Microeconomic Analysis Problem Set 1

1 $x_i = \alpha_i m / p_i$, $\sum_{i=1}^{n} \alpha_i = 1$

$[\vec{x}(\vec{p}) - \vec{x}(\vec{p}')] \cdot [\vec{p} - \vec{p}']$

$$= \left(\begin{pmatrix} \alpha_1 m/p_1 \\ \alpha_2 m/p_2 \\ \vdots \\ \alpha_n m/p_n \end{pmatrix} - \begin{pmatrix} \alpha_1 m/p_1' \\ \alpha_2 m/p_2' \\ \vdots \\ \alpha_n m/p_n' \end{pmatrix} \right) \cdot \left(\begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix} - \begin{pmatrix} p_1' \\ p_2' \\ \vdots \\ p_n' \end{pmatrix} \right)$$

$$= \begin{pmatrix} \alpha_1 m(1/p_1 - 1/p_1') \\ \alpha_2 m(1/p_2 - 1/p_2') \\ \vdots \\ \alpha_n m(1/p_n - 1/p_n') \end{pmatrix} \cdot \begin{pmatrix} p_1 - p_1' \\ p_2 - p_2' \\ \vdots \\ p_n - p_n' \end{pmatrix}$$

$= \alpha_1 m(1/p_1 - 1/p_1')(p_1 - p_1') + \alpha_2 m(1/p_2 - 1/p_2')(p_2 - p_2') + \cdots$
$+ \alpha_n m(1/p_n - 1/p_n')(p_n - p_n')$

$= \sum_{i=1}^{n} \alpha_i m (1/p_i - 1/p_i')(p_i - p_i')$

$= m \sum_{i=1}^{n} \alpha_i (p_i' - p_i / p_i p_i')(p_i - p_i')$

$= -m \sum_{i=1}^{n} \alpha_i (1/p_i p_i')(p_i - p_i')^2 \cancel{\neq 0}$

< 0 since given Cobb-Douglas demands, $m > 0$ and $\forall i: \alpha_i \cancel{\neq 0} > 0$, and $p_i, p_i' > 0$

$\therefore$ The Law of Demand holds for Cobb-Douglas demands

2a $x + 3y - 2z = 2$

$x + 2y + z = 1$

$x + 5y + \alpha z = \beta$

The system of linear equations can ~~eqt~~ be represented as

$$\begin{pmatrix} 1 & 3 & -2 \\ 1 & 2 & 1 \\ 1 & 5 & \alpha \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ \beta \end{pmatrix}$$

Solving by Gauss-Jordan elimination

$$\left(\begin{array}{ccc|c} 1 & 3 & -2 & 2 \\ 1 & 2 & 1 & 1 \\ 1 & 5 & \alpha & \beta \end{array} \right) \underset{\substack{R_2 - R_1 \\ R_3 - R_1}}{\sim} \left(\begin{array}{ccc|c} 1 & 3 & -2 & 2 \\ 0 & -1 & 3 & -1 \\ 0 & 2 & \alpha+2 & \beta-2 \end{array} \right)$$

$$\underset{\substack{\cancel{R_3 + 2R} \\ R_3 + 2R_2}}{\sim} \left(\begin{array}{ccc|c} 1 & 3 & -2 & 2 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & \alpha+8 & \beta-4 \end{array} \right)$$

$(\alpha+8)z = \beta - 4$

If $\alpha \neq -8$, $z = (\beta-4)/(\alpha+8)$

$-y + 3z = -1$, $y = 1 + 3z = (1 + 3(\beta-4)/(\alpha-8)$

$x + 3y - 2z = 2$, $x = 2 - 3y + 2z = -1 - 7z = -1 - 7(\beta-4)/(\alpha+8)$

If $\alpha = -8$, then if $\beta \neq 4$

there are no solutions

If $\alpha = -8$, then if $\beta = 4$

there are infinitely many solutions since $\forall z \in \mathbb{R}$: $(\alpha+8)z = \beta \cancel{\neq} 4$, the solutions are

$x = -1 - 7z$, $y = 1 + 3z$, $z \in \mathbb{R}$

b $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$, and $\begin{pmatrix} -2 \\ 1 \\ \alpha \end{pmatrix}$ span $\begin{pmatrix} 2 \\ 1 \\ \beta \end{pmatrix}$ iff $\exists x, y, z \in \mathbb{R}$:

$$x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} + z \begin{pmatrix} -2 \\ 1 \\ \alpha \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ \beta \end{pmatrix} \text{ iff}$$

$$\begin{pmatrix} 1 & 3 & -2 \\ 1 & 2 & 1 \\ 1 & 5 & \alpha \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ \beta \end{pmatrix}$$ has at least one solution iff $\alpha \neq -8$ or $\beta = 4$

3a $\vec{u} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $\vec{v} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

Suppose that $W \neq \text{span}[\vec{u}, \vec{v}]$ ①

then $\exists \vec{w} \in W: \vec{w} \notin \text{span}[\vec{u}, \vec{v}]$ ② ④

or $\exists \vec{w} \in \text{span}[\vec{u}, \vec{v}]: \vec{w} \notin W$ ③

From ②, $\text{span}[\vec{u}, \vec{v}] = \text{span}[\vec{u}, \vec{v}, \vec{w}]$ ⑤

and $\vec{w} \notin \text{span}[\vec{u}, \vec{v}]$ ⑥

~~From ⑤,~~

By definition of span, since $\vec{w} = 0\vec{u} + 0\vec{v} + 1\vec{w}$, $\vec{w} \in \text{span}[\vec{u}, \vec{v}, \vec{w}]$ ⑦

From ⑤ and ⑦, $\vec{w} \in \text{span}[\vec{u}, \vec{v}]$ ⑧

By reductio, from ⑥ and ⑧, not ② ⑨

From ③, $\vec{w} \in \text{span}[\vec{u}, \vec{v}]$ ⑩

and $\text{span}[\vec{u}, \vec{v}] \neq \text{span}[\vec{u}, \vec{v}, \vec{w}]$ ⑪

From ⑩, $\exists \alpha, \beta \in \mathbb{R}: \vec{w} = \alpha\vec{u} + \beta\vec{v}$ ⑫

~~From ⑫, $\forall \vec{a}: \exists \alpha', \beta', \gamma' \in \mathbb{R}: \vec{a} = \alpha'\vec{u} + \beta'\vec{v} + \gamma'\vec{w}$~~

~~for all $\vec{a}$ $\forall \alpha', \beta' \in \mathbb{R}: \exists \alpha'', \beta'', \gamma'' \in \mathbb{R}:$~~

~~$\alpha'\vec{u} + \beta'\vec{v} = \alpha''\vec{u} + \beta''\vec{v} + \gamma''$~~

$\text{span}[\vec{u}, \vec{v}] = \text{span}[\vec{u}, \vec{v}, \vec{w}]$ since ~~all~~ ~~$\forall \alpha', \beta' \in \mathbb{R}: \exists$~~ for all $\alpha', \beta' \in \mathbb{R}$:

$\alpha'\vec{u} + \beta'\vec{v} = \alpha'\vec{u} + \beta'\vec{v} + 0\vec{w}$ and for all $\alpha'', \beta'', \gamma'' \in \mathbb{R}$: $\alpha''\vec{u} + \beta''\vec{v} + \gamma''\vec{w} = (\alpha'' + \gamma''\alpha)\vec{u} + (\beta'' + \gamma''\beta)\vec{v}$. ⑬

By reductio, from ⑪ and ⑬, not ③ ⑭

~~By~~ By reductio, from ⑨ and ⑭, not ①

$\therefore W = \text{span}[\vec{u}, \vec{v}]$

Intuitively, W coincides with $\text{span}[\vec{u}, \vec{v}]$ since only linear combinations of $\vec{u}$ and $\vec{v}$ do not expand the span.

c $\vec{z} \notin W$, then $\vec{z} \notin \text{span}[\vec{u}, \vec{v}]$. By definition of span, $\vec{z}$ is linearly independent of $\vec{u}$ and $\vec{v}$. By inspection $\vec{u}$ and $\vec{v}$ are linearly independent. Then, $\vec{u}, \vec{v}$ and $\vec{z}$ are linearly independent. By definition of ~~span~~ a basis, $\vec{u}, \vec{v}, \vec{z}$ span $\mathbb{R}^3$

4a Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}$ $B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1l} \\ b_{21} & b_{22} & \cdots & b_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{ml} \end{pmatrix}$

$A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1m} & a_{2m} & \cdots & a_{nm} \end{pmatrix}$ $B^T = \begin{pmatrix} b_{11} & b_{21} & \cdots & b_{m1} \\ b_{12} & b_{22} & \cdots & b_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1l} & b_{2l} & \cdots & b_{ml} \end{pmatrix}$

$$A\cdot B = \begin{pmatrix} \sum_{i=1}^m a_{1i}b_{i1} & \sum_{i=1}^m a_{1i}b_{i2} & \cdots & \sum_{i=1}^m a_{1i}b_{il} \\ \sum_{i=1}^m a_{2i}b_{i1} & \sum_{i=1}^m a_{2i}b_{i2} & \cdots & \sum_{i=1}^m a_{2i}b_{il} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^m a_{ni}b_{i1} & \sum_{i=1}^m a_{ni}b_{i2} & \cdots & \sum_{i=1}^m a_{ni}b_{il} \end{pmatrix}$$

$$(A\cdot B)^T = \begin{pmatrix} \sum_{i=1}^m a_{1i}b_{i1} & \sum_{i=1}^m a_{2i}b_{i1} & \cdots & \sum_{i=1}^m a_{ni}b_{i1} \\ \sum_{i=1}^m a_{1i}b_{i2} & \sum_{i=1}^m a_{2i}b_{i2} & \cdots & \sum_{i=1}^m a_{ni}b_{i2} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^m a_{1i}b_{il} & \sum_{i=1}^m a_{2i}b_{il} & \cdots & \sum_{i=1}^m a_{ni}b_{il} \end{pmatrix}$$

$$B^T\cdot A^T \quad \cancel{A^T\cdot B^T} = \begin{pmatrix} \sum_{i=1}^m b_{i1}a_{1i} & \sum_{i=1}^m b_{i1}a_{2i} & \cdots & \sum_{i=1}^m b_{i1}a_{ni} \\ \sum_{i=1}^m b_{i2}a_{1i} & \sum_{i=1}^m b_{i2}a_{2i} & \cdots & \sum_{i=1}^m b_{i2}a_{ni} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^m b_{il}a_{1i} & \sum_{i=1}^m b_{il}a_{2i} & \cdots & \sum_{i=1}^m b_{il}a_{ni} \end{pmatrix}$$

By inspection, $(A\cdot B)^T = B^T \cdot A^T$

b $\det A = 1 \det \begin{pmatrix} 2 & 3 & \cdots & n \\ 0 & 3 & \cdots & n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n \end{pmatrix} = 1\times 2 \det \begin{pmatrix} 3 & 4 & \cdots & n \\ 0 & 4 & \cdots & n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n \end{pmatrix}$

$= 1\times 2\times 3 \det \begin{pmatrix} 4 & 5 & \cdots & n \\ 0 & 5 & \cdots & n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n \end{pmatrix} = \cdots = 1\times 2\times 3\times\cdots\times n$

$= n!$

5a $\begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 1 & 3 & 2 & 0 & 4 \\ 1 & 2 & 1 & 1 & 1 \end{pmatrix}$

Row 1 = row 3, the maximal number of linearly independent rows is 2.

By inspection, each column is a linear combination of $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, so the maximal number of linearly independent columns is 2

$\therefore \text{Rank} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 1 & 3 & 2 & 0 & 4 \\ 1 & 2 & 1 & 1 & 1 \end{pmatrix} = 2$

b The vectors do not span $\mathbb{R}^3$

By inspection, each vector is a linear combination of $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Then, $\text{span}[\vec{u}_1, \dots, \vec{u}_5] = \{\vec{w} : \vec{w} = \alpha_1\vec{u}_1 + \dots + \alpha_5\vec{u}_5\}$

$= \{\vec{w} : \vec{w} = \alpha_1(\beta_1^1\vec{v}_1 + \beta_2^1\vec{v}_2) + \dots + \alpha_5(\beta_1^5\vec{v}_1 + \dots + \beta_2^5\vec{v}_2)\}$

where $\vec{u}_i = \beta_1^i\vec{v}_1 + \beta_2^i\vec{v}_2$ for all i $= \{\vec{w} : \vec{w} = \gamma_1\vec{v}_1 + \gamma_2 v_2\}$

$= \text{span}[\vec{v}_1, \vec{v}_2] \neq$ ~~span~~ $\mathbb{R}^3$ since at least 3 vectors are required to span $\mathbb{R}^3$

The vectors not spanned by $\vec{u}_1, \dots, \vec{u}_5$, are those not spanned by $\vec{v}_1$ and $\vec{v}_2$, which are those whose first element and third element are not equal.

c From result in (a), the rank of any such matrix is 2, ~~so~~ any such matrix is not full rank, and has determinant 0.