Information Transmission Notes

Milgrom-Roberts Reputation Model

- Parameters (Selten Chain Store Model)
  - In each period $t \in \{1, \ldots, T\}$ entrant E^t chooses either In (to enter local market t) or Out (not to enter local market t). If E^t chose In, then common incumbent I chooses either A (to accommodate entry) or F (to prey on E^t). If E^t chooses Out, then E^t's payoff is 0 and I's payoff in period (market) t is 2. If E^t and I choose (In, A), then their payoffs are $1, 0$. This models the case where entrants are more efficient than incumbents (so fare better upon entry) and where competition decreases total profit. If E^t and I choose (In, F), then their payoffs are $-1, -1$. I's total payoff is the undiscounted sum of payoffs in each period. Periods play out sequentially and publicly.
- Result
  - The Selten chain store model can be solved by backward induction. The unique subgame perfect equilibrium is such that each E^t chooses In (regardless of past play) and I chooses A in each period.
- Parameters (Simplified Milgrom-Roberts Reputation Model)
  - Consider the Selten chain store model. Suppose that each E^t entertains some non-zero prior probability ϵ^1 that I is "strong" and can only play F. "Weak" I is rational and has actions and payoffs as in the Selten chain store model. Suppose for simplicity that $T = 2$.
- Analysis
  - Let ϵ^2 denote the probability consistent with Bayesian beliefs at period 2 (where E^2 is the active player) that I is strong. E^2 chooses In if $\epsilon^2 < \frac{1}{2}$, Out if $\epsilon^2 > \frac{1}{2}$, and is indifferent iff $\epsilon^2 = \frac{1}{2}$.
    - It is trivial that weak I always plays A, and by construction strong I always plays F. Then, E^2's expected payoff from In is $\epsilon^2(-1) + (1 - \epsilon^2)(1)$. E^2's expected payoff from Out is 0. $\epsilon^2(-1) + (1 - \epsilon^2)(1) \gtreqless 0 \Leftrightarrow \epsilon^2 \lesseqgtr \frac{1}{2}$. By rationality, E^2 chooses In if $\epsilon^2 < \frac{1}{2}$, Out if $\epsilon^2 > \frac{1}{2}$, and is indifferent iff $\epsilon^2 = \frac{1}{2}$.
  - Suppose that $\epsilon^1 \geq \frac{1}{2}$, then the perfect Bayesian equilibrium is such that E^1 plays Out and (weak) I plays F if E^1 plays In.
    - Given that both weak and strong I play F with certainty if E^1 plays In, by Bayesian beliefs, if weak I played F, then $\epsilon^2 = \epsilon^1 \geq \frac{1}{2}$ and if weak I played A then $\epsilon^2 = 0$. Then, from the above, E^2's playing Out upon observing F in $t = 1$ and In upon observing A in $t = 1$ is sequentially rational. Given that both weak and strong I play F with certainty if E^1 plays In, E^1's playing Out is sequentially rational. Given that E^2 plays Out upon observing F in $t = 1$ and In upon observing A in $t = 2$, only F is sequentially rational for I in $t = 1$. The corresponding assessment (strategy profile-beliefs pair) is a perfect Bayesian equilibrium.
  - Suppose that $\epsilon^1 < \frac{1}{2}$, then the perfect Bayesian equilibrium is such that (weak) I mixes between F and A in $t = 1$.
    - Suppose that $\epsilon^1 < \frac{1}{2}$. Suppose further for reductio that at equilibrium, I plays F with certainty in $t = 1$, then by Bayesian beliefs, on the equilibrium path, $\epsilon^2 = \epsilon^1 < \frac{1}{2}$. By sequential rationality, E^2 chooses In if $\epsilon^2 < \frac{1}{2}$, Out if $\epsilon^2 > \frac{1}{2}$, and is indifferent iff $\epsilon^2 = \frac{1}{2}$, and in $t = 2$, I always plays A. Then, on the equilibrium path, I's payoff is -1. Deviation to A in each period yields payoff 0, then F with certainty in $t = 1$ is not sequentially rational. By reductio, supposing that $\epsilon^1 < \frac{1}{2}$, at equilibrium, I does not play F with certainty in $t = 1$.
    - Suppose that $\epsilon^1 < \frac{1}{2}$. Suppose further for reductio that at equilibrium, I plays A with certainty in $t = 1$, then by Bayesian beliefs, on the equilibrium path, $\epsilon^2 = \epsilon^1 < \frac{1}{2}$.By sequential rationality, E^2 chooses In if $\epsilon^2 < \frac{1}{2}$, Out if $\epsilon^2 > \frac{1}{2}$, and is indifferent iff $\epsilon^2 = \frac{1}{2}$, and in $t = 2$, I always plays A. Then, on the equilibrium path, I's payoff is 0. By Bayesian beliefs, off the equilibrium path, upon observing F in $t = 1$, $\epsilon^2 = 1$. Then, deviation to F in $t = 1$ (and A in $t = 2$) yields payoff 1, so A with certainty in $t = 1$ is not sequentially rational. By reductio, supposing that $\epsilon^1 < \frac{1}{2}$, at equilibrium, I does not play A with certainty in $t = 1$.
    - Intuitively, weak I only partially imitates strong I because perfect imitation renders F as a signal uninformative.
  - Continue to suppose that $\epsilon^1 < \frac{1}{2}$, then the perfect Bayesian equilibrium is such that if E^1 plays In, then if I plays A in $t = 1$, E^2 plays In with certainty in $t = 2$, and if I plays F in $t = 1$, E^2 plays In with probability $\frac{1}{2}$ and Out with probability $\frac{1}{2}$.
    - Suppose that I plays A in $t = 1$, then by Bayesian beliefs $\epsilon^2 = 0$, and by sequential rationality E^2 plays In with certainty. I's playing A reveals I as weak.
    - Suppose that I plays F in $t = 1$. Suppose further for reductio that this induces E^2 to play In with certainty in $t = 2$. Then I's strategy fails to the deviation under which I plays A with certainty in $t = 1$, and I's strategy is not sequentially rational. By reductio, at equilibrium, F in $t = 1$ does not induce In with certainty in $t = 2$.

        - Suppose that I plays F in $t = 1$. Suppose further for reductio that this induces E^2 to play Out with certainty in $t = 2$. Then I's strategy fails to the deviation under which I plays F with certainty in $t = 1$, and I's strategy is not sequentially rational. By reductio, at equilibrium, F in $t = 1$ does not induce Out with certainty in $t = 2$.
        - Intuitively, if imitation is never rewarded, weak types have no incentive to imitate, hence do not mix, and if imitation is always rewarded, weak types have strict incentive to imitate, hence do not mix. Weak types always mix at equilibrium, so at equilibrium, imitation is rewarded but not with certainty.
        - Given that at equilibrium I mixes in $t = 1$, I's payoff from A, (0) is equal to I's payoff from F $(-1 + 2m)$ where m is the probability that E^2 plays Out. Then $m = \frac{1}{2}$.
    - Continue to suppose that $\epsilon^1 < \frac{1}{2}$, then the perfect Bayesian equilibrium is such that I mixes in $t = 1$ such that $\epsilon^2 = \frac{1}{2}$ and E^2 is indifferent between In and Out.
        - By Bayes' rule, I plays F with probability $q = \frac{\epsilon^1}{1-\epsilon^1}$.
    - Continue to suppose that $\epsilon^1 < \frac{1}{2}$, then the perfect Bayesian equilibrium is such that E^1 plays In iff $\epsilon^1 \leq \frac{1}{4}$ such that the total probability that I plays F in $t = 1$ is $\leq \frac{1}{2}$.
    - Analysis of (the interesting $\epsilon^1 < \frac{1}{2}$ case of) the reputation game proceeds in five steps. First, establish that weak types imperfectly imitate strong types in early periods. Second, establish that signals of strength are not rewarded with certainty. Third, compute the rewarding player's optimal mixing probabilities given that the "reputation player" is indifferent between imitating and not. Fourth, compute the imitating player's optimal mixing probabilities given that the rewarding player is indifferent between rewarding and not. Fifth, compute all other players' optimal strategies.
- Parameters (Milgrom-Roberts Reputation Model)
    - Consider the general Milgrom-Roberts reputation model with T periods.
- Result
    - E^1 plays Out iff $\epsilon^1 > (\frac{1}{2})^T$. As $T \to \infty$, this threshold converges to 0.
    - If E^1 plays In, I plays F with certainty iff $\epsilon^1 > (\frac{1}{2})^{T-1}$.
    - In every subsequent period, if I has not previously played A, the entrant plays Out with positive probability, and if I has previously played A, the entrant plays In with certainty.
- Theorem (Fudenberg and Levine)
    - If the long-lived player ($P1$) has each possible commitment type with positive prior probability, then in the unique perfect Bayesian equilibrium of the T period repeated game against short-lived opponents, the average payoff of $P1$ converges to $max_{a_1} u(a_1, BR_2(a_1))$, where $BR_2(a_1)$ is the best response function of the short-lived player.

Crawford-Sobel Cheap Talk Game

- Parameters
    - There are two players $P1, P2$ and an infinite number of states $\theta \in [0, 1]$. $P1$ receives a perfect signal of the state, $P2$ receives no signal. In the first stage, $P1$ publicly chooses message $m \in M$ (where M is rich). In the second stage, $P2$ chooses policy y. $P2$'s receives greater payoff for choosing y closer to θ, and $P1$ receives greater payoff for y closer to $\theta + b$ where $b > 0$ is some bias. Formally, $u_1(y, \theta) = -(\theta + b - y)^2$ and $u_2(y, \theta) = -(\theta - y)^2$.
- Definition (Induced Action)
    - An action y is induced in state θ iff some message m is sent with positive probability such that $s_2(m) = y$, where $s_2(m)$ is $P2$'s strategy. An action is induced if it is induced in any state θ.
- Result (Finite Number of Induced Actions)
    - Suppose that (at perfect Bayesian equilibrium) there are (at least) two induced actions y', y'', where $y' < y''$ without loss of generality. Let $\bar{\theta} = \frac{y'+y''}{2}$ and $\hat{\theta} = \bar{\theta} - b$. Then $\theta \gtreqqless \hat{\theta} \Leftrightarrow -(\theta + b - y'')^2 = -(\bar{\theta} - y'')^2 \gtreqqless -(\theta + b - y')^2 = -(\bar{\theta} - y')^2$, i.e. iff $\theta > \hat{\theta}$, then y'' yields greater payoff to $P1$ than y' (and analogously if $\theta < \hat{\theta}$ or $\theta = \hat{\theta}$).
    - By sequential rationality, $\sigma_1(\theta)$ is such that for all $\theta > \hat{\theta}$, for all $m \in supp(\sigma_1(\theta))$ (i.e. for all m played with non-zero probability given θ), $s_2(m) \neq y'$. In other words, by sequential rationality, given $\theta > \hat{\theta}$, (at perfect Bayesian equilibrium) $P1$ never induces y'. Then if $P1$ induces y' (plays some m such that $s_2(m) = y'$), $P2$ infers that $\theta \leq \hat{\theta}$, then y' is sequentially rational for $P2$ only if $y' \leq \hat{\theta}$. Then $y' \leq \hat{\theta} < \bar{\theta} = \frac{y'+y''}{2} < y''$. By generalisation, any two induced actions, $y' < y''$ are such that $y' + b < y''$. Then, in any such game, there are fewer than $\frac{1}{b}$ induced actions.