Quantitative Spatial Models

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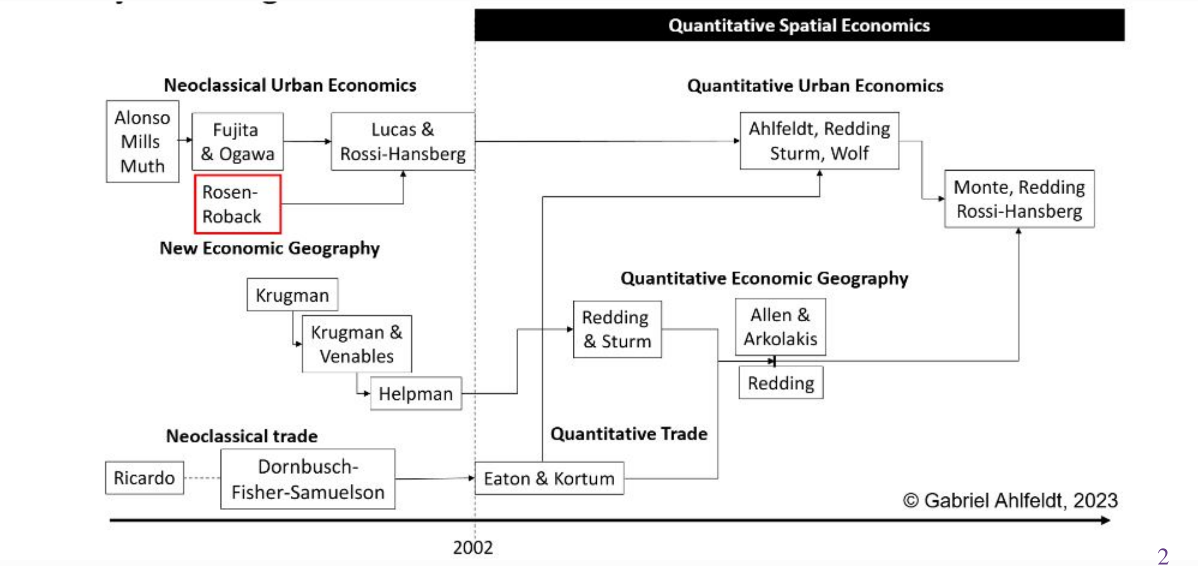
Quantitative spatial models (QSMs)

- In the 2010s, QSMs have began taking the field of spatial economics
  - inspired by trade models, especially Eaton and Kortum (2002)
  - extended to allow for spatial mobility and increasing returns
- Key idea: functional forms that lead to the gravity equation

$$log\left(X_{ij}\right) = \eta_j + \gamma_i - \beta\tau_{ij} + u_{ij}$$

  - τ_{ij}: bilateral iceberg trade costs
  - β: trade elasticity
- Still an ongoing boom (but for how long?)
- How does it differ from NEG, and which assumptions are key here?

Tree of QSMs



Quantitative Spatial Economics
Neoclassical Urban Economics
Quantitative Urban Economics
Alonso Mills Muth
Fujita & Ogawa
Lucas & Rossi-Hansberg
Ahlfeldt, Redding Sturm, Wolf
Rosen-Roback
Monte, Redding Rossi-Hansberg
New Economic Geography
Quantitative Economic Geography
Krugman
Krugman & Venables
Helpman
Redding & Sturm
Allen & Arkolakis
Redding
Neoclassical trade
Quantitative Trade
Ricardo
Dornbusch-Fisher-Samuelson
Eaton & Kortum
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2002

Space: welfare

- Following Allen and Arkolakis (2017), "Modern Spatial Economics: A Primer"
- Finite set of locations: $i \in S = \{1, ..., N\}$
- Arlington's assumption: each location produces a differentiated tradable good using labor
- Welfare in location i:

$$W_i = \left(\sum_{i \in S} q_{ji}^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} u_i$$

  - q_{ji}: per capita consumption in i of good from j
  - $\sigma > 1$: elasticity of substitution (across locations)
  - u_i: local amenity of the locality

Consumer problem

- Given location, each consumer maximizes W_i by choosing consumption according to a budget constraint defined by import prices p_{ji} and wages w_i:

$$\sum_j p_{ji} q_{ji} = w_i$$

- Given population L_j, j's expenditure on the good from i is

$$X_{ij} = p_{ij}^{1-\sigma} P_j^{\sigma-1} w_j L_j \tag{1}$$

    - the DS price index in j: $P_j = \left(\sum_i p_{ij}^{1-\sigma}\right)^{\frac{1}{1-\sigma}}$

Production and trade technologies

- Let A_i be the productivity of labor in i
- Iceberg transport costs: $\tau_{ij} \geq 1$ units must be shipped from i so that 1 unit arrives to destination j
- Competitive markets: imply that prices follow the marginal cost, summing production and trade costs:

$$p_{ij} = \frac{w_i}{A_i}\tau_{ij}$$

- Assume agglomeration economies, with agglomeration elasticity α:

$$A_i = \overline{A}_i L_i^{\alpha}$$

  - isomorphic to the NEG framework with differentiated varieties

The gravity equation (for trade)

- Replacing prices in equation 1, we obtain a gravity equation:

$$X_{ij} = \tau_{ij}^{1-\sigma} \times \left(\frac{w_i}{A_i}\right)^{1-\sigma} \times P_j^{\sigma-1} w_j L_j \tag{2}$$

  - $1-\sigma$ is the trade elasticity

Migration

- L_i^o workers "are born" in i, and the utility of such worker when living in j depends on real wages, amenities and migration frictions ν_{ij}:

$$
\begin{aligned}
W_{ij} &= \frac{w_j}{P_j} u_j \cdot \nu_{ij} \\
&= \frac{w_j}{P_j} u_j \cdot \left(\frac{L_i j}{L_i^o} \right)^{-\beta} \frac{1}{\mu_{ij}}
\end{aligned}
$$

  - $L_i j$: number of workers migrating from i to j
  - μ_{ij}: exogenous migration friction

The gravity equation (for migration)

- In equilibrium, all agents originating from i have the same utility:

$$W_i = W_{ij} = \frac{w_j}{P_j}\frac{u_j}{\mu_{ij}}\left(\frac{L_{ij}}{L_i^o}\right)^{-\beta}$$

- Inverting the equation, we obtain a gravity equation:

$$L_{ij} = \mu_{ij}^{-\frac{1}{\beta}} \times W_i^{-\frac{1}{\beta}} L_i^o \times \left(\frac{w_j}{P_j}u_j\right)^{\frac{1}{\beta}} \tag{3}$$

Goods market clearing conditions

- Equality between payments received and income

$$w_i L_i = \sum_{j \in S} X_{ij}$$

- Equality between payments disbursed and income

$$w_i L_i = \sum_{j \in S} X_{ji}$$

Price index

- Using the gravity equation 2 on the last equality, we have

$$w_i L_i = \sum_{j \in S} X_{ji} = \sum_j \tau_{ij}^{1-\sigma} \times \left(\frac{w_i}{A_i}\right)^{1-\sigma} \times P_j^{\sigma-1} w_j L_j$$

- Rearranging, we obtain an analytical solution to the price index

$$P_i = \left[\sum_j \tau_{ji}^{1-\sigma} \left(\frac{w_j}{A_j}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}} \tag{4}$$

- The price index increases with remoteness and distance weighted marginal cost shifts

Labor market clearing conditions

- All labor distributes across the locations (closed economy):

$$L_i^o = \sum_{j \in S} L_{ij}$$

- Replacing the migration gravity equation 3:

$$L_i^o = \sum_{j \in S} \mu_{ij}^{-\frac{1}{\beta}} \times W_i^{-\frac{1}{\beta}} L_i^o \times \left(\frac{w_j}{P_j} u_j \right)^{\frac{1}{\beta}}$$

Welfare according to place of birth

- By inverting the last equation, we have that welfare depends on migration-friction weighted real wages and amenities:

$$W_i = \Big[\sum_{j \in S} \big(\frac{w_j}{P_j} \frac{u_j}{\mu_{ij}} \big)^{\frac{1}{\beta}} \Big]^{\beta}$$

- Note that, by plugging back into the migration gravity equation, destinations are somewhat similar to a multinomial logit model:

$$\frac{L_{ij}}{L_i^o} = \frac{\left(\frac{w_j}{P_j} \frac{u_j}{\mu_{ij}} \right)^{\frac{1}{\beta}}}{\sum_k \left(\frac{w_k}{P_k} \frac{u_k}{\mu_{ik}} \right)^{\frac{1}{\beta}}}$$

Defining a geography in this economy

- A geography consists of:
  1. trade costs: $\{\tau_{ij}\}$
  2. migration frictions: $\{\mu_{ij}\}$
  3. productivities: $\{\overline{A}_i\}$
  4. amenities: $\{u_i\}$
  5. initial population: $\{L_i^o\}$
- The geography and the trade (σ), agglomeration (α) and migration (β) elasticities determine the outcome of the model
  - these are all the parameters that have to be estimated ($2N^2 + 3N + 3$)

Equilibrium

- Given a geography and the elasticities (σ, α, β), an equilibrium consists of lists of labor L_i, wages w_i, price indexes P_i and welfare W_i (in total, $4N$ equilibrium variables) that satisfy the following ($4N$) equations:

$$w_i L_i = \sum_{j \in S} \tau_{ij}^{1-\sigma} \times \left(\frac{w_i}{A_i}\right)^{1-\sigma} \times P_j^{\sigma-1} w_j L_j$$

$$P_i^{1-\sigma} = \sum_{j \in S} \tau_{ji}^{1-\sigma} \left(\frac{w_j}{A_j}\right)^{1-\sigma}$$

$$L_i = \sum_{j \in S} \mu_{ij}^{-\frac{1}{\beta}} \times W_i^{-\frac{1}{\beta}} L_i^o \times \left(\frac{w_j}{P_j} u_j\right)^{\frac{1}{\beta}}$$

$$W_i^{\frac{1}{\beta}} = \sum_{j \in S} \left(\frac{w_j}{P_j} \frac{u_j}{\mu_{ij}}\right)^{\frac{1}{\beta}}$$

When is the equilibrium unique?

- The same reinforcing dynamics present in the CP model are also present in QSMs
- Hence, there may be equilibrium multiplicity
- But, to quantitatively solve the model, we need the equilibrium to be unique (and so invertible)
- Allen and Arkolakis (QJE 2014) show that uniqueness hold when:
  - geography is sufficiently varied (heterogeneity in A_i and u_i)
  - dispersion forces (β, σ) are high relative to agglomeration forces (α)
- To allow for multiplicity, two possible solutions:
  - Allen and Donaldson (2022): consider dynamics, with multiple steady-states but determined equilibrium path (history matters)
  - Ouazad (2024): consider homotopies, functions connecting complex economies with simpler economies (where equilibria are easier to characterize)

Special cases

1. **Trade model:** workers cannot move ($\mu_{ij} = \infty$ if $i \neq j$), so $L_i = L_i^o$
   - only the first two set of conditions matter
   - welfare is simple: $W_i = \frac{w_i}{P_i} u_i$
2. **Spatial equilibrium:** no migration frictions ($\mu_{ij} = 1$)
   - welfare is equalized across space $W = W_i = \frac{w_i}{P_i} u_i$
   - β is better interpreted as housing costs, and the population residing in i follows:

$$\frac{L_i}{\bar{L}} = \frac{\left(\frac{w_i u_i}{P_i}\right)^{\frac{1}{\beta}}}{\sum_j \left(\frac{w_j u_j}{P_j}\right)^{\frac{1}{\beta}}}$$

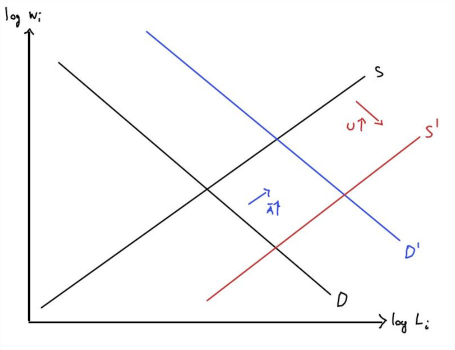
The Rosen-Roback model

- A canonical model of urban economics, the Rosen-Roback model, is a special case of the QSM framework
- This model focuses on small cities within a larger economy, with free movement of population and a composite perfectly tradable good
- In the QSM framework, $\tau_{ij} = 1$ and $\mu_{ij} = 1$, and $N \to \infty$
- Same welfare and price index across all cities, which simplifies half of the equilibrium variables
- The Rosen-Roback model is a supply and demand model of city-level wages w_i and population L_i

The Rosen-Roback in log (and graphical) form

Supply: $\quad log\ L_i + \kappa^S + \frac{1}{\beta} log\ u_i + \frac{1}{\beta} log\ w_i$

Demand: $\quad log\ L_i + \kappa^D + \frac{\sigma - 1}{1-(1-\sigma)\alpha} log\ \overline{A}_i - \frac{\sigma}{1-(1-\sigma)\alpha} log\ w_i$



Estimation of the model in two steps

1. Estimate model elasticities by using log equations and assuming that trade and migration frictions are a function of distance:

$$log\left(X_{ij}\right) = \eta_j + \gamma_i - \theta logDist_{ij} + u_{ij}$$

   - Note that only some flow data can be enough to estimate these elasticities
   - Challenge: what to do with zeros? Silva and Tenreyro (2006), "The log of gravity"
   - Challenge 2: endogeneity in distances? Instrument (e.g. with proposed transportation networks)

2. Once we have the elasticities, the system of equations can be computationally inverted to back out the geography (amenities and productivities)

   - Influential computational toolkit: `https://github.com/Ahlfeldt/MRRH2018-toolkit`

The making of the modern metropolis: Evidence from London

Heblich, Redding, Sturm

The Quarterly Journal of Economics

2020

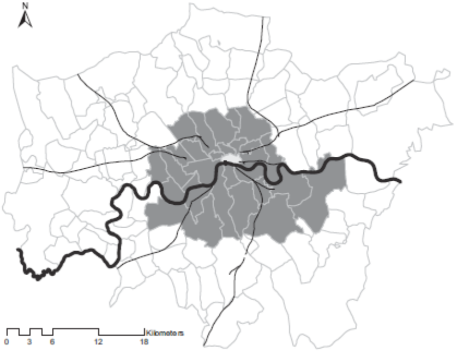
Motivation

- Big metropolitan areas require the spatial separation of residential and work locations
  - commuting is an important aspect of resident lives
  - but to commute is slow, especially in dense regions
- This paper: investigate how metropolitan railroad transportation led to metropolitan growth
  - context: London (1801-1921)
  - focus: neighborhood specialization as railroad lines are built
  - rich historical data, used to discipline a QSM

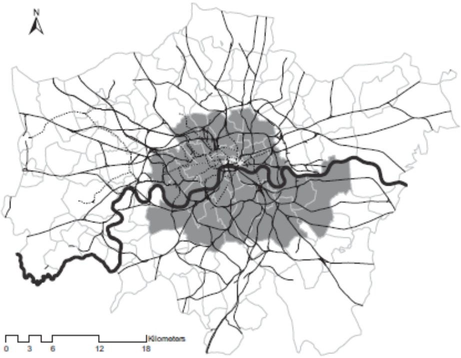
Historical background

- Before railways: compact, walkable city (1 million people in 1801)
- Steam railway (1836 onward): reduced travel times, enabled suburbanization
- By 1921, Greater London population exceeded 7 million
- How much does railroad improve commute speeds?
  - walking: 3 mph
  - horse omnibus, stagecoaches and trams: 5–6 mph
  - steam railroads: 21 mph

Railroad expansion over the sample period



(B) Railway Network 1841



(D) Railway Network 1921

Data

- Population and employment data from UK Census (1801-1921)
  - constant parish population data roughly every 10 years
  - employment from day censuses of the City of London (1866, 1881, 1891, 1911)
- Commuting flows from the 1921 Census travel-to-work matrix
- Rateable land values from historical tax records
- Historical railway maps digitized from archival sources

London’s population grew but downtown’s did not

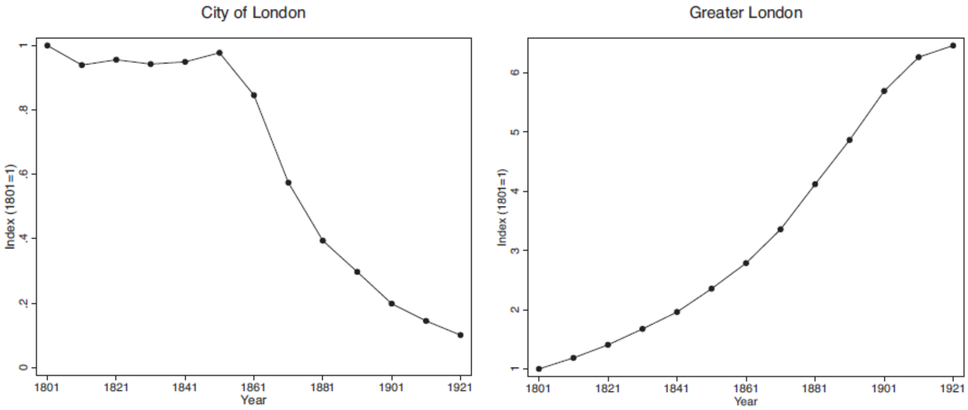
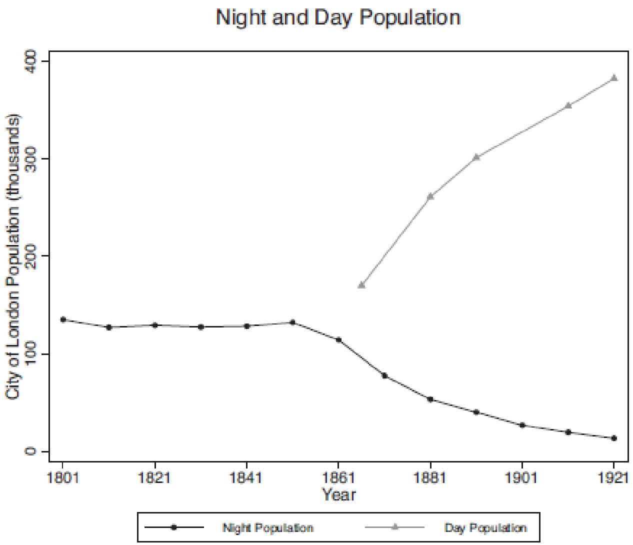


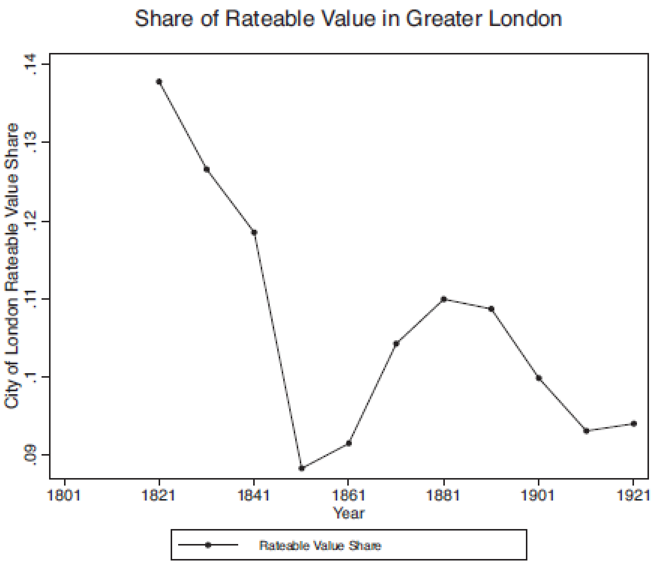
FIGURE II

Population Indexes over Time (City of London and Greater London, 1801 = 1)

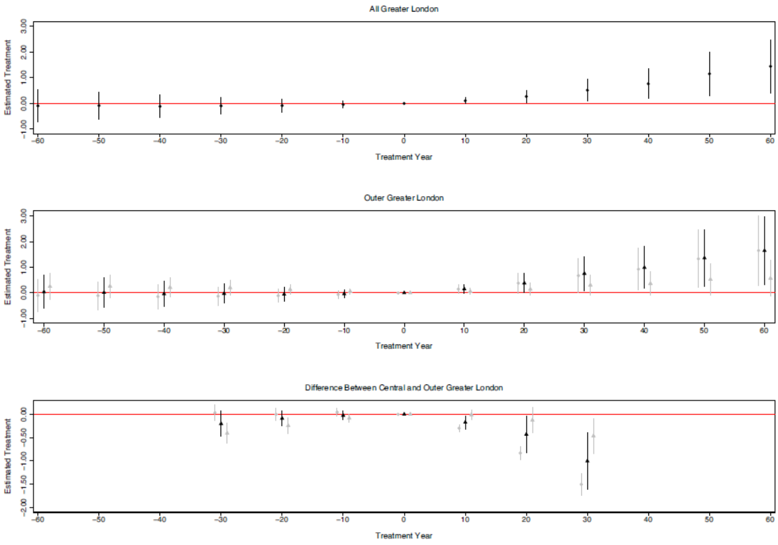
Downtown specialized in day jobs instead,



and its share in real estate value fell, to later grow



Event study effects of railroad stations on population



Theoretical model

- N locations in London, $M - N$ locations in the rest of the economy
  - assume prohibitive commuting costs out of London
- Exogenous mass of workers in each period: L_{Mt}
- Choose pairs of residence n and employment i to maximize:

$$U_{ni} = \frac{B_n b_{ni} w_i}{\kappa_{ni} P_n^{\alpha} Q_n^{1-\alpha}}$$

  - B_n: amenity of residential location n
  - b_{ni}: idiosyncratic Frechet shock with scale 1 and shape $\varepsilon > 1$
  - w_i: wage earned in employment location i
  - P_n: local price of goods
  - Q_n: local price of housing

Location choice

- Given the Frechet specification, the probability of choosing (n, i) (conditional on living and working in London) is

$$\lambda_{ni} = \frac{L_{ni}}{L_N} = \frac{\left(B_n w_i\right)^{\varepsilon}\left(\kappa_{ni} P_n^{\alpha} Q_n^{1-\alpha}\right)^{-\varepsilon}}{\sum_{i' \in N} \sum_{n' \in N} \left(B_{n'} w_{i'}\right)^{\varepsilon}\left(\kappa_{n'i'} P_{n'}^{\alpha} Q_{n'}^{1-\alpha}\right)^{-\varepsilon}}$$

- Summing up across i (n) we find the share of the population residing (working) in each location

$$\lambda_n^R = \frac{R_n}{L_N} = \frac{\sum_i \left(B_n w_i\right)^{\varepsilon}\left(\kappa_{ni} P_n^{\alpha} Q_n^{1-\alpha}\right)^{-\varepsilon}}{\sum_{i' \in N} \sum_{n' \in N} \left(B_{n'} w_{i'}\right)^{\varepsilon}\left(\kappa_{n'i'} P_{n'}^{\alpha} Q_{n'}^{1-\alpha}\right)^{-\varepsilon}}$$

$$\lambda_i^L = \frac{L_i}{L_N} = \frac{\sum_n \left(B_n w_i\right)^{\varepsilon}\left(\kappa_{ni} P_n^{\alpha} Q_n^{1-\alpha}\right)^{-\varepsilon}}{\sum_{i' \in N} \sum_{n' \in N} \left(B_{n'} w_{i'}\right)^{\varepsilon}\left(\kappa_{n'i'} P_{n'}^{\alpha} Q_{n'}^{1-\alpha}\right)^{-\varepsilon}}$$

Labor supply to London

- The expected utility of a worker in the economy is

$$\overline{U} = \Gamma(1-\frac{1}{\varepsilon})\left(\sum_{i'\in N\cup M}\sum_{n'\in N\cup M}\left(B_{n'}w_{i'}\right)^{\varepsilon}\left(\kappa_{n'i'}P_{n'}^{\alpha}Q_{n'}^{1-\alpha}\right)^{-\varepsilon}\right)^{\frac{1}{\varepsilon}}$$

- Using the choice of N, we have that ε defines the elasticity of labor supply to London

$$\overline{U}\left(\frac{L_N}{L_{N\cup M}}\right)^{\frac{1}{\varepsilon}} = \Gamma(1-\frac{1}{\varepsilon})\left(\sum_{i'\in N}\sum_{n'\in N}\left(B_{n'}w_{i'}\right)^{\varepsilon}\left(\kappa_{n'i'}P_{n'}^{\alpha}Q_{n'}^{1-\alpha}\right)^{-\varepsilon}\right)^{\frac{1}{\varepsilon}}$$

Production of consumption goods

- Cobb-Douglas technology using labor L_i, machinery inputs M_i and commercial space (land) H_i^L
- Expenditures are constant shares of revenue X_i

$$w_i L_i = \beta^L X_i,\ q_i H_i = \beta^H X_i,\ rM = \beta^M X_i$$

- Note that payments for commercial floor space are proportional to labor income
- Allow distortions in land markets (e.g. due to zoning): $q_i = \xi_i Q_i$
- For counterfactual analysis, the authors consider a structure with both traded and non-traded goods

Market clearing conditions

- Land market clearing:

$$Q_n H_n^R + q_n H_n^L = (1-\alpha)\left(\sum_{i\in N} \lambda_{ni|n}^R w_i\right) R_n + \frac{\beta^H}{\beta^L} w_n L_n \tag{5}$$

- Commuter market clearing:

$$L_i = \sum_{n\in N} \lambda_{ni|n}^R R_n \tag{6}$$

Exact hat algebra

- Fix $\tau = 1921$ as the reference year
- All equations can be rewritten as a function of the variables in the baseline year and the ratio

$$\hat{Y}_{nt} = \frac{Y_{nt}}{Y_{n\tau}}$$

- Useful because we can find the values for observable variables in the baseline year (e.g. commuting flows) and invert the system to obtain their values for other years, given changes in observable variables (e.g. rateable values)
- Part of the reason for the success of QSEs, as they are run well with limited data, due to the assumed structure
- This is also why the authors do not need to make assumptions on housing markets and production technologies for estimating the model

Model estimation

- **Step 1:** using 1921 data, find residential population, employment, commuting probabilities and rateable values for the baseline year.
- **Step 2:** using the 1921 equilibrium, invert the equilibrium conditions to find the wages in the locations. For this, we need assumptions on parameters:
  - $\alpha = 0.75$ (based on a parliamentary survey of workers)
  - from historical papers, $\beta^L = 0.55$, $\beta^M = 0.20$ and $\beta^H = 0.25$

Model estimation (cont.)

- **Step 3:** estimate commuting costs for all years. This requires the assumption that commuting costs are a function of distance

$$-\varepsilon log \kappa_{nit} = -\varepsilon \phi log d^W_{nit} + u^R_{nt} + u^L_{it}$$

  - use assumed speeds and the rail network to find distances d^W_{nit} using the LCPs obtained through Djiikstra's algorithm
  - for $t = 1921$, we observe the commuting flows, and this assumption plus the Frechet model implies that commuting flows follow

$$log \lambda_{nit} = -\varepsilon \phi log d^W_{nit} + \gamma_{it} + \gamma_{nt} + u_{nit}$$

  - it can be estimated with OLS, or using linear distances as IVs (since railroads are endogenous)
  - ignore pairs without commuting, assuming prohibitive costs (so basically the model will capture intensive margins)
  - using changes in the network and the estimates, find the exact hat changes in bilateral commuting costs for other years

Gravity equation estimates for 1921

TABLE I

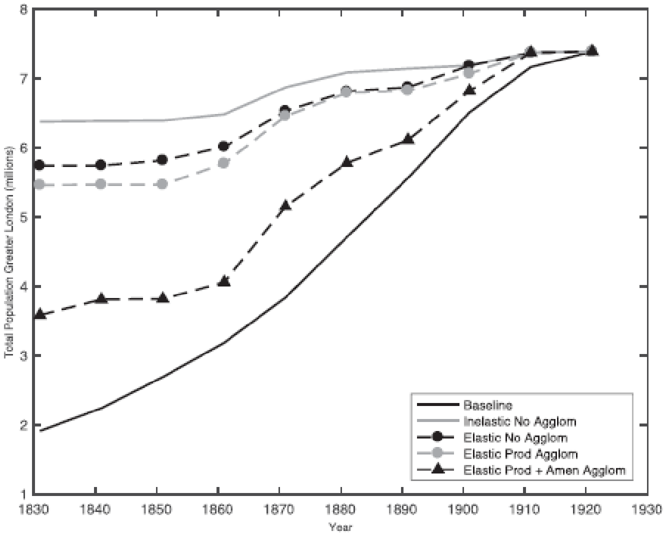
GRAVITY EQUATION ESTIMATION USING 1921 BILATERAL COMMUTING DATA

	(1)	(2)
Second-stage regression		
	$\log \lambda_{nit}$	$\log \lambda_{nit}$
$\log d^{W}_{nit}$	−4.899***	−5.203***
	(0.062)	(0.069)
Workplace fixed effects	yes	yes
Residence fixed effects	yes	yes
Kleibergen-Paap (p-value)		.000
Estimation	OLS	IV
Observations	3,023	3,023
R-squared	0.851	—
First-stage regression		
		$\log d^{W}_{nit}$
$\log d^{S}_{ni}$		0.429***
		(0.003)
Workplace and residence fixed effects		yes
First-stage F-statistic		22,235
Observations		3,023
R-squared		0.949

Model estimation

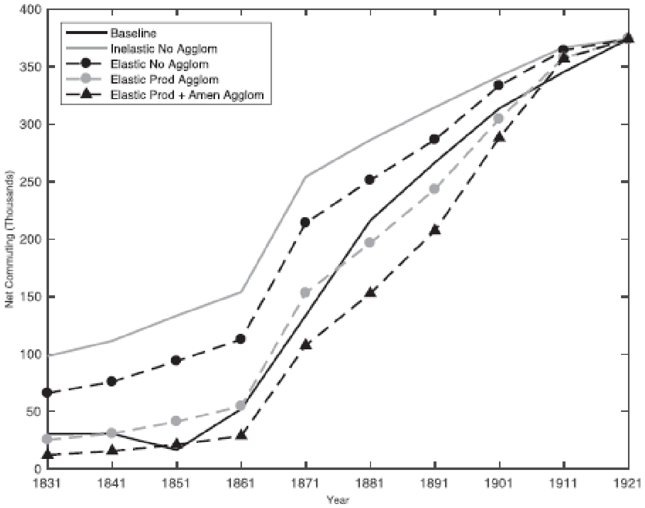
- **Step 4:** find the Frechet parameter and employment for other years.
  - for other years, we know employment only for the City of London
  - select the Frechet parameter to minimize the squared residuals for this subset, $\epsilon = 5.25$
  - it is then possible to find employment estimates across all years
- Note that amenities and productivity do not enter here
  - because the analysis is conditional on employment flows in 1921
  - for counterfactuals, the authors include (much) more structure

Counterfactual: removing all railroads



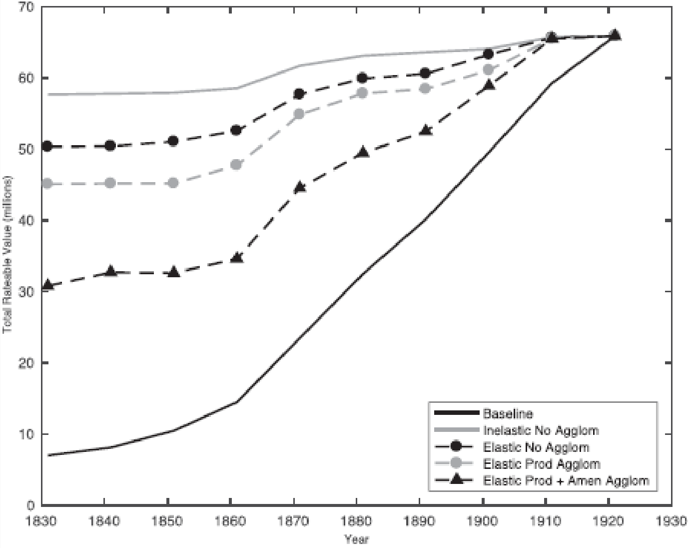
(C) Population Greater London (All Rail)

Counterfactual: removing all railroads (cont.)



(A) Net Commuting into City of London (All Rail)

Counterfactual: removing all railroads (cont.)



(E) Rateable Value Greater London (All Rail)

Evaluating the metropolitan railroads

Floor space supply elasticity	$\mu = 0$	$\mu = 1.83$	$\mu = 1.83$	$\mu = 1.83$
Production agglomeration force	$\eta^L = 0$	$\eta^L = 0$	$\eta^L = 0.086$	$\eta^L = 0.086$
Residential agglomeration force	$\eta^R = 0$	$\eta^R = 0$	$\eta^R = 0$	$\eta^R = 0.172$
Panel A: Removing the entire overground and underground railway network				
Economic impact				
Rateable value	–£8.24m	–£15.55m	–£20.78m	–£35.07m
NPV rateable value (3%)	–£274.55m	–£518.26m	–£692.76m	–£1,169.05m
NPV rateable value (5%)	–£164.73m	–£310.96m	–£415.66m	–£701.43m
Construction costs				
Cut-and-cover underground		–£9.96m		
Bored-tube underground		–£22.90m		
Overground railway		–£33.19m		
Total all railways		–£66.05m		
Ratio economic impact to construction cost				
$\frac{\text{NPV rateable value (3\%)}}{\text{Construction cost}}$	4.16	7.85	10.49	17.70
$\frac{\text{NPV rateable value (5\%)}}{\text{Construction cost}}$	2.49	4.71	6.29	10.62