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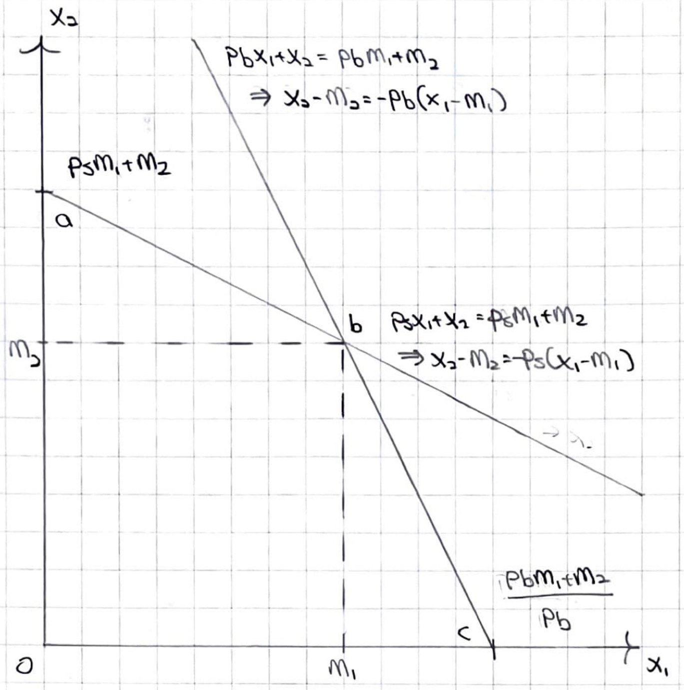

The budget set is represented by area 0abc. p_s is the price, in units of good 2, at which the household can sell good 1. p_b is the price, in units of good 2, at which the household can buy good 1.

The household's optimisation problem is

$\max_{x_1,x_2} \; u(x_1,x_2)=x_1^\beta+x_2^\beta$ st

$g_1(x_1,x_2)=x_1\ge 0$, $g_2(x_1,x_2)=x_2\ge 0$

$g_s(x_1,x_2)=p_sx_1+x_2\le p_sm_1+m_2$

$g_b(x_1,x_2)=p_bx_1+x_2\le p_bm_1+m_2$

where $0<\beta<1$; $m_1,m_2>0$; $0<p_s<p_b$

$$Dg=\begin{pmatrix}\partial g_1/\partial x_1 & \partial g_1/\partial x_2\\ \partial g_2/\partial x_1 & \partial g_2/\partial x_2\\ \partial g_s/\partial x_1 & \partial g_s/\partial x_2\\ \partial g_b/\partial x_1 & \partial g_b/\partial x_2\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\\ p_s & 1\\ p_b & 1\end{pmatrix}$$

~~Suppose that~~ The constraint qualification holds iff the number of binding constraints $e\le 2$. Suppose that both g_s and g_b bind, then, given that $m_1,m_2>0$, $x_1=m_1>0$ and $x_2=m_2>0$, so neither g_1 nor g_2 bind. Suppose that both g_1 and g_2 bind, then $x_1=x_2=0$, $p_sx_1+x_2=0<p_sm_1+m_2$ and $p_bx_1+x_2=0<p_bm_1+m_2$, so neither g_s nor g_b bind. So $e\le 2$, and the constraint qualification holds. The Kuhn-Tucker first-order conditions and complementary slackness conditions are necessary conditions for all optima.

$L(x_1,x_2;\mu_1,\mu_2;\lambda_s,\lambda_b)$
$=x_1^\beta+x_2^\beta+\mu_1x_1+\mu_2x_2$
$-\lambda_s(p_sx_1+x_2-(p_sm_1+m_2))-\lambda_b(p_bx_1+x_2-(p_bm_1+m_2))$

FOCx_1: $\beta x_1^{\beta-1} + \mu_1 - \lambda_s p_s - \lambda_b p_b = 0$

FOCx_2: $\beta x_2^{\beta-1} + \mu_2 - \lambda_s - \lambda_b = 0$

CSμ_1 : $\mu_1 \geq 0$, $x_1 \geq 0$, $\mu_1 x_1 = 0$

CSμ_2 : $\mu_2 \geq 0$, $x_2 \geq 0$, $\mu_2 x_2 = 0$

CSλ_s : $\lambda_s \geq 0$, $p_s x_1 + x_2 \leq p_s m_1 + m_2$,

$\lambda_s(p_s x_1 + x_2 - (p_s m_1 + m_2)) = 0$

CSλ_b : $\lambda_b \geq 0$, $p_b x_1 + x_2 \leq p_b m_1 + m_2$,

$\lambda_b(p_b x_1 + x_2 - (p_b m_1 + m_2)) = 0$

Suppose both g_1 and g_2 bind, i.e. $x_1 = x_2 = 0$. Then, there are zero degrees of freedom, there are no (x_1, x_2) around $(x_1=0, x_2=0)$ such that both these constraints bind, and it is not necessary to verify that the second order condition holds at $(x_1=0, x_2=0)$

Suppose ~~that only~~ g_1 and not g_2 binds, i.e. $x_1 = 0$. Then, $x_1 = 0$, $\mu_1 > 0$, $x_2 > 0$, $\mu_2 = 0$. $u(x_1=0, x_2) = x_2^{\beta}$. ~~By this~~ Given ~~*~~ $0<\beta<1$, by inspection, $u(x_1=0, x_2)$ is increasing in x_2, for all x_2. So $\max_{x_2} = \operatorname{argmax}_{x_2} u(x_1=0, x_2)$. Only $(x_1=0, x_2 = \cancel{p_s}\, p_s m_1 + m_2)$ satisfies the FOCs and CSs.

$$\cancel{at}\ D_x L = (\partial L/\partial x_1, \partial L/\partial x_2) = \cancel{(\beta x_1^{\beta-1} + \mu_1 - \lambda_s p_s}$$
$$= (\beta x_1^{\beta-1} + \mu_1 - \lambda_s p_s - \lambda_b p_b,\ \beta x_2^{\beta-1} + \mu_2 - \lambda_s - \lambda_b)$$

$$D_x^2 L = \begin{pmatrix} \partial^2 L/\partial x_1^2 & \partial^2 L/\partial x_1 \partial x_2 \\ \partial^2 L/\partial x_2 \partial x_1 & \partial^2 L/\partial x_2^2 \end{pmatrix} = \begin{pmatrix} \beta(\beta-1)x_1^{\beta-2} & 0 \\ 0 & \beta(\beta-1)x_2^{\beta-2} \end{pmatrix}$$

The relevant bordered Hessian

$$H = \begin{pmatrix} 0 & 1 & 0 \\ 1 & \beta(\beta-1)x_1^{\beta-2} & 0 \\ 0 & 0 & \beta(\beta-1)x_2^{\beta-2} \end{pmatrix}$$

$$|H| = -1 \begin{vmatrix} 1 & 0 \\ 0 & \beta(\beta-1)x_2^{\beta-2} \end{vmatrix} = -\beta(\beta-1)x_2^{\beta-2} > 0$$

By the bordered Hessian test, $(x_1=0, x_2 = p_s m_1 + m_2)$ is a strict local maximum.

Analogously, supposing that g_2 and not g_1 binds, only $(x_1 = p_b m_1 + m_2 / p_b, x_2 = 0)$ satisfies the FOCs and CSs.

The relevant bordered Hessian

$$H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \beta(\beta-1)x_1^{\beta-2} & 0 \\ 1 & 0 & \beta(\beta-1)x_2^{\beta-2} \end{pmatrix}$$

$$|H| = \begin{vmatrix} 0 & \beta(\beta-1)x_1^{\beta-2} \\ 1 & 0 \end{vmatrix} = -\beta(\beta-1)x_1^{\beta-2} > 0$$

By the bordered Hessian test, $(x_1 = p_b m_1 + m_2 / p_b, x_2 = 0)$ is a strict local maximum.

Suppose only g_s binds, then $x_1, x_2 > 0$, $\mu_1 = \mu_2 = \lambda_b = 0$, $\lambda_s > 0$

$p_s x_1 + x_2 = p_s m_1 + m_2$

$\beta x_1^{\beta-1} - \lambda_s p_s = 0$, $\beta x_2^{\beta-1} - \lambda_s = 0$

$\lambda_s = \beta/p_s\, x_1^{\beta-1} = \beta x_2^{\beta-1}$, $x_1^{\beta-1} = p_s x_2^{\beta-1}$, $x_1 = p_s^{1/\beta-1} x_2$

$p_s p_s^{1/\beta-1} x_2 + x_2 = p_s m_1 + m_2$

$x_2 = (p_s m_1 + m_2)/(p_s^{\beta/\beta-1} + 1)$

$x_1 = (p_s m_1 + m_2)/\cancel{p_s}(p_s + p_s^{-1/\beta-1})$

The relevant bordered Hessian

$$H = \begin{pmatrix} 0 & p_s & 1 \\ p_s & \beta(\beta-1)x_1^{\beta-2} & 0 \\ 1 & 0 & \beta(\beta-1)x_2^{\beta-2} \end{pmatrix}$$

$$|H| = -p_s \begin{vmatrix} p_s & 0 \\ 1 & \beta(\beta-1)x_2^{\beta-2} \end{vmatrix} + \begin{vmatrix} p_s & \beta(\beta-1)x_1^{\beta-2} \\ 1 & 0 \end{vmatrix}$$

$$= -p_s^2\beta(\beta-1)x_2^{\beta-2} - \beta(\beta-1)x_1^{\beta-2} > 0$$

By the bordered Hessian test, $\left(x_1 = (p_s m_1+m_2)/(p_s+p_s^{-1/\beta-1}),\ x_2 = (p_s m_1+m_2)/(p_s^{\beta/\beta-1}+1)\right)$ is a strict local maximum.

Suppose only g_b binds, then $x_1, x_2 > 0$, $\mu_1=\mu_2=\lambda_s=0$, $\lambda_b>0$

$p_b x_1 + x_2 = p_b m_1 + m_2$

$\beta x_1^{\beta-1} - \lambda_b p_b = 0$, $\beta x_2^{\beta-1} - \lambda_b = 0$

$\lambda_b = 1/p_b\ \beta x_1^{\beta-1} = \beta x_2^{\beta-1}$, $x_1^{\beta-1} = p_b x_2^{\beta-1}$, $x_1 = p_b^{1/\beta-1} x_2$

$p_b^{\beta/\beta-1} x_2 + x_2 = p_b m_1 + m_2$

$x_2 = (p_b m_1+m_2)/(p_b^{\beta/\beta-1}+1)$

$x_1 = (p_b m_1+m_2)/(p_b+p_b^{-1/\beta-1})$

By an analogous bordered Hessian test, $\left(x_1 = (p_b m_1+m_2)/(p_b+p_b^{-1/\beta-1}),\ x_2 = (p_b m_1+m_2)/(p_b^{\beta/\beta-1}+1)\right)$ is a strict local maximum.

Suppose that only g_s and g_b bind, then $x_1 = m_1$ and $x_2 = m_2$ and it is not necessary to verify the SOC ~~are~~ is satisfied.

The candidate optima are

$(x_1=0,\ x_2=0)$ ①

$(x_1=0,\ x_2 = p_s m_1+m_2)$ ②

$(x_1 = p_b m_1+m_2/p_b,\ x_2=0)$ ③

$\left(x_1 = (p_s m_1+m_2)/(p_s+p_s^{-1/\beta-1}),\ x_2 = (p_s m_1+m_2)/(p_s^{\beta/\beta-1}+1)\right)$ ④

$\left(x_1 = (p_b m_1+m_2)/(p_b+p_b^{-1/\beta-1}),\ x_2 = (p_b m_1+m_2)/(p_b^{\beta/\beta-1}+1)\right)$ ⑤

$(x_1 = m_1,\ x_2 = m_2)$ ⑥

~~By ins~~ Since u is increasing in x_1, x_2, $(x_1=0, x_2=0)$ is not a maximum.

~~Each of the remaining candidate optima correspond to one of the following points~~

~~It can be proven graphically that~~ $(x_1=0,\ x_2=p_s m_1+m_2)$ and $(x_1 = p_b m_1+m_2/p_b,\ x_2=0)$ are not optima because at these points, $MRS = -MU_1/MU_2 = -(x_1/x_2)^{\beta-1} \neq MRT$, which is ~~either~~ equal to p_s at the former and p_b at the latter.

$MRS|_m = -(m_1/m_2)^{\beta-1}$

~~The optimum is ④ iff~~

The optimum is ④ iff $-MRS|_m < p_s < p_b$

The optimum is ⑤ iff $-MRS|_m > p_b > p_s$

The optimum is ⑥ iff $p_s < MRS|_m < p_b$