Game Theory Problem Set 5

A

1a $u_A(x_A, m_A) = x_A + m_A$

$u_B(x_B, m_B) = 3x_B + m_B$

$x_A + x_B = 100 \Rightarrow x_B = 100 - x_A$

$m_A + m_B = 200 \Rightarrow m_B = 200 - m_A$

where u_c, x_c, and m_c denote c's utility, c's share of farmland in acres and c's share of cash respectively for $c \in \{A, B\}$.

$X = \{(x_A, m_A) : x_A \in [0, 100], m_A \in [0, 200]\}$,

$D = (x_A = 50, m_A = 100)$

$U = \cancel{\{u_A(x_A, m_A}$

$\{(u_A(x_A, m_A), u_B(x_B = 100 - x_A, m_B = 200 - m_A)) : (x_A, m_A) \in X\}$

$d = (u_A(x_A = 50, m_A = 100) = 150, u_B(x_B = 100 - x_A, m_B = 200 - m_A)$
$= u_B(x_B = 50, m_B = 100) = 250)$

where X is the set of possible agreements, D is the disagreement outcome, U is the set of payoff pairs corresponding to X and d is the payoff pair corresponding to D.

~~Suppose $x_A = 100, m_A = 200$. then $u_A =$~~
~~Let~~

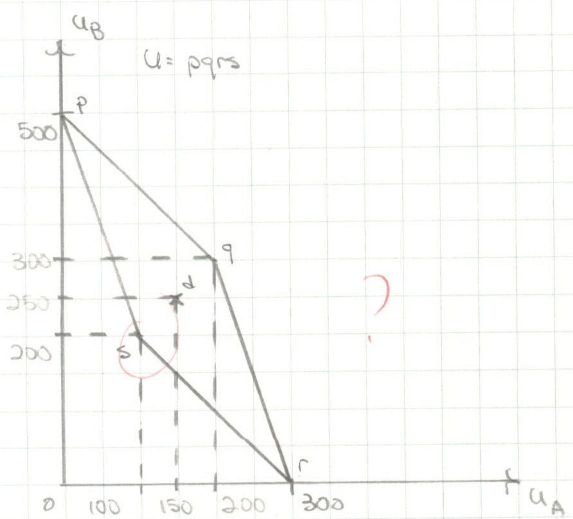
$u_A(x_A = 100, m_A = 200) = 300$, $u_B(x_A = 100, m_A = 200) = 0$

$u_A(x_A = 100, m_A = 0) = 100$, $u_B(x_A = 100, m_A = 0) = 200$

$u_A(x_A = 0, m_A = 200) = 200$, $u_B(x_A = 0, m_A = 200) = 300$

$u_A(x_A = 0, m_A = 0) = 0$, $u_B(x_A = 0, m_A = 0) = 500$

All other $x \in X$ are weighted averages of these possible agreements. Then, because utilities are linear, ~~utilit~~ for all other $x \in X$, the payoff pair corresponding to x is a weighted average of these four possible agreements.



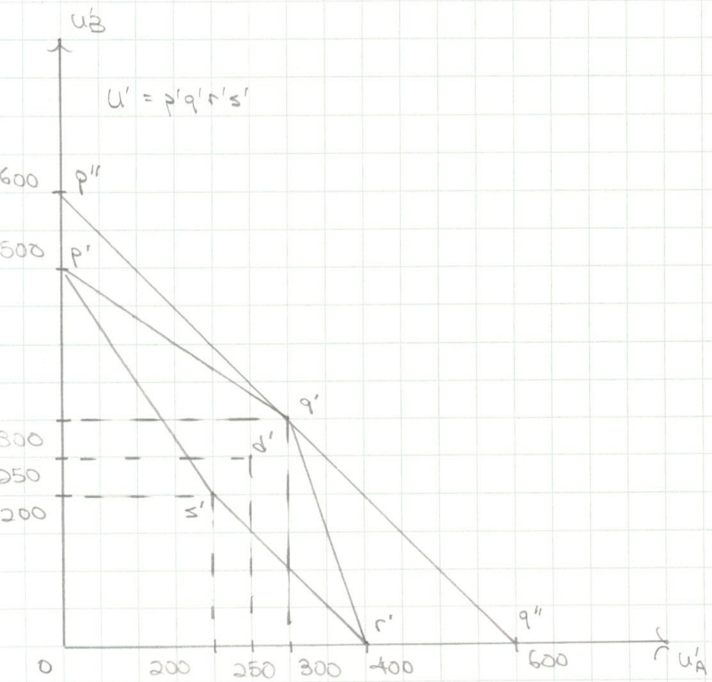
?

Assume they can "throw stuff away" but not that they can borrow, i.e. lower bound on total x, m is 0.

b Let $F(U, d) = u$ be a bargaining solution that satisfies the Nash axioms.

~~Let $u_A'(x_A, m_A) = 2u_A(x_A, m_A) - 100$, and $u_B'(x_B, m_B) = u_B(x_B, m_B)$,~~
~~$U = \{(u_A'(x_A, m_A), u_B'(x_B = 100 - x_A, m_B = 200 - m_A) : (x_A, m_A) \in X\}$,~~
~~and $d = \{(u_A'(x_A = 50, m_A = 100), u_B'(x_B = 100 - x_A = 50, m_B = 200 - x_A = 100))$~~
$= ($

Let $u_A'(x_A, m_A) = u_A(x_A, m_A) + 100$, $u_B'(x_B, m_B) = u_B(x_B, m_B)$,
$U' = \{(u_A'(x_A, m_A), u_B'(x_B = 100 - x_A, m_B = 200 - m_A) : (x_A, m_A) \in X\}$,
$d' = (u_A', u_B')|_0 = (250, 250)$



Consider the bargaining problem (U'', d'), where $U'' = \{(u_A', u_B') : u_A' + u_B' \in [0, 600]\}$. U'' is represented by area $0p''q''$. By inspection, $U'' \subset U'$.

Suppose that $(u_1', u_2') \in U''$, then $u_1' + u_2' \in [0, 600]$, then $(u_2', u_1') \in U''$. ~~So~~ $d_1' = 250 = d_2'$, so the bargaining problem (U'', d') is symmetric.

Let $u'' = F(U'', d')$. By symmetry, $u'' = (u_A^{*''}, u_B^{*''})$ is such that $u_A^{*''} = u_B^{*''}$. Suppose for reductio that $u_A'' + u_B'' < 600$. ~~Then~~ Let $\varepsilon = 600 - u_A'' - u_B''$. Then $u_A'' + \varepsilon/2 + u_B'' + \varepsilon/2 = 600$, ~~so~~ $(u_A'' + \varepsilon/2, u_B'' + \varepsilon/2) \in U''$ and $u_A'' + \varepsilon/2 > u_A''$ and $u_B'' + \varepsilon/2 > u_B''$. By weak Pareto efficiency, $u'' \neq F(U'', d')$. By reductio $u_A'' + u_B'' = 600$. Solving simultaneously, $u_A'' = u_B'' = 300$. By independence of irrelevant alternatives, $u' = F(U', d') = u'' = (300, 300)$. By invariance to equivalent payoff representations, $u = F(U, d) = (u_A' - 100, u_B') = (200, 300)$. This corresponds to the distribution $x_A = 0$, $m_A = 200$, $x_B = 100$, $m_B = 0$.

c. Suppose that A is the divider. ~~Sup~~ Denote the portions chosen by A as ~~(x_A, m_A) an~~ (x_1, m_1) and $(x_2 = 100 - x_1, m_2 = 200 - m_1)$. In the second stage subgame, B's maximisation problem is
$\max_{i \in \{1,2\}} u_B(x_i, m_i) = 3x_i + m_i$.

B chooses (x_1, m_1) ~~iff~~ if $3x_1 + m_1 \not> 3x_2 + m_2 = 3(100 - x_1) + 200 - m_1$,
$3x_1 + m_1 \not> 500 - 3x_1 - m_1$,
$3x_1 + m_1 \not> 250$

In the first stage, A's payoff reduces to
$u_A(x_1, m_1) = x_1 + m_1$ iff $3x_1 + m_1 \not> 250$, $(100 - x_1) + (200 - m_1)$ otherwise

Just Max
$(u_1 - d_1)(u_2 - d_2)$

Just max $(u_1 - d_1)(u_2 - d_2)$
convex fn so if gradient of $(u_1 - d_1)(u_2 - d_2)$ is intermediate between slopes of the two frontier edges

With Lagrangian optimisation, have to check λ multiplier sign to make sure it "hangs together"

The relevant constraints are $x_A, x_B, m_A, m_B \geq 0$ and $x_A + x_B \leq 100$, $m_A + m_B \leq 200$

$\mathcal{L} = (u_A - d_A)(u_B - d_B) - \lambda_1(u_A + u_B - 500) - \lambda_2(\frac{3u_A}{~~u_A~~} + u_B - 900)$

If $\lambda > 0$, the constraint binds

Easier to argue geometrically

Or argue that solution lies on frontier, then write objective function in terms of one variable, e.g. u_A then track along frontier

"Find and prove that it is so" simply means find the Nash solution and show your method

Suppose, without loss of generality, that A intends to induce B to choose (x_2, m_2). Then A's maximisation problem is $\max_{x_1, m_1} x_1+m_1$ subject to $3x_1+m_1 \leq 250$, $x_1 \in [0,100]$, and $m_1 \in [0,200]$

~~By inspection~~

By inspection, A chooses $(x_1 = 50/3, m_1 = 200)$, then $(x_2 = 250/3, m_2 = 0)$, and B (weakly) prefers (x_2, m_2) to (x_1, m_1).

The SPE is the strategy profile such that A's strategy is to offer $(x_1 = 50/3, m_1 = 200)$ and $(x_2 = 250/3, m_2 = 0)$ and B's strategy is to choose (x_1, m_1) iff $3x_1 + m_1 > 250$.

Suppose that B is the divider. In the second stage subgame, A's maximisation problem is $\max_{i \in \{1,2\}} u_A(x_i, m_i) = x_i + m_i$. ~~A~~ A chooses (x_1, m_1) if $u_A(x_1, m_1) = x_1 + m_1 > u_A(x_2, m_2) = x_2 + m_2 = (100 - x_1) + (200 - m_1)$ $x_1 + m_1 > 150$. In the first stage, B's ~~fns~~ payoff function reduces to $u_B(x_1, m_1) = 3x_1 + m_1$ iff $x_1 + m_1 \leq 150$, $3(100 - x_1) + (200 - m_2)$ otherwise. Suppose wlog that B intends to induce A to choose (x_2, m_2). Then B's maximisation problem is $\max_{x_1, m_1} 3x_1 + m_1$ subject to $x_1 + m_1 \leq 150$, $x_1 \in [0,100]$, $m_1 \in [0,200]$. ~~By inspec~~ inspection, ~~$x_1 = 100$, $m_1 = 50$~~ B chooses $(x_1 = 100, m_1 = 50)$, then $(x_2 = 100 - x_1 = 0, m_2 = 200 - m_1 = 150)$, and A (weakly) prefers (x_2, m_2) to (x_1, m_1). The SPE is the strategy profile such that B's strategy is to offer $(x_1 = 100, m_1 = 50)$ and $(x_2 = 0, m_2 = 150)$, and A's strategy is to choose (x_1, m_1) iff $x_1 + m_1 > 150$.

d Suppose that A is the divider, then at SPE, ~~$u_A = 50/3$~~
$u_A = 50/3 + 200$, $u_B = 250$
Suppose that B is the divider, then at SPE,
$u_A = 150$, $u_B = 350$
Expected utilities
$U_A = (\cancel{(5} \, 50/3 + 200 + 150)/2 = 1100/6$, ~~$U_B =$~~
$U_B = (\cancel{350 +} \, 250 + 350)/2 = 300$

This solution is not Pareto optimal because it is Pareto-dominated by $(x_A, m_A) = (0, 200)$, where $U_A = 200$ and $U_B = 300$.

Graphically, each of the "pure" divide and choose outcomes is some point on the ppr frontier that does not coincide with q since each first mover divides portions such as to advantage itself. By the convexity of U, any weighted average of the "pure" outcomes lies within the frontier, and is not Pareto optimal.

Date

2a IIA: If $\mathcal{U}' \subseteq \mathcal{U}$ and $d' = d$ and $F(\mathcal{U}, d) \in \mathcal{U}'$, then ~~F(U'~~
$F(\mathcal{U}', d) = F(\mathcal{U}, d)$

consider $\mathcal{U}' = \{(u_1, u_2) : u_1, u_2 \in [0,1]\}$, $\mathcal{U} = \mathcal{U}' \cup \{(0,10)\}$, and $d = d' = (0,0)$. Then $u_1'^{+} = u_2'^{+} = u_1^{+} = 1$ and $u_2^{+} = 10$.

$\mathcal{U} = \{(u_1, u_2) : u_1 + u_2 = 1\}$
$\mathcal{U}' = \{(u_1, u_2) : u_1 + u_2 = 1,\ u_2 \leq 0.5\}$
~~Possible to plot~~
Show graphically
KS: "one player is in a weaker position"

This solution is not ideal because it violates convexity.

$\max u_1/u_1^{+}$ subject to $u_1/u_1^{+} = u_2/u_2^{+}$ and $(u_1, u_2) \in \mathcal{U}$
coincides with
$\max u_1$ subject to $u_1 = u_2/10$ and $(u_1, u_2) \in \mathcal{U}$
~~= max u_1 subject to ≠~~
~~$(u_1^*$~~
$= \max u_1$ subject to $(u_1, 10u_1) \in \mathcal{U}$
$= 1/10$
$(u_1^*, u_2^*) = (1/10, 1)$

$\max u_1'/u_1'^{+}$ subject to $u_1'/u_1'^{+} = u_2'/u_2'^{+}$ and $(u_1', u_2') \in \mathcal{U}'$
coincides with
$\max u_1'$ subject to $u_1' = u_2'$ and $(u_1', u_2') \in \mathcal{U}'$
$= \max u_1'$ subject to $(u_1', u_1') \in \mathcal{U}'$
$= 1$
$(u_1'^*, u_2'^*) = (1,1)$

Consequently, KS solution does not satisfy IIA.

b $u_1(x_1) = x_1$
$u_2(x_2) = x_2^{\rho}$ for $0 < \rho < 1$
~~$x_1 + x_2 = 1$~~
$\mathcal{U} = \{(u_1(x_1), u_2(x_2)) : x_1 + x_2 = 1\}$
$u_1^{+} = \max u_1$ st $(u_1, u_2) \in \mathcal{U} = 1$
$u_2^{+} = \max u_2$ st $(u_1, u_2) \in \mathcal{U} = 1$

Again assume feasible, costless disposal
$x_1 + x_2 = 1$ follows from efficiency which is one of the KS axioms. Have to argue that solution is $x_1 + x_2 = 1$

~~max u_1/u_1^{+} st $u_1/u_1^{+} = u_2/u_2^{+}$, $(u_1, u_2) \in \mathcal{U}$~~
~~= max u_1 st $u_1 = u_2$, $(u_1, u_2) \in \mathcal{U}$~~
~~= max u_1 st $(u_1, u_1) \in \mathcal{U}$~~

~~Suppose that $u_1/u_1^{+} = u_2/u_2^{+}$, then $u_1 = u_2$, $x_1 = x_2^{\rho}$,~~
~~$x_1 = (1-x_1)^{\rho}$, $x_1^{1/\rho} + x_1 - 1 = 0$, $x_1 > x_2$ since $0 < \rho < 1$~~
~~and $x_1, x_2 \in [0,1]$~~
~~max u_1/u_1^{+} st $u_1/u_1^{+} = u_2/u_2^{+}$, $(u_1, u_2) \in \mathcal{U}$~~
~~= max u_1 st $u_1 = u_2$, $(u_1, u_2) \in \mathcal{U}$~~
~~= max x_1 st $x_1^{1/\rho} + x_1 - 1 = 0$, $x_1 \in [0,1]$~~
~~EOC~~

Suppose that $u_1/u_1^{+} = u_2/u_2^{+}$ and $(u_1, u_2) \in \mathcal{U}$, then $u_1 = u_2$, $x_1 = x_2^{\rho}$.
Suppose for reductio that $x_2 = 1$, then $x_1' = 1$, then $x_1 = 1$, then $(u_1, u_2) = (1,1) \notin \mathcal{U}$. By reductio, $x_2 \neq 1$.
Then $x_2 \in [0,1)$ ~~and so, since~~ consequently, ~~since 0~~ given $0 < \rho < 1$, $x_1 > x_2$.

Acceptable argument

c Let $\mathcal{U}' = \{(u_1(x_1), u_2(x_2)) : x_1 + x_2 = s\}$, ~~$u_1'^{+} = u_2'^{+} = s$~~
$u_1'^{+} = s$, $u_2'^{+} = s^{\rho}$
~~Suppose that $u_1'^*/u_1'^{+} = u_2'^*/u_2'^{+}$ and $(u_1'^*, u_2'^*) \in \mathcal{U}$~~
~~$x_1 = x_2^{\rho}/s^{\rho}$, $(s - x_2) = (x_2/s)^{\rho}$~~
~~$x_1/s = (x_2/s)^{\rho}$, $1 - x_2/s = (x_2/s)^{\rho}$~~

Date

$\cancel{\max u_1/u_1^+ \text{ subject to } u_1/u_1^+ = u_2/u_2^+, (u_1,u_2)\in U}$

$\cancel{= \max x_1/s \text{ st } x_1/s = x_2^\rho/s^\rho, (x_1, x_2^\rho)\in U}$

$\cancel{= \max x_1/s \text{ st } x_2^\rho = x_1 s^{\rho-1}, (x_1, x_2^\rho)\in U}$

$\cancel{= \max x_1/s \text{ st } (x_1, x_1 s^{\rho-1})\in U}$

~~coincides with~~

$\cancel{\max x_1 \text{ st } (x_2 s^{1-\rho}, x_2^\rho)\in U}$

$\cancel{= \max x_1 \text{ st } x_2 s^{1-\rho} + x_2 = s}$

$\cancel{= \max x_1}$

$\cancel{\max x_2 s^{1-\rho} \text{ st } (x_2 s^{1-\rho}, x_2^\rho)\in U}$

$\cancel{= \max x_2 s^{1-\rho} \text{ st } x_2 s^{1-\rho} + x_2 = s}$

$\cancel{= \max x_2 s^{1-\rho} \text{ st } x_2 = s/(1+s^{1-\rho})}$

$u_1^*/u_1^+ = u_2^*/u_2^+$ and $(u_1^*, u_2^*)\in U$

$x_1/s = x_2^\rho/s^\rho$

$x_1 = x_2^\rho s^{1-\rho}$

$(x_1, x_2^\rho)\in U$

$\cancel{(x_1,}\ (x_2^\rho s^{1-\rho}, x_2^\rho)\in U$

$x_2^\rho s^{1-\rho} + x_2 = s$

$x_2^\rho s^{-\rho} + x_2^1 s^{-1} = 1$

$\cancel{(x_2/s)^\rho + (x_2/s)^{-1} = 1}$

Differentiating implicitly wrt s

$\cancel{x_2^\rho s}$

$-\rho x_2^\rho s^{-\rho-1} + \rho x_2^{\rho-1}\, dx_2/ds\, s^{-\rho}$

$- x_2 s^{-2} + dx_2/ds\, s^{-1} = 0$

$(\rho x_2^{\rho-1} s^{-\rho} + s^{-1})\, dx_2/ds = \rho x_2^\rho s^{-\rho-1} + x_2 s^{-2}$

Given $0<\rho<1$, $x_2>0$, $s>0$,

$dx_2/ds > 0$

$dx_1/dx_2 = \rho x_2^{\rho-1} s^{1-\rho} > 0$

$dx_1/ds > 0$

consequently for ~~s>1~~ x_1 and x_2 each increase when s increases from 1 to some $s>1$

Alternate solution by relabeling $z_1 = x_1/s$
$z_2 = x_2/s$

The point is that KS solution has some intuitive properties

3a In the second stage subgame, given that P1 chose x_2, P2 chooses A (accept) or R (reject) to maximise

$$u_2 = \begin{cases} x_2^\rho - s x_1^\rho = x_2^\rho - s(1-x_2)^\rho & \text{if } a_2 = A \\ 0 & \text{if } a_2 = R \end{cases}$$

By backward induction

$u_2(a_2, x_2)$

where a_2 denotes P2's action. P2 chooses A iff $x_2^\rho - s x_1^\rho \geq 0$, $x_2^\rho \geq s x_1^\rho$, $x_2 \geq s^{1/\rho} x_1$, $x_2 \geq s^{1/\rho}(1-x_2)$, $(1+s^{1/\rho})x_2 \geq s^{1/\rho}$, $x_2 \geq s^{1/\rho}/1+s^{1/\rho}$. Then P1's ~~maximi~~ payoff function in the first stage reduces to

s is some envy coefficient

$$u_1(x_2) = \begin{cases} (1-x_2)^\rho & \text{if } x_2 \geq s^{1/\rho}/1+s^{1/\rho} \\ 0 & \text{otherwise} \end{cases}$$

then P1's maximisation problem is $\max_{x_2} u_1(x_2)$ which has solution $x_2 = s^{1/\rho}/1+s^{1/\rho}$.

since $u_1(x_2)$ is decreasing in x_2

The SPE is the strategy profile such that P1's strategy is to offer $x_2 = s^{1/\rho}/1+s^{1/\rho}$ and P2's strategy is to A iff $x_2 \geq s^{1/\rho}/1+s^{1/\rho}$ and R otherwise

b Supposing that s is known to P2, P2's payoff function, maximisation problem, and best response function are unchanged from (a). P2 chooses A iff $x_2 \geq s^{1/\rho}(1-x_2)$, $s^{1/\rho} \leq x_2/1-x_2$, $s \leq (x_2/1-x_2)^\rho$.

P1's expected payoff function reduces to

$$\pi_1(x_2) = (1-x_2)^\rho P(s \leq (x_2/1-x_2)^\rho) + 0^\rho P(s > (x_2/1-x_2)^\rho)$$
$$= (1-x_2)^\rho F((x_2/1-x_2)^\rho)$$
$$= (1-x_2)^\rho (x_2/1-x_2)^{2\rho}$$

P1's maximisation problem is $\max_{x_2} \pi_1(x_2)$ ~~which coincides with $\max_{x_2} (1-x_2)(x_2/1-x_2)^2$ since $0<\rho<1$ so raising to the power $1/\rho$ is a strictly increasing monotonic transformation.~~

~~FOC: $\partial/\partial x_2 \, (1-x_2)(x_2/1-x_2)^2$~~
~~$= \partial/\partial x_2 \, x_2^2/1-x_2$~~
~~$= 2x_2/1-x_2 + x_2^2/(1-x_2)^2 = 0$~~
~~$2x_2(1-x_2) + x_2^2 = 0$~~
~~$= 2x_2 - x_2^2 = 0$~~
~~$x_2 = 0$ (reject) or $x_2 = 2$ (reject)~~

FOC: $\partial/\partial x_2 \; x_2^{2\rho}/(1-x_2)^\rho$

$= 2\rho x_2^{2\rho-1}/(1-x_2)^\rho + x_2^{2\rho}(-\rho)(1-x_2)^{-\rho-1}(-1)$

$= 2\rho x_2^{2\rho-1}/(1-x_2)^\rho + \rho x_2^{2\rho}/(1-x_2)^{1+\rho} = 0$

~~$2/1-x_2 + x_2/$~~

~~$2 + x_2/1-x_2 = 0$~~

$2/(1-x_2)^\rho + x_2/(1-x_2)^{\rho+1} = 0$

$2 + x_2/1-x_2 = 0$

$x_2 = 2$ (reject)

There are no interior solutions. The constraints are $x_2 \in [0,1]$ and $F(s) \in [0,1] \Rightarrow (x_2/1-x_2)^{2\rho} \in [0,1] \Rightarrow x_2 \in [0, 1/2]$

~~At $x_2 = 0$,~~ $\pi_1(x_2=0) = 0$, $\pi_1(x_2 = 1/2) = (1/2)^\rho > 0$ ~~P1 chooses $x_2 = 1/2$~~

The SPE is such that P1's strategy is to offer $x_2 = 1/2$ and P2's strategy is to choose A iff $s \leq (x_2/1-x_2)^\rho$. At SPE, P2 chooses A with probability 1.

Interesting result: always offer sth acceptable even if unsure, i.e. ~~match the most envious~~ satisfy the most envious player

$E(s) = \int sF'(s)\,ds = \int 2s^2\,ds \; [2/3 s^3]_0^1$

c Expected total utility $U = (1-x_2)^\rho + E(x_2^\rho - s(1-x_2)^\rho)$

$= x_2^\rho + (1-E(s))(1-x_2)^\rho = x_2^\rho + (1-2/3)(1-x_2)^\rho$

$= (1/2)^\rho + 1/3(1/2)^\rho = 4/3(1/2)^\rho$

Date