Velocity and Acceleration Constrained Trajectory Planning by Smoothing Splines

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***Abstract*—We develop a trajectory planning method of mobile robots in a two-dimensional plane. Trajectories are constructed as vector smoothing splines using B-splines as the basis functions. We plan trajectory for road-like path specified by piecewise linear boundaries as treated in our recent paper, and our specific focus in this paper is on introducing quadratic inequality constraints on the trajectory as well as on its derivatives. It is shown that such constraints are systematically included in the smoothing spline settings and the problem is formulated as quadratically constrained quadratic programming problem. This problem can be solved numerically by MATLAB based software CVX efficiently. Such a quadratic constraint can be used, e.g. to limit the maximum speed and acceleration of the robot motion, and the usefulness and effectiveness of the developed method are confirmed by numerical experiments.**

***Keywords*-trajectory planning; B-spline; quadratic constraints; velocity and acceleration limit; QCQP problem**

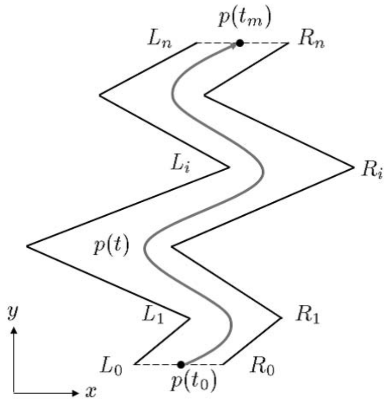

Figure 1. A path with piecewise linear boundaries.

I. Introduction

In a typical problem of trajectory planning for mobile robots and robotic arms, we are required to construct a function of time that satisfies various constraints. First, the initial and terminal constraints are usually needed, and via points constraints may be treated by interpolation or approximation or smoothing depending on whether the points need to be passed exactly or approximately.

The problem of obstacle avoidance is one of the key problems in trajectory planning and various methods have been developed. They are often treated as nonlinear programming problem introducing the cost function such as the distance to obstacles, the arc length of the trajectory, etc. and the optimization problems with constraints are solved numerically. Trajectories are expressed by cubic splines [15], [18], [17], quartic splines [1], or by sums of harmonics [9]. In [1], obstacles are treated by linear inequality constraints and the trajectory with minimum curvature is obtained.

For such trajectory planning problems, spline functions are obviously very suited since they are piecewise polynomials and easy to design locally [2], [6], [14]. Moreover considerable effort has been devoted to develop splines with constraints, including monotone splines, shape preserving splines, etc. The authors have been developing a framework based on B-splines for designing smoothing splines where various types of constraints can be incorporated systematically [10]. The constraints include those at isolated points, those over intervals, those on derivatives and integral values, etc. [11], [12], [8].

Based on our studies on constrained splines, we developed a trajectory planning method for road like path with piecewise linear boundaries as shown in Fig. 1 [13]. The problem was to design a smooth trajectory $p(t)$ between given initial and final positions while guaranteeing to remain within the path. We showed that boundary constraints of the path can be very well treated in our B-spline approach, and formulated as convex quadratic programming problem.

In this paper, we further study the trajectory planning problem along the line of [13]. In particular, here we develop a method of including quadratic inequality constraints on the trajectory as well as on its derivatives. We will see that the problem is formulated as quadratically constrained quadratic programming (QCQP) problem [3], and can be solved numerically by using MATLAB based software CVX [5]. Such a quadratic constraint can be used, e.g. to limit the maximum speed and acceleration of the robot motion. Quadratic constraints have not been considered in our work as [11]-[13], and to the authors knowledge, such a treatment and developments are novel.

This paper is organized as follows. Section II provides preliminaries on construction of 2D vector smoothing splines using B-splines. Section III describes trajectory planning problem, where we first review constraints for road-like path and then develop algorithms for quadratic constraints on trajectory and its derivatives. The results of

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numerical experiment are given in Section IV. Concluding remarks are given in Section V.

The following symbols are used throughout the paper: $\otimes$ denotes the Kronecker product, and 'vec' the vec-function, i.e. for a matrix $A = [a_1\ a_2 \cdots\ a_n] \in \mathbf{R}^{m\times n}$ with $a_i \in \mathbf{R}^m$, vec $A = [a_1^T\ a_2^T\ \cdots\ a_n^T]^T \in \mathbf{R}^{mn}$ (see e.g. [16]). For real symmetric matrix S, $S > 0$ and $S \geq 0$ respectively show that S is positive-definite and nonnegative-definite. Vector norms are $\|v\| = \sqrt{v^Tv}$ and $\|v\|_S = \sqrt{v^TSv}$ for $v \in \mathbf{R}^n$ and $S \geq 0$ with $S \in \mathbf{R}^{n\times n}$.

II. Preliminaries

We review B-spline formulation for vector smoothing splines [8], [13].

A. Construction of Splines

We construct a trajectory $p(t) \in \mathbf{R}^2$ in the xy-plane

$$p(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \tag{1}$$

for given time interval $[t_0, t_m]$ by

$$p(t) = \sum_{i=-k}^{m-1} \tau_i B_k(\alpha(t - t_i)). \tag{2}$$

Here $B_k(t)$ is normalized uniform B-spline of degree k,

$$B_k(t) = \begin{cases} N_{k-j,k}(t-j) & j \leq t < j+1, \\ & j = 0, 1, \cdots, k \\ 0 & t < 0 \quad \text{or} \quad t \geq k+1, \end{cases} \tag{3}$$

$\tau_i \in \mathbf{R}^2$ are weighting coefficients called control points, and $\alpha (> 0)$ is a constant for scaling the interval between equally-spaced knot points t_i with

$$t_{i+1} - t_i = \frac{1}{\alpha}. \tag{4}$$

In (3), $N_{j,k}(t)$ $(j = 0, 1, \cdots, k)$, $0 \leq t \leq 1$, are the basis elements derived recursively [4] and satisfy

$$N_{j,k}(t) \geq 0, \quad \sum_{j=0}^{k} N_{j,k}(t) = 1, \quad \forall t \in [0, 1]. \tag{5}$$

Cubic splines are most frequently used, and we show $N_{j,3}(t)$ and its derivatives in Table I.

Table I
$N_{j,3}(t)$ $(j = 0, 1, 2, 3)$ and its derivatives

j	$(3!)N_{j,3}(t)$	$(2!)N_{j,3}^{(1)}(t)$	$N_{j,3}^{(2)}(t)$	$N_{j,3}^{(3)}(t)$
0	$(1-t)^3$	$-(1-t)^2$	$1-t$	-1
1	$4-6t^2+3t^3$	$-4t+3t^2$	$-2+3t$	3
2	$1+3t+3t^2-3t^3$	$1+2t-3t^2$	$1-3t$	-3
3	t^3	t^2	t	1

Smoothing splines are used to determine the control points τ_i, or the control point matrix $\tau \in \mathbf{R}^{2\times M}$ $(M = m + k)$

$$\tau = [\ \tau_{-k} \quad \tau_{-k+1} \quad \cdots \quad \tau_{m-1}\ ]. \tag{6}$$

B. Smoothing Splines

For our problem of road-like path, we assume that a function $f(t) \in \mathbf{R}^2$ is given and consider smoothing splines with the following cost function,

$$J(\tau) = \lambda \int_{t_0}^{t_m} \left\| p^{(l)}(t) \right\|_\Lambda^2 dt + \int_{t_0}^{t_m} \|p(t) - f(t)\|^2 dt. \tag{7}$$

Here $\lambda (> 0)$ is a smoothing parameter, and $\Lambda \in \mathbf{R}^{2\times 2}$ $(\Lambda > 0)$ is a weight matrix. We take the integer l as $l = 2$ for cubic spline $(k = 3)$ and $l = 3$ for quintic spline $(k = 5)$.

Now, let $\hat{\tau} \in \mathbf{R}^{2M}$ be the vector version of τ as

$$\hat{\tau} = \text{vec}\ \tau. \tag{8}$$

Then, following the similar procedure as in [10], $J(\tau)$ in (7) is expressed as a quadratic function $J(\hat{\tau})$ in $\hat{\tau}$,

$$J(\hat{\tau}) = \hat{\tau}^T G \hat{\tau} - 2g^T\hat{\tau} + g_c \tag{9}$$

where $G \in \mathbf{R}^{2M\times 2M}$ and $g \in \mathbf{R}^{2M}$ are given by

$$G = Q \otimes \Lambda + Q_0 \otimes I_2 \tag{10}$$

$$g = \int_{t_0}^{t_m} b(t) \otimes f(t) dt, \tag{11}$$

and g_c is a scalar, independent of $\hat{\tau}$. Here $b(t) \in \mathbf{R}^M$ is a vector of shifted B-splines defined by

$$b(t) = \left[\ B_k(\alpha(t - t_{-k})) \quad B_k(\alpha(t - t_{-k+1})) \quad \cdots \quad B_k(\alpha(t - t_{m-1}))\ \right]^T, \tag{12}$$

and $Q, Q_0 \in \mathbf{R}^{M\times M}$ are Gram matrices defined by

$$Q = \int_{t_0}^{t_m} \frac{d^l b(t)}{dt^l} \frac{d^l b^T(t)}{dt^l} dt \tag{13}$$

$$Q_0 = \int_{t_0}^{t_m} b(t) b^T(t) dt. \tag{14}$$

Note that $Q_0 > 0$ holds, and since $Q \geq 0$, the matrix G in (10) is always positive-definite. Moreover Q and Q_0 are banded matrices and can be set up easily once the degree k is given (see [7], for details).

III. Trajectory with Quadratic Constraints

We treat trajectory planning problem for road-like path considered in [13]. For this purpose, in Section III-A, we briefly review the method developed in [13] and then describe quadratic constraints in Section III-B.

A. Path with Piecewise Linear Boundaries

We consider the path with piecewise linear boundaries as shown in Fig. 1. These boundaries are simply defined by a pair of corner points (R_i, L_i), $i = 0, 1, \cdots, n$ on the xy-plane, and let the coordinates be given as

$$R_i = \begin{bmatrix} x_i \\ y_i \end{bmatrix}, \ L_i = \begin{bmatrix} x'_i \\ y'_i \end{bmatrix}. \tag{15}$$

In order to plan trajectories for this path, the entire time interval $[t_0, t_m]$ is divided into n subintervals $[s_i, s_{i+1}]$, $i = 0, 1, \cdots, n-1$ in accordance with n pairs of boundary segments R_iR_{i+1} and L_iL_{i+1}. These subintervals are all

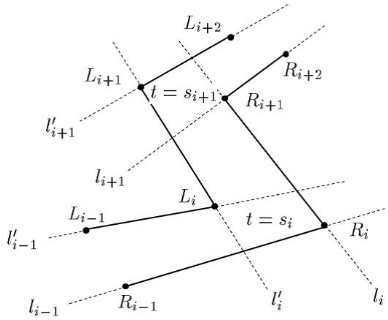

Figure 2. Corner points (R_i, L_i) and boundary lines (l_i, l'_i).

taken as knot point intervals, i.e. s_i's are taken as knot points t_j with $s_0 = t_0$ and $s_n = t_m$.

The trajectory $p(t)$ is then planned in such a way that $p(t)$ for $t \in [s_i, s_{i+1}]$ stays between the two straight lines l_i and l'_i passing through R_i, R_{i+1} and L_i, L_{i+1}, respectively (see Fig. 2, [13]). Denoting $[s_i, s_{i+1}]$ by knot point t_j as $[s_i, s_{i+1}] = [t_\kappa, t_\mu]$ with κ, μ some integer satisfying $\kappa < \mu$, it is shown in [13] that $p(t)$ remains for all $t \in [t_\kappa, t_\mu]$ between the two lines l_i and l'_i, if the control points τ in (6) satisfy

$$A\tau_j \le d, \ \kappa - k \le j \le \mu - 1. \tag{16}$$

Here A and d are defined from the coordinates of $R_i, R_{i+1}, L_i, L_{i+1}$ by

$$A = \begin{bmatrix} -y_i + y_{i+1} & x_i - x_{i+1} \\ y'_i - y'_{i+1} & -x'_i + x'_{i+1} \end{bmatrix}, \tag{17}$$

$$d = \begin{bmatrix} x_i y_{i+1} - x_{i+1} y_i \\ -x'_i y'_{i+1} + x'_{i+1} y'_i \end{bmatrix}. \tag{18}$$

Moreover the inequality (16) on τ_i is expressed in terms of $\hat{\tau}$ in (8) as

$$F_i \hat{\tau} \le h_i \tag{19}$$

where $F_i \in \mathbf{R}^{2(\mu-\kappa+k)\times 2M}$ and $h_i \in \mathbf{R}^{2(\mu-\kappa+k)}$ are defined by $F_i = E^T_{\kappa,\mu} \otimes A$ and $h_i = \mathbf{1}_{\mu-\kappa+k} \otimes d$, with

$$E_{\kappa,\mu} = \begin{bmatrix} 0_{\mu-\kappa+k,\kappa} & I_{\mu-\kappa+k} & 0_{\mu-\kappa+k,M-\mu-k} \end{bmatrix}^T, \tag{20}$$

and $\mathbf{1}_{\mu-\kappa+k} = [1 \ 1 \ \cdots \ 1]^T \in \mathbf{R}^{\mu-\kappa+k}$.

For planning trajectory $p(t), \ t_0 \le t \le t_m$ using the cost function $J(\hat{\tau})$ in (9), we impose the inequality constraint (19) for all $i = 0, 1, \cdots, n-1$.

Remark 1: Such an idea of treating the trajectory for each pairs of right and left boundaries (R_i, R_{i+1}) and (L_i, L_{i+1}) in the time subinterval $[s_i, s_{i+1}]$ is easily realized by the spline $p(t)$ in (2) since $p(t)$ itself is a piecewise polynomial. The time instants s_i for each i can be determined by employing e.g. cord length distribution or centripetal distribution method (see [2]).

B. Quadratic Constraints

We develop quadratic constraints on the trajectory and its derivatives. Specifically we derive conditions on control point vector $\hat{\tau}$ for constraints of the form

$$\|p^{(l)}(t)\|_\Gamma \le c \tag{21}$$

for given $l (l = 0, 1, 2)$, $c (> 0)$ and $\Gamma = \Gamma^T > 0$. Letting $\Gamma = I_2$, (21) is rewritten as

$$\begin{cases} l = 0: & \sqrt{x(t)^2 + y(t)^2} \le c \\ l = 1: & \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2} \le c \\ l = 2: & \sqrt{\ddot{x}(t)^2 + \ddot{y}(t)^2} \le c \end{cases} \tag{22}$$

Thus when $l = 0$, this constraint restricts the trajectory to lie within the circle of radius c on the xy-plane, and the velocity and acceleration can be constrained by letting $l = 1$ and $l = 2$, respectively.

1) The case $l = 0$: We consider (21) for $l = 0$, first in the unit knot point interval $[t_j, t_{j+1}]$, which can then be generalized to larger intervals. Note that, by the definition of $B_k(t)$ in (3), $p(t)$ is written for $[t_j, t_{j+1}]$ as

$$p(t) = \sum_{i=0}^{k} \tau_{j-k+i} N_{i,k}(\alpha(t - t_j)), \quad t \in [t_j, t_{j+1}], \tag{23}$$

and it depends on the $k+1$ weights τ_i for $j - k \le i \le j$. Introducing a new variable $u = \alpha(t - t_j)$, we see that $[t_j, t_{j+1}]$ in t is normalized to $[0, 1]$ in u and $p(t)$ is written as $\hat{p}(u)$,

$$\hat{p}(u) = \sum_{i=0}^{k} \tau_{j-k+i} N_{i,k}(u), \quad u \in [0, 1]. \tag{24}$$

We then have the following results.

Proposition 1: The following inequality on $p(t)$

$$\|p(t)\|_\Gamma \le c \quad \forall t \in [t_j, t_{j+1}] \tag{25}$$

holds if control point τ_i satisfies

$$\|\tau_i\|_\Gamma \le c, \quad j - k \le i \le j \tag{26}$$

or if expressed in terms of the control point vector $\hat{\tau}$

$$\|\hat{\tau}\|_{\Gamma_i} \le c, \quad j + 1 \le i \le j + k + 1 \tag{27}$$

where $\Gamma_i = (e_i e_i^T) \otimes \Gamma$ and $e_i \in \mathbf{R}^M$ contains one at its i-th position and zeros elsewhere.

(Proof) For $p(t)$, $t \in [t_j, t_{j+1}]$ or for $\hat{p}(u)$, for $u \in [0, 1]$ in (24), it holds that

$$\begin{aligned} \|p(t)\|_\Gamma &= \|\hat{p}(u)\|_\Gamma = \|\sum_{i=0}^{k} \tau_{j-k+i} N_{i,k}(u)\|_\Gamma \\ &\le \sum_{i=0}^{k} \|\tau_{j-k+i} N_{i,k}(u)\|_\Gamma \\ &= \sum_{i=0}^{k} \|\tau_{j-k+i}\|_\Gamma N_{i,k}(u) \end{aligned} \tag{28}$$

where $N_{i,k}(u) \ge 0$ is used. Thus using (5) and (26), we get

$$\|p(t)\|_\Gamma \le \sum_{i=0}^{k} c N_{i,k}(u) = c \tag{29}$$

and (25) holds. Next we derive (27). By (6), τ_i is written as $\tau_i = \tau e_{i+k+1}$, or $\tau_i = \tau e_{i'}$ with $i' = i + k + 1$ and hence

$$\begin{aligned}\|\tau_i\|_\Gamma^2 &= \|\tau e_{i'}\|_\Gamma^2 = \text{tr}\left(e_{i'}^T \tau^T \Gamma \tau e_{i'}\right) \\ &= \text{tr}\left(\tau^T \Gamma \tau e_{i'} e_{i'}^T\right) \\ &= (\text{vec } \tau)^T \left((e_{i'} e_{i'}^T) \otimes \Gamma\right) (\text{vec } \tau) \\ &= \|\hat{\tau}\|_{\Gamma_{i'}}^2 \end{aligned} \tag{30}$$

where we used the relation $\text{tr}(A^T BCD^T) = (\text{vec } A)^T (D \otimes B)(\text{vec } C)$ that holds for matrices A, B, C, D of compatible dimensions. Thus $\|\tau_i\|_\Gamma = \|\hat{\tau}\|_{\Gamma_{i'}}$, and noting $j+1 \le i' \le j+k+1$ when $j-k \le i \le j$, (27) is derived. (QED)

This results on the constraint $\|p(t)\|_\Gamma \le c$ for the interval $[t_j, t_{j+1}]$ can be extended to any knot point interval. The case of entire interval $[t_0, t_m]$ is of particular interest, and can be derived by letting $j = 0, 1, \cdots, m-1$. Namely we see that

$$\|p(t)\|_\Gamma \le c \;\; \forall t \in [t_0, t_m] \tag{31}$$

holds if

$$\|\tau_i\|_\Gamma \le c, \quad -k \le i \le m-1 \tag{32}$$

or in terms of $\hat{\tau}$, with $M = m + k$,

$$\|\hat{\tau}\|_{\Gamma_i} \le c, \quad 1 \le i \le M \tag{33}$$

2) The cases $l = 1$ and $l = 2$: Next we consider the case of constraints on derivatives. For $l = 1$, the 1st derivative $p^{(1)}(t)$ of $p(t)$ in (2) can be described by B-splines of degree $k-1$ as

$$p^{(1)}(t) = \sum_{i=-(k-1)}^{m-1} \tau_i' B_{k-1}(\alpha(t - t_i)), \tag{34}$$

where the control point τ_i' is related to τ_i in (2) by

$$\tau_i' = \alpha(\tau_i - \tau_{i-1}). \tag{35}$$

(see e.g. [11]). Since (34) is in the same form as for $p(t)$ in (2), the results obtained for the case $l = 0$ can be extended to the case of derivative.

For the entire time interval $[t_0, t_m]$, in particular, the following constraint,

$$\|p^{(1)}(t)\|_\Gamma \le c \;\; \forall t \in [t_0, t_m] \tag{36}$$

holds if the control point τ_i' in (34) satisfy

$$\|\tau_i'\|_\Gamma \le c, \quad -(k-1) \le i \le m-1. \tag{37}$$

Moreover (37) can be written in terms of $\hat{\tau}$ in (8) as

$$\|\hat{\tau}\|_{\Gamma_i'} \le c, \quad 2 \le i \le M \tag{38}$$

where $\Gamma_i' = (e_i' e_i'^T) \otimes \Gamma$ and $e_i' \in \mathbf{R}^M$ is defined by

$$e_i' = \alpha(e_i - e_{i-1}). \tag{39}$$

The case of $l = 2$ follows similarly, by noting that the 2nd derivative $p^{(2)}(t)$ of $p(t)$ in (2) can be described as

$$p^{(2)}(t) = \sum_{i=-(k-2)}^{m-1} \tau_i'' B_{k-2}(\alpha(t - t_i)), \tag{40}$$

where the control point τ_i'' is defined by

$$\tau_i'' = \alpha^2(\tau_i - 2\tau_{i-1} + \tau_{i-2}). \tag{41}$$

Then we easily see that

$$\|p^{(2)}(t)\|_\Gamma \le c \;\; \forall t \in [t_0, t_m] \tag{42}$$

holds if τ_i'' in (40) satisfy

$$\|\tau_i''\|_\Gamma \le c, \quad -(k-2) \le i \le m-1, \tag{43}$$

or equivalently, if the control point vector $\hat{\tau}$ satisfies

$$\|\hat{\tau}\|_{\Gamma_i''} \le c, \quad 3 \le i \le M \tag{44}$$

where $\Gamma_i'' = (e_i'' e_i''^T) \otimes \Gamma$ and $e_i'' \in \mathbf{R}^M$ is defined by

$$e_i'' = \alpha^2(e_i - 2e_{i-1} + e_{i-2}). \tag{45}$$

C. Smoothing Spline Trajectory

Now that we developed the method for imposing quadratic constraints on the trajectory $p(t)$ and its derivatives $p^{(l)}(t)$, we summarize the procedure for trajectory planning. Namely, our problem reduces to quadratically constrained quadratic programming (QCQP) problem described in terms of the control point vector $\hat{\tau}$, where we minimize the quadratic cost function,

$$\min_{\hat{\tau} \in \mathbf{R}^{2M}} J(\hat{\tau}) = \frac{1}{2} \hat{\tau}^T G \hat{\tau} - g^T \hat{\tau} \tag{46}$$

subject, typically, to the linear equality and inequality constraints of the form

$$A_{eq} \hat{\tau} = d_{eq}, \;\; A_{in} \hat{\tau} \le d_{in}, \tag{47}$$

and the quadratic inequality constraints

$$\hat{\tau}^T P_i \hat{\tau} \le c, \;\; i = 1, 2, \cdots, N_q. \tag{48}$$

In (46), G and g are given in (10) and (11) respectively, and $A_{in}\hat{\tau} \le d_{in}$ is set up as collection of inequalities in (19). The total number of constraints resulting from (19) for $0 \le i \le n-1$ becomes $2M + 2(n-1)k$. The equality $A_{eq}\hat{\tau} = d_{eq}$ typically is used to specify initial and final conditions of $p(t)$ ($t_0 \le t \le t_m$) as

$$\begin{aligned} p(t_0) = p_0, \;\; \dot{p}(t_0) = 0, \;\; \ddot{p}(t_0) = 0 \\ p(t_m) = p_m, \;\; \dot{p}(t_m) = 0, \;\; \ddot{p}(t_m) = 0 \end{aligned} \tag{49}$$

in which case $A_{eq}\hat{\tau} = d_{eq}$ consists of 12 equations (see e.g. [10]).

The quadratic constraints (48) can be used to limit the magnitude of $p(t)$ as $\|p(t)\|_\Gamma \le c$ or magnitude of velocity $\|\dot{p}(t)\|_\Gamma \le c$ and/or acceleration $\|\ddot{p}(t)\|_\Gamma \le c$. The number of quadratic constraints is $N_q = M, M-1$ and $M-2$ for (33), (38) and (44), respectively.

Such a QCQP problem can be solved numerically by using MATLAB-based software CVX [3], [5].

IV. Numerical Example

We examine the effect of introducing quadratic inequality constraints to the trajectory planning problem for a path with piecewise linear boundaries. In particular, limits are imposed on the magnitudes of both velocity and acceleration. For numerical computations, cubic splines are used, i.e. $k = 3$ and $l = 2$ in (2) and (7), and CVX is used for numerical solutions.

A. Experimental Settings

We plan trajectory $p(t)$ in time interval $[t_0, t_m] = [0, 10]$ for the path with the 13 pairs of corner points shown in Table II. Letting the midpoints of each pair of right and left corners be C_i, i.e. $C_i = (R_i + L_i)/2$, we set the initial and final conditions for $p(t)$ as given in (49) with $p_0 = C_0$ and $p_m = C_{12}$. As the function $f(t)$ in the cost function $J(\tau)$ in (7), we employ the piecewise linear centerline of the path, namely we set

$$f_i(t) = q_i t + r_i, \;\; t \in [s_i, s_{i+1}]. \tag{50}$$

The parameters $q_i, r_i \in \mathbf{R}^2$ are determined so that $f_i(s_i) = C_i$ and $f_i(s_{i+1}) = C_{i+1}$. The total length L_c of the centerline for this path is $L_c = 97.97$ and hence the average speed is estimated as $v = L_c/(t_m - t_0) \simeq 9.8$.

We set the number of control points m as $m = 200$ and the smoothing parameter in (7) as $\lambda = 0.001$. We impose constraints on both velocity and acceleration simultaneously for the entire time interval $[0, 10]$, specifically, in (36) we let $\Gamma = I_2$ and $c = 12$ for $l = 1$ (velocity) and $c = 40$ for $l = 2$ (acceleration) in (42), i.e.

$$\begin{aligned} v(t) &= \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2} \leq 12 \\ a(t) &= \sqrt{\ddot{x}(t)^2 + \ddot{y}(t)^2} \leq 40 \end{aligned} \tag{51}$$

for all $t \in [0, 10]$. Note that this constraints require considerably more control points, $m = 200$ in above, than the case without quadratic constraints, say $m = 80$.

B. Experimental Results

Constructed smoothing spline trajectory $(x(t), y(t))$ is plotted in thick blue line in Fig. 3 and we see that the trajectory closely follows the centerline denoted by red dotted line, and it stays within the path shown in black lines. The trajectory planned without the quadratic constraints (51) is denoted by $(x0(t), y0(t))$ and plotted in thick red dotted line.

Velocity and acceleration profiles, $v(t)$ and $a(t)$, are plotted in Fig. 4 and 5 respectively. Those of $(x(t), y(t))$ and $(x0(t), y0(t))$ are shown, respectively in blue line and red dotted line. We see that the constraint on the speed $v(t) \leq 12$ is realized successfully for trajectory $(x(t), y(t))$, while the maximum value of $v(t)$ for $(x0(t), y0(t))$ was 13.52. Similarly the acceleration limit $a(t) \leq 40$ is achieved in $(x(t), y(t))$, while the maximum value without this constraint was max $a(t) = 68.57$. Thus in particular, we could reduce maximum acceleration considerably, although these two trajectories are similar in the xy plane as we see in Fig. 3.

Moreover, $x(t)$ and $y(t)$ are plotted in Fig. 6 in blue and red lines respectively and those without the quadratic constraints are shown together in respective dotted lines. Similarly the first derivative and second derivative of $x(t)$ and $y(t)$ are shown in Figs. 7 and 8 respectively. Obviously the magnitudes of the first and second derivatives of $x(t)$ and $y(t)$ remain within the specified limits of 12 and 40.

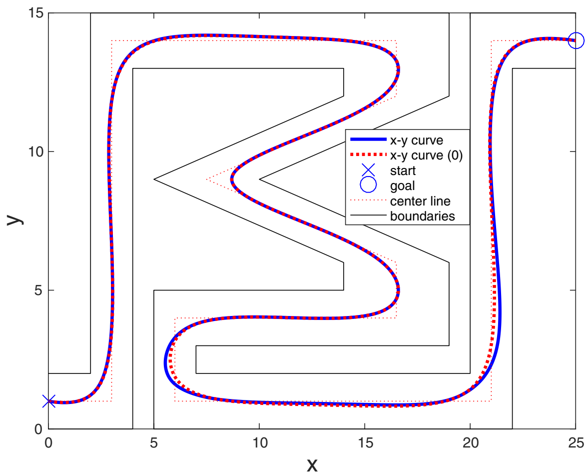

Figure 3. Constructed spline trajectories $(x(t), y(t))$ and $(x0(t), y0(t))$ plotted in the xy plane with and without the quadratic constraints in (51).

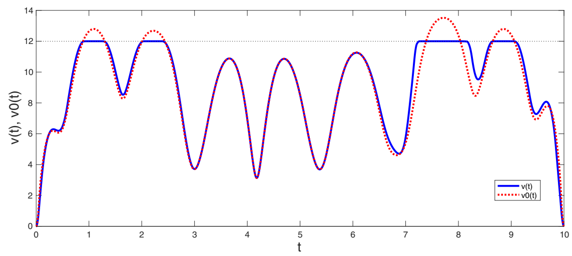

Figure 4. Velocity profiles $v(t) = \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}$ for $(x(t), y(t))$ and $(x0(t), y0(t))$ plotted in blue and red lines respectively.

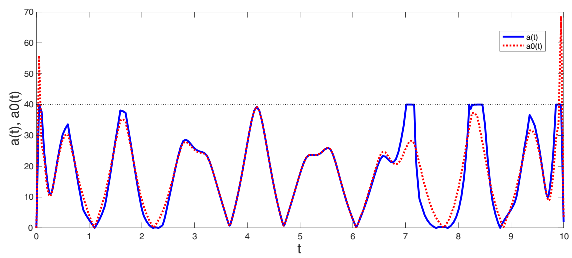

Figure 5. Acceleration profiles $a(t) = \sqrt{\ddot{x}(t)^2 + \ddot{y}(t)^2}$ for $(x(t), y(t))$ and $(x0(t), y0(t))$ plotted in blue and red lines respectively.

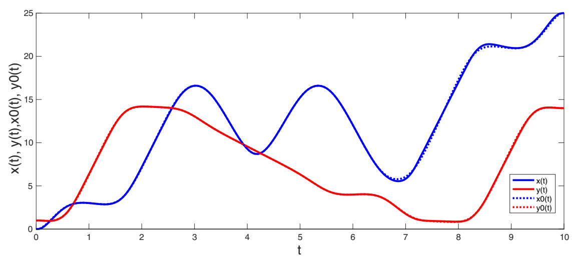

Figure 6. Positions $x(t)$ and $y(t)$ plotted in blue and red lines respectively, and $x0(t)$ and $y0(t)$ plotted in dotted lines.

V. Concluding Remarks

We developed a trajectory planning method of mobile robots in a two-dimensional plane. Trajectories are

Table II
THE COORDINATES R_i AND L_i OF RIGHT AND LEFT CORNERS FOR $i = 0, 1, \cdots, n$ WITH $n = 12$.

	0	1	2	3	4	5	6	7	8	9	10	11	12
R_i	0	4	4	14	14	5	14	14	5	5	22	22	25
	0	0	13	13	12	9	6	5	5	0	0	13	13
L_i	0	2	2	19	19	10	19	19	7	7	20	20	25
	2	2	15	15	12	9	6	3	3	2	2	15	15

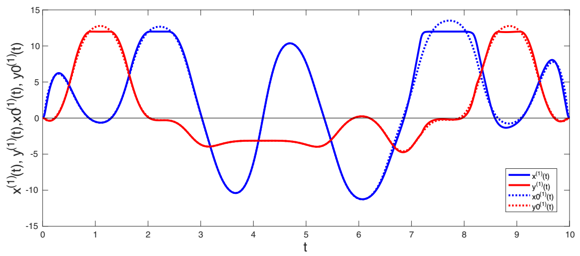

Figure 7. Velocities $x^{(1)}x(t)$ and $y^{(1)}(t)$ plotted in blue and red lines respectively, and $x0^{(1)}(t)$ and $y0^{(1)}(t)$ plotted in dotted lines.

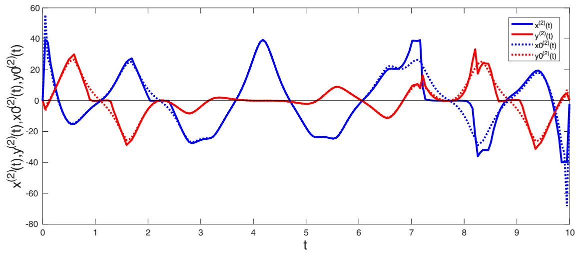

Figure 8. Accelerations $x^{(2)}(t)$ and $y^{(2)}(t)$ plotted in blue and red lines respectively, and $x0^{(2)}(t)$ and $y0^{(2)}(t)$ plotted in dotted lines.

constructed as two-dimensional vector smoothing splines using B-splines as the basis functions. In particular we introduced quadratic inequality constraints on the trajectory as well as on its derivatives, which can be used e.g. to limit the maximum speed and acceleration of the robot motion. Such constraints are applied here to plan trajectories for path with piecewise linear boundaries as treated in our recent paper [13]. The problem reduced to a QCQP problem described in terms of the control point vector. MATLAB-based software CVX can be used effectively to solve the problem numerically. We showed the usefulness and effectiveness by a numerical simulation example.

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