ТФКП. Контрольное 1.

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Рената. БФЗ182.

1. $\left(-\frac{1}{2}+\frac{\sqrt{3}\,i}{2}\right)^n$, где $n = 1826$.

$$\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^2 = -\frac{1}{2}-\frac{\sqrt{3}}{2}i$$

$$\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^3 = \frac{1}{4}-\frac{i\sqrt{3}}{4}+\frac{i\sqrt{3}}{4}+\frac{3}{4} = 1.$$

$$\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^4 = -\frac{1}{2}+\frac{\sqrt{3}i}{2} = \left(-\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)^1$$

т.к. $n = 1826 = 608\cdot 3 + 2$

$$\left(-\frac{1}{2}+\frac{\sqrt{3}\,i}{2}\right)^n = \boxed{-\frac{1}{2}-\frac{\sqrt{3}}{2}i.}$$

2. $f(z) = \dfrac{\sin(2z)-1+e^{-2z}}{z^2(\cos(z)-1)^2}$ $\qquad \operatorname*{Res}_{z=0} f(z) = ?$

$z = 0$ — полюс 3-го порядка.

$$\operatorname*{Res}_{z=0} = \frac{1}{(3-1)!}\lim_{z\to 0}\left(\frac{z^3(\sin(2z)-1+e^{-2z})}{\cancel{z^2}(\cos(z)-1)^2}\right)'' =$$

$$= \frac{1}{2}\lim_{z\to 0}\left(2(2\cos(2x)-2\exp(-2x))\right)\left(\frac{2\sin x}{x(\cos x)-1)^3}-\frac{1}{x^2(\cos x-1)^2}\right)+$$

$$+\left(e^{-2x}+\sin(2x)-1\right)^{\to 0}\left(\frac{2}{x^3(\cos x-1)^2}-\frac{4\sin x}{x^2(\cos-1)^3}+\frac{\frac{2\cos x}{(\cos(x)-1)^3}+\frac{6\sin^2 x}{(\cos x-1)^4}}{x}\right)+$$

$$+\frac{4e^{-2x}-4\sin 2x}{x(\cos(x)-1)^2} = \frac{1}{2}\lim_{z\to 0} f = 4.$$

3. $\dfrac{1}{z^2-2z-3}$, $z=3$ - полюс 1-го порядка.

$$\frac{1}{(z-3)(z+1)} \Rightarrow C_{-1} = \frac{1}{4(z-3)}$$

$$\frac{1}{x+1} = \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} - \frac{1}{x^4} \ldots \Rightarrow$$

$$\left(\frac{1}{z-3} - \frac{1}{z+1}\right) = \left(-\frac{1}{3-z} - \frac{1}{z+1}\right) =$$

$$\frac{1}{z^2-2z-3} = \sum_{n=-1}^{\infty} \frac{(-1)^{n+1}}{(4)^{n+2}} (z-3)^n = \boxed{\frac{1}{4(z-3)} - \frac{1}{16}}$$

4. $\dfrac{\sin(z)\tan\left(\frac{1}{z}\right)}{z-1}$

$z=0$ - един. особая точка.

т.к $\lim\limits_{z\to 0} \dfrac{\sin(z)\tan\left(\frac{1}{z}\right)}{z-1} = \varnothing$

$z=1$ - полюс первого порядка.

$$\lim_{z\to 1} \frac{\sin(z)\tan\left(\frac{1}{z}\right)}{z-1} = 1$$

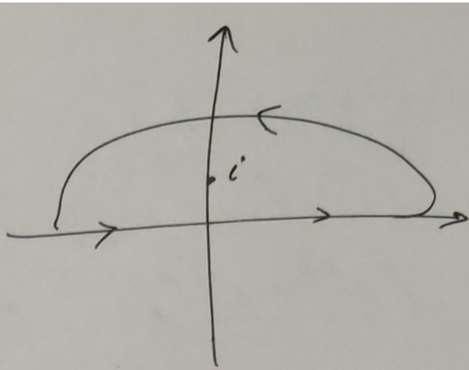
5. $\displaystyle\int \frac{\bar{z}}{(|z|+1)^3}\,dz$ $\quad r=2$, $z=0$ - центр.

$z = |z|e^{i\varphi}$, $|z|=2$.

$\bar{z} = |z|e^{-i\varphi}$. $\quad dz = ie^{i\varphi}d\varphi \cdot |z|$

$$\int_0^{2\pi} \frac{|z|\cancel{e^{-i\varphi}}|z| i \cancel{e^{i\varphi}}\,d\varphi}{(|z|+1)^3} = \frac{i|z|^2 2\pi}{(|z|+1)^3} = \frac{i4\cdot 2\pi}{3^3} = \boxed{\frac{i8\pi}{27}}.$$

6. $\int\limits_{-\infty}^{\infty} \frac{\sin(x)}{(x-i)^2}\, dx$.



$x \to z$

$\int\limits_{-\infty}^{\infty} \frac{\sin(z)}{(z-i)^2}\, dz$. $z=i$ – полюс 2-го порядка.

$\Rightarrow \int\limits_{-\infty}^{\infty} \frac{\sin(z)\,dz}{(z-i)^2} = \operatorname*{Res}_{z=i} f(z) = \lim\limits_{z\to i}\left(\frac{\cancel{(z-i)^2}\sin(z)}{\cancel{(z-i)^2}}\right)' =$

$= \lim\limits_{z\to i} \cos(z) = \boxed{\cos(i)}$

8. $I = \oint \frac{z^2 e^{\frac{1}{2z}}}{(z+3)(z+4)}\, dz$. $r = \frac{7}{2}$

$z = 0$ центр.

$\Rightarrow$ $z = -3$ – полюс 1-го порядка.

$z = -4$ – полюс, но за областью интегр.

$z = 0$ сущ. особая точка.

$I = \operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=-3} f(z) = -\operatorname*{Res}_{z=-4} f(z)$

$\Rightarrow I = -\frac{z^2 e^{\frac{1}{2z}}}{z+3+z+4} = -\frac{16\cdot e^{-\frac{1}{8}}}{-8+7} = \boxed{16 e^{-\frac{1}{8}}}$

7. $\int\limits_{0}^{\infty} \frac{4\sin(2x) - 2\sin(4x)}{x^3}\, dx$

$\sin(4x) = 2\sin(2x)\cos(2x)$

$4\sin(2x)(1-\cos(2x)) = -16\cos^3 x \sin x$.

$$I = \int_0^{\infty} -16\frac{\cos^3 x \sin x}{x^3}\,dx \qquad x=0 \text{ - полюс 2-го порядка.}$$

$$\cos^3 x = \left(\frac{e^{ix}+e^{-ix}}{2}\right)^3 = \frac{e^{3ix}+e^{-3ix}+3e^{ix}+3e^{-ix}}{8}$$

$$\sin x = \frac{e^{ix}-e^{-ix}}{2i}$$

$$\cos^3 x \sin x = \frac{1}{16i}\left(e^{4ix}-e^{-4ix}+2e^{2ix}-2e^{-2ix}\right)$$

$$I = -\frac{1}{i}\int_0^{\infty}\frac{e^{4ix}-e^{-4ix}+2e^{2ix}-2e^{-2ix}}{x^3}\,dx$$

$$\cancel{x} \to \cancel{z} = \rho e^{i\varphi}.$$

$$I_1 = -\frac{1}{i}\int_0^{\infty}\frac{e^{4ix}}{x^3}dx = -\frac{1}{i}\int_0^{\frac{\pi}{2}}\frac{\rho i e^{i\varphi}d\varphi\, e^{4i\rho e^{i\varphi}}}{\rho^3 e^{3i\varphi}}\,d\varphi =$$

$$= -\int_0^{\frac{\pi}{2}}\frac{1}{\rho^2}\exp(4i\rho e^{i\varphi}-2i\varphi)\,d\varphi.$$

$$\exp(4i\rho(\cos\varphi+i\sin\varphi)-2i\varphi) = \exp(-4\rho\sin\varphi)$$

при $\rho\to\infty$ т.к. $x\in(0;+\infty)$

$$\Rightarrow \exp(-4\rho\sin\varphi)=0.$$

$$\Rightarrow I_1 = 0$$

Аналогично $I_2 = I_3 = I_4 = 0 \Rightarrow \boxed{I=0}$