Equivalent continuum for viscoelastic metamaterials

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Appendix A. Frequency-dependent effective material properties

In this appendix, $\underset{\sim}{\xi}$ is eliminated first followed by elimination of $\underset{\sim}{\mathbf{v}}$. Dropping the subscript ${}^2\underset{\sim}{\boldsymbol{\mu}}_{\mathrm{M}} = {}^2\underset{\sim}{\boldsymbol{\mu}}$ for clarity, the effective constitutive model for momentum rate can be written by replacing the column with a summation. The first equation (33) become

$$\dot{\boldsymbol{\pi}}_{\mathrm{M}} = \sum_{k=1}^{3} \boldsymbol{\mu}_k \cdot \dot{\mathbf{v}}_k + \frac{1}{V} \sum_{j=1}^{\mathrm{N}} \mathbf{a}_j^{(1)} \dot{\xi}_j. \tag{A.1}$$

The first equation (34) is also rewritten with a summation notation for each corner $k = 1, 2, 3$ and local resonance modes $j = 1, ..., \mathrm{N}$. Hence, it reads for each column entry $i = 1, 2, 3$

$$\boldsymbol{\mu}_i \cdot \dot{\mathbf{u}}_{\mathrm{M}} - \frac{1}{V} \sum_{k=1}^{3} \mathbf{M}_{ik}^{(\mathrm{lw})} \cdot \mathbf{v}_k + \frac{1}{V} \sum_{j=1}^{\mathrm{N}} \left(\mathbf{b}_j \dot{\xi}_j - \mathbf{a}_j^{(1)} \xi_j \right) = \mathbf{0}. \tag{A.2}$$

Next, the second equation (34) can expressed for modes $j = 1, ..., \mathrm{N}$. Since the $\underline{I}^{(\mathrm{lr})}$ and $\underline{\Lambda}^{(\mathrm{lr})}$ are diagonal each column entry can be written separately

$$\boldsymbol{\alpha}_j \cdot \dot{\mathbf{u}}_{\mathrm{M}} + \sum_{k=1}^{3} (\mathbf{b}_{jk} \cdot \dot{\mathbf{v}}_k - \mathbf{a}_{jk} \cdot \mathbf{v}_k) + \dot{\xi}_j - \mathrm{i}\omega_j \xi_j = 0. \tag{A.3}$$

Equation (A.3) in the frequency domain reads

$$\xi_j = \frac{-\boldsymbol{\alpha}_j \cdot \mathrm{i}\omega \mathbf{u}_{\mathrm{M}} + \sum_{k=1}^{3} (-\mathrm{i}\omega \mathbf{b}_{jk} + \mathbf{a}_{jk}) \cdot \mathbf{v}_k}{(\mathrm{i}\omega - \mathrm{i}\omega_j)}, \quad \text{for } j = 1, 2, ..., \mathrm{N}. \tag{A.4}$$

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and this expression is used to eliminate ξ_j from equations (A.1), leading to

$$
\begin{aligned}
\boldsymbol{\pi}_{\mathrm{M}} =& \overbrace{\left[\frac{1}{V}\sum_{j=1}^{\mathrm{N}}\frac{-\boldsymbol{\alpha}_j\otimes\boldsymbol{\alpha}_j}{(\mathrm{i}\omega-\mathrm{i}\omega_j)}\right]}^{\boldsymbol{\delta}_1(\omega)\equiv}\cdot\mathrm{i}\omega\mathbf{u}_{\mathrm{M}} \\
&+\sum_{k=1}^{3}\overbrace{\left[\boldsymbol{\mu}_k+\frac{1}{V}\sum_{j=1}^{\mathrm{N}}\left(\frac{-\mathrm{i}\omega\boldsymbol{\alpha}_j\otimes\mathbf{b}_{jk}+\boldsymbol{\alpha}_j\otimes\mathbf{a}_{jk}}{\mathrm{i}\omega-\mathrm{i}\omega_j}\right)\right]}^{\text{k-th entry of column } \underset{\sim}{\boldsymbol{\delta}}_2^{\mathrm{T}}(\omega)}\cdot\mathbf{v}_k \\
&=\boldsymbol{\delta}_1(\omega)\cdot\mathrm{i}\omega\mathbf{u}_{\mathrm{M}}+\underset{\sim}{\boldsymbol{\delta}}_2^{\mathrm{T}}(\omega)\cdot\underset{\sim}{\mathbf{v}}
\end{aligned}
\tag{A.5}
$$

and the elimination of the enrichment variables ξ_j in the constitutive model (A.2) reads for corners i=1,2,3

$$
\mathbf{0}=\overbrace{\left[\boldsymbol{\mu}_i+\frac{1}{V}\sum_{j=1}^{\mathrm{N}}\left(\frac{-\mathrm{i}\omega\mathbf{b}_{ji}\otimes\boldsymbol{\alpha}_j+\mathbf{a}_{ji}\otimes\boldsymbol{\alpha}_j}{\mathrm{i}\omega-\mathrm{i}\omega_j}\right)\right]}^{\text{i-th entry of column } \underset{\sim}{\boldsymbol{\delta}}_2(\omega)}\cdot\mathrm{i}\omega\mathbf{u}_{\mathrm{M}} \tag{A.6}
$$

$$
-\sum_{k=1}^{3}\overbrace{\left[\frac{1}{V}\mathbf{M}_{ik}^{(\mathrm{lw})}+\frac{1}{V}\sum_{j=1}^{\mathrm{N}}\left(\frac{-\omega^2\mathbf{b}_{ji}\otimes\mathbf{b}_{jk}-\mathrm{i}\omega(\mathbf{a}_{ji}\otimes\mathbf{b}_{jk}+\mathbf{b}_{ji}\otimes\mathbf{a}_{jk})+\mathbf{a}_{ji}\otimes\mathbf{a}_{jk}}{\mathrm{i}\omega-\mathrm{i}\omega_j}\right)\right]}^{(i,k)\text{-th entry of matrix } \underline{\boldsymbol{\delta}}_4(\omega)}\cdot\mathbf{v}_k \tag{A.7}
$$

or, all together

$$
\underset{\sim}{\mathbf{0}}=\underset{\sim}{\boldsymbol{\delta}}_2(\omega)\cdot\mathrm{i}\omega\mathbf{u}_{\mathrm{M}}+\underline{\boldsymbol{\delta}}_4(\omega)\cdot\underset{\sim}{\mathbf{v}}. \tag{A.8}
$$

Finally, eliminating $\mathbf{v}_k$, the effective density and effective 4th-order elastic tensor are

$$
\boldsymbol{\pi}_{\mathrm{M}}=\overbrace{\left(\boldsymbol{\delta}_1(\omega)+\underset{\sim}{\boldsymbol{\delta}}_2^{\mathrm{T}}(\omega)\cdot\underline{\boldsymbol{\delta}}_4^{-1}(\omega)\cdot\underset{\sim}{\boldsymbol{\delta}}_2(\omega)\right)}^{{}^2\boldsymbol{\rho}^{\mathrm{eff}}(\omega)\equiv}\cdot\mathrm{i}\omega\mathbf{u}_{\mathrm{M}} \tag{A.9}
$$

$$
{}^2\boldsymbol{\sigma}_{\mathrm{M}}=\overbrace{\left({}^4\mathbf{C}_{\mathrm{M}}+\mathrm{i}\omega\,{}^4\boldsymbol{\eta}_{\mathrm{M}}\right)}^{\mathbf{C}^{\mathrm{eff}}(\omega)}:\boldsymbol{\nabla}_{\mathrm{M}}\mathbf{u}_{\mathrm{M}}. \tag{A.10}
$$

This is an alternative expression for the effective material properties given in Liupekevicius et al. [1] and carried out in the scripts in the supporting software dataset [2].

References

[1] Liupekevicius, R., Dommelen, J., Geers, M. & Kouznetsova, V. Equivalent continuum for viscoelastic metamaterials. *Computer Methods In Applied Mechanics And Engineering.* **445** pp. 118160 (2025)

[2] Liupekevicius, R., Van Dommelen, J.A.W., Geers, M.G.D. & Kouznetsova, V.G. MATLAB script and COMSOL models of the article 'Equivalent continuum for viscoelastic metamaterials' [Internet]. 4TU.ResearchData; 2025. Available from: https://data.4tu.nl/datasets/e1e12991-b957-4ca6-bd21-c59fe9d165b5/1