Microeconomic Analysis Problem Set 1

Admin: lecturers run the classes for their respective weeks

1 $x_i = \alpha_i m / p_i$, $\sum_{i=1}^{n} \alpha_i = 1$

$$[\vec{x}(\vec{p}) - \vec{x}(\vec{p}')] \cdot [\vec{p} - \vec{p}']$$

$$= \left(\begin{pmatrix} \alpha_1 m/p_1 \\ \vdots \\ \alpha_n m/p_n \end{pmatrix} - \begin{pmatrix} \alpha_1 m/p_1' \\ \vdots \\ \alpha_n m/p_n' \end{pmatrix} \right) \cdot \left(\begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix} - \begin{pmatrix} p_1' \\ \vdots \\ p_n' \end{pmatrix} \right)$$

$$= \begin{pmatrix} \alpha_1 m (1/p_1 - 1/p_1') \\ \vdots \\ \alpha_n m (1/p_n - 1/p_n') \end{pmatrix} \cdot \begin{pmatrix} p_1 - p_1' \\ \vdots \\ p_n - p_n' \end{pmatrix}$$

$$= \alpha_1 m (1/p_1 - 1/p_1')(p_1 - p_1') + \dots + \alpha_n m (1/p_n - 1/p_n')(p_n - p_n')$$

$$= \cancel{\sum_{i=1}^{n} \alpha_i m (1/p_n - 1/p_n')(p_n - p_n')}$$

$$= \cancel{m \sum_{i=1}^{n} \alpha_i (1/p_i p_i'}$$

$$= \cancel{\sum_{i=1}^{n} \alpha_i m (p_i}$$

$$= \sum_{i=1}^{n} \alpha_i m (1/p_i - 1/p_i')(p_i - p_i')$$

$$= m \sum_{i=1}^{n} \alpha_i (1/p_i p_i')(p_i' - p_i)(p_i - p_i')$$

$$= -m \sum_{i=1}^{n} \alpha_i (1/p_i p_i')(p_i - p_i')^2$$

$[\vec{x}(\vec{p}) - \vec{x}(\vec{p}')] \cdot [\vec{p} - \vec{p}'] < 0$ for all $\vec{p}, \vec{p}' \in \mathbb{R}^n_+$, $\vec{p} \neq \vec{p}'$ if $m > 0$ and $\forall i: \alpha_i > 0$. The Law of Demand holds for Cobb-Douglas demands if $m > 0$ and $\forall i: \alpha_i > 0$.

Given Cobb-Douglas preferences, $m > 0$ and $\forall i: \alpha_i > 0$, so ...

$x+3y-2z=2$ ①
$x+2y+z=1$ ②
$x+5y+\alpha z=\beta$ ③

From ①
$x=2-3y+2z$ ④
Sub ④ into ②
$2-3y+2z+2y+z=1$
$1-y+3z=0$
$y=1+3z$ ⑤
Sub ⑤ into ④
$x=2-3(1+3z)+2z=-1-7z$ ⑥
Sub ⑤ and ⑥ into ③
$(-1-7z)+5(1+3z)+\alpha z=\beta$
$4+8z+\alpha z=\beta$ ⑦
~~$(8+\alpha)z=\beta$~~
$(\alpha+8)z=\beta-4$
If $\alpha\neq-8$,
$z=(\beta-4)/(\alpha+8)$ ⑧
Sub ⑧ into ⑥ and ⑤
$x=-1-7(\beta-4)/(\alpha+8)$ ⑨
$y=1+3(\beta-4)/(\alpha+8)$ ⑩
If $\alpha=-8$,
then if $\beta=4$,
~~there are infinitely many solutions, gi~~
⑦ reduces to $0=0$
~~then~~ if instead $\beta\neq4$
⑦ has no solution.

So if $\alpha=-8$ and $\beta=4$, the linear system has infinitely many solutions given by $x=-1-7z$, $y=1+3z$, $z\in\mathbb{R}$. If $\alpha=-8$ and $\beta\neq4$, the linear system has no solutions. If $\alpha\neq-8$, the linear system has a unique solution $x=-1-7(\beta-4)/(\alpha+8)$, $y=1+3(\beta-4)/(\alpha+8)$, $z=(\beta-4)/(\alpha+8)$

The linear system can be written as $A\cdot\vec{x}=\vec{b}$ where
$$A=\begin{pmatrix}1&3&-2\\1&2&1\\1&5&\alpha\end{pmatrix}\quad \vec{x}=\begin{pmatrix}x\\y\\z\end{pmatrix}\text{ and }\vec{b}=\begin{pmatrix}2\\1\\\beta\end{pmatrix}$$

$\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}3\\2\\5\end{pmatrix},\begin{pmatrix}-2\\1\\\alpha\end{pmatrix}$ span $\vec{b}$ iff $A\cdot\vec{x}=\vec{b}$ has at least one solution iff $\alpha\neq-8$ or $\beta=4$.

Better to use Gauss-Jordan Elim

$$\left(\begin{array}{ccc|c}1&3&-2&2\\1&2&1&1\\1&5&\alpha&\beta\end{array}\right)\underset{\substack{R_2-R_1\\R_3-R_1}}{\sim}\left(\begin{array}{ccc|c}1&3&2&2\\0&-1&3&-1\\0&2&\alpha+2&\beta-2\end{array}\right)$$
$$\underset{R_3+2R_2}{\sim}\left(\begin{array}{ccc|c}1&3&2&2\\0&-1&3&-1\\0&0&\alpha+8&\beta-4\end{array}\right)$$

$(\alpha+8)z=\beta-4$, $\alpha\neq-8$: $z=(\beta-4)/(\alpha+8)$
$y=3z+1$
$x=$ ~~$1-7z$~~ $2+2z-3y=1-7z$

Important to go over $\alpha=-8$ case
then either ∞ many solutions or 0 solutions

By definition of span, the vectors in A span $\mathbb{R}^3$ iff $\forall\vec{b}\in\mathbb{R}^3:\exists\beta_1,\beta_2,\beta_3\in\mathbb{R}:\vec{b}=\beta_1(\cdot)+\beta_2(\cdot)+\beta_3(\cdot)$ iff $\exists\vec{x}:A\vec{x}=\vec{b}$, so the vectors span $\mathbb{R}^3$ if the above linear system has at least one solution

3 $W = \text{span}[\vec{u}, \vec{v}]$

Suppose for reductio that
$W \neq \text{span}[\vec{u}, \vec{v}]$ (1)
Then, ~~by definition of W,~~
$\exists \vec{w} \in W : w \notin \text{span}[\vec{u}, \vec{v}]$ (2)
or $\exists \vec{x} \in \text{span}[\vec{u}, \vec{v}] : x \notin W$ (3)
$\Big\}$ (4)

Suppose for reductio that
$\vec{\omega} \in W$ (5)
and $\vec{\omega} \notin \text{span}[\vec{u}, \vec{v}]$ (6)
$\Big\}$ (7)
By definition of span, since $\vec{\omega} = 1\vec{\omega} + 0\vec{u} + 0\vec{v}$,
$\vec{\omega} \in \text{span}[\vec{u}, \vec{v}, \vec{\omega}]$ (8)
From (5), by definition of W,
$\text{span}[\vec{u}, \vec{v}] = \text{span}[\vec{u}, \vec{v}, \vec{\omega}]$ (9)
From (8) and (9)
$\vec{\omega} \in \text{span}[\vec{u}, \vec{v}]$ (10)
~~(6) and (10) contradict so~~
By reductio, since (6) contradicts (10) so (6)
contradicts (5), reject (7)
~~Not (5)~~ Then, reject (2)

Suppose for reductio that
$\vec{K} \in \text{span}[\vec{u}, \vec{v}]$ (11)
and $\vec{K} \notin W$ (12)
$\Big\}$ (13)
By definition of span,
$\vec{K} \in \text{span}[\vec{u}, \vec{v}, \vec{K}]$ (14)
From (11) and (14), by definition of W
$\vec{K} \in W$ (15)
By reductio, since (15) contradicts (12) so (12)
contradicts (4), reject (13)
Then reject (3)

Then reject (4)
By reductio, reject (1)

is this sort of proof required?

Alternate argument
$\text{span}[\vec{u}, \vec{v}, \cancel{\vec{w}}] \subseteq \text{span}[\vec{u}, \vec{v}, \vec{w}]$
since, by def of span,
$\forall \vec{x} \in \text{span}[\vec{u}, \vec{v}]$ ...

if $u \notin \text{span}[\vec{u}, \vec{v}]$, $\text{span}[\vec{u}, \vec{v}] \neq \text{span}[\vec{u}, \vec{v}, \vec{u}]$

~~if $\vec{x} \in \text{span}[\vec{u}, \vec{v}]$, then by def^n span,~~
~~$\vec{x} \in$ span~~
~~if $\vec{\omega} \in$~~
$\forall \vec{\omega} \in \text{span}[\vec{u}, \vec{v}] : \exists \alpha, \beta \in \mathbb{R} : \vec{\omega} = \alpha\vec{u} + \beta\vec{v}$
$\forall \vec{z} \in \text{span}[\vec{u}, \vec{v}, \vec{\omega}]$:

can argue that
$\forall w \in W : w \in \text{span}[\vec{u}, \vec{v}]$
$\forall w \in \text{span}[\vec{u}, \vec{v}] : w \in W$
(this is an alternative way to prove the bidirectional)

$W = \text{span}[\vec{u}, \vec{v}] = \{\vec{w} : \vec{w} = \alpha_1\vec{u} + \alpha_2\vec{v}, \ \alpha_1, \alpha_2 \in \mathbb{R}\}$

~~W = $\vec{w}$~~ W coincides with $\text{span}[\vec{u}, \vec{v}]$ because $\text{span}[\vec{u}, \vec{v}] = \text{span}[\vec{u}, \vec{v}, \vec{w}]$ iff ~~any linear~~ all and only linear combinations of $\vec{u}$ and $\vec{v}$ are linear combinations of $\vec{u}, \vec{v}$ and $\vec{w}$, ~~iff $\forall \alpha_1, \alpha_2$: $\exists \alpha_3, \alpha_4, \alpha_5 : \alpha_1\vec{u} + \alpha_2\vec{v} = \alpha_3\vec{u} + \alpha_4\vec{v} + \alpha_5\vec{w}$ iff~~ iff $\vec{w}$ is a linear combination of $\vec{u}$ and $\vec{v}$, ~~i.e. $\vec{w} = \alpha$ $\exists \alpha_6, \alpha_7 : \vec{w} = \alpha_6\vec{u} + \alpha_7\vec{v}$ then $\alpha_1\vec{u} + \alpha_2\vec{v} = (\alpha_3 + \alpha_5\alpha_6)\vec{u} + (\alpha_4 + \alpha_5\alpha_7)\vec{v}$~~ iff $\vec{w} \in \text{span}[\vec{u}, \vec{v}]$.

$\vec{z} \notin W$, then by the result above, $\vec{z} \notin \text{span}[\vec{u}, \vec{v}]$, then by definition of ~~linear independence,~~ span, $\vec{z}$ is linearly independent of $\vec{u}$ and $\vec{v}$. By inspection, $\vec{u}$ and $\vec{v}$ are linearly independent. Then $\vec{u}$ is linearly independent of $\vec{v}$ and $\vec{z}$, and $\vec{v}$ is linearly independent of $\vec{u}$ and $\vec{z}$. Then $\vec{u}, \vec{v}$ and $\vec{z}$ are a basis of $\mathbb{R}^3$ < by definition of span

is this level of detail sufficient?

$A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}$ $B = \begin{pmatrix} b_{11} & \cdots & b_{1l} \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{ml} \end{pmatrix}$

$A^T = \begin{pmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & & \vdots \\ a_{1m} & \cdots & a_{nm} \end{pmatrix}$ $B^T = \begin{pmatrix} b_{11} & \cdots & b_{m1} \\ \vdots & & \vdots \\ b_{1l} & \cdots & b_{ml} \end{pmatrix}$

$A \cdot B = \begin{pmatrix} a_{11}b_{11} + \ldots + a_{1m}b_{m1} & \cdots & a_{11}b_{1l} + \ldots + a_{1m}b_{ml} \\ \vdots & & \vdots \\ a_{n1}b_{11} + \ldots + a_{nm}b_{m1} & \cdots & a_{n1}b_{1l} + \ldots + a_{nm}b_{ml} \end{pmatrix}$

$B^T \cdot A^T = \begin{pmatrix} b_{11}a_{11} + \ldots + b_{m1}a_{1m} & \cdots & b_{11}a_{n1} + \ldots + b_{m1}a_{nm} \\ \vdots & & \vdots \\ b_{1l}a_{11} + \ldots + b_{ml}a_{1m} & & b_{1l}a_{n1} + \ldots + b_{ml}a_{nm} \end{pmatrix}$

$(A \cdot B)^T = \begin{pmatrix} a_{11}b_{11} + \ldots + a_{1m}b_{m1} & \cdots & a_{n1}b_{11} + \ldots a_{nm}b_{m1} \\ \vdots & & \vdots \\ a_{11}b_{1l} + \ldots a_{1m}b_{ml} & \cdots & a_{n1}b_{1l} + \ldots + a_{nm}b_{ml} \end{pmatrix}$

$(A \cdot B)^T = B^T \cdot A^T$

show $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}$

This very manual method is right

write as $\sum_{i=1}^{m} a_{1i}b_{i1}$ $\sum_{i=1}^{m} a_{1i}b_{i2} \cdots \sum_{i=1}^{m} a_{1i}b_{i1}$

May be quicker to prove by taking a generic element of the result

~~let $A_m = a_{mm} \cdots a_{m}$~~

Let the value in the ith ~~column~~ row and jth column of A be a_{ij}

Then let $A_m = \begin{pmatrix} a_{mm} & \cdots & a_{mn} \\ \vdots & & \vdots \\ a_{nm} & \cdots & a_{nn} \end{pmatrix}$

The trick was to proceed down column 1 rather than across row 1

So $A_n = (n)$, $A_{n-1} = \begin{pmatrix} n-1 & n \\ 0 & n \end{pmatrix}$, $A_{n-2} = \begin{pmatrix} n-2 & n-1 & n \\ 0 & n-1 & n \\ 0 & 0 & n \end{pmatrix}$ and

$A_1 = A$

$\det A_n = n$, $\det A_{n-1} = (n-1)n$

~~$\det A_{n-2} = (n-2)\det A_{n-1} - (n-1)\det \begin{pmatrix} 0 & n \\ 0 & n \end{pmatrix} + n \det \begin{pmatrix} 0 & n-1 \\ 0 & 0 \end{pmatrix}$~~

~~$= (n-2)\det A_{n-1} = (n-2)(n-1)n$~~

~~$\det A_{n-3} = (n-3)\det A_{n-2} - (n-2)\det \begin{pmatrix} 0 & n-1 & n \\ 0 & n-1 & n \\ 0 & 0 & n \end{pmatrix}$~~

~~$+ (n-1)\det \begin{pmatrix} 0 & n-2 & n \\ 0 & 0 & n \\ 0 & 0 & n \end{pmatrix} - n\det \begin{pmatrix} 0 & n-2 & n-1 \\ 0 & 0 & n-1 \\ 0 & 0 & 0 \end{pmatrix}$~~

Let B_n be ~~a~~ any n-square matrix where all values in the first column are 0.

So $B_1 = (0)$, $B_2 = \begin{pmatrix} 0 & b_{12} \\ 0 & b_{22} \end{pmatrix}$, $B_3 = \begin{pmatrix} 0 & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & b_{32} & b_{33} \end{pmatrix}$, etc.

~~Clai~~ ~~$\forall n$: det~~ $\forall B_n: \det B_n = 0$

Base case(s):

By definition of det, $\det B_1 = 0$ and

$\det B_2 = 0 \times b_{22} - 0 \times b_{12} = 0$

Induction hypothesis:

Assume $\forall k \leq n$ ~~det~~ $: \forall B_k : \det B_k = 0$

Induction step:

~~$\det B_{n+1} = 0 \det$~~

Let $B_{n+1} = \begin{pmatrix} 0 & b_{12} & b_{13} & \cdots & b_{1\,n+1} \\ 0 & b_{22} & b_{23} & \cdots & b_{2\,n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & b_{n+1\,2} & b_{n+1\,3} & \cdots & b_{n+1\,n+1} \end{pmatrix}$

By definition of det,

$$\det B_{n+1} = 0 \det \begin{pmatrix} b_{22} & b_{23} & \cdots & b_{2\,n+1} \\ \vdots & \vdots & & \vdots \\ b_{n+1\,2} & b_{n+1\,3} & & b_{n+1\,n+1} \end{pmatrix}$$

$$- b_{12} \det \begin{pmatrix} 0 & \cdots \\ \vdots & \cdots \\ 0 & \cdots \end{pmatrix} + b_{13} \begin{pmatrix} 0 & \cdots \\ \vdots & \cdots \\ 0 & \cdots \end{pmatrix} - \cdots$$

$= 0$ (by induction hypothesis)

~~By~~ By induction, $\forall B_n : \det B_n = 0$

Intuitively, all matrix transformations B_n collapse the first dimension of $\mathbb{R}^n$.

Then ~~det~~

$\det A_1 = 1 \det A_2 - 2(0) + 3(0) - \ldots = 1 \det A_2$

$= 1\left[2 \det A_3 - 3(0) + 4(0) - \ldots\right] = 1 \times 2 \det A_3$

$\vdots$

$= n!$

Date

5 Rank $\begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 1 & 3 & 2 & 0 & 4 \\ 1 & 2 & 1 & 1 & 1 \end{pmatrix} = 2$

Is working required?
How should working be shown?

Since the rank of this matrix is 2, no 3 of the vectors forming this matrix are linearly independent. So the vectors do not span $\mathbb{R}^3$

How can this be more fully spelled out?

By inspection, each of $\vec{u}_1, \dots, \vec{u}_5$ is a linear combination of ~~(1 0 1) and (0 1 0)~~. $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\vec{v}_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

~~write this~~ in

~~So the span of $\vec{u}_1$~~ So span $[\vec{u}_1, \dots, \vec{u}_5]$ ~~$= \{\vec{w} : \vec{w} = a_1\vec{v}_1 + a_2\vec{v}_2\}$~~
$= \{\vec{w} : \vec{w} = a_1\vec{v}_1 + a_2\vec{v}_2\}$

By inspection, the vectors not spanned by $\vec{u}_1, \dots, \vec{u}_5$ are all and only vectors $\vec{w}' = \begin{pmatrix} w_1' \\ w_2' \\ w_3' \end{pmatrix}$ such that

~~$w_1' \neq w_1^3$~~: $w_1' \neq w_3'$.

Since ~~any 3 of~~ the vectors ~~are linearly dependent~~ have rank 2, the matrix transformation composed by any 3 of the vectors ~~has~~ has rank 2 and collapses $\mathbb{R}^3$ into $\mathbb{R}^2$, so has determinant 0.

Is this adequate?

+ any 3 vectors are linearly dependent
+ not full rank

① can be spelled out with Gauss-Jordan ~~e-eli~~ elimination

Solve simult eqns, inverting matrices by Gauss-Jordan elim

Expected computations
Invert matrices
Compute det
Compute eigen
Transpose matrices
Matrix products

Learn Cramer's rule for inverting matrices