Game Theory Supplementary Exercises 1

1a

	L	C	R
T	$\underline{4}$ / $\underline{4}$	1 / 0	2 / $\underline{3}$
M	0 / 1	$\underline{1}$ / $\underline{1}$	-1 / 1
B	$\underline{3}$ / 2	1 / -1	2 / 2

Best responses underlined

~~A st~~

Player i's ~~strat~~ action a_i is rationalisable, by definition, iff it is a best response against some potentially correlated mix of other players' actions. Equivalently, player i's action is rationalisable, by definition, iff it survives iterated elimination of ~~strat~~ actions that are not best responses against some potentially correlated mix of other players' actions.

~~$L >_2 R$, so R is never a best response~~

By inspection, $L >_2 R$, then by definition of strict dominance and BR, R is never a BR.

Likewise, $T >_1 B$, so B is never a BR.

~~Each r~~ Then, each remaining action is a BR to some action of the other player, so iterated elimination of never-BRs terminates here. T, M, L, and C only are rationalisable.

Since ~~CKR implies~~ players play only rationalisable actions, we consider the reduced game involving only rationalisable actions

	L	C
T	$\underline{4}$ / $\underline{4}$	1 / 0
M	0 / 1	$\underline{1}$ / $\underline{1}$

Best responses underlined

By inspection, (T, L) and (M, C) are the only pure NE where players play mutual best responses.

By inspection there are no hybrid NE since no player is ever indifferent between his actions when the other plays some fixed strategy.

~~Suppose there is some~~ consider mixed ~~NE~~ strategy profile s^* ~~under, then~~ at s^* Pl.1 plays T with probability p and M with probability 1-p and Pl.2 plays L with probability q and C with probability 1-q for $p, q \in (0,1)$. If s^* is a mixed NE, then $\pi_1(T, s_2^*) = \pi_1(M, s_2^*)$,

$4q = q + (1-q)$, $q = 1/4$ and

$\pi_2(L, s_1^*) = \pi_2(C, s_1^*)$,

$4p = p + (1-p)$, $p = 1/4$.

so the only mixed NE is $s^* = (1/4\,T + 3/4\,M,\ 1/4\,L + 3/4\,C)$

b

	L	C	R
T	$\underline{4}$ / $\underline{4}$	1 / $\underline{0}$	2 / $\underline{3}$
M	$\underline{0}$ / 1	$\underline{0}$ / $\underline{0}$	-1 / 1
B	$\underline{3}$ / 2	1 / -1	2 / 2

Best responses underlined

By inspection, $L \geq_2 R$ and $T >_1 B$.

~~Since rational~~ ~~Since CKR~~

~~the consider~~ We consider ~~only~~ the reduced game involving only strategies that survive ISD.

	L	C
T	$\underline{4}$ / $\underline{4}$	1 / $\underline{0}$
M	$\underline{0}$ / 1	$\underline{0}$ / $\underline{0}$

By inspection, (T, L) and (M, C) are the only pure NE where players play mutual best responses.

Consider some strategy profile s^* such that Pl.1 mixes T and M, i.e. $s_1^* = (pT + (1-p)M)$, $p \in (0,1)$

If ~~that~~ s^* is an NE, Pl.1 has no profitable deviation, so $\pi_1(T, s_2^*) = \pi_1(M, s_2^*)$, $4q = 1q$ where $s_2^* = (qL + (1-q)C)$ for $q \in [0,1]$, so $q = 0$ and $s_2^* = C$

If s^* is an NE, Pl.2 has no profitable deviation, so $\pi_2(C, s_1^*) \geq \pi_2(L, s_1^*)$, ~~4p~~ $p \geq 4p$, $p \leq 0$.

This contradicts the supposition that $p \in (0,1)$, so by reductio, there is no NE where Pl.1 mixes T and M. By symmetry, there is no NE where Pl.2 mixes L and C.

A mixed NE does not exist because each player has a weakly ~~rational~~ dominant ~~strateg~~ action, so any mixed ~~strat~~ strategy fails to the deviation which reallocates probability mass

2a

	L	C	R
T	$\underline{15}$ / 10	11 / 16	12 / $\underline{8}$
M	2 / 12	1 / $\underline{20}$	$\underline{5}$ / 5
B	7 / 11	10 / 19	$\underline{11}$ / 4

Best responses underlined

Player i's action a_i is rationalisable, by definition, iff it is a best response against some potentially correlated mix of other players' actions. ~~By~~

Equivalently, player i's action a_i is rationalisable iff it survives iterated elimination of actions that are not best responses to any potentially correlated mix of other players' yet uneliminated actions.

By inspection $M >_1 B$, so ~~B is~~ by definition of strict dominance and BR, B is never a best response against any ~~potentially correlated~~ mix of Pl.2's strategies. Similarly, $R >_2 C$ so C is never a best response against any mix of Pl.1's strategies so eliminate B and C. Then the game reduces to

	L	R
T	$\underline{15}$, 10	12, $\underline{8}$
M	2, $\underline{12}$	$\underline{5}$, 5

By inspection, each remaining strategy is a best response to some ~~action~~ strategy of the other player, so iterated elimination stops here. T, M, L, and R only are rationalisable.
Let s^* be the unique NE. By definition of strict dominance and NE, only strategies that survive ISD are played in NE, so B and C are not played in NE. Then, given that s^* is fully mixed, $s^* = (pT + (1-p)M, ~~qR+(~~ qL + (1-q)R)$ for some $p, q \in (0,1)$.
By definition of NE, at s^*, there is no profitable deviation, so ~~$\pi_1(T, s_2^*) = \pi_1(B, s_2^*)$ and $\pi_2(L, s_1^*) = \pi_2(R, s_1^*)$ since otherwise some player can profitably deviate by reallocating probability mass to the action with higher payoff~~ and $\pi_1(T, s_2^*) = \pi_1(M, s_2^*) \geq \pi_1(B, s_2^*)$ and $\pi_2(L, s_1^*) = \pi_2(R, s_1^*) \geq \pi_2(C, s_1^*)$ since otherwise some player can profitably deviate by reallocating probability mass to the action with the higher payoff.

$10q + 8(1-q) = 12q + 5(1-q) \geq 11q + 4(1-q)$
$3 - 3q = 2q,\ 5q = 3,\ q = 3/5$
$15p + 2(1-p) = 12p + 5(1-p) \geq 11p + 1(1-p)$
$3p = 3(1-p),\ 6p = 3,\ p = 1/2$
$s^* = (1/2\, T + 1/2\, M, 3/5\, L + 2/5\, R)$ is the unique NE.

Each correlated equilibrium of this game

$$p = \begin{pmatrix} p_{TL} & p_{TC} & p_{TR} \\ p_{ML} & p_{MC} & p_{MR} \\ p_{BL} & p_{BC} & p_{BR} \end{pmatrix}$$

is a joint distribution that satisfies the incentive compatibility constraints
Pl.1 finds it optimal to play T when so instructed, to ~~$10p_{TL} + 16p$ Pl.2 finds it optimal~~ play M when so instructed, ~~to~~ and to play B when so instructed. Similarly for Pl.2
Since $M >_1 B$, ~~player~~ Pl.1 never finds it optimal to play B when so instructed, since it is always better to play M. Likewise, Pl.2 never finds it optimal to play C since $R >_2 C$, so ~~ea~~ each correlated equilibrium never instructs Pl.1 to play B and never instructs Pl.2 to play C.
$p_{TC} = p_{MC} = p_{BL} = p_{BC} = p_{BR} = 0$. So ~~each~~ each correlated equilibrium reduces to

$$p = \begin{pmatrix} p_{TL} & p_{TR} \\ p_{ML} & p_{MR} \end{pmatrix}$$

The incentive constraints are
Pl.1 finds it optimal to play T when so instructed
$10p_{TL} + 8p_{TR} \geq 12p_{TL} + 5p_{TR}$
Pl.1 finds it optimal to play M when so instructed
$12p_{ML} + 5p_{MR} \geq 10p_{ML} + 8p_{MR}$
For ~~player~~ Pl.2,
$15p_{TL} + 2p_{ML} \geq 12p_{TL} + 5p_{ML}$
$12p_{TR} + 5p_{MR} \geq 15p_{TR} + 2p_{MR}$
Simplifying,
$3p_{TR} \geq 2p_{TL},\quad p_{TR} \geq 2/3\, p_{TL}$
$2p_{ML} \geq 3p_{MR},\quad p_{ML} \geq 3/2\, p_{MR}$
$3p_{TL} \geq 3p_{ML},\quad p_{TL} \geq p_{ML}$
$3p_{MR} \geq 3p_{TR},\quad p_{MR} \geq p_{TR}$
Let $p_{TL} = x$, then one solution is
$p_{TL} = x,\ p_{TR} = 2/3x,\ p_{MR} = 2/3x,\ p_{ML} = x$
$p_{TL} + p_{TR} + p_{ML} + p_{MR} = 1$, ~~so~~ $10/3x = 1,\ x = 3/10$
$p_{TL} = 3/10,\ p_{TR} = 2/10,\ p_{ML} = 3/10,\ p_{MR} = 2/10$
This corresponds to the mixed NE
$(1/2\, T + 1/2\, M, 3/5\, L + 2/5\, R)$
By inspection of the simplified incentive compatibility constraints,
~~$\uparrow p_{TL}$ (from the given solution) $\Rightarrow \uparrow p_{TR} \Rightarrow \uparrow p_{MR} \Rightarrow \uparrow p_{ML} \Rightarrow \uparrow(p_{TL}+p_{TR}+p_{ML}+p_{MR}) > 1$, so by reductio there is no solution where $p_{TL} >$~~
$p_{TL} > 3/10 \Rightarrow p_{TR} > 2/10 \Rightarrow p_{MR} > 2/10 \Rightarrow p_{ML} > 3/10$,
collectively $\Rightarrow (p_{TL} + p_{TR} + p_{ML} + p_{MR}) > 1$
$p_{TL} < 3/10 \Rightarrow p_{TR} < 2/10 \Rightarrow p_{MR} < 2/10 \Rightarrow p_{ML} < 3/10$,
collectively $\Rightarrow (p_{TL} + p_{TR} + p_{ML} + p_{MR}) < 1$
By ~~reductio~~ definition of a correlated equilibrium,
$p_{TL} + p_{TR} + p_{ML} + p_{MR} = 1$
By reductio, there is no correlated equilibrium where $p_{TL} \neq 3/10$. Similarly, there is no correlated equilibrium where $p_{TR} \neq 2/10$, $p_{ML} \neq 3/10$, or $p_{MR} \neq 2/10$.
~~The unique~~ The correlated equilibrium is unique and coincides with the unique Nash equilibrium.
This correlated equilibrium both maximises and minimises ~~each players'~~ the sum of players' payoffs.

State space $\Omega = \{w, x, y, z\}$
Probability distribution over Ω, π such that
$\pi(w) = \pi(y) = 3/10,\ \pi(x) = \pi(z) = 2/10$
Partition for ~~player~~ Pl.1
$P_1 = \{\{w, x\}, \{y, z\}\}$
Partition for Pl.2
$P_2 = \{\{w, y\}, \{x, z\}\}$
Strategy for Pl.1
σ_1 such that $\sigma_1(w) = \sigma_1(x) = T,\ \sigma_1(y) = \sigma_1(z) = M$
Strategy for Pl.2
σ_2 such that $\sigma_2(w) = \sigma_2(y) = L,\ \sigma_2(x) = \sigma_2(z) = R$

3a Set of players $N=\{1,2\}$

Set of actions of each player i

~~$A_i = \mathbb{R}^+$~~ $B_i = \mathbb{R}^+$

Payoff of each player i

$u_i(b_i, b_{-i}) = -b_i$ if $b_i < b_{-i}$

$\quad a/2 - b_i$ if $b_i = b_{-i}$

$\quad a - b_i$ if $b_i > b_{-i}$

Let $BR_i(b_{-i})$ denote player i's best response given player $-i$ plays b_{-i}

Suppose $b_{-i} = a$, then

$u_i(b_i=0, b_{-i}) = 0$

$u_i(b_i\in(0,b_{-i}), b_{-i}) = -b_i < 0$

$u_i(b_i=b_{-i}, b_{-i}) = a/2 - b_{-i} = -a/2 < 0$

$u_i(b_i > b_{-i}, b_{-i}) = a - b_i < 0$

So $BR_i(b_{-i}=a) = 0$

Suppose $b_{-i} > a$, then

$u_i(b_i=0, b_{-i}) = 0$

$u_i(b_i\in(0,b_{-i}), b_{-i}) = -b_i < 0$

$u_i(b_i=b_{-i}, b_{-i}) = a/2 - b_{-i} < 0$

$u_i(b_i > b_{-i}, b_{-i}) = a - b_i < 0$

So $BR_i(b_{-i}=a) = 0$

Suppose $b_{-i}<a$, then

$b_i < b_{-i}$ fails to deviation $b_i' = b_{-i}+\varepsilon$ for sufficiently small $\varepsilon>0$ since $u_i(b_i<b_{-i}, b_i) = \cancel{a}-b_i \le 0$ ~~and~~ $< u_i(b_i'=b_{-i}+\varepsilon, b_{-i}) = a - b_i' = (a-b_{-i})-\varepsilon > 0$ for $\varepsilon < (a-b_{-i})$

$b_i=b_{-i}$ fails to a similar deviation since $u_i(b_i=b_{-i},b_{-i}) = a/2 - b_i < u_i(b_i'=b_{-i}+\varepsilon, b_{-i}) = a-b_{-i}-\varepsilon$ for $\varepsilon < a/2$

$b_i>b_{-i}$ fails to deviation $b_i' = b_i + (b_i - b_{-i}/2)$ since $u_i(b_i>b_{-i}, b_{-i}) = a-b_i < u_i(b_i', b_{-i}) = a - b_i'$ where ~~$b_i' > b_i$ and b_i b_{-i}~~ $b_{-i}<b_i'<b_i$.

So $BR_i(b_{-i}<a) = \emptyset$

$$BR_i(b_{-i}) = \begin{cases} 0 & \text{if } b_{-i}\ge a \\ \emptyset & \text{if } b_{-i}<a \end{cases}$$

By definition of NE and BR, players play mutual best responses at NE.

~~$b_1 = BR_1(b_2)$~~ Suppose that some ~~strategy~~ pure strategy profile b^* is a NE, then

$b_1^* = BR_1(b_2^*) = 0$ if $b_2^*\ge a$

$\quad \emptyset$ if $b_2^* < a$

By definition of NE and strategy profile, $b_1^*, b_2^* \ne \emptyset$, so $b_1^*=b_2^*=0$. then $b_2^*<a$ then $b_1^*=\emptyset$. By reductio, there is no pure NE b^*.

b Suppose there is a mixed NE where each ~~firm~~ player i mixes a finite number of actions. Let $\bar b_i$ be the highest bid ~~on~~ which firm i ~~places pos~~ plays with positive probability.

Suppose that ~~$\bar b_i \ne \bar b_j$~~ $\bar b_i \ne \bar b_j$, then $\bar b_i > \bar b_j$ for some i,j. Then player i can profitably deviate by reallocating probability mass from $\bar b_i$ to $\bar b_i' \in (\bar b_j, \bar b_i)$ since $u_i(\bar b_i, b_j) = a - \bar b_i < u_i(\bar b_i', b_j) = a - \bar b_i'$ for all b_j, so $E(u_i(\bar b_i, b_j)) < E(u_i(\bar b_i', b_j))$.

Suppose instead that $\bar b_1 = \bar b_2$. Then either player i can profitably deviate by reallocating ~~probab~~ probability mass from $\bar b_i$ to $\bar b_i' = \bar b_i + \varepsilon$ for sufficiently small $\varepsilon>0$. ~~This reallocation yields an increase in expected payoff from a greater Then player on the probability $\bar p_j$ that player j plays $\ne \bar b_j$, under the prior strategy profile Then where previously there was a $\bar p_i \bar p_j$ probability that $b_i=\bar b_i=\bar b_j=b_j$ so $u_i(b_i,b_j) = a/2 - \bar b_i$, under the deviated strategy profile, there is a $\bar p_i\bar p_j$ probability that $b_i = \bar b_i' > \bar b_j = b_j$ so $u_i(b_i,b_j) = a - \bar b_i' = a - \bar b_i - \varepsilon$.~~

Suppose further that under the candidate equilibrium, $\bar b_i$ is played with probability $\bar p_i$ and $\bar b_j$ is played with probability $\bar p_j$. ~~Only the change in u_i when~~ Under the proposed deviation, when i plays $\bar b_i'$ instead of $\bar b_i$ with probability $\bar p_i$, ~~i's payoff decreases by ε but if~~ $u_i(\bar b_i', b_j) = u_i(\bar b_i, b_j) - \varepsilon$ for $b_j \ne \bar b_j$ and $u_i(\bar b_i', \bar b_j) = u_i(\bar b_i, \bar b_j) + a/2 - \varepsilon$ for $b_j = \bar b_j$. So ~~$E(u_i$~~ i's expected payoff increases if $\varepsilon < \bar p_j a/2$. Informally, this deviation is profitable for i if the cost of the higher bid is smaller than the ~~p~~ value of winning the prize with greater probability.

So if each player mixes over a finite number of actions, some player has a profitable deviation, this is not a NE.

c Suppose there is a symmetric mixed NE, b^*, ~~where then by the result~~ where each player i plays mixed strategy b_i^* which is a probability ~~over~~ distribution over B_i. Let ~~$F(b)$ denote~~ Let $F(b)$ denote the common cdf of b_1^* and b_2^*.

There are no gaps in the support of $F(b)$. Suppose that there is a gap $[\hat b, \check b]$ in the support of $F(b)$, ~~then~~ i.e. ~~and~~ players never play an action $b_i \in [\hat b, \check b]$. Then, each player can profitably deviate by reallocating probability mass from ~~actions to~~ actions "just above" this ~~interval the~~ $\check b$ to actions "just above" $\hat b$ since this reallocation has no effect on ~~a player's~~ the probability of each player's making the winning bid but reduces the ~~expected~~ cost of bidding.

By definition of NE, at b^*, there is no profitable deviation, so ~~eac the payoff to each~~ for each player, the payoff of each action in the support of $F(b)$ must be equal, since otherwise ~~e~~ some player can profitably deviate by reallocating probability mass from actions in the support with lower expected payoff to actions with higher payoff, i.e.
$E(u_i(b_i, b^*_{-i})) = F(b_i)a - b_i$ is constant for all b_i in the support of $F(b)$.

~~0 is in the~~

0 is in the support of $F(b)$. Suppose that 0 is not in the support of $F(b)$. Then given that there are no gaps in the support of $F(b)$, the support of $F(b)$ is ~~sub~~ $[\underline{b}, \bar{b}]$ for some $0 < \underline{b} < \bar{b} \in \mathbb{R}^+$. Then, each player can ~~profitabili~~ profitably deviate by reallocating probability mass from actions ~~at the bottom of this~~ $b_i \in [\underline{b}, \underline{b} + \varepsilon]$ to $b_i = 0$ since the former actions have payoff $-\underline{b} < 0$ in the limit as ε becomes small while $b_i = 0$ has payoff 0.

~~$0 = F(b_i$~~ $F(0)a - 0 = F(b_i)a - b_i = 0$.
~~$b_i = F($~~ $F(b_i) = a/b_i$
$F(b_i)$ ~~correspon~~ is the cdf of $U[0, a]$.
At b^*, each player plays the mixed strategy b^*_i which is a uniform probability distribution over $[0, a]$. Each player's expected payoff is 0. In first-price all-pay auctions, ~~buyers~~ (at least in the two-player case), all surplus is captured by the seller.