Game Theory Problem Set 1

1

| | a | b | c | d |
|---|---|---|---|---|
| A | 1, 1 | 0, $\underline{2}$ | $\underline{3}$, 1 | 4, 0 |
| B | 3, 2 | 1, 1 | 1, 3 | 0, $\underline{4}$ |
| C | 1, $\underline{6}$ | 2, 5 | 0, 4 | $\underline{6}$, 5 |
| D | $\underline{4}$, 3 | $\underline{6}$, $\underline{4}$ | 2, 1 | 5, 2 |

Best responses underlined.
$D \succ_1 B$, then $a \succ_2 d$ and $b \succ_2 c$, then $D \succ_1 A$ and $D \succ_1 C$, then $b \succ_2 a$. So only D and b survive ISD.

~~A strategy~~ Player i's strategy $a_i$ is rationalisable iff it is a best response to some (potentially correlated) mix of other players' strategies. Equivalently, player i's strategy ~~is~~ $a_i$ is rationalisable if it survives iterated elimination of strategies that are not best responses to any (potentially mixed) strategies of other players.
Pearce's Lemma: in a two-player finite game, a pure strategy is strictly dominated by another pure or mixed strategy iff the former is not a best response ~~so~~ to any (potentially ~~?~~ mixed) strategies of other players. By definition of rationalisability and Pearce's Lemma, ~~all an~~ rationalisability coincides with ISD. So only D and b are rationalisable.
If a ~~strat~~ unique strategy profile survives ISD, that strategy profile is the unique NE. ~~If a unique strategy profile survives ISD but~~ Suppose for reductio that a unique strategy profile survives ISD but is not ~~the unique~~ an NE. Then, by definition of NE, some player has some profitable deviation from this strategy profile. ~~Then, the profitable deviation~~ that ~~is~~ (weakly) maximises his payoff. Then, this deviation is not strictly dominated, so there are multiple strategy profiles that survive ISD. By reductio, if a unique strategy profile survives ISD, it is a ~~the~~ NE. ~~By~~ By definition of NE and SD, all NE survive ISD. So if a unique strategy profile survives ISD, it is the unique NE. So the unique NE is (B,d)
~~ISD~~ Rationalisability and ISD follow from CKR, since rational players never play strictly dominated strategies or strategies that are never best responses. NE "extends" rationalisability and ISD with correct beliefs.

4

| | L | C | R |
|---|---|---|---|
| T | 0, $\underline{3}$ | $\underline{3}$, 0 | 1, 1 |
| M | $\underline{3}$, 0 | 0, $\underline{3}$ | 1, 1 |
| B | 1, $\underline{1}$ | 1, $\underline{1}$ | $\underline{6}$, 0 |

Best responses underlined
$(\frac{1}{2}L + \frac{1}{2}C) \succ_2 R$, then $(\frac{1}{2}T + \frac{1}{2}M) \succ_1 B$. So only T, M, L, and C survive ISD.
By definition of rationalisability and by Pearce's Lemma, only T, M, L, and C are rationalisable.
The unique Nash equilibrium is $(\frac{1}{2}T + \frac{1}{2}M, \frac{1}{2}L + \frac{1}{2}C)$. At this NE, ~~expected payoff to each~~ for each player, expected payoff of each action he mixes is equal ~~(ISD)~~ ($\frac{3}{2}$) and greater than that of the actions excluded (1), so no player can profitably deviate. By inspection, there are no mutual best responses in fixed strategies, so there are no fixed NE. By inspection there are no hybrid NE. There are no other mixed NE since if either player plays one action ~~more~~ with greater probability than the other, the other can profitably deviate to some fixed strategy.

2 Let $R^k = [\underline{x}^k, \bar{x}^k]$ denote the interval of yet uneliminated pure strategies after k iterations of IWD.
$R^0 = [\underline{x}^0, \bar{x}^0]$, $\underline{x}^0 = 0$, $\bar{x}^0 = 100$.
For $x_i > 100p$, $\forall x_i' \in [0, 100p]: \forall x_{-i}: \pi_i(x_i, x_{-i}) = 0 \leq \pi_i(x_i', x_{-i})$ and ~~$\exists x_i' \in [0, 100p]$~~ $\forall x_i' \in [0, 100p]$: $\exists x_{-i}: \pi_i(x_i, x_{-i}) = 0 < \pi_i(x_i', x_{-i})$, i.e. $x_i > 100p$ is weakly dominated by each $x_i' \in [0, 100p]$.
$R^1 = [0, 100p]$
Then, by a similar argument, $x_i > 100p^2$ is weakly dominated by each $x_i \in [0, 100p^2]$
$R^2 = [0, 100p^2]$
In the limit as $k \to \infty$
$R^k = [0, 0]$
So the only strategy profile that survives IWD is such that each player i plays $x_i = 0$.

Player i maximises his payoff if ~~at~~ $x_i = p\bar{x}$
$= p(\frac{1}{N}) \sum_{i=1}^{N} x_i = \frac{p}{N}(N-1)\bar{x}_{-i} + \frac{p}{N} x_i$, where $\bar{x}$ is the mean ~~of~~ action of all players and $\bar{x}_{-i}$ is the mean action of all players other than i.
$(1 - \frac{p}{N}) x_i = \frac{p}{N}(N-1)\bar{x}_{-i}$
$x_i = \cancel{\frac{Np-p}{N-Np}\bar{x}_{-i}}$ $\frac{Np-p}{N-p}\bar{x}_{-i}$
Then, ignoring cases where player i receives positive payoff by ~~being~~ choosing $x_i$ closest to but not equal to $p\bar{x}$, player i's best response $B_i(x_{-i}) = \frac{Np-p}{N-p}\bar{x}_{-i}$.

~~At~~ NE, all players choose a common action. Suppose for reductio that at some NE, players do not choose a common action, then some player receives zero payoff and would receive greater payoff if he chooses $x_i = B_i(\bar{x}_{-i})$. ~~So at all.~~ Then ~~By reduc~~ the outcome is not a NE. By reductio, at NE all players choose a common action. Then, at NE $x_i^* = \bar{x}_{-i}^*$. By definition of NE, $x_i^* = B_i(x_{-i}^*)$ so $x_i^* = {}^{Np-p}/_{N-p}\, x_i^*$, ${}^{N-Np}/_{N-p}\, x_i^* = 0$. Since ${}^{N-Np}/_{N-p} \neq 0$, $x_i^* = 0$. So at the unique NE, all players choose 0.

3

| | L | C | R |
|---|---|---|---|
| T | 2, 0 | 0, 2 | 0, 0 |
| M | 0, 2 | 2, 0 | 0, 0 |
| B | 0, 0 | 0, 0 | 1, 1 |

Best responses underlined

By inspection, (B,R) is the unique pNE, where players play mutual best responses.

Suppose $\exists$ mNE ~~st p1 mixes T,M,B. Then~~ $s^*$ st p1 mixes T, M, B. Then $\pi_1(T, s_2^*) = \pi_1(M, s_2^*) = \pi_1(B, s_2^*)$, $2p_L = 2p_C = p_B$. By def$^n$ of mixed strategy, $p_L + p_C + p_R = 1$. Solving simultaneously, ~~$p_L = p_C$~~ $p_L = p_C = 1/4$, $p_R = 1/2$. So p2 mixes L, C, R. Then $\pi_2(L, s_1^*) = \pi_2(C, s_1^*) = \pi_2(R, s_1^*)$, $2p_T = 2p_M = p_B$. $p_T + p_M + p_B = 1$. Solving simultaneously, $p_T = p_M = 1/4$, $p_B = 1/2$. $(1/4T + 1/4M + 1/2B, 1/4L + 1/4C + 1/2R)$ is the unique mNE where p1 mixes T, M, B. By symmetry, it is also the unique mNE where p2 mixes L, C, R.

Suppose $\exists$ mNE $s^*$ st p1 mixes T, M only. Then $\pi_1(T, s_2^*) = \pi_1(M, s_2^*)$, $2p_L = 2p_C$, $p_R = 0$, $p_L = p_C = 1/2$. ~~Then~~ So p2 mixes L, C only. Then $\pi_2(L, s_1^*) = \pi_2(C, s_1^*)$ ~~$2p_T = 2$~~ $2p_T = 2p_M$, $p_B = 0$, $p_T = p_M = 1/2$. Then $\pi_1(B, s_2^*) = 0$, $\pi_2(R, s_1^*) = 0$. $(1/2T + 1/2M, 1/2L + 1/2C)$ is the unique ~~Nash eq.~~ NE where p1 mixes T, M only. By symmetry, it is the unique NE where ~~p1 mixe~~ p2 mixes L, C only.

Suppose $\exists$ mNE $s^*$ st p1 mixes T, B only, then $\pi_1(T, s_2^*) = \pi_1(B, s_2^*)$, $2p_C = p_R$, $p_L = 0$. So $\pi_2(C, s_1^*) = \pi_2(R, s_1^*)$, $2p_M = p_B$, $p_T = 0$. This contradicts the supposition, so $\nexists$ mNE $s^*$ st p1 mixes T, B only. By symmetry, $\nexists$ mNE st p1 mixes M, B only or p2 mixes L, R only or p2 mixes C, R only.

By inspection $\nexists$ hNE since each player has a strict unique best response to each action of the other.

The NE are

(B,R)

~~(1/2T + 1/2M ...)~~

$(1/4T + 1/4M + 1/2B, 1/4L + 1/4C + 1/2R)$

$(1/2T + 1/2M, 1/2L + 1/2C)$

At the first NE, each player has expected payoff 1. At the second, each player has expected payoff 1/2. At the third, each player has expected payoff 1.

Players have lower payoff at the second NE due to the possibility of miscoordination.

| | L | C | R |
|---|---|---|---|
| T | 1, 1 | 1, 1 | 0, 0 |
| B | 0, 0 | 0, 0 | 1, 1 |

Best responses underlined

By inspection there are 3 pNE, namely (T,L), (T,C) (B,R), where players play mutual BRs.

$\exists$ hNE ~~st~~ $(T, pL + (1-p)C)$. ~~$\pi_1(B$~~

$\pi_1(T, pL + (1-p)C) = 1 > \pi_1(B, pL + (1-p)C) = 0$, so p1 has no profitable deviation. $\pi_2(T, L) = \pi_2(T, C) = 1 > \pi_2(T, R) = 0$, so p2 has no profitable deviation.

By inspection, $\nexists$ other hNE.

Suppose $\exists$ mNE $s^*$, then p1 mixes T, B, then $\pi_1(T, s_2^*) = \pi_1(B, s_2^*)$, $p_L + p_C = p_R$. By def$^n$ mix ~~#~~ strategy, $p_L + p_C + p_R = 1$. $p_R = 1/2$ $p_L = p/2$, $p_C = {}^{1-p}/_2$. So p2 mixes R with L and/or C. $\pi_2(R, s_1^*) = \pi_2(L, s_1^*)$ and/or $\pi_2(R, s_1^*) = \pi_2(C, s_1^*)$. The two conditions turn out to be equivalent: $p_T = p_B$. $p_T + p_B = 1$, $p_T = p_B = 1/2$

So $\exists$ mNE $(1/2T + 1/2B, p/_2L + {}^{1-p}/_2C + 1/2R)$ $p \in [0,1]$

The NE are

(T,L), (T,C), (B,R)

$(T, pL + (1-p)C)$ $p \in [0,1]$

$(1/2T + 1/2B, p/_2L + {}^{1-p}/_2C + 1/2R)$ $p \in [0,1]$

Expected payoff of each player is 1 in pNE and hNE, and 1/2 in mNE because of the possibility of miscoordination.

G1

| | L | R |
|---|---|---|
| T | 1,2,3 | 0,0,0 |
| B | 0,0,0 | 0,0,0 |

G2

| | L | R |
|---|---|---|
| T | 0,0,0 | 0,0,0 |
| B | 0,0,0 | 3,2,1 |

Best responses underlined

By inspection, there are two pNE (T,L,G1) and (B,R,G2) where players play mutual best responses.

~~By inspection $\nexists$ hNE. By inspecti~~

By inspection, $\nexists$ hNE

Suppose $\exists$ mNE $s^*$ then p1 mixes T, B, then $\pi_1(T, s_{-1}^*) = \pi_1(B, s_{-1}^*)$, $p_L p_{G1} = 3 p_R p_{G2}$. And p2 mixes L, R, then $\pi_2(L, s_{-2}^*) = \pi_2(R, s_{-2}^*)$, $2 p_T p_{G1} = 2 p_B p_{G2}$. And p3 mixes G1, G2, then $\pi_3(G1, s_{-3}^*) = \pi_3(G2, s_{-3}^*)$, $3 p_T p_L = p_B p_R$

$p_T p_{G1} = p_B p_{G2} = (1-p_T)(1-p_{G1}) = 1 + p_T p_{G1} - p_T - p_{G1}$,

$p_T + p_{G1} = 1$, $p_T = 1 - p_{G1} = p_{G2}$, $p_B = p_{G1}$

$p_L p_B = 3 p_R p_T$, $p_L(1-p_T) = 3p_T(1-p_L)$, $p_L - 3p_T + 2p_L p_T = 0$

$3p_T p_L = (1-p_T)(1-p_L)$, $p_L + p_T + 2p_L p_T - 1 = 0$

$4p_T - 1 = 0$, $p_T = 1/4$

$p_T = 1/4$, $p_B = 3/4$, $p_{G1} = 3/4$, $p_{G2} = 1/4$, $p_L = 1/2$, $p_R = 1/2$

Since for each player, each action he mixes yields equal payoff, and each player mixes over all his actions, $(1/4T + 3/4B, 1/2L + 1/2R, 3/4G1 + 1/4G2)$ is a mNE. The expected payoff to each player is 3/8 due to the possibility of miscoordination.