There's always a prime between n and $2n$

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1 Introduction

In 1845, Joseph Bertrand, first postulated that there's always a prime between n and $2n$. He also verified his statement for $n < 3\times10^6$. In 1852, his conjecture was completely proved by Chebyshev, so the postulate is also known as Chebyshev's theorem.

The Prime Number Theorem, first proved by Hadamard and de Vallee Poussin in 1896, states that

$$\pi(x) \sim \frac{x}{logx} \text{ as } x \to \infty$$

where $\pi(x)$ is the number of primes less than or equal to x. As a result of PNT, we can state that there exists $n(\epsilon) > 0$ for $\epsilon > 0$, such that for all $n > n(\epsilon)$, there is always a prime in the interval $(n, (1+\epsilon)n]$, that is

$$n < p \leq (1+\epsilon)n$$

Thus, we have

$$\pi((1+\epsilon)n) - \pi(n) \sim \frac{(1+\epsilon)n}{log(1+\epsilon)n} - \frac{n}{logn} \to \infty \text{ as } n \to \infty$$

With an error estimate, we can use the information from the PNT and give

Bertrand's Postulate : *For all $n \geq 1$, there is a prime between n and $2n$*

There have been several generalisations of the Bertrand's postulate. In 1919, Ramanujan gave a simple proof using the properties of Gamma function [3]. With this paper, the concept of Ramanujan primes also arose. In 1973, Denis

Hanson proved that there exist a prime between $3n$ and $4n$ [4]. In 2006, El Bacharoui proved that there are exists a prime between $2n$ and $3n$ [5]. In 2011, Andy Loo proved that as n tends to infinity, the primes between $3n$ and $4n$ also tend to infinity [6], thus generalising both Erdos's and Ramanujan's result.

In this paper, we will discuss Erdos's proof of the postulate (using [1], Chapter 2). He gave a beautiful elementary proof using basic facts of binomial coefficients and the Chebyshev function.

2 Erdos's Proof

We consider the middle binomial coefficient $\binom{2n}{n} = \frac{(2n)!}{(n!)^2}$. An easy lower bound is

$$\tbinom{2n}{n} > \tfrac{4^n}{2n+1}$$

Thus, $\binom{2n}{n}$ is the largest term in $(2n+1)$-term sum $\sum_{i=0}^{2n} \binom{2n}{i} = (1+1)^{2n} = 4^n$. Erdos's proof then shows that if there is no prime between n and $2n$, we can out an upper bound on $\binom{2n}{n}$ which is smaller than $4^n/(2n+1)$ unless n is small.

For a prime p and an integer n, we define $o_p(n)$ to be the largest exponent of p that divides n. We note that $o_p(ab) = o_p(a) + o_p(b)$ and $o_p(a/b) = o_p(a) - o_p(b)$. We also note that

$$o_p(n!) = \sum_{i \geq 1} \left[\frac{n}{p^i} \right]$$

The essence of the proof lies in the fact that if $n \geq 3$ and $\frac{2}{3} < p \leq n$ and then $o_p\left(\binom{2n}{n}\right) = 0$

So if $n \geq 3$ such that there is no prime p between n and $2n$, then all the factors of $\binom{2n}{n}$ lie between 2 and $2n/3$. We will prove that each of these factors appears only to small exponent, forcing $\binom{2n}{n}$ to be small.

Lemma 2.1 : *If $p|\binom{2n}{n}$, then*

$$p^{0_p\left(\binom{2n}{n}\right)} \leq 2n$$

Proof : Let $r(p)$ be such that $p^{r(p)} < 2n < p^{r(p)+1}$. We have

$$o_p\left(\binom{2n}{n}\right) = o_p((2n)!) - 2o_p(n!) = \sum_{i=1}^{r(p)}\left[\frac{2n}{p^i}\right] - 2\sum_{i=1}^{r(p)}\left[\frac{n}{p^i}\right]$$

Thus,

$$o_p\left(\binom{2n}{n}\right) \leq r(p)$$

And so we can say that

$$o_p\left(\binom{2n}{n}\right) \leq p^{r(p)} \leq 2n$$

□

Corollary 2.2 : *The number of primes dividing* $\binom{2n}{n}$ *is at least* $log_2\binom{2n}{n}/log_2(2n)$

Proof : Let $p_1, ..., p_l$ be distinct primes dividing $\binom{2n}{n}$. We have

$$\binom{2n}{n} = \prod_{i}^{l} p^{o_p\left(\binom{2n}{n}\right)} \leq (2n)^l$$

Now using Lemma 2.1 and the above inequality, we take logarithms on both side.Thus, the result follows.

□

We will also need one more result before we proceed to proving the Bertrand's postulate.

Lemma 2.3 : *For all* $n \geq 2$, $\prod_{p \leq n} p \leq 4^n$, *where the product is over primes*

Proof: We proceed by induction on n. For smaller values of n, it is trivial. But for larger values of n, we have

$$\prod_{p \leq n} p = \prod_{p \leq n-1} p \leq 4^{n-1} \leq 4n$$

The equality follows because n is even, thus it is not prime and the first inequality follows from inductive hypothesis. For larger odd n, say $n = 2m + 1$, we have

$$\prod_{p \leq n} p = \prod_{p \leq m+1} p \prod_{m+2 \leq p \leq 2m+1} p \leq 4^{\binom{2m+1}{m}}$$

We use the inductive hypothesis to bound $\prod_{p \leq m+1} p$ and we bound $\prod_{m+2 \leq p \leq 2m+1} p$ by observing that all primes between $m+2$ and $2m+1$ divide $\binom{2m+1}{m}$. It implies that

$$\prod_{p \leq n} p \leq 4^{m+1} 2^{2m} = 4^{2m+1} = 4^n$$

We bound $\binom{2m+1}{m} \leq 2^{2m}$ by noting that $\sum_{i=0}^{2m+1} \binom{2m+1}{i} = 2^{2m+1}$.

□

Now we have sufficient results to prove the postulate. Let $n \geq 3$ be such that there is no prime between n and $2n$. Then we have

$$\binom{2n}{n} \leq (2n)^{\sqrt{2n}} \prod_{\sqrt{2n} < p \leq 2n/3} p \leq \prod_{p \leq 2n/3} p$$

We have used the simple fact that $\binom{2n}{n}$ has at most $\sqrt{2n}$ prime factors that do not exceed $\sqrt{2n}$, and by Lemma 2.1, each of these these prime factors contributes at most $2n$ to $\binom{2n}{n}$. Next, we have used that all prime factors p of $\binom{2n}{n}$ satisfy $p \leq 2n/3$, and the fact that each such p with $p > \sqrt{2n}$ appears in $\binom{2n}{n}$ with exponent 1. Thus suing Lemma 2.3, we have

$$\binom{2n}{n} \leq (2n)^{\sqrt{2n}} 4^{2n/3}$$

Combining the inequality obtained above with the lower bound earlier stated, we obtain

$$\frac{4^n}{2n+1} \leq (2n)^{\sqrt{2n}} 4^{2n/3}$$

This inequality can hold only for small values of n. Indeed, for any $\epsilon > 0$ the left handed side of the above inequality grows faster than $(4-\epsilon)^n$ whereas the right-hand side grows more slowly than $(4^{2/3}-\epsilon)^n$. We can check that in fact, the above inequality fails at $n \geq 468$. To verify Bertrand's postulate for all $n < 468$, we illustrate the primes

$$2, 3, 5, 7, 13, 23, 43, 83, 163, 317, 631$$

Here each term of which is less than twice the term preceding it. It follows that every interval $n+1, \ldots.., 2n$ with $n < 468$ contains one of these 11 primes. Thus, this proves the Bertrand's postulate.

3 References

[1] M. Aigner and G. M. Ziegler, Proofs from the book (4th ed.), Springer, 2010

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