



A Space-time Adaptive, Higher-order Finite Element Method for Sonic Boom

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Motivation For Sonic Boom Study

Ultimate goal:

Enable supersonic commercial flights overland.



Source: lockheedmartin.com

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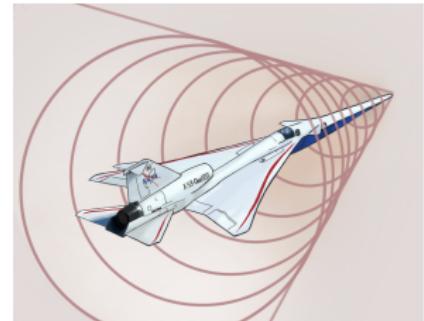
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Main challenge:

- Negative impact of sonic boom loudness on humans, other animals, and structures.



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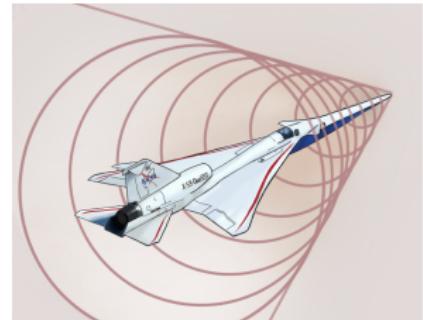
- Negative impact of sonic boom loudness on humans, other animals, and structures.

Study sonic boom to:

- Predict loudness sensitivities to airplane geometry.
- Perform airplane shape optimization to reduce loudness at ground.



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Solution Approach: Broad Picture

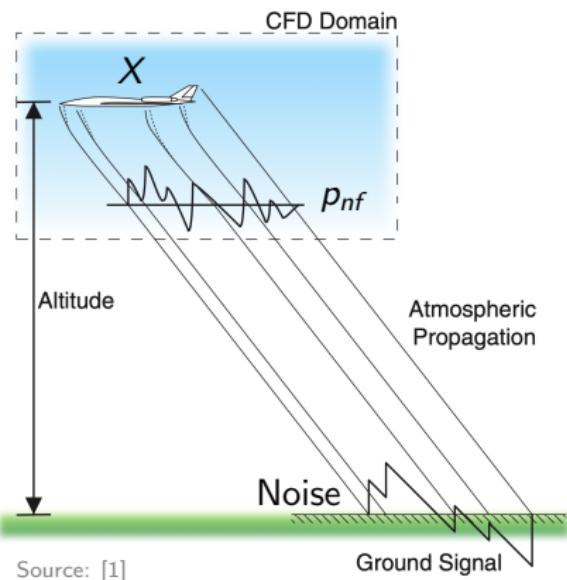
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- CFD solver.
 - Euler/Navier-Stokes in 3D.
 - Uniform atmosphere.
- *Sonic boom propagation tool*
 - 2D problem.
 - Weakly non-linear.
 - Species relaxation.
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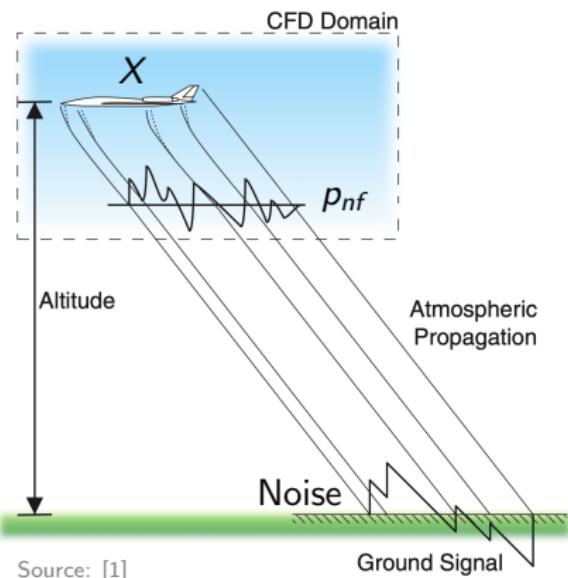


¹D. L. Rodriguez et. al. 2025

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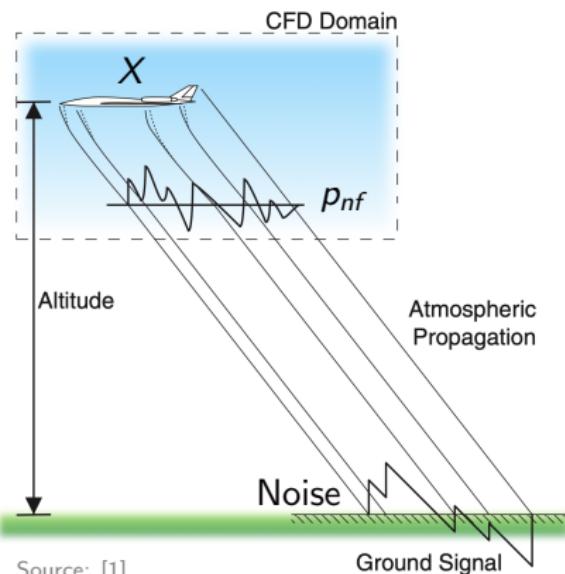
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Compute noise at ground and



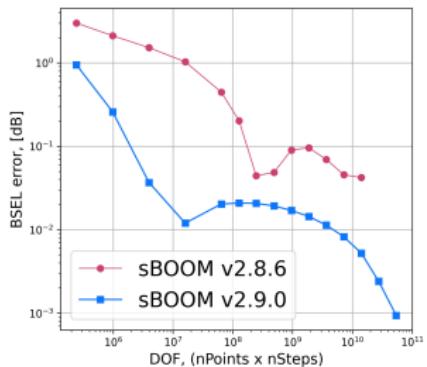
$$\frac{d(\text{Noise})}{dX} = \frac{d(\text{Noise})}{dp_{nf}} \frac{dp_{nf}}{dX} .$$

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Propagation Problem: Motivation for Mesh Adaptation

Results for standard time-marching methods²:

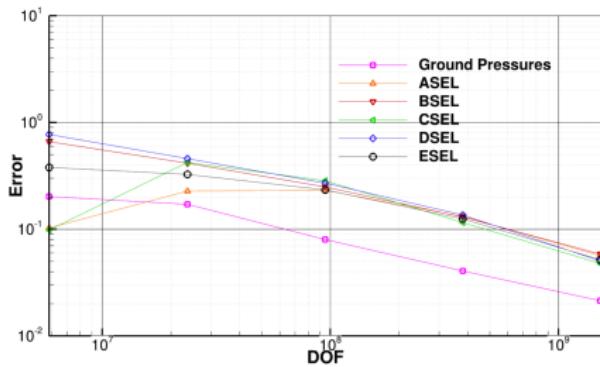
Error in Noise



Source: [2]

($\approx 10^{11}$ DOF for 10^{-3} error)

Error in $|d(\text{Noise})/dp_{nf}|$



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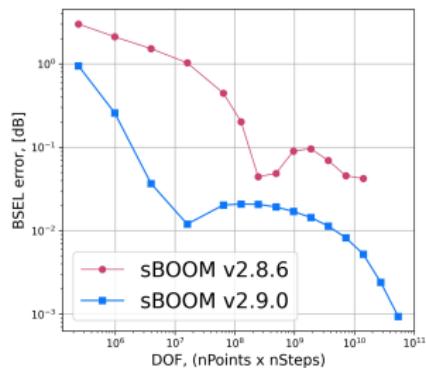
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²S. K. Rallabhandi et. al. 2023

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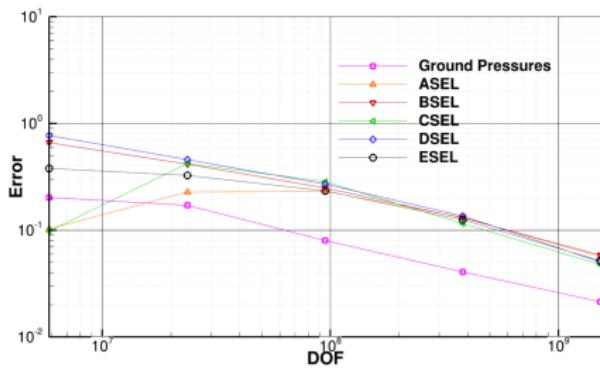
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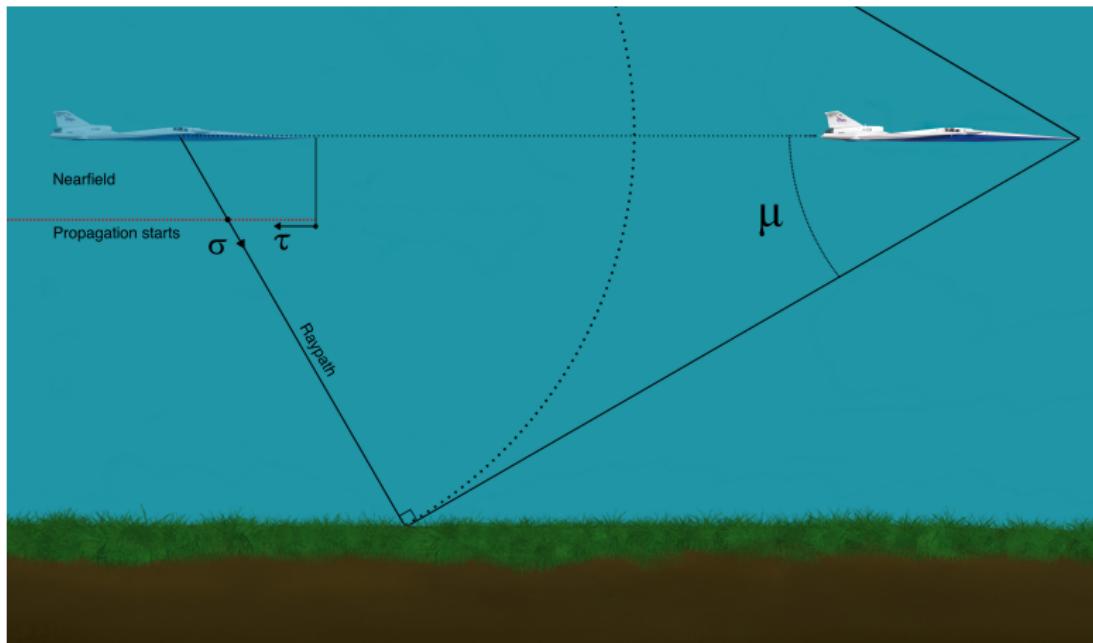
Our goal: Reduce the significant computational cost involved. Enable efficient, automated high accuracy predictions of boom propagation and design sensitivities through adaptive control of numerical error.

²S. K. Rallabhandi et. al. 2023

Boom Propagation Modeling and Adaptive Approach

Coordinate System

Airplane at cruise altitude and cruise Mach number (M_a):



$$\text{Mach cone angle: } \mu = \sin^{-1}(1/M_a).$$

Augmented Burgers System

To model sonic boom propagation we use the augmented Burgers system of equations, for the states $(P, \tilde{P}_{O_2}, \tilde{P}_{N_2})$:

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$$-\frac{\partial \tilde{P}_{\nu}}{\partial \tau} + \frac{P - \tilde{P}_{\nu}}{\theta_{\nu}} = 0 \text{ on } \Omega, \quad \nu = \{O_2, N_2\}, \quad (2)$$

which includes:

- Thermoviscous diffusion.
- Atmospheric absorption by relaxation species (O_2 and N_2).
- Ray tube area variation.

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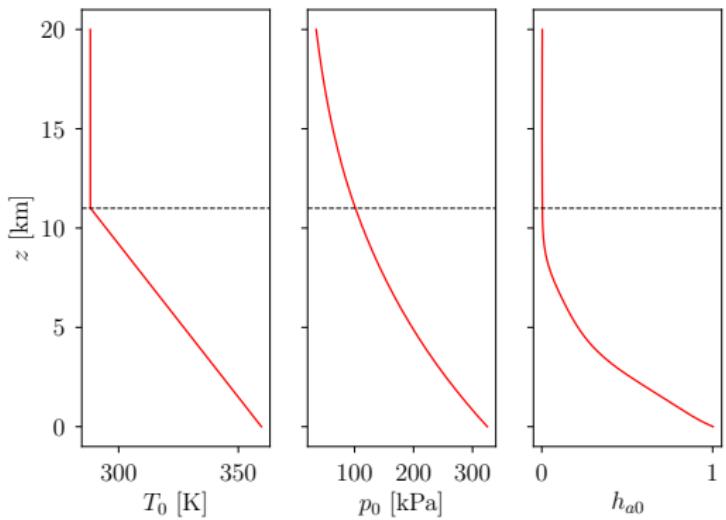
Remarks:

- Eq (1) is parabolic, with σ the time-like direction.
- It is typically solved with a time-marching scheme.

Atmosphere Model: Standard

Atmosphere model refers to how the atmospheric properties depend on altitude.

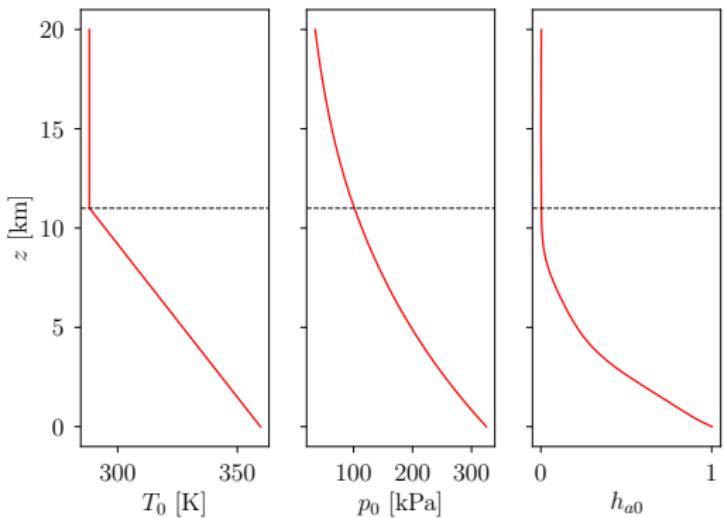
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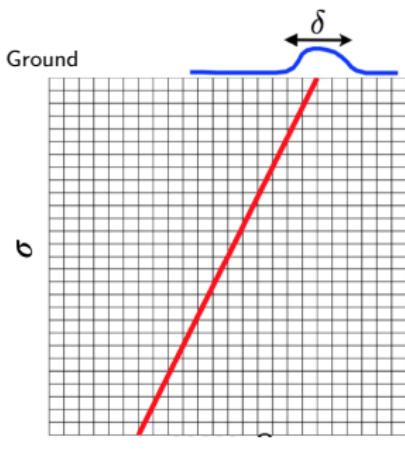
Standard model: First two layers:



Additionally, models for:

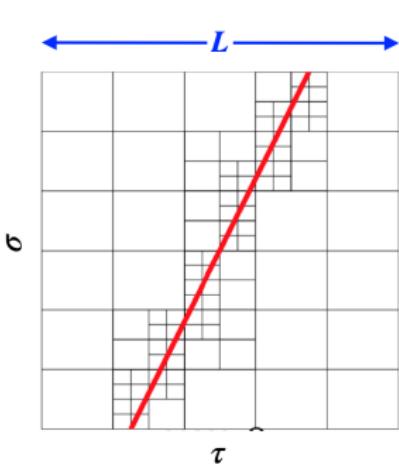
- Density and speed of sound: ρ_0, c_0 ,
 - Gol'berg number: Γ ,
 - Relaxation coefficients: C_ν, θ_ν ,
 - Ray tube area: A_{n0} ,
- as functions of atmospheric properties.

Space-time Adaptive Method



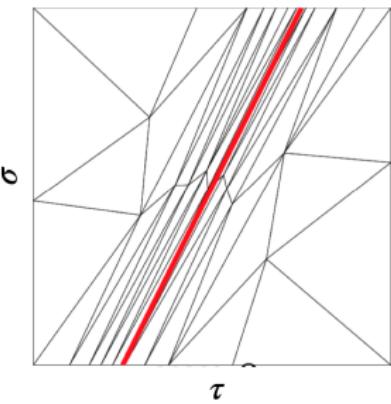
(a) uniform

$$\text{DOF} = O((L/\delta)^2)$$



(b) space-time tensor-product

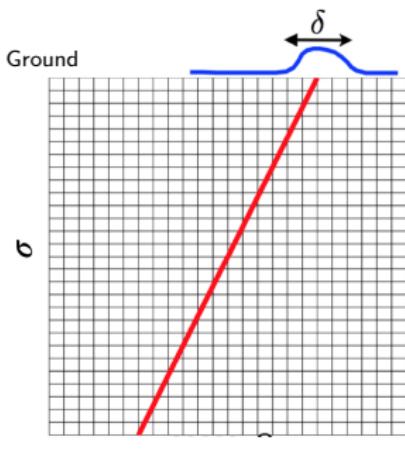
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(c) space-time unstructured

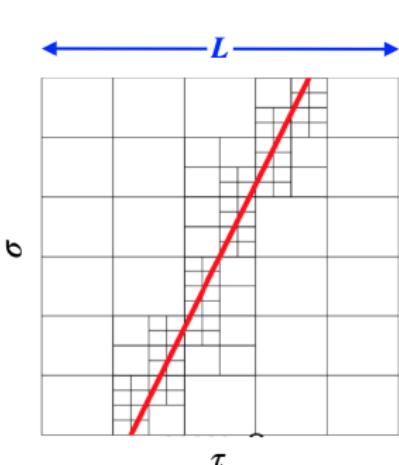
$$\text{DOF} = O(1)$$

Space-time Adaptive Method



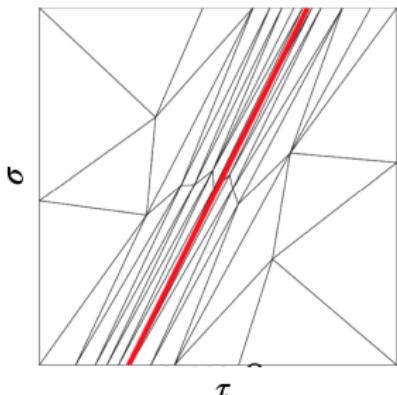
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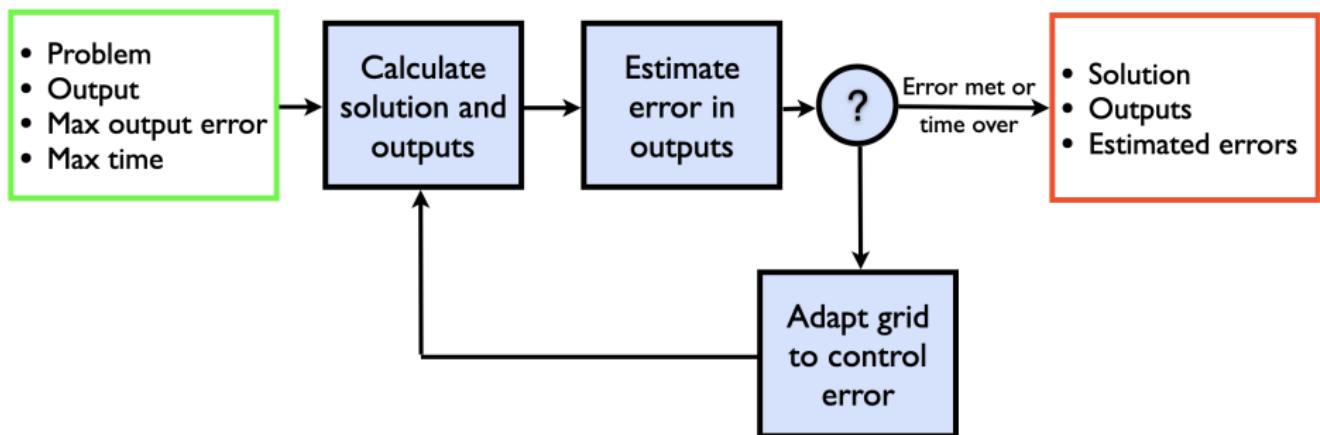
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But, space-time unstructured requires solving an optimization problem for the mesh.

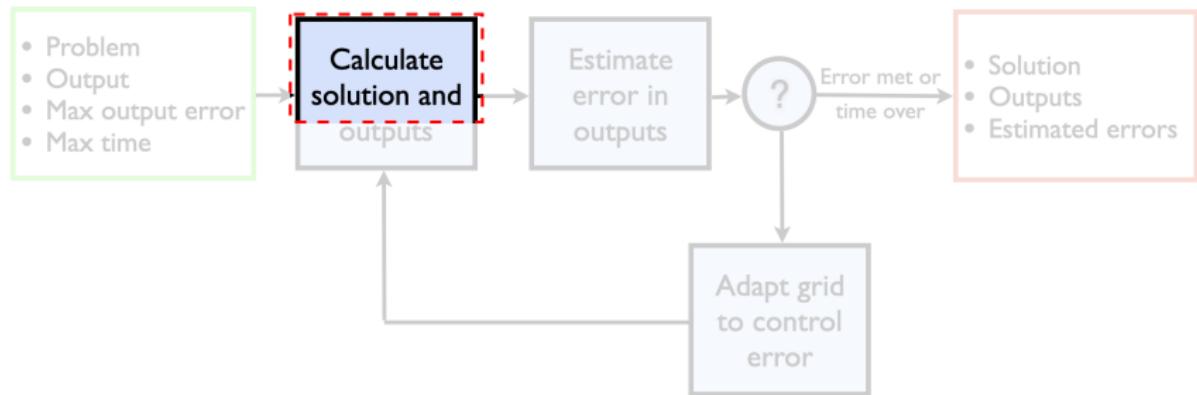
Output-based Adaptation Cycle

Output of interest: Loudness at ground.

Adaptive cycle scheme:



Discretization and Shock Capturing



Continuous Galerkin type FEM

CG weak statement:

Find $\mathbf{u}_h \in \mathcal{V}_{h,p}$ such that:

$$\mathcal{R}(\mathbf{v}_h, \mathbf{u}_h) = 0, \quad \forall \mathbf{v}_h \in \mathcal{V}_{h,p}, \quad (3)$$

where the space $\mathcal{V}_{h,p}$ contains polynomials of order p in Ω .

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Remarks:

- A discontinuous subscale is used for stabilization, and the resulting method is known as Variational Multiscale with Discontinuous Subscales.
- The discretization is adjoint consistent.

The Need for Artificial Viscosity

Challenge: Discontinuities (shocks) in the solutions, leading to unstable numerical solves and lack of convergence.

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Add extra diffusion term in Burgers equation:

$$\frac{\partial P}{\partial \sigma} - \frac{1}{2} \frac{\partial \ln(\rho_0 c_0 / A_{n0})}{\partial \sigma} P - \frac{1}{2} \frac{\partial P^2}{\partial \tau} - \frac{1}{\Gamma} \frac{\partial^2 P}{\partial \tau^2} - \frac{\partial}{\partial \tau} \left(\sum_{\nu} C_{\nu} \frac{\partial \tilde{P}_{\nu}}{\partial \tau} \right) - \underbrace{\frac{\partial}{\partial \tau} \left(\epsilon_{AV} \frac{\partial P}{\partial \tau} \right)}_{\text{extra term}} = 0, \quad (4)$$

with ϵ_{AV} as:

$$\epsilon_{AV} := \underbrace{\frac{1}{2} \frac{H_{\tau\tau}}{p} |P| s}_{AV_{\max}}. \quad (5)$$

PDE-based Shock Sensor³

Shock sensor design requirements:

- $s \approx 1$ in shock areas, of order $\mathcal{O}(h)$.
- $s \approx 0$ away from shocks, of order $\mathcal{O}(h^p)$.
- s smooth.

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Define PDE for shock sensor s :

$$\underbrace{s - C_1 s_{\text{grad}}}_{\text{source term}} = 0 \text{ on } \Omega. \quad (6)$$

- s_{grad} : **Shock indicator** based on pressure solution gradient.

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$$\underbrace{s - C_1 s_{\text{grad}}}_{\text{source term}} + \underbrace{C_2 \nabla \cdot \left(\frac{H^T H}{p^2} \nabla s \right)}_{\text{diffusion term}} = 0 \text{ on } \Omega. \quad (6)$$

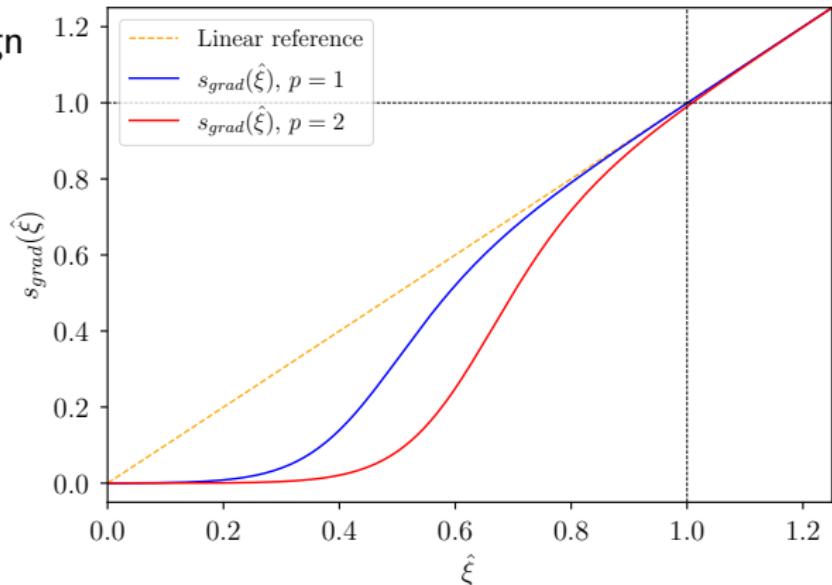
- s_{grad} : **Shock indicator** based on pressure solution gradient.
- Diffusion term: to have a smooth sensor solution.
- H : element size field.

Shock Indicator s_{grad}

Identify pressure changes in τ direction:

$$\xi := \frac{H_{\tau\tau}}{p} \left| \frac{\partial P}{\partial \tau} \right|, \quad \hat{\xi} = \frac{\xi}{\xi_1}, \quad (7)$$

Define $s_{\text{grad}}(\hat{\xi})$ to meet design requirements



Test With Smooth Problem

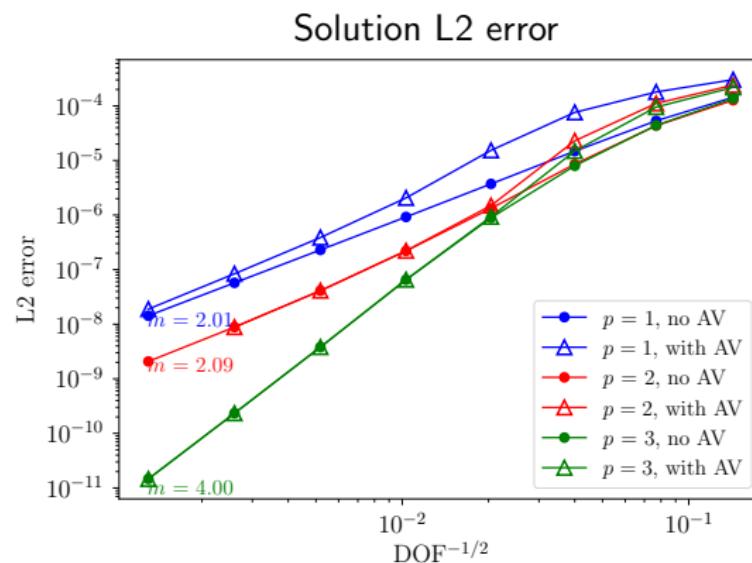
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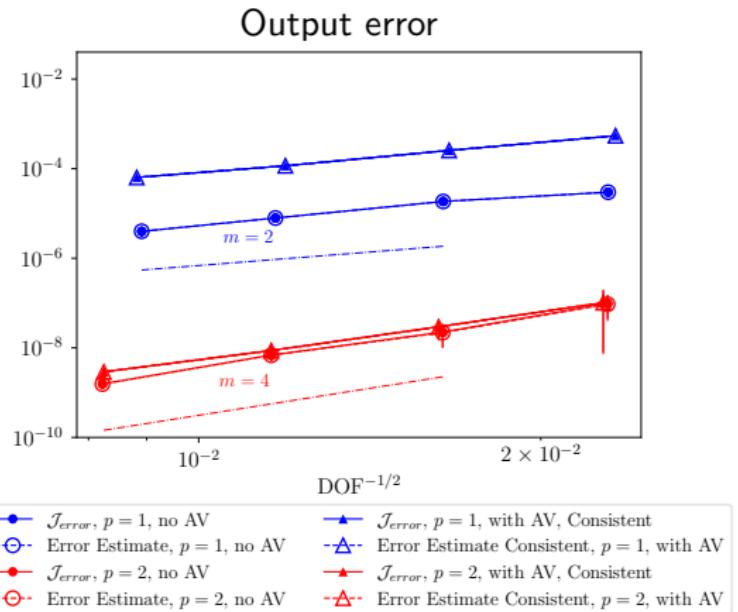
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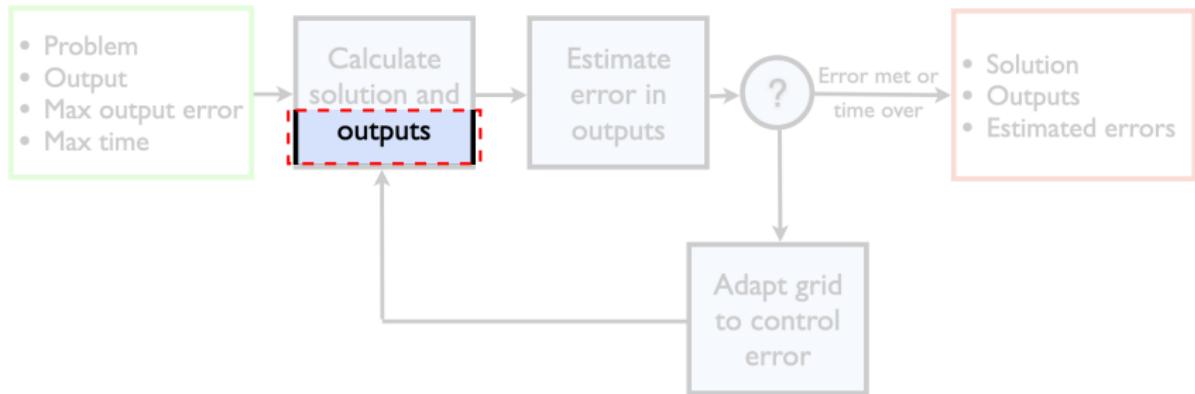
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Ground Signal Filtering



At Ground: Relevant Loudness Metrics

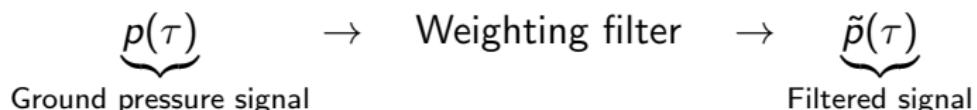
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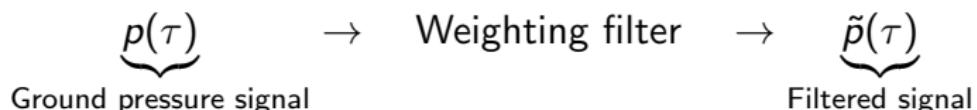
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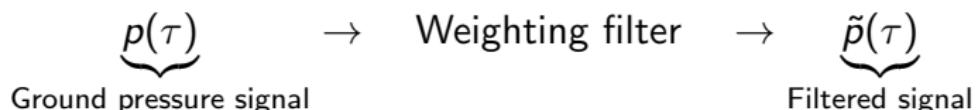
Sound exposure:

$$E = \frac{1}{\omega_{\text{ref}}} \int_{\tau_0}^{\tau_f} [\tilde{p}(\tau)]^2 d\tau. \quad (8)$$

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Sound exposure:

$$E = \frac{1}{\omega_{\text{ref}}} \int_{\tau_0}^{\tau_f} [\tilde{p}(\tau)]^2 d\tau. \quad (8)$$

Loudness level in dB:

$$\text{Loudness} = 10 \log_{10} \left(\frac{E}{E_0} \right), \quad E_0 = 400 \text{ } (\mu\text{Pa})^2\text{s}. \quad (9)$$

B-SEL Metric

We focus on the B-SEL curve, and the approach can be generalized to any other.

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Transfer function in the complex frequency domain:

$$H_B(\chi) = \frac{\tilde{P}(\chi)}{P(\chi)} = \frac{c_B \chi^3}{(\chi + 2\pi f_1)^2 (\chi + 2\pi f_{2B})(\chi + 2\pi f_4)^2}, \quad (10)$$

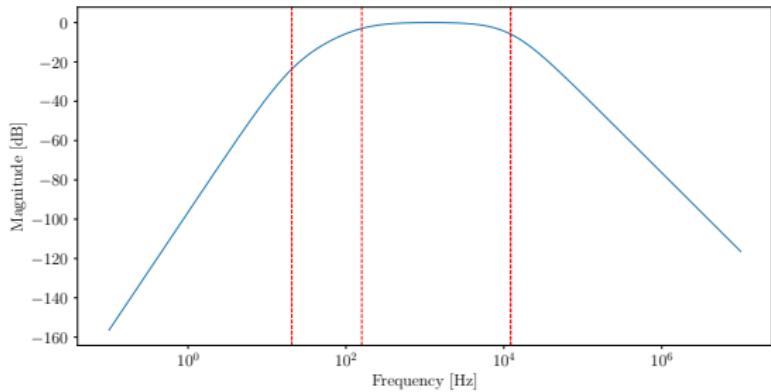
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- $c_B = 5.99185 \times 10^9$
- $f_1 = 20.598997 \text{ Hz}$
- $f_{2B} = 158.48932 \text{ Hz}$
- $f_4 = 12194.217 \text{ Hz}$



Filter Application at Ground: ODE Approach

Common filtering techniques not suitable for our unstructured grid.

Convert transfer function in complex frequency domain:

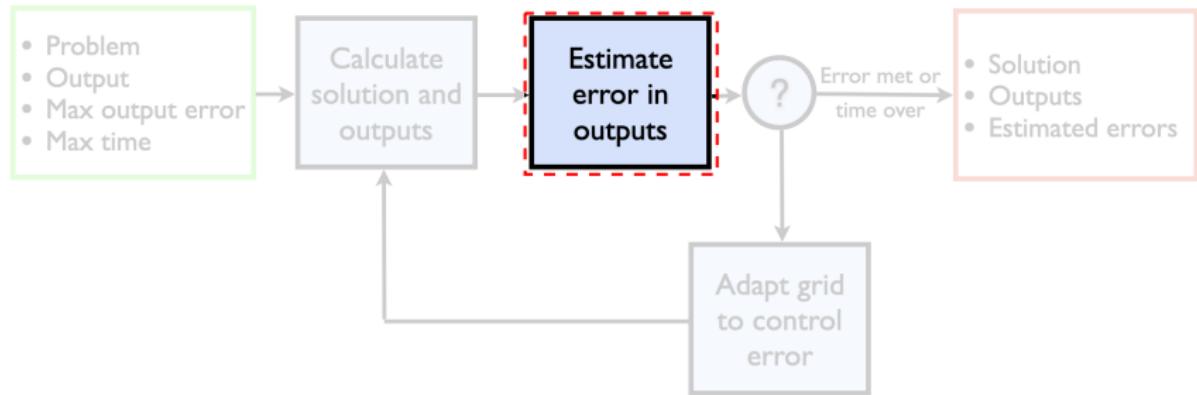
$$\tilde{P}(\chi) = P(\chi) \frac{K^{1/3}}{(\chi + a)^2} \frac{K^{1/3}\chi^2}{(\chi + c)^2} \frac{K^{1/3}\chi}{(\chi + b)}, \quad (11)$$

to a **system of ODE's** in the time domain (to solve in *ground* boundary):

$$\frac{d\bar{u}}{d\tau} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -a^2 & -2a & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -K^{1/3}a^2 & -2K^{1/3}a & -c^2 & -2c & 0 \\ 0 & 0 & 0 & K^{1/3} & -b \end{pmatrix} \bar{u} + \begin{pmatrix} 0 \\ K^{1/3}p(\tau) \\ 0 \\ K^{2/3}p(\tau) \\ 0 \end{pmatrix}, \quad (12)$$

where $\bar{u} = (u_0, u_1, u_2, u_3, \tilde{p})^T$, with homogeneous initial conditions.

Output Error Estimation



Output Functional and Error

In general, consider output functional of the form:

$$\mathcal{J}(\mathbf{u}) := \int_{\Omega} g_v(\mathbf{u}) dV + \int_{\partial\Omega} g_b(\mathbf{u}) dS. \quad (13)$$

(In our case $\mathcal{J}(\mathbf{u}) = \int_{\text{ground}} [\tilde{p}(\tau)]^2 d\tau$)

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We define output error as:

$$\varepsilon(\mathbf{u}_h) := \mathcal{J}(\mathbf{u}) - \mathcal{J}(\mathbf{u}_h). \quad (14)$$

For general nonlinear problem, the output error can be approximated using the **dual weighted residual** (DWR) method.

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Needs **correction**⁴ for artificial viscosity term (asymptotically consistent).

⁴B. Couchman 2020

Residual Consistency

- We say the residual form \mathcal{R} is **consistent** if:

$$\mathcal{R}(\mathbf{v}_h, \mathbf{u}) = 0, \quad \forall \mathbf{v}_h \in \mathcal{V}_h, \tag{15}$$

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- We say the residual form \mathcal{R} is **asymptotically consistent** if:

$$\mathcal{R}(\mathbf{v}_h, \mathbf{u}) = \mathcal{O}(h^\alpha), \quad \forall \mathbf{v}_h \in \mathcal{V}_h, \tag{16}$$

where \mathbf{u} is the exact solution, $\alpha > 0$, and h is a characteristic element size in \mathcal{T}_h .

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In our situation:

$$\mathcal{R}(\mathbf{v}_h, \mathbf{u}_h) = \mathcal{R}^C(\mathbf{v}_h, \mathbf{u}_h) + \underbrace{\mathcal{R}^A(\mathbf{v}_h, \mathbf{u}_h)}_{\text{AV term}}. \quad (17)$$

Dual Problem and Error for Linear Case

We assume linear residual and output functional, and define the **dual** (adjoint) problem as:

Find $\psi \in \mathcal{W}$ such that:

$$\mathcal{R}(\psi, \mathbf{w}) - \mathcal{J}(\mathbf{w}) = 0, \quad \forall \mathbf{w} \in \mathcal{W}. \quad (18)$$

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From there:

$$\begin{aligned}\varepsilon(\mathbf{u}_h) &= \mathcal{J}(\mathbf{u}) - \mathcal{J}(\mathbf{u}_h) \quad (\Delta\mathbf{u} := \mathbf{u} - \mathbf{u}_h) \\ &= \mathcal{J}(\Delta\mathbf{u})\end{aligned}$$

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Dual Problem and Error for Linear Case

We assume linear residual and output functional, and define the **dual** (adjoint) problem as:

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Approximations: DWR Error Estimate

So far:

$$\varepsilon(\mathbf{u}_h) = -[\mathcal{R}(\psi, \mathbf{u}_h) - \mathcal{R}^A(\psi, \mathbf{u})]. \quad (20)$$

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- Residual and output functional are in general nonlinear.
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First approximation:

$$\mathcal{R}^A(\psi, \mathbf{u}) \approx \mathcal{R}^A(\psi, \mathbf{u}_h), \quad (21)$$

justified on a shock dominated problem with AV.

Approximations: DWR Error Estimate

Second approximation:

ψ is approximated with a numerical adjoint $\psi_{\hat{h}}$ defined by⁵:

⁵M. Yano and D. L. Darmofal 2012

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where:

- $\mathcal{R}'[\mathbf{u}_h]$ and $\mathcal{J}'[\mathbf{u}_h]$ are the linearizations of \mathcal{R} and \mathcal{J} about \mathbf{u}_h .
- $\tilde{\mathbf{u}}$ represents a perturbation from \mathbf{u}_h .
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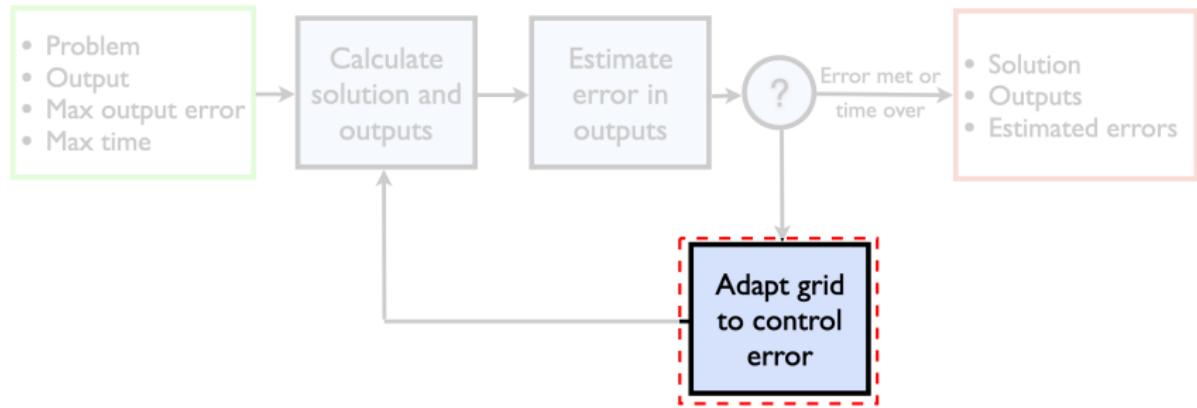
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Final DWR error estimate:

$$\varepsilon(\mathbf{u}_h) \approx - \left[\mathcal{R}(\psi_{\hat{h}}, \mathbf{u}_h) - \mathcal{R}^A(\psi_{\hat{h}}, \mathbf{u}_h) \right]. \quad (23)$$

⁵M. Yano and D. L. Darmofal 2012

Mesh Adaptation



Continuous Optimization: Mesh-Metric Duality

Want mesh producing the smallest output error indicator \mathcal{E} :

$$\hat{\mathcal{T}}_h = \arg \inf_{\mathcal{T}_h \in \mathbb{T}(\Omega)} \mathcal{E}(\mathcal{T}_h), \quad \mathcal{C}(\mathcal{T}_h) < C. \quad (24)$$

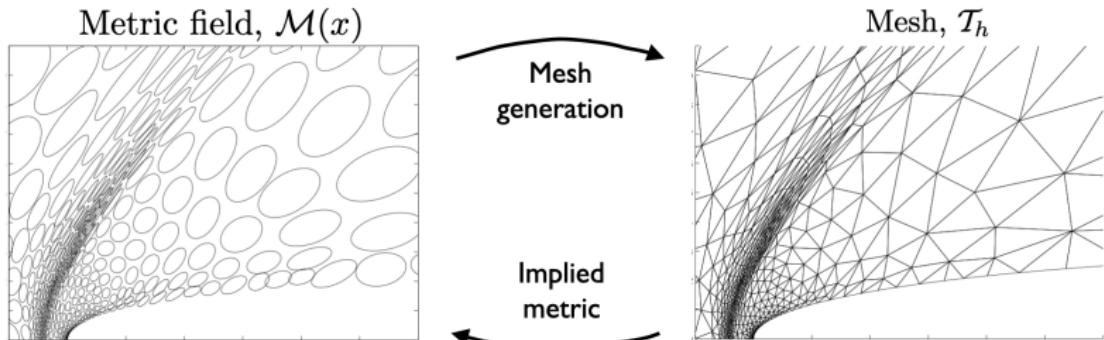
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Continuous relaxation⁶to address intractability of discrete problem.

$$\hat{\mathcal{M}} = \arg \inf_{\mathcal{M} \in \mathbb{M}(\Omega)} \mathcal{E}(\mathcal{M}), \quad \mathcal{C}(\mathcal{M}) < C \quad (25)$$



⁶A. L'oseille and E. Alauzet 2011

MOESS⁸: Error Sampling and Synthesis

Need model for error indicator $\mathcal{E}(\mathcal{M})$: How \mathcal{E} changes with \mathcal{M} .

⁷T. Richter and T. Wick 2015

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$$\varepsilon_v = -\mathcal{R}(\phi_v \psi_{\hat{h}}, \mathbf{u}_h) + \mathcal{R}^A(\phi_v \psi_{\hat{h}}, \mathbf{u}_h), \quad (26)$$

$$\eta_v := |\mathcal{R}(\phi_v \psi_{\hat{h}}, \mathbf{u}_h)| + |\mathcal{R}^A(\phi_v \psi_{\hat{h}}, \mathbf{u}_h)|, \quad \eta := \sum_v \eta_v \equiv \mathcal{E} \quad (27)$$

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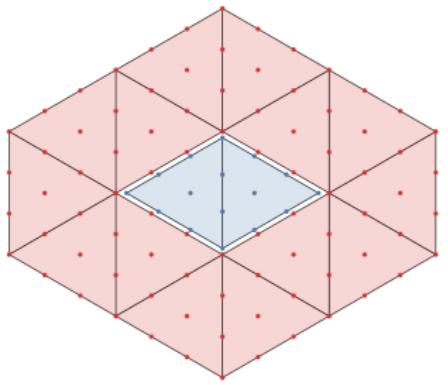
Next: Recompute solution and nodal indicators in refined *local patches* in the mesh.

The comparison of the nodal error indicators before and after the local solves gives information about how \mathcal{E} changes with \mathcal{M} .

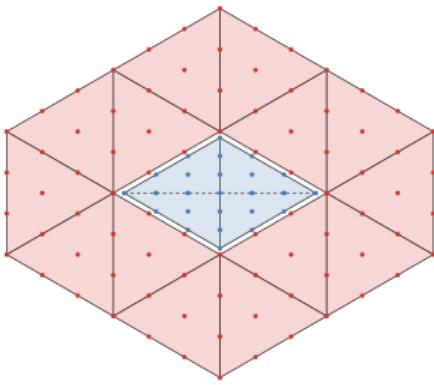
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MOESS: Error Sampling and Synthesis



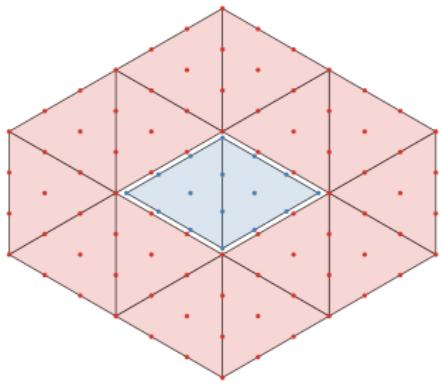
(a) Unsplit



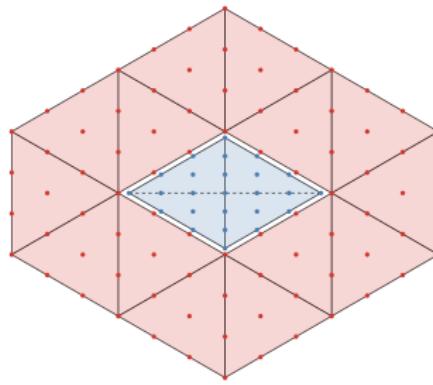
(b) Split

MOESS: Error Sampling and Synthesis

Local solves: Loop over edges in the mesh and solve for \mathbf{u}_h^ϵ in refined local patch. $\mathcal{R}_{\text{local}}(\mathbf{v}_h^\epsilon, \mathbf{u}_h^\epsilon) = 0, \quad \forall \mathbf{v}_h^\epsilon \in \mathcal{V}_{h,p}^\epsilon$.



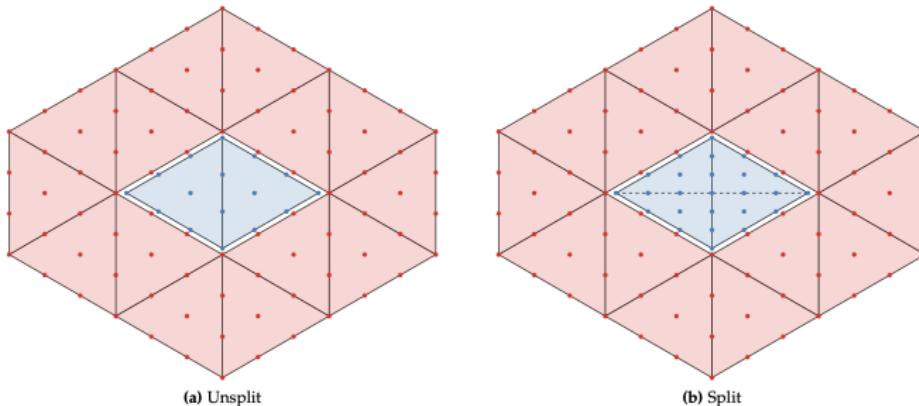
(a) Unsplit



(b) Split

MOESS: Error Sampling and Synthesis

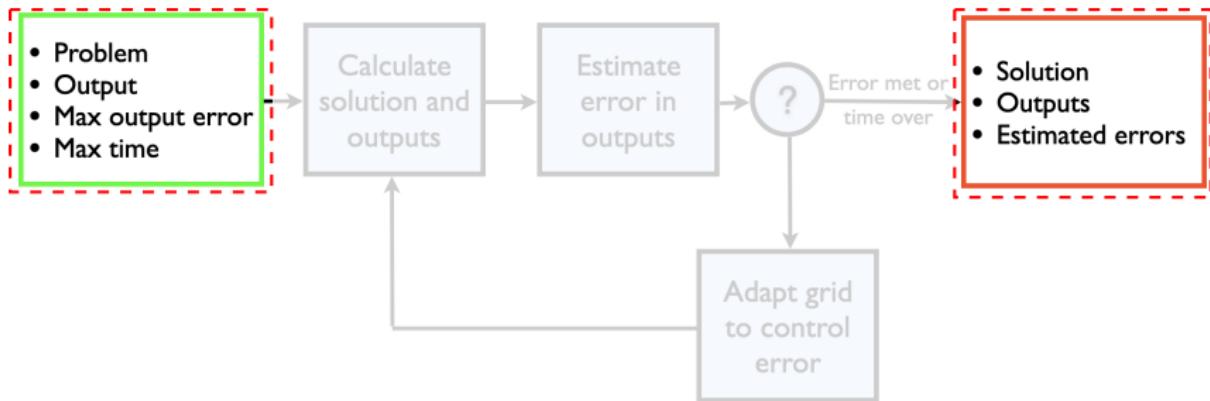
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Nodal indicator for vertices in inner patch:

$$\eta_v^\epsilon = |\mathcal{R}_{\text{local}}(\phi_v \psi_{\hat{h}}, \mathbf{u}_h^\epsilon)| + |\mathcal{R}_{\text{local}}^A(\phi_v \psi_{\hat{h}}, \mathbf{u}_h^\epsilon)|. \quad (28)$$

Results for Practical Case



Preliminary: Implementation Notes

Software: Solution Adaptive Numerical Simulator (SANS)⁹

- C++ framework to numerically solve partial differential equations.
- Extensive use of templates for efficient yet general code.
- Supports several CG and DG discretizations, with output-based mesh adaptation.
- Automatic differentiation via operator overloading.
- MPI parallelization.
- Unit testing and continuous integration.
- Open source.

⁹Galbraith et. al. 2015

Preliminary: Run Summary

① Set:

- Case parameters/conditions.
- Initial 2D mesh.
- Target DOF.
- Number of adaptive iterations (N).

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② Loop, $i \in \{1, \dots, N\}$:

- Extract *ground* boundary and form 1D mesh for filter ODE.
- Solve primal problem:
 - [Burgers system + shock sensor] in 2D mesh.
 - Filter ODE in 1D mesh (ground boundary).
- Solve adjoint problem:
 - Adjoint ODE in 1D mesh (ground boundary).
 - [Burgers system + shock sensor] adjoint in 2D mesh.
- Nodal error sampling and synthesis.
- Adapt mesh.

Case Description

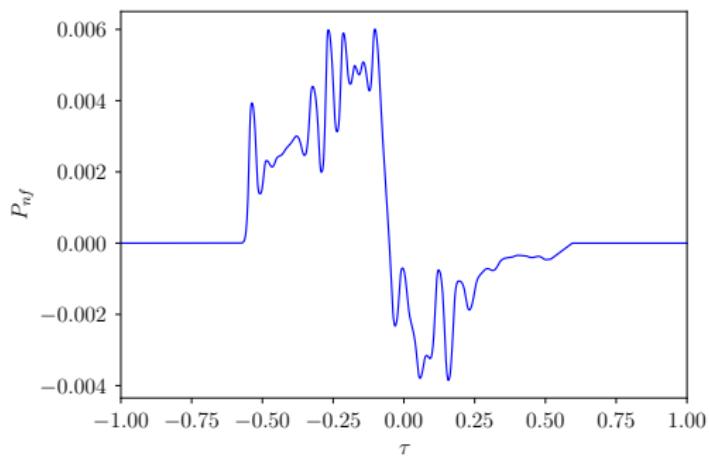
- Airplane Mach number: $M_a = 1.4$.
- Airplane altitude: $z_a = 16459.2$ m.
- Ground altitude: 110 m.
- Ground reflection factor: 1.9.



Source: lockheedmartin.com

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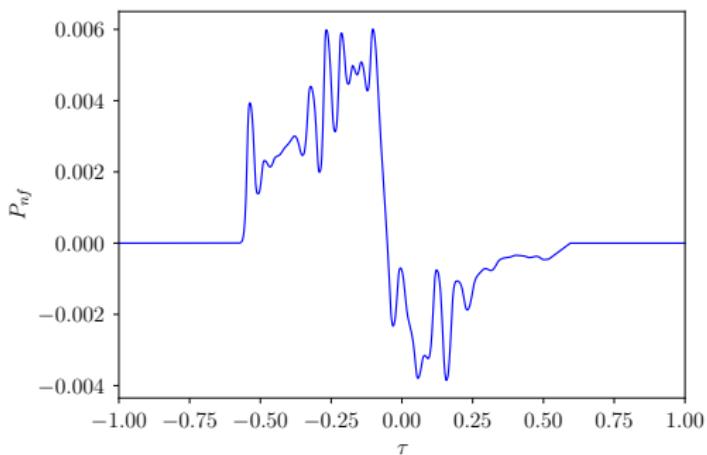
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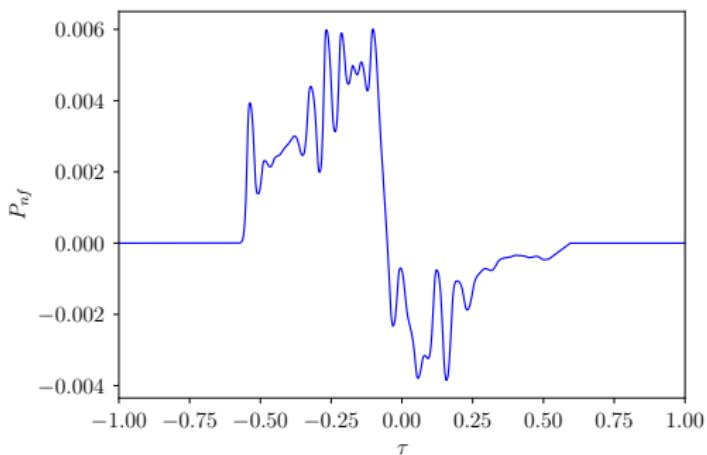


Source: lockheedmartin.com

- Domain dimensions:
 $\Omega = [-1, 2] \times [0, 257]$
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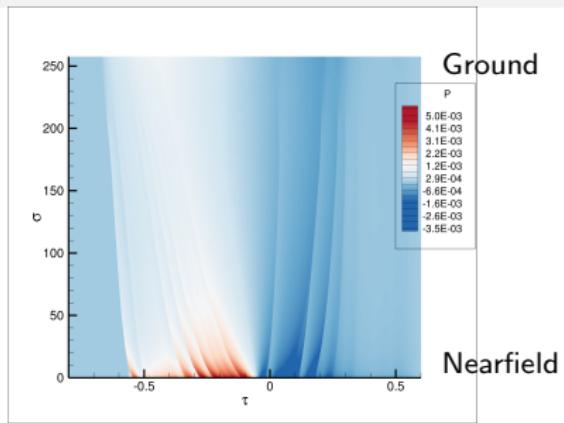
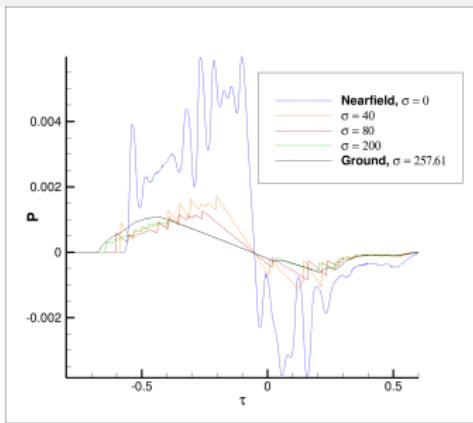


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- Also for comparison:
 $\mathcal{J}_P = \int_{\text{ground}} [p(t)]^2 dt$

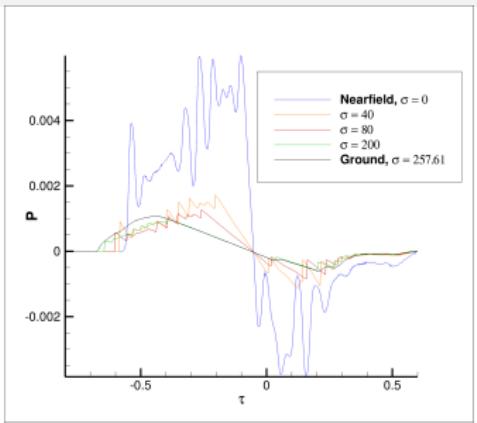
Propagation: Pressure Perturbation Solution

$p = 1$

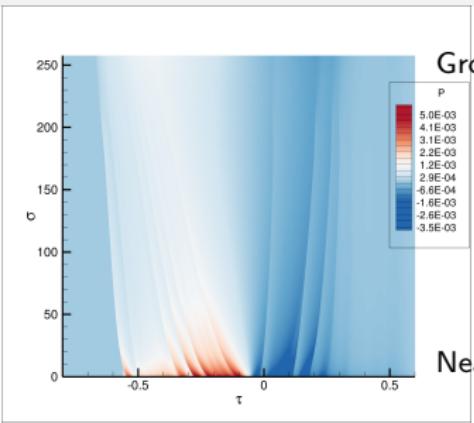
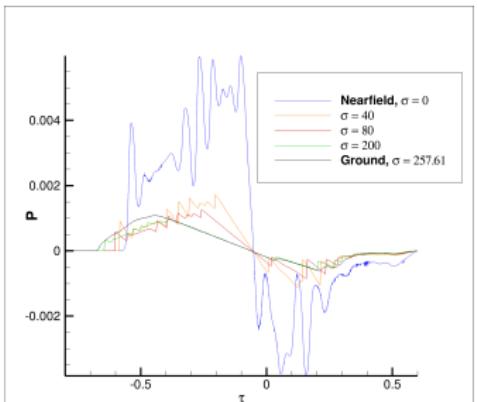


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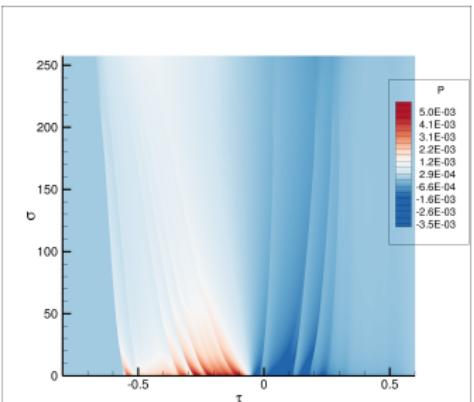
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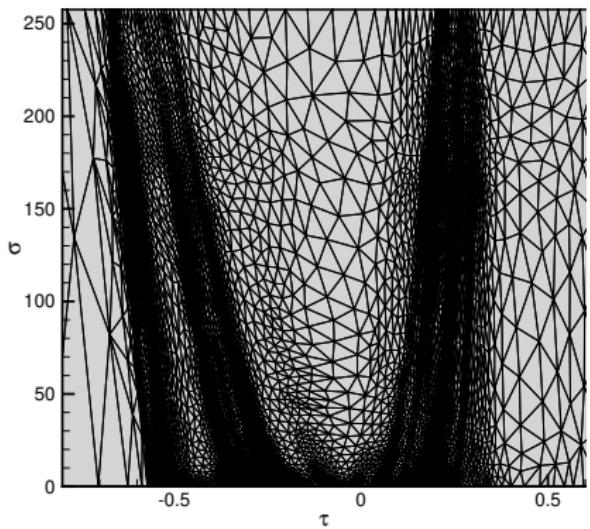
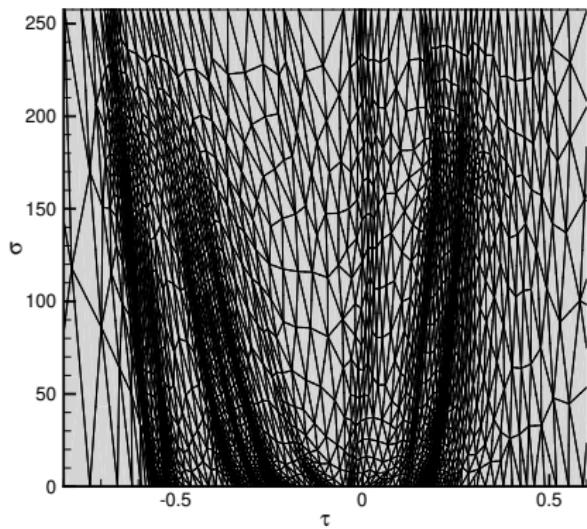
$p = 2$



Nearfield



Propagation: Final Adapted Mesh for 8K Target DOF

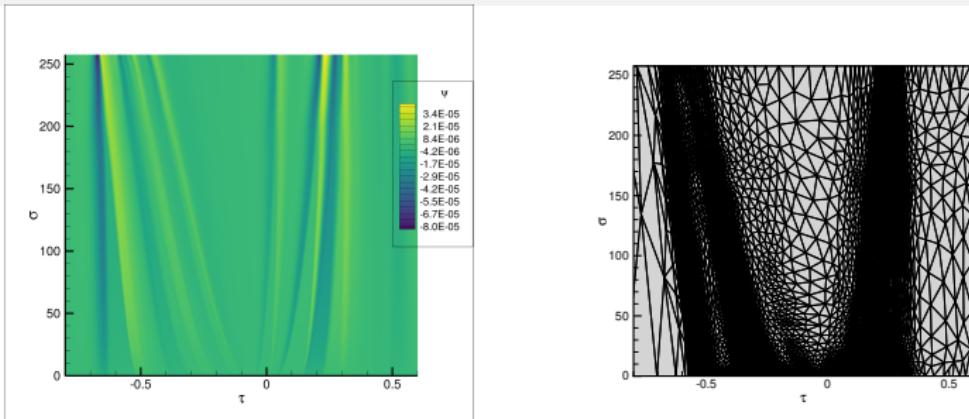
 $p = 1$  $p = 2$ 

Propagation: Evolution Over Adaptive Cycle, 8K Target DOF

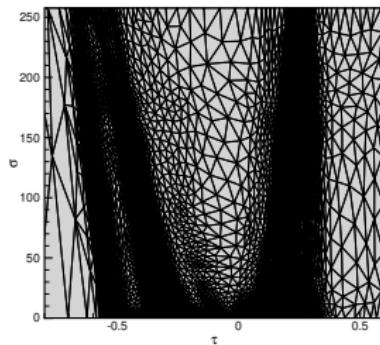
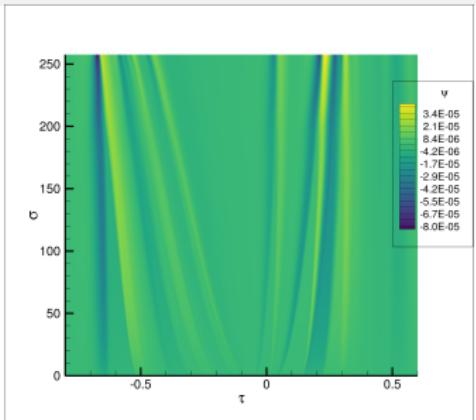
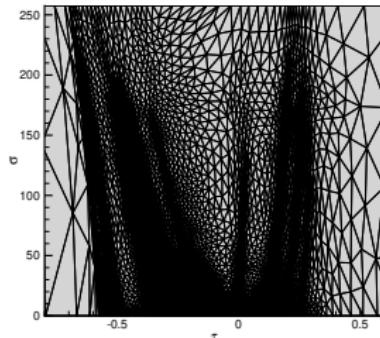
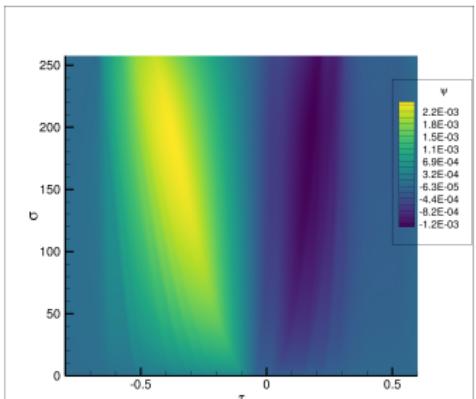


Different Adaptation Outputs and Their Adjoints

$\mathcal{J}_{\text{BSEL}}$

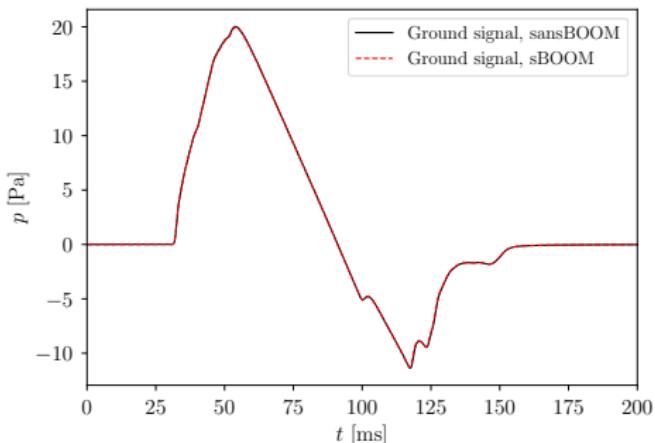


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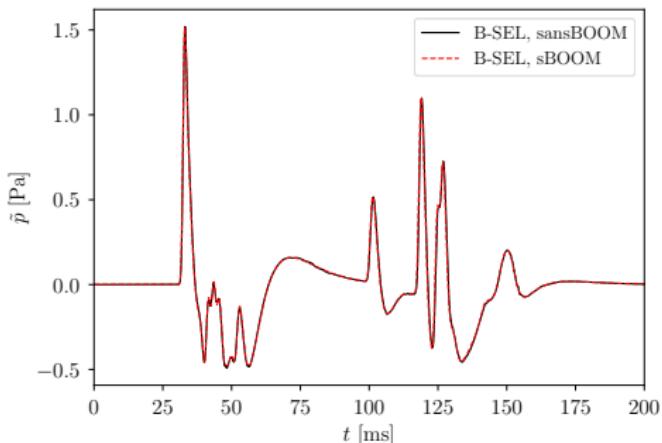
 $\mathcal{J}_{\text{BSEL}}$  \mathcal{J}_P 

At Ground: Pressure Signal and Its Filtering

Pressure Signal

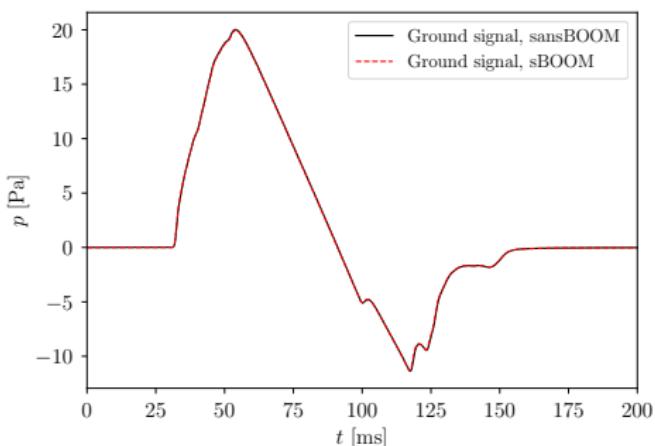


B-SEL Filter Output

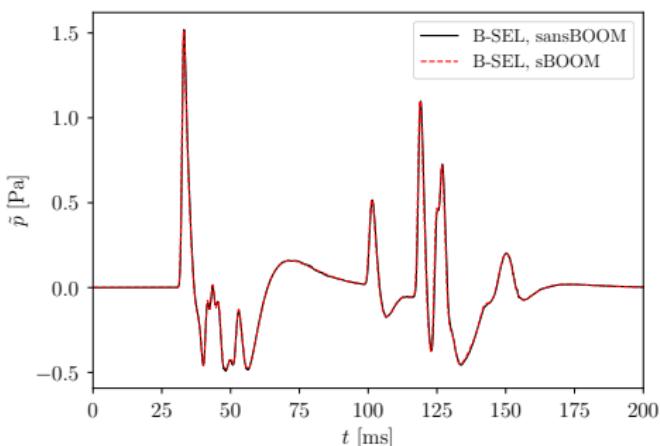


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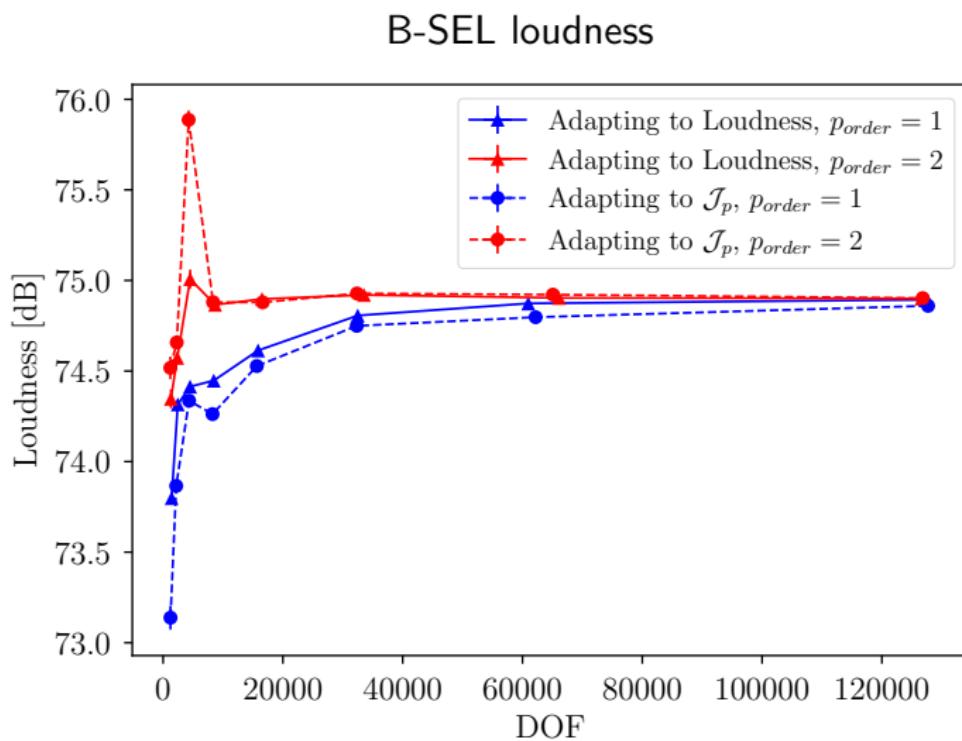


Comparison with NASA *sBOOM* code¹⁰:

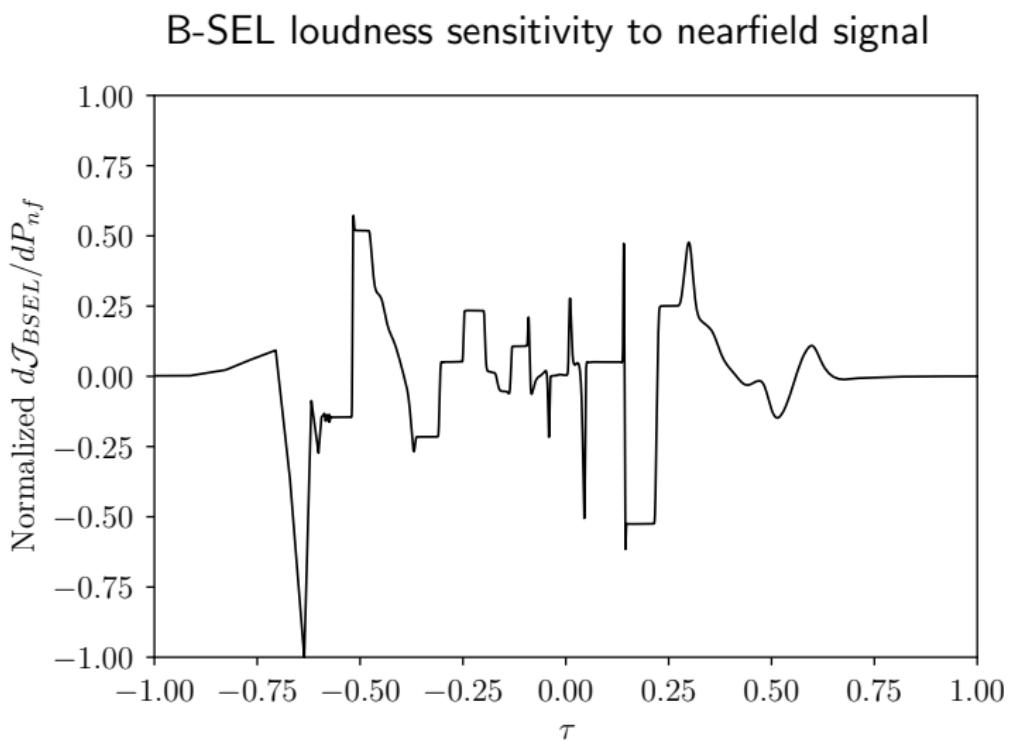
- sansBOOM: 128K DOF in total (space-time).
- sBOOM:
 - 32K DOF in τ direction.
 - 39K steps (marching) in σ direction.
 - 1.2B DOF in total (space-time).

¹⁰S. K. Rallabhandi et. al. 2023

At Ground: Loudness Convergence with Mesh Refinement



At Nearfield: Loudness Sensitivity



Concluding Remarks

Work completed:

- Higher-order FEM to solve sonic boom propagation problem.
- Unstructured space-time mesh adaptation.
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Ongoing effort:

- Study convergence of loudness sensitivity to nearfield signal.

Thanks for the attention!
Questions?

Variational Multiscale with Discontinuous Subscales (VMSD) Method

Discretization of Ω :

$\mathcal{T}_h := \{\kappa\}_{\kappa=1}^K$ is a triangulation of the domain Ω into K elements.

Propose solution:

$$\mathbf{u}_h := \bar{\mathbf{u}}_{h,p} + \mathbf{u}'_{h,p'}, \quad \bar{\mathbf{u}}_{h,p} \in \bar{\mathcal{V}}_{h,p}, \quad \mathbf{u}'_{h,p'} \in \mathcal{V}'_{h,p'}.$$

VMSD solution spaces:

$$(Coarse scale) \quad \bar{\mathcal{V}}_{h,p} := \{\mathbf{v} \in [C^0(\Omega)]^m : \mathbf{v}|_\kappa \in [\mathcal{P}^p(\kappa)]^m, \forall \kappa \in \mathcal{T}_h\}, \quad (29)$$

$$(Fine scale) \quad \mathcal{V}'_{h,p'} := \{\mathbf{v} \in [L^2(\Omega)]^m : \mathbf{v}|_\kappa \in [\mathcal{P}^{p'}(\kappa)]^m, \forall \kappa \in \mathcal{T}_h\}. \quad (30)$$

Variational Multiscale with Discontinuous Subscales (VMSD) Method

Weak statement:

Find $(\bar{\mathbf{u}}_{h,p}, \mathbf{u}'_{h,p'}) \in \bar{\mathcal{V}}_{h,p} \times \mathcal{V}'_{h,p'}$ **such that:**

$$\mathcal{R}(\bar{\mathbf{v}}_{h,p}, \mathbf{v}'_{h,p'}; \bar{\mathbf{u}}_{h,p}, \mathbf{u}'_{h,p'}) = 0, \quad \forall (\bar{\mathbf{v}}_{h,p}, \mathbf{v}'_{h,p'}) \in \bar{\mathcal{V}}_{h,p} \times \mathcal{V}'_{h,p'}. \quad (31)$$

Remarks:

- $\mathbf{u}'_{h,p'}$ DOFs are element-wise decoupled. Thus, they can be static condensed and the total cost becomes the same as a CG method.
- For same accuracy requirement, more efficient (less DOFs) than CG and DG.
- Adjoint consistent.

Shock Indicator s_{grad}

Starting point:

$$\xi := \frac{H_{\tau\tau}}{p} \left| \frac{\partial P}{\partial \tau} \right|, \quad \xi_1 \simeq \max_{x \in \Omega} \xi(x), \quad \hat{\xi} = \frac{\xi}{\xi_1}, \quad (32)$$

then:

$$s_{\text{grad}} := s_{\text{grad}}(\hat{\xi}) = \frac{\hat{\xi} [\tanh(p^2 \hat{\xi})]^{p-1}}{1 + \exp \left[-k (\hat{\xi} - \alpha(p)) \right]}, \quad (33)$$

where $k, \alpha(p) \in \mathbb{R}$.

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