3y^*<9$, then some (x',y^*) ~~is and in the constra~~ such that $x'>x^*$ is in the constraint set, namely $x' = 9-3y^*/2$. Given that $y^*>0$, by inspection, u is increasing in x, so $u(x',y^*) > u(x^*,y^*)$, i.e. (x^*,y^*) is not a constrained maximum. By reductio, $2x^*+3y^* \geq 9$. Given that $2x^*+3y^* \leq 9$, $2x^*+3y^*=9$.

ii $L(x,y;\lambda) = x^2y - \lambda(2x+3y-9)$

$FOC_x = 2xy - \lambda(2) = 0$ ①

$FOC_y = x^2 - \lambda(3) = 0$ ②

$C: 2x+3y=9$ ③

From ①, $\lambda = xy$ ④

Substituting ④ into ②

$x^2 - 3xy = 0$, $x=0$ (reject) or $x=3y$ ⑤

Substituting ⑤ into ③

$6y+3y=9$, $y=1$, $x=3$

The only candidate optimum is $(3,1)$

iii $Du = (\partial u/\partial x \;\; \partial u/\partial y) = (2xy \;\; x^2)$

$D_{x,y}L = (\partial L/\partial x \;\; \partial L/\partial y) = (2xy - 2\lambda \;\; x^2-3\lambda)$

$$D^2_{x,y}L = \begin{pmatrix} \partial^2 L/\partial x\partial x & \partial^2 L/\partial x\partial y \\ \partial^2 L/\partial y\partial x & \partial^2 L/\partial y\partial y \end{pmatrix} = \begin{pmatrix} 2y & 2x \\ 2x & 0 \end{pmatrix}$$

Let H denote the bordered Hessian

$$H = \begin{pmatrix} 0 & 2xy & x^2 \\ 2xy & 2y & 2x \\ x^2 & 2x & 0 \end{pmatrix}$$

There are two variables and there is one constraint, so there is one degree of freedom and one leading principal minor to compute for the bordered Hessian test, namely $|H|$.

$$|H| = -2xy\left|\begin{pmatrix} 2xy & 2x \\ x^2 & 0 \end{pmatrix}\right| + x^2\left|\begin{pmatrix} 2xy & 2y \\ x^2 & 2x \end{pmatrix}\right|$$

$$= -2xy(-2x^3) + x^2(4x^2y - 2x^2y)$$

$$= 4x^4y + 2x^4y$$

$$= 6x^4y$$

At $(x,y)=(3,1)$, $|H| = 6(3)^4(1) > 0$

By the bordered Hessian test, $(x,y)=(3,1)$ is a strict local maximum.

iv The constraint set is $\{(x,y): 2x+3y \leq 9, x \geq 0, y \geq 0\}$.
By inspection, the constraint set is compact (closed and bounded). By the Weierstrass extreme value theorem, a global maximum and a global minimum exist. So the ~~#~~ unique local maximum is also a global maximum.

4 $\min_{K,L} rK+wL$ st $K,L \geq 0, Y(K,L) = (K+1)^{1-\alpha}(L+1)^{\alpha} - 1 \geq y$

Let $c(K,L) = rK+wL$.

$Y(K,L)$ is increasing in ~~the constraint set K and L~~ each of K and L in the constraint set, given that $K,L \geq 0$ in the constraint set and $0<\alpha<1$. Suppose for reductio that the constraint $Y(K,L) \geq y$ is not binding at the global minimum (K^*,L^*), i.e. $Y(K^*,L^*) > y$. Then, there exists (K',L') such that $Y(K',L') = y$ and $K' \leq K^*$, $L' \leq L^*$, and $K'<K^*$ or $L'<L^*$ so $c(K',L') < c(K^*,L^*)$, then (K^*,L^*) is not a minimum. By reductio, the constraint $Y(K,L) \leq y$ is binding at the global minimum. The minimisation problem reduces to

$\min_{K,L} rK+wL$ st $K,L \geq 0, Y(K,L) = y$

$\cancel{DY = (\partial Y/\partial K \ \ \partial Y/\partial L) = ((L+1)^{\alpha}(1-\alpha)(K+1)^{-\alpha}}$
$\cancel{= ((L+1)^{\alpha}}$

which is equivalent to
$\min_{K,L} rK+wL$ st $K,L \geq 0$, $Y'(K,L) = y'$
where $Y'(K,L) = (1-\alpha)\ln(K+1) + \alpha\ln(L+1)$ and
$y' = \ln(y+1)$

$DY' = (\partial Y/\partial K \ \ \partial Y/\partial L) = \left(\frac{1-\alpha}{K+1} \ \ \frac{\alpha}{L+1}\right)$

$$D(Y', g_1, g_2) = \begin{pmatrix} \partial Y/\partial K & \partial Y/\partial L \\ \partial g_1/\partial K & \partial g_1/\partial L \\ \partial g_2/\partial K & \partial g_2/\partial L \end{pmatrix} = \begin{pmatrix} \frac{1-\alpha}{K+1} & \frac{\alpha}{L+1} \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

~~By inspecti~~ Given that $0<\alpha<1$ and in the constraint set $K,L \geq 0$, in the constraint set $\frac{1-\alpha}{K+1}, \frac{\alpha}{L+1} > 0$. Then, by inspection, ~~in the constraint set~~, the constraint qualification holds for all points in the constraint set since at most two constraints are binding.

Then, the Kuhn-Tucker first order conditions are necessary for an optima.

$\mathcal{L} = rK+wL - \lambda_Y((1-\alpha)\ln(K+1) + \alpha\ln(L+1) - y') - \mu_K K - \mu_L L$
$FOC_K: r - \lambda_Y\left(\frac{1-\alpha}{K+1}\right) - \mu_K = 0$
$FOC_L: w - \lambda_Y\left(\frac{\alpha}{L+1}\right) - \mu_L = 0$
$C_Y: (1-\alpha)\ln(K+1) + \alpha\ln(L+1) = \ln(y+1)$
equivalently $(K+1)^{1-\alpha}(L+1)^{\alpha} - 1 = y$
$CS_K: \mu_K \geq 0, K \geq 0, \mu_K K = 0$
$CS_L: \mu_L \geq 0, L \geq 0, \mu_L L = 0$

Suppose that neither positivity constraint is binding.
Then $\mu_K = \mu_L = 0$ (and $K,L>0$)
By substitution into FOC_K, FOC_L,
$r - \lambda_Y\left(\frac{1-\alpha}{K+1}\right) = 0$, $w - \lambda_Y\left(\frac{\alpha}{L+1}\right) = 0$
$r = (1-\alpha)\lambda_Y/(K+1)$, $w = \alpha\lambda_Y/(L+1)$
$K+1 = (1-\alpha)\lambda_Y/r$, $L+1 = \alpha\lambda_Y/w$
$L+1 = \frac{\alpha}{1-\alpha}\frac{r}{w}(K+1)$

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By substitution into C_Y,

$(K+1)^{1-\alpha}\left(\frac{\alpha}{1-\alpha}\frac{r}{w}\right)^{\alpha}(K+1)^{\alpha} = y+1$

$K+1 = \left(\frac{1-\alpha}{\alpha}\frac{w}{r}\right)^{\alpha}(y+1)$

$L+1 = \left(\frac{\alpha}{1-\alpha}\frac{r}{w}\right)\left(\frac{1-\alpha}{\alpha}\frac{w}{r}\right)^{\alpha}(y+1)$

$\quad = \left(\frac{\alpha}{1-\alpha}\frac{r}{w}\right)^{1-\alpha}(y+1)$

$(K,L) = \left(\left(\frac{1-\alpha}{\alpha}\frac{w}{r}\right)^{\alpha}(y+1),\ \left(\frac{\alpha}{1-\alpha}\frac{r}{w}\right)^{1-\alpha}(y+1)\right)$

~~The relevant bordered Hessian is~~

~~H=~~

$\cancel{D_{K,L}\mathcal{L} = \left(\partial\mathcal{L}/\partial K \quad \partial\mathcal{L}/\partial L\right) = (r-\lambda_Y(1-\alpha/K+1)-\mu_K,}$

$\cancel{= (r-\lambda_Y(1-\alpha/K+1)-\mu_K,\ w-\lambda_Y(\alpha/L+1)-\mu_L)}$

$\cancel{D^2_{K,L}\mathcal{L} = \begin{pmatrix} \partial^2\mathcal{L}/\partial K^2 & \partial^2\mathcal{L}/\partial K\partial L \\ \partial^2\mathcal{L}/\partial L\partial K & \partial^2\mathcal{L}/\partial L^2 \end{pmatrix}}$

Suppose that only the positivity constraint on K binds,
then $\mu_K > 0$, $K=0$, $\mu_L=0$, $L \neq 0$.

By substitution into FOC_K, FOC_L,

$r-\lambda_Y(1-\alpha)-\mu_K = 0$, $\quad w-\lambda_Y(\alpha/L+1) = 0$

By substitution into C_Y,

$(L+1)^{\alpha} = y+1$, $L = (y+1)^{1/\alpha}-1$

$(K,L) = (0,\ (y+1)^{1/\alpha}-1)$

Suppose that only the positivity constraint on L binds,
then $L=0$.

By substitution into C_Y

$(K+1)^{1-\alpha} = y+1$, $K = (y+1)^{1/1-\alpha}-1$

$(K,L) = ((y+1)^{1/1-\alpha}-1,\ 0)$

~~By inspection, the constraint set is closed~~

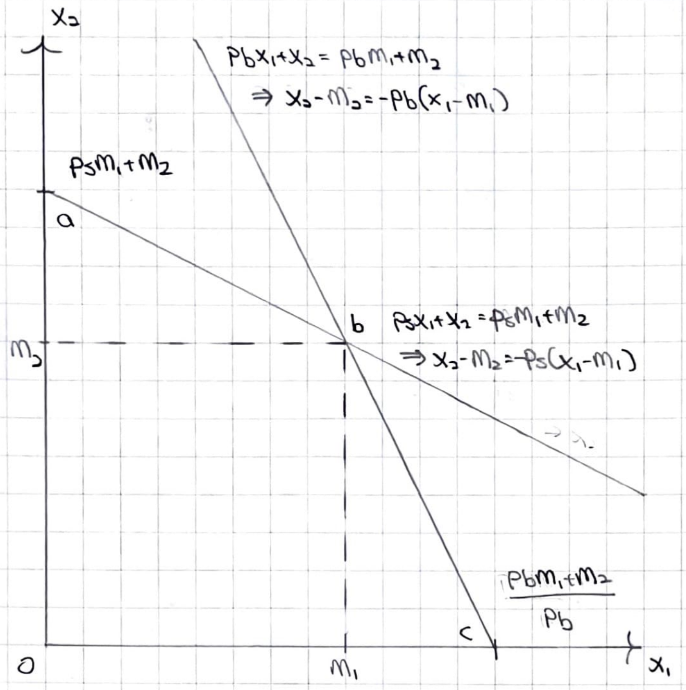
~~Evaluating c(K,L) at the three candidate optima,~~

$\cancel{c(K_1,L_1) = r\left(\frac{1-\alpha}{\alpha}\frac{w}{r}\right)^{\alpha}(y+1)+w\left(\frac{\alpha}{1-\alpha}\frac{r}{w}\right)^{1-\alpha}(y+1)}$

$\cancel{= (y+1)\left[\frac{1-\alpha}{\alpha}\, w/r^{1-1/\alpha} + \frac{\alpha}{1-\alpha}\, r/w^{1-1/\alpha}\right]}$

$\cancel{c(K_2,L_2) =}$

5



The budget set is represented by area 0abc. P_s is the price, in units of good 2, at which the household can sell good 1. P_b is the price, in units of good 2, at which the household can buy good 1.

The household's optimisation problem is

$\max_{x_1, x_2} \; u(x_1, x_2) = x_1^{\beta} + x_2^{\beta}$ st

$g_1(x_1, x_2) = x_1 \geq 0$, $g_2(x_1, x_2) = x_2 \geq 0$

$g_s(x_1, x_2) = P_s x_1 + x_2 \leq P_s m_1 + m_2$

$g_b(x_1, x_2) = P_b x_1 + x_2 \leq P_b m_1 + m_2$

where $0 < \beta < 1$; $m_1, m_2 > 0$; $0 < P_s < P_b$

$$Dg = \begin{pmatrix} \partial g_1/\partial x_1 & \partial g_1/\partial x_2 \\ \partial g_2/\partial x_1 & \partial g_2/\partial x_2 \\ \partial g_s/\partial x_1 & \partial g_s/\partial x_2 \\ \partial g_b/\partial x_1 & \partial g_b/\partial x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ P_s & 1 \\ P_b & 1 \end{pmatrix}$$

~~Suppose that~~ The constraint qualification holds iff the number of binding constraints $e \leq 2$. Suppose that both g_s and g_b bind, then, given that $m_1, m_2 > 0$, $x_1 = m_1 > 0$ and $x_2 = m_2 > 0$, so neither g_1 nor g_2 bind. Suppose that both g_1 and g_2 bind, then $x_1 = x_2 = 0$, $P_s x_1 + x_2 = 0 < P_s m_1 + m_2$ and $P_b x_1 + x_2 = 0 < P_b m_1 + m_2$, so neither g_s nor g_b bind. So $e \leq 2$, and the constraint qualification holds. The Kuhn-Tucker first-order conditions and complementary slackness conditions are necessary conditions for all optima.

$$\begin{aligned} &L(x_1, x_2; \mu_1, \mu_2; \lambda_s, \lambda_b) \\ &= x_1^{\beta} + x_2^{\beta} + \mu_1 x_1 + \mu_2 x_2 \\ &\quad - \lambda_s(P_s x_1 + x_2 - (P_s m_1 + m_2)) - \lambda_b(P_b x_1 + x_2 - (P_b m_1 + m_2)) \end{aligned}$$

FOCx_1: $\beta x_1^{\beta-1} + \mu_1 - \lambda_s p_s - \lambda_b p_b = 0$

FOCx_2: $\beta x_2^{\beta-1} + \mu_2 - \lambda_s - \lambda_b = 0$

CSμ_1 : $\mu_1 \geq 0,\ x_1 \geq 0,\ \mu_1 x_1 = 0$

CSμ_2 : $\mu_2 \geq 0,\ x_2 \geq 0,\ \mu_2 x_2 = 0$

CSλ_s : $\lambda_s \geq 0,\ p_s x_1 + x_2 \leq p_s m_1 + m_2,$
$\lambda_s (p_s x_1 + x_2 - (p_s m_1 + m_2)) = 0$

CSλ_b : $\lambda_b \geq 0,\ p_b x_1 + x_2 \leq p_b m_1 + m_2,$
$\lambda_b (p_b x_1 + x_2 - (p_b m_1 + m_2)) = 0$

Suppose both g_1 and g_2 bind, i.e. $x_1 = x_2 = 0$. Then, there are zero degrees of freedom, there are no (x_1, x_2) around $(x_1 = 0, x_2 = 0)$ such that both these constraints bind, and it is not necessary to verify that the second order condition holds at $(x_1 = 0, x_2 = 0)$

Suppose ~~that only~~ g_1 and not g_2 binds, i.e. $x_1 = 0$. Then, $x_1 = 0, \mu_1 > 0, x_2 > 0, \mu_2 = 0$. $u(x_1 = 0, x_2) = x_2^{\beta}$. ~~By this~~ Given ~~$\neq$~~ $0 < \beta < 1$, by inspection, $u(x_1 = 0, x_2)$ is increasing in x_2, for all x_2. So $\max_{x_2} = \operatorname{argmax}_{x_2} u(x_1 = 0, x_2)$. Only $(x_1 = 0, x_2 = \cancel{p_s}\ p_s m_1 + m_2)$ satisfies the FOCs and CSs.

$\cancel{\partial L}\ D_x L = (\partial L/\partial x_1,\ \partial L/\partial x_2) = \cancel{(\beta x_1^{\beta-1} + \mu_1 - \lambda_s p_s}$
$= (\beta x_1^{\beta-1} + \mu_1 - \lambda_s p_s - \lambda_b p_b,\ \beta x_2^{\beta-1} + \mu_2 - \lambda_s - \lambda_b)$

$$D_x^2 L = \begin{pmatrix} \partial^2 L/\partial x_1^2 & \partial^2 L/\partial x_1 \partial x_2 \\ \partial^2 L/\partial x_2 \partial x_1 & \partial^2 L/\partial x_2^2 \end{pmatrix} = \begin{pmatrix} \beta(\beta-1)x_1^{\beta-2} & 0 \\ 0 & \beta(\beta-1)x_2^{\beta-2} \end{pmatrix}$$

The relevant bordered Hessian

$$H = \begin{pmatrix} 0 & 1 & 0 \\ 1 & \beta(\beta-1)x_1^{\beta-2} & 0 \\ 0 & 0 & \beta(\beta-1)x_2^{\beta-2} \end{pmatrix}$$

$$|H| = -1 \begin{vmatrix} 1 & 0 \\ 0 & \beta(\beta-1)x_2^{\beta-2} \end{vmatrix} = -\beta(\beta-1)x_2^{\beta-2} > 0$$

By the bordered Hessian test, $(x_1 = 0, x_2 = p_s m_1 + m_2)$ is a strict local maximum.

Analogously, supposing that g_2 and not g_1 binds, only $(x_1 = p_b m_1 + m_2 / p_b,\ x_2 = 0)$ satisfies the FOCs and CSs.
The relevant bordered Hessian

$$H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \beta(\beta-1)x_1^{\beta-2} & 0 \\ 1 & 0 & \beta(\beta-1)x_2^{\beta-2} \end{pmatrix}$$

$$|H| = \begin{vmatrix} 0 & \beta(\beta-1)x_1^{\beta-2} \\ 1 & 0 \end{vmatrix} = -\beta(\beta-1)x_1^{\beta-2} > 0$$

By the bordered Hessian test, $(x_1 = p_b m_1 + m_2 / \cancel{p_b}\ p_b,\ x_2 = 0)$ is a strict local maximum.

Suppose only g_s binds, then $x_1, x_2 > 0$, $\mu_1 = \mu_2 = \lambda_b = 0$, $\lambda_s > 0$

$p_s x_1 + x_2 = p_s m_1 + m_2$

$\beta x_1^{\beta-1} - \lambda_s p_s = 0,\quad \beta x_2^{\beta-1} - \lambda_s = 0$

$\lambda_s = \beta/p_s\, x_1^{\beta-1} = \beta x_2^{\beta-1},\quad x_1^{\beta-1} = p_s x_2^{\beta-1},\ x_1 = p_s^{1/\beta-1} x_2$

$p_s\, p_s^{1/\beta-1} x_2 + x_2 = p_s m_1 + m_2$

$x_2 = (p_s m_1 + m_2)/(p_s^{\beta/\beta-1} + 1)$

$x_1 = (p_s m_1 + m_2)/\cancel{p_s}(p_s + p_s^{-1/\beta-1})$

The relevant bordered Hessian

$$H = \begin{pmatrix} 0 & p_s & 1 \\ p_s & \beta(\beta-1)x_1^{\beta-2} & 0 \\ 1 & 0 & \beta(\beta-1)x_2^{\beta-2} \end{pmatrix}$$

$$|H| = -p_s \begin{vmatrix} p_s & 0 \\ 1 & \beta(\beta-1)x_2^{\beta-2} \end{vmatrix} + \begin{vmatrix} p_s & \beta(\beta-1)x_1^{\beta-2} \\ 1 & 0 \end{vmatrix}$$

$$= -p_s^2\beta(\beta-1)x_2^{\beta-2} - \beta(\beta-1)x_1^{\beta-2} > 0$$

By the bordered Hessian test, $\left(x_1 = (p_s m_1 + m_2)/(p_s + p_s^{-1/\beta-1}),\ x_2 = (p_s m_1 + m_2)/(p_s^{\beta/\beta-1}+1)\right)$ is a strict local maximum.

Suppose only g_b binds, then $x_1, x_2 > 0$, $\mu_1 = \mu_2 = \lambda_s = 0$, $\lambda_b > 0$

$p_b x_1 + x_2 = p_b m_1 + m_2$

$\beta x_1^{\beta-1} - \lambda_b p_b = 0$, $\beta x_2^{\beta-1} - \lambda_b = 0$

$\lambda_b = 1/p_b\ \beta x_1^{\beta-1} = \beta x_2^{\beta-1}$, $x_1^{\beta-1} = p_b x_2^{\beta-1}$, $x_1 = p_b^{1/\beta-1} x_2$

$p_b^{\beta/\beta-1} x_2 + x_2 = p_b m_1 + m_2$

$x_2 = (p_b m_1 + m_2)/(p_b^{\beta/\beta-1} + 1)$

$x_1 = (p_b m_1 + m_2)/(p_b + p_b^{-1/\beta-1})$

By an analogous bordered Hessian test, $\left(x_1 = (p_b m_1 + m_2)/(p_b + p_b^{-1/\beta-1}),\ x_2 = (p_b m_1 + m_2)/(p_b^{\beta/\beta-1}+1)\right)$ is a strict local maximum.

Suppose that only g_s and g_b bind, then $x_1 = m_1$ and $x_2 = m_2$ and it is not necessary to verify the SOC ~~are~~ is satisfied.

The candidate optima are

$(x_1 = 0,\ x_2 = 0)$ 1

$(x_1 = 0,\ x_2 = p_s m_1 + m_2)$ 2

$(x_1 = p_b m_1 + m_2/p_b,\ x_2 = 0)$ 3

$\left(x_1 = (p_s m_1 + m_2)/(p_s + p_s^{-1/\beta-1}),\ x_2 = (p_s m_1 + m_2)/(p_s^{\beta/\beta-1}+1)\right)$ 4

$\left(x_1 = (p_b m_1 + m_2)/(p_b + p_b^{-1/\beta-1}),\ x_2 = (p_b m_1 + m_2)/(p_b^{\beta/\beta-1}+1)\right)$ 5

$(x_1 = m_1,\ x_2 = m_2)$ 6

~~By this~~ Since u is increasing in x_1, x_2, $(x_1 = 0, x_2 = 0)$ is not a maximum.

~~Each of the remaining candidate optima correspond to one of the following points~~

~~It can be proven graphically that~~ $(x_1 = 0,\ x_2 = p_s m_1 + m_2)$ and $(x_1 = p_b m_1 + m_2/p_b,\ x_2 = 0)$ are not optima because at these points, $MRS = -MU_1/MU_2 = -(x_1/x_2)^{\beta-1} \neq MRT$, which is ~~either~~ equal to p_s at the former and p_b at the latter.

$MRS|_m = -(m_1/m_2)^{\beta-1}$

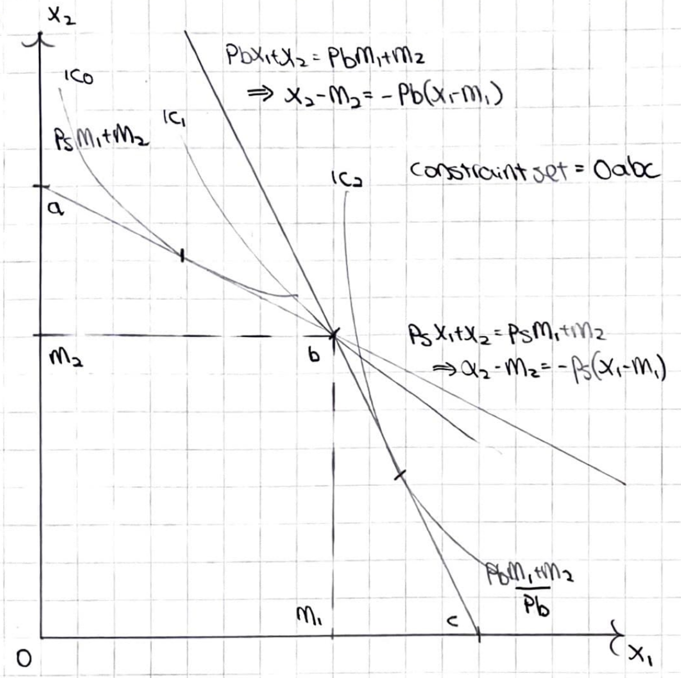
~~The optimum is 4 iff~~

The optimum is 4 iff $-MRS|_m < p_s < p_b$

The optimum is 5 iff $-MRS|_m > p_b > p_s$

The optimum is 6 iff $p_s < MRS|_m < p_b$

5



The household's optimisation problem is

$\max_{x_1,x_2} u(x_1,x_2) = x_1^{\beta} + x_2^{\beta}$ where $0<\beta<1$

subject to

$g_1(x_1,x_2) = x_1 \geq 0$

$g_2(x_1,x_2) = x_2 \geq 0$

$g_s(x_1,x_2) = p_s x_1 + x_2 \leq p_s m_1 + m_2$, ~~where m_1~~

$g_b(x_1,x_2) = p_b x_1 + x_2 \leq p_b m_1 + m_2$, where $m_1, m_2 > 0$; $0<p_s<p_b$

p_s is the price, in ~~terms~~ units of good 2, at which the household can sell good 1. p_b is the price, in units of good 2, at which the household can buy good 1.

~~$\partial u/\partial x_1 = \beta x_1^{\beta-1}$~~

By inspection of u, u is increasing in each of x_1 and x_2 for $x_1, x_2 \geq 0$. So only (x_1,x_2) that maximise some weighted sum ~~of x_1~~ $\alpha x_1 + (1-\alpha)x_2$, i.e. that lie on the frontier abc maximise u.

$MRS = -MU_1/MU_2 = -(x_1/x_2)^{\beta-1} = -(x_2/x_1)^{1-\beta}$

At a, ~~MRS = 0~~ MRS is undefined. The household would substitute ~~all~~ any ~~all~~ marginal amount of x_2 for a marginal unit of x_1. $MRT = -p_s$, so the household can substitute p_s ~~unit~~ marginal units of x_2 for one marginal unit of x_1, and this substitution leaves the household with greater utility. So a is not an optimum.

At c, MRS is zero, $MRT = -p_b$, so the household can substitute one marginal unit of x_1 for p_b marginal units of x_2 and this substitution leaves the household better off. So c is not an optimum.

Suppose that some point x on line ab exclusive is the optimum. Then $MRS|_x = MRT = -p_s$.

$-(x_1/x_2)^{\beta-1} = -p_s$, $x_1^{\beta-1} = p_s x_2^{\beta-1}$, $x_1 = p_s^{1/\beta-1} x_2$

By substitution into $y_s = p_s m_1 + m_2$,

$p_s^{\beta/\beta-1} x_2 + x_2 = p_s m_1 + m_2$, $x_2 = p_s m_1 + m_2 / 1 + p_s^{\beta/\beta-1}$

$x_1 = \cancel{p_s^{1/\beta-1}} \cdot p_s m_1 + m_2 / p_s^{1/1-\beta} + p_s = 1/p_s \; p_s m_1 + m_2 / 1 + p_s^{\beta/1-\beta}$

Suppose that some point x on line bc exclusive is the optimum. Then $MRS|_x = -p_b$. By an analogous argument,

$x_1 = 1/p_b \; p_b m_1 + m_2 / 1 + p_b^{\beta/1-\beta}$, $x_2 = p_b m_1 + m_2 / 1 + p_b^{\beta/\beta-1}$

b is an optimum iff $p_s \le -MRS|_b = (m_1/m_2)^{\beta-1} \le p_b$

if $(m_1/m_2)^{\beta-1} < p_s < p_b$, then the optimum is

$(x_1 = p_s m_1 + m_2 / p_s^{1/1-\beta} + p_s, \; x_2 = p_s m_1 + m_2 / 1 + p_s^{\beta/1-\beta})$

if $p_s < p_b < (m_1/m_2)^{\beta-1}$, then the optimum is

$(x_1 = p_b m_1 + m_2 / p_b^{1/1-\beta} + p_b, \; x_2 = p_b m_1 + m_2 / 1 + p_b^{\beta/1-\beta})$