Microeconomic Analysis Problem Set 6

1a ~~The~~ Given that it is optimal to induce $e=1$, the principal P's optimisation problem is
$\min_{w_1, w_2, w_3} E(w|e=1)$ subject to
Individual Rationality: $E(\sqrt{w}-e|e=1) \geq \bar{u}$ and
Incentive compatibility: $E(\sqrt{w}-e|e=1) \geq E(\sqrt{w}-e|e=0)$
This is equivalent to
$\min_{w_1, w_2, w_3} w_1/4 + w_2/2 + w_3/4$ subject to
$1/4\sqrt{w_1} + 1/2\sqrt{w_2} + 1/4\sqrt{w_3} - 1 \geq 10$ and
$1/4\sqrt{w_1} + 1/2\sqrt{w_2} + 1/4\sqrt{w_3} - 1 \geq 1/3(\sqrt{w_1}+\sqrt{w_2}+\sqrt{w_3})$

Solving by Lagrangian optimisation,
$L = w_1/4 + w_2/2 + w_3/4 - \lambda_1(\sqrt{w_1}/4 + \sqrt{w_2}/2 + \sqrt{w_3}/4 - 11)$
$-\lambda_2(\sqrt{w_1}/4 + \sqrt{w_2}/2 + \sqrt{w_3}/\cancel{24} \, 1 - 1/3(\sqrt{w_1}+\sqrt{w_2}+\sqrt{w_3}))$
FOCw_1: $1/4 - \lambda_1(1/8)w_1^{-1/2} - \lambda_2(-1/24)w_1^{-1/2} = 0$
FOCw_2: $1/2 - \lambda_1(1/4)w_2^{-1/2} - \lambda_2(1/12)w_2^{-1/2} = 0$
FOCw_3: $1/4 - \lambda_1(1/8)w_3^{-1/2} - \lambda_2(-1/24)w_3^{-1/2} = 0$

Is there any argument for either IR binding or IC binding?

Do the KT FOCs include the CSs?

Is it necessary to check the CSs? If not, why not?

~~FOCw_1, FOCw_3 ⇒ $w_1^* = w_3^*$~~
~~FOCw_1, FOCw_3 ⇒ $w_1 = w_3$~~
~~Then, FOCw_1 is satisfied iff FOCw_3 is satisfied. Given that KT FOCs are necessary and sufficient for optimality FOCw_1 and FOCw_2 are together necessary and sufficient for optimality.~~

~~For w_1~~
CSλ_1: $\lambda_1 \geq 0$ ~~$\sqrt{w_1}/2 + \sqrt{w_2}/4$~~ $\sqrt{w_1}/4 + \sqrt{w_2}/2 + \sqrt{w_3}/4 - 1 \geq 10$,
$\lambda_1(\sqrt{w_1}/4 + \sqrt{w_2}/2 + \sqrt{w_3}/4 - (11)) = 0$
CSλ_2: $\lambda_2 \geq 0$, $\sqrt{w_1}/4 + \sqrt{w_2}/2 + \sqrt{w_3}/4 - 1 \geq 1/3(\sqrt{w_1} + \sqrt{w_2} + \sqrt{w_3})$
$\lambda_2(-\sqrt{w_1}/12 + \sqrt{w_2}/6 - \sqrt{w_3}/12 - 1) = 0$

IR binds. Suppose for reductio that IR does not bind, ~~i.e.~~ then $\sqrt{w_1}/4 + \sqrt{w_2}/2 + \sqrt{w_3}/4 - 1 > 10$. For sufficiently small ε, $w_1, w_2' = w_2 - \varepsilon, w_3$ satisfies IR and IC and $E(w'|e=1) > E(w|e=1)$

$\lambda_1 > 0$, $\sqrt{w_1}/4 + \sqrt{w_2}/2 + \sqrt{w_3}/4 - 1 = 10$, $\sqrt{w_1}/2 + \sqrt{w_2}/2 = 11$,
$\sqrt{w_1} + \sqrt{w_2} = 22$

Suppose that IC does not bind, then $\lambda_2 = 0$. ~~FOCw_1 and FOCw_2 reduce~~
FOCw_1 ⇒ $1/4 = \lambda_1(1/8)w_1^{-1/2}$ ⇒ $\sqrt{w_1} = \lambda_1/2$
FOCw_2 ⇒ $1/2 = \lambda_1(1/4)w_2^{-1/2}$ ⇒ $\sqrt{w_2} = \lambda_1/2$
⇒ $w_1 = w_2 = w_3$ ⇒ $\sqrt{w_1}/4 + \sqrt{w_2}/2 + \sqrt{w_3}/4 - 1 < 1/3(\sqrt{w_1} + \sqrt{w_2} + \sqrt{w_3})$
CSλ_2 fails. By reductio, IC binds.

What is the argument for IC binding

$\lambda_2 > 0$, $\sqrt{w_1}/4 + \sqrt{w_2}/2 + \sqrt{w_3}/4 - 1 = \sqrt{w_1}/2 + \sqrt{w_2}/2 - 1 = 10 = 1/3(\sqrt{w_1} + \sqrt{w_2} + \sqrt{w_3})$
$\sqrt{w_1} + \sqrt{w_2} = 22$, $\sqrt{w_1} + \sqrt{w_2} + \sqrt{w_3} = 30$, ⇒ $\sqrt{w_3} = 8$ ⇒
$\sqrt{w_1} = 8$ ⇒ $\sqrt{w_2} = 14$
FOCw_1: $1/4 - \lambda_1/64 + \lambda_2/192 = 0$
FOCw_2: $1/2 - $ ~~$\lambda_1/32$~~ $\lambda_1/56 - \lambda_2/168 = 0$
⇒ $\lambda_1 = 22$, $\lambda_2 = 18$, CS hold

Is this check necessary?

The optimal contract $(w_1^*, w_2^*, w_3^*) = (64, 196, 64)$

b: i. It is in the agent's interest to secretly destroy ~~reve~~ revenue in the outcome π_3 since wage is higher at the lower revenue π_2.

ii. It is never in the agent's interest to destroy revenue iff $w_1 \leq w_2 \leq w_3$.

At the optimum, at least one of these inequalities is strict. Otherwise IC fails. If both inequalities hold with equality, the contract is equivalent to a fixed wage and it is in the agent's interest to choose $e = 0$.

Suppose that $w_1 = w_2$. Then $w_1 = w_2 < w_3$ $\Rightarrow$ ~~then $\sqrt{w_1} = \sqrt{w_2}$~~
$\sqrt{w_1} = \sqrt{w_2} < \sqrt{w_3} \Rightarrow \frac{1}{4}\sqrt{w_1} + \frac{1}{2}\sqrt{w_2} + \frac{1}{4}\sqrt{w_3} = \frac{3}{4}\sqrt{w_1} + \frac{1}{4}\sqrt{w_3}$
$< \frac{1}{3}\sqrt{w_1} + \frac{1}{3}\sqrt{w_2} + \frac{1}{3}\sqrt{w_3} = \frac{2}{3}\sqrt{w_1} + \frac{1}{3}\sqrt{w_3} \Rightarrow$
$\frac{1}{4}\sqrt{w_1} + \frac{1}{2}\sqrt{w_2} + \frac{1}{4}\sqrt{w_3} - 1 < \frac{1}{3}(\sqrt{w_1} + \sqrt{w_2} + \sqrt{w_3})$ i.e. IC fails.
By reductio $w_1 < w_2$.

~~P's optimisation problem is~~
~~$\min_{w_1, w_2, w_3} w_1/4 + w_2/2 + w_3/4$ subject to~~
~~$\sqrt{w_1}/4 + \sqrt{w_2}/2 + \sqrt{w_3}/4 - 1 \geq 10$~~
~~$\sqrt{w_1}/4 + \sqrt{w_2}/2 + \sqrt{w_3}/4 - 1 \geq 1/3$~~
~~$w_1 \leq w_2$~~
~~$w_2 \leq w_3$~~

~~Solving by Lagrangian optimisation~~,

IR binds. Suppose for reductio that IR does not bind, i.e. $\sqrt{w_1}/4 + \sqrt{w_2}/4 + \sqrt{w_3}/4 - 1 > 10$. For sufficiently small ε, $w_1, w_2' = w_2 - \varepsilon, w_3$ satisfy IR, IC, ~~and~~ and $w_1 \leq w_2 \leq w_3$ given that $w_1 < w_2$, and $E(w|e=1) > E(w'|e=1)$. By reductio, IR binds.

At the optimum, $w_2 = w_3$. Otherwise some $w_2^{*\prime} > w_2$, $w_3' < w_3$ ~~satisf~~ satisfies IR, IC, ~~and~~ $w_1 \leq w_2 \leq w_3$, and ~~dec~~ $E(w|e=1) > E(w'|e=1)$.

~~$w_1 < w_2 = w_3 \Rightarrow \sqrt{w_1}/4 + \sqrt{w_2}/2 + \sqrt{w_3}/4 = \sqrt{w_1}/4 + 3\sqrt{w_2}/4 > \sqrt{w_1}/3 + 2\sqrt{w_2}/3$~~

P's optimisation problem ~~reduces to~~ is
$\min_{w_1, w_2, w_3} w_1/4 + w_2/2 + w_3/4$ subject to
$\sqrt{w_1}/4 + \sqrt{w_2}/2 + \sqrt{w_3}/4 - 1 \geq 10$
$\sqrt{w_1}/4 + \sqrt{w_2}/2 + \sqrt{w_3}/4 - 1 \geq \frac{1}{3}(\sqrt{w_1} + \sqrt{w_2} + \sqrt{w_3})$
$w_1 \leq w_2$, $w_2 \leq w_3$

Solving by Lagrangian optimisation
$L = w_1/4 + w_2/2 + w_3/4 - \lambda_1(\sqrt{w_1}/4 + \sqrt{w_2}/2 + \sqrt{w_3}/4 - 11)$
$\quad - \lambda_2(-\sqrt{w_1}/12 + \sqrt{w_2}/6 - \sqrt{w_3}/12 - 1) + \mu_1(w_1 - w_2) + \mu_2(w_2 - w_3)$
FOCw_1: $1/4 - \lambda_1/8\, w_1^{-1/2} + \lambda_2/24\, w_1^{-1/2} + \mu_1 = 0$
FOCw_2: $1/2 - \lambda_1/4\, w_2^{-1/2} - \lambda_2/12\, w_2^{-1/2} - \mu_1 + \mu_2 = 0$
FOCw_3: $1/4 - \lambda_1/8\, w_3^{-1/2} + \lambda_2/24\, w_3^{-1/2} - \mu_2 = 0$

CS_{λ_1}: $\lambda \geq 0$ $\sqrt{w_1}/4 + \sqrt{w_2}/2 + \sqrt{w_3}/4 - 1 \geq 10$

$\lambda_1(\sqrt{w_1}/4 + \sqrt{w_2}/2 + \sqrt{w_3}/4 - 1 - 10) = 0$

CS_{λ_2}: $\lambda_2 \geq 0$ $-\sqrt{w_1}/12 + \sqrt{w_2}/6 - \sqrt{w_3}/12 - \cancel{b} \geq 0$

$\lambda_2(-\sqrt{w_1}/12 + \sqrt{w_2}/6 - \sqrt{w_3}/12) = 0$

CS_{μ_1}: $\mu_1 \geq 0$, $w_1 \leq w_2$, $\mu_1(w_1 - w_2) = 0$

CS_{μ_2}: $\mu_2 \geq 0$, $w_2 \leq w_3$, $\mu_2(w_2 - w_3) = 0$

$w_1 < w_2 \Rightarrow \mu_1 = 0$

$w_2 = w_3 \Rightarrow \mu_2 \geq 0$

IR binds $\Rightarrow \lambda_1 > 0$, $\sqrt{w_1}/4 + \sqrt{w_2}/2 + \sqrt{w_3}/4 = 11$

Suppose that IC binds, then $\lambda_2 > 0$,

$\sqrt{w_1}/4 + \sqrt{w_2}/2 + \sqrt{w_3}/4 - 1 = 1/3(\sqrt{w_1} + \sqrt{w_2} + \sqrt{w_3}) = 10$

$\sqrt{w_1}/4 + 3\sqrt{w_2}/4 = 11 \Rightarrow \sqrt{w_1} + 3\sqrt{w_2} = 44$

$\sqrt{w_1}/3 + 2\sqrt{w_2}/3 = 10 \Rightarrow \sqrt{w_1} + 2\sqrt{w_2} = 30$

$\Rightarrow \sqrt{w_2} = 14$, $\sqrt{w_1} = 2$, $\sqrt{w_3} = 14$

FOC_{w_1}: $1/4 - \lambda_1/16 + \lambda_2/48 = 0$

FOC_{w_2}: $1/2 - \lambda_1/56 - \lambda_2/168 + \mu_2 = 0$

~~FOC_{w_3}: $1/4 - \lambda_1/16 + \lambda_2/48 - \mu_2 = 0$~~

FOC_{w_3}: $1/4 - \lambda_1/112 + \lambda_2/336 - \mu_2 = 0$

$\lambda_1 = 22$, $\lambda_2 = 54$, $\mu_2 = 3/14$

~~Given that th~~

Suppose that IC does not bind, then $\lambda_2 = 0$

FOC_{w_1}: $1/4 - \lambda_1/8 \, w_1^{-1/2} = 0$

FOC_{w_2}: $1/2 - \lambda_1/4 \, w_2^{-1/2} + \mu_2 = 0$

FOC_{w_3}: $1/4 - \lambda_1/8 \, w_3^{-1/2} - \mu_2 = 0 \Rightarrow 1/4 - \lambda_1/8 \, w_2^{-1/2} - \mu_2 = 0$

$\Rightarrow 1/2 - \lambda_1/4 \, w_2^{-1/2} - 2\mu_2 = 0$

$\Rightarrow \mu_2 = 0$

~~By reductio~~ There is no optimum such that IC does not bind

The optimal contract is $(w_1^* = 4, w_2^* = 196, w_3^* = 196)$

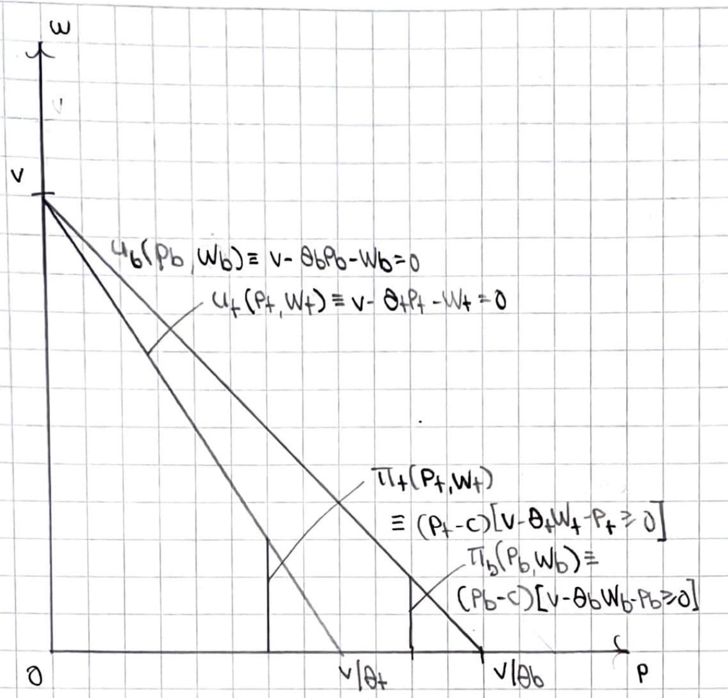
Is this check necessary?

c The wage required to induce $e=1$ when effort is observable, w^* is such that $\sqrt{w^*} - 1 = \bar{u}$, $w^* = 122$

In (a), $E(w) = 1/4(64) + 1/2(196) + 1/4(64) = 130$, $AC = 8$

In (b), $E(w) = 1/4(4) + 1/2(196) + 1/4(196) = 148$, $AC = 28$

Agency cost is greater if the agent can secretly destroy revenue. This is because, where secret revenue destruction is possible, the principal must increase the wage in the event of high revenue to disincentivise revenue destruction. Then, the principal has incentive to decrease wage in the low-revenue outcome until IR and IC again bind. The new wage schedule is more risky, ~~so the cost of~~ and ~~compensating~~ the principal must compensate the agent for greater risk bearing, so agency cost increases.



The firm's optimisation problem is

$\max_{p_b, w_b, p_t, w_t} (1-\lambda)(p_b - c) + \lambda(p_t - c)$ subject to

$PC_b: v - \theta_b p_b - w_b \geq 0$

$PC_t: v - \theta_t p_t - w_t \geq 0$

$IC_b: v - \theta_b p_b - w_b \geq v - \theta_b p_t - w_t$

$IC_t: v - \theta_t p_t - w_t \geq v - \theta_t p_b - w_b$

$p_b^+: p_b \geq 0$, $p_t^+: p_t \geq 0$, $w_b^+: w_b \geq 0$, $w_t^+: w_t \geq 0$

supposing without loss of generality that consumers have reservation utility 0

b ~~Foc~~

$v - \theta_b p_b - w_b \geq_1 v - \theta_b p_t - w_t \geq_2 v - \theta_t p_t - w_t \geq_3 0$,

$\geq_1$ follows from IC_b, $\geq_2$ from $\theta_b < \theta_t$ (supposing $p_t \geq 0$)

$\geq_3$ from PC_t

$\Rightarrow v - \theta_b p_b - w_b > v - \theta_t p_t - w_t$, i.e. PC_b does not bind.

Suppose for reductio that at the optimum $\hat{p}_b, \hat{w}_b, \hat{p}_t, \hat{w}_t$, PC_t does not bind, i.e. $v - \theta_t \hat{p}_t - \hat{w}_t > 0$. Then, for sufficiently small ε (such that PC_b and PC_t continues to hold), $p_b' = \hat{p}_b + \varepsilon$, $p_t' = \hat{p}_t + \varepsilon$ is such that all constraints hold and $\pi' > \hat{\pi}$. So $\hat{p}_b, \hat{w}_b, \hat{p}_t, \hat{w}_t$ is not an optimum. By reductio, PC_t binds at the optimum, and tourists are indifferent between travelling and not travelling.

c Suppose for reductio that at the optimum, IC_b does not bind, i.e. $v - \theta_b \hat{p}_b - \hat{w}_b > v - \theta_b \hat{p}_t - \hat{w}_t$. Then, for sufficiently small ε (such that PC_b and IC_b continue to hold), $p_b' = \hat{p}_b + \varepsilon$ is such that all constraints hold and $\pi' > \hat{\pi}$ ~~So the candidate~~ By reductio, IC_b binds at the optimum, and business men are indifferent between buying at w_b and at w_t.

~~Suppose for reductio that $\hat{w}_b \neq 0$.~~ IC_b binds, i.e. $v - \theta_b \hat{p}_b - \hat{w}_b = v - \theta_b \hat{p}_t - \hat{w}_t \Leftrightarrow \theta_b \hat{p}_b + \hat{w}_b = \theta_b \hat{p}_t + \hat{w}_t \Leftrightarrow$ $\theta_b(\hat{p}_b - \hat{p}_t) = \hat{w}_t - \hat{w}_b \Rightarrow \theta_t(\hat{p}_b - \hat{p}_t) > \hat{w}_t - \hat{w}_b \Rightarrow$ $v - \theta_t \hat{p}_b - \hat{w}_b < v - \theta_t \hat{p}_t - \hat{w}_t$, i.e. IC_t does not bind.

Suppose for reductio that $\hat{w}_b \neq 0$. Then for sufficiently small ε (such that IC_t continues to hold), $w_b' = \hat{w}_b - \varepsilon$, $p_b' = \hat{p}_b + \varepsilon/\theta_b$ is such that ~~$v-\theta_b \hat{p}_b$~~ $v - \theta_b \hat{p}_b - \hat{w}_b = v - \theta_b p_b' - w_b'$, so it is trivial that PC_b, PC_t, and IC_b continue to hold (and IC_t holds by construction of ε), and $\pi' > \hat{\pi}$. By reductio, $\hat{w}_b = 0$.

Businessmen buy at $\hat{w}_b = 0$ and are indifferent ~~to~~ between buying at this time and buying when tourists do at $\hat{w}_t$.

d

d Given that PC_t binds, $v - \theta_t \hat{P}_t - \hat{w}_t = 0$, $(\hat{P}_t, \hat{w}_t)$ lies on the indifference curve $u_t(P_t, w_t) = 0$. Given that IC_b binds, $v - \theta_b \hat{P}_b - \hat{w}_b = v - \theta_b \hat{P}_t - \hat{w}_t$, $(\hat{P}_b, \hat{w}_b)$ lies on the indifference $u_b(P_b, w_b) = u_b(\hat{P}_t, \hat{w}_t)$, i.e. the indifference curve that crosses $(\hat{P}_t, \hat{w}_t)$. Given that $\hat{w}_b = 0$, $(\hat{P}_b, \hat{w}_b)$ lies on the intersection of that indifference curve with the P axis.

How could we know this is the right method rather than Lagrangian?

$\hat{w}_t = v - \theta_t \hat{P}_t$

$v - \theta_b \hat{P}_b - \hat{w}_b = v - \theta_b \hat{P}_t - \hat{w}_t \Rightarrow \theta_b \hat{P}_b + \hat{w}_b = \theta_b \hat{P}_t + \hat{w}_t$

$\Rightarrow \theta_b \hat{P}_b = \theta_b \hat{P}_t + v - \theta_t \hat{P}_t \Rightarrow \hat{P}_b = \hat{P}_t + v/\theta_b - \theta_t/\theta_b \hat{P}_t$

$\hat{\pi} = (1-\lambda)\hat{\pi}_b + \lambda \hat{\pi}_t$

$= (1-\lambda)(\hat{P}_t + v/\theta_b - \theta_t/\theta_b \hat{P}_t - c) + \lambda(\hat{P}_t - c)$

~~At the optimum, the following FOC holds~~

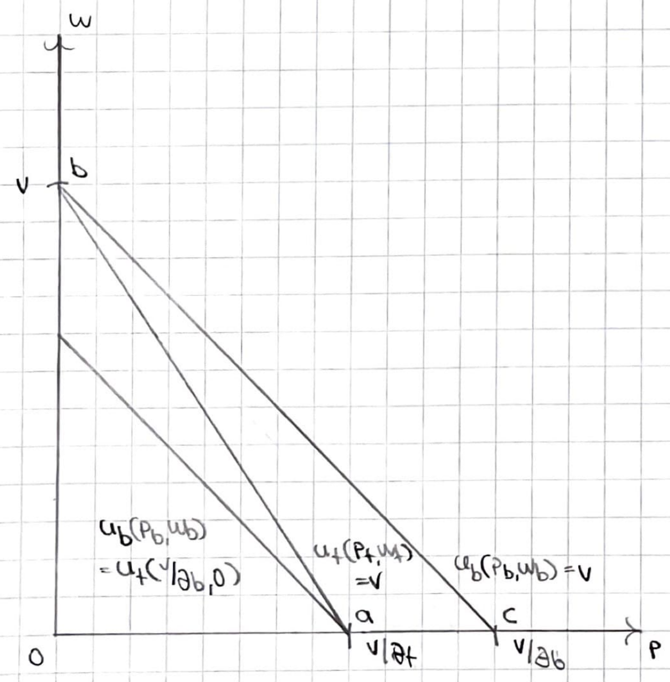
$d\hat{\pi}/d\hat{P}_t = (1-\lambda)(1 - \theta_t/\theta_b) + \lambda = 0 \Rightarrow$

$1 - (1-\lambda)(\theta_t/\theta_b) = 0 \Rightarrow$

$1 - \lambda = \theta_b/\theta_t \Rightarrow \lambda = 1 - \theta_b/\theta_t$

Suppose $\lambda > 1 - \theta_b/\theta_t$, then $d\hat{\pi}/d\hat{P}_t > 0$, the firm maximises profit by choosing the maximum feasible (i.e. subject to the above constraints) $\hat{P}_t$, namely v/θ_t. Then $\hat{w}_t = 0$, $\hat{P}_b = \hat{P}_t = v/\theta_t$, $\hat{w}_b = 0$. This is the pooling equilibrium. $\hat{\pi} = (v/\theta_t - c)$

Suppose $\lambda < 1 - \theta_b/\theta_t$, then $d\hat{\pi}/d\hat{P}_t > 0$, the firm maximises profit by choosing the minimum feasible $\hat{P}_t$, namely 0. Then $\hat{w}_t = v$, $\hat{P}_b = v/\theta_b$, $\hat{w}_b = 0$. This is the separating equilibrium. Suppose that indifferent tourists do not buy. $\hat{\pi} = (1-\lambda)(v/\theta_b - c)$



The pooling eqm is a, the separating eqm is points b and c.

e If $c > v/\theta_b$, then $c > v/\theta_t$, $\hat{\pi} < 0$ in either eqm, the firm should choose high P_b, P_t, w_b, w_t such that no consumers buy.

If $v/\theta_t < c < v/\theta_b$, then only the separating eqm is profitable, the firm should choose $p_b = v/\theta_b$, $w_b = 0$ and high p_t, w_t such that no tourists buy.

If $c < v/\theta_t < v/\theta_b$ the airline does not serve tourists iff

$(1-\lambda)(v/\theta_b - c) > (v/\theta_t - c) \Leftrightarrow$

$(1-\lambda)v/\theta_b - v/\theta_t > -c + (1-\lambda)c \Leftrightarrow$

$(1-\lambda)v/\theta_b - v/\theta_t > -\lambda c \Leftrightarrow$

$c > \lambda - 1/\lambda \; v/\theta_b + v/\lambda\theta_t \Leftrightarrow$

$c > \lambda-1/\lambda \; v/\theta_b + 1/\lambda \; v/\theta_t$

Marginal cost exceeds some weighted average of the maximum price each type is willing and able to pay