ТФКП.

Контрольная работа №2.

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Рената БФЗ192.

1. $f(z) = \ln(\sqrt{z^4+1})$

$z^4 + 1 = 0$

$z^2 = \pm i$

$z = \pm(\pm i)^{\frac{1}{2}}$

$z = \pm \frac{1 \pm i}{\sqrt{2}}$ – 4 точки ветвлен.

Проверим сущ. на бескон., заменим $z = \frac{1}{\omega}$

$F(\omega) = \ln\left(\sqrt{\frac{1}{\omega^4}+1}\right) = \ln\left(\frac{\sqrt{1+\omega^4}}{\omega^2}\right) = \ln(\sqrt{1+\omega^4}) - 2\ln(\omega)$

При $\omega = 0$ $\ln(\sqrt{1+\omega^4})$ определен, а $\ln(\omega)$ – имеет точку ветвлен.

$\Rightarrow z = \infty$ точка ветвления.

Ответ: $\boxed{z = \pm \frac{1 \pm i}{\sqrt{2}}, \infty.}$

2. $f(z) = \ln(\sqrt{z+1})$, $z = 0$

$f(z=0) = \ln(g(z)\, e^{2\pi i n}) = 2\pi i n + \ln\left(\sqrt{z+1}\, e^{\frac{i\pi n 2}{2}}\right) =$

$= 2\pi i n + i\pi n + \ln(\sqrt{0+1}) = 3\pi i n.$

Ответ: $3\pi i n$.

3. $f(z) = \frac{(z+1)^3}{\sqrt{z}}$ $z = 0$ – точка ветвлен.

Проверим сущ. точки ветвления на бесконеч., проведя замену $z = \frac{1}{\omega}$. $F(\omega) = \left(\frac{1}{\omega}+1\right)^3 \omega^{\frac{1}{2}} =$

$F(\omega) = (1+\omega)^3\omega^{-\frac{5}{2}} \Rightarrow$ т.к. $\omega = 0$ - точка ветвления

2) $z = \infty$ - точка ветвления.

При обходе по окружности $z = 0$ аргумент изменится на $e^{i\pi}$

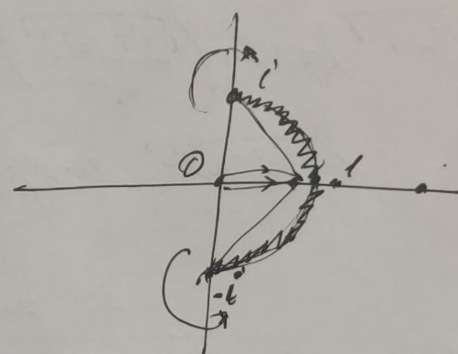
При обходе по окружности $z = \infty$. аргумент изменяется на $e^{\frac{i\pi 5}{2}}$.

4. $f(z) = \sqrt{z^2+1}$

$f(0) = 1$

$f(1-0)$ - ?

$f(1+0)$ - ?



$f(z) = \sqrt{(z+i)(z-i)}$

$$f(1-0) = f(0)\sqrt{\left|\frac{1+i\cdot 0}{i}\right|}\sqrt{\left|\frac{1-i+0}{-i}\right|}\, e^{-\frac{i\pi}{4\cdot 2}} \cdot e^{\frac{i\pi}{4\cdot 2}}$$

$$= 1\cdot 2^{\frac{1}{4}}\, 2^{\frac{1}{4}} = \boxed{\sqrt{2}}$$

$$f(1+0) = f(0)\sqrt{\left|\frac{1+i+0}{i}\right|}\sqrt{\left|\frac{1-i-0}{-i}\right|}\, e^{\frac{\pi i}{2}} e^{-\frac{\pi i}{2}} = \boxed{\sqrt{2}}$$

Ответ: $\boxed{f(1-0) = f(1+0) = \sqrt{2}}$.

5. $\int \frac{\bar{z}}{(|z|+2)^3}\,dz$ $\qquad z = e^{i\varphi}\rho \quad \bar{z} = e^{-i\varphi}\rho$

$|z| = \rho \quad dz = i\rho e^{i\varphi}d\varphi$.

$$\int_0^{2\pi} \frac{e^{-i\varphi}\, i e^{i\varphi} \rho\, d\varphi}{(\rho+2)^2} = |\rho = 4| = \int \frac{4 i\, d\varphi}{6^3} = \frac{i 2\pi 2^2}{2^3 3^3} = \frac{i\pi}{27}.$$

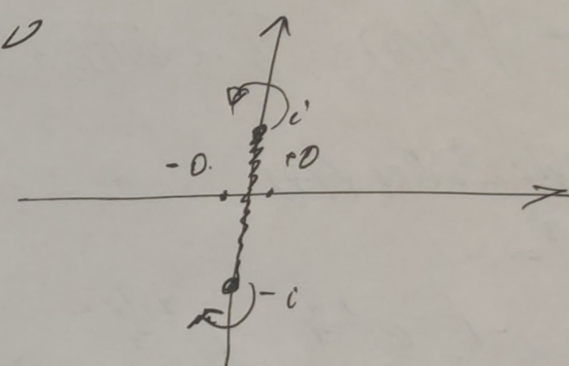
Ответ: $\boxed{\frac{i\pi}{27}}$

6. $f(z) = (1+iz)^{\nu}(1-iz)^{1-\nu}$

$f(+0) = 1$

$f(-0) - ?$

$\operatorname*{res}_{z=\infty} f(z) - ?$



$$f(-0) = f(+0)\cdot\left(\frac{1-i0}{1+i0}\right)^{\nu}\left|\frac{1+i0}{1-i0}\right|^{1-\nu} e^{2\pi i\nu} e^{-2\pi i(1-\nu)} =$$

$$= e^{4\pi i\nu} e^{-2\pi i} = \underline{e^{4\pi i\nu}}$$

$$(1+iz)^{\nu}(1-iz)^{1-\nu} = z^{\nu}\cdot z^{1-\nu}\left(\frac{1}{z}+i\right)^{\nu}\left(\frac{1}{z}-i\right)^{1-\nu} =$$

$$= z\left(i^{\nu} + \frac{i^{(\nu-1)}\nu}{z} - \frac{i^{\nu}(\nu-1)\nu}{2z^2}\right)\left((-i)^{1-\nu} - \frac{(-i)^{-\nu}(\nu-1)}{z} + \frac{i(-i)^{-\nu}(\nu-1)\nu}{2z^2}\right) =$$

$$= z\left(-ie^{i\pi\nu} - \frac{e^{i\pi\nu}\nu}{z} + \frac{ie^{i\pi\nu}(\nu-1)\nu}{2z^2} - \frac{e^{i\pi\nu}(\nu-1)}{z} + \frac{ie^{i\pi\nu}\nu(\nu-1)}{z^2} +\right.$$

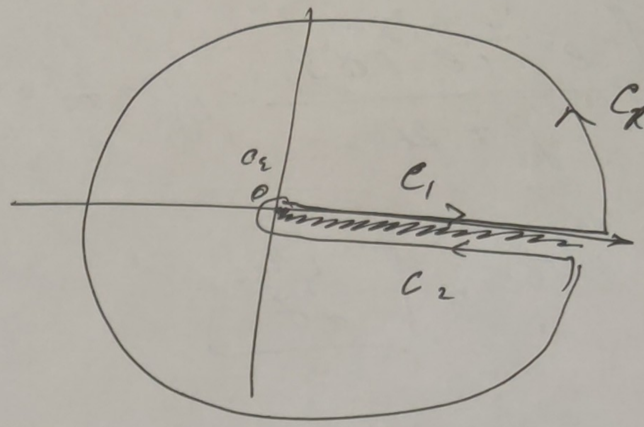
$$\left. + \frac{e^{i\pi\nu}(\nu-1)^2\nu}{2z^3} + \frac{i(\nu-1)\nu e^{i\pi\nu}}{2z^2} + \frac{e^{i\pi\nu}\nu^2(\nu-1)}{2z^3} - \frac{ie^{i\pi\nu}(\nu-1)^2\nu^2}{4z^4}\right) =$$

$$= z\left(\frac{\cancel{8}^{2} i(\nu-1)\nu e^{i\pi\nu}}{8z^2} + \ldots\right) = \frac{2i(\nu-1)\nu e^{i\pi\nu}}{z} + \ldots$$

$$\Rightarrow c_{-1} = \operatorname*{res}_{z=\infty} f(z) = \underline{2i\nu(\nu-1)e^{i\pi\nu}}$$

Ответ: $\boxed{2i\nu(\nu-1)e^{i\pi\nu}}$

7. $I = \int_0^\infty \frac{\ln^2(x)}{x^3+1}\,dx$



$$\oint = \int_{C_R}^{\to 0} + \int_{C_\varepsilon}^{\to 0} + \int_{C_1}^{I} + \int_{C_2} =$$

$$= I + \int f(x) = I + 4\pi^2 I = I(1+4\pi^2).$$

$$\oint = 2\pi i \sum \operatorname{Res} f(z)$$

$$z^3 = -1$$

$$z = -1,\ e^{\frac{i\pi}{3}},\ -e^{\frac{2\pi i}{3}}$$

$$I(1+4\pi^2) = 2\pi i \sum \operatorname{Res} f(z) =$$

$$= 2\pi i\left(\frac{\ln^2(-1)}{3} + \frac{\ln^2(e^{\frac{i\pi}{3}})}{3e^{\frac{2i\pi}{3}}} + \frac{\ln^2(-e^{\frac{2\pi i}{3}})}{3e^{\frac{4\pi i}{3}}}\right) =$$

$$= 2\pi i\left(-\frac{\pi^2}{3} - \frac{\pi^2 e^{-\frac{2\pi i}{3}}}{27} \mp \frac{(i\pi + \frac{2\pi i}{3})^2}{3e^{\frac{4\pi i}{3}}}\right) =$$

$$= 2\pi i\left(-\frac{\pi^2}{3} - \frac{\pi^2 e^{-\frac{2\pi i}{3}}}{27} - \frac{25\pi^2 e^{-\frac{4\pi i}{3}}}{27}\right) =$$

$$= -2\pi^3 i\left(\frac{1}{3} + \frac{e^{-\frac{2\pi i}{3}}}{3^3} + \frac{5^2 e^{-\frac{4\pi i}{3}}}{3^3}\right).$$

$$e^{-\frac{2\pi i}{3}} = -\frac{1+i\sqrt{3}}{2} \qquad e^{-\frac{4\pi i}{3}} = -\frac{1-i\sqrt{3}}{2}$$

$$I = \frac{-2\pi^3 i}{(1+4\pi^2)}\left(\frac{24i\sqrt{3} - 8}{27\cdot 2}\right) = -\frac{\pi^3 i\, 8(3i\sqrt{3} - 1)}{(1+4\pi^2)27}.$$

Ответ: $\operatorname{Re} I = \frac{\pi^3 24\sqrt{3}}{27(1+4\pi^2)}$

$$\boxed{\frac{\pi^3 24\sqrt{3}}{27(1+4\pi^2)}}$$

8. $f(z)=\sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1}z^{n+1}}{n}$ $\quad z=0.$

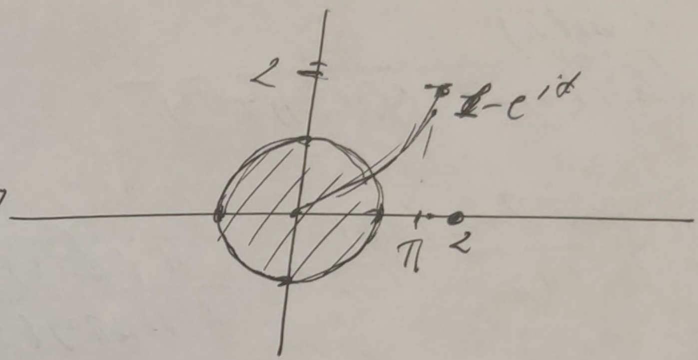
$$R=\lim_{n\to\infty}\left(\frac{(-1)^{n+1}}{n}\right)^{-\frac{1}{n}}=\lim_{n\to\infty}(-1)^{-1-\frac{1}{n}}n^{-\frac{1}{n}}=1.$$

$R=1.$

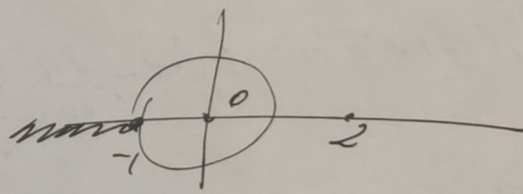
$f(2)-?$

$g(t)=1-e^{it},\ t\in[0,\pi]$

$g(0)=0\quad g(\pi)=2.$



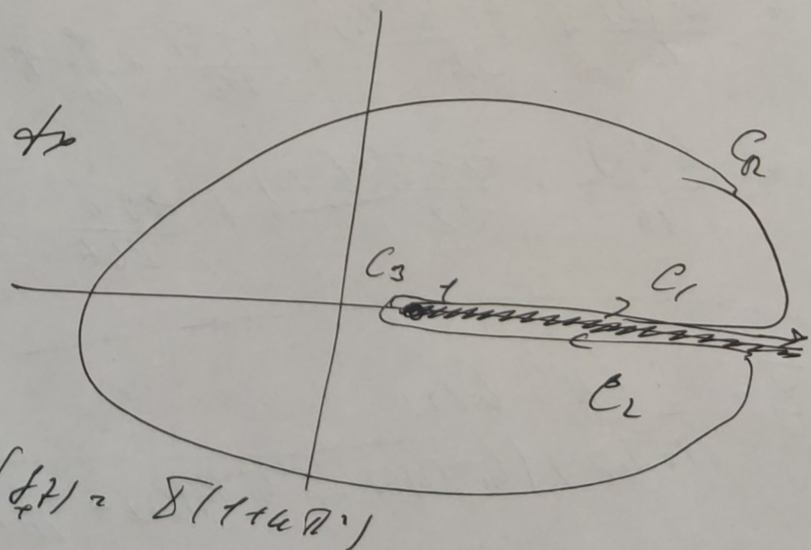
$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}z^{n+1}}{n}=z\ln(z+1)$$



$f(2)=2\ln(2)=$

Ответ: $\boxed{2\ln(2)}$

9.
$$I=\int_1^{\infty}\frac{1}{(x^2+1)(\ln^2(x-1)+\pi^2)}\,dx$$



$$\oint=\int_{C_R}^{\to 0}+\int_{C_3}^{\to 0}+\int_{C_1}^{I}+\int_{C_2}=$$

$$=I-\int f_-(z)=I+4\pi^2\int f_+(z)=I(1+4\pi^2)$$

$$\oint=2\pi i\sum\operatorname{res}f(z)\qquad z=\pm i,\ e^{\pm i\pi}+1.$$

$$\sum\operatorname{res}f(z)=\frac{1}{2i(\ln^2(i-1)+\pi^2)}-\frac{1}{2i(\ln^2(-i-1)+\pi^2)}+\frac{1}{(e^{2\pi i}+2e^{i\pi}+1+1)\left(\frac{2\ln(e^{i\pi})}{e^{i\pi}}\right)}$$

$$+\frac{1}{\left(e^{-2\pi i}+2e^{-i\pi}+(+1)\right)\left(\frac{2\ln\left(e^{-i\pi}\right)}{-e^{-i\pi}}\right)^2}$$

$$=\frac{1}{2i}\frac{\overbrace{\ln\left(\frac{-i-1}{i-1}\right)}^{=\frac{i\pi}{2}}\overbrace{\ln\left((-i-1)(i-1)\right)}^{=\ln(2)}}{\left(\ln^2(i-1)+\pi^2\right)\left(\ln^2(-i-1)+\pi^2\right)}+-\frac{1}{2i\pi}+\frac{1}{2i\pi}=$$

$$=\frac{\pi\ln(2)}{4\left(\ln^2(i-1)+\pi^2\right)\left(\ln^2(-i-1)+\pi^2\right)}$$

$$\sum=\frac{2\pi i\,\Sigma\operatorname{Res}(f(z))}{1+4\pi^2}=\frac{\pi i\ln(2)}{2(1+4\pi^2)\left(\ln^2(i-1)+\pi^2\right)\left(\ln^2(-i-1)+\pi^2\right)}$$

$$\ln^2(i-1)+\pi^2=\frac{\ln^2(2)}{4}-\frac{9\pi^2}{16}+\pi^2+\frac{3}{4}i\pi\ln(2)$$

$$\ln^2(-i-1)+\pi^2=\frac{\ln^2(2)}{4}-\frac{9\pi^2}{16}+\pi^2-\frac{3}{4}i\pi\ln(2)$$

$$\left(\ln^2(i-1)+\pi^2\right)\left(\ln^2(-i-1)+\pi^2\right)=\left(\frac{\ln^2(2)}{4}+\frac{7\pi^2}{16}\right)^2+\frac{9}{16}\pi^2\ln^2(2)=$$

$$=\frac{\ln^4(2)}{16}+\frac{7\pi^2\ln^2(2)}{32}+\frac{49\pi^4}{256}+\frac{9}{16}\pi^2\ln^2(2)=$$

$$=\frac{\ln^4(2)}{16}+\frac{25\pi^2\ln^2(2)}{32}+\frac{49\pi^4}{256}$$

$$\boxed{\sum=\frac{8\pi^2 i\ln(2)}{(1+4\pi^2)\left(\ln^4(2)+\frac{25}{2}\pi^2\ln^2(2)+\frac{49}{16}\pi^4\right)}}$$