Microeconomic Analysis Paper 180607

1a i $\vec{u}_1 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \vec{u}_2 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, \vec{u}_3 = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}, \vec{u}_4 = \begin{pmatrix} -6 \\ 5 \\ 1 \end{pmatrix}$

Let U be the matrix formed by these vectors. Compute the rank of U by Gauss-Jordan elimination.

$$\begin{pmatrix} -1 & 3 & 1 & -6 \\ 2 & 1 & 5 & 5 \\ 1 & 2 & 4 & 1 \end{pmatrix} \xrightarrow[R_3 \leftarrow R_3 + R_1]{R_2 \leftarrow R_2 + 2R_1} \begin{pmatrix} -1 & 3 & 1 & -6 \\ 0 & 7 & 7 & -7 \\ 0 & 5 & 5 & -5 \end{pmatrix}$$

$$\xrightarrow[R_3 \leftarrow R_3 - 5/7 R_2]{R_2 \leftarrow 1/7 R_2} \begin{pmatrix} -1 & 3 & 1 & -6 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

There are two non-zero rows in this row echelon matrix, ~~#~~ U has rank 2.

At most two of these vectors are linearly independent. Three linearly independent vectors are necessary to span $\mathbb{R}^3$. These do not span $\mathbb{R}^3$.

ii $\vec{v}_\alpha = \begin{pmatrix} -5 \\ 3 \\ \alpha \end{pmatrix}$

At most two of the four vectors are linearly independent. By inspection, $\vec{u}_1$ and $\vec{u}_2$ are linearly independent, so each of $\vec{u}_3$ and $\vec{u}_4$ is a linear combination of $\vec{u}_1$ and $\vec{u}_2$. Then, span $[\vec{u}_1, \dots, \vec{u}_4]$ = span $[\vec{u}_1, \vec{u}_2]$. So $\vec{v}_\alpha \notin$ span $[\vec{u}_1, \dots, \vec{u}_4]$ iff $\vec{v}_\alpha \notin$ span $[\vec{u}_1, \vec{u}_2]$ which is iff $\vec{v}_\alpha$ is linearly independent of $\vec{u}_1$ and $\vec{u}_2$ which is iff the matrix whose columns are $\vec{v}_\alpha, \vec{u}_1, \vec{u}_2$ is full rank, which is iff that matrix has non-zero determinant.

$$\det \begin{pmatrix} -5 & -1 & 3 \\ 3 & 2 & 1 \\ \alpha & 1 & 2 \end{pmatrix} = \alpha \det \begin{pmatrix} -1 & 3 \\ 2 & 1 \end{pmatrix} - 1 \det \begin{pmatrix} -5 & 3 \\ 3 & 1 \end{pmatrix} + 2 \det \begin{pmatrix} -5 & -1 \\ 3 & 2 \end{pmatrix}$$
$$= \alpha(-7) - 1(-14) + 2(-7)$$
$$= -7\alpha$$
$$= 0 \Leftrightarrow \alpha = 0$$

$\vec{v}_\alpha \in$ span $[\vec{u}_1, \dots, \vec{u}_4]$ iff $\alpha = 0$

$\vec{v}_0 = \vec{u}_4 - \vec{u}_1$

iii By inspection, $\vec{u}_1$ and $\vec{u}_4$ are linearly independent. From the above, span $[\vec{u}_1, \vec{u}_4]$ = span $[\vec{u}_1, \dots, \vec{u}_4]$ = span $[\vec{u}_1, \vec{u}_2]$, so $\vec{v}_\alpha$ is ~~iff~~ linearly independent of $\vec{u}_1, \vec{u}_4$ iff $\vec{v}_\alpha \notin$ span $[\vec{u}_1, \vec{u}_4]$ = span $[\vec{u}_1, \vec{u}_2]$ which is iff (from above) $\alpha \neq 0$. So $\vec{v}_0$ is ~~iff~~ not linearly independent of $\vec{u}_1, \vec{u}_4$ but $\vec{v}_1$ is, so ~~th~~ $\{\vec{u}_1, \vec{u}_4, \vec{v}_1\}$ spans $\mathbb{R}^3$ and is a basis of $\mathbb{R}^3$ but $\{\vec{u}_1, \vec{u}_4, \vec{v}_0\}$ does not and is not. By inspection, $\vec{v}_0$ and $\vec{v}_1$ are linearly ~~inde~~ independent and each of $\vec{u}_1$ and $\vec{u}_4$ is linearly independent of $\vec{v}_0$ and $\vec{v}_1$ (their first two values are not in a $-5/3$ ratio. So each of $\{\vec{u}_1, \vec{v}_0, \vec{v}_1\}$ and $\{\vec{u}_4, \vec{v}_0, \vec{v}_1\}$ spans $\mathbb{R}^3$ and is a basis of $\mathbb{R}^3$. Three linearly independent vectors are necessary to span $\mathbb{R}^3$ so ~~each of~~ there are no other possibilities.

Any ~~subset of~~ three-membered subset except $\{\vec{u}_1, \vec{u}_4, \vec{v}_0\}$ is a basis of $\mathbb{R}^3$.

6 $f(x,y) = \begin{cases} (x-y)^2/x^2+y^2 & \text{for } (x,y) \neq (0,0) \\ 1 & \text{otherwise} \end{cases}$

For $(x,y) \neq (0,0)$, partial derivatives exist and can be found by direct differentiation.

$$f_x(0,0) = \lim_{h \to 0} f(h,0) - f(0,0)/h$$
$$= \cancel{[h^2/h^2 - 1]/h} \lim_{h \to 0} [h^2/h^2 - 1]/h$$
$$= \lim_{h \to 0} 0$$
$$= 0$$

$$f_y(0,0) = \lim_{h \to 0} f(0,h) - f(0,0)/h$$
$$= \lim_{h \to 0} [h^2/h^2 - 1]/h$$
$$= \lim_{h \to 0} 0$$
$$= 0$$

Partial derivatives of f exist at all points in the domain of f.

$(1/n, 1/n)$ converges to $(0,0)$ as n becomes large. $f(1/n, 1/n) = 0$ fails to converge to $f(0,0) = 1$ as n becomes large. f is not continuous, let alone differentiable or C^1.

It is not possible to apply the IFT at the origin because f is not continuous at the origin, so it is not C^1 in an open ball around the origin, which is necessary for the IFT to be applicable at the origin.

Date

2ai Real-valued function $f: S \to \mathbb{R}$ where $S \subseteq \mathbb{R}^n$ is concave iff $\forall \vec{x}, \vec{x}' \in S$, $\forall t \in (0,1)$:
$f(t\vec{x} + (1-t)\vec{x}') \geq tf(\vec{x}) + (1-t)f(\vec{x}')$

An optimisation problem is concave iff it has the following form.
$\max_{\vec{x}} f(\vec{x})$ s.t. $\vec{g}(\vec{x}) \leq \vec{b}$, ~~where~~ where f is concave, and each of $g_1, \ldots, g_m$ is convex (~~a~~ the definition of a convex function is obtained by reversing the sign of the inequality ~~above~~ in the definition above).

If an optimisation problem is concave, then the K-T FOCs are sufficient for ~~an optimisation~~ maximum. If, in addition, the constraint set is non-empty, then the K-T FOCs are also necessary.

An optimisation problem is convex iff it has the following form.
$\min_{\vec{x}} f(\vec{x})$ s.t. $\vec{g}(\vec{x}) \geq \vec{b}$, where f is convex and each of $g_1, \ldots, g_m$ is concave

If an optimisation problem is convex, then again the K-T FOCs are sufficient for a minimum.

ii Given concave f ~~g~~ and g, the function $f+g$ is concave.

Concave $f \Rightarrow \forall \vec{x}, \vec{x}' \in \mathbb{R}^n, \forall t \in (0,1)$:
$f(t\vec{x} + (1-t)\vec{x}') \geq tf(\vec{x}) + (1-t)f(\vec{x}')$
Concave $g \Rightarrow \forall \vec{x}, \vec{x}' \in \mathbb{R}^n, \forall t \in (0,1)$:
$g(t\vec{x} + (1-t)\vec{x}') \geq tg(\vec{x}) + (1-t)g(\vec{x}')$
$\Rightarrow$
$\forall \vec{x}, \vec{x}' \in \mathbb{R}^n, \forall t \in (0,1)$: $f(t\vec{x} + (1-t)\vec{x}') + g(t\vec{x} + (1-t)\vec{x}') \geq t(f(\vec{x}) + g(\vec{x})) + (1-t)(f(\vec{x}') + g(\vec{x}'))$
$\Rightarrow$
$\forall \vec{x}, \vec{x}' \in \mathbb{R}^n, \forall t \in (0,1)$: $(f+g)(t\vec{x} + (1-t)\vec{x}') \geq t(f+g)(\vec{x}) + (1-t)(f+g)(\vec{x}')$
$\Rightarrow$ $f+g$ is concave.

Concavity of f and g do not imply concavity of $f-g$.

Counterexample: $f(x) = -x^2/2$, $g(x) = -x^2$
$\Rightarrow (f-g)(x) = x^2/2$

Concavity of f and g do not imply concavity of fg.

Counterexample: $f(x) = x^{2/3}$, $g(x) = x^{2/3}$,
$(fg)(x) = x^{4/3}$

bi $f(x,y) = \alpha \ln x + \beta \ln y$ $\quad (x, y > 0)$
$Df(x,y) = \begin{pmatrix} \alpha/x & \beta/y \end{pmatrix}$
$D^2f(x,y) = \begin{pmatrix} -\alpha x^{-2} & 0 \\ 0 & -\beta y^{-2} \end{pmatrix}$

$\text{tr}\, D^2f(x,y) = -\alpha x^{-2} - \beta y^{-2} < 0$
$\det D^2f(x,y) = \alpha\beta x^{-2} y^{-2} > 0$
Both eigenvalues of the Hessian are negative, the Hessian is negative definite, f is concave.

ii $g(x,y) = x^\alpha y^\beta$
$Dg(x,y) = \begin{pmatrix} \alpha x^{\alpha-1} y^\beta & \beta x^\alpha y^{\beta-1} \end{pmatrix}$
$D^2g(x,y) = \begin{pmatrix} \alpha(\alpha-1)x^{\alpha-2}y^\beta & \alpha\beta x^{\alpha-1}y^{\beta-1} \\ \alpha\beta x^{\alpha-1}y^{\beta-1} & \beta(\beta-1)x^\alpha y^{\beta-2} \end{pmatrix}$
$\text{tr}\, D^2g(x,y) = \alpha(\alpha-1)x^{\alpha-2}y^\beta + \beta(\beta-1)x^\alpha y^{\beta-2}$
$= x^\alpha y^\beta [\alpha(\alpha-1)x^{-2} + \beta(\beta-1)y^{-2}]$
~~$\det D^2g(x,y) =$~~
$\text{tr}\, D^2g(x,y) > 0$ for some x, y (which depends on the magnitude of α, β), then for such x, y, at least one eigenvalue of the Hessian is positive, so the Hessian is not negative definite or negative semi-definite, so g is not concave.

iii $h(x,y) = -2x^2 + kxy - 3y^2$
$Dh(x,y) = \begin{pmatrix} -4x + ky & -6y + kx \end{pmatrix}$
$D^2h(x,y) = \begin{pmatrix} -4 & k \\ k & -6 \end{pmatrix}$

$\text{tr}\, D^2h(x,y) = -10 < 0$
$\det D^2h(x,y) = 24 - k^2 \geq 0 \Leftrightarrow k^2 \leq 24 \Leftrightarrow$
$k \in [-\sqrt{24}, \sqrt{24}]$
For such k (and only such k), both eigenvalues of the Hessian are ~~negative~~ weakly negative, the Hessian is negative definite or negative semi-definite, and ~~the~~ h is concave.

ci $MU_x = \partial u/\partial x = \delta/x$
$MU_y = \partial u/\partial y = 1-\delta/y$
Marginal utilities approach infinity as the amount of each respective good approaches ~~a~~ zero, so the positivity constraints will not bind.

ii $\max_{x,y}$ ~~s.t.~~ $\delta \ln x + (1-\delta)\ln y$ s.t.
BC: $p_x x + p_y y \leq m$
$\mathcal{L} = \delta \ln x + (1-\delta)\ln y - \lambda_B(p_x x + p_y y - m)$
FOC_x: $\delta/x - \lambda_B p_x = 0$
FOC_y: $1-\delta/y - \lambda_B p_y = 0$
FOC_{λ_B}: $p_x x + p_y y - m = 0$
$FOC_x \Rightarrow x = 1/\lambda_B \; \delta/p_x$
$FOC_y \Rightarrow y = 1/\lambda_B \; 1-\delta/p_y$
$FOC_{\lambda_B} \Rightarrow p_x x + p_y y = m$
In words, expenditure on each good is proportionate to the respective coefficient and income is exhausted. This is the familiar

cobb-Douglas result.
$x^* = \delta m/p_x, \quad y^* = (1-\delta)m/p_y$

iv From earlier result,
$x_A^F = \frac{4}{5}100/2 = 40, \quad y_A^F = \frac{1}{5}100/1 = 20$
(Multiplying utility by 1/5 is a monotonic transformation that does not affect the optimum).
$x_B^F = \frac{2}{5}100/2 = 20, \quad y_B^F = \frac{3}{5}100/1 = 60$

$x_A^T = \frac{4}{5}180/2 = 72, \quad y_A^T = \frac{1}{5}180/3 = 12$
$x_B^T = \frac{2}{5}180/2 = 36, \quad y_B^T = \frac{3}{5}180/3 = 36$

~~$u^A(x^A, y^A$~~
$u^A(x_A^F, y_A^F) = 4\ln 40 + \ln 20 = \ln 40^4 20$
$= \ln 512\times 10^5$
$u^A(x_A^T, y_A^T) = 4\ln 72 + \ln 12 = \ln 72^4 12$
$= \ln 322486272$
$< u^A(x_A^F, y_A^F)$

$u^B(x_B^F, y_B^F) = 3\ln 20 + 2\ln 60 = \ln 20^3 60^2$
$= \ln 288\times 10^5$
$u^B(x_B^T, y_B^T) = 3\ln 36 + 2\ln 36 = \ln 36^5$
$= \ln 60466176$ ~~$= u^B$~~
$> u^B(x_B^F, y_B^F)$

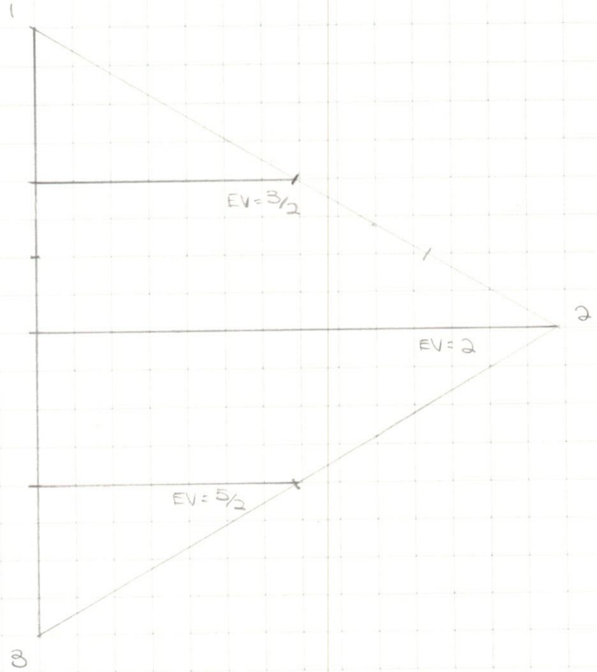
~~Both are better off under only TC rather than under only BC~~ A better off under only FC than under only TC because the TC ~~bun~~ optimal bundle is ~~available~~ affordable but not chosen under FC. B better off under TC than under FC because the FC ~~bun~~ optimal bundle is affordable ~~under TC~~ but not chosen under TC.

v From the above, A is made worse off because A's TC-optimal bundle is no longer affordable, B is made no worse off because B's TC-optimal bundle remains affordable.

vi From the above, B is made worse off because B's FC-optimal bundle is no longer affordable, A is made no worse off because A's FC-optimal bundle remains affordable.

Necessary to relax TC such that B's FC-optimal bundle just remains affordable under TC. $2(20) + 3(60) = 220$, ~~the diff~~ TC must be relaxed by 40 units. This does not benefit A because ~~A~~ FC binds for A, and TC does not initially bind for A.

4ai



Lotteries with the same mean lie on the same horizontal (in the above diagram) line. For ~~examp~~ example, for all lotteries L (with these three outcomes) such that $EV(L) = 3/2$, one point of this line lies exactly halfway along the 12 edge, the other lies exactly $1/4$ ~~along~~ of the way from 1 to 3. Trigonometrically, it can be proved that all such lines are horizontal.

ii The only possible mean preserving spread of any lottery with exactly these three outcomes is by reallocating probability mass from outcome 2 evenly between outcomes 1 and 3. If L' is a mean-preserving spread of L, it lies on the same horizontal ~~EQV~~ iso-EV line, but lies to the left of L, i.e. further from the 2 vertex but closer to each of the 1 and 3 vertices.

iii For a risk-neutral agent, indifference curves are simply iso-EV lines. So the indifference curve through any given lottery L is simply the horizontal (in the above diagram) line through L. A risk-neutral EU maximiser ~~#~~ has zero risk premium hence CE equal to EV, hence constant EU = u(CE) for all lotteries with equal EV. Equivalently, a risk-neutral EU maximiser has linear Bernoulli utility hence EU equal to some affine transformation of EV, and so is indifferent between lotteries with equal EV. The slope of the indifference curve through L is horizontal.

For a risk-averse agent, indifference curves are upward-sloping and lotteries on lower (i.e. closer to the ~~23 edge~~ 3 vertex and further from the 1 vertex) indifference curves are preferred to lotteries on higher indifference curves.

Consider arbitrary lottery L. Lotteries L' to the left of L are mean-preserving spreads of L. These have the same expected value but higher risk premium for risk averse expected utility maximisers. Lotteries L' to the left of L must lie on (graphically) lower iso-EV lines and have higher EV such that $CE(L) = CE(L')$ and the agent is indifferent between the two. More succinctly, a riskier lottery L' to the (graphical) left of L must have higher EV and so be (graphically) below L in order for a risk ~~neutral~~ averse agent to be indifferent between the two, so indifference curve will be upward sloping.

bi
$$V(x) = E(U(c,x)) = E(u(c) - x) = E(u(\tilde{w}x) - x)$$

FOC:

$V'(x) = 0 \iff$

$E(u'(\tilde{w}x)\tilde{w} - 1) = 0 \iff$

$E(u'(\tilde{w}x)\tilde{w}) = 1$

SOC:

$V''(x) = E(u''(\tilde{w}x)\tilde{w}^2) < 0$

given that u is strictly concave hence $u''(\tilde{w}x) < 0$ so the relevant SOC holds

ii $u(c) = 2c^{1/2} \Rightarrow u'(c) = c^{-1/2}$

FOC:

$E(\tilde{w}^{-1/2} x^{-1/2} \tilde{w}) = 1 \iff$

$E(\tilde{w}^{1/2} x^{-1/2}) = 1 \iff$

$E(\tilde{w}^{1/2}) = x^{1/2} \iff$

$x = (E(\tilde{w}^{1/2}))^2$

From the above SOC is satisfied for a maximum. $x^* = (E(\tilde{w}^{1/2}))^2$ uniquely solves the agents' EU maximisation problem.

By concavity of exponentiation by $1/2$, as $\tilde{w}$ becomes more risky, $E(\tilde{w}^{1/2})$ decreases hence optimal work x^* decreases.

5a $u(w,e) = \sqrt{w} - 2e$

$P(w=81 \mid e=1) = 2/3$, $P(w=0 \mid e=1) = 1/3$

$P(w=81 \mid e=0) = 1/3$, $P(w=0 \mid e=0) = 2/3$

$E[u(w,e) \mid e=1] = 2/3(\sqrt{81}-2) + 1/3(\sqrt{0}-2) = 4$

$E[u(w,e) \mid e=0] = 1/3(\sqrt{81}-0) + 2/3(\sqrt{0}-0) = 3$

$e=1$ yields higher expected utility, so agent A will choose $e=1$, which yields expected utility $\bar{u}=4$. This is A's reservation utility. Any contract that yields lower ~~a~~ expected utility will be rejected.

b Principal P optimally offers full insurance where effort is observable. At any optimum, ~~the~~ A's participation constraint binds. At any candidate optimum such that ~~A's~~ PC does not bind, P has profitable deviation ~~by to offering~~ by decreasing f or increasing s by small amount ε such that PC remains satisfied. Then, given that PC binds at ~~the~~ any optimum, P optimally offers full insurance. Any candidate optimum ~~such~~ where P does not offer full insurance fails to deviation consisting in (1) a mean preserving ~~con~~ contraction of 81-s and f, and (2) a small ε increase in s. ~~(1)~~ By concavity of u in w, (1) increases expected utility and loosens PC, it has no effect on expected profit. Given that PC no longer binds, for sufficiently small ε, (2) continues to satisfy PC and increases ~~profit~~. expected profit.

To induce low effort, ~~81-s=f. Denote this value w_0.~~ P offers full insurance. $81-s=f=w_0$. PC binds.

$1/3(\sqrt{w_0}-0) + 2/3(\sqrt{w_0}-0) = \bar{u} \iff w_0 = 16 \implies s = 65, f = 16 \iff \pi = 1/3\, 65 - 2/3\, 16 = 11$

To induce high effort,

$81-s=f=w_1$, $2/3(\sqrt{w_1}-2) + 1/3(\sqrt{w_1}-2) = \bar{u} \iff w_1 = 36$

$\iff s=45, f=36 \iff \pi = 2/3\, 45 - 1/3\, 36 = 18 > 11$

~~P optimal~~ It is optimal for P to ~~induce~~ require high effort and to do so by offering $s=45$, $f=36$

c To induce high effort, P must offer a contract that satisfies both PC and incentive constraint IC. At any optimum, both constraints bind. ~~If~~ Any candidate optimum such that PC does not bind fails to deviation by decreasing f by small amount ε. For sufficiently small ε, PC remains satisfied. IC remains satisfied ~~be~~ because A has less incentive to choose low effort. Expected profit increases. Any candidate optimum such that IC does not bind fails to deviation ~~by~~ consisting in (1) a small mean-preserving contraction of 81-s and f w.p. 2/3, 1/3, and (2) a small ~~increas~~ decrease in f. By concavity of u in w, (1) increases expected utility and loosens PC. For a small mean-preserving contraction, IC remains satisfied. π is unchanged. Then, given PC is loose, ~~(2)~~ for small decrease in f, (2) satisfies PC. IC remains satisfied and π increases.

PC: $2/3(\sqrt{81-s}-2) + 1/3(\sqrt{0+f}-2) \geq \bar{u} = 4$

IC: $2/3(\sqrt{81-s}-2) + 1/3(\sqrt{0+f}-2) \geq$
$1/3(\sqrt{81-s}-0) + 2/3(\sqrt{0+f})$

Let $w^H = 81-s$, $w^L = f$, $v^H = \sqrt{81-s}$, $v^L = \sqrt{f}$

$2/3(v^H-2) + 1/3(v^L-2) = 4 \iff 2/3 v^H + 1/3 v^L = 6$

$2/3(v^H-2) + 1/3(v^L-2) = 1/3 v^H + 2/3 v^L \iff$

$1/3 v^H - 2 = 1/3 v^L \iff 1/3 v^H - 1/3 v^L = 2$

$\implies$

$v^H = 8$, $v^L = 2$

$\implies$

$w^H = 64$, $w^L = 4$

$\implies$

$s = 17$, $f = 4$

$\implies$

$\pi = 2/3(17) - 1/3(4) = 10$

If P induces $e=1$, P optimally does so by offering $s=17$, $f=4$

To induce $e=0$, P offers the same "fixed wage" full insurance contract as in (b). This satisfies IC because effort is costly and has no effect on final wealth under this contract, so A has strict incentive to choose $e=0$. From before, this contract yields $\pi = 11$.

It is optimal for P to induce $e=0$ with the contract $s=65$, $f=16$. Then A optimally chooses $e=0$.

Agency cost is so high ~~that it~~ $(18-10=8)$ that it is no longer optimal to induce $e=1$. P incurs an agency cost in inducing $e=1$ because a variable wage scheme is necessary to make $e=1$ incentive compatible for A where effort is unobservable. Then, A bears some risk under this variable wage, partial insurance scheme, so A must be offered higher expected wage (than under observable effort) for participation to remain optimal. So P offers higher expected wage and has lower expected profit in inducing $e=1$. In inducing $e=0$, the full insurance contract is still optimal so there is no agency cost and expected profit is unchanged.

Date