Microeconomic Analysis Problem set 4

1a Let x_C denote the contestant's initial choice, ~~and~~ x_H denote the host's choice, i.e. the door the host opens, and x_P denote the prize choice, i.e. the door that hides the prize.

Suppose that $x_P = A$, suppose further that $x_C = A$, then ~~the~~ $x_H = B$ with probability ½ and $x_H = C$ with probability ½, i.e. ~~$P(x_P = B)$~~
$P(x_H = B \mid x_P = A, x_C = A) = 1/2$ and $P(x_H = C \mid x_P = A, x_C = A) = 1/2$

~~Suppose~~ continue to suppose that $x_P = A$, suppose further instead that ~~$x_P =$~~ $x_C = B$, then $x_H = C$ with certainty, i.e. $P(x_H = C \mid x_P = A, x_C = B) = 1$ ~~and $P(x_H =$~~. By symmetry, $P(x_H = B \mid x_P = A, x_C = C) = 1$.

~~continue to suppose that $x_P = A$.~~
~~$P(x$~~
~~$P(x_P = A \mid x_C = B, x_H = C)$~~
~~$= P(x_P = A \cap x_C = B \cap x_H = C) / P(x_C = B$~~
~~suppose that~~
~~$P(x_P = A \mid x_C = A, x_H = B)$~~
~~$= P(x_P = A \cap x_H = B \mid x_C = A) / P(x_H = B \mid x_C = A)$~~
~~$= 1/6 \,/\, 1/2$~~
~~$= 1/3$~~

~~$P(x_P = C \mid x_C = A, x_H = B)$~~
~~$= P(x_P = C \cap x_H = B \mid x_C = A) / P(x_H = B \mid x_C = A)$~~
~~$= 1/3 \,/\, 1/2$~~
~~$= 2/3$~~

$P(x_P = A \mid x_C = A, x_H = B)$
$= P(x_P = A \cap x_H = B \mid x_C = A) / P(x_H = B \mid x_C = A)$
$= \cancel{(1/3 \times 1/2) / 1/2}$
$P(x_P = A \mid x_C = A) P(x_H = B \mid x_P = A, x_C = A) / \cancel{P(x_H = B, x_C = A)/P}$ $P(x_H = B \mid x_C = A)$
$= (1/3 \times 1/2) / (1/2)$
where $P(x_H = B \mid x_C = A) = 1/2$ follows by symmetry from B's being chosen arbitrarily, ~~so~~ i.e. $P(x_H = C \mid x_C = A) = P(x_H = B \mid x_C = A)$ and given $P(x_H = A \mid x_C = A) = 0$, $P(x_H = B \mid x_C = A) = P(x_H = C \mid x_C = A) = 1/2$
$= 1/3$

$P(x_P = C \mid x_C = A, x_H = B)$
$= 1 - P(x_P = A \mid x_C = A, x_H = B)$
given that $P(x_P = B \mid x_C = A, x_H = B) = 0$ ~~and $x_P \in$~~
(and $x_P \in \{A, B, C\}$)
$= 2/3$

Since ~~x_C, x_H, and x_P~~ x_C and x_H were chosen arbitrarily, by generalization,
$P(x_P = x_C \mid x_C, x_H \neq x_C) = 1/3$
$P(x_P \neq x_C \mid x_C, x_H \neq x_C) = 2/3$
Switching doubles the contestant's probability of winning the prize.

Takeaway: probability models are subjective, the explanation in b is not strictly wrong. If ~~#~~ contestant always picks X and host has a choice rule: open left-most door. Specifically C strategy is to choose A, H strategy is to go left to right. Switch is only optimal if H chooses randomly. No particular objective is consistent with some rule.

The host's opening one door "collapses" the probability mass that the prize is behind ~~that door into the~~ ~~one of~~ the two unchosen doors onto the remaining unchosen and unopened door, rather than distributing the probability that the prize is behind the opened door evenly between the two unopened doors.

b $x_H = B$ is more informative than merely revealing that $x_P \neq B$. The probability that $x_H = B$ given that ~~x_A~~ $x_C = A$ ~~is~~ and $x_P = A$ is $1/2$ while the probability that $x_H = B$ given that $x_C = A$ and $x_P = C$ is 1. Given that ($x_C = A$ and $x_P = A$) and ($x_C = A$ and $x_P = C$) ~~are equally likely~~ have equal prior probability, by Bayes rule, $x_H = B$ should result ~~in a positive~~ an increase in the posterior probability of $x_P = C$ and a decrease in the (relative) posterior probability of $x_P = A$

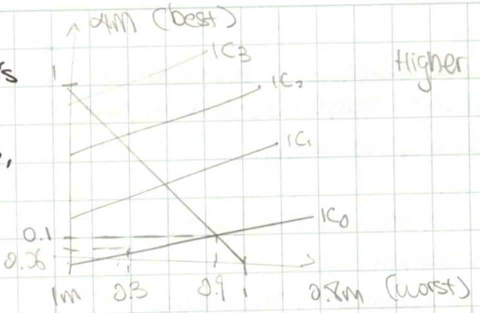
2a let $L_1 = [0.9, 0.1; 0.8, 4]$, $L_2 = [1; 1]$

let u ~~de~~ be the utility representation of ~~A's~~ A's preferences $\succsim$ over lotteries.

Given that A has expected utility preferences,
$u(L) = \sum_{i=1}^{n} p_i u(x_i)$, where $L = [p_1, \dots, p_n; x_1, \dots, x_n]$

$u(L_2) = u(L_1) = 0.9u(0.8) + 0.1u(4) = 0.1$

$u(L_2) = 1u(1) = 0.1$, $u(1) = 0.1$



Higher is better

a. find expected value of the [0.9, 0.1; 0.8, 4] then show RP

b ~~$4u(1) = 0.4 < u(4) = 1$~~

$\frac{15}{16} u(0.8) + \frac{1}{16} u(4) = \frac{1}{16}$

$u(\frac{15}{16} \times 0.8 + \frac{1}{16} \times 4) = u(1) = 0.1$ ~~$\frac{1}{16}$~~

$\frac{15}{16} u(0.8) + \frac{1}{16} u(4) < u(\frac{15}{16} 0.8 + \frac{1}{16} 4)$

~~u is strictly concave, A is risk averse~~

consider $L' = [\frac{15}{16}, \frac{1}{16}; 0.8, 4]$

$EV(L') = \frac{15}{16} 0.8 + \frac{1}{16} 4 = 1$

~~From (a), $u(L') = \frac{1}{16}$~~

$u(L') = \frac{1}{16} < u([1; EV(L')]) = 0.1$

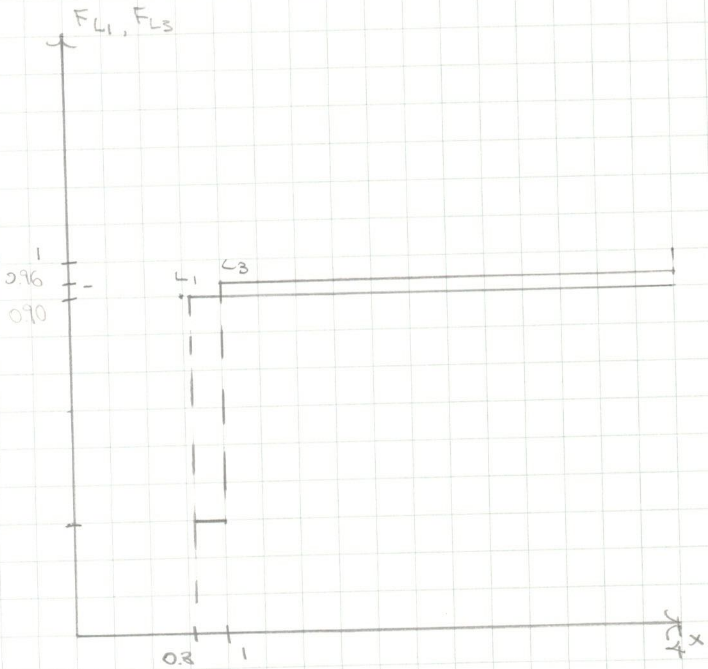
$[1; EV(L')] \succ L'$

~~It is not possible to tell whether A is risk averse or not since~~

Assuming that A's Bernoulli utility is well-behaved, ~~A is ris~~ it is strictly concave, and A is risk averse.

is this assumption implicit in EU preferences?

c let $L_3 = [0.3, 0.64, 0.06; 0.8, 1, 4]$



By inspection of the cumulative distribution functions F_{L_1} and F_{L_3}, neither ~~$L_1 \succsim_{FOSD}$~~ $L_1 \succsim_{FOSD} L_3$ nor $L_3 \succsim_{FOSD} L_1$ since neither F_{L_1} lies entirely below F_{L_3} nor F_{L_3} lies entirely below F_{L_1}. $L_3 \succsim_{SOSD} L_1$ since F_{L_3} crosses F_{L_1} once from below.

Equal EV : no FOSD

write that one is a mean-preserving spread and show how

$L_3 \succsim_{SOSD} L_1 \sim L_2$

d let $L_4 = [1-p, p; 1, 4]$ ~~$[p, 1-p]$~~

$u(L_4) = pu(4) + (1-p)u(1) =$ ~~0.4~~ $0.1 + 0.9p$

$u(L_3) = 0.3u(0.8) + 0.64u(1) + 0.06u(4) = 0.064 + 0.06 = 0.124$

$p = 0.024/0.9 = 2/75$

3 ~~a~~

Let F and G be cumulative distribution functions on $[a,b] \subset \mathbb{R}$, i.e. $F(a)=G(a)=0$, ~~F(a)=~~ $F(b)=G(b)=1$, and each of F and G is weakly increasing.

Suppose that $F \succ_{LR} G$ in the domain $[a,b]$
then by definition of LR dominance,
$F'(x)/G'(x)$ is weakly increasing in the domain $[a,b]$
then $\forall$ ~~x<z~~ $x>z \in [a,b]: F'(x)/G'(x) \geq F'(z)/G'(z)$
Given that each of F and G is strictly increasing,
~~F'(x)~~ $\forall x \in \mathbb{R}$, $F'(x) \geq 0$ and $G'(x) \geq 0$.
so $\forall x>z \in [a,b] : F'(x)G'(z) \geq F'(z)G'(x)$
$\int_{z=a}^{x} F'(x)G'(z)\, dz = F'(x)\int_{z=a}^{x} G'(z)\, dz = F'(x)G(x)$
$\int_{z=a}^{x} F'(z)G'(x)\, dz = G'(x)\int_{z=a}^{x} F'(z)\, dz = G'(x)F(x)$
then $F'(x)G(x) \geq G'(x)F(x)$
$\int_{x=z}^{b} F'(x)G'(z)\, dx = G'(z)\int_{x=z}^{b} F(x)\, dx = G'(z)(F(b)-F(z))$
$= G'(z)(1-F(z))$
$\int_{x=z}^{b} G'(x)F'(z)\, dx = F'(z)\int_{x=z}^{b} G'(x)\, dx = F'(z)(G(b)-G(z))$
$= F'(z)(1-G(z))$
then $G'(z)(1-F(z)) \geq F'(z)(1-G(z))$

Rearranging,
$F(x) \leq G(x)\, F'(x)/G'(x)$
$1-F(z) \geq (1-G(z))\, F'(x)/G'(x)$
Suppose that $F'(x)/G'(x) < 1$, then $F(x) < G(x)$
Suppose that $F'(x)/G'(x) = 1$, then
$F(x) \leq G(x)$ and $1-F(x) \geq 1-G(x) \Rightarrow F(x) = G(x)$
Suppose that $F(x)/G'(x) > 1$, then
$1-F(x) \geq 1-G(x) \Rightarrow F(x) < G(x)$
so for all $x \in [a,b]$ $F(x) \leq G(x)$ and for some such x
$F(x) < G(x)$, so $F \succ_{FOSD} G$

LR dominance relevant to principal agent problems for >2 outcomes ~~sa~~.
only if effort yields LR dominance ~~does effort~~ then it is optimal to reward effort
FOSD is not sufficient ~~to reward~~ for it to be optimal to reward effort.

Be explicit about the lemma
Suppose $f(x) \geq g(x)$ for $x \in [a,b]$: $\int_a^b f(x)dx \geq \int_a^b g(x)dx$

4a Let $\succsim$ denote A's preferences over lotteries. Given that A has EU preferences, there is some function U that represents $\succsim$ such that $U(L)=\sum_{i=1}^{n} p_i u(x_i)$ for some u, where $L=[p_1,\dots,p_n; x_1,\dots,x_n]$. Let $L(x)$ denote the lottery given x.

$\cancel{Lx}\ L(x) = [\alpha, 1-\alpha; w+2x, w-x]$

$U(L(x)) = \alpha u(w+2x) + (1-\alpha)u(w-x)$

A's maximisation problem is $\max_x U(L(x))$

FOC: $\partial U(L(x))/\partial x|_{x=x^*} = 2\alpha u'(w+2x^*) - (1-\alpha)u'(w-x^*) = 0$

~~By implicit differentiation wrt α~~

~~$2u'(w+2x^*)$~~

By implicit differentiation wrt α

$2u'(w+2x^*) + 4\alpha u''(w+2x^*)\partial x^*/\partial\alpha$

$+u''(w-x^*)\partial x^*/\partial\alpha$

$+u'(w-x^*) - \alpha u''(w-x^*)\partial x^*/\partial\alpha$

$=0$

Rearranging,

$[4\alpha u''(w+2x^*) + (1-\alpha)u''(w-x^*)]\partial x^*/\partial\alpha$

$= -2u'(w+2x^*) - u'(w-x^*)$

Supposing that A prefers more wealth to less, u is strictly increasing, i.e. ~~$u'(x)>0$~~ $\forall x: u'(x)>0$. Given that A is risk averse, $\forall x: u''(x)<0$. $\alpha\in[0,1]$

Then $\partial x^*/\partial\alpha > 0$

SOC: $\partial^2 U(L(x))/\partial x^2|_{x=x^*} = 4\alpha u''(w+2x^*) + (1-\alpha)u''(w-x^*) < 0$

~~The FOC is satisfied at~~ The FOC is satisfied at a local maximum.

$x^* = \arg\max_x U(L(x))$ increases with increasing α, the higher the value of α, the higher the value x A optimally chooses.

Very important to check SOC
Sometimes adequate to write "I checked SOC" or function is concave in x

b $\forall r\in(0,1)$: CRRA$(r) \Rightarrow u(x) = x^{1-r}$

$u'(x) = (1-r)x^{-r}$

FOC: $\partial U(L(x))/\partial x|_{x=x^*} = 2\alpha(1-r)(w+2x^*)^{-r} - (1-\alpha)(1-r)(w-x^*)^{-r} = 0$

$2\alpha(w+2x^*)^{-r} = (1-\alpha)(w-x^*)^{-r}$

$(2\alpha)^{-1/r}(w+2x^*) = (1-\alpha)^{-1/r}(w-x^*)$

$[2(2\alpha)^{-1/r} + (1-\alpha)^{-1/r}]x^* = [-(2\alpha)^{-1/r} + (1-\alpha)^{-1/r}]w$

$x^* = [(1-\alpha)^{-1/r} - (2\alpha)^{-1/r}]w / [(1-\alpha)^{-1/r} + 2(2\alpha)^{-1/r}]$

~~$\uparrow r \Rightarrow \downarrow 1/r \Rightarrow \uparrow -1/r \Rightarrow \uparrow(1-\alpha)^{-1/r}, \uparrow(2\alpha)^{-1/r}$~~

$u''(x) = -r(1-r)x^{-1-r} < 0$, so the SOC holds and the FOC is satisfied at a local maximum.

For $\alpha \leq 1/3$, $(w+2x^*)^{-r} \geq (w-x^*)^{-r}$, $w+2x^* \leq w-x^*$, $x^* \leq 0$.

Given that $x\in[0,w]$, for $\alpha<1/3$, A's maximisation problem has a corner solution, $x^*=0$. For such α, x^* (weakly) ~~decreases~~ decreases with increasing r.

This is critical

For $\alpha > 1/3$, ~~$(w+2x^*)^{-r} < (w-x^*)^{-r}$~~

$(2\alpha)^{-1/r}(w+2x^*) = (1-\alpha)^{-1/r}(w-x^*)$,

$w+2x^*/w-x^* = \cancel{(2\alpha/1-\alpha)^{-1/r}}\ (1-\alpha/2\alpha)^{-1/r}$

~~$2\alpha/1-\alpha > 1$~~, $1-\alpha/2\alpha < 1$

~~so $\uparrow r \Rightarrow \downarrow 1/r \Rightarrow \uparrow -1/r \Rightarrow \uparrow(2\alpha/1-\alpha)^{-1/r} \Rightarrow \uparrow w+2x^*/w-x^* \Rightarrow \uparrow x^*$~~

~~x^*~~

so $\uparrow r \Rightarrow \downarrow 1/r \Rightarrow \uparrow -1/r \Rightarrow \downarrow(1-\alpha/2\alpha)^{-1/r} \Rightarrow \downarrow w+2x^*/w-x^* \Rightarrow \downarrow x^*$

x^* strictly decreases with increasing r

CRRA(1) $\Rightarrow u(x) = \ln x$

$u'(x) = \cancel{\#\#}\, 1/x, \quad u''(x) = -x^{-2} < 0$ for all x

FOC: $\cancel{\partial(u}\ \partial u(c(x))/\partial x^* \big|_{x=x^*} = 2\alpha/(w+2x^*) - (1-\alpha)/(w-x^*) = 0$

$2\alpha(w-x^*) = (1-\alpha)(w+2x^*)$

$\cancel{-2\alpha x^* + 2\alpha x^* = -2\alpha w + (1-\alpha)w}$

$-2\alpha x^* - (1-\alpha)2x^* = -2\alpha w + (1-\alpha)w$

$-2x^* = (1-3\alpha)w$

$x^* = \frac{3\alpha-1}{2} w$

As $r \to 1$ from 0, $-1/r \to -1$ from $-\infty$

$x^* = [(1-\alpha)^{-1/r} - (2\alpha)^{-1/r}]w / [(1-\alpha)^{-1/r} + 2(2\alpha)^{-1/r}]$

$\to [(1-\alpha)^{-1} - (2\alpha)^{-1}]w / [(1-\alpha)^{-1} + 2(2\alpha)^{-1}]$

$= [2\alpha - (1-\alpha)/2\alpha(1-\alpha)]w / [2\alpha + 2(1-\alpha)/2\alpha(1-\alpha)]$

$= \frac{3\alpha-1}{2} w$

~~So x* decreases with increasing r for~~

So $x^*(r)$ is continuous ~~for $r \in (0,1]$.~~ over the domain $r \in (0,1]$, and x^* decreases with increasing r in this domain.

5a ~~$V(\alpha)=$~~

Let F_X and F_Y denote the cdf of X and the cdf of Y respectively.

~~$V(\alpha) = \int$~~

~~Let D_X and D_Y~~ Let D_X and D_Y denote the domains of F_X and the domain of F_Y respectively.

~~$V(\alpha) = \int_{x\in D_X} \int_{y \in D_Y} \alpha x + (1-\alpha) y F_X'(x)dx F_Y'(y)$~~

~~$V(\alpha) = \int_{x\in D_X} \int_{y\in D_Y} \alpha x + (1-\alpha)y F_X'(x)dx F_Y'(y) dy$~~

~~$V'(\alpha)=0$~~

$V(\alpha) = \int_{x\in D_X} \int_{y\in D_Y} u(\alpha x + (1-\alpha)y) F_X'(x) dx F_Y'(y) dy$

Alternate notation in expectations

$V'(\alpha) = 0$

b $V'(\alpha) = \int_x \int_y u'(\alpha x + (1-\alpha)y)(x-y) F_X'(x) dx F_Y'(y) dy$

$= \int_x \int_y xu'(\alpha x + (1-\alpha)y) - yu'(\alpha x + (1-\alpha)y) F_X'(x) dx F_Y'(y) dy$

$= E(xu'(\alpha x + (1-\alpha)y) - yu'(\alpha x + (1-\alpha)y))$

$= E(xu'(\alpha x + (1-\alpha)y)) - E(yu'(\alpha x + (1-\alpha)y))$

All x and y from here on should instead be X and Y respectively

Suppose ~~for reductio~~ that $\alpha = 0$

then $V'(\alpha) = E(xu'(y)) - E(yu'(y))$

$=_1 E(x)E(u'(y)) - E(yu'(y))$

~~since x and y are independently distributed~~

$>_2 E(x)E(u'(y)) - E(y)E(u'(y))$

~~since the agent is risk-averse so $u''(y)<0$ then~~

~~$u'(y)$ is a decreasing function of y.~~

$=_3 0$

where $=_1$ follows from $X \perp Y$, which implies $X \perp u'(Y)$, $>_2$ follows from the agent's being risk averse which implies $u''<0$, $u'(Y)$ is a decreasing function of Y, $\operatorname{cov}(Y, u'(Y)) = E(Yu'(Y)) - E(Y)E(u'(Y)) < 0$, and $=_3$ follows from $E(X) = E(Y)$.

~~By re-~~ Consequently, if $\alpha = 0$, $V'(\alpha) \neq 0$

By symmetry, if $\alpha = 1$, $V'(\alpha) \neq 0$

Then, $\alpha \in (0,1)$.

Technique ~~$V'(\alpha)>0$, $V'(\alpha)$~~ $V'(0)>0$, $V'(1)<0$. By continuity of V' in α, $V'(\alpha)=0$ for some $\alpha\in(0,1)$

5a ~~There is a discounted utility model that explains the given preferences since the given preferences do not violate any of completeness, transitivity, continuity, monotonicity in the first parameter and stationarity.~~

There is no discounted utility model that explains the given preferences since the given preferences violate stationarity. $(9,1) \sim (4,0)$ but $(9,2) \succ (4,1)$

b $u(9,1) = u(4,0)$

$\frac{1}{1+\gamma}\sqrt{9} = \frac{1}{1+\gamma 0}\sqrt{4}$

$\frac{3}{1+\gamma} = \frac{2}{2}$

$\gamma = 2$

Verifying that $u(x,t) = \frac{1}{1+\gamma t}u(x)$ represents the given preferences

$u(9,0) = 3/2$

$u(9,1) = 3/3 = 1$

$u(4,0) = 2/2 = 1$

$u(9,2) = 3/5$

$u(4,1) = 2/3$

$u(4,2) = 2/5$

$u(1,0) = 1/2$

$u(1,1) = 1/3$

$u(1,2) = 1/5$

Useful in some case to suppose wlog that $u(\text{best}) = 1$, $u(\text{worst}) = 0$

$u(4,2) = u(1,0)$

$\frac{1}{1+\gamma^2}\sqrt{4} = \frac{1}{1+\gamma 0}\sqrt{1}$

$\frac{2}{1+\gamma^2} = \frac{1}{2}$

$1+\gamma^2 = 4$, $\gamma^2 = 3$, $\gamma = \sqrt{3}$

Hyperbolic discounting, non geometric

"tmr feels closer to today than the day after feels to tmr"

~~it does not represent the given preferences~~

misunderstanding γ times t not γ^t, then $\gamma = \frac{1}{2}$