Moving least square curve and surface fitting with interpolation conditions

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Abstract—**This paper presents a method for moving least square curve and surface fitting with interpolation conditions. The method is firstly proposed for solving the problem of the curve fitting with interpolation conditions. It has more advantages including that the degree of fitting function is low and the fitting computation is convenient. Then, the method is extended to solve the surface fitting problem. We use moving least square approximation to solve the surface fitting with interpolation conditions. The experimental results show that it obtains satisfactory fitting effect.**

Keywords - interpolation; curve fitting; surface fitting; least square approximation; moving least square approximation

I. INTRODUCTION

Data fitting is commonly used in dealing with large amounts of data involved in scientific fields, and it has many important applications in biological, chemical, signal processing, computer graphics, statistics and so on [1]. We need to find the hidden rules for some of the data which are related in the experiment, and we usually use the curve or surface fitting method to approach these discrete data. The fitting curve or surface can be constructed by the least square method which does not need to go through all the data points [2-3]. Recently, moving least square method is proposed for fitting the data. Compared to traditional least square method, it has more advantages and has broadly been used in the data fitting and analysis [5-6].

In some cases when fitting the data, the fitting curve or surface need to go through some key points, we need to consider the interpolation conditions for the least square or moving least square method. R[4] introduced the least square method with interpolation conditions and provided the construction formula. In this paper, we propose a new least square method with interpolation conditions. It has more advantages, for example, the degree of fitting function is low and the construction computation is convenient. Then this method is extended for moving least square fitting with interpolation conditions. It can also obtain satisfying fitting effect.

II. LEAST SQUARE CURVE FITTING WITH INTERPOLATION CONDITIONS

A. New fitting formula construction

Suppose that a set of scattered nodes are (x_i, y_i), $i = 1, 2, \cdots, n$, and the interpolation conditions are (x_s, y_s), $s = 1, 2, \cdots, t$, $t \le n$. If $P(x) = a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \cdots + a_1 x + a_0$ is the fitting function, the least square fitting curve with interpolation conditions can be written as

$$y = P(x) - \sum_{s=1}^{t} l_s(x)\delta_s, \tag{1}$$

where $\delta_s = P(x_s) - y_s$

$$= a_{m-1}x_s^{\,m-1} + a_{m-2}x_s^{\,m-2} + \cdots + a_1 x_s + a_0 - y_s .$$

We define

$$\lambda = \sum_{j=1}^{n} [y_j - P(x_j) + \sum_{s=1}^{t} l_s(x_j)\delta_s]^2,$$

where λ is a function with independent variable a_i, which are coefficients of $P(x)$. The problem is converted into solve the minimum of λ. We use the following equations

$$\lambda_{a_i}(a_0, a_1, \cdots, a_{m-1}) = 0,\ \ i = 1, 2, \cdots, m-1$$

to evaluate a_i, i.e.

$$\begin{cases} \sum_{j=1}^{n}[y_j - (a_{m-1}x_j^{\,m-1} + \cdots + a_0) + \sum_{s=1}^{t} l_s(x_j)\delta_s]\cdot[-1 + \sum_{s=1}^{t} l_s(x_j)] = 0 \\ \sum_{j=1}^{n}[y_j - (a_{m-1}x_j^{\,m-1} + \cdots + a_0) + \sum_{s=1}^{t} l_s(x_j)\delta_s]\cdot[-x_j + \sum_{s=1}^{t} l_s(x_j)x_s] = 0 \\ \cdots \\ \sum_{j=1}^{n}[y_j - (a_{m-1}x_j^{\,m-1} + \cdots + a_0) + \sum_{s=1}^{t} l_s(x_j)\delta_s]\cdot[-x_j^{\,m-1} + \sum_{s=1}^{t} l_s(x_j)x_s^{\,m-1}] = 0. \end{cases}$$

We calculate $a_0, a_1, \cdots, a_{m-1}$ from above equations and substitute them into Equ.(1) to get the least square curve fitting with interpolation conditions.

We notice that if the degree of $P(x)$ is $m-1$ and the number of interpolation points is *t*, then the degree of the least square fitting curve with interpolation conditions is Max{*m*-1，*t*-1}. The degree is lower than the function of the least square fitting provided in R[4]. In addition, we can regard the new formula as the modification of the least square fitting method where the correction term is $\sum_{s=1}^{t} l_s(x)\delta_s$.

Therefore, we firstly calculate the curve fitting by the least

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square method, then combine it with the correction term $\sum_{s=1}^{t} l_s(x)\delta_s$ to obtain the fitting curve $P(x)$ with interpolation conditions. It can be seen as an effective replacement of the least square fitting with interpolation conditions, but the whole construction is convenient.

B. *Curve fitting examples*

Given a set of scattered nodes x = [1, 2.5, 4.5, 6, 7, 8, 9, 10], y = [1, 2, 2.5, 3, 4, 5, 5.5, 7], we respectively use our least square method with interpolation conditions and the method in R[4] to fit these points. The interpolation nodes are (1, 1), (10, 7). Fig.1 is the result of the least square fitting with interpolation conditions by R[4]. When the empirical function is a polynomial of degree 3, the degree of fitting polynomial is 5. The curve represented by solid line in Fig.2 is the fitting result of our method. When the empirical function is a polynomial of degree 3, the degree of fitting polynomial is also 3. The curve represented by dotted line is the fitting result of our method by calculating the correction term. This shows that we can use lower degree of empirical function to construct the least square fitting with interpolation conditions while keeping satisfactory fitting effect. In addition, we find there is little difference of fitting results between our original method and the method by calculating the correction term.

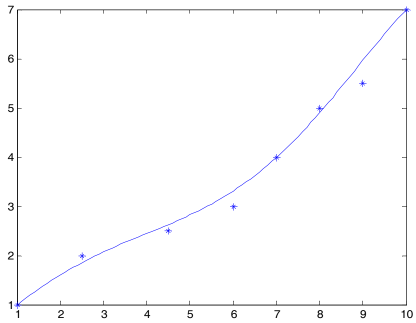

Figure 1. Result by R[4]

（The degree of fitting polynomial is 5）

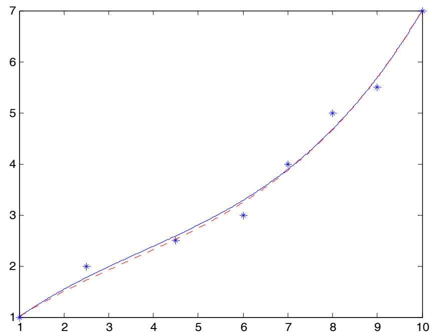

Figure 2. Result by our method

（The degree of fitting polynomial is 3）

III. Moving Least Square Curve Fitting With Interpolation Conditions

A. *Moving least square method introduction*

The fitting function by moving least square method can be written as [3]

$$f(x) = \sum_{i=1}^{m} p_i(x)a_i(x) = p^T(x)a(x), \tag{2}$$

where

$$a(x) = (a_1(x), a_2(x), \cdots, a_m(x))^T$$

is a set of coefficients, and

$$p(x) = (p_1(x), p_2(x), \cdots, p_m(x))^T$$

is the basis function. We commonly use the complete polynomial basis, such as

Linear base $\quad p(x) = (1, x, y)^T \quad (m = 3)$

Quadratic base $\quad p(x) = (1, x, y, x^2, xy, y^2)^T. \quad (m = 6)$

We need to make the weighted square of difference between local approximation $f(x_i)$ and point value y_i be the minimum to obtain a more accurate local approximation, so the discrete weighted L_2 norm of residual is

$$J = \sum_{i=1}^{n} w(x - x_i)[f(x) - y_i]^2 = \sum_{i=1}^{n} w(x - x_i)[p^T(x_i)a(x) - y_i]^2, \tag{3}$$

where n is the number of points in the solving domain, and $f(x)$ is the fitting function, $w(x - x_i)$ is the weight function of the point x_i.

The weight function should be non-negative, and monotonically decrease when $\|x - x_i\|_2$ increases. The weight function should also be with compact support, which means that the function isn't equal to 0 in the support domain, while it is equal to 0 outside the support domain. We usually choose the circle as the support domain, and the radius is r. And we normally construct the spline function as the weight function. Let $s' = x - x_i$, $s = \frac{s'}{r}$, then the cubic spline function is

$$\omega(s) = \begin{cases} \frac{2}{3} - 4s^2 + 4s^3 & s \le \frac{1}{2} \\ \frac{4}{3} - 4s + 4s^2 - \frac{4}{3}s^3 & \frac{1}{2} < s \le 1 \\ 0 & s > 1. \end{cases} \tag{4}$$

Firstly, we make J be minimum to evaluate $a(x)$. Matrix form of Equ.(3) is as follows

$$J = (Pa(x) - Y)^T W(x)(Pa(x) - Y),$$

where

$$Y = (y_1, y_2, \cdots, y_n)^T,$$

$$W(x) = diag(w_1(x), w_2(x), \cdots, w_n(x)), \; w_i(x) = w(x - x_i),$$

$$P = \begin{bmatrix} p_1(x_1) & p_2(x_1) & \cdots & p_m(x_1) \\ p_1(x_2) & p_2(x_2) & \cdots & p_m(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ p_1(x_n) & p_2(x_n) & \cdots & p_m(x_n) \end{bmatrix}.$$

We use the least square method to obtain $a(x)$

$$a(x) = A^{-1}(x)B(x)Y,$$

where $A(x) = P^T W(x)P$, $B(x) = P^T W(x)$.

We then substitute it in Equ. (2) to get the fitting function

$$f(x) = \sum_{i=1}^{n} \phi_i^k(x) y_i = \psi^k(x)Y,$$

where $\psi^k(x)$ is the shape function, and k is the order of basis function.

$$\psi^k(x) = [\phi_1^k, \phi_2^k, \cdots, \phi_n^k] = p^T(x)A^{-1}(x)B(x).$$

B. *Moving least square curve fitting with interpolation conditions*

When the fitting curve is required to go through some key points, we can use above moving least square method with interpolation conditions to construct the fitting formula.

Supposing there are a set of scattered nodes (x_i, y_i), $i = 1, 2, \cdots, n$, and the interpolation conditions are (x_s, y_s), $s = 1, 2, \cdots, t$, $t \le n$. We firstly construct the fitting curve $f(x)$ by moving least square method, then the moving least square fitting curve with interpolation conditions can be obtained by

$$y = f(x) - \sum_{s=1}^{t} l_s(x)\delta_s, \quad (5)$$

where $l_s(x) = \prod_{\substack{j=1 \\ s \ne j}}^{t} \frac{(x - x_j)}{(x_s - x_j)}, s = 1, 2, \cdots, t$, $\delta_s = f(x_s) - y_s$.

This formula can also be considered as the modification of moving least square fitting formula where the correction function is $\sum_{s=1}^{t} l_s(x)\delta_s$. So the main calculation steps are

a) Firstly, we obtain moving least square fitting curve without interpolation conditions;

b) Secondly, we calculate the deviation at each interpolation points $\delta_s = f(x_s) - y_s$;

c) Finally, we calculate $l_s(x)$ and obtain moving least square fitting curve with interpolation conditions by Equ.(5).

We do the experiment to compare our moving least square method with interpolation conditions to traditional moving least square method. There are a set of scattered nodes x = [1, 2.5, 4.5, 6, 7, 8, 9, 10], y = [1.5, 2, 2.2, 3, 4, 5.5, 6.5, 7]. When we use moving least squares method to fit the curve, we choose linear base $p(x) = (1, x, y)^T$ and set the cubic spline function (Equ.(4)) as the weight function. Fig.3 is the fitting result of traditional moving least square method, and Fig.4 is the fitting result of our moving least square method with interpolation conditions which go through two nodes (4.5, 2.2), (9, 6.5). We compare the sum of all error values on scattered nodes between our method and traditional method. We find the error of our method is smaller than that of traditional method.

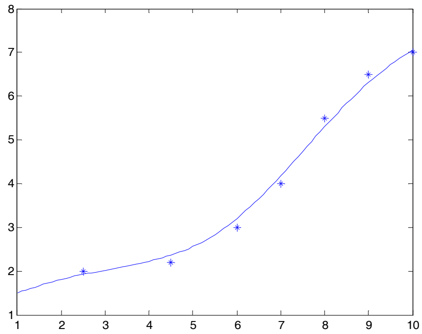

Figure 3. Moving least square fitting result

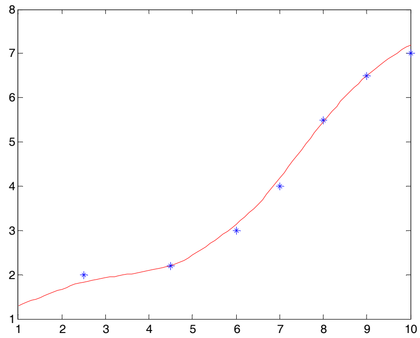

Figure 4. Moving least square fitting result with interpolation conditions

IV. SURFACE FITTING WITH INTERPOLATION CONDITIONS

The curve fitting method with interpolation conditions can be extended to the surface fitting in three-dimensional space. When the fitting surface is required to go through some key points, we apply the least square method or moving least square method with interpolation conditions to construct the fitting surface.

Supposing there are a set of scattered nodes (x_i, y_i, z_i), $i = 1, 2, \cdots, n$ and the interpolation conditions are (x_s, y_s, z_s), $s = 1, 2, \cdots, t$, $t \le n$. We denote $P(x, y)$ is the fitting surface constructed by the least square method or moving least square method, then the fitting surface with interpolation conditions can be written as

$$z = P(x, y) - \sum_{s=1}^{t} l_s(x, y)\delta_s, \quad (5)$$

where $l_s(x)=\prod_{\substack{j=1\\ s\neq j}}^{t}\frac{(x-x_j)}{(x_s-x_j)}\cdot\frac{(y-y_j)}{(y_s-y_j)}, s=1,2,\cdots,t,$

$\delta_s=P(x_s,y_s)-z_s$.

A. Surface fitting examples

We do the experimental examples to compare the surface fitting between traditional least square method and the least square method with interpolation conditions.

Supposing a set of scattered nodes are x = [-1, -1.3, -2.1, 1, 1.2, 0.2, -0.2, -2.2, 1.3, 2.3, -0.3, 1.4, -2.4, 0.4, 1.5, -0.5, 2.5, 0.6, -0.6, -2.6, -1.3, -0.7, 1.7, 0.8, 1.8, 2.8, 0.9, 1, -2, 1.3];
y = [1, -2, -2.4, 0.6, 1.2, 0.5, -0.7, -2.9, -0.3, 2.4, 0.8, -1.9, 2.4, 0.7, 2.6, -2.8, 2, 2.3, -2.3, -1.6, -2.8, 1.5, 0.2, 2.5, 1.5, 0.8, -1.9, -0.3, -1.4, 0.8];
z = [1.0140, -0.3882, 0.0006, -0.3180, 1.1635, 0.2996, 1.4301, 0.0007, -0.9486, 0.0118, 1.5846, -0.9528, 0.0071, 0.8560, 0.1439, -0.3905, 0.0086, 2.2630, -1.6072, 0.0014, -0.0491, 4.8470, -0.8791, 0.9932, 0.2269, -0.0247, -2.9919, -0.6862, 0.0424, -0.2365].

The fitting surface realized by traditional least square method is shown in Fig 5. If we want the fitting surface to go through some interpolation nodes which are (-1, 1, 1.0140), (0.2, 0.5, 0.2996), (0.6, 2.3, 2.263), (0.6, 2.3, 2.263), the fitting surface can be constructed by our method, which is shown in Fig 6.

We also do experimental examples to compare the surface fitting between traditional moving least square method and moving least square method with interpolation conditions. The scattered nodes and interpolation nodes are same as above. We respectively use traditional moving least square method and moving least square method with interpolation conditions to fit these scattered nodes, the fitting surfaces are shown in Fig 7 and Fig 8. From these experimental results, we find the fitting surface with interpolation conditions is more accurate in the interpolation area and more effectively reflects the geometric shape of scattered nodes.

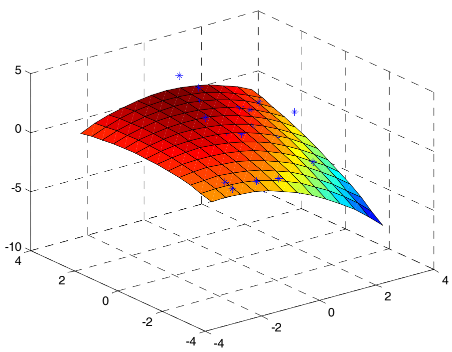

Figure 5. Result by the least square method

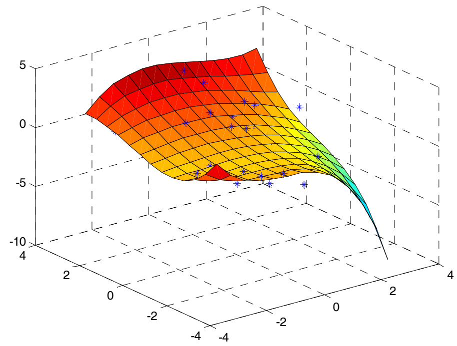

Figure 6. Result by the least square method with interpolation conditions

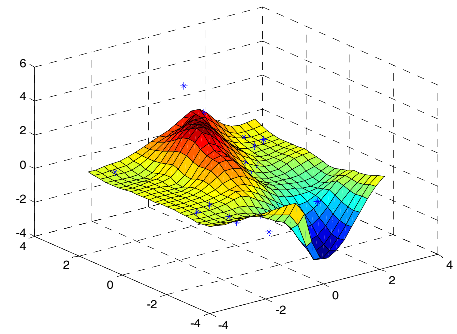

Figure 7. Result by moving least square method

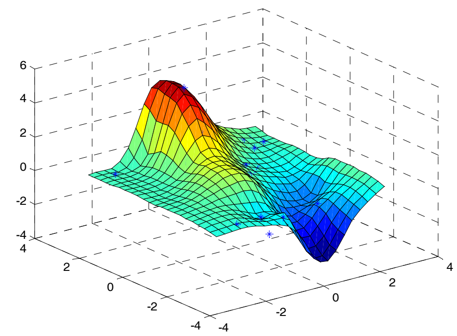

Figure 8. Result by moving least square method with interpolation conditions

V. CONCLUSION AND FUTURE WORK

In this paper, we provide a new least square curve fitting method with interpolation conditions. It has more advantages including that the degree of fitting function is low and the computation is convenient. It can also be applied for the curve fitting with interpolation conditions by moving least square method. Finally, we extend it to the surface fitting with interpolation conditions by the least square method and moving least square method. The fitting surface is more accurate around the interpolation area.

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