A Space-time Adaptive, Higher-order Finite Element Method for Sonic Boom

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Motivation For Sonic Boom Study

Ultimate goal:

Enable supersonic commercial flights overland.

Source: lockheedmartin.com

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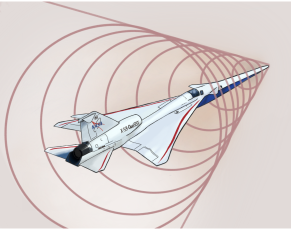

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Main challenge:

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Study sonic boom to:

- Predict loudness sensitivities to airplane geometry.
- Perform airplane shape optimization to reduce loudness at ground.

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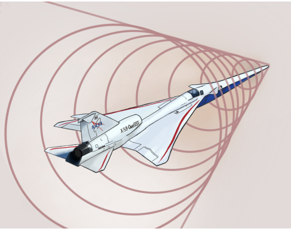

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Solution Approach: Broad Picture

Multidisciplinary design optimization[1]:

[1] D. L. Rodriguez et. al. 2025

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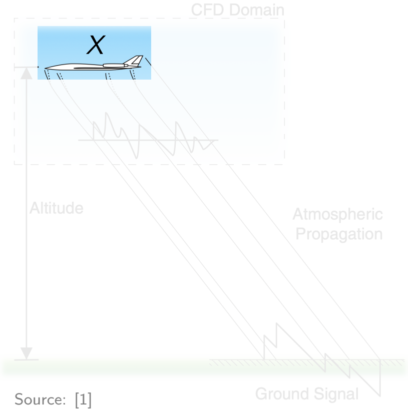

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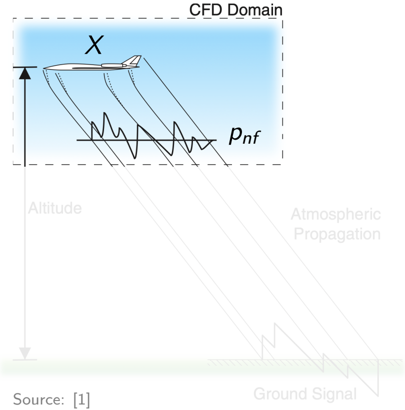

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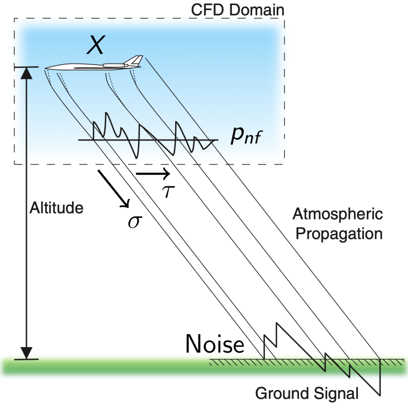

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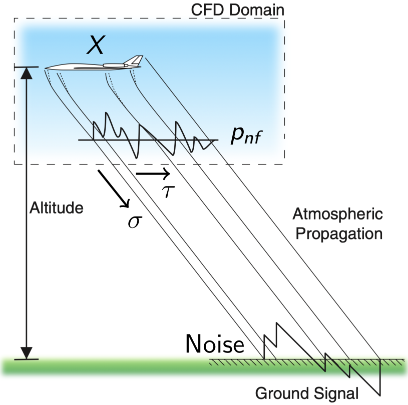

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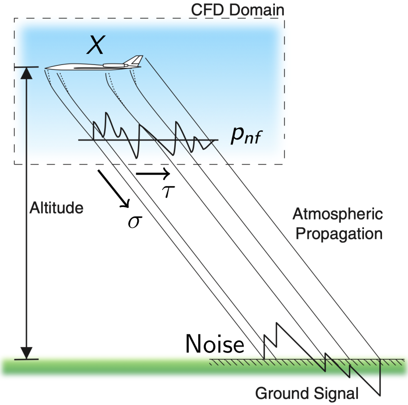
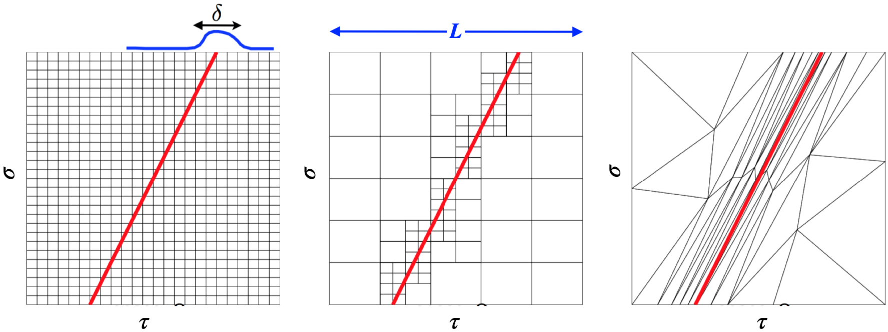


$\boxed{\text{Compute noise at ground}}$ and $$\frac{d(\text{Noise})}{dX} = \boxed{\frac{d(\text{Noise})}{dp_{nf}}} \frac{dp_{nf}}{dX}.$$

[1] D. L. Rodriguez et. al. 2025

Propagation Problem: Mesh Adaptation



(a) uniform $\quad$ (b) space-time tensor-product $\quad$ (c) space-time unstructured

$\mathrm{DOF} = O\left((L/\delta)^2\right)$ $\quad$ $\mathrm{DOF} = O\left(L/\delta\right)$ $\quad$ $\mathrm{DOF} = O\left(1\right)$

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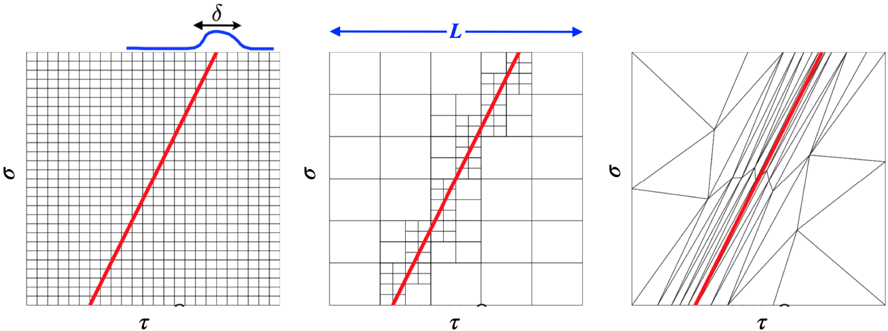


(a) uniform (b) space-time tensor-product (c) space-time unstructured

DOF $= \mathcal{O}\left((L/\delta)^2\right)$ DOF $= \mathcal{O}(L/\delta)$ DOF $= \mathcal{O}(1)$

But, space-time unstructured requires 2D solve instead of marching in σ, and solving an optimization problem for the mesh.

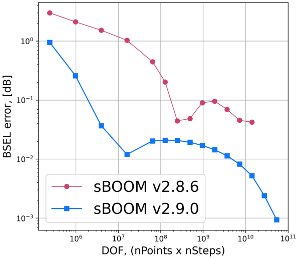
Propagation Problem: Motivation for Mesh Adaptation

[2] S. K. Rallabhandi et. al. 2023

Propagation Problem: Motivation for Mesh Adaptation

Results for standard time-marching methods[2]:

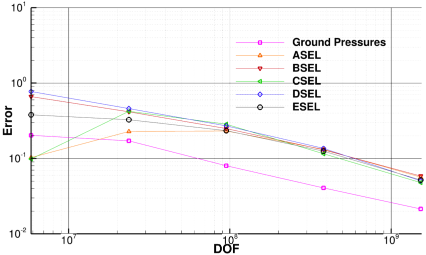
Error in Noise



Source: [2]

($\approx 10^{11}$ DOF for 10^{-3} error)

Error in $|d(\text{Noise})/dp_{nf}|$



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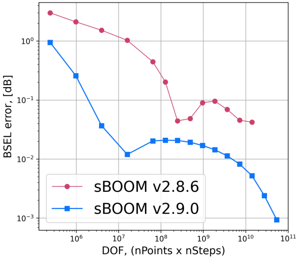

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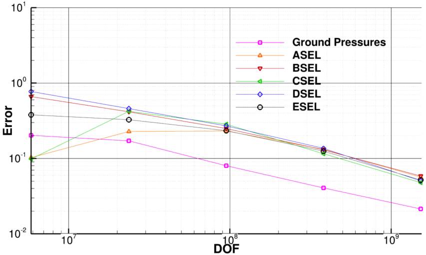

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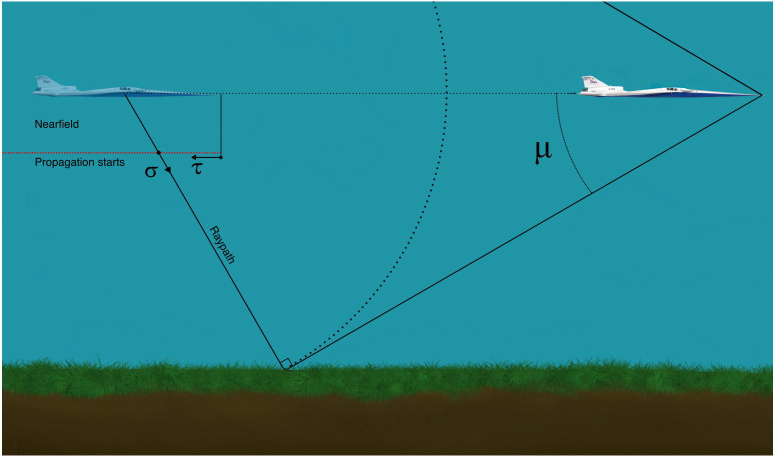
Our goal: Reduce the significant computational cost involved. Enable efficient, automated high accuracy predictions of boom propagation and design sensitivities through adaptive control of numerical error.

[2] S. K. Rallabhandi et. al. 2023

Boom Propagation Modeling and Adaptive Approach

Coordinate System

Airplane at cruise altitude and cruise Mach number (M_a):



Mach cone angle: $\mu = \sin^{-1}(1/M_a)$.

Augmented Burgers System

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$$-\frac{\partial \tilde{P}_{\nu}}{\partial \tau} + \frac{P - \tilde{P}_{\nu}}{\theta_{\nu}} = 0 \text{ on } \Omega, \qquad \nu = \{O_2, N_2\}, \quad (2)$$

which includes:

- Thermoviscous diffusion.
- Atmospheric absorption by relaxation species (O_2 and N_2).
- Ray tube area variation.

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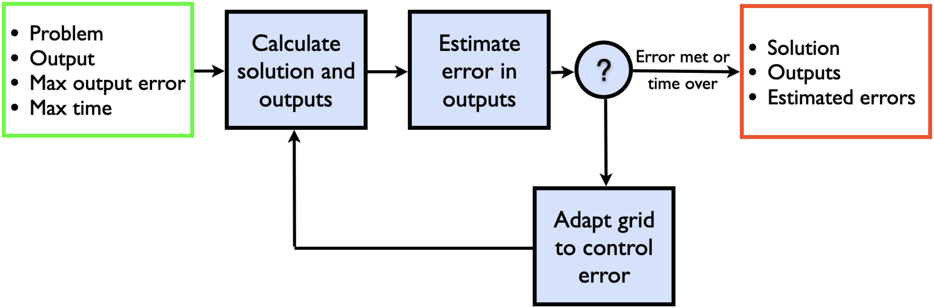
Remarks:

- Eq (1) is parabolic, with σ the time-like direction.
- It is typically solved with a time-marching scheme.

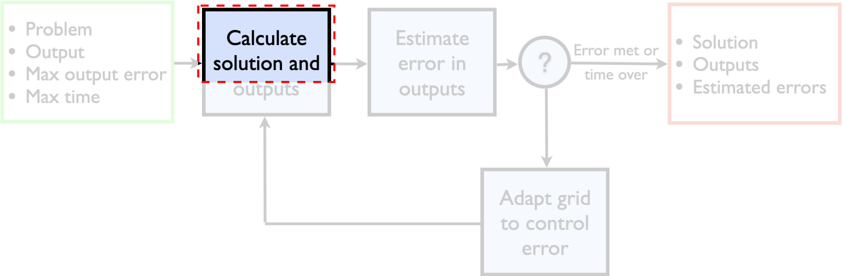
Output-based Adaptation Cycle

Adaptive cycle scheme:



Output of interest: Loudness at ground.

Discretization and Shock Capturing



Continuous Galerkin type FEM

CG weak statement:

Find $\boldsymbol{u}_h \in \mathcal{V}_{h,p}$ such that:

$$\mathcal{R}(\boldsymbol{v}_h, \boldsymbol{u}_h) = 0, \quad \forall \boldsymbol{v}_h \in \mathcal{V}_{h,p}, \tag{3}$$

where the space $\mathcal{V}_{h,p}$ contains polynomials of order p in Ω.

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Remarks:

- A discontinuous subscale is used for stabilization, and the resulting method is known as Variational Multiscale with Discontinuous Subscales (VMSD).
- The discretization is adjoint consistent.

The Need for Artificial Viscosity

Challenge: Discontinuities (shocks) in the solutions, leading to unstable numerical solves and lack of convergence.

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Add extra diffusion term in Burgers equation:

$$\frac{\partial P}{\partial \sigma} - \frac{1}{2}\frac{\partial \ln(\rho_0 c_0 / A_{n0})}{\partial \sigma} P - \frac{1}{2}\frac{\partial P^2}{\partial \tau} - \frac{1}{\Gamma}\frac{\partial^2 P}{\partial \tau^2} - \frac{\partial}{\partial \tau}\left(\sum_{\nu} C_{\nu}\frac{\partial \tilde{P}_{\nu}}{\partial \tau}\right) \underbrace{- \frac{\partial}{\partial \tau}\left(\epsilon_{\text{AV}}\frac{\partial P}{\partial \tau}\right)}_{\text{extra term}} = 0, \quad (4)$$

with ϵ_{AV} as:

$$\epsilon_{\text{AV}} := \underbrace{\frac{1}{2}\frac{H_{\tau\tau}}{p}|P|}_{\text{AV}_{\text{max}}} s. \quad (5)$$

PDE-based Shock Sensor[3]

Shock sensor design requirements:

- $s \approx 1$ in shock areas.
- $s \approx 0$ away from shocks, of order $\mathcal{O}(h^p)$.
- s smooth.

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[2] G. E. Barter and D. L. Darmofal 2007

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- s_{grad}: **Shock indicator** based on pressure solution gradient.
- Diffusion term: to have a smooth sensor solution.
- H: element size field.

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Shock Indicator s_{grad}

Identify pressure changes in τ direction:

$$\xi := \frac{H_{\tau\tau}}{p} \left| \frac{\partial P}{\partial \tau} \right|, \quad \hat{\xi} := \frac{\xi}{\xi_1}, \tag{7}$$

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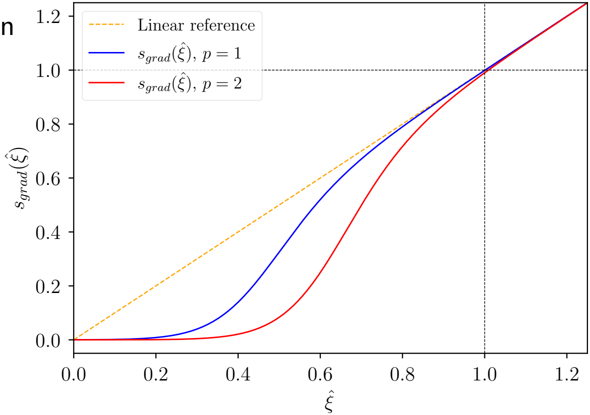

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Smooth problem with available exact solution:

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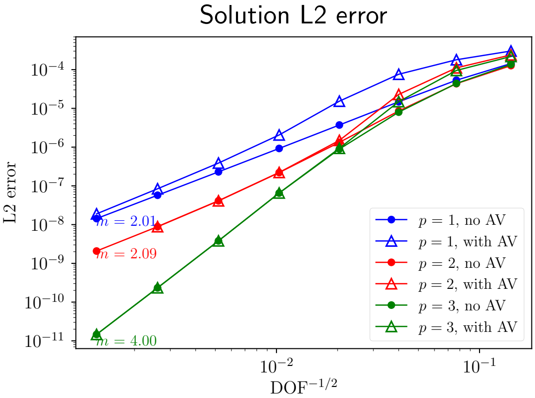

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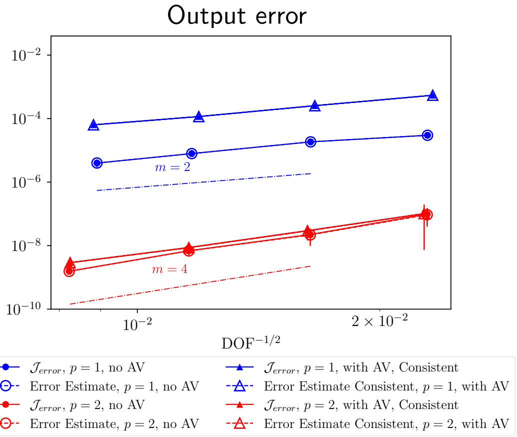
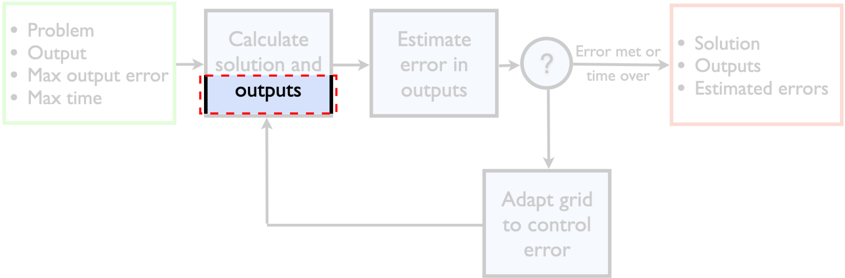


Ground Signal Filtering



At Ground: Relevant Loudness Metrics

The human ear is less sensitive to low audio frequencies.

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Loudness level in dB:

$$\text{Loudness} = 10 \log_{10} \left(\frac{E}{E_0} \right), \quad E_0 = 400 \ (\mu\text{Pa})^2 s. \tag{9}$$

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Transfer function in the complex frequency (χ) domain:

$$H_B(\chi) = \frac{\tilde{P}(\chi)}{P(\chi)} = \frac{c_B \chi^3}{(\chi + 2\pi f_1)^2 (\chi + 2\pi f_{2B})(\chi + 2\pi f_4)^2}, \qquad (10)$$

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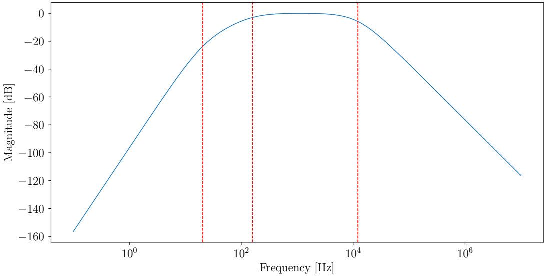
- $c_B = 5.99185 \times 10^9$
- $f_1 = 20.598997$ Hz
- $f_{2B} = 158.48932$ Hz
- $f_4 = 12194.217$ Hz



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$$\tilde{P}(\chi) = P(\chi)\frac{K\chi^3}{(\chi + a)^2(\chi + b)(\chi + c)^2}, \tag{11}$$

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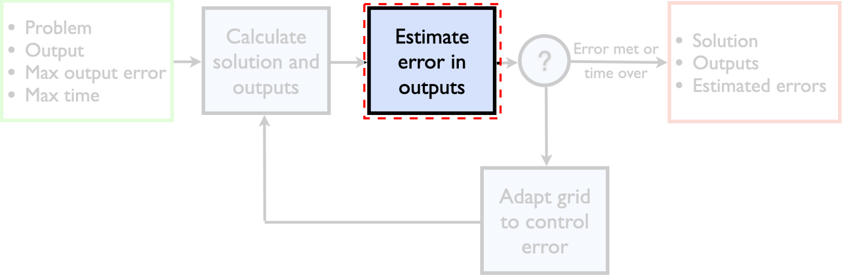
$$\frac{d\bar{u}}{d\tau} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -a^2 & -2a & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -K^{1/3}a^2 & -2K^{1/3}a & -c^2 & -2c & 0 \\ 0 & 0 & 0 & K^{1/3} & -b \end{pmatrix} \bar{u} + \begin{pmatrix} 0 \\ K^{1/3}p(\tau) \\ 0 \\ K^{2/3}p(\tau) \\ 0 \end{pmatrix}, \tag{12}$$

where $\bar{u} = (u_0, u_1, u_2, u_3, \tilde{p})^T$, with homogeneous initial conditions.

Output Error Estimation



Output Functional and Error

In general, consider output functional of the form:

$$\mathcal{J}(\boldsymbol{u}) := \int_{\Omega} g_v(\boldsymbol{u}) dV + \int_{\partial\Omega} g_b(\boldsymbol{u}) dS. \tag{13}$$

(In our case $\mathcal{J}(\boldsymbol{u}) = \int_{\text{ground}} [\tilde{p}(\tau)]^2 \, d\tau$)

[4] B. Couchman 2020

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We define output error as:

$$\varepsilon(\boldsymbol{u}_h) := \mathcal{J}(\boldsymbol{u}) - \mathcal{J}(\boldsymbol{u}_h). \tag{14}$$

For general nonlinear problem, the output error can be approximated using the **dual weighted residual** (DWR) method.

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Needs **correction**[4] for artificial viscosity term (asymptotically consistent).

[4] B. Couchman 2020

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  where $\boldsymbol{u}$ is the exact solution, $\alpha > 0$, and h is a characteristic element size in $\mathcal{T}_h$.

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  where $\boldsymbol{u}$ is the exact solution, $\alpha > 0$, and h is a characteristic element size in $\mathcal{T}_h$.

In our situation:

$$\mathcal{R}(\boldsymbol{v}_h, \boldsymbol{u}_h) = \mathcal{R}^C(\boldsymbol{v}_h, \boldsymbol{u}_h) + \underbrace{\mathcal{R}^A(\boldsymbol{v}_h, \boldsymbol{u}_h)}_{\text{AV term}}. \tag{17}$$

Dual Problem and Error for Linear Case

We assume linear residual and output functional, and define the **dual** (adjoint) problem as:

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Approximations: DWR Error Estimate

So far:

$$\varepsilon(\boldsymbol{u}_h) = -\Big[\mathcal{R}(\psi, \boldsymbol{u}_h) - \mathcal{R}^A(\psi, \boldsymbol{u})\Big]. \tag{20}$$

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First approximation:

$$\mathcal{R}^A(\psi, \boldsymbol{u}) \approx \mathcal{R}^A(\psi, \boldsymbol{u}_h), \tag{21}$$

justified on a shock dominated problem with AV.

Approximations: DWR Error Estimate

Second approximation:

ψ is approximated with a numerical adjoint $\psi_{\hat{h}}$ defined by[5]:

[5] M. Yano and D. L. Darmofal 2012

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where:

- $\mathcal{R}'[\boldsymbol{u}_h]$ and $\mathcal{J}'[\boldsymbol{u}_h]$ are the linearizations of $\mathcal{R}$ and $\mathcal{J}$ about $\boldsymbol{u}_h$.
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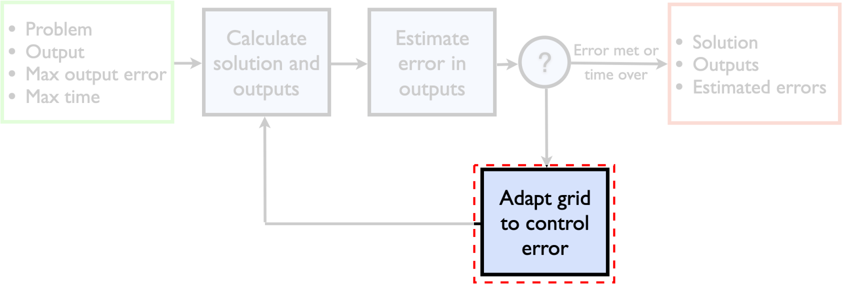
Final DWR error estimate:

$$\varepsilon(\boldsymbol{u}_h) \approx -\Big[\mathcal{R}(\psi_{\hat{h}}, \boldsymbol{u}_h) - \mathcal{R}^A(\psi_{\hat{h}}, \boldsymbol{u}_h)\Big]. \tag{23}$$

[5] M. Yano and D. L. Darmofal 2012

Mesh Adaptation



Continuous Optimization: Mesh-Metric Duality

Want mesh producing the smallest output error indicator $\mathcal{E}$:

$$\hat{\mathcal{T}}_h = \arg \inf_{\mathcal{T}_h \in \mathbb{T}(\Omega)} \mathcal{E}(\mathcal{T}_h), \quad \mathcal{C}(\mathcal{T}_h) < C. \tag{24}$$

[6] A. Loseille and F. Alauzet 2011

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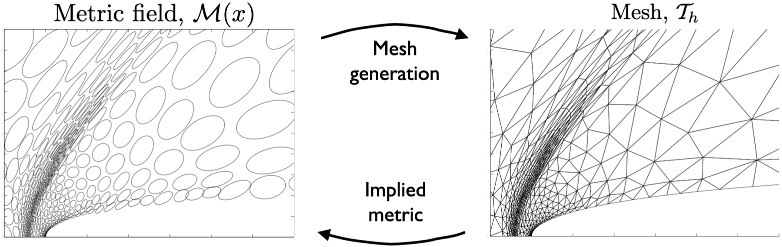
Continuous relaxation[6]to address intractability of discrete problem.

$$\hat{\mathcal{M}} = \arg \inf_{\mathcal{M} \in \mathbb{M}(\Omega)} \mathcal{E}(\mathcal{M}), \quad \mathcal{C}(\mathcal{M}) < C \tag{25}$$



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Need model for error indicator $\mathcal{E}(\mathcal{M})$: How $\mathcal{E}$ changes with $\mathcal{M}$.

[7] T. Richter and T. Wick 2015

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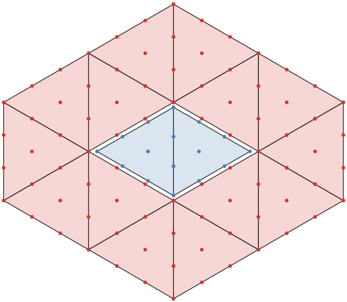
Next: Recompute solution and nodal indicators in refined *local patches* in the mesh.

The comparison of the nodal error indicators before and after the local solves gives information about how $\mathcal{E}$ changes with $\mathcal{M}$.

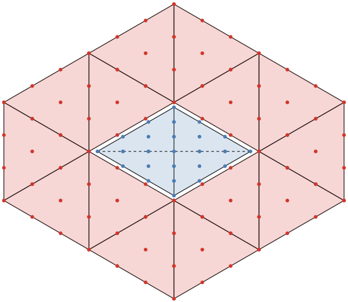
[7] T. Richter and T. Wick 2015

[8] M. Yano and D. L. Darmofal 2012

MOESS: Error Sampling and Synthesis



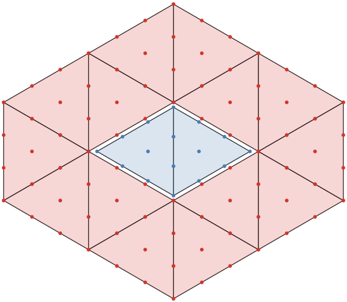
(a) Unsplit



(b) Split

MOESS: Error Sampling and Synthesis

Local solve: Solve for $\boldsymbol{u}_h^\epsilon \in \mathcal{V}_{h,p}^\epsilon$ s.t. $\mathcal{R}_{\text{local}}(\boldsymbol{v}_h^\epsilon, \boldsymbol{u}_h^\epsilon) = 0, \quad \forall \boldsymbol{v}_h^\epsilon \in \mathcal{V}_{h,p}^\epsilon.$



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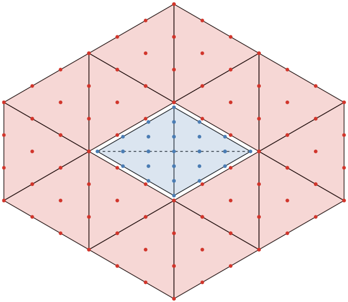

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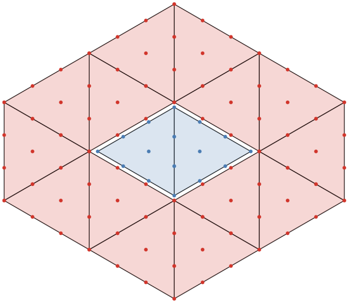
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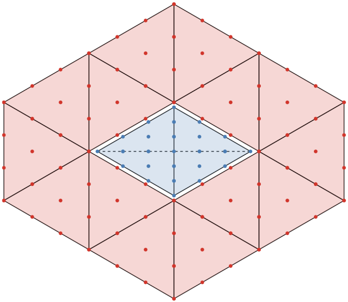
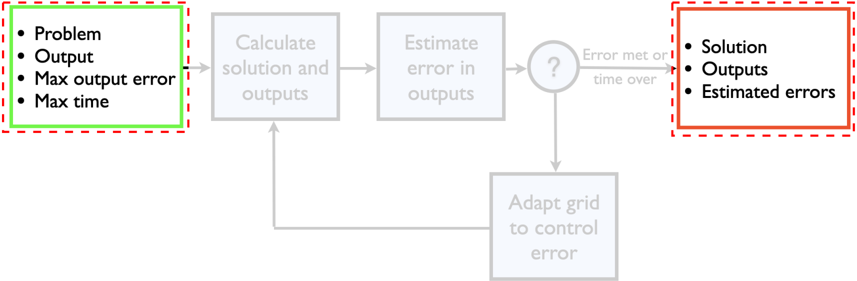

(b) Split

Nodal indicator for vertices in inner patch:

$$\eta_v^\epsilon = |\mathcal{R}_{\text{local}}(\phi_v \psi_{\hat{h}}, \boldsymbol{u}_h^\epsilon)| + |\mathcal{R}_{\text{local}}^A(\phi_v \psi_{\hat{h}}, \boldsymbol{u}_h^\epsilon)|. \tag{28}$$

Results for Practical Case



Preliminary: Implementation Notes

Software: Solution Adaptive Numerical Simulator (SANS)[9]

- C++ framework to numerically solve partial differential equations.
- Supports several CG and DG discretizations, with output-based mesh adaptation.
- MPI parallelization.
- Unit testing and continuous integration.
- Open source.

[9] Galbraith et. al. 2015

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3. **Average results:**
   - Average any quantity of interest (e.g. output value) over last n adaptive iterations.

Case Description

- Airplane Mach number: $M_a = 1.4$.
- Airplane altitude: $z_a = 16459.2$ m.
- Ground altitude: 110 m.

Source: lockheedmartin.com

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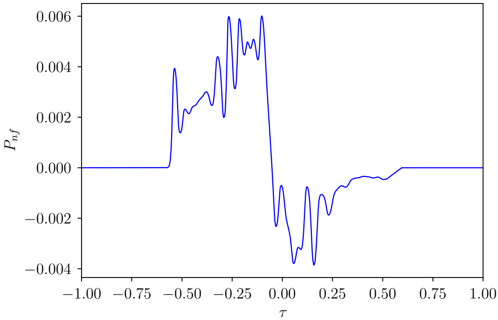

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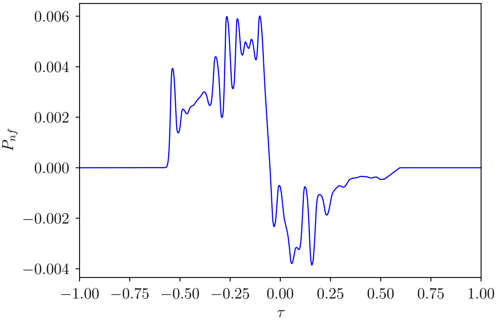

Source: lockheedmartin.com

- Domain dimensions:
  $\Omega = [-1, 2] \times [0, 257]$
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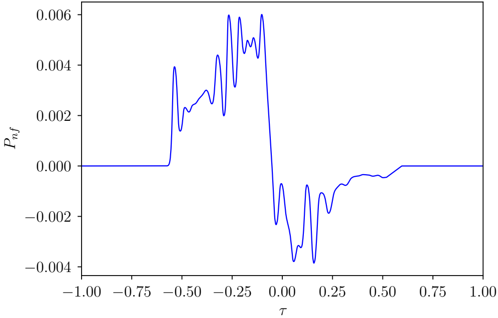

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- Also for comparison:
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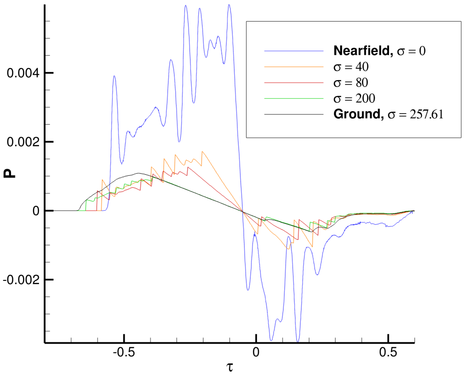
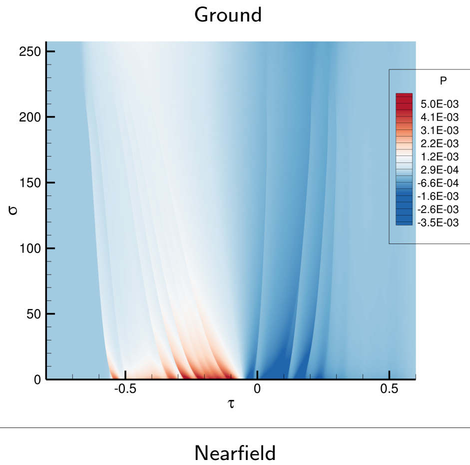
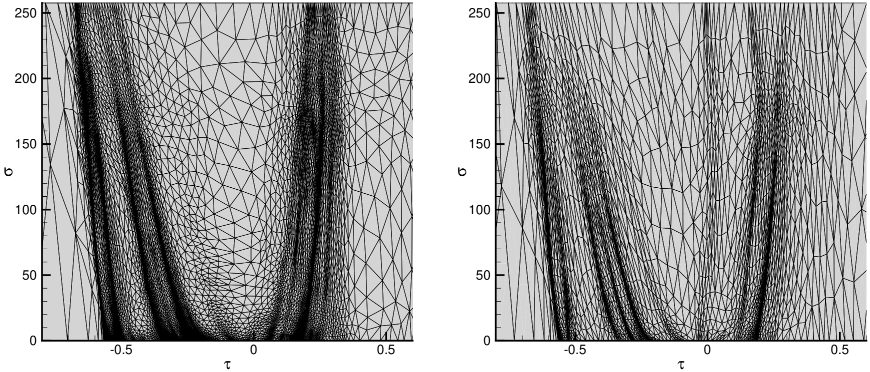
Propagation: Pressure Perturbation Solution





Propagation: Final Adapted Mesh for 8K Target DOF

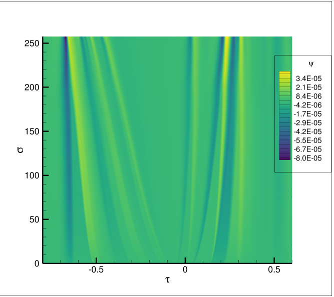
$p = 1$ $\qquad\qquad$ $p = 2$

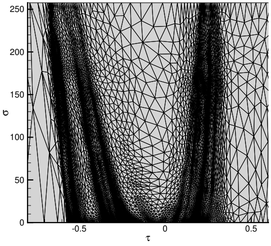


Propagation: Evolution Over Adaptive Cycle, 8K Target DOF

Different Adaptation Outputs and Their Adjoints

$\mathcal{J}_{\text{BSEL}}$





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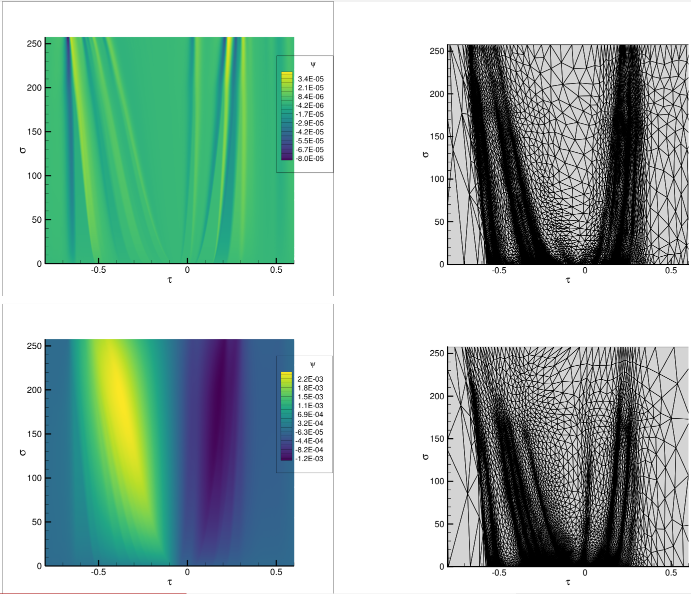
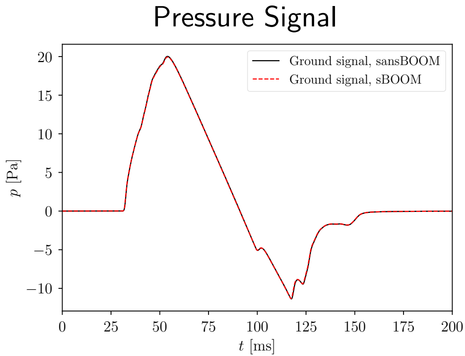
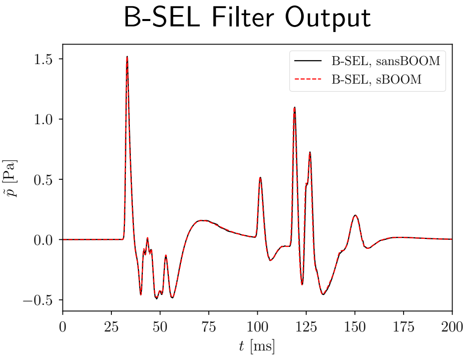
$\mathcal{J}_p$



At Ground: Pressure Signal and Its Filtering

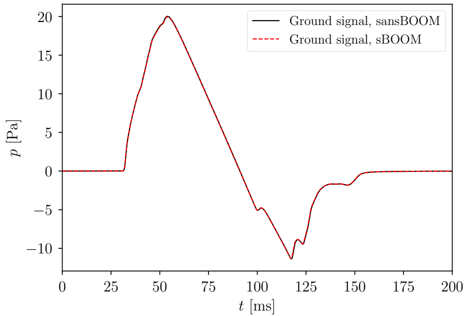




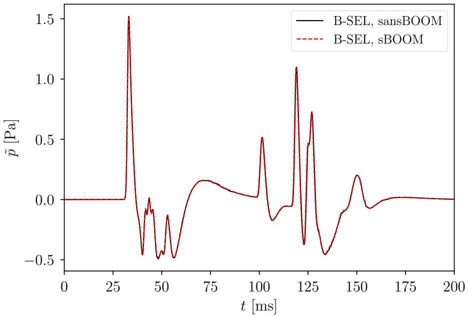
[10] S. K. Rallabhandi et. al. 2023

At Ground: Pressure Signal and Its Filtering

Pressure Signal



B-SEL Filter Output



Comparison with NASA *sBOOM* code[10]:

sansBOOM:

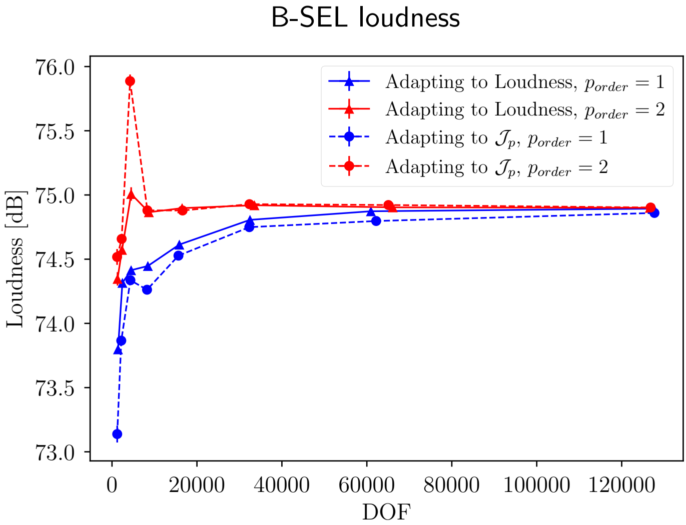
- Target of 128K DOF in total (space-time).
- Total DOF across the 40 adaptive iterations ≈ 3.2M DOF.

sBOOM (NASA, time-marching):

- 32K DOF in τ direction.
- 39K steps (marching) in σ direction.
- 1.2B DOF in total (space-time).

[10] S. K. Rallabhandi et. al. 2023

At Ground: Loudness Convergence with Mesh Refinement



Concluding Remarks

Work completed:

- Higher-order FEM to solve sonic boom propagation problem.
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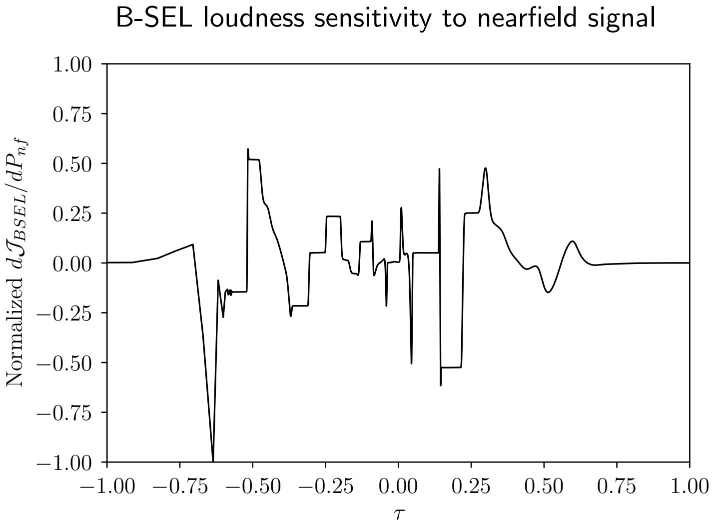
Ongoing effort:

- Study convergence of loudness sensitivity to nearfield signal.

Ongoing Effort: Loudness Sensitivity to Nearfield Signal



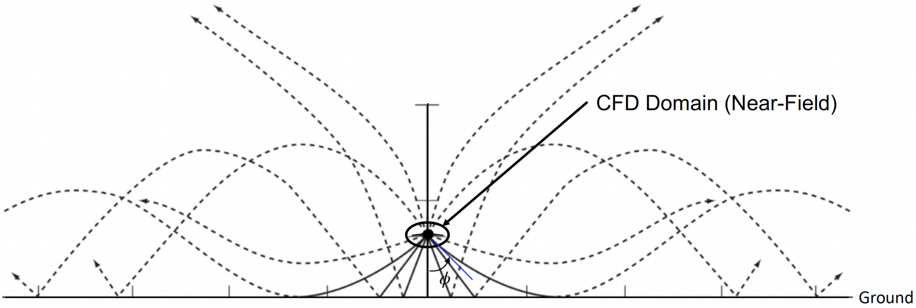
Acknowledgments

- This work was supported by funding from NASA (#80NSSC22K0193) with technical monitor Dr. Sriram Rallabhandi.
- The authors gratefully acknowledge technical assistance of NASA specifically from Sriram Rallabhandi, Michael Aftosmis, Marian Nemec, and David Rodriguez.

Thanks for the attention!

Questions?

Sonic Boom Carpet: Front View



Variational Multiscale with Discontinuous Subscales (VMSD) Method

Discretization of Ω:

$\mathcal{T}_h := \{\kappa\}_{\kappa=1}^{K}$ is a triangulation of the domain Ω into K elements.

Propose solution:

$\boldsymbol{u}_h := \bar{\boldsymbol{u}}_{h,p} + \boldsymbol{u}'_{h,p'}, \qquad \bar{\boldsymbol{u}}_{h,p} \in \overline{\mathcal{V}}_{h,p}, \quad \boldsymbol{u}'_{h,p'} \in \mathcal{V}'_{h,p'}.$

VMSD solution spaces:

$$(\textit{Coarse}\text{ scale})\ \overline{\mathcal{V}}_{h,p} := \{\boldsymbol{v} \in [C^0(\Omega)]^m : \boldsymbol{v}|_\kappa \in [\mathcal{P}^p(\kappa)]^m, \forall \kappa \in \mathcal{T}_h\}, \quad (29)$$

$$(\textit{Fine}\text{ scale})\ \mathcal{V}'_{h,p'} := \{\boldsymbol{v} \in [L^2(\Omega)]^m : \boldsymbol{v}|_\kappa \in [\mathcal{P}^{p'}(\kappa)]^m, \forall \kappa \in \mathcal{T}_h\}. \quad (30)$$

Variational Multiscale with Discontinuous Subscales (VMSD) Method

Weak statement:

Find $(\bar{\boldsymbol{u}}_{h,p},\ \boldsymbol{u}'_{h,p'}) \in \overline{\mathcal{V}}_{h,p} \times \mathcal{V}'_{h,p'}$ **such that:**

$$\mathcal{R}(\bar{\boldsymbol{v}}_{h,p}, \boldsymbol{v}'_{h,p'}; \bar{\boldsymbol{u}}_{h,p}, \boldsymbol{u}'_{h,p'}) = 0, \quad \forall (\bar{\boldsymbol{v}}_{h,p}, \boldsymbol{v}'_{h,p'}) \in \overline{\mathcal{V}}_{h,p} \times \mathcal{V}'_{h,p'}. \tag{31}$$

Remarks:

- $\boldsymbol{u}'_{h,p'}$ DOFs are element-wise decoupled. Thus, they can be static condensed and the total cost becomes the same as a CG method.
- For same accuracy requirement, more efficient (less DOFs) than CG and DG.
- Adjoint consistent.

Shock Indicator s_{grad}

Starting point:

$$\xi := \frac{H_{\tau\tau}}{p}\left|\frac{\partial P}{\partial \tau}\right|, \quad \hat{\xi} = \frac{\xi}{\xi_1}, \tag{32}$$

then:

$$s_{\text{grad}} := s_{\text{grad}}(\hat{\xi}) = \frac{\hat{\xi}[\tanh(p^2\hat{\xi})]^{p-1}}{1+\exp\left[-k\left(\hat{\xi}-\alpha(p)\right)\right]}, \tag{33}$$

where $k, \alpha(p) \in \mathbb{R}$.

Shock Indicator s_{grad}

$$s_{\text{grad}} := s_{\text{grad}}(\hat{\xi}) = \frac{\hat{\xi}[\tanh(p^2\hat{\xi})]^{p-1}}{1 + \exp\left[-k\left(\hat{\xi} - \alpha(p)\right)\right]}, \tag{34}$$