A Loudness-based Adaptive, Higher-order Finite Element Method for Sonic Boom Propagation

R. Trono Figueras

David L. Darmofal, Marshall C. Galbraith, Steven R. Allmaras

MIT, Department of Aeronautics and Astronautics

Funded by NASA (#80NSSC22K0193)

April 11, 2025

Motivation and Overview

Motivation For Sonic Boom Study

Ultimate goal:
Enable supersonic commercial flights overland.

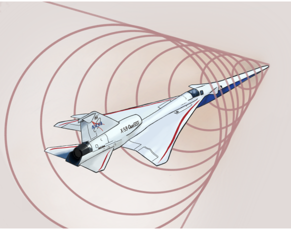
Main challenge:

- Negative impact of sonic boom loudness on humans, other animals, and structures.

Study sonic boom to:

- Predict loudness sensitivities to airplane geometry.
- Perform airplane shape optimization to reduce loudness at ground.

Source: lockheedmartin.com

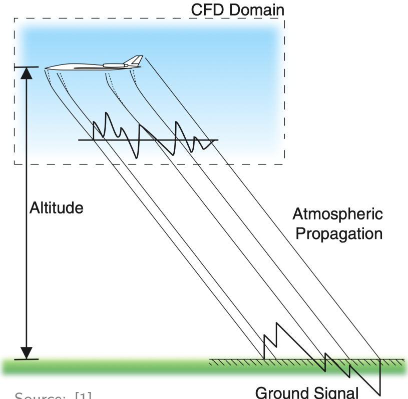


Source: michigandaily.com

Solution Approach: Broad Picture

Multidisciplinary design optimization[1]:

- Parametric geometry generator.
- CFD solver.
  - Euler/Navier-Stokes in 3D.
  - Uniform atmosphere.
- *Sonic boom propagation tool*
  - 2D problem.
  - Weakly non-linear.
  - Species relaxation.
  - Non-uniform atmosphere.
- Numerical optimizer.



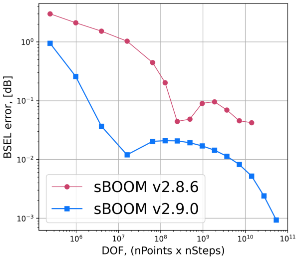
Source: [1]

All together: $$\frac{d(\text{Noise})}{dX} = \frac{\partial(\text{Noise})}{\partial p_{gs}} \frac{\partial p_{gs}}{\partial p_{ns}} \frac{dp_{ns}}{dQ} \frac{dQ}{dT} \frac{dT}{dX}.$$

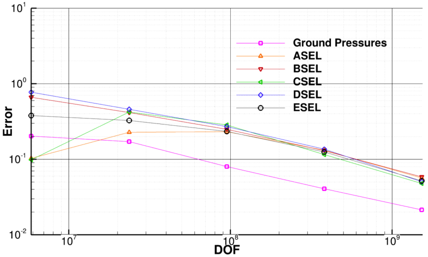
[1] D. L. Rodriguez et. al. 2025

Propagation Problem: Motivation for Space-time Mesh Adaptation

Results for standard time-marching methods[2]:



Source: [2]

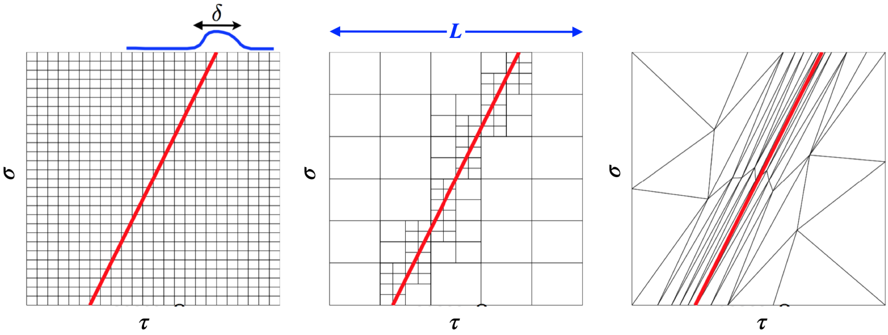


Source: [2]

Our goal: Reduce the significant computational cost involved. Enable efficient, automated high accuracy predictions of boom propagation and design sensitivities through adaptive control of numerical error.

[2] S. K. Rallabhandi et. al. 2023

Propagation Problem: Space-time Adaptive Method



(a) uniform $\text{DOF} = O\left((L/\delta)^2\right)$

(b) space-time tensor-product $\text{DOF} = O(L/\delta)$

(c) space-time unstructured $\text{DOF} = O(1)$

But, space-time unstructured requires coupled solve over entire space-time domain.

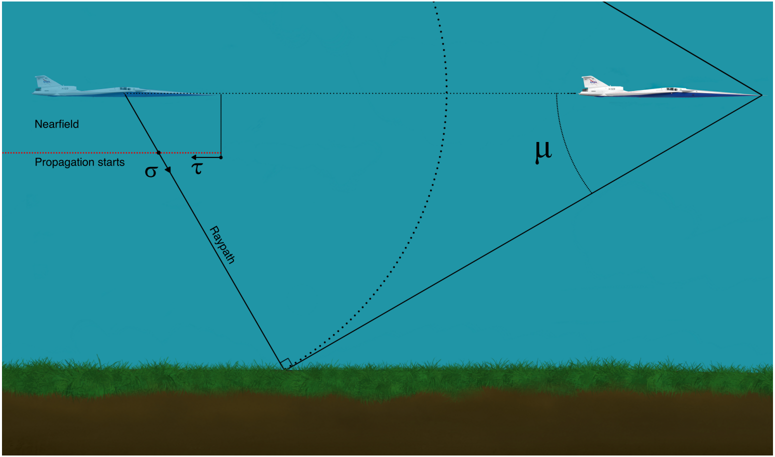
Propagation Problem: Methodology Overview

1. Boom Propagation Modeling.
2. Shock Capturing.
3. Ground Signal Filtering.
4. FEM Discretization and Output Error Estimation.
5. Output-based Mesh Adaptation.

Boom Propagation Modeling

Coordinate System

Airplane at cruise altitude and cruise Mach number (M_a):



Mach cone angle: $\mu = \sin^{-1}(1/M_a)$.

Augmented Burgers System

To model sonic boom propagation we use the augmented Burgers system of equations:

$$\frac{\partial P}{\partial \sigma} - \frac{1}{2}\frac{\partial \ln(\rho_0 c_0/A_{n0})}{\partial \sigma}P - \frac{1}{2}\frac{\partial P^2}{\partial \tau} - \frac{1}{\Gamma}\frac{\partial^2 P}{\partial \tau^2} - \frac{\partial}{\partial \tau}\left(\sum_\nu C_\nu \frac{\partial \tilde{P}_\nu}{\partial \tau}\right) = 0, \quad (1)$$

$$-\frac{\partial \tilde{P}_\nu}{\partial \tau} + \frac{P - \tilde{P}_\nu}{\theta_\nu} = 0, \qquad \nu = \{O_2, N_2\}, \quad (2)$$

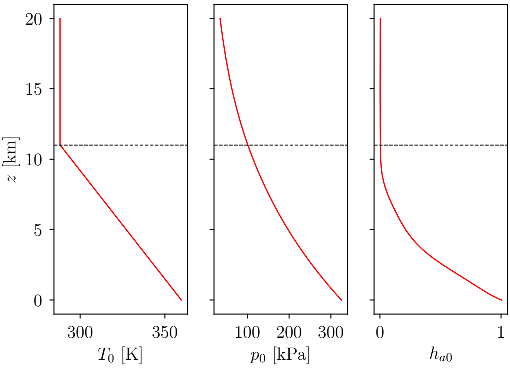
which includes:

- Thermoviscous diffusion.
- Atmospheric absorption by relaxation species (O_2 and N_2).
- Ray tube area variation.

Atmosphere Model: Standard

Atmosphere model refers to how the atmospheric properties depend on altitude.

Standard model: First two layers:



Additionally, models for:

- Density and speed of sound: ρ_0, c_0,
- Gol'berg number: Γ,
- Relaxation coefficients: C_ν, θ_ν,
- Ray tube area: A_{n0},

as functions of atmospheric properties.

Shock Capturing

The Need for Artificial Viscosity

Challenge: Discontinuities (shocks) in the solutions, leading to unstable numerical solves and lack of convergence.

Goal: Smoothen discontinuities to improve stability, while not modifying the already smooth areas.

Solution Approach: Employ a shock sensor, s, to keep track of discontinuities, and use that information to add localized artificial viscosity.

- Design requirements:
  - $s \approx 1$ in shock areas.
  - $s \approx 0$ away from shocks.
  - Order $\mathcal{O}(h^p)$.
  - s smooth.

PDE-based Shock Sensor[3]

The shock sensor becomes a state itself, s, with its own equation:

$$\underbrace{s - C_1 s_{\text{grad}}}_{\text{source term}} + \underbrace{\frac{C_2}{p^2}\nabla \cdot \left(H_{00}^2 \frac{\partial s}{\partial \tau} + H_{01}^2 \frac{\partial s}{\partial \sigma}, H_{10}^2 \frac{\partial s}{\partial \tau} + H_{11}^2 \frac{\partial s}{\partial \sigma} \right)}_{\text{diffusion term}} = 0. \qquad (3)$$

- s_{grad}: Shock indicator based on pressure solution gradient.
- p: polynomial solution order.
- H: element size field.
- Diffusion term: to have a smooth sensor solution.

[2] G. E. Barter and D. L. Darmofal 2007

Shock Indicator s_{grad}

Starting point:

$$\xi := \frac{H_{00}}{p}\left|\frac{\partial P}{\partial \tau}\right|, \qquad \xi_1 \simeq \max_{x\in\Omega} \xi(x), \quad \hat{\xi} = \frac{\xi}{\xi_1}, \tag{4}$$

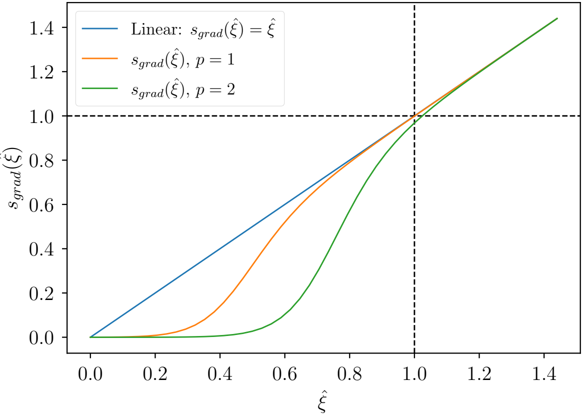
then:

$$s_{\mathrm{grad}} := s_{\mathrm{grad}}(\hat{\xi}) = \frac{\hat{\xi}[\tanh(p^2\hat{\xi})]^{p-1}}{1+\exp\left[-k\left(\hat{\xi}-\alpha(p)\right)\right]}, \tag{5}$$

where $k, \alpha(p) \in \mathbb{R}$.

Shock Indicator s_{grad}

$$s_{\text{grad}} := s_{\text{grad}}(\hat{\xi}) = \frac{\hat{\xi}[\tanh(p^2\hat{\xi})]^{p-1}}{1 + \exp\left[-k\left(\hat{\xi} - \alpha(p)\right)\right]}, \tag{6}$$



Addition of Artificial Viscosity to the Burgers Equation

Compute artificial viscosity coefficient ϵ_{AV}:

$$\epsilon_{\mathrm{AV}} := \frac{1}{2}\frac{H_{00}}{p}|P|s. \tag{7}$$

Add extra diffusion term in Burgers equation:

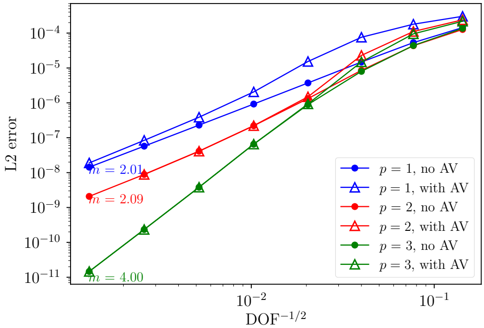
$$\frac{\partial P}{\partial \sigma} - \frac{1}{2}\frac{\partial \ln(\rho_0 c_0/A_{n0})}{\partial \sigma}P - \frac{1}{2}\frac{\partial P^2}{\partial \tau} - \frac{1}{\Gamma}\frac{\partial^2 P}{\partial \tau^2} - \frac{\partial}{\partial \tau}\left(\sum_{\nu} C_{\nu}\frac{\partial \tilde{P}_{\nu}}{\partial \tau}\right) \underbrace{-\frac{\partial}{\partial \tau}\left(\epsilon_{\mathrm{AV}}\frac{\partial P}{\partial \tau}\right)}_{\text{extra term}} = 0. \tag{8}$$

Test With Smooth Problem

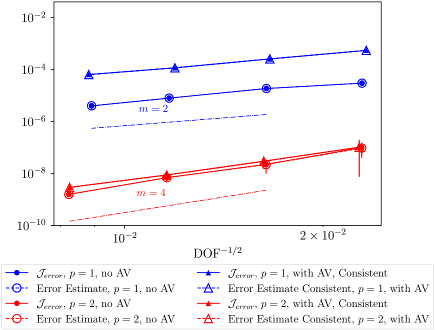
Smooth problem with available exact solution:

- Solve without artificial viscosity.
- Solve with artificial viscosity and see how it affects convergence.

Solution L2 error



Output error



Ground Signal Filtering

Relevant Loudness Metrics

The human ear is less sensitive to low audio frequencies.

There is a family of weighting filter curves that account for this relative loudness perceived by humans: A/B/C/D/-SEL curves.

(Ground pressure signal) $p(t) \rightarrow$ Weighting filter $\rightarrow \tilde{p}(t)$ (Filtered signal)

Sound exposure:

$$E = \int_{t_0}^{t_f} \tilde{p}(t)\ dt. \tag{9}$$

Loudness level in dB:

$$\text{Loudness} = 10 \log_{10} \left(\frac{E}{E_0} \right), \quad E_0 = 400\ (\mu\text{Pa})^2 s. \tag{10}$$

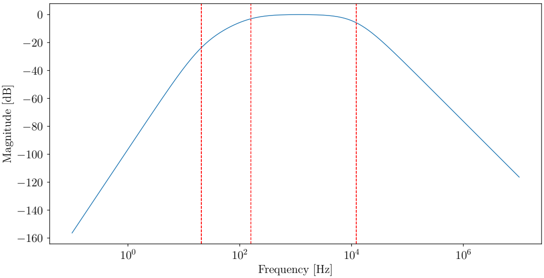
B-SEL Metric

We focus on the B-SEL curve, and the approach can be generalized to any other.

Transfer function in the complex frequency domain:

$$H_B(s) = \frac{\tilde{P}(s)}{P(s)} = \frac{c_B s^3}{(s + 2\pi f_1)^2 (s + 2\pi f_{2B})(s + 2\pi f_4)^2}, \tag{11}$$

- $c_B = 5.99185 \times 10^9$
- $f_1 = 20.598997$ Hz
- $f_{2B} = 158.48932$ Hz
- $f_4 = 12194.217$ Hz



Filter Application: ODE Approach

Common filtering techniques not suitable for our unstructured grid.

Convert transfer function in complex frequency domain:

$$\tilde{P}(s) = P(s)\frac{K^{1/3}}{(s+a)^2}\frac{K^{1/3}s^2}{(s+c)^2}\frac{K^{1/3}s}{(s+b)}, \tag{12}$$

to a **system of ODE's** in the time domain (to solve in *ground* boundary):

$$\frac{d\bar{u}}{dt} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -a^2 & -2a & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -K^{1/3}a^2 & -2K^{1/3}a & -c^2 & -2c & 0 \\ 0 & 0 & 0 & K^{1/3} & -b \end{pmatrix} \bar{u} + \begin{pmatrix} 0 \\ K^{1/3}p(t) \\ 0 \\ K^{2/3}p(t) \\ 0 \end{pmatrix}, \tag{13}$$

where $\bar{u} = (u_0, u_1, u_2, u_3, \tilde{p})^T$, with homogeneous initial conditions.

FEM Discretization and Output Error Estimation

Variational Multiscale with Discontinuous Subscales (VMSD) Method

Discretization of Ω:

$\mathcal{T}_h := \{\kappa\}_{\kappa=1}^{K}$ is a triangulation of the domain Ω into K elements.

Propose solution:

$\boldsymbol{u}_h := \bar{\boldsymbol{u}}_{h,p} + \boldsymbol{u}'_{h,p'}, \qquad \bar{\boldsymbol{u}}_{h,p} \in \overline{\mathcal{V}}_{h,p}, \quad \boldsymbol{u}'_{h,p'} \in \mathcal{V}'_{h,p'}.$

VMSD solution spaces:

$$(\textit{Coarse} \text{ scale}) \ \overline{\mathcal{V}}_{h,p} := \{\boldsymbol{v} \in [C^0(\Omega)]^m : \boldsymbol{v}|_\kappa \in [\mathcal{P}^p(\kappa)]^m, \forall \kappa \in \mathcal{T}_h\}, \quad (14)$$

$$(\textit{Fine} \text{ scale}) \ \mathcal{V}'_{h,p'} := \{\boldsymbol{v} \in [L^2(\Omega)]^m : \boldsymbol{v}|_\kappa \in [\mathcal{P}^{p'}(\kappa)]^m, \forall \kappa \in \mathcal{T}_h\}. \quad (15)$$

Variational Multiscale with Discontinuous Subscales (VMSD) Method

Weak statement:

Find $(\bar{\boldsymbol{u}}_{h,p},\ \boldsymbol{u}'_{h,p'}) \in \overline{\mathcal{V}}_{h,p} \times \mathcal{V}'_{h,p'}$ **such that:**

$$\mathcal{R}(\bar{\boldsymbol{v}}_{h,p}, \boldsymbol{v}'_{h,p'}; \bar{\boldsymbol{u}}_{h,p}, \boldsymbol{u}'_{h,p'}) = 0, \quad \forall (\bar{\boldsymbol{v}}_{h,p}, \boldsymbol{v}'_{h,p'}) \in \overline{\mathcal{V}}_{h,p} \times \mathcal{V}'_{h,p'}. \tag{16}$$

Remarks:

- $\boldsymbol{u}'_{h,p'}$ DOFs are element-wise decoupled. Thus, they can be static condensed and the total cost becomes the same as a CG method.
- For same accuracy requirement, more efficient (less DOFs) than CG and DG.
- Adjoint consistent.

Output Functional

In general, consider output functional of the form:

$$\mathcal{J}(\boldsymbol{u}) := \int_{\Omega} g_v(\boldsymbol{u})dV + \int_{\partial\Omega} g_b(\boldsymbol{u})dS. \tag{17}$$

We define output error as:

$$\varepsilon(\boldsymbol{u}_h) := \mathcal{J}(\boldsymbol{u}) - \mathcal{J}(\boldsymbol{u}_h). \tag{18}$$

For general nonlinear problem, the output error can be approximated using the **dual weighted residual** (DWR) method.

Needs **correction**[4] for asymptotically consistent problems.

[4] B. Couchman 2020

Residual Consistency

- We say the residual form $\mathcal{R}$ is consistent if:

$$\mathcal{R}(\boldsymbol{v}_h, \boldsymbol{u}) = 0, \quad \forall \boldsymbol{v}_h \in \mathcal{V}_h, \tag{19}$$

  where $\boldsymbol{u}$ is the exact solution.
- We say the residual form $\mathcal{R}$ is asymptotically consistent if:

$$\mathcal{R}(\boldsymbol{v}_h, \boldsymbol{u}) = \mathcal{O}(h^\alpha), \quad \forall \boldsymbol{v}_h \in \mathcal{V}_h, \tag{20}$$

  where $\boldsymbol{u}$ is the exact solution, $\alpha > 0$, and h is a characteristic element size in $\mathcal{T}_h$.

In our situation:

$$\mathcal{R}(\boldsymbol{v}_h, \boldsymbol{u}_h) = \mathcal{R}^C(\boldsymbol{v}_h, \boldsymbol{u}_h) + \underbrace{\mathcal{R}^A(\boldsymbol{v}_h, \boldsymbol{u}_h)}_{\text{AV term}}. \tag{21}$$

Dual Problem: Mean Value Linearization

We define the mean value linearizations of the residual and output functional as:

$$\overline{\mathcal{R}}'[\boldsymbol{u}_h, \boldsymbol{u}](\boldsymbol{v}, \boldsymbol{w}) := \int_0^1 \mathcal{R}'[\boldsymbol{u}_h + \theta\underbrace{(\boldsymbol{u} - \boldsymbol{u}_h)}_{\Delta \boldsymbol{u} :=}](\boldsymbol{v}, \boldsymbol{w})\ d\theta, \tag{22}$$

$$\overline{\mathcal{J}}'[\boldsymbol{u}_h, \boldsymbol{u}](\boldsymbol{w}) := \int_0^1 \mathcal{J}'[\boldsymbol{u}_h + \theta(\boldsymbol{u} - \boldsymbol{u}_h)](\boldsymbol{w})\ d\theta. \tag{23}$$

Then, the **dual** (adjoint) problem is:

Find $\psi^{\mathrm{mv}} \in \mathcal{W} \cup \mathcal{V}_h$ such that:

$$\overline{\mathcal{R}}'[\boldsymbol{u}_h, \boldsymbol{u}](\psi^{\mathrm{mv}}, \boldsymbol{w}) - \overline{\mathcal{J}}'[\boldsymbol{u}_h, \boldsymbol{u}](\boldsymbol{w}) = 0, \quad \forall \boldsymbol{w} \in \mathcal{W} \cup \mathcal{V}_h. \tag{24}$$

DWR From Mean Value Linearization

We start from:

$$\begin{aligned}\varepsilon(\boldsymbol{u}_h) = \mathcal{J}(\boldsymbol{u}) - \mathcal{J}(\boldsymbol{u}_h) &= -\overline{\mathcal{J}}'[\boldsymbol{u}_h, \boldsymbol{u}](\Delta \boldsymbol{u}) \\ &= -\overline{\mathcal{R}}'[\boldsymbol{u}_h, \boldsymbol{u}_h + \Delta \boldsymbol{u}](\psi^{\text{mv}}, \Delta \boldsymbol{u}).\end{aligned} \tag{25}$$

After some work, we obtain the corrected DWR error expression:

$$\varepsilon(\boldsymbol{u}_h) = -\Big[\mathcal{R}(\psi^{\text{mv}}, \boldsymbol{u}_h) - \mathcal{R}^A(\psi^{\text{mv}}, \boldsymbol{u})\Big]. \tag{26}$$

Two issues:

- Primal exact solution $\boldsymbol{u}$ is not available.
- The mean value adjoint ψ^{mv} is not computationally tractable.

Approximations: DWR Estimate

First approximation:

$$\mathcal{R}^A(\psi^{\mathrm{mv}}, \boldsymbol{u}) \approx \mathcal{R}^A(\psi^{\mathrm{mv}}, \boldsymbol{u}_h), \tag{27}$$

justified on a shock dominated problem with AV.

Second approximation: Tractable adjoint:

ψ^{mv} is approximated with a numerical adjoint $\psi_{\hat{h}}$ defined by[5]:
Find $\psi_{\hat{h}} \in \mathcal{V}_{\hat{h}}$ such that:

$$\mathcal{J}'[\boldsymbol{u}_h](\tilde{\boldsymbol{u}}) - \mathcal{R}'[\boldsymbol{u}_h](\psi_{\hat{h}}, \tilde{\boldsymbol{u}}) = \mathcal{J}^*(\psi_{\hat{h}}) - \mathcal{R}^*(\tilde{\boldsymbol{u}}, \psi_{\hat{h}}), \quad \forall \tilde{\boldsymbol{u}} \in \mathcal{V}_{\hat{h}}. \tag{28}$$

DWR error estimate:

$$\varepsilon(\boldsymbol{u}_h) \approx -\Big[\mathcal{R}(\psi_{\hat{h}}, \boldsymbol{u}_h) - \mathcal{R}^A(\psi_{\hat{h}}, \boldsymbol{u}_h)\Big]. \tag{29}$$

[5] M. Yano and D. L. Darmofal 2012

Output-based Mesh Adaptation

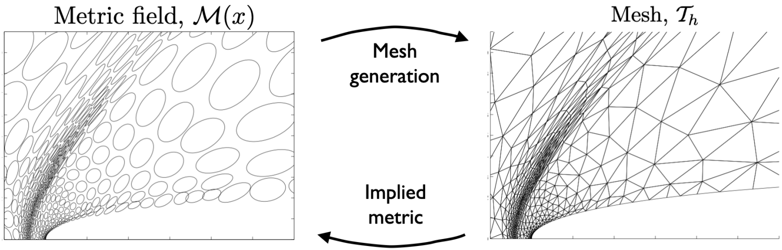
Continuous Optimization: Mesh-Metric Duality

Want mesh producing the smallest output error indicator:

$$\hat{\mathcal{T}}_h = \arg \inf_{\mathcal{T}_h \in \mathbb{T}(\Omega)} \mathcal{E}(\mathcal{T}_h), \quad \mathcal{C}(\mathcal{T}_h) < C. \tag{30}$$

Continuous relaxation[6] to address intractability of discrete problem.

$$\hat{\mathcal{M}} = \arg \inf_{\mathcal{M} \in \mathbb{M}(\Omega)} \mathcal{E}(\mathcal{M}), \quad \mathcal{C}(\mathcal{M}) < C \tag{31}$$

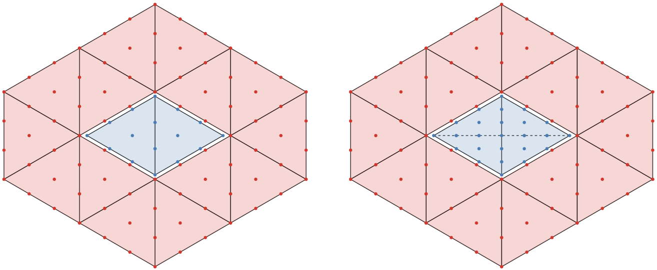


[6] A. Loseille and F. Alauzet 2011

MOESS[8]: Error Sampling and Synthesis

Need model for error indicator $\mathcal{E}(\mathcal{M})$.

Loop over edges in the mesh and compute nodal[7] DWR error estimates in *enriched* local patches.



(a) Unsplit **(b)** Split

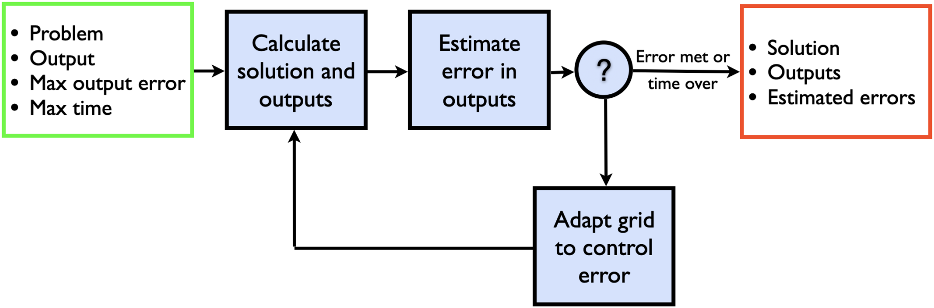
$$\text{Nodal error estimate: } \varepsilon_v^\epsilon = -\mathcal{R}_{\text{local}}(\phi_v \psi_{\hat{h}}, \boldsymbol{u}_h^\epsilon) + \mathcal{R}_{\text{local}}^A(\phi_v \psi_{\hat{h}}, \boldsymbol{u}_h^\epsilon), \quad (32)$$

$$\text{Nodal error indicator: } \eta_v^\epsilon = |\mathcal{R}_{\text{local}}(\phi_v \psi_{\hat{h}}, \boldsymbol{u}_h^\epsilon)| + |\mathcal{R}_{\text{local}}^A(\phi_v \psi_{\hat{h}}, \boldsymbol{u}_h^\epsilon)|. \quad (33)$$

[7] T. Richter and T. Wick 2015

[8] M. Yano and D. L. Darmofal 2012

Adaptation Cycle



Results for Practical Case

Preliminary: Implementation Notes

Software: Solution Adaptive Numerical Simulator (SANS)[9]

- C++ framework to numerically solve partial differential equations.
- Extensive use of templates for efficient yet general code.
- Supports several CG and DG discretizations, with output-based mesh adaptation.
- Automatic differentiation via operator overloading.
- MPI parallelization.
- Unit testing and continuous integration.
- Open source.

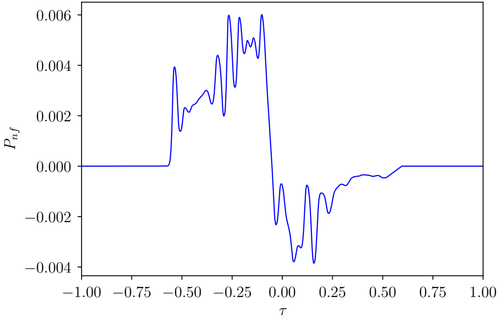
[9] Galbraith et. al. 2015

Preliminary: Run Summary

1. **Set:**
    - Case parameters/conditions.
    - Initial 2D mesh.
    - Target DOF.
    - Number of adaptive iterations (N).
2. **Loop,** $i \in \{1, \ldots, N\}$**:**
    - Extract *ground* boundary and form 1D mesh for filter ODE.
    - Solve primal problem:
        - [Burgers system + shock sensor] in 2D mesh.
        - Filter ODE in 1D mesh (ground boundary).
    - Solve adjoint problem:
        - Adjoint ODE in 1D mesh (ground boundary).
        - [Burgers system + shock sensor] adjoint in 2D mesh.
    - Nodal error sampling and synthesis.
    - Adapt mesh.

Case Description

- Airplane Mach number: $M_a = 1.4$.
- Airplane altitude: $z_a = 16459.2$ m.
- Ground altitude: 110 m.
- Ground reflection factor: 1.9.
- Nearfield signal (initial condition):

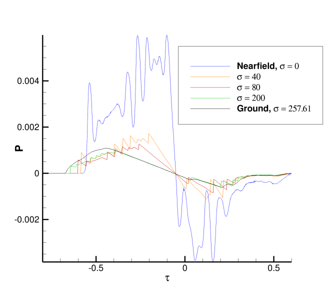


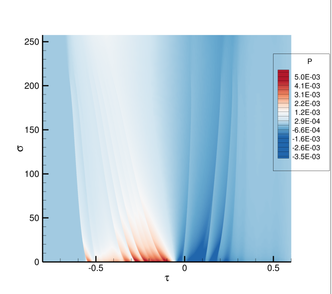
Source: lockheedmartin.com

- Domain dimensions: $\Omega = [-1, 2] \times [0, 257]$
- Output for adaptation: $\mathcal{J}_{\text{BSEL}} = \int_{\text{ground}} [\tilde{p}(t)]^2 \, dt$
- Also for comparison: $\mathcal{J}_p = \int_{\text{ground}} [p(t)]^2 \, dt$

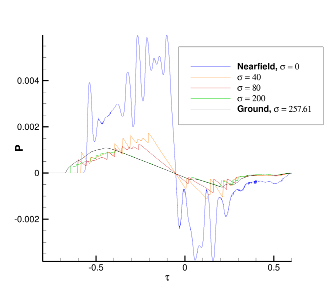
Propagation: Pressure Solution

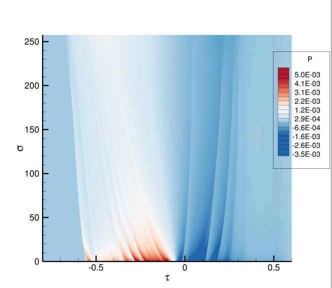
$p = 1$





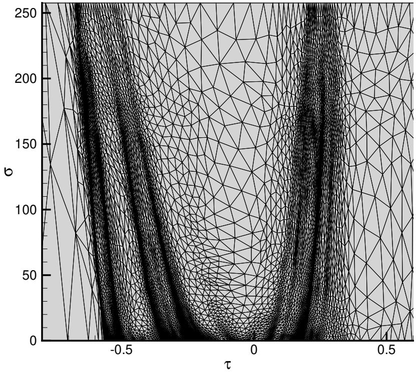
$p = 2$

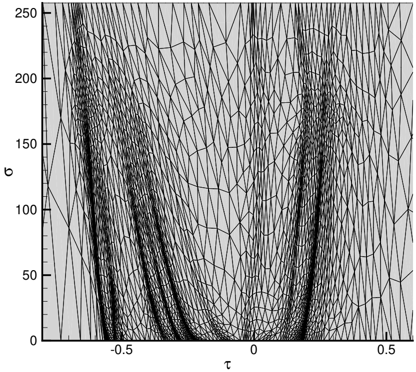




Propagation: Final Adapted Mesh for 8K Target DOF

$p = 1$ $\qquad$ $p = 2$

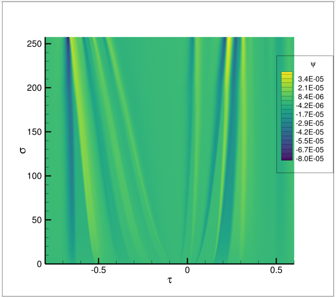


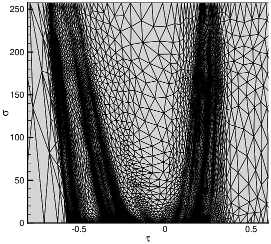


Propagation: Evolution Over Adaptive Cycle, 8K Target DOF

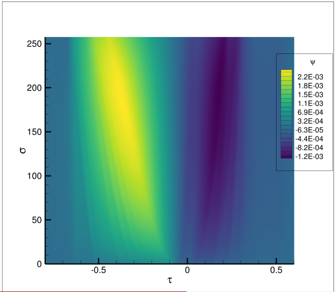
Different Adaptation Outputs and Their Adjoints

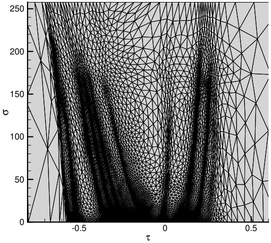
$\mathcal{J}_{\text{BSEL}}$





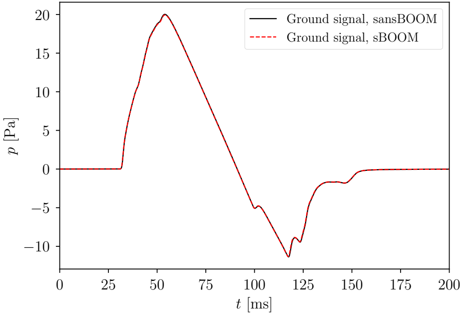
$\mathcal{J}_p$



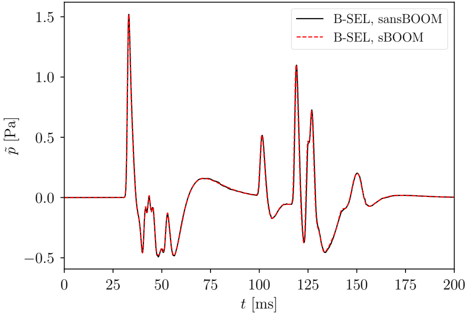


At Ground: Pressure Signal and Its Filtering

Pressure Signal



B-SEL Filter Output

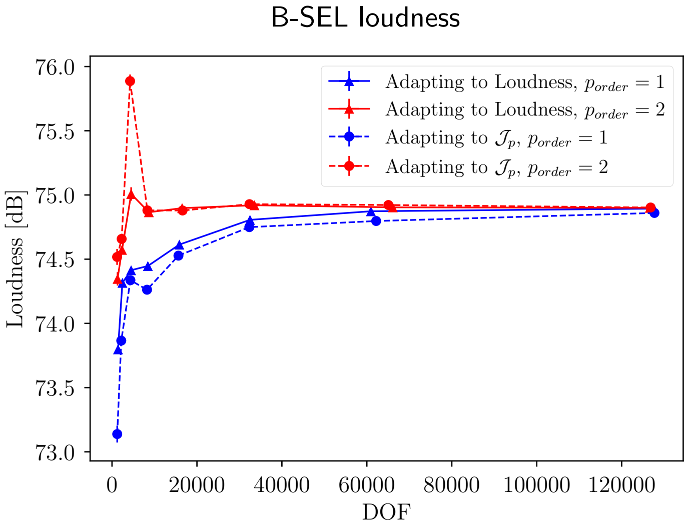


Comparison with NASA *sBOOM* code[10]:

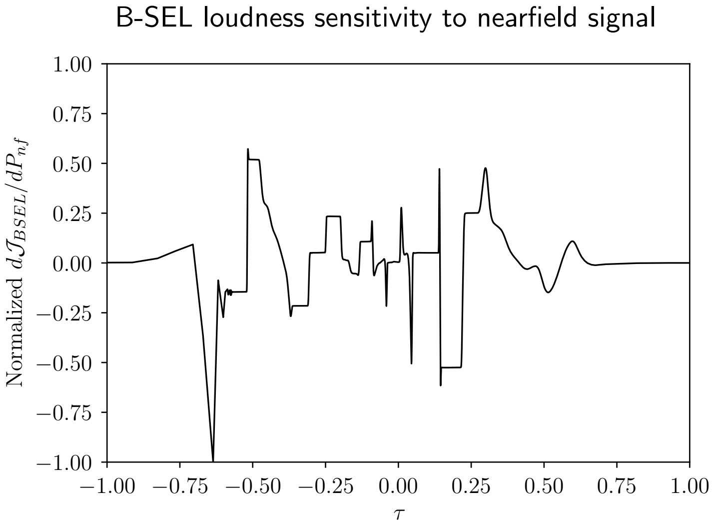
- sansBOOM: 128K DOF in total (space-time).
- sBOOM:
  - 32K DOF in τ direction.
  - 39K steps (marching) in σ direction.
  - 1.2B DOF in total (space-time).

[10] S. K. Rallabhandi et. al. 2023

At Ground: Loudness Convergence with Mesh Refinement



At Nearfield: Loudness Sensitivity



Concluding Remarks

Work completed:

- Higher-order FEM to solve sonic boom propagation problem.
- Unstructured space-time mesh adaptation.
- Loudness error estimate driving mesh adaptation.

Outcome:

- Significant reduction in space-time DOF count, at the expense of solving the dual problem.
- The above is highlighted when using higher-order solutions, as quadratic solutions converge the output faster than linear solutions.

Ongoing effort:

- Study convergence of loudness sensitivity to nearfield signal.

Thanks for the attention!

Questions?