Sider Predicate Logic Exercises

1 consider arbitrary PC-model $M = \langle D, I \rangle$.

Prove by induction that for all PC-wffs ϕ, for all variable assignments g and h for M, if g and h agree on the free variables in ϕ, then $V_{M,g}(\phi) = V_{M,h}(\phi)$.

Base case

~~C(φ)~~ consider arbitrary PC wff ϕ such that $c(\phi)=0$. Suppose that g and h agree on the free variables in ϕ. ϕ is a basic wff $\pi\alpha_1 \dots \alpha_n$. Each variable in ϕ is free. $|\pi|_{M,g} = I(\pi) = |\pi|_{M,h}$. For $\alpha \in \{\alpha_1, \dots, \alpha_n\}$, $|\alpha|_{M,g} = I(\alpha) = |\alpha|_{M,h}$ if α is a constant, $|\alpha|_{M,g} = g(\alpha) = h(\alpha) = |\alpha|_{M,h}$ if α is a variable. $V_{M,g}(\phi)=1$ iff $\langle |\alpha_1|_{M,g}, \dots, |\alpha_n|_{M,g} \rangle \in |\pi|_{M,g}$ iff $\langle |\alpha_1|_{M,h}, \dots, |\alpha_n|_{M,h} \rangle \in |\pi|_{M,h}$ iff $V_{M,h}(\phi)=1$. By conditional proof, for $c(\phi)=0$ if g and h agree on the free variables in ϕ, then $V_{M,g}(\phi) = V_{M,h}(\phi)$.

Induction Hypothesis.
Given n, for all $m<n$, for all PC-wff ϕ such that $c(\phi)=m$, if g and h agree on the free variables in ϕ, then $V_{M,g}(\phi) = V_{M,h}(\phi)$.

Induction Step
consider arbitrary PC wff ϕ such that $c(\phi)=n$. Suppose that g and h agree on the free variables in ϕ. $\phi = \sim\psi$, $\psi \to \chi$, or $\forall\alpha\psi$.

Suppose that $\phi = \forall\alpha\psi$. $c(\psi) = n-1$. Every ~~variable~~ ~~varia~~ if g and h agree on free variables in ϕ, then they agree on free variables in ψ except at most α, then g^α_u and h^α_u agree on free variables in ψ. ~~VmF~~ $V_{M,g}(\phi)=1$ iff $\forall u \in D$: ~~Vm~~ $V_{M,g^\alpha_u}(\psi)=1$ iff^{IH} $\forall u \in D$: ~~Vm,h(~~ $V_{M,h^\alpha_u}(\psi)=1$ iff $V_{M,h}(\phi)=1$. ~~So~~ By conditional proof, if g and h agree on the free variables in ϕ, then $V_{M,g}(\phi) = V_{M,h}(\phi)$.

Suppose that $\phi = \sim\psi$. $c(\psi) = n-1$. ~~if g and h~~ Every free variable in ψ is free in ϕ. So if g and h agree on the free variables in ϕ, they also agree on the free variables in ψ. Suppose g and h agree on the free variables in ϕ. $V_{M,g}(\phi)=1$ iff $V_{M,g}(\psi)=0$ iff by IH $V_{M,h}(\psi)=0$ iff $V_{M,g}(\phi)=1$. ~~So if~~ By conditional proof, if g and h agree on the free variables in ϕ then $V_{M,g}(\phi) = V_{M,h}(\phi)$.

Similarly for $\phi = \psi \to \chi$.

By cases, for all ϕ such that $c(\phi)=n$, if g and h agree on the free variables in ϕ then $V_{M,g}(\phi) = V_{M,h}(\phi)$.

By induction, for all ϕ, if g and h agree on the free variables in ϕ then $V_{M,g}(\phi) = V_{M,h}(\phi)$.

Then, ~~if~~ for all ϕ containing no free variables, for all g and h, $V_{M,g}(\phi) = \cancel{V_{M,g}(}\ V_{M,h}(\phi)$.

2a consider arbitrary PC-model $M = \langle D, I \rangle$ and variable assignment g for M.

Suppose for reductio that

(1) $V_{M,g}(\forall x(Fx \to (Fx \vee Gx))) = 0$
 (1), $\forall \Rightarrow$
(2) $\cancel{\forall}\ \exists u \in D: V_{M,g^x_u}(Fx \to (Fx \vee Gx)) = 0$
 (2), $\to \Rightarrow$
(3) $\exists u \in D: V_{M,g^x_u}(Fx) = 1,\ V_{M,g^x_u}(Fx \vee Gx) = 0$
~~(4) ∃u∈D: Vm,g^x_u(Fx∨Gx)=0~~
(3)~~(4)~~, $\vee \Rightarrow$
~~(5) ∃u∈D: Vm,g^x_u(Fx)=0~~
~~(6)~~
(4) $\exists u \in D: V_{M,g^x_u}(Fx)=1,\ V_{M,g^x_u}(Fx)=0$
 (4), reductio $\Rightarrow$
(5) $V_{M,g}(\forall x(Fx \to (Fx \vee Gx))) = 1$
 (5), generalisation, definition of $\vDash_{PC}$
(6) $\vDash_{PC} \forall x(Fx \to (Fx \vee Gx))$

b consider ...

Suppose for reductio that

(1) $V_{M,g}(\forall x(Fx \wedge Gx) \to (\forall x Fx \wedge \forall x Gx)) = 0$
 (1), $\to \Rightarrow$
(2) $V_{M,g}(\forall x(Fx \wedge Gx)) = 1$
(3) $V_{M,g}(\forall x Fx \wedge \forall x Gx) = 0$
 (2), $\forall \Rightarrow$
(4) $\forall u \in D: V_{M,g^x_u}(Fx \wedge Gx) = 1$
 (4), $\wedge \Rightarrow$
(5) $\forall u \in D: V_{M,g^x_u}(Fx) = 1$
(6) $\forall u \in D: V_{M,g^x_u}(Gx) = 1$
 (5), $\forall \Rightarrow$
(7) $V_{M,g}(\forall x Fx) = 1$
~~(8) Vm,~~
 (6), $\forall \Rightarrow$
(8) $V_{M,g}(\cancel{\forall x F}\ \forall x Gx) = 1$
 (7), (8), $\wedge \Rightarrow$
(9) $V_{M,g}(\forall x Fx \wedge \forall x Gx) = 1$
 (3), (9), reductio $\Rightarrow$
(10) $V_{M,g}(\forall x(Fx \wedge Gx) \to (\forall x Fx \wedge \forall x Gx)) = 1$
 (10), generalisation, definition of $\vDash_{PC}$
(11) $\vDash_{PC} \forall x(Fx \wedge Gx) \to (\forall x Fx \wedge \forall x Gx)$

c Consider ...

Suppose for ~~reductio that~~ conditional proof that

(1) $\cancel{\forall x(Fx \to Gx)}$ $V_{M,g}(\forall x(Fx \to Gx))=1$

(2) $\cancel{\forall x(Gx \to Hx)}$ $V_{M,g}(\forall x(Fx \to Gx))=1$

Suppose for reductio that

(3) $V_{M,g}(\forall x(Fx \to Hx))=0$

(1), $\forall \Rightarrow$

(4) $\forall u \in D: V_{M,g^x_u}(Fx \to Gx)=1$

(2), $\forall \Rightarrow$

(5) $\forall u \in D: V_{M,g^x_u}(Gx \to Hx)=1$

(3), $\forall \Rightarrow$

(6) $\exists u \in D: V_{M,g^x_u}(Fx \to Hx)=0$

(6), $\to \Rightarrow$

(7) $\exists u \in D: V_{M,g^x_u}(Fx)=1,\ V_{M,g^x_u}(Hx)=0$

~~(8) ∃~~

~~(7)~~ (4), (5), (7) $\Rightarrow$

(8) $\exists u \in D: V_{M,g^x_u}(Fx)=1,\ V_{M,g^x_u}(Hx)=0,\ V_{M,g^x_u}(Fx \to Gx)=1,\ V_{M,g^x_u}(Gx \to Hx)=1$

(8), $\to \Rightarrow$

(9) $\exists u \in D: V_{M,g^x_u}(Hx)=1,\ V_{M,g^x_u}(Hx)=0$

(9), reductio $\Rightarrow$

(10) $V_{M,g}(\forall x(Fx \to Hx))=1$

~~(11)~~ (10), ~~conditional~~ conditional proof, generalisation, definition of $\vDash_{PC} \Rightarrow$

(11) $\forall x(Fx \to Gx),\ \forall x(Gx \to Hx) \vDash_{PC} \forall x(Fx \to Hx)$

d Consider ...

Suppose for reductio that

(1) $V_{M,g}(\exists x \forall y Rxy \to \forall y \exists x Rxy)=0$

(1), $\to \Rightarrow$

(2) $V_{M,g}(\exists x \forall y Rxy)=1$

(3) $V_{M,g}(\forall y \exists x Rxy)=0$

(2), $\exists \Rightarrow$

(4) $\exists u \in D: V_{M,g^x_u}(\forall y Rxy)=1$

~~(5)~~ (4), $\forall \Rightarrow$

(5) $\exists u \in D: \forall v \in D: V_{M,g^{x\,y}_{u\,v}}(Rxy)=1$

(3), $\forall \Rightarrow$

(6) $\exists v \in D: V_{M,g^y_v}(\exists x Rxy)=0$

(6), $\exists \Rightarrow$

(7) $\exists v \in D: \forall u \in D: V_{M,g^{y\,x}_{v\,u}}(Rxy)=0$

~~$= \forall u \in D: \exists v \in D: V_{M,g^{x\,y}_{u\,v}}(Rxy)=0$~~

~~$= \forall u \in D: \forall v \in D: V_{M,g^{x\,y}_{u\,v}}(Rxy)\neq 1$~~

~~$= \nexists u \in D: \forall v \in D: V_{M,g^{x\,y}_{u\,v}}(Rxy)=1$~~

~~(4), (7), reductio ⇒~~

~~(8) $V_{M,g}(\exists x \forall y Rxy \to \forall y \exists x Rxy)=1$~~

~~(8), generalisation, definition of $\vDash_{PC}$~~

~~(9) $\vDash_{PC} \exists x \forall y Rxy \to \forall y \exists x Rxy$~~

(5) $\Rightarrow$

(8) $\exists u \in D$, namely u^*: $\forall v \in D: V_{M,g^{x\,y}_{u\,v}}(Rxy)=1$

~~(6) ⇒~~ (7) $\Rightarrow$

(9) $\exists v \in D$, namely v^*: $\forall u \in D: V_{M,g^{x\,y}_{u\,v}}(Rxy)=0$

(8) $\Rightarrow$

(10) $V_{M,g^{x\,y}_{u^*v^*}}(Rxy)=1$

(9) $\Rightarrow$

(11) $V_{M,g^{x\,y}_{u^*v^*}}(Rxy)=0$

(10), (11), reductio $\Rightarrow$

(12) $V_{M,g}(\exists x \forall y Rxy \to \forall y \exists x Rxy)=1$

(12), generalisation, definition of $\vDash_{PC} \Rightarrow$

(13) $\vDash_{PC} \exists x \forall y Rxy \to \forall y \exists x Rxy$.

3a Consider the countermodel $M=\langle D, I \rangle$

$D=\{0,1\}$

$I(F)=\{0\}$ $I(G)=\{0,1\}$

consider ~~the~~ arbitrary variable assignment g

$V_{M,g}(\forall x(Fx \to Gx) \to \forall x(Gx \to Fx))=0 \Rightarrow$

$\nvDash_{PC} \forall x(Fx \to Gx) \to \forall x(Gx \to Fx)$

b Consider the countermodel $M=\langle D, I \rangle$

$D=\{0,1\}$

$I(F)=\{0\}$, $I(G)=\{0\}$

~~for~~ For arbitrary variable assignment g,

$V_{M,g}(\forall x(Fx \lor \sim Gx) \to (\forall x Fx \lor \sim \exists x Gx))=0 \Rightarrow$

$\nvDash_{PC} \forall x(Fx \lor \sim Gx) \to (\forall x Fx \lor \sim \exists x Gx)$

c Consider the countermodel $M=\langle D, I \rangle$

$D=\{0,1\}$

$I(R)=\{\langle 0,1 \rangle\}$, $I(a)=0$, $I(b)=1$

d Consider the counterexample $M=\langle D, I \rangle$, g.

$D=\{0,1\}$

$I(F)=\{0\}$

$g(x)=0$

e Consider the countermodel $M=\langle D, I \rangle$

$D=\mathbb{N}$

$I(R)=\{\langle n_1, n_2 \rangle : n_1 < n_2\}$

5 ~~A~~ Let $T=\{\cancel{\emptyset}, \sim \exists_\infty F, \exists_1 F, \exists_2 F, \exists_3 F, \dots\}$

~~Consider T's.~~

Consider arbitrary $T' \subset T$. ~~such that T' is finite~~ Suppose that T' is finite. Then, there is some $\exists_n F \in T'$ such that there is no $\exists_{n'} F \in T'$ such that $n' > n$. Then the ~~domain~~ model $M=\langle D, I \rangle$, where $D=\{1, \dots, n\}$ and $I(F)=D$ satisfies T'. By conditional proof, generalisation, T' is finitely satisfiable.

~~Consider arbitrary model M=⟨D,I⟩. The size of D is countable, so if it is finite it is some natural number, then it is either infinite or some n.~~ In any model $M=\langle D, I \rangle$, the number of F things is countable, so it is either some natural number n or infinite. Then either $\exists_{n+1} F$ is false or $\sim \exists_\infty F$ is false, so T is not satisfiable.

1a consider arbitrary $PC_=$ model $M = \langle D, I \rangle$ and arbitrary variable assignment g for M.

suppose for conditional proof that

(1) $V_{M,g}(Fab) = 1$

suppose for reductio that

(2) $V_{M,g}(\forall x\,(x=a \rightarrow Fxb)) = 0$

(2), $\forall$ $\Rightarrow$

~~(3) $\forall u \in D: V_{M,g^x_u}(x=a \rightarrow Fxb) = 0$~~

~~(3) $\Rightarrow$~~

~~(4) $V_{M,g^x_\alpha}(x=a \rightarrow Fxb) = 0$~~

~~where $\alpha = I(a)$~~

~~(4), $\rightarrow$ $\Rightarrow$~~

~~(5) $V_{M,g^x_\alpha}(x=a) = 1$~~

~~(3) $\nexists u \in D: V_{M,g^x_u}(x=a \rightarrow Fxb) = 1$~~

(3) $\exists u \in D: V_{M,g^x_u}(x=a \rightarrow Fxb) = 0$

(1) $\Rightarrow$

(4) $V_{M,g^x_\alpha}(Fxb) = 1$

where $\alpha = I(a)$

$\cancel{\Leftarrow}$ (4), $\rightarrow$ $\Rightarrow$

(5) $V_{M,g^x_\alpha}(x=a \rightarrow Fxb) = 1$

(6) $\forall \beta \neq I(a): V_{M,g^x_\beta}(x=a) = \cancel{*}\,0$

(6), $\rightarrow$ $\Rightarrow$

(7) $\forall \beta \neq I(a): V_{M,g^x_\beta}(x=a \rightarrow Fxb) = 1$

(5), (7) $\Rightarrow$

(8) $\forall u \in D: V_{M,g^x_u}(x=a \rightarrow Fxb) = 1$

(8) $\Rightarrow$

(9) $\nexists u \in D: V_{M,g^x_u}(x=a \rightarrow Fxb) = 0$

(3), (9), reductio $\Rightarrow$

(10) $V_{M,g}(\forall x\,(x=a \rightarrow Fxb)) = \cancel{*}\,1$

(10), conditional proof, reductio, definition of $\cancel{\vDash}$

~~(11) Fab~~ $\vDash_{PC_=}$ $\Rightarrow$

~~(12)~~

(11) $Fab \vDash_{PC_=} \forall x\,(x=a \rightarrow Fxb)$

b consider the ~~follow~~ countermodel $M = \langle D, I \rangle$

$D = \{0, 1, 2\}$

$I(F) = \{0, 1, 2\}$

$I(G) = \varnothing$

$I(H) = \{0, 1, 2\}$

2a ~~$\forall x\,(Px \rightarrow (\exists y\,(x \neq y \wedge Lxy)) \rightarrow$~~

$\forall x\,(Px \rightarrow (\exists y\,(Py \wedge x \neq y \wedge Lxy) \rightarrow \forall z\,(Pz \rightarrow Lxz)))$

P: ... is a person

L: ... loves ...

b $\forall x\,(Gx \leftrightarrow x=a)$

G: ... is a truly great player in the NBA

a: Allan Iverson

c ~~$\forall x\,(Px \rightarrow (\exists y\,(Gy$~~

$\forall x\,(Px \rightarrow (\exists y \exists z\,(Gy \wedge Cz \wedge \cancel{\exists} Sxyz \rightarrow Ixz \wedge Iyz$

$\wedge \sim\exists x'\,(x' \neq x \wedge x' \neq y \wedge Px' \wedge I\cancel{y}x'z)\,\cancel{\wedge \sim\exists x'}\,)))$

P: ... is a person

G: ... is a guard

C: ... is a cell

S: ... and ... share ...

I: ... is in ...

d $\exists x_1 \exists x_2 \exists x_3 \exists x_4 \exists x_5\,(Dx_1 \wedge Dx_2 \wedge Dx_3 \wedge Dx_5$

$\wedge x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge x_1 \neq x_4 \wedge x_1 \neq x_5$

$\wedge x_2 \neq x_3 \wedge x_2 \neq x_4 \wedge x_2 \neq x_5$

$\wedge x_3 \neq x_4 \wedge x_3 \neq x_5$

$\wedge x_4 \neq x_5)$

D: ... is a dinosaur

5a Consider arbitrary PC_ι model $M = \langle D, I \rangle$ and variable assignment g for M.

Suppose for reductio that

~~$V_{M,g}(\forall x L xy\ \forall x\,(L x\,\iota y Fxy \rightarrow \forall x \exists y\,(Lxy) = 0$~~

(1) $V_{M,g}(\forall x\, L x\, \iota y Fxy \rightarrow \forall x \exists y\, Lxy) = 0$

(1), $\rightarrow$ $\Rightarrow$

(2) $V_{M,g}(\forall x\, L x\, \iota y Fxy) = 1$

(3) $V_{M,g}(\forall x \exists y\, Lxy) = 0$

(2), $\forall$ $\Rightarrow$

(4) ~~$V_{M,g}$~~ $\forall u \in D: V_{M,g^x_u}(L x\, \iota y Fxy) = 1$

(4) $\Rightarrow$

(5) $\forall u \in D: \langle |x|^{M,g^x_u}, |\iota y Fxy|^{M,g^x_u} \rangle \in |L|^{M,g^x_u}$

(5) $\Rightarrow$

(6) ~~$\forall u \in D: \langle u, |\iota y Fxy|, v \rangle \in$~~

$\forall u \in D: \exists v \in D: \langle u, v \rangle \in |L|^{M,g^x_u}$

(6) $\Rightarrow$

(7) $\forall u \in D: \exists v \in D: V_{M,g^{x\,y}_{u\,v}}(Lxy) = 1$

(7), $\exists$, $\forall$ $\Rightarrow$

(8) ~~$\forall x \exists y$~~ $V_{M,g}(\forall x \exists y\, Lxy) = 1$

(3), (8), reductio $\Rightarrow$

(9) $V_{M,g}(\forall x\, L x\, \iota y Fxy \rightarrow \forall x \exists y\, Lxy) = 1$

(9), generalisation, definition of $\vDash_{PC_\iota}$ $\Rightarrow$

(10) $\vDash_{PC_\iota} \forall x\, L x\, \iota y Fxy \rightarrow \forall x \exists y\, Lxy$

b consider ...

Suppose for conditional proof that

(1) ~~$F \iota x \forall y$~~ ~~$F\, \iota x \forall y Lxy$~~

$V_{M,g}(F\, \iota x \forall y Lxy) = 1$

Suppose for reductio that

(2) $V_{M,g}(\forall x \forall y\,((\forall z Lxz \wedge \forall z Lyz) \rightarrow x=y)) = 0$

(1) $\Rightarrow$

(3) $|\iota x \forall y Lxy|^{M,g} \in |F|^{M,g}$

(3) $\Rightarrow$

(4) $|\iota x \forall y Lxy|^{M,g}$ is defined

(4) $\Rightarrow$

(5) $\exists$ unique $u \in D: V_{M,g^x_u}(\forall x Lxy) = 1$

(2), $\forall$ $\Rightarrow$

(6) ~~$\exists x$~~ $\exists u \in D: \exists v \in D: V_{M,g^{x\,y}_{u\,v}}(\forall z Lxz \wedge \forall z Lyz \rightarrow x=y)$

$= 0$

(6), $\rightarrow$, $\wedge$ $\Rightarrow$

(7) $\exists u \in D: \exists v \in D: V_{M,g^{x\,y}_{u\,v}}(\forall z Lxz) = 1$,

$V_{M,g^{x\,y}_{u\,v}}(\forall y L\cancel{yy}) = 1$

$V_{M,g^{x\,y}_{u\,v}}(x=y) = 0$

~~(5, 7)~~ (5), (7), reductio.

(8) $V_{M,g}(\forall x \forall y\,((\forall z Lxz \wedge \forall z Lyz) \rightarrow x=y)) = 1$

(8), conditional proof, generalisation, ~~the~~ definition of $\vDash_{PC_i} \rightarrow$

(9) $F\,\iota x \forall y Lxy \vDash_{PC_i} \forall x \forall y ((\forall z Lxz \wedge \forall z Lyz) \rightarrow x = y)$

c Consider the counter model $M = \langle D, I \rangle$

$D = \{0, 1\}$

$I(F) = \{0\}$

$I(G) = \{0, 1\}$

7a $\forall x (Px \wedge Cx \rightarrow W\,\iota y (Jy \wedge Syx))$

P: ... is a person

C: ... ~~is~~ commits a crime

W: ... wears a wig

J: ... is a judge

S: ... sentences ...

b $S\,\iota x (Sx \wedge \forall y (Sy \wedge x \neq y \rightarrow Txy))$

S: ... is a spy

T: ... is taller than ...

8 $\sim H\,\iota x Tx$

H: ... is happy

T: ... is ten feet tall

$\exists x [Tx \wedge \forall x' (Tx' \leftrightarrow x = x') \wedge \sim Hx]$

$\sim \exists x [Tx \wedge \forall x' (Tx' \leftrightarrow x = x') \wedge Hx]$

The latter formalisation is equivalent to the ι formalisation.

The former is true iff there exists a unique ~~te~~ ten foot tall man and he is not happy. The latter is true iff it is not the case that there exists a unique ten foot tall man and he is happy. The two formalisations agree when a unique ten foot tall man exists, but not when he does not, then the former is false but the latter is true.