Predicate Logic Philosophical Summary

Second-Order Logic

Set Theory

- **Second-order logic is incomplete**, i.e. there is no axiomatic proof system for second-order logic under which every second-order logically valid wff is provable.
  - That second-order logic is incomplete is not sufficient grounds for thinking that second-order logic is set theory (or anything else) rather than logic. **Decidability is as significant a property as completeness**. A logic is decidable iff there exists an "effective method", like constructing a truth table for determining logical truths. Such methods require rigour but not ingenuity, terminate after a finite number of steps, and are in some sense mechanical. First-order logic is not decidable, but some fragments of both first-order logic and second-order logic are decidable. There seems to be no reason to favour completeness over decidability as grounds for thinking that some theory if logic rather than set theory (or anything else)
- **Quine: second-order logic is set theory in sheep's clothing because second-order logic has substantial ontological commitments.** According to Quine, **quantification over predicates is inappropriate because predicates are not names of either sets of attributes, and it is predicates that predicate letters stand in place of.** In quantifying over ordinary variables, these variables stand in place of names. To write "$\exists x : x \text{ walks}$" is to write that the thing named by some name that x stands in the place of walks. To quantify over predicate letters, then, seems to be to think that predicates, that predicate letters stand in place of, are names of certain things. For example, "$\forall X : \text{Aristotle is } X$" seems to treat X as standing in place of some name of something that Aristotle is or has. **The friend of second-order logic would respond that to quantify over X is not to treat X as standing in place of a name but to treat X as having some range. Then, X is a value-taking variable rather than a substitution-taking variable.** Presumably, the values that X takes are either sets or attributes (these are the two natural candidates). **Attributes are inadequately individuated, because of their intensionality.** So, the friend of second-order logic will **treat X as ranging over sets.** This is where the allegation of being "set theory in sheep's clothing" gets a grip. **Because $\exists X \forall x : \neg Xx$ and $\exists X \forall x : Xx$ are second-order logical validities, second-order logic seems committed to the existence of the empty set and the universal set.** According to Quine, this violates the topic-neutrality of logic. **Logic should remain silent on the existence of the empty set and the universal set, as it remains silent on the existence of Mars and Venus.**
  - The ontological commitments of second-order logic are limited. **First, the logical validity of $\exists X \forall x : Xx$ means only that for every domain, there is a set to which all members of that domain belong** (namely the domain itself). Second, the logical validity of $\exists X \forall x : \neg Xx$ means only that for every domain, there is a set to which no members of the domain belong. **None of this commits second-order logic to any Russell's paradox-like claims.** Second-order logic is committed to the existence of a set of non-self-membered sets only if there is some model whose domain is the set of all sets. But there is no reason to think such a domain exists, and plenty of reason to doubt it. Because second-order logic is silent on the domain (apart from that it should be non-empty), such claims can remain topic-neutral.
  - **Third, second-order logic is silent on many set-theoretic truths.** For example, $\exists X \exists y \exists z : (Xy \wedge Xz \wedge y \neq z)$ is not a second-order logical validity (because this evaluates as false under a model with a one-membered domain), so second-order logic is not committed to the existence of a two-membered set. It would be unreasonable to consider second-order logic to be set theory if it fails to recognise such a basic set theoretic truth.

Logic

- Second-order logic **naturally extends first-order logic** to yield greater expressive power. On Sider's account, the definition of a model for second-order logic is identical to the definition of a model for first-order logic. The definition of a variable assignment (and a variant assignment) is naturally extended to assign an extension to predicate variables. The definition of a valuation function is naturally extended by amending the clause for basic wffs and quantification. The definitions of logical validity and semantic consequence are also identical to those for first-order logic.
- Second-order logic has **greater expressive power than first-order logic**. Notions of finitude, identity, and ancestral, for example, can be formalised in second-order logic but not first-order logic. So second-order logic can recognise **"palpably" logical inconsistencies** that first-order logic cannot. For example, second-order logic can recognise the inconsistency in {'there are finitely many things', 'there is more than one thing', 'there are more than two things', ...} and in {'Smith is an ancestor of Jones', 'Smith is not a parent of Jones', 'Smith is not a grandparent of Jones', ...}. These are "palpably" logical in the sense that their inconsistency seems to be the same sort as the inconsistency in {'there are fewer

than two things', 'there is more than one thing'}. Second-order logic can also **recognise validities that first-order logic cannot.** For example, "there is some predicate whose extension contains all the things in the domain" can be formalised as "$\exists X \forall x : Xx$", which is valid, in second-order logic, but cannot be formalised in first-order logic. This greater expressive power constitutes reason to think that second-order logic is logic because, plausibly, **one of the functions of or motivations for logic is to systematically recognise such things as logical validities and logical inconsistencies**.

Middle Ground

- We should think that second-order logic is in some sense intermediate between first-order logic and set theory. One reason for thinking this is that second-order validity can be defined in terms of set theoretic truth but not the reverse, and first-order validity can be defined in terms of second-order validity but not the reverse. **So second-order logic is stronger than first-order logic but not as strong as set theory.** Then, the question of whether second-order logic is logic or set theory seems to be a question of whether the line between logic and set theory should be drawn to the "left" or "right" of second-order logic. It is reasonable to wonder what we are after in drawing such a line. **I conclude as Boolos begins, that whether second-order logic is logic or set theory is inconsequential, and any point in asking this question lies in the clarification of the relevant reasons on each side that this exercise necessitates. So our purpose here is complete.**

Predicate Logic with Identity

- **"Logical truths" of predicate logic with identity (that Quine refers to as truths of identity theory) are falsifiable by substituting the identity predicate with other predicates.** In contrast, we think that (truly) logical truths are not so falsifiable. For example, (uniformly) substituting some sentence (not sentence letter) for another in propositional logic, does not falsify the truths of propositional logic. Such a substitution involves only considering a different "instantiation" of a sentence letter. Similarly, substituting some predicate (not predicate letter) or some constant (not constant term) for another in simple predicate logic does not falsify the truths of simple predicate logic. Again, such a substitution involves only considering different "instantiations" of predicate letters and constant terms (which schematise over the predicates and constants that instantiate them).
- **The contrast between generalities that can be expressed in the object language (English, for our purposes) and generalities that "call for" semantic ascent "marks a conspicuous and tempting place at which to draw the line between the other sciences and logic".** The generality "all men are mortal" in English, generalises on "Tom is mortal", "Dick is mortal", and so on. This generalisation can be expressed in English. In contrast, generalisation on "Tom is mortal or Tom is not mortal" and "Snow is white or snow is not white" requires semantic ascent. We so generalise by saying "Every sentence of the form p or not p is true", this generalisation involves talk of the truth of sentences, while the earlier generalisation does not. Among the sentences generalised on, in the former case, it is names that are changed, whereas in the latter case, it is sentences or clauses that are changed. We generalise on "Tom is Tom", "Dick is Dick", and so on in English by saying "everything is itself". If the identity predicate is "proper to" logic, then it is unique in that its generalities can be expressed in English without semantic ascent.
- Godel proved that **complete proof procedures are available for predicate logic with identity**, but also that complete proof procedures are not available for elementary number theory.
- **A "serviceable facsimile" of identity can be constructed in a language with truth functions ($\neg$, $\rightarrow$, etc.) and quantification ($\forall$, $\exists$) and a finite lexicon of predicates.** The idea is to construct a facsimile of $x = y$ as a sentence that "reads" x As iff y As, and Bs iff y Bs, ..., and for all a, x Ps a iff y Ps a, ..., and so on. This "simulation" of identity "means" that x and y are indistinguishable by any sentences that can be phrased in the language. If x and y are indeed distinct, then this simulation fails to capture such distinctness, but such failure is "unobservable from within the language". In predicate logic, schematic predicate letters (F, G) are used in place of actual predicates ("is red", "is the parent of"). In the same spirit, **the notation $x = y$ can be viewed schematically as standing in place of the compound sentence (in the object language, English), whose counterpart is the construct above.**