Predicate Logic Rough Notes

Exam Technique

- For "impossible" questions, reasonable credit will be awarded for some attempt. For example, by testing notable cases like the empty set or the universal relation.
- For semantic proofs in second-order logic, "it is acceptable to use words once in the set-theoretic part of the proof, but the form of the proof is otherwise similar to the form of proofs in PL, MPL, PC, SC, LC".
- The ancestral relation R^*ab abbreviates $\forall X[\forall x(Rax \to Xx) \land \forall y_1 \forall y_2(Xy_1 \land Ry_1y_2 \to Xy_2) \to Xb]$.

Classical Predicate Logic

Syntax

- Definition (PC-Term)
  - $\neg, \to, \forall$ are the PC-connectives, $x, y, x_1, y_1, x_2, \ldots$ are the PC-variables, $F, G, F_1, G_1, F_2, \ldots$ are the n-place PC-predicates, $a, b, a_1, b_1, a_2, \ldots$ are the PC-constants. If α is a PC-variable or a PC-constant, then α is a PC-term.
- Definition (PC-wff)
  - If Π is a n-place PC-predicate, and each of $\alpha_1, \ldots, \alpha_n$ is a PC-term, then $\Pi\alpha_1 \ldots \alpha_n$ is a PC-wff.
  - If each of ϕ, ψ is a PC-wff, and α is a PC-variable, then each of $\neg\phi$, $(\phi \to \psi)$, and $\forall\alpha\phi$ is a PC-wff.
  - Only strings that can be shown to be PC-wffs by the above clauses are PC-wffs.
  - PC-term and PC-wff are defined simultaneously and recursively.
- Abbreviations
  - "$\exists\alpha\phi$" abbreviates "$\neg\forall\alpha\neg\phi$.
  - Abbreviations for "$\phi \land \psi$", "$\phi \lor \psi$", and "$\phi \leftrightarrow \psi$" are introduced in the familiar way.
  - The familiar bracketing conventions apply.
- Definition (Free Variable Occurrence)
  - An occurrence of PC-variable α in PC-wff ϕ is bound iff it occurs in a subformula of the form $\forall\alpha\psi$, an occurrence of α is free otherwise.

Semantics

- Definition (PC-Model)
  - A PC-model $\mathcal{M}$ is a pair $\langle \mathcal{D}, \mathcal{I} \rangle$ such that:
    - $\mathcal{D}$ is a non-empty set, the domain, and
    - $\mathcal{I}$ is a function on the set of constants and predicates, the interpretation function, such that:
      - for all constants α, $\mathcal{I}(\alpha) \in \mathcal{D}$, and
      - for all n-place predicates Π, $\mathcal{I}(\Pi)$ is some n-place relation over $\mathcal{D}$.
- Definition (PC-Variable Assignment)
  - A PC-variable assignment g for model $\mathcal{M} = \langle \mathcal{D}, \mathcal{I} \rangle$ is a function on the set of variables such that for all variables α, $g(\alpha) \in \mathcal{D}$.
- Definition (PC-Variant Assignment)
  - A PC-variant assignment g_u^α, where α is some variable and $u \in \mathcal{D}$, of variable assignment g for model $\mathcal{M} = \langle \mathcal{D}, \mathcal{I} \rangle$ is the variable assignment such that $g_u^\alpha(\beta) = \begin{cases} u & \text{if } \beta = \alpha \\ g(\beta) & \text{otherwise} \end{cases}$. In words, g_u^α is the variable assignment that differs from g only in assigning u to α.
- Definition (PC-Term Denotation)
  - The PC-term denotation $[\alpha]_{\mathcal{M},g}$ of term α for model $\mathcal{M}$ and variable assignment g for $\mathcal{M}$ is such that $[\alpha]_{\mathcal{M},g} = \begin{cases} \mathcal{I}(\alpha) & \text{if } \alpha \text{ is a constant} \\ g(\alpha) & \text{if } \alpha \text{ is a variable} \end{cases}$.
- Definition (PC-Valuation)
  - The PC-valuation $V_{\mathcal{M},g}$ for model $\mathcal{M} = \langle \mathcal{D}, \mathcal{I} \rangle$ and variable assignment g for $\mathcal{M}$ is the unique function from the set of PC-wffs to the set of truth values $\{0, 1\}$ such that:
    - $V_{\mathcal{M},g}(\Pi\alpha_1 \ldots \alpha_n) = 1$ iff $\langle [\alpha_1]_{\mathcal{M},g}, \ldots, [\alpha_n]_{\mathcal{M},g} \rangle \in \mathcal{I}(\Pi)$, for all n-place predicates Π and terms $\alpha_1, \ldots, \alpha_n$,

        - $V_{\mathcal{M},g}(\forall\alpha\phi) = 1$ iff for all $u \in \mathcal{D}$, $V_{\mathcal{M},g_u^\alpha}(\phi) = 1$, for all variables α and PC-wffs ϕ,
        - $V_{\mathcal{M},g}(\neg\phi) = 1$ iff $V_{\mathcal{M},g}(\phi) = 0$, and
        - $V_{\mathcal{M},g}(\phi \to \psi) = 1$ iff $V_{\mathcal{M},g}(\phi) = 0$ or $V_{\mathcal{M},g}(\psi) = 1$.
    - The definition of PC-valuation and the definition of $\exists\alpha\phi$ together imply $V_{\mathcal{M},g}(\exists\alpha\phi) = 1$ iff for some $u \in \mathcal{D}$, $V_{\mathcal{M},g_u^\alpha}(\phi) = 1$, for all variables α and PC-wffs ϕ.
- Definition (PC-Validity)
    - PC-wff ϕ is valid iff for all models $\mathcal{M}$ for all variable assignments g for $\mathcal{M}$, $V_{\mathcal{M},g}(\phi) = 1$.
- Definition (PC-Semantic Consequence)
    - PC-wff ϕ is a PC-semantic consequence of set of wffs $\Gamma = \{\gamma_1, \gamma_2, \ldots\}$ iff for all models $\mathcal{M}$ for all variable assignments g for $\mathcal{M}$, if for all $\gamma \in \Gamma$, $V_{\mathcal{M},g}(\gamma) = 1$, then $V_{\mathcal{M},g}(\phi) = 1$.

Predicate Logic with Identity

Syntax

- The syntax of $PC_=$ is exactly analogous to that of PC except in:
    - including the connective $=$,
    - including the additional clause in the definition of a wff:
        - if each of α, β is a term, then $\alpha = \beta$ is a wff.

Semantics

- Definition ($PC_=$-Valuation)
    - The definition of $PC_=$-valuation is exactly analogous to that of PC-valuation except in including the additional clause:
        - $V_{\mathcal{M},g}(\alpha = \beta) = 1$ iff $[\alpha]_{\mathcal{M},g} = [\beta]_{\mathcal{M},g}$, for all terms α, β.

Predicate Logic with Complex Terms

Syntax

- The syntax of PC_ι is exactly analogous to that of PC except in:
    - including the connective ι,
    - including the additional clause in the definition of a term:
        - if α is a variable and ϕ is a wff, then $\iota\alpha\phi$ is a term.
- Note that PC_ι-term and PC_ι-wff are defined simultaneously and recursively, in the (non-trivial) sense that the definition of PC_ι-term refers to PC_ι-wffs and the definition of PC_ι-wff refers to PC_ι-terms.

Semantics

- Definition (PC_ι-Term Denotation)
    - The definition of PC_ι-term denotation is exactly analogous to that of PC-term denotation except in including the additional case:
        - $[\iota\alpha\phi]_{\mathcal{M},g} = \begin{cases} \text{the unique } u \in \mathcal{D} \text{ such that } V_{\mathcal{M},g_u^\alpha}(\phi) = 1 & \text{if such } u \text{ exists} \\ \text{undefined} & \text{otherwise} \end{cases}$.
- Definition (PC_ι-Valuation)
    - The definition of PC_ι-valuation is exactly analogous to that of PC-valuation except in modifying the clause for elementary wffs as follows:
        - $V_{\mathcal{M},g}(\Pi\alpha_1 \ldots \alpha_n) = 1$ iff each of $[\alpha_1]_{\mathcal{M},g}, \ldots, [\alpha_n]_{\mathcal{M},g}$ is defined and $\langle [\alpha_1]_{\mathcal{M},g}, \ldots, [\alpha_n]_{\mathcal{M},g} \rangle \in \mathcal{I}(\Pi)$, for all n-place predicates Π and terms $\alpha_1, \ldots, \alpha_n$.
    - This modification is simply to account for instances of undefined complex terms.
- Note that PC_ι-term denotation and PC_ι are defined simultaneously and recursively in a non-trivial sense.

Predicate Logic with Complex Predicates

Syntax

- The syntax of PC_λ is exactly analogous to that of PC except in:

- including the additional clause in the definition of a n-place predicate:
  - if α is a variable and ϕ is a wff, then $\lambda\alpha\phi$ is a 1-place predicate.

Semantics

- Definition (Extension of a Complex Predicate)
  - The extension $\phi^{\mathcal{M},g,\alpha}$ of a complex predicate $\lambda\alpha\phi$ is given by $\phi^{\mathcal{M},g,\alpha} = \{u \in \mathcal{D} : V_{\mathcal{M},g_u^\alpha}(\phi) = 1\}$.
- Definition (PC_λ-Valuation)
  - The definition of PC_λ-valuation is exactly analogous to that of PC-valuation except in:
    - modifying the clause for elementary wffs as follows:
      - $V_{\mathcal{M},g}(\Pi\alpha_1 \dots \alpha_n) = 1$ iff $\langle [\alpha_1]_{\mathcal{M},g}, \dots, [\alpha_n]_{\mathcal{M},g} \rangle \in \mathcal{I}(\Pi)$, for all n-place simple predicates Π and terms $\alpha_1, \dots, \alpha_n$,
    - including the additional clause:
      - $V_{\mathcal{M},g}((\lambda\alpha\phi)(\beta)) = 1$ iff $[\beta]_{\mathcal{M},g} \in \phi^{\mathcal{M},g,\alpha}$ for all variables α, terms β, and wffs ϕ.

Second-Order Logic

Syntax

- The syntax of SOL is exactly analogous to that of PC except in:
  - including the following definition of predicate variables:
    - $X, Y, X_1, Y_1, X_2, \dots$ are the SOL-predicate variables,
  - including the additional clauses in the definition of a SOL-wff:
    - if π is a n-place predicate variable, and each of $\alpha_1, \dots, \alpha_n$ is a term, then $\pi\alpha_1 \dots \alpha_n$ is a wff,
    - if π is a n-place predicate variable and ϕ is a wff, then $\forall\pi\phi$ is a wff.

Semantics

- Definition (SOL-Model)
  - Th definition of a SOL model is identical to that of a PC model.
- Definition (SOL-Variable Assignment)
  - A SOL-variable assignment g for model $\mathcal{M} = \langle \mathcal{D}, \mathcal{I} \rangle$ is a function on the set of variables and predicate variables such that for each variable α, $g(\alpha) \in \mathcal{D}$ and for each n-place predicate variable π, $g(\pi)$ is a n-place relation over $\mathcal{D}$.
- Definition (SOL-Variant Assignment)
  - A SOL-variant assignment g_u^α, where α is some variable and $u \in \mathcal{D}$, of variable assignment g for model $\mathcal{M} = \langle \mathcal{D}, \mathcal{I} \rangle$ is the variable assignment such that $g_u^\alpha(\beta) = \begin{cases} u & \text{if } \beta = \alpha \\ g(\beta) & \text{otherwise} \end{cases}$. In words, g_u^α is the variable assignment that differs from g only in assigning u to α.
  - A SOL-variant assignment g_U^π, where π is some n-place predicate variable and U is some n-place relation over $\mathcal{D}$, of variable assignment g for model $\mathcal{M} = \langle \mathcal{D}, \mathcal{I} \rangle$ is the variable assignment such that $g_U^\pi(\rho) = \begin{cases} U & \text{if } \rho = \pi \\ g(\rho) & \text{otherwise} \end{cases}$. In words, g_U^π is the variable assignment that differs from g only in assigning U to π.
- Definition (SOL-Valuation)
  - The definition of SOL-valuation is exactly analogous to that of PC-valuation except in including the additional clauses:
    - $V_{\mathcal{M},g}(\pi\alpha_1 \dots \alpha_n) = 1$ iff $\langle [\alpha_1]_{\mathcal{M},g}, \dots, [\alpha_n]_{\mathcal{M},g} \rangle \in g(\pi)$,
    - $V_{\mathcal{M},g}(\forall\pi\phi) = 1$ iff for every n-place relation U over $\mathcal{D}$, $V_{\mathcal{M},g_U^\pi}(\phi) = 1$.