Reflections Within a Circle

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1 Introduction

Suppose one has an infinitesimally small ball, which is hit in some direction within the boundary of a circle, with friction ignored. This means that the collision with a point on the circle can be treated as specular reflection. The goal of this project is to model the various effects on the trajectory of the ball based on the initial conditions. The Diagrams were produced by the GeoGebra online calculator as well as the Mathemaica Model.

2 Properties

Observe the rotational symmetry of a circle. No matter the starting position or the direction, the ball will collide with the circle, at some angle θ against the boundary, producing a specular reflection. We will define θ as the acute angle constructed from a specular reflection.

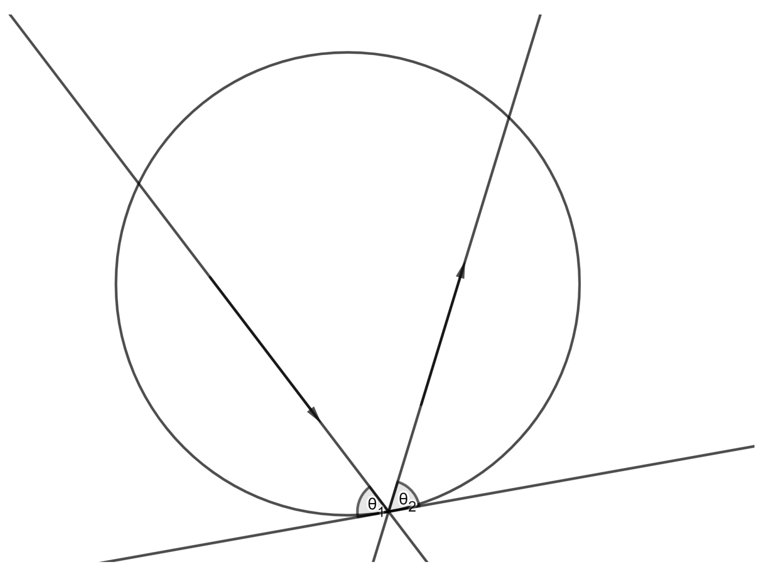

Figure 1: Collision where θ_1 and θ_2 are equal by Law of Reflection

The trajectory of the ball will solely depend on this angle, as every possible displacement will result with a collision with the boundary at some angle θ, which determines the trajectory of the ball. A trajectory will be unique to angle θ: the only way to produce a congruent trajectory will be to reflect off with θ at another point. Additionally, we will only consider θ from 0 to $\frac{\pi}{2}$, as the circle is symmetric as: any angle $\frac{\pi}{2} < \theta < \pi$ will result in a equal trajectory from an angle $0 < \theta < \frac{\pi}{2}$. Additionally, each reflection against the boundary at angle θ will produce another reflection at the same angle. This can be shown by using the symmetric property of the circle, by bisecting the circle by a chord produced by the reflections.

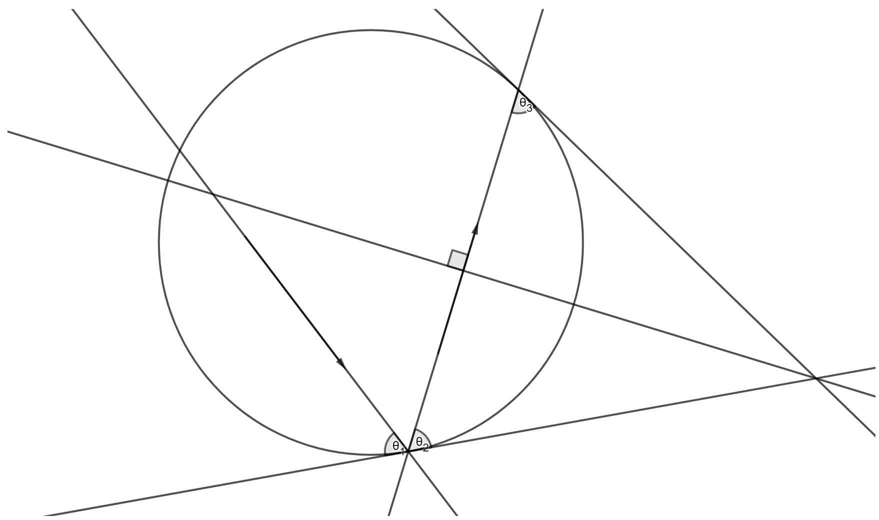

Figure 2: θ_2 and θ_3 are equal by symmetry

Additionally, because every chord is launched from the same angle, they are rotationally symmetric and congruent. The length of the chord will be evaluated by constructing a triangle with side lengths of r, the radius of the circle, along with the chord as a base.

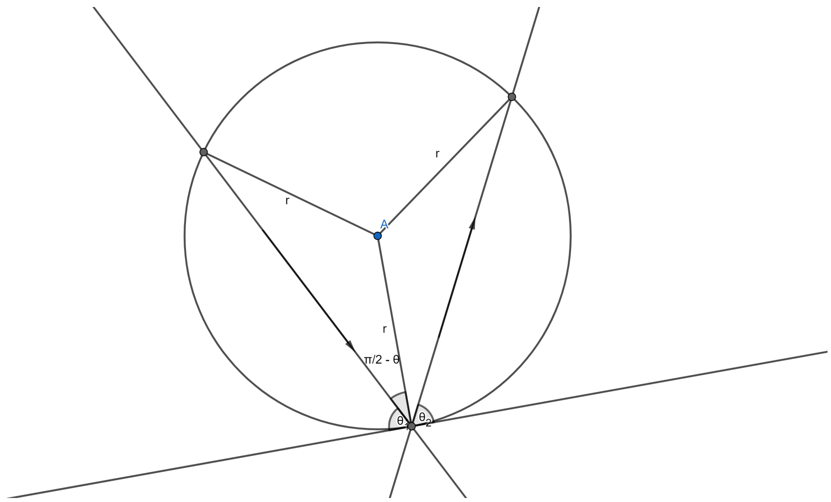

Figure 3: Isosceles triangles constructed by drawing radii from the origin to instances of collision

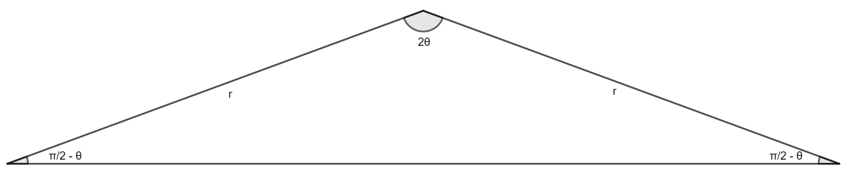

Figure 4: triangle taken from the circle

Figure 5: Triangle from Figure 4, bisected by the altitude, where the length of the chord is c.

With the triangle shown in Figure 5, it is possible to relate θ, r, and c, to find that

$$c = 2r \sin\theta$$

3 Model

It is possible to model the trajectory by taking a vector with length $2r\sin\theta$ and repeatedly applying the rotation matrix: A matrix that, when left multiplied by a column vector, gives the vector rotated by an angle. This gives the displacements of the ball after each collision. This matrix for θ is given by:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

A displacement vector will be found by rotating the previous displacement by 2θ around the tail of the vector:

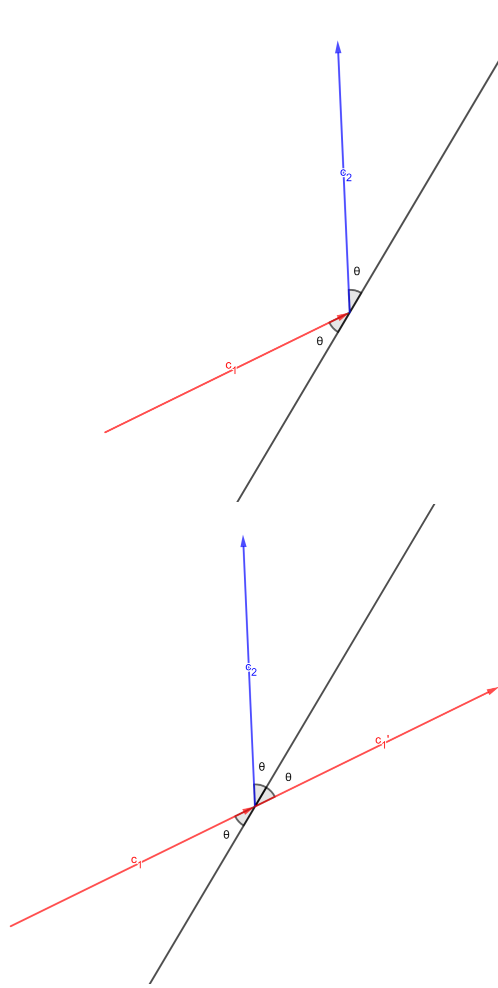

Figure 6: Initial displacement vector c_1 adjusted to show an angle of 2θ needed for rotation

Mathematica was used to model this trajectory, with an adjustable value of θ. Images in 3.1 and 3.2 were produced with this model.

```
In[1]:= v[x_, a_] :=
         If[x == 0, {Sin[2 a], -2 Sin[a]^2}, RotationMatrix[2 a].v[x - 1, a]]
        f[x_, a_] :=
         If[x == 0, {0, -1}, f[x - 1, a] + v[x, a]]
        P = Circle[{0, 0}, 1]
        g[Iterations_, Angle_] := Graphics[{P, Line[Table[f[i, Angle], {i, 0, Iterations}]]}]
        Manipulate[g[Iterations, Angle], {{Iterations, 10}, 0, 300, 1}, {{Angle, 1}, 0, Pi / 2}]
```

Figure 7: Mathematica Code

3.1 Polygons and Stars

All of a polygon's outer angles will add up to 2π regardless the number of sides. If an angle formed by the trajectory of the reflection were to be an angle of a polygon, the exterior angle would be 2θ. Consequently, a normal polygon with n sides will have exterior angles of $\frac{2\pi}{n}$. Because the exterior angle is given by 2θ, a polygon with n sides will be produced by an instance of the model with a θ value of $\frac{\pi}{n}$ for $n > 2$.

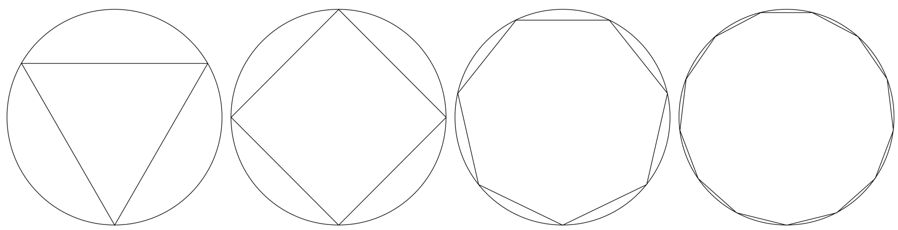

Figure 8: A triangle, square, heptagon, and 13-gon produced by θ values of $\frac{\pi}{3}$, $\frac{\pi}{4}$, $\frac{\pi}{7}$, and $\frac{\pi}{13}$.

These polygons only "cover" the circle once: the arcs corresponding to the chords of the trajectory add up to one circle. If we consider the arcs corresponding to add up to m circles, the outer angles should add up to $2m\pi$. A star with n sides and m circles covered will be produced by an instance of the model with a θ value of $\frac{m\pi}{n}$, where $\frac{m}{n}$ is in lowest form.

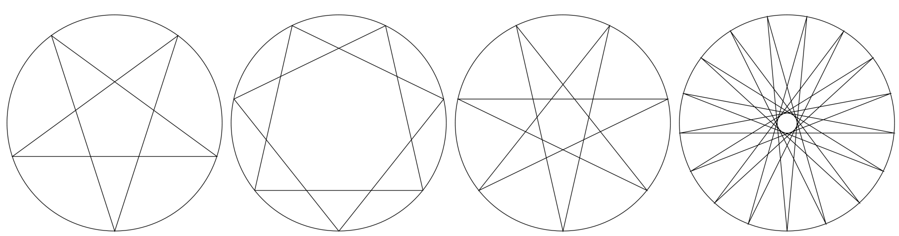

Figure 9: Four stars produced by θ values of $\frac{2\pi}{5}$, $\frac{2\pi}{7}$, $\frac{3\pi}{7}$, and $\frac{8\pi}{17}$

3.2 Filling Trajectories

When given a θ value of $\frac{m\pi}{n}$, the outer angle is 2θ and therefore $\frac{2m\pi}{n}$. after n iterations, the total outer angle is $2m\pi$ and a multiple of 2π, resulting in a return to the starting point.

However, when the ratio between θ and π is irrational, then the ball will not return to its starting position, as no multiple of θ will ever add up to a multiple of 2π by the definition of irrationality. Additionally, because every new collision is a reflection with angle θ, the trajectory of the ball will never return to a point with the boundary it has already collided with.

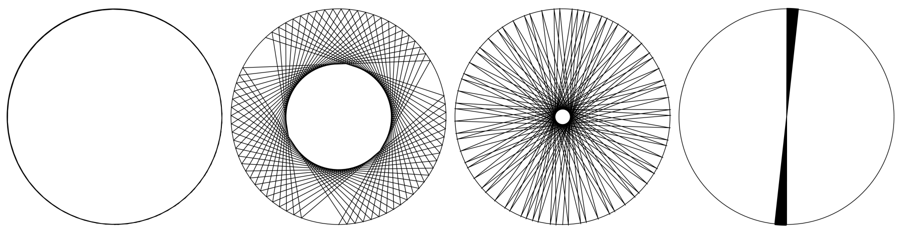

Figure 10: Four stars produced by θ values of $0.1, 1.06, 1.5$, and 1.57 radians, after 70 reflections.

No matter the angle θ, an inner circle appears where the trajectory will not pass through. For θ values where $\frac{\theta}{\pi}$ is rational, chords will form a polygon or a star, which will be repeated infinitely as the number of reflections goes to infinity.

Observe that the inner circle is comprised of the set of points on the trajectory that are closest to the origin. Knowing this, it is possible to conclude that the radius of the circle will be the distance from the midpoint of the chord to the origin. Referencing Figure 5, the radius p of the inner circle is shown to be

$$p = r \cos \theta$$

4 Future Work

Results of running the Mathematica model for θ values where $\frac{\theta}{\pi}$ is irrational suggests the circle formed by the chord should approach the circle as the number of reflections goes to infinity, and that the entire rest of the circle should be filled.

Based on this, when $\frac{\theta}{\pi}$ is irrational taking the limit $\lim_{\theta \to \frac{\pi}{2}}$ results in the trajectory approaching a disk, where the entirety is filled, as the inner circle goes to 0. Additionally, when $\frac{\theta}{\pi}$ is irrational taking the limit $\lim_{\theta \to 0}$ results in the trajectory approaching the circular boundary, as the inner circle approaches the boundary circle.

However, this is not obvious. It's certainly possible for systems to hit infinitely many points but not reach every single one to fill a space. In the future, I would like to study measure theory to rigorously prove that this reflection is ergodic (the entire space is covered), something that is intuitive but difficult to confirm.