Задание 2.

1. (1) $\int_C \frac{z e^z}{\tan(z^2)} dz$ полюс 2. порядка.

$$\operatorname{Res} f(z) = \lim_{z\to 0}\left(\frac{z e^z z^2}{\tan(z^2)}\right)' = \lim_{z\to 0}\left(\frac{z^3 e^z}{\tan(z^2)}\right)' =$$

$$= \lim_{z\to 0}\left(\frac{(3z^2 e^z + e^z z^3)\tan(z^2) - \frac{1}{\cos^2(z^2)} 2z\, z^3 e^z}{\tan^2(z^2)}\right) =$$

$$= \lim_{z\to 0} e^z z^2\left(\frac{(3+z)\tan(z^2) - \frac{2z^2}{\cos^2(z^2)}}{\tan^2(z^2)}\right) = \lim_{z\to 0} e^z z^2\left(\frac{3+z}{\tan(z^2)} - \frac{2z^2}{\sin^2(z^2)}\right) =$$

$$= \lim_{z\to 0} \frac{e^z z^2}{\sin^2(z^2)}\left(\cos^2(z^2)(z+3) - 2z^2\right) = 1$$

$$I = 2\pi i \cdot \operatorname{Res} = \underline{+2\pi i}$$

(2) $\int_C e^{-\frac{1}{z}} \sin\left(\frac{1}{z}\right) dz$

$\exp\left(-\frac{1}{z}\right) = 1 - \frac{1}{z} \Rightarrow \int \frac{1}{z} - \frac{1}{z^2} dz = 2\pi i$

$\sin\left(\frac{1}{z}\right) = \frac{1}{z}$

$$\operatorname{Res}_{z\to\infty}\left(e^{-\frac{1}{z}} \sin\left(\frac{1}{z}\right) z\right) = 1.$$

$\operatorname{Res} = \lim z^2\left(e^{-\frac{1}{z}}\sin\left(\frac{1}{z}\right)\right)' = -1.$

$$I = \underline{2\pi i}$$

$I = -\operatorname{Res} 2\pi i = 2\pi i$

(3) $\int_C \frac{e^z}{z^n}$ $\cdot$ $n \in \mathbb{N}$

$$\operatorname{Res} f(z) = \frac{1}{(n-1)!} \lim_{z\to 0}\left(z^n \frac{e^z}{z^n}\right)^{(n-1)} =$$

$$= \frac{1}{(n-1)!} \lim_{z\to 0} e^z = \frac{1}{(n-1)!}$$

$$I = \underline{\frac{2\pi i}{(n-1)!}}$$

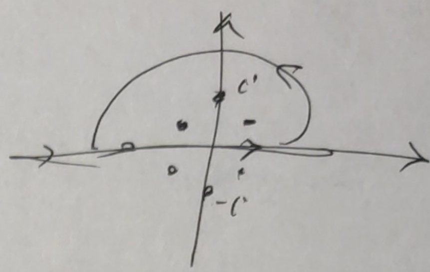
5. 1) $\int_{-\infty}^{\infty} \frac{x^4}{1+x^6}\,dx$

$1+x^6=0$

$x = i$

$x = \frac{\sqrt{3}}{2}+\frac{i}{2}$

$x = -\frac{\sqrt{3}}{2}+\frac{i}{2}$




$I = 2\pi i \sum_{j=1}^{3} \operatorname{Res} f(z)$

$\operatorname{Res} f(z) = \frac{x^4}{6x^5} = \frac{1}{6x_i}$

$\Rightarrow I = \frac{2\pi i}{6}\sum_i x_i = \frac{\pi i}{3}\left(\frac{1}{i} + \frac{2}{\sqrt{3}+i} + \frac{2}{-\sqrt{3}+i}\right) = \frac{2\pi}{3}$

2) $\int_0^{2\pi}\frac{\cos 2\theta}{2+\cos\theta}\,d\theta.$ $\cos 2\theta = 2\cos^2\theta - 1$, $\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$, $d\theta = \frac{dz}{iz}$

$\int_{|z|=1} \frac{(z^4+1)\,2z\,dz}{2z^2\, iz\,(z^2+4z+1)} = \int_{|z|=1} \frac{1}{i}\,\frac{(z^4+1)\,dz}{z^2(z^2+4z+1)}$

$z=0$: $\operatorname{Res}_{z=0}(f(z)) = \lim_{z\to 0}\left(\frac{z^2(z^4+1)}{z^2(z^2+4z+1)}\right)' =$

$= \lim_{z\to 0}\frac{4z^3(z^2+4z+1)-(2z+4)(z^4+1)}{(z^2+4z+1)^2} = -4.$

$z^2+4z+1=0$

$z = -2+\sqrt{3}.$

$\operatorname{Res}_{z=-2+\sqrt{3}} f(z) = \frac{(z^4+1)}{2z(z^2+4z+1)+z^2(2z+4)} =$

$= \frac{z^4+1}{4z^3+12z^2+2z} = \frac{7}{\sqrt{3}}$

$I = 2\pi\left(\frac{7}{\sqrt{3}}-4\right)$

(3) $\int\limits_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)^2}$ $\quad x \to z.$

$x^2 = -a^2 \Rightarrow x = ia.$

$$\operatorname*{Res}_{z=ia}(f(z)) = \frac{1}{2x(b^2+x^2)(2a^2+b^2+3x^2)} = \frac{1}{2ia(b^2-a^2)(b^2-a^2)^2}$$

$$a>0 \quad = \frac{1}{2ia(b^2-a^2)^2} = -\frac{i}{2a(b^2-a^2)^2}$$

$$\operatorname*{Res}_{z=ib}(f(z)) = \lim_{z\to ib}\left(\frac{(x+ib)^2}{(x^2+a^2)(x^2+b^2)^2}\right)'_z$$

$$= \lim_{z\to ib} = -\frac{2i(a^2+x(2x+ib))}{(a^2+x^2)^2(b-ix)^3} = \frac{2i(a^2+ib(3ib))}{2b^3(a^2-b^2)^2} =$$

$$b>0 \quad = -\frac{2i(a^2-3b^2)}{2b^3(a^2-b^2)^2}$$

$$I = 2\pi i \sum \operatorname{Res}(f(z)) = \pi\left(\frac{2(a^2-3b^2)}{\cancel{2}b^3(a^2-b^2)^2} + \frac{1}{a(b^2-a^2)^2}\right) =$$

$$= \frac{2a^3 - 6ab^2 + 2b^3}{(b^2-a^2)^2 b^3 a} =$$

$a, b, \in \mathbb{Z}$ $\quad I = \pi i\left[\frac{|a| + 2|b|}{2|a||b|^3(|a|+|b|)^2}\right].$

6. $\int_C \frac{z^5 dz}{1+z^6}$ $\quad z^6 = \pm i, \pm\frac{\sqrt{3}}{2} \pm \frac{i}{2}$

$\text{Res} = \frac{z^5}{6z^5} = \frac{1}{6}$ - не зависит от $x_i$.

$|z|^2 \to \infty, \Rightarrow \frac{z^5}{1+z^6} = \frac{1}{z} \to 0 \Rightarrow I = \int\limits_0^{2\pi} d\varphi\, i = \underline{2\pi i}$

7. $\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2(x^2+1)}dx = \int_{-\infty}^{+\infty} \frac{\sin^2 x}{x^2}dx - \int_{-\infty}^{+\infty} \frac{\sin^2 x}{x^2+1}dx =$

$= \pi - \frac{1}{2}\int_{-\infty}^{\infty} \frac{1-\cos(2x)}{x^2+1}dx = \pi - \frac{\pi}{2} + \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos(2x)}{x^2+1}$

$I_1 = \operatorname{Re}\int_{-\infty}^{\infty} \frac{e^{2ix}}{x^2+1}dx = 2\pi i \lim_{x\to i} \frac{e^{2ix}}{x+i} = \frac{\pi}{e^2}$ $\quad I_1^4$

$\Rightarrow I = \frac{\pi}{2} + \frac{\pi}{2e^2} = \boxed{\frac{\pi}{2}\left(1+\frac{1}{e^2}\right)}$

10. $\int_{-\infty}^{\infty} \frac{e^{-iz}}{z^2+9}dz$ $\qquad z = \pm i3$.

1-полюс перво пор.

$I = 2\pi i \operatorname{Res}_{z=i3} f(z) = 2\pi i \frac{e^{-3}}{2\,3i} = \boxed{\frac{\pi e^{-3}}{3}}$

14. $\lim_{R\to\infty}\int_{C_R} e^{iz}dz = \lim_{R\to\infty}\int e^{i|z|e^{i\varphi}}\, ie^{i\varphi}|z|\,d\varphi =$

$= \lim_{R\to\infty}\int_{C_R, 0}^{\pi} e^{iRe^{i\varphi}}\, ie^{i\varphi}R\,d\varphi \ldots \varnothing \ldots = \int e^{i(Re^{i\varphi}+\varphi)}iR\,d\varphi.$

$\lim_{R\to\infty}\int_{C_R} e^{iz^2}dz = \lim_{R\to\infty}\int_{C_R,0}^{\frac{\pi}{4}} e^{iR^2e^{2i\varphi}}\,\varphi i \cancel{R}e^{i\varphi}d\varphi =$

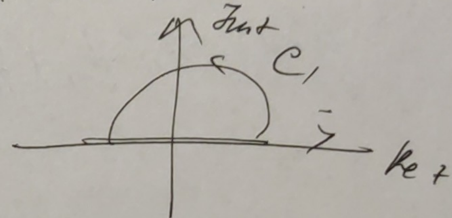
$= \int_0^{\frac{\pi}{4}} ie^{i(R^2e^{2i\varphi}+\varphi)}\varphi R\,d\varphi$

8. $\int_0^\infty \frac{x\sin(ax)}{x^2+h^2}\,dx$

$\sin(ax) = \frac{e^{iax}-e^{-iax}}{2i}$

$I = \frac{1}{i2}\left[\int_{-\infty}^{\infty}\frac{e^{iax}x}{x^2+h^2} - \int_{-\infty}^{\infty}\frac{e^{-iax}x}{x^2+h^2}\right] =$

$\frac{1}{2}\,\mathrm{Im}\int_{-\infty}^{\infty}\frac{x e^{iax}}{x^2+h^2}$




$x \to z$

Оцениваем интегр при $a>0$

$\left|\int_{C_1}\frac{z e^{iaz}}{z^2+h^2}\right| < \int_{C_1}\left|\frac{z e^{iaz}}{z^2+h^2}\right|dz < \int_{C_1}\frac{dz}{|z|} =$

$= \int_{C} e^{iaz} f(z)\,dz \to 0, \quad a>0, \quad |f|<M.$

$\int_C \frac{z e^{iaz}\,dz}{z^2+h^2} = 2\pi i\,\frac{ih\,e^{iah}}{2ih} = \pi i\,e^{-ah}$

$I = \frac{1}{2}\,\mathrm{Im}\int f(z)\,dz = \frac{1}{2}\pi e^{-ah}, \quad a>0,\ h>0$

$\forall a,h \Rightarrow \boxed{I = \frac{\pi}{2}\,e^{-|a||h|}\,\mathrm{sign}(a)}$

11. $f(z) = z^3 \cos\left(\frac{1}{z-2}\right)$

$z = \infty$

$$\operatorname*{Res}_{z=\infty} f(z) = \frac{1}{(3-1)!} \lim_{z\to\infty} \left(\cos\left(\frac{1}{z-2}\right)\right)^{(2)} =$$

$$= \frac{1}{2} \lim_{z\to\infty} \left(+\sin\left(\frac{1}{z-2}\right) \cdot \frac{1}{(z-2)^2}\right)' =$$

$$= \frac{1}{2} \lim_{z\to\infty} \left(-\cos\left(\frac{1}{z-2}\right)\frac{1}{(z-2)^4} + \frac{2}{(z-2)^3}\sin\left(\frac{1}{z-2}\right)\right) = \frac{143}{24}$$

$$f(z) = z^3 - \frac{z}{2} - 2 - \frac{143}{24z} + O\left(\frac{1}{z^2}\right)$$

12. $f(z) = \frac{1}{z^3 - z^5}$

$$\operatorname*{Res}_{z=-1} f(z) = \frac{1}{3z^2 - 5z^4} = -\frac{1}{2}$$

$$\operatorname*{Res}_{z=0} f(z) = \lim_{z\to 0} \left(\frac{z^3}{z^3 - z^5}\right)'' = \lim_{z\to 0}\left(\frac{1}{1-z^2}\right)'' = \lim_{z\to 0}\left(\frac{2z}{(1-z^2)^2}\right)' =$$

$$= \lim_{z\to 0} \frac{8z^2}{(1-z^2)^3} + \frac{2}{(1-z^2)^2} = 2$$

$$\operatorname*{Res}_{z=1} f(z) = \frac{1}{3z^2 - 5z^4} = -\frac{1}{2}$$

$$\operatorname*{Res}_{z=\infty} f(z) = \lim_{|z|\to\infty} \frac{z^8}{(1-z^2)^2} = 0.$$