Game Theory Problem Set 1

1. Game 1

	a	b	c	d
A	1, 1	0, $\underline{2}$	$\underline{3}$, 1	4, 0
B	3, 2	1, 1	1, 3	0, $\underline{4}$
C	1, $\underline{6}$	2, 5	0, 4	$\underline{6}$, 5
D	$\underline{4}$, 3	$\underline{6}$, $\underline{4}$	2, 1	5, 2

Best responses underlined.

$R_1^0 = \{A,B,C,D\}$, $R_2^0 = \{a,b,c,d\}$
$D >_1 B$
$R_1^1 = \{A,C,D\}$, $R_2^1 = \{a,b,c,d\}$
$\cancel{b >_2 c}$, $a >_2 d$, $b >_2 c$
$R_1^2 = \{A,C,D\}$, $R_2^2 = \{a,b\}$
$D >_1 A$, $D >_1 C$
$R_1^3 = \{D\}$, $R_2^3 = \{a,b\}$
$b >_2 a$
$R_1^4 = \{D\}$, $R_2^4 = \{b\}$
$R_1^\infty = \{D\}$, $R_2^\infty = \{b\}$

Only strategies D and b survive IESDS.

Some player's ~~*~~ pure strategy is rationalisable iff it is a best response to some potentially correlated mix of the opponents' rationalisable strategies. Equivalently, some player's pure strategy is rationalisable if it survives iterated elimination of pure strategies that are not a best response to some potentially correlated mix of opponents' yet uneliminated pure strategies.

Pearce's Lemma: In a two-player finite game, a pure strategy is ~~dominated~~ strictly dominated by another pure or mixed strategy iff it is not a best response to any pure or mixed strategy of the opponent.

Then, all and only pure strategies that survive IESDS are rationalisable. So only D and b are rationalisable.

The strategy profile (D,b) is the unique pure strategy Nash equilibrium, where both players play mutual best responses.

Since only one ~~strategy~~ ~~pure strategy profit~~ action of each player survives IESDS, ~~there are no~~ the corresponding strategy profile is the unique Nash equilibrium. There are no mixed or hybrid equilibria

why is this?

Game 2

	L	C	R
T	$\underline{3}$ 0	0 $\underline{3}$	1 1
M	0 $\underline{3}$	$\underline{3}$ 0	1 1
B	$\underline{1}$ 1	$\underline{1}$ 1	0 $\underline{6}$

Best responses underlined.

$R_1^0 = \{T, M, B\}$, $R_2^0 = \{L, C, R\}$

$0.5 \times L + 0.5 \times C >_2 R$

$R_1^1 = \{T, M, B\}$, $R_2^1 = \{L, C\}$

$0.5 \times T + 0.5 \times M >_1 B$

$R_1^2 = \{T, M\}$, $R_2^2 = \{L, C\}$

$R_1^\infty = \{T, M\}$, $R_2^\infty = \{L, C\}$

Only strategies T, M, L, C survive IESDS

By definition of rationalisability and by Pearce's Lemma, all and only pure strategies that survive IESDS are rationalisable. So only T, M, L, C are rationalisable.

By inspection, there are no pure strategy Nash equilibria.

Considering only strategies that survive IESDS, the game is a matching pennies game. The unique Nash equilibrium is a mixed strategy Nash equilibrium $(0.5 \times T + 0.5 \times M,\ 0.5 \times L + 0.5 \times C)$.

If player 1 plays B with positive probability, he fails to maximise his expected payoff. Likewise for player 2 and R. If either player does not play his two remaining actions with equal probability, the other player ~~sea~~ maximises his payoff by playing some fixed strategy, then the first player~~'s st~~ fails to maximise his payoff.

2 ~~Let $R^k = [\underline{x}^k, \overline{x}^k]$ and $\underline{x}^k = 0$, $\overline{x}^k = 100$~~

Let $R^k = [\underline{x}^k, \overline{x}^k]$ denote the interval of yet uneliminated pure strategies after k iterations of elimination of weakly dominated strategies

$R^0 = [0, 100]$

Then, for $x_i > 100p$,

~~$\forall x_{-i}: \forall x_i' \in [0, 100p]: \pi_i(x_i, x_{-i}) = 0 \leq \pi_i(x_i', x_{-i})$~~

~~and $\exists x_i' \in [0, 100p]: \exists x_{-i}$~~

$\forall x_i' \in [0, 100p]: \forall x_{-i}: \pi_i(x_i, x_{-i}) = 0 \leq \pi_i(x_i', x_{-i})$

~~and $\forall x_i' \in [0, 100p]: \exists x_{-i}: \pi_i(x_i, x_{-i}) < \pi_i$~~

and $\forall x_i' \in [0, 100p]: \exists x_{-i}: \pi_i(x_i, x_{-i}) = 0 < \pi_i(x_i', x_{-i})$

In other words, any strategy $x_i > 100p$ is weakly dominated by each other strategy $x_i \in [0, 100p]$ since the former always yields zero payoff while the latter always yields non-negative payoff and sometimes yields positive payoff.

$R^1 = [0, 100p]$

Then, any strategy $x_i > 100p^2$ is weakly dominated by each other strategy $x_i' \in [0, 100p^2]$.

$R^2 = [0, 100p^2]$

$\lim_{k \to \infty} \overline{x}^k = 0$

$R^\infty = [0, 0]$

So the only strategy profile that survives IEWDS is such that all players choose 0.

~~Player i's best response~~

~~$B_i(x_{-i}) = P/N \sum x_j$~~ ~~$P/N \sum_{x_j \in x_{-i}} x_j$~~

~~Let x^* denote the Nash equilibrium~~

~~By definition of Nash equilibrium, $\forall i: x_i^* = B_i(x_{-i}^*)$~~

~~$x_i^* = P/N \sum_{x_j \in x_{-i}^*} x_j^*$~~

~~By symmetry, $\forall j: x_j^* = x_i^*$~~

~~All and only player~~

Player i maximises his payoff iff

$x_i = P/N \sum_j x_j = P/N \sum_{k \neq i} x_k + px_i/N$

$(1 - P/N) x_i = P/N \sum_{k \neq i} x_k$

$x_i = P/(N-p) \sum_{x_k \in x_{-i}} x_k$

Player i's best response

$B_i(x_{-i}) = P/(N-p) \sum_{x_k \in x_{-i}} x_k$

At Nash equilibrium x^*,

by definition of Nash equilibrium and definition of best response functions,

$\forall i: x_i^* = B_i(x_{-i}^*) = P/(N-p) \sum_{x_k^* \in x_{-i}^*} x_k^*$

By symmetry $x_i^* = x_j^*$ for all players i, j

~~$x_i^* = N-1/N-p\, x_i^*$~~

$x_i^* = (P/(N-p))(N-1) x_i^* = \cancel{Np-P/N-p}\, (Np-P/(N-p))\, x_i^*$

$N(1-p) x_i^* = 0$

$x_i^* = 0$ since $N \neq 0$ and $1-p \neq 0$

$\therefore x^*$ is such that $\forall i: x_i^* = 0$

3 Game 1

	L	C	R
T	$0, \underline{2}$	$\underline{2}, 0$	$0, 0$
M	$\underline{2}, 0$	$0, \underline{2}$	$0, 0$
B	$0, 0$	$0, 0$	$\underline{1}, \underline{1}$

Best responses underlined.

The unique pure strategy Nash equilibrium is (B,R), where both players play mutual best responses.

Since each player has a strict best response to each pure strategy of the opponent, there are no hybrid equilibria.

Is "unique" best response also necessary?

Suppose that there is some mixed strategy Nash equilibrium where player 1 mixes T, M, and B. Then expected payoff ~~from T, M, and B are~~ to player 1 from T, M, and B are equal. Let $P_T, P_M, P_B, P_L, P_C, P_R$ denote the probabilities of the respective actions. ~~$2P_C =$~~ $2P_C = 2P_L = P_R$, $P_L + P_C + P_R = 1$, $P_L = 1/4$, $P_C = 1/4$, $P_R = 1/4$. So player 2 mixes L, C, and R. Then expected payoff to player 2 from L, C, and R are equal. $2P_T = 2P_M = P_B$, $P_T + P_M + P_B = 1$, $P_T = 1/4$, $P_M = 1/4$, $P_B = 1/2$. At $(1/4 \times T + 1/4 \times M + 1/2 \times B, 1/4 \times L + 1/4 \times C + 1/2 \times R)$, no player has incentive to deviate since ~~each act~~ for each player, each action has equal expected payoff. This strategy profile ~~is a mixed~~ the unique mixed strategy Nash equilibrium given player 1 mixes T, M, and B and/or player 2 mixes L, C, and R.

Suppose that there is some mixed strategy Nash equilibrium where player 1 mixes T and B only. Then player 2 never plays C since C is strictly dominated by some mix of L and R. Then player 1 never plays T since T is ~~s~~ strictly dominated by some mix of M and B. By reductio there is no such mixed strategy Nash equilibrium. By symmetry, there are no mixed strategy Nash equilibrium where player 1 mixes T and B only or M and B only, or where player 2 mixes L and R only or C and R only.

Suppose that there is some mixed strategy Nash equilibrium where player 1 mixes T and M only and player 2 mixes L and C only. Then expected payoff to player 1 from T and M are equal. $2P_C = 2P_L$, $P_C + P_L = 1$, $P_C = P_L = 1/2$. And expected payoff to player 2 from L and C are equal. $2P_M = 2P_T$, $P_M + P_T = 1$, $P_M = P_T = 1/2$. At $(1/2 \times T + 1/2 \times M, 1/2 \times L + 1/2 \times C)$, each player has no incentive to deviate. This strategy profile is the unique mixed strategy Nash equilibrium given that player 1 mixes T and M only and/or player 2 mixes L and C only.

There are 3 Nash equilibria:

(B, R)

($\frac{1}{4} \times T + \frac{1}{4} \times M + \frac{1}{2} \times B$, $\frac{1}{4} \times L + \frac{1}{4} \times C + \frac{1}{2} \times R$)

($\frac{1}{2} \times T + \frac{1}{2} \times M$, $\frac{1}{2} \times L + \frac{1}{2} \times C$)

At the first equilibrium, each player has payoff 1.

At the second equilibrium, each player has expected payoff ½.

At the third equilibrium, each player has expected payoff 1.

Players have lower expected payoff at the second equilibrium due to the non-zero probability of miscoordination.

Game 2

	L	C	R
T	$\underline{1}$, $\underline{1}$	$\underline{1}$, $\underline{1}$	0, 0
B	0, 0	0, 0	$\underline{1}$, $\underline{1}$

Best responses underlined.

There are three pure strategy Nash equilibria, where both players play mutual best responses, (T, L), (T, C), (B, R).

Since both L and C are best responses to T, the strategy profile (T, pL + (1-p)C) is a hybrid Nash equilibrium for $\forall p \in (0,1)$.

Suppose that there is a mixed strategy Nash equilibrium. Then player 1 mixes T and B. So expected payoff to player 1 from T and B are equal. $P_L + P_C = P_R$, $P_L + P_C + P_R = 1$, $P_R = 1/2$, $P_L + P_C = 1/2$. ~~Then expected payoff to player 2~~ Player 2 mixes R with at least one of L and C. If player 2 mixes R and L, expected payoff to player 2 from R and L are equal. $P_T = P_B$, $P_T + P_B = 1$, $P_T = P_B = 1/2$. By symmetry, if player 2 mixes R and C, $P_T = P_B = 1/2$. Then, if player 2 mixes L, C, and R, $P_T = P_B = 1/2$. So the strategy profile $(1/2 \times T + 1/2 \times B,\ p \times L, (1/2 - p) \times C + 1/2 \times R)$ is a mixed strategy Nash equilibrium for $\forall p \in (0, 1/2)$

At each pure strategy Nash equilibrium, each player has payoff 1.

At each hybrid Nash equilibrium, each player has payoff 1.

At each mixed strategy Nash equilibrium, each player has expected payoff 1/2

Players have lower expected payoff when mixing due to the non-zero probability of miscoordination

Game 3

	L	R	L	R
T	~~0,0,0~~ 1,2,3	0,0,	T	
B			B	

	L	R
T	1,2,3	0,0,0
B	0,0,0	0,0,0

G1

	L	R
T	0,0,0	0,0,0
B	0,0,0	3,2,1

G2

Best responses underlined.

There are two pure strategy Nash equilibria where all players play mutual best responses, (T, L, G1) and (B, R, G2).

Suppose that there is a mixed strategy Nash equilibrium. Then player 1 mixes T and B. So expected payoff to player 1 from T and B are equal. $P_L P_{G1} = 3P_R P_{G2}$. Player 2 mixes L and R. So expected payoff to player 2 from L and R are equal. $2P_T P_{G1} = 2P_B P_{G2}$, $P_T P_{G1} = P_B P_{G2}$, $P_T + P_B = 1$, $P_{G1} + P_{G2} = 1$, $P_T P_{G1} = (1-P_T)(1-P_{G1}) = 1 + P_T P_{G1} - P_T - P_{G1}$, $P_T + P_{G1} = 1$. Similarly for player 3, $3P_T P_L = P_B P_R$.

Solving simultaneously, $\cancel{P_L(1-P_T) = 3P_R P_T - 3(1-P_L)P_T}$

$P_L(1-P_T) = 3P_R P_T = 3(1-P_L)P_T$,

$P_L - P_L P_T = 3P_T - 3P_L P_T$, $2P_L P_T = 3P_T - P_L$

$3P_T P_L = P_B P_R = (1-P_T)(1-P_L) = 1 + P_T P_L - P_T - P_L$,

$2P_T P_L = 1 - P_T - P_L$

$3P_T - P_L = 1 - P_T - P_L$

$3P_T = 1 - P_T$

$P_T = 1/4$, $P_B = 3/4$

$P_{G1} = 3/4$, $P_{G2} = 1/4$

$P_L = 1/2$, $P_R = 1/2$

The strategy profile $(1/4 \times T + 3/4 \times B,\ 1/2 \times L + 1/2 \times R,\ 3/4 \times G1 + 1/4 \times G2)$ is a mixed strategy Nash equilibrium since no player has incentive to deviate since for each player, each action has equal expected payoff.

How can we show that there are no hybrid equilibria?

At the pure strategy Nash equilibria, payoffs are (1,2,3) or (3,2,1)

At the mixed strategy Nash equilibrium, expected payoffs are (3/8, 3/8, 3/8).

Players have lower expected payoffs at the mixed strategy Nash equilibrium because of the high probability of miscoordination.

4 Player 1's pure strategy T is strictly dominant iff no other pure strategy of player 1, namely B, ever yields a higher payoff for player 1, given any action of other players, namely player 2.
$a > c$ and $b > d$

The matching pennies game does not have a pure strategy Nash equilibrium.

~~L R~~

~~T a= $\alpha = 0$ $\gamma = 0$~~

~~$a=1$ $b=1$~~

~~B $\beta = 1$~~

~~$c = 0$~~

	L	R
T	0	$\underline{1}$
	$\underline{1}$	0
B	$\underline{1}$	0
	0	$\underline{1}$

By definition of ~~the pure strategy~~ Nash equilibrium and best response, a pure strategy Nash equilibrium exists iff some pure strategy profile consists of mutual best responses.

$T \in B_1(L)$ iff $a \geq c$ ①
$B \in B_1(L)$ iff $c \geq a$ ②
$T \in B_1(R)$ iff $b \geq d$ ③
$B \in B_1(R)$ iff $d \geq b$ ④
$L \in B_2(T)$ iff $\alpha \geq \gamma$ ⑤
$R \in B_2(T)$ iff $\gamma \geq \alpha$ ⑥
$L \in B_2(B)$ iff $\beta \geq \delta$ ⑦
$R \in B_2(B)$ iff $\delta \geq \beta$ ⑧

Then no pure strategy Nash equilibrium exists iff

~~$(a \geq c$ and $\alpha < \gamma)$ or $(a < c$ and $\alpha \geq \gamma)$~~
~~and $[(b \geq d$ and $\beta < \delta)$ or $(b < d$ and $\beta \geq \delta)]$~~
~~and $[(c \geq a$ and $\gamma < \alpha)$ or $(c < a$ and $\gamma \geq \alpha)$~~

not (① and ⑤) and not (② and ⑦)
and not (③ and ⑥) and not (④ and ⑧)
iff
$(a < c$ or $\alpha < \gamma)$ and $(c < a$ or $\beta < \delta)$
and $(b < d$ or $\gamma < \alpha)$ and $(d < b$ or $\delta < \beta)$
iff
$(a < c$ and $\beta < \delta$ and $d < b$ and $\gamma < \alpha)$ or
$(\alpha < \gamma$ and $b < d$ and $\delta < \beta$ and $c < a)$

At a mixed strategy Nash equilibrium, each player's expected payoff from each action he mixes is equal, otherwise, a player could increase expected payoff by reallocating probability mass to the action with higher expected payoff.

$\pi_1(T) = \pi_1(B)$, $a p_L + b p_R = c p_L + d p_R$
$\pi_2(L) = \pi_2(R)$, $\alpha p_T + \beta p_B = \gamma p_T + \delta p_B$

$(a-c)p_L = (d-b)p_R$, $p_R = {}^{a-c}/_{d-b}\, p_L$
$(\alpha-\gamma)p_T = (\delta-\beta)p_B$, $p_B = {}^{\alpha-\gamma}/_{\delta-\beta}\, p_T$
$p_L + p_R = 1$, ${}^{d-b+a-c}/_{d-b}\, p_L = 1$
$p_L = {}^{d-b}/_{d-b+a-c}$, $p_R = {}^{a-c}/_{d-b+a-c}$
$p_T + p_B = 1$, ${}^{\delta-\beta+\alpha-\gamma}/_{\delta-\beta}\, p_T = 1$
$p_T = {}^{\delta-\beta}/_{\delta-\beta+\alpha-\gamma}$, $p_B = {}^{\alpha-\gamma}/_{\delta-\beta+\alpha-\gamma}$

Suppose for reductio that there is no Nash equilibrium ①
Then, there is no pure strategy Nash equi equilibrium ②
And there is no mixed strategy Nash. equilibrium ③
From ②, by the result in (2),
$\left.\begin{array}{l}(a<c \text{ and } d<b \text{ and } \gamma<\alpha \text{ and } \beta<\delta) \text{ ④ or} \\ (c<a \text{ and } b<d \text{ and } \alpha<\gamma \text{ and } \delta<\beta) \text{ ⑤}\end{array}\right\}$ ⑥
If ④, then by the result in (3),
$p_L, p_R, p_T, p_B \in (0,1)$ ⑦
~~From ⑦~~, then
there is some mixed strategy Nash equilibrium,
namely $(p_T \times T + p_B \times B,\ p_L \times L + p_R \times R)$ ⑧
so if ④ then ⑧ } ⑨
If ⑤, then by the result in (3),
$p_L, p_R, p_T, p_B \in (0,1)$ ⑩
then
there is some mixed strategy Nash equilibrium,
namely $(p_T \times T + p_B \times B,\ p_L \times L + p_R \times R)$ ⑪
so if ⑤ then ⑪ } ⑫
From ⑥, ⑨, ⑫
there is some mixed strategy Nash equilibrium ⑬
③ and ⑬ contradict, so by reductio, ① is false, there is some Nash equilibrium.