Lecture 14 - Graph Laplacians

Renan Monteiro Barbosa

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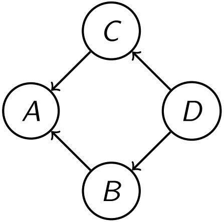
Graph Laplacians

Definition:

Let G be a graph. Let i be a vertex of G, the degree of i is the number of edges incident to vertex i, it is denoted d_i



Graph Laplacians

Definition:

Let $D_G \in \mathbb{R}^{nxn}$ be the diagonal matrix whose i^{th} diagonal entry is d_i.

Recall the adjacency matrix $A_G = (a_{ij}) \in \mathbb{R}^{nxn}$ is defined by:

$$a_{ij} = \begin{cases} 1, & \text{if } [i,j] \text{ is an edge in G} \\ 0, & \text{if not} \end{cases}$$

Definition:
The Laplacian of G is:

$$L_G = D_G - A_G = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix}$$

Graph Laplacians

Facts:

- L_G is PSD (Positive Semi-Definite).
- $\lambda = 0$ is always an eigenvalue of L_G with eigenvector $\vec{1}$.

$$\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Every row of L_G sums to 0 by construction.

$$L_G\vec{1} = \begin{pmatrix} \text{sum of row 1} \\ \cdots \\ \cdots \\ \text{sum of row n} \end{pmatrix} = \vec{0}$$

Why is L_G PSD ?

We will show that $L_G = B^T B$ for some matrix B.

Positive Semi-Definite Characterization

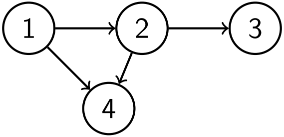
Definition:

A symmetric matrix $A \in \mathbb{R}^{nxn}$ is positive semi-definite, denoted $A \geq 0$, if any of the following equivalent conditions hold:

1. $\vec{x}^T A\vec{x} \geq 0$, $\forall \vec{x} \in \mathbb{R}^n$. (the graph of $g_A(\vec{x}) = \vec{x}^T A\vec{x}$ is on or above the domain, $\mathbb{R}^n$).
2. All eigenvalues of A are non-negative.
3. There exists some amtrix B s.t. $A = BTB$.
4. All principal minors of A are non-negative.

Define BG

Define the matrix B_G, the "directed" node-edge incidence matrix of G.



pick a direction for each edge
rows indexed by nodes
columns indexed by edges of G

$$B_G = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 12 & 14 & 23 & 24 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix} \end{array}$$

In the column indexed by the directed edge ij, we put 1 in row i and -1 in row j.

$L_G = B^T B$

Define BG

$L_G = B_G \cdot B_G^T$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix}$$

If you get i^{th} row and i^{th} column, 1^2 or $(-1)^2$ for every edge incident to i. So the sum d_i.
i^{th} row and j^{th} column (if $i \neq j$) if ij is on edge, you will get $1(-1) = -1$
Therefore, L_G is PSD.

Lemma

Lemma:

If G is a graph, the Laplacian L_G is PSD and its eigenvalues are:

$$0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n$$

and $\vec{1}$ is an eigenvector of L_G with eigenvalue 0.
Since L_G is PSD, the $\vec{x}^T L_G \vec{x} \geq 0$, $\forall \vec{x} \in \mathbb{R}^n$

Lemma

PwP:

Let $G = ([n], E)$ be the graph with vertices [n] and edge set E. If $\vec{x} = (x_1, \cdots, x_n)^T$.

$$\vec{x}^T L_G \vec{x} = \sum_{\{i,j\} \in E} (x_i - x_j)^2$$

Proof

We know that $L_G = B_G B_G^T$. So $\vec{x}^T L_G \vec{x} = \vec{x}^T B_G B_G^T \vec{x}$.

Lets understand $\vec{x}^T B_G$.

So the entries of $\vec{x}^T B_G$ are all $x_i - xj$ when ij is an edge of G.

Lemma

PwP:

So $\vec{x}^T L_G \vec{x} = \left(\vec{x}^T B_G\right) \cdot \left(B_G^T \vec{x}\right) = \sum_{\{i,j\} \in E} (x_i - x_j)^2$.

Do this for the example:

Connectivity

Definition:

A Graph is conencted if there is a way to walk from any vertex to any other vertex along the edges of G.

Theorem:

G is connected if and only if $\lambda_2 > 0$.

Proof:

If G is not connected the Graph Laplacian will look like this:

Connectivity

We sort of proved if G is not connected, then $\lambda_2 = 0$. (if G has k components, $\lambda_1 = \lambda_2 = \cdots \lambda_4 = 0$)

We still need t show that if G is connected, $\lambda_2 > 0$.
For a symmetric matrix, $AM(\lambda) = GM(\lambda)$ for an eigenvalues.
So we need to show $GM(0) = 1$. Let $\vec{u}$ be an eigenvector with eigenvalue of 0. Then $L_G \vec{u} = \vec{0}$.

$$0 = \vec{u}^T L_G \vec{u} = \sum_{\{i,j\} \in E} (u_i - u_j)^2$$

If $u_1 = C$ and 1,k is an edge, the $u_k = c$. for all edge $ij \in E$.
Similarly if kj is an edge $u_j = c$. Since G is connected $\vec{u} = c\vec{1}$.
AM, algebraic multiplicity.

GM, geometric multiplicity.

Conclusion

Slight Generalization:

The number of times 0 is on eigenvalue of L_G counts the number of conencted components of G.
Fact:

The second eigenvalue λ_2 is called the spectra gap or the Fiedler value of G.

Testing Latex stuff

$$\alpha = \begin{array}{c} \\ k_1 \\ k_2 \\ k_3 \\ k_4 \end{array} \begin{array}{c} \begin{array}{ccc} \bar{f}_1 & \bar{f}_2 & \bar{f}_3 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

$$\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix}$$

Testing Latex stuff

$$\beta = \begin{array}{c c} & \begin{array}{ccc} \bar{f}_1 & \bar{f}_2 & \bar{f}_3 \end{array} \\ \begin{array}{c} f_1 \\ f_2 \\ f_3 \end{array} & \begin{pmatrix} 3 & 2 & 0 \\ 2 & 4 & 2 \\ 5 & 3 & 1 \end{pmatrix} \end{array}$$

$$\begin{array}{c c} & \begin{array}{cccc} C_1 & C_2 & \dots & C_n \end{array} \\ \begin{array}{c} N_1 \\ N_2 \\ \\ N_n \end{array} & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \end{array}$$

$$\gamma = \begin{array}{c c c} & \begin{array}{cccc} \bar{f}_1 & \bar{f}_2 & \dots & \bar{f}_n \end{array} & \\ \begin{array}{c} k_1 \\ k_2 \\ \\ k_n \end{array} & \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} & \begin{array}{c} = 1 \\ = 1 \\ \\ = 1 \end{array} \end{array}$$