ADM 2303 - Assignment 2

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```
##
## Attaching package: 'dplyr'
## The following objects are masked from 'package:stats':
##
##     filter, lag
## The following objects are masked from 'package:base':
##
##     intersect, setdiff, setequal, union
```

1. Next Generation Aircraft

In order to reduce carbon emissions from commercial passenger aircraft, research is underway on two different methods of powering them: electric motors or hydrogen turbines. The electricity and hydrogen would be obtained from renewable power sources such as wind and solar. The hydrogen would be produced by electrolysis of water which is currently only 80% efficient. Electric aircraft can use multiple motors driving propellers along the length of the wings increasing the airflow over the wing, thus reducing the size and cost of the wing and improving efficiency. Hydrogen turbines can be integrated into the airframe, also improving efficiency and reducing capital cost.

A technology forecasting company needs to advise investors as to the probability that new designs of aircraft will become available. It analyses future scenarios as follows. With 80% efficient electrolysis (which is already available today), the probability of a hydrogen aircraft being Economicalal in 5 years is 0.6. The efficiency of electrolysis can be improved to 90% within 5 years with a probability of 0.7. With 90% efficient electrolysis, the probability of a hydrogen aircraft being Economicalal in 5 years is 0.8.

1.1 Conditional Probability

> Which of the probabilities in the above description is/are conditional?

The probability of a hydrogen aircraft being Economicalal in 5 years is conditional because it is dependent on whether the electrolysis efficiency (80% or 90%) of the hydrogen aircraft.

1.2 Probability Tree

> Draw a probability tree for the situation above (showing probabilities of electrolysis efficiency and an Economicalal hydrogen aircraft in 5 years).

```
tree <- Node$new("Hydrogen Aircraft") # Initializing the probability tree by creating a new node
# Adding the different branches of the probability tree and their description

efficient_80_percent <-
  tree$AddChild("80% Efficient Aircraft")
Economical_5_years <-
```

```r
  efficient_80_percent$AddChild("Economical in 5 Years")
efficient_80_percent_and_Economical_5_years <-
  Economical_5_years$AddChild("80% Efficient Aircraft and Economical in 5 Years")
not_Economical_5_years <-
  efficient_80_percent$AddChild("Not Economical in 5 Years")
efficient_80_percent_and_not_Economical_5_years <-
  not_Economical_5_years$AddChild("80% Efficient Aircraft and Not Economical in 5 Years")
efficient_90_percent <-
  tree$AddChild("90% Efficient Aircraft")
Economical_5_years <-
  efficient_90_percent$AddChild("Economical in 5 Years")
efficient_90_percent_and_Economical_5_years <-
  Economical_5_years$AddChild("90% Efficient Aircraft and Economical in 5 Years")
not_Economical_5_years <-
  efficient_90_percent$AddChild("Not Economical in 5 Years")
efficient_90_percent_and_not_Economical_5_years <-
  not_Economical_5_years$AddChild("90% Efficient Aircraft and Not Economical in 5 Years")

# Adding the probabilities of the branches in the tree
tree$`80% Efficient Aircraft`$p <- 0.3
tree$`80% Efficient Aircraft`$`Economical in 5 Years`$p <- 0.6
tree$`80% Efficient Aircraft`$`Economical in 5 Years`$`80% Efficient Aircraft and Economical in 5 Years
  tree$`80% Efficient Aircraft`$p *
  tree$`80% Efficient Aircraft`$`Economical in 5 Years`$p # Joint probability
tree$`80% Efficient Aircraft`$`Not Economical in 5 Years`$p <- 0.4
tree$`80% Efficient Aircraft`$`Not Economical in 5 Years`$`80% Efficient Aircraft and Not Economical in
  tree$`80% Efficient Aircraft`$p *
  tree$`80% Efficient Aircraft`$`Not Economical in 5 Years`$p # Joint probability
tree$`90% Efficient Aircraft`$p <- 0.7
tree$`90% Efficient Aircraft`$`Economical in 5 Years`$p <- 0.8
tree$`90% Efficient Aircraft`$`Economical in 5 Years`$`90% Efficient Aircraft and Economical in 5 Years
  tree$`90% Efficient Aircraft`$p *
  tree$`90% Efficient Aircraft`$`Economical in 5 Years`$p # Joint probability
tree$`90% Efficient Aircraft`$`Not Economical in 5 Years`$p <- 0.2
tree$`90% Efficient Aircraft`$`Not Economical in 5 Years`$`90% Efficient Aircraft and Not Economical in
  tree$`90% Efficient Aircraft`$p *
  tree$`90% Efficient Aircraft`$`Not Economical in 5 Years`$p # Joint probability

print(tree, "p")
```

```
##                                                          levelName    p
## 1  Hydrogen Aircraft                                                 NA
## 2   ¦--80% Efficient Aircraft                                      0.30
## 3   ¦   ¦--Economical in 5 Years                                   0.60
## 4   ¦   ¦   °--80% Efficient Aircraft and Economical in 5 Years     0.18
## 5   ¦   °--Not Economical in 5 Years                               0.40
## 6   ¦       °--80% Efficient Aircraft and Not Economical in 5 Years 0.12
## 7   °--90% Efficient Aircraft                                      0.70
## 8       ¦--Economical in 5 Years                                   0.80
## 9       ¦   °--90% Efficient Aircraft and Economical in 5 Years     0.56
## 10      °--Not Economical in 5 Years                               0.20
## 11          °--90% Efficient Aircraft and Not Economical in 5 Years 0.14
```

1.3 Joint Probabilities

What is the probability that an Economicalal hydrogen aircraft will be developed in 5 years? If an Economicalal hydrogen aircraft is developed in 5 years, research into electric aircraft will be cut back so that the probability of an Economicalal electric aircraft being developed in 7 years is 0.5. If an Economicalal hydrogen aircraft is not developed in 5 years, the probability of an Economicalal electric aircraft being developed in 7 years is 0.9.

```
p_economic_5_years <-
  tree$`80% Efficient Aircraft`$`Economic in 5 Years`$`80% Efficient Aircraft and Economic in 5 Years`$p
  tree$`90% Efficient Aircraft`$`Economic in 5 Years`$`90% Efficient Aircraft and Economic in 5 Years`$p
  # Probability that an economical hydrogen aircraft will be developed in 5 years
p_economic_5_years
```

```
## integer(0)
```

1.4 Extended Probability Tree

Extend the probability tree from 1.3 to also include probabilities of an Economicalal electric aircraft in 7 years.

```
tree_new <- Node$new("Hydrogen Aircraft") # Initializing another probability tree by creating a new nod
# Adding the different branches of the probability tree and their description
  efficient_80_percent <- tree_new$AddChild("80% Efficient Aircraft")
    economic_5_years <- efficient_80_percent$AddChild("Economical in 5 Years (Hydrogen)")
      economic_7_years <- economic_5_years$AddChild("Economical in 7 Years (Electric)")
        efficient_80_percent_and_economic_5_years_and_economic_7_years <- economic_7_years$AddChild("80%
      not_economic_7_years <- economic_5_years$AddChild("Not Economical in 7 Years (Electric)")
        efficient_80_percent_and_economic_5_years_and_not_economic_7_years <- not_economic_7_years$AddCh
    not_economic_5_years <- efficient_80_percent$AddChild("Not Economical in 5 Years (Hydrogen)")
      economic_7_years <- not_economic_5_years$AddChild("Economical in 7 Years (Electric)")
        efficient_80_percent_and_not_economic_5_years_and_economic_7_years <- economic_7_years$AddChild
      not_economic_7_years <- not_economic_5_years$AddChild("Not Economical in 7 Years (Electric)")
        efficient_80_percent_and_not_economic_5_years_and_not_economic_7_years <- not_economic_7_years$A
  efficient_90_percent <- tree_new$AddChild("90% Efficient Aircraft")
    economic_5_years <- efficient_90_percent$AddChild("Economical in 5 Years (Hydrogen)")
      economic_7_years <- economic_5_years$AddChild("Economical in 7 Years (Electric)")
        efficient_90_percent_and_economic_5_years_and_economic_7_years <- economic_7_years$AddChild("90%
      not_economic_7_years <- economic_5_years$AddChild("Not Economical in 7 Years (Electric)")
        efficient_90_percent_and_economic_5_years_and_not_economic_7_years <- not_economic_7_years$AddCh
    not_economic_5_years <- efficient_90_percent$AddChild("Not Economical in 5 Years (Hydrogen)")
      economic_7_years <- not_economic_5_years$AddChild("Economical in 7 Years (Electric)")
        efficient_90_percent_and_not_economic_5_years_and__economic_7_years <- economic_7_years$AddChild
      not_economic_7_years <- not_economic_5_years$AddChild("Not Economical in 7 years (Electric)")
        efficient_90_percent_and_not_economic_5_years_and_not_economic_7_years <- not_economic_7_years$A

# Adding the probabilities of the tree's branches and calculating the joint probabilities
tree_new$`80% Efficient Aircraft`$p <- 0.3
  tree_new$`80% Efficient Aircraft`$`Economical in 5 Years (Hydrogen)`$p <- 0.6
    tree_new$`80% Efficient Aircraft`$`Economical in 5 Years (Hydrogen)`$`Economical in 7 Years (Electri
      tree_new$`80% Efficient Aircraft`$`Economical in 5 Years (Hydrogen)`$`Economical in 7 Years (Elec
    tree_new$`80% Efficient Aircraft`$`Economical in 5 Years (Hydrogen)`$`Not Economical in 7 Years (El
      tree_new$`80% Efficient Aircraft`$`Economical in 5 Years (Hydrogen)`$`Not Economical in 7 Years (
  tree_new$`80% Efficient Aircraft`$`Not Economical in 5 Years (Hydrogen)`$p <- 0.4
    tree_new$`80% Efficient Aircraft`$`Not Economical in 5 Years (Hydrogen)`$`Economical in 7 Years (El
      tree_new$`80% Efficient Aircraft`$`Not Economical in 5 Years (Hydrogen)`$`Economical in 7 Years (
```

```r
    tree_new$`80% Efficient Aircraft`$`Not Economical in 5 Years (Hydrogen)`$`Not Economical in 7 Years
      tree_new$`80% Efficient Aircraft`$`Not Economical in 5 Years (Hydrogen)`$`Not Economical in 7 Yea
tree_new$`90% Efficient Aircraft`$p <- 0.7
  tree_new$`90% Efficient Aircraft`$`Economical in 5 Years (Hydrogen)`$p <- 0.8
    tree_new$`90% Efficient Aircraft`$`Economical in 5 Years (Hydrogen)`$`Economical in 7 Years (Electri
      tree_new$`90% Efficient Aircraft`$`Economical in 5 Years (Hydrogen)`$`Economical in 7 Years (Elec
    tree_new$`90% Efficient Aircraft`$`Economical in 5 Years (Hydrogen)`$`Not Economical in 7 Years (El
      tree_new$`90% Efficient Aircraft`$`Economical in 5 Years (Hydrogen)`$`Not Economical in 7 Years (
  tree_new$`90% Efficient Aircraft`$`Not Economical in 5 Years (Hydrogen)`$p <- 0.2
    tree_new$`90% Efficient Aircraft`$`Not Economical in 5 Years (Hydrogen)`$`Economical in 7 Years (El
      tree_new$`90% Efficient Aircraft`$`Not Economical in 5 Years (Hydrogen)`$`Economical in 7 Years (
    tree_new$`90% Efficient Aircraft`$`Not Economical in 5 Years (Hydrogen)`$`Not Economical in 7 years
      tree_new$`90% Efficient Aircraft`$`Not Economical in 5 Years (Hydrogen)`$`Not Economical in 7 Yea

print(tree_new, "p")
```

```
##                                                                                              levelName
## 1  Hydrogen Aircraft
## 2   ¦--80% Efficient Aircraft
## 3   ¦   ¦--Economical in 5 Years (Hydrogen)
## 4   ¦   ¦   ¦--Economical in 7 Years (Electric)
## 5   ¦   ¦   ¦   °--80% Efficient Aircraft and Economical in 5 and 7 Years; Hydrogen, Electric
## 6   ¦   ¦   °--Not Economical in 7 Years (Electric)
## 7   ¦   ¦       °--80% Efficient Aircraft and Economical in 5 but not 7 years; Hydrogen, Electric
## 8   ¦   °--Not Economical in 5 Years (Hydrogen)
## 9   ¦       ¦--Economical in 7 Years (Electric)
## 10  ¦       ¦   °--80% efficient aircraft and not economical in 5 but in 7 years; Hydrogen, Electric
## 11  ¦       °--Not Economical in 7 Years (Electric)
## 12  ¦           °--80% efficient Aicraft and not economical in 5 nor 7 years; Hydrogen, Electric
## 13  °--90% Efficient Aircraft
## 14      ¦--Economical in 5 Years (Hydrogen)
## 15      ¦   ¦--Economical in 7 Years (Electric)
## 16      ¦   ¦   °--90% Efficient Aircraft and economical in 5 and 7 years; Hydrogen, Electric
## 17      ¦   °--Not Economical in 7 Years (Electric)
## 18      ¦       °--90% Efficient Aircraft and economical in 5 but not 7 years; Hydrogen, Electric
## 19      °--Not Economical in 5 Years (Hydrogen)
## 20          ¦--Economical in 7 Years (Electric)
## 21          ¦   °--90% Efficient Aircraft and not economical in 5 but in 7 years; Hydrogen, Electric
## 22          °--Not Economical in 7 years (Electric)
## 23              °--90% efficient Aicraft and not economical in 5 nor 7 years; Hydrogen, Electric
##        p
## 1     NA
## 2  0.300
## 3  0.600
## 4  0.500
## 5  0.090
## 6  0.500
## 7  0.090
## 8  0.400
## 9  0.100
## 10 0.012
## 11 0.900
## 12 0.108
## 13 0.700
```

```
## 14 0.800
## 15 0.500
## 16 0.280
## 17 0.500
## 18 0.280
## 19 0.200
## 20 0.100
## 21 0.014
## 22 0.900
## 23    NA
```

1.5 Bayes' Theorem

What is the probability that an Economicalal electric aircraft will be developed in 7 years?

```
p_economic_electric_7_years <-
  tree_new$`80% Efficient Aircraft`$`Economical in 5 Years (Hydrogen)`$`Economical in 7 Years (Electric
  tree_new$`80% Efficient Aircraft`$`Not Economical in 5 Years (Hydrogen)`$`Economical in 7 Years (Elec
  tree_new$`90% Efficient Aircraft`$`Economical in 5 Years (Hydrogen)`$`Economical in 7 Years (Electric
  tree_new$`90% Efficient Aircraft`$`Not Economical in 5 Years (Hydrogen)`$`Economical in 7 Years (Elec
# Calculating the probability that an economical electric aircraft will be developed in 7 years

revised_p_economic_7_years <- p_economic_electric_7_years / p_economic_5_years
# Calculating the revised probability that an economical electric aircraft will be developed in 7 years
revised_p_economic_7_years
```

```
## numeric(0)
```

2. Bank Charges

For one type of bank account, a bank allows 4 transactions per month for free and charges 1 dollar per transaction for the 5th and subsequent transactions. (E.g. if I make 7 transactions I will pay (7-4)*1 = $3.) The number of transactions I make each month has a mean of 4 and a variance of 2.4 and follows a Binomial distribution. I need to estimate how much I expect to pay each month in bank charges.

2.1 Finding p and n

Calculate the probability, p, and the number, n, that define the Binomial distribution of the number of transactions each month. [Hint: Use the formulas for the mean and variance of the Binomial Distribution. Mean = np, Variance = np(1-p) = Mean*(1-p).]

$$\mu = 4\sigma^2 = 2.4$$

Step 1: Finding p using the variance formula:

$$\sigma^2 = n \cdot p \cdot q\sigma^2 = 2.4n \cdot p \cdot q = 2.4\mu \cdot q = 2.44 \cdot q = 2.4\frac{4 \cdot q}{4} = \frac{2.4}{4}q = 0.6p + q = 1p + 0.6 = 1p = 1 - 0.6p = 0.4$$

Step 2: Finding n using median formula

$$n \cdot p = \mu n \cdot p = 4n \cdot 0.4\ 4\frac{n \cdot 0.4}{0.4} = \frac{4}{0.4}n = 10$$

The probability, p, and the number, n, that define the Binomial distribution of the number of transactions each month are $p = 0.4$ and $n = 10$.

2.2 Binomial Distribution

Calculate the probabilities of making 5, 6, 7, ... transactions in a month.

Let X = the number of transactions in a month, C = Charge per transaction (in $).

```
p_x_5 <- dbinom(5,10,0.4) # P(X=5)
p_x_6 <- dbinom(6,10,0.4) # P(X=6)
p_x_7 <- dbinom(7,10,0.4) # P(X=7)
p_x_8 <- dbinom(8,10,0.4) # P(X=8)
p_x_9 <- dbinom(9,10,0.4) # P(X=9)
p_x_10 <- dbinom(10,10,0.4) # P(X=10)

# Creating a dataframe to summarize the probability data from above
binomial_prob_summary <- data.frame(X = 5:10,
                                    C = 1:6,
                                    "P(x)" = c(p_x_5,p_x_6,p_x_7,p_x_8,p_x_9,p_x_10))
binomial_prob_summary
```

```
##    X C         P.x.
## 1  5 1 0.2006581248
## 2  6 2 0.1114767360
## 3  7 3 0.0424673280
## 4  8 4 0.0106168320
## 5  9 5 0.0015728640
## 6 10 6 0.0001048576
```

2.3 Expected Value of Binomial Distribution

Calculate the expected total bank charges for one month.

```
binomial_prob_summary <-
  binomial_prob_summary %>%
  mutate("E(x)"=binomial_prob_summary$P.x. * binomial_prob_summary$C)
# Mutating the dataframe with pipe operator to add new column for expected charge per transaction
binomial_prob_summary
```

```
##    X C         P.x.         E(x)
## 1  5 1 0.2006581248 0.2006581248
## 2  6 2 0.1114767360 0.2229534720
## 3  7 3 0.0424673280 0.1274019840
## 4  8 4 0.0106168320 0.0424673280
## 5  9 5 0.0015728640 0.0078643200
## 6 10 6 0.0001048576 0.0006291456
```

```
expected_total_charges <-
  sum(binomial_prob_summary$`E(x)`)
# Calculating the expected total charges by summing the individual expected charges of each level of tr
expected_total_charges
```

```
## [1] 0.6019744
```

The expected total bank charges for one month are 0.6020, so we can expect to pay $0.6020 per transaction a month.

3. Letter Carriers

A Canada Post letter carrier delivers mail to houses on a street. The number of mail items to be delivered to an individual house follows a Poisson distribution with mean = 1.3 and is independent from one house to another. The time taken to walk past a house for which there are no items requiring delivery is 15 seconds. The time taken to deliver items to a house is 15 seconds (plus the 15 seconds to walk past) independent of the number of items requiring delivery. When deciding how many houses to assign to a letter carrier, we need to estimate how much uncertainty there is in the time required to deliver mail to a lot of houses, in relation to the expected time required.

3.1 Poisson Distribution (1)

For a given house, what is the probability that there are no items requiring delivery?

```
p_x_0 <- dpois(0,1.3) # P(X=0)
p_x_0
```

```
## [1] 0.2725318
```

3.2 Poisson Distribution (2)

What is the expected time taken by the letter carrier for a single house?

```
p_x_1 <- dpois(1,1.3) # P(X=1)
expected_time_1_house <- 15 * p_x_0 + 30 * p_x_1 # Formula from laws of expectated values
expected_time_1_house
```

```
## [1] 14.71672
```

3.3 Coefficient of Variation for 1 House

What is the coefficient of variation of the time taken by the letter carrier for a single house?

```
variance_1_house <-
  ((15 - expected_time_1_house)^2 * p_x_0^2) + ((30 - expected_time_1_house)^2 * p_x_1^2)
# Law of V(X) formula, multiplying by constants

sd_1_house <- sqrt(variance_1_house) # Standard deviation

coeff_of_var_1_house <- sd_1_house / expected_time_1_house # Coefficient of variation
coeff_of_var_1_house
```

```
## [1] 0.3679683
```

3.4 Expected Value of Poisson Distribution

What is the expected time taken by the letter carrier for a street of 100 houses?

```
probabilities_100_houses <-
  data.frame(X = 0:100) # Creating the dataframe
probabilities_100_houses <-
  probabilities_100_houses %>% mutate(t = c(probabilities_100_houses$X * 30))
  # Creating the column for time taken for number of houses requiring delivery
probabilities_100_houses[1,2] <-
  15 # t = 0 does not follow the same rule as the other entries so it is manually adjusted
probabilities_100_houses <-
  probabilities_100_houses %>% mutate("P(t)" = c(dpois(probabilities_100_houses$X,1.3)))
```

```r
  # Calculating the probability for each row using Poisson distribution with pipe operator and mutate()
probabilities_100_houses <-
  probabilities_100_houses %>% mutate("E(t)" = c(probabilities_100_houses$`P(t)` * probabilities_100_hou
  # Calculating the expected time take for each number of houses

expected_time_100_houses <- sum(probabilities_100_houses$`E(t)`) # Calculating the total expected time
expected_time_100_houses
```

```
## [1] 43.08798
```

3.5 Coefficient of Variation for 100 Houses

> What is the coefficient of variation of the time taken by the letter carrier for a street of 100 houses?

```r
variance_100_houses <-
  sum((probabilities_100_houses$t - probabilities_100_houses$`E(t)`)^2 * probabilities_100_houses$`P(t)`
  # calculating the variance of 100 houses
sd_100_houses <- sqrt(variance_100_houses) # Calculating the standard deviation

coeff_of_var_100_houses <- sd_100_houses / expected_time_100_houses # Coefficient of variation of 100 h
coeff_of_var_100_houses
```

```
## [1] 0.3671692
```