



A Loudness-based Adaptive, Higher-order Finite Element Method for Sonic Boom Propagation

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Motivation and Overview

Motivation For Sonic Boom Study

Ultimate goal:

Enable supersonic commercial flights overland.

Main challenge:

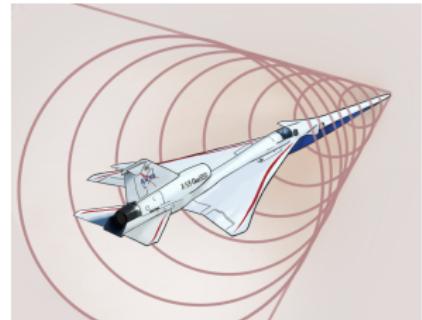
- Negative impact of sonic boom loudness on humans, other animals, and structures.

Study sonic boom to:

- Predict loudness sensitivities to airplane geometry.
- Perform airplane shape optimization to reduce loudness at ground.



Source: lockheedmartin.com



Source: michigandaily.com

Solution Approach: Broad Picture

Multidisciplinary design optimization¹:

- Parametric geometry generator.
- CFD solver.
 - Euler/Navier-Stokes in 3D.
 - Uniform atmosphere.

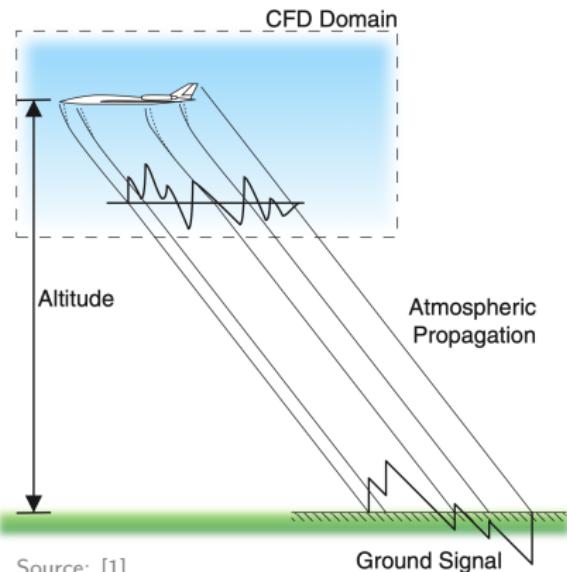
- *Sonic boom propagation tool*

- 2D problem.
- Weakly non-linear.
- Species relaxation.
- Non-uniform atmosphere.

- Numerical optimizer.

All together:

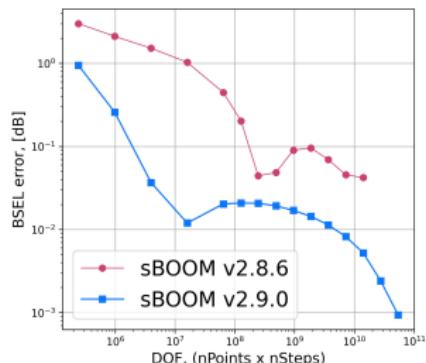
$$\frac{d(\text{Noise})}{dX} = \frac{\partial(\text{Noise})}{\partial p_{gs}} \frac{\partial p_{ns}}{\partial Q} \frac{dp_{ns}}{dT} \frac{dQ}{dT} \frac{dT}{dX}.$$



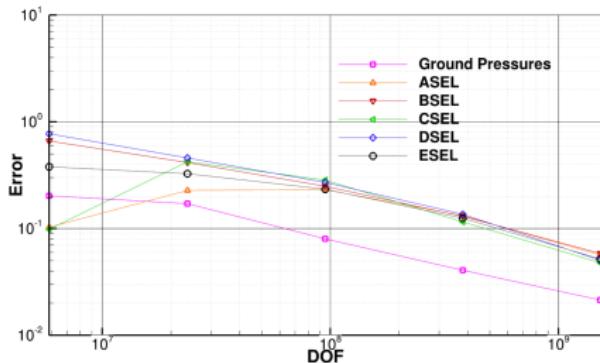
¹D. L. Rodriguez et. al. 2025

Propagation Problem: Motivation for Space-time Mesh Adaptation

Results for standard time-marching methods²:



Source: [2]

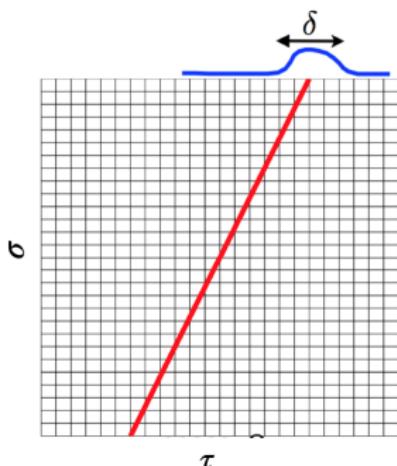


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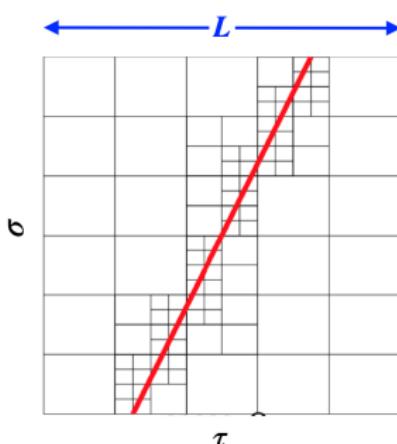
Our goal: Reduce the significant computational cost involved. Enable efficient, automated high accuracy predictions of boom propagation and design sensitivities through adaptive control of numerical error.

²S. K. Rallabhandi et. al. 2023

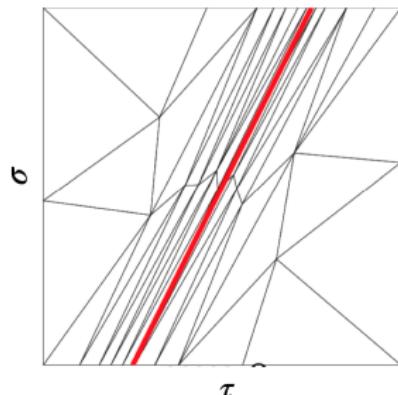
Propagation Problem: Space-time Adaptive Method



$$\text{DOF} = O((L/\delta)^2)$$



$$\text{DOF} = O(L/\delta)$$



$$\text{DOF} = O(1)$$

But, space-time unstructured requires coupled solve over entire space-time domain.

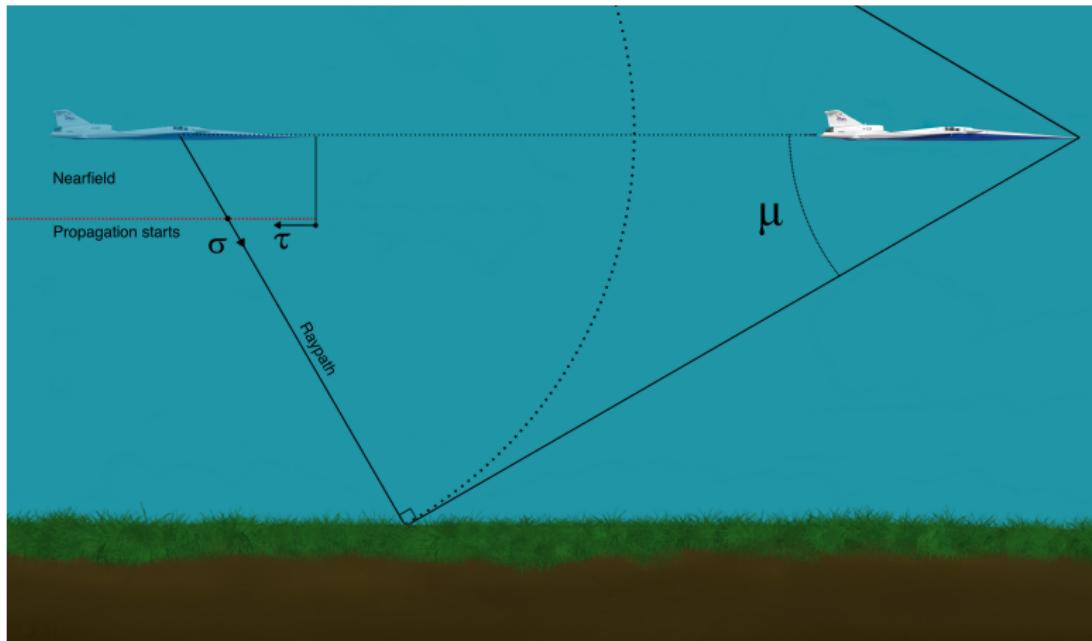
Propagation Problem: Methodology Overview

- ① Boom Propagation Modeling.
- ② Shock Capturing.
- ③ Ground Signal Filtering.
- ④ FEM Discretization and Output Error Estimation.
- ⑤ Output-based Mesh Adaptation.

Boom Propagation Modeling

Coordinate System

Airplane at cruise altitude and cruise Mach number (M_a):



$$\text{Mach cone angle: } \mu = \sin^{-1}(1/M_a).$$

Augmented Burgers System

To model sonic boom propagation we use the augmented Burgers system of equations:

$$\frac{\partial P}{\partial \sigma} - \frac{1}{2} \frac{\partial \ln(\rho_0 c_0 / A_{n0})}{\partial \sigma} P - \frac{1}{2} \frac{\partial P^2}{\partial \tau} - \frac{1}{\Gamma} \frac{\partial^2 P}{\partial \tau^2} - \frac{\partial}{\partial \tau} \left(\sum_{\nu} C_{\nu} \frac{\partial \tilde{P}_{\nu}}{\partial \tau} \right) = 0, \quad (1)$$

$$-\frac{\partial \tilde{P}_{\nu}}{\partial \tau} + \frac{P - \tilde{P}_{\nu}}{\theta_{\nu}} = 0, \quad \nu = \{O_2, N_2\}, \quad (2)$$

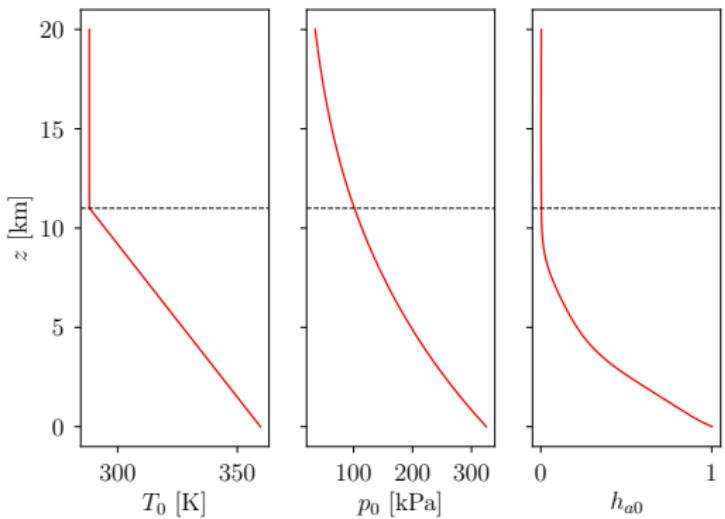
which includes:

- Thermoviscous diffusion.
- Atmospheric absorption by relaxation species (O_2 and N_2).
- Ray tube area variation.

Atmosphere Model: Standard

Atmosphere model refers to how the atmospheric properties depend on altitude.

Standard model: First two layers:



Additionally, models for:

- Density and speed of sound: ρ_0, c_0 ,
 - Gol'berg number: Γ ,
 - Relaxation coefficients: C_ν, θ_ν ,
 - Ray tube area: A_{n0} ,
- as functions of atmospheric properties.

Shock Capturing

The Need for Artificial Viscosity

Challenge: Discontinuities (shocks) in the solutions, leading to unstable numerical solves and lack of convergence.

Goal: Smoothen discontinuities to improve stability, while not modifying the already smooth areas.

Solution Approach: Employ a shock sensor, s , to keep track of discontinuities, and use that information to add localized artificial viscosity.

- Design requirements:
 - $s \approx 1$ in shock areas.
 - $s \approx 0$ away from shocks.
 - Order $\mathcal{O}(h^P)$.
 - s smooth.

PDE-based Shock Sensor³

The shock sensor becomes a state itself, s , with its own equation:

$$\underbrace{s - C_1 s_{\text{grad}}}_{\text{source term}} + \underbrace{\frac{C_2}{p^2} \nabla \cdot \left(H_{00}^2 \frac{\partial s}{\partial \tau} + H_{01}^2 \frac{\partial s}{\partial \sigma}, H_{10}^2 \frac{\partial s}{\partial \tau} + H_{11}^2 \frac{\partial s}{\partial \sigma} \right)}_{\text{diffusion term}} = 0. \quad (3)$$

- s_{grad} : Shock indicator based on pressure solution gradient.
- p : polynomial solution order.
- H : element size field.
- Diffusion term: to have a smooth sensor solution.

Shock Indicator s_{grad}

Starting point:

$$\xi := \frac{H_{00}}{p} \left| \frac{\partial P}{\partial \tau} \right|, \quad \xi_1 \simeq \max_{x \in \Omega} \xi(x), \quad \hat{\xi} = \frac{\xi}{\xi_1}, \quad (4)$$

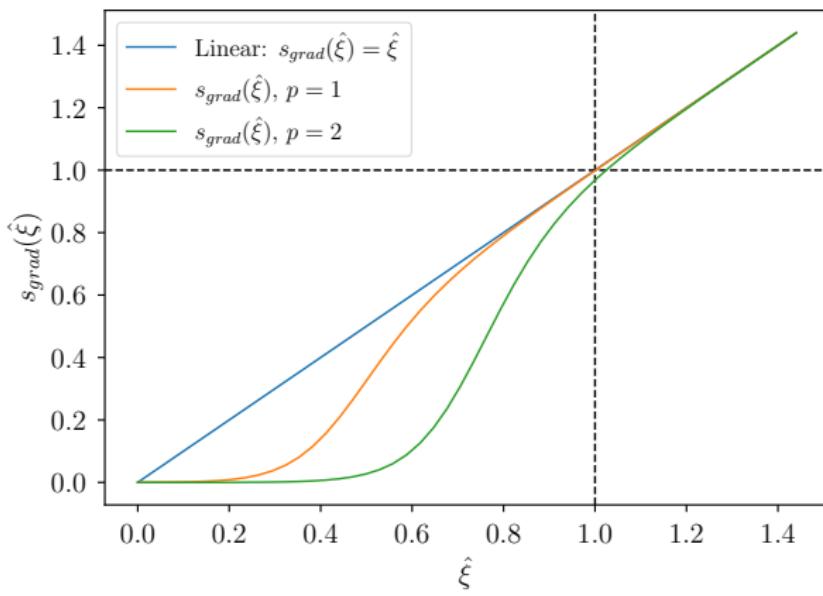
then:

$$s_{\text{grad}} := s_{\text{grad}}(\hat{\xi}) = \frac{\hat{\xi} [\tanh(p^2 \hat{\xi})]^{p-1}}{1 + \exp \left[-k (\hat{\xi} - \alpha(p)) \right]}, \quad (5)$$

where $k, \alpha(p) \in \mathbb{R}$.

Shock Indicator s_{grad}

$$s_{\text{grad}} := s_{\text{grad}}(\hat{\xi}) = \frac{\hat{\xi}[\tanh(p^2\hat{\xi})]^{p-1}}{1 + \exp \left[-k \left(\hat{\xi} - \alpha(p) \right) \right]}, \quad (6)$$



Addition of Artificial Viscosity to the Burgers Equation

Compute artificial viscosity coefficient ϵ_{AV} :

$$\epsilon_{AV} := \frac{1}{2} \frac{H_{00}}{p} |P| s. \quad (7)$$

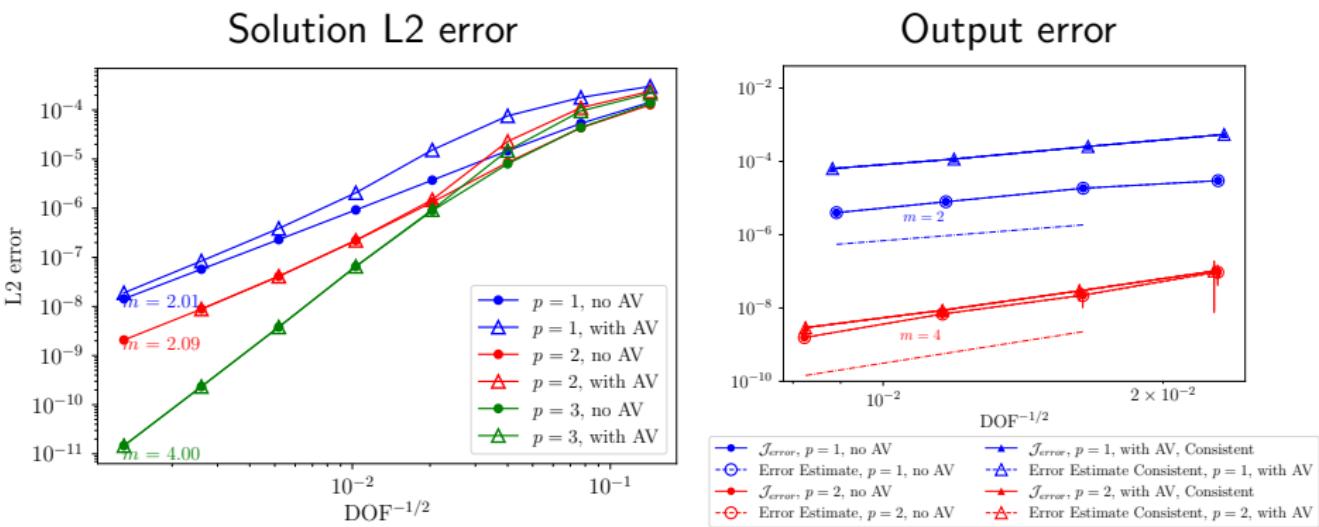
Add extra diffusion term in Burgers equation:

$$\frac{\partial P}{\partial \sigma} - \frac{1}{2} \frac{\partial \ln(\rho_0 c_0 / A_{n0})}{\partial \sigma} P - \frac{1}{2} \frac{\partial P^2}{\partial \tau} - \frac{1}{\Gamma} \frac{\partial^2 P}{\partial \tau^2} - \frac{\partial}{\partial \tau} \left(\sum_{\nu} C_{\nu} \frac{\partial \tilde{P}_{\nu}}{\partial \tau} \right) - \underbrace{\frac{\partial}{\partial \tau} \left(\epsilon_{AV} \frac{\partial P}{\partial \tau} \right)}_{\text{extra term}} = 0. \quad (8)$$

Test With Smooth Problem

Smooth problem with available exact solution:

- Solve without artificial viscosity.
- Solve with artificial viscosity and see how it affects convergence.



Ground Signal Filtering

Relevant Loudness Metrics

The human ear is less sensitive to low audio frequencies.

There is a family of weighting filter curves that account for this relative loudness perceived by humans: A/B/C/D/-SEL curves.

(Ground pressure signal) $p(t)$ → Weighting filter → $\tilde{p}(t)$ (Filtered signal)

Sound exposure:

$$E = \int_{t_0}^{t_f} \tilde{p}(t) \, dt. \quad (9)$$

Loudness level in dB:

$$\text{Loudness} = 10 \log_{10} \left(\frac{E}{E_0} \right), \quad E_0 = 400 \text{ } (\mu\text{Pa})^2\text{s}. \quad (10)$$

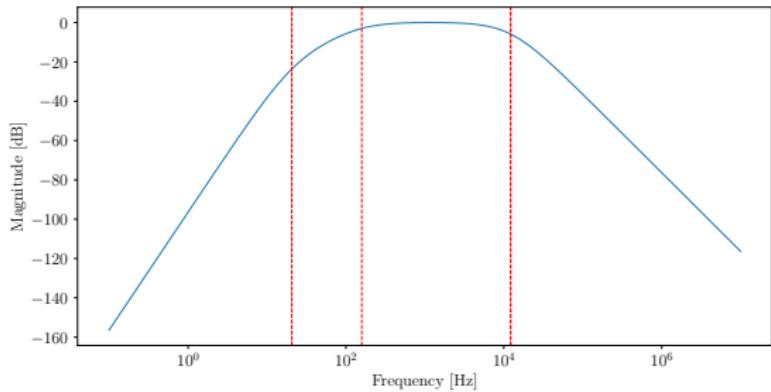
B-SEL Metric

We focus on the B-SEL curve, and the approach can be generalized to any other.

Transfer function in the complex frequency domain:

$$H_B(s) = \frac{\tilde{P}(s)}{P(s)} = \frac{c_B s^3}{(s + 2\pi f_1)^2 (s + 2\pi f_{2B}) (s + 2\pi f_4)^2}, \quad (11)$$

- $c_B = 5.99185 \times 10^9$
- $f_1 = 20.598997 \text{ Hz}$
- $f_{2B} = 158.48932 \text{ Hz}$
- $f_4 = 12194.217 \text{ Hz}$



Filter Application: ODE Approach

Common filtering techniques not suitable for our unstructured grid.

Convert transfer function in complex frequency domain:

$$\tilde{P}(s) = P(s) \frac{K^{1/3}}{(s+a)^2} \frac{K^{1/3}s^2}{(s+c)^2} \frac{K^{1/3}s}{(s+b)}, \quad (12)$$

to a **system of ODE's** in the time domain (to solve in *ground* boundary):

$$\frac{d\bar{u}}{dt} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -a^2 & -2a & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -K^{1/3}a^2 & -2K^{1/3}a & -c^2 & -2c & 0 \\ 0 & 0 & 0 & K^{1/3} & -b \end{pmatrix} \bar{u} + \begin{pmatrix} 0 \\ K^{1/3}p(t) \\ 0 \\ K^{2/3}p(t) \\ 0 \end{pmatrix}, \quad (13)$$

where $\bar{u} = (u_0, u_1, u_2, u_3, \tilde{p})^T$, with homogeneous initial conditions.

FEM Discretization and Output Error Estimation

Variational Multiscale with Discontinuous Subscales (VMSD) Method

Discretization of Ω :

$\mathcal{T}_h := \{\kappa\}_{\kappa=1}^K$ is a triangulation of the domain Ω into K elements.

Propose solution:

$$\mathbf{u}_h := \bar{\mathbf{u}}_{h,p} + \mathbf{u}'_{h,p'}, \quad \bar{\mathbf{u}}_{h,p} \in \bar{\mathcal{V}}_{h,p}, \quad \mathbf{u}'_{h,p'} \in \mathcal{V}'_{h,p'}$$

VMSD solution spaces:

$$(Coarse scale) \quad \bar{\mathcal{V}}_{h,p} := \{\mathbf{v} \in [C^0(\Omega)]^m : \mathbf{v}|_\kappa \in [\mathcal{P}^p(\kappa)]^m, \forall \kappa \in \mathcal{T}_h\}, \quad (14)$$

$$(Fine scale) \quad \mathcal{V}'_{h,p'} := \{\mathbf{v} \in [L^2(\Omega)]^m : \mathbf{v}|_\kappa \in [\mathcal{P}^{p'}(\kappa)]^m, \forall \kappa \in \mathcal{T}_h\}. \quad (15)$$

Variational Multiscale with Discontinuous Subscales (VMSD) Method

Weak statement:

Find $(\bar{\mathbf{u}}_{h,p}, \mathbf{u}'_{h,p'}) \in \bar{\mathcal{V}}_{h,p} \times \mathcal{V}'_{h,p'}$ **such that:**

$$\mathcal{R}(\bar{\mathbf{v}}_{h,p}, \mathbf{v}'_{h,p'}; \bar{\mathbf{u}}_{h,p}, \mathbf{u}'_{h,p'}) = 0, \quad \forall (\bar{\mathbf{v}}_{h,p}, \mathbf{v}'_{h,p'}) \in \bar{\mathcal{V}}_{h,p} \times \mathcal{V}'_{h,p'}. \quad (16)$$

Remarks:

- $\mathbf{u}'_{h,p'}$ DOFs are element-wise decoupled. Thus, they can be static condensed and the total cost becomes the same as a CG method.
- For same accuracy requirement, more efficient (less DOFs) than CG and DG.
- Adjoint consistent.

Output Functional

In general, consider output functional of the form:

$$\mathcal{J}(\mathbf{u}) := \int_{\Omega} g_v(\mathbf{u}) dV + \int_{\partial\Omega} g_b(\mathbf{u}) dS. \quad (17)$$

We define output error as:

$$\varepsilon(\mathbf{u}_h) := \mathcal{J}(\mathbf{u}) - \mathcal{J}(\mathbf{u}_h). \quad (18)$$

For general nonlinear problem, the output error can be approximated using the **dual weighted residual** (DWR) method.

Needs **correction**⁴ for asymptotically consistent problems.

Residual Consistency

- We say the residual form \mathcal{R} is consistent if:

$$\mathcal{R}(\mathbf{v}_h, \mathbf{u}) = 0, \quad \forall \mathbf{v}_h \in \mathcal{V}_h, \quad (19)$$

where \mathbf{u} is the exact solution.

- We say the residual form \mathcal{R} is asymptotically consistent if:

$$\mathcal{R}(\mathbf{v}_h, \mathbf{u}) = \mathcal{O}(h^\alpha), \quad \forall \mathbf{v}_h \in \mathcal{V}_h, \quad (20)$$

where \mathbf{u} is the exact solution, $\alpha > 0$, and h is a characteristic element size in \mathcal{T}_h .

In our situation:

$$\mathcal{R}(\mathbf{v}_h, \mathbf{u}_h) = \mathcal{R}^C(\mathbf{v}_h, \mathbf{u}_h) + \underbrace{\mathcal{R}^A(\mathbf{v}_h, \mathbf{u}_h)}_{\text{AV term}}. \quad (21)$$

Dual Problem: Mean Value Linearization

We define the mean value linearizations of the residual and output functional as:

$$\overline{\mathcal{R}}'[\mathbf{u}_h, \mathbf{u}](\mathbf{v}, \mathbf{w}) := \int_0^1 \mathcal{R}'[\mathbf{u}_h + \theta(\underbrace{\mathbf{u} - \mathbf{u}_h}_{\Delta \mathbf{u}})](\mathbf{v}, \mathbf{w}) d\theta, \quad (22)$$

$$\overline{\mathcal{J}}'[\mathbf{u}_h, \mathbf{u}](\mathbf{w}) := \int_0^1 \mathcal{J}'[\mathbf{u}_h + \theta(\mathbf{u} - \mathbf{u}_h)](\mathbf{w}) d\theta. \quad (23)$$

Then, the **dual** (adjoint) problem is:

Find $\psi^{\text{mv}} \in \mathcal{W} \cup \mathcal{V}_h$ such that:

$$\overline{\mathcal{R}}'[\mathbf{u}_h, \mathbf{u}](\psi^{\text{mv}}, \mathbf{w}) - \overline{\mathcal{J}}'[\mathbf{u}_h, \mathbf{u}](\mathbf{w}) = 0, \quad \forall \mathbf{w} \in \mathcal{W} \cup \mathcal{V}_h. \quad (24)$$

DWR From Mean Value Linearization

We start from:

$$\begin{aligned}\varepsilon(\mathbf{u}_h) &= \mathcal{J}(\mathbf{u}) - \mathcal{J}(\mathbf{u}_h) = -\overline{\mathcal{J}}'[\mathbf{u}_h, \mathbf{u}](\Delta \mathbf{u}) \\ &= -\overline{\mathcal{R}}'[\mathbf{u}_h, \mathbf{u}_h + \Delta \mathbf{u}](\psi^{\text{mv}}, \Delta \mathbf{u}).\end{aligned}\tag{25}$$

After some work, we obtain the corrected DWR error expression:

$$\varepsilon(\mathbf{u}_h) = -\left[\mathcal{R}(\psi^{\text{mv}}, \mathbf{u}_h) - \mathcal{R}^A(\psi^{\text{mv}}, \mathbf{u}) \right].\tag{26}$$

Two issues:

- Primal exact solution \mathbf{u} is not available.
- The mean value adjoint ψ^{mv} is not computationally tractable.

Approximations: DWR Estimate

First approximation:

$$\mathcal{R}^A(\psi^{\text{mv}}, \mathbf{u}) \approx \mathcal{R}^A(\psi^{\text{mv}}, \mathbf{u}_h), \quad (27)$$

justified on a shock dominated problem with AV.

Second approximation: Tractable adjoint:

ψ^{mv} is approximated with a numerical adjoint $\psi_{\hat{h}}$ defined by⁵:

Find $\psi_{\hat{h}} \in \mathcal{V}_{\hat{h}}$ such that:

$$\mathcal{J}'[\mathbf{u}_h](\tilde{\mathbf{u}}) - \mathcal{R}'[\mathbf{u}_h](\psi_{\hat{h}}, \tilde{\mathbf{u}}) = \mathcal{J}^*(\psi_{\hat{h}}) - \mathcal{R}^*(\tilde{\mathbf{u}}, \psi_{\hat{h}}), \quad \forall \tilde{\mathbf{u}} \in \mathcal{V}_{\hat{h}}. \quad (28)$$

DWR error estimate:

$$\varepsilon(\mathbf{u}_h) \approx - \left[\mathcal{R}(\psi_{\hat{h}}, \mathbf{u}_h) - \mathcal{R}^A(\psi_{\hat{h}}, \mathbf{u}_h) \right]. \quad (29)$$

⁵M. Yano and D. L. Darmofal 2012

Output-based Mesh Adaptation

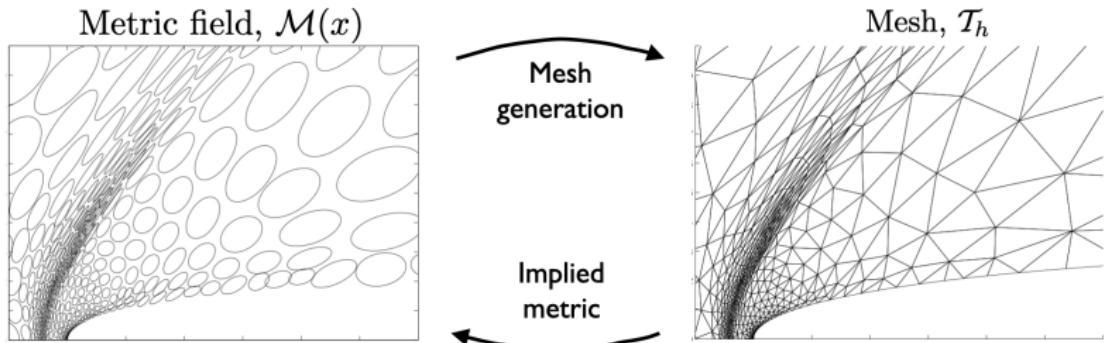
Continuous Optimization: Mesh-Metric Duality

Want mesh producing the smallest output error indicator:

$$\hat{\mathcal{T}}_h = \arg \inf_{\mathcal{T}_h \in \mathbb{T}(\Omega)} \mathcal{E}(\mathcal{T}_h), \quad \mathcal{C}(\mathcal{T}_h) < C. \quad (30)$$

Continuous relaxation⁶ to address intractability of discrete problem.

$$\hat{\mathcal{M}} = \arg \inf_{\mathcal{M} \in \mathbb{M}(\Omega)} \mathcal{E}(\mathcal{M}), \quad \mathcal{C}(\mathcal{M}) < C \quad (31)$$

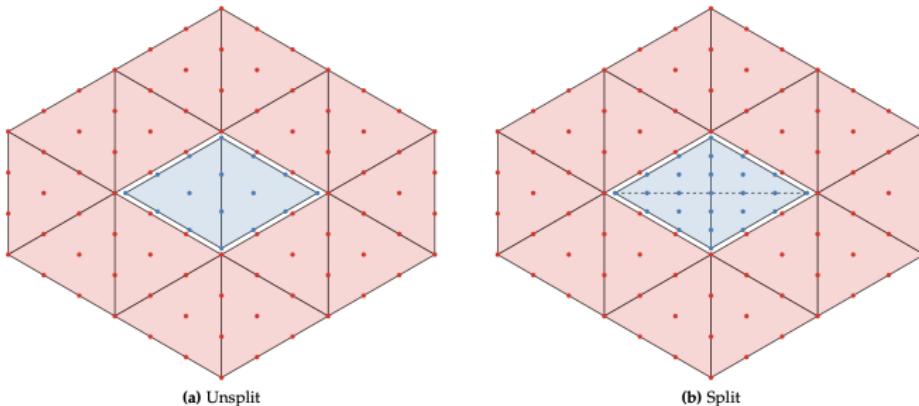


⁶A. Loseille and F. Alauzet 2011

MOESS⁸: Error Sampling and Synthesis

Need model for error indicator $\mathcal{E}(\mathcal{M})$.

Loop over edges in the mesh and compute nodal⁷ DWR error estimates in *enriched* local patches.



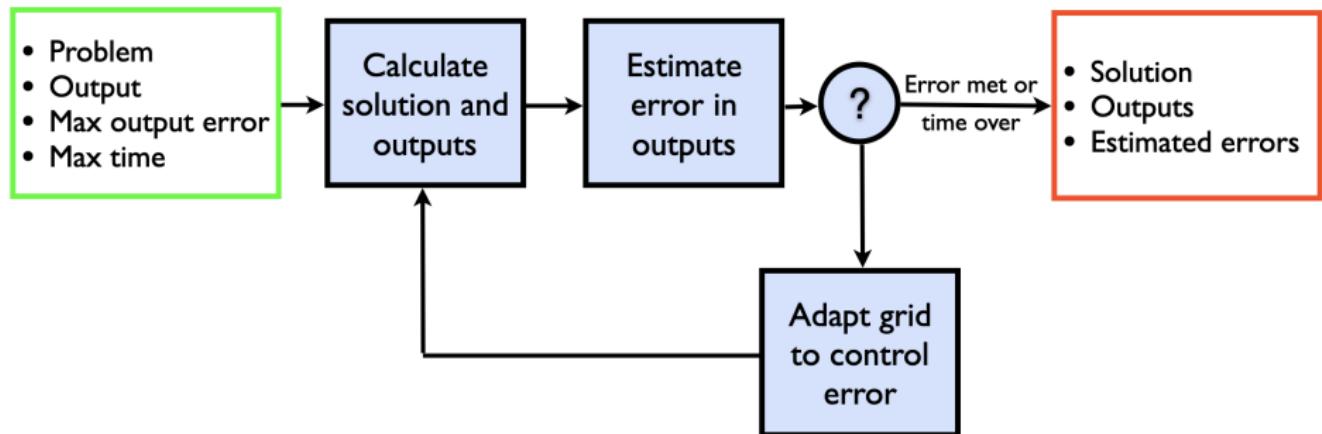
$$\text{Nodal error estimate: } \varepsilon_v^\epsilon = -\mathcal{R}_{\text{local}}(\phi_v \psi_{\hat{h}}, \mathbf{u}_h^\epsilon) + \mathcal{R}_{\text{local}}^A(\phi_v \psi_{\hat{h}}, \mathbf{u}_h^\epsilon), \quad (32)$$

$$\text{Nodal error indicator: } \eta_v^\epsilon = |\mathcal{R}_{\text{local}}(\phi_v \psi_{\hat{h}}, \mathbf{u}_h^\epsilon)| + |\mathcal{R}_{\text{local}}^A(\phi_v \psi_{\hat{h}}, \mathbf{u}_h^\epsilon)|. \quad (33)$$

⁷T. Richter and T. Wick 2015

⁸M. Yano and D. L. Darmofal 2012

Adaptation Cycle



Results for Practical Case

Preliminary: Implementation Notes

Software: Solution Adaptive Numerical Simulator (SANS)⁹

- C++ framework to numerically solve partial differential equations.
- Extensive use of templates for efficient yet general code.
- Supports several CG and DG discretizations, with output-based mesh adaptation.
- Automatic differentiation via operator overloading.
- MPI parallelization.
- Unit testing and continuous integration.
- Open source.

⁹Galbraith et. al. 2015

Preliminary: Run Summary

① Set:

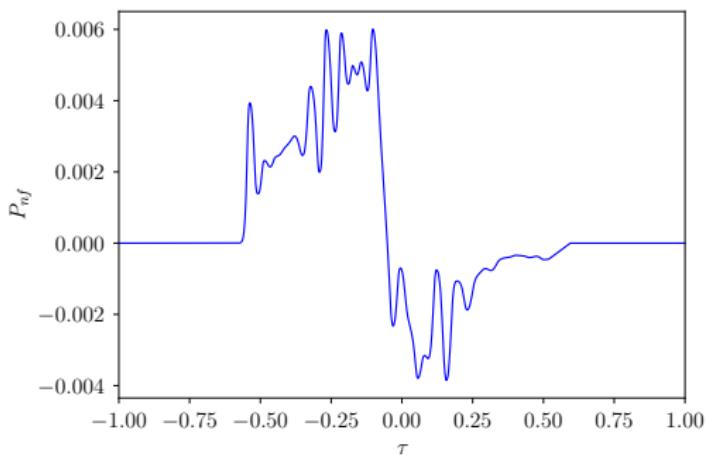
- Case parameters/conditions.
- Initial 2D mesh.
- Target DOF.
- Number of adaptive iterations (N).

② Loop, $i \in \{1, \dots, N\}$:

- Extract *ground* boundary and form 1D mesh for filter ODE.
- Solve primal problem:
 - [Burgers system + shock sensor] in 2D mesh.
 - Filter ODE in 1D mesh (ground boundary).
- Solve adjoint problem:
 - Adjoint ODE in 1D mesh (ground boundary).
 - [Burgers system + shock sensor] adjoint in 2D mesh.
- Nodal error sampling and synthesis.
- Adapt mesh.

Case Description

- Airplane Mach number: $M_a = 1.4$.
- Airplane altitude: $z_a = 16459.2$ m.
- Ground altitude: 110 m.
- Ground reflection factor: 1.9.
- Nearfield signal (initial condition):

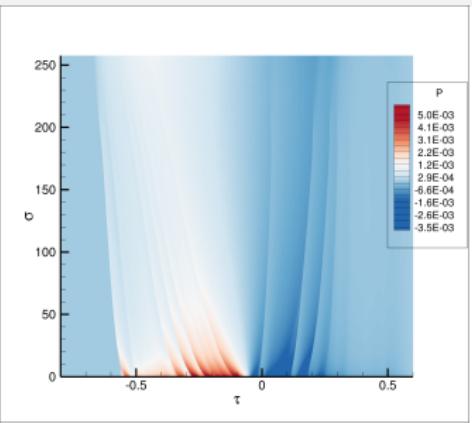
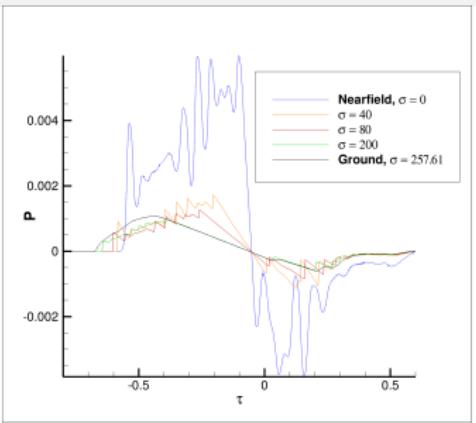


Source: lockheedmartin.com

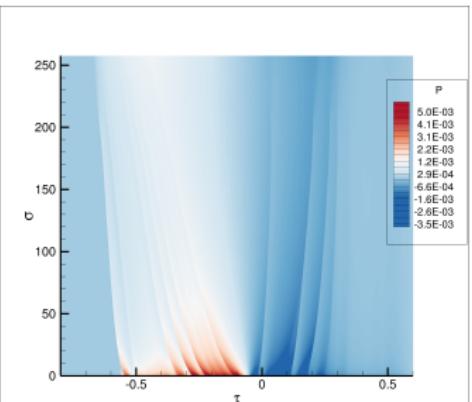
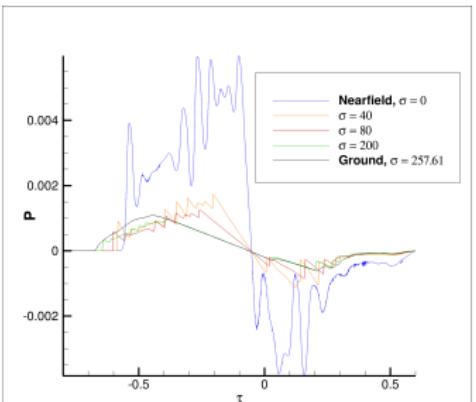
- Domain dimensions:
 $\Omega = [-1, 2] \times [0, 257]$
- Output for adaptation:
 $\mathcal{J}_{BSEL} = \int_{\text{ground}} [\tilde{p}(t)]^2 dt$
- Also for comparison:
 $\mathcal{J}_P = \int_{\text{ground}} [p(t)]^2 dt$

Propagation: Pressure Solution

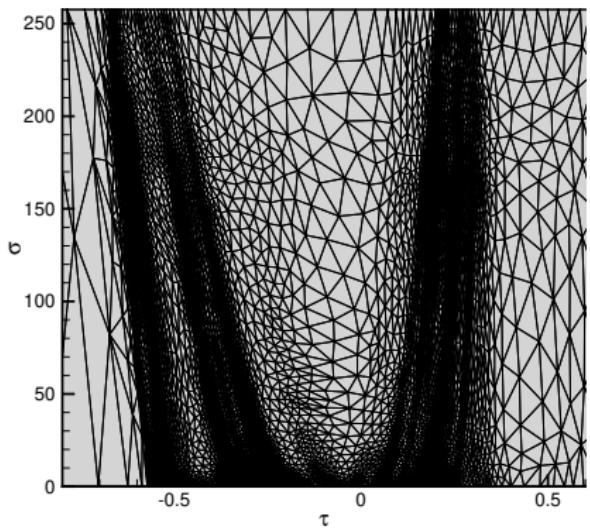
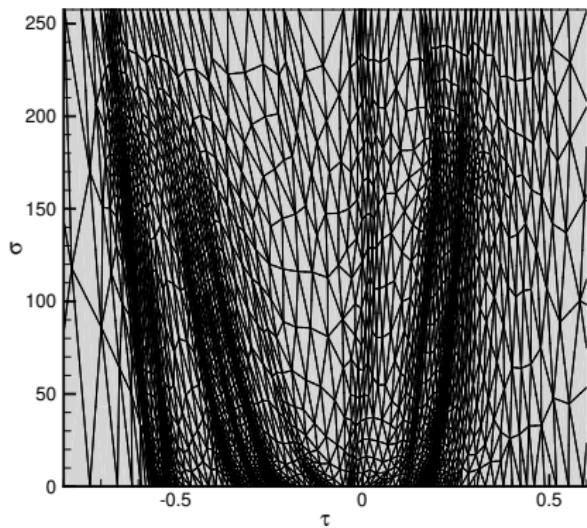
$p = 1$



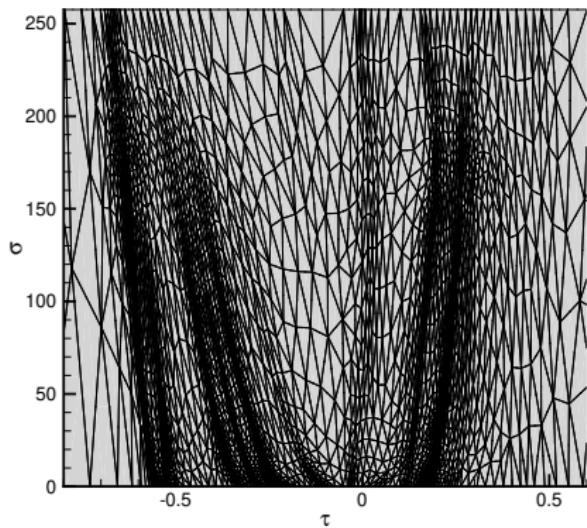
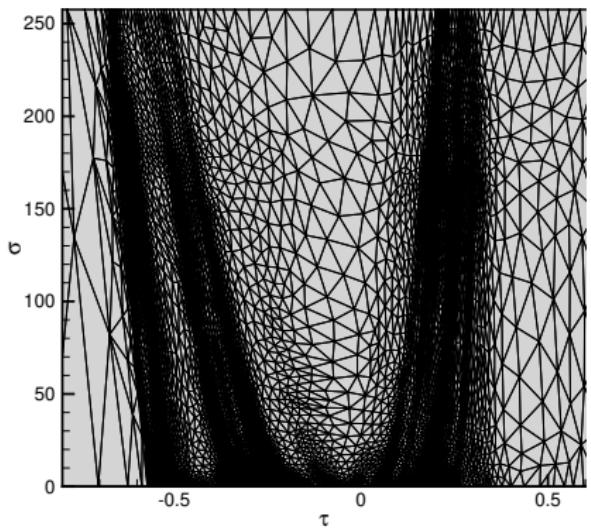
$p = 2$



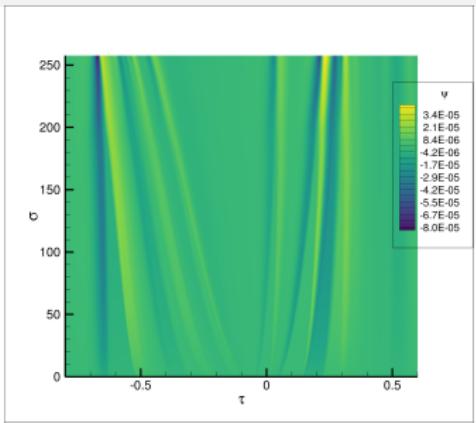
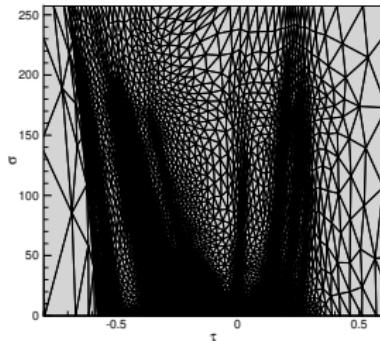
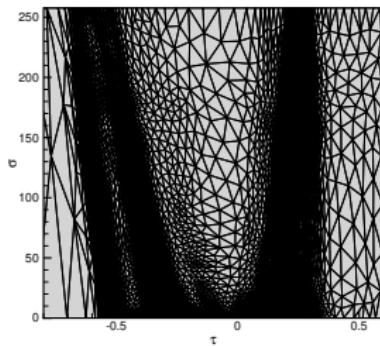
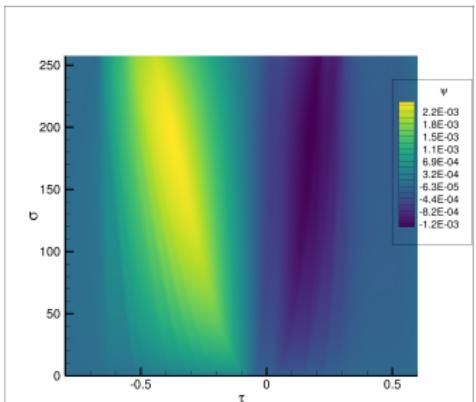
Propagation: Final Adapted Mesh for 8K Target DOF

 $p = 1$  $p = 2$ 

Propagation: Evolution Over Adaptive Cycle, 8K Target DOF

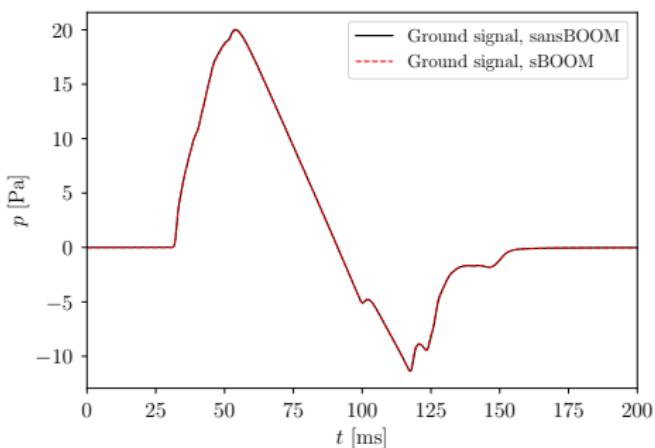


Different Adaptation Outputs and Their Adjoints

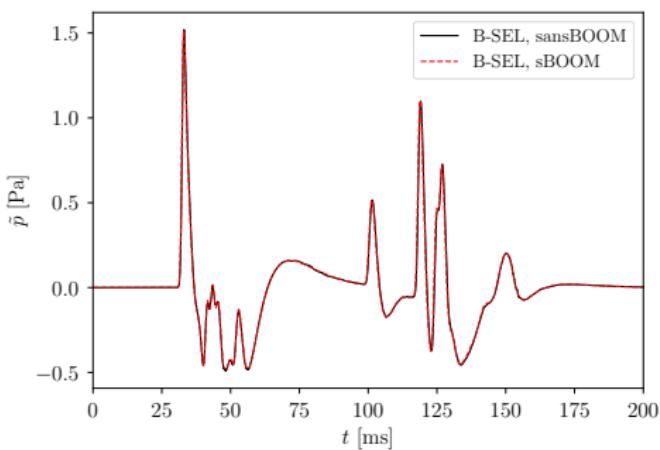
 $\mathcal{J}_{\text{BSEL}}$  \mathcal{J}_P 

At Ground: Pressure Signal and Its Filtering

Pressure Signal



B-SEL Filter Output

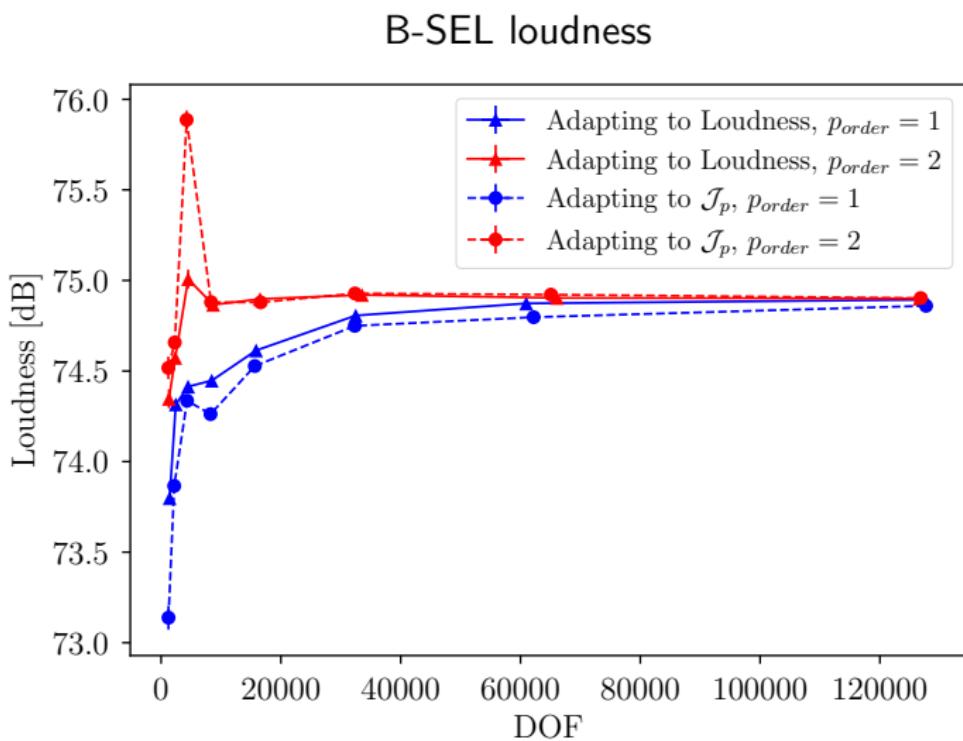


Comparison with NASA *sBOOM* code¹⁰:

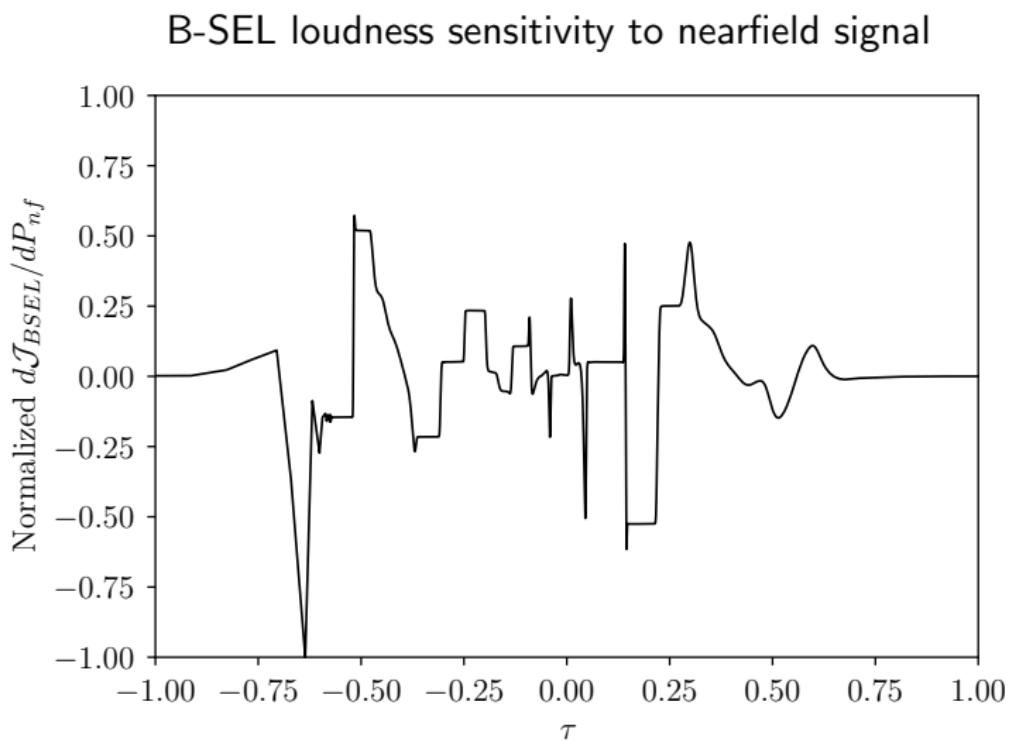
- sansBOOM: 128K DOF in total (space-time).
- sBOOM:
 - 32K DOF in τ direction.
 - 39K steps (marching) in σ direction.
 - 1.2B DOF in total (space-time).

¹⁰S. K. Rallabhandi et. al. 2023

At Ground: Loudness Convergence with Mesh Refinement



At Nearfield: Loudness Sensitivity



Concluding Remarks

Work completed:

- Higher-order FEM to solve sonic boom propagation problem.
- Unstructured space-time mesh adaptation.
- Loudness error estimate driving mesh adaptation.

Outcome:

- Significant reduction in space-time DOF count, at the expense of solving the dual problem.
- The above is highlighted when using higher-order solutions, as quadratic solutions converge the output faster than linear solutions.

Ongoing effort:

- Study convergence of loudness sensitivity to nearfield signal.

Thanks for the attention!

Questions?