Elec4A - Traitement du Signal

Frequential Signal Analysis

Renato Martins, ICB UMR CNRS - Univ. Bourgogne
UFR Sciences & Techniques - IEM, 2024

Agenda

- **Part I**: Fourier Decomposition
  - 1D Fourier Decomposition: Series and Transform
  - Sampling and Aliasing
  - 2D Images and Discrete Fourier Transform
- **Part II**: Applications & Advanced topics
  - Applications to filtering and denoising
  - Fourier features (NeurIPS 2020)
  - Spherical harmonics

Acknowledgments

- The course slides are based on materials generously made publicly available by many other people and lectures:

  Derek Hoiem (Illinois), James Hays (Georgia Tech), Andrew Zisserman (Oxford), Alexei Efros (UC Berkeley), Fabrice Meriaudeau (uB), Olivier Laligant (uB), Steve Seitz (UW)

- I might not have credits on every slide (which is bad, sorry).

Part I :

Decomposition of signals - Fourier!

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Decomposition of signals - Fourier!

Motivation & Recap of Spatial Filtering

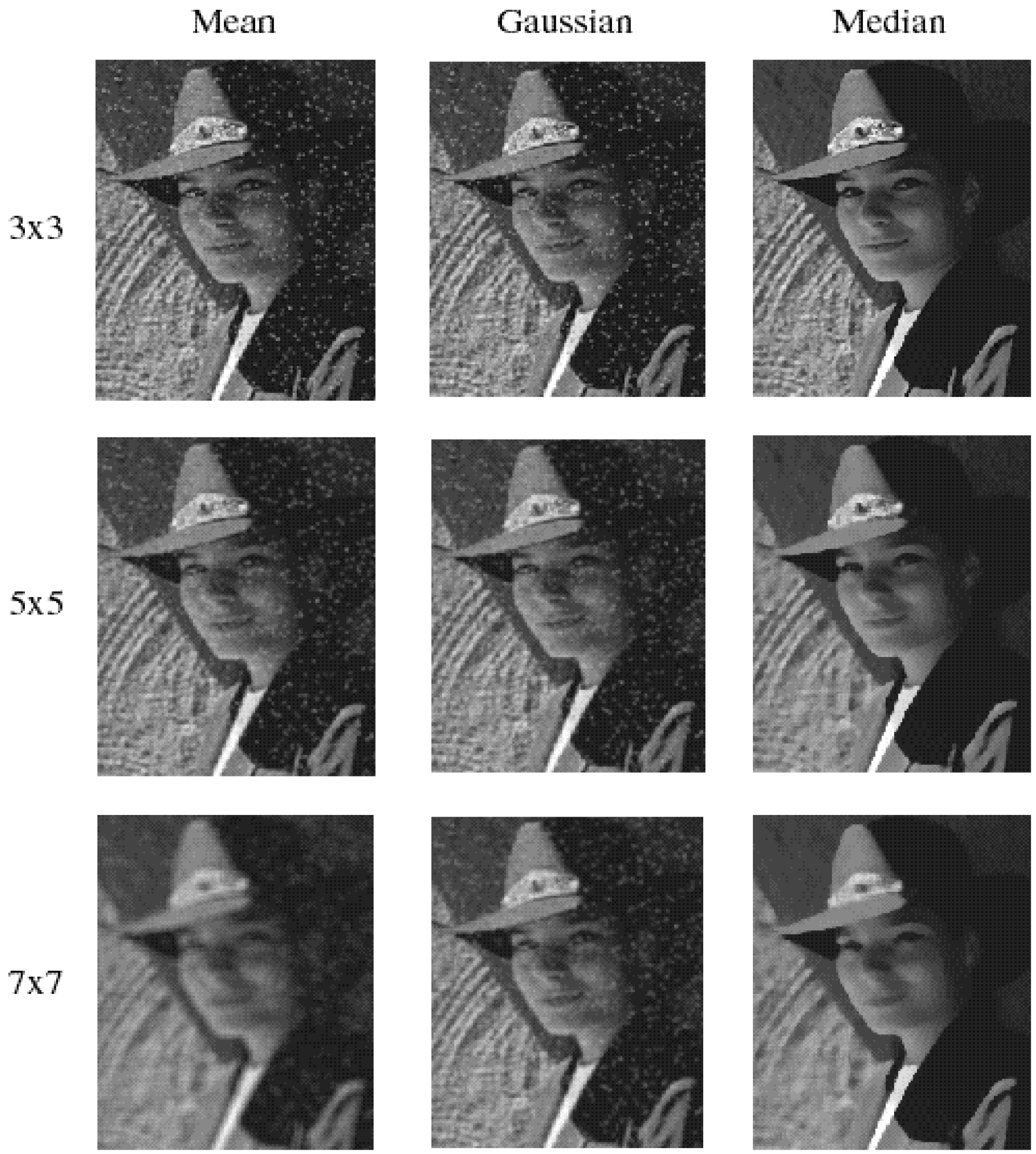
- Linear filtering: function is a weighted sum/difference of pixel values

- Really important!
  - Enhance images
    - Denoise, smooth, increase contrast, etc.
  - Extract information from images
    - Texture, edges, distinctive points, etc.
  - Detect patterns
    - Template matching

Original

Salt and pepper noise

Comparison: salt and pepper noise



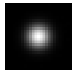
Slide: Steve Seitz

Motivation & Recap of Spatial Filtering

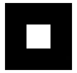
Motivation & Recap of Spatial Filtering

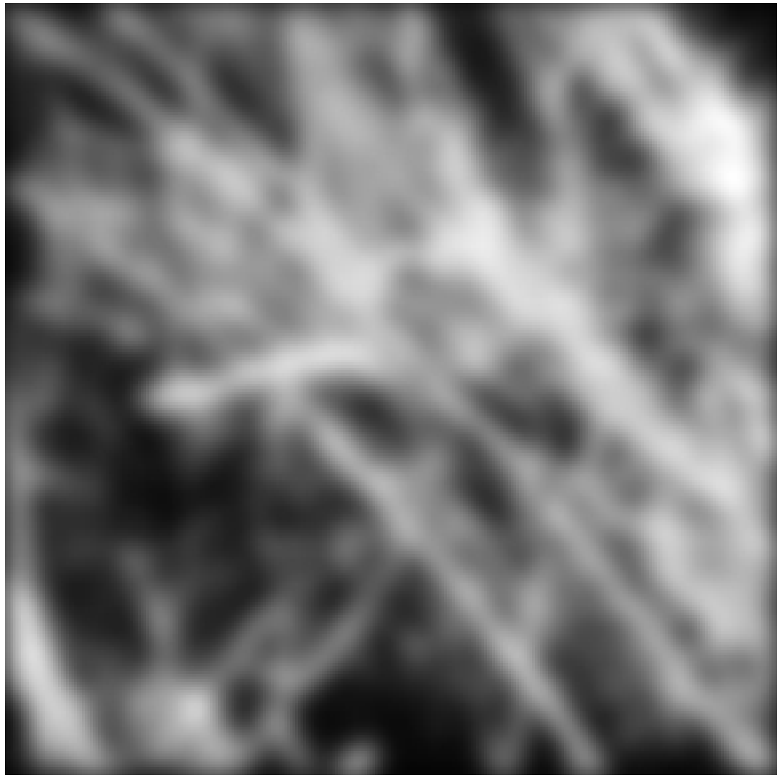
Why does the Gaussian give a nice smooth image, but the square filter give edgy artifacts?

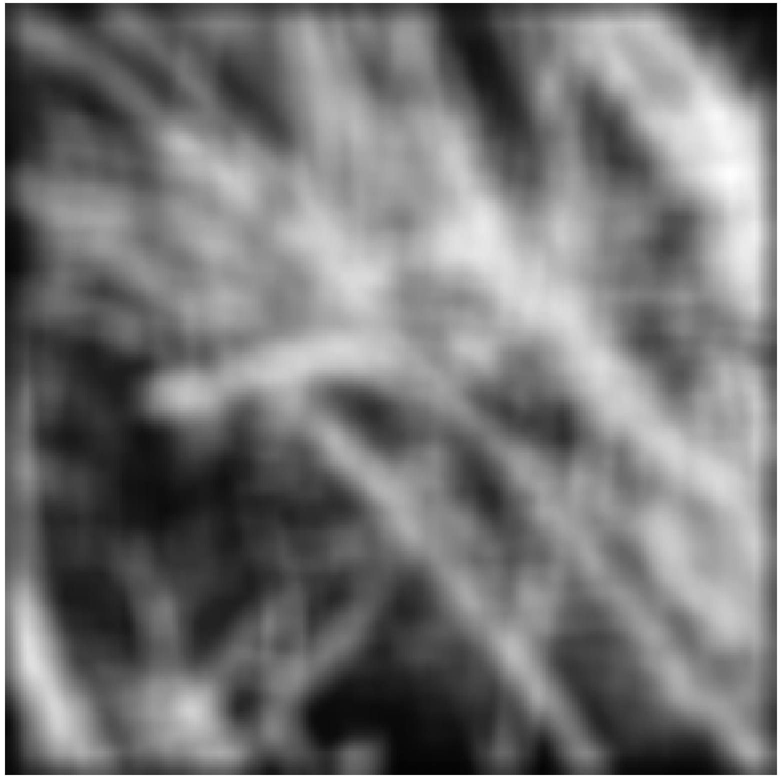
Gaussian



Box filter







Motivation & Recap of Spatial Filtering

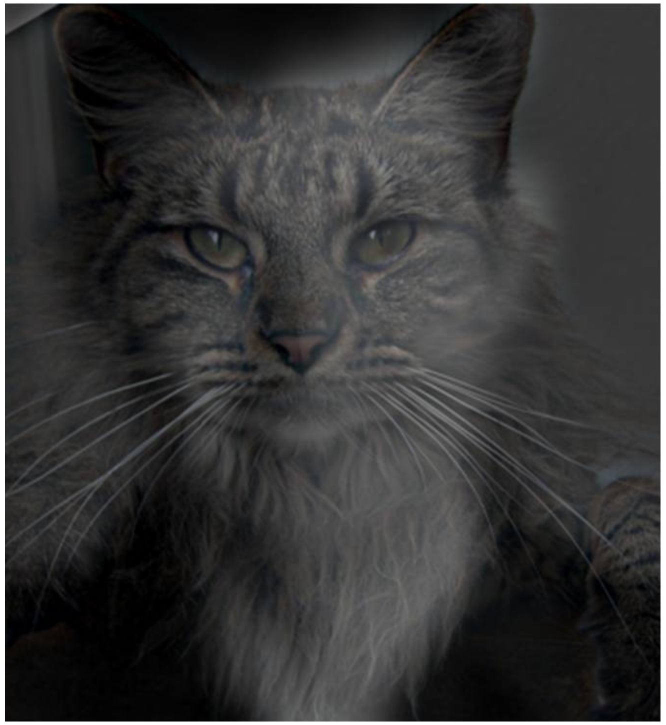
Why does a lower resolution image still make sense to us? What do we lose?

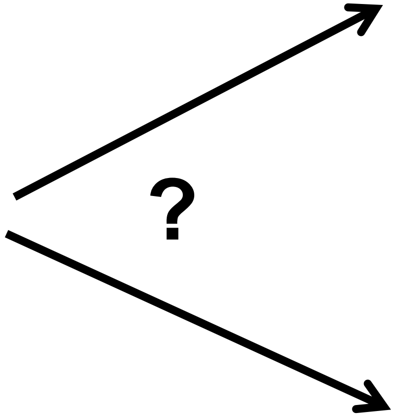
Image: http://www.flickr.com/photos/igorms/136916757/

Slide: Hoiem

Motivation & Recap of Spatial Filtering

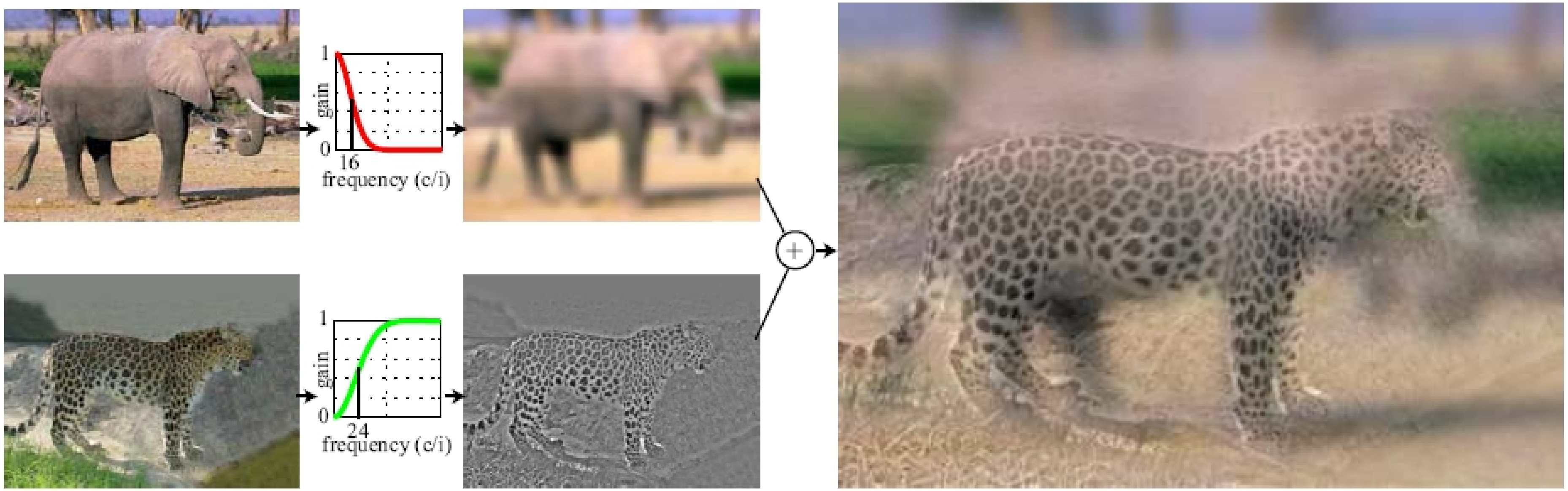
Why do we get different, distance-dependent interpretations of hybrid images?





Slide: Hoiem

Motivation & Recap of Spatial Filtering



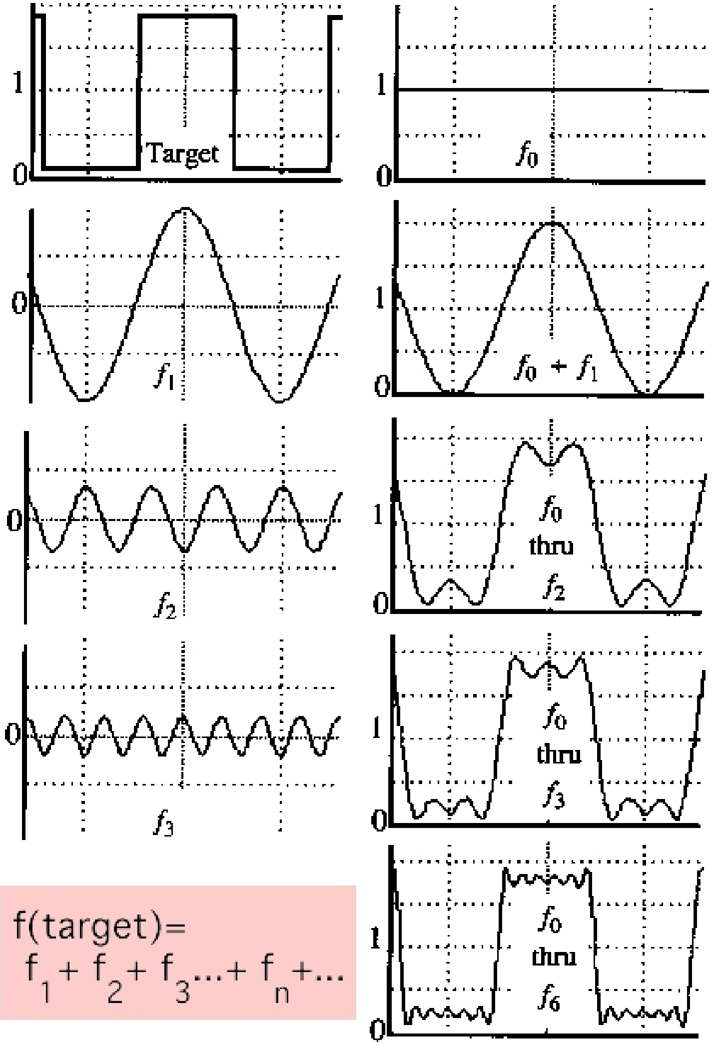
A. Oliva, A. Torralba, P.G. Schyns, [“Hybrid Images,”]() SIGGRAPH 2006

A Sum of Sines

Building block of periodic functions:

$$A\sin(\omega x+\phi)$$

Add enough of them to get any signal *g(x)* you want!



f(target)=
$f_1 + f_2 + f_3 ... + f_n + ...$

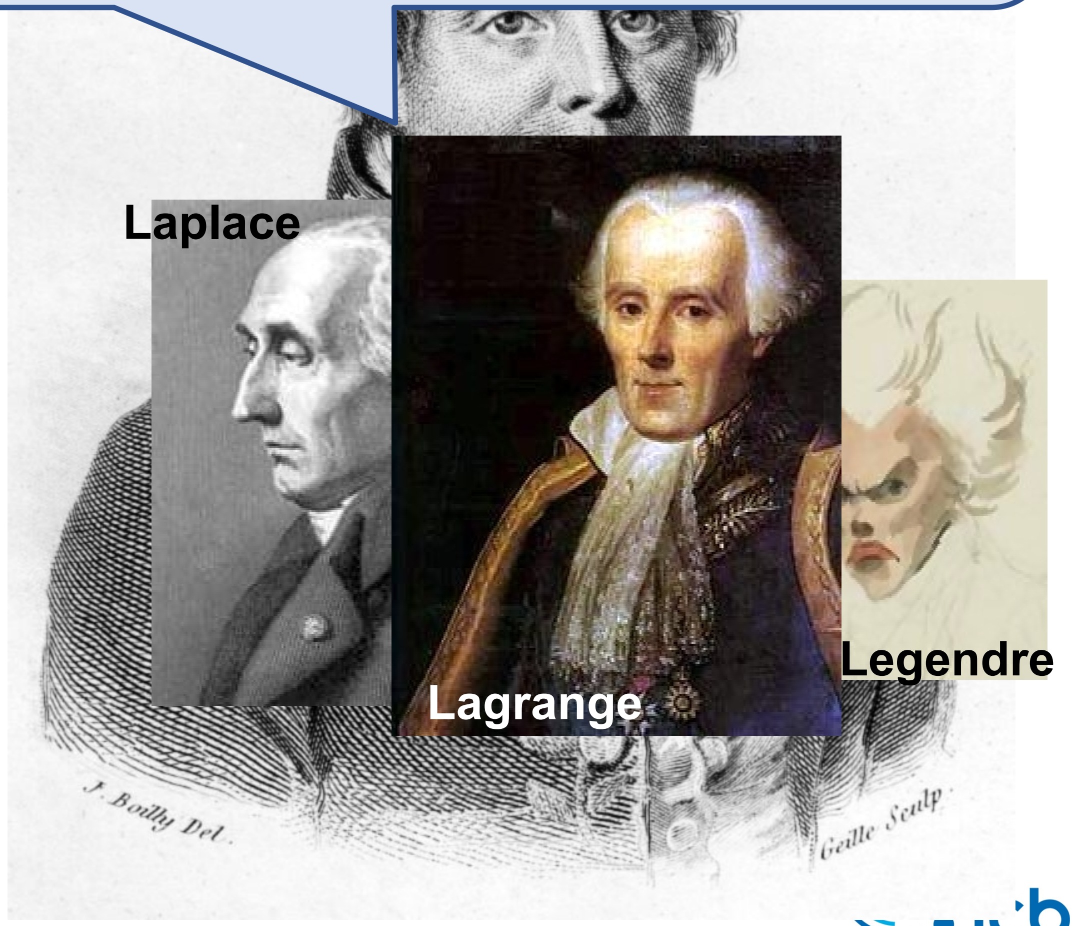
Jean Baptiste Joseph Fourier (1768-1830)

had crazy idea (1807):

> *Any univariate function can be rewritten a weighted sum of sines and cosines of different frequencies.*

...the manner in which the author arrives at these equations is not exempt of difficulties and...his analysis to integrate them still leaves something to be desired on the score of generality and even rigour.

- Don’t believe it?
  - Neither did Lagrange, Laplace, Poisson and other big wigs
  - Not translated into English until 1878!
- But it’s (mostly) true!
  - called Fourier Series
  - there are some subtle restrictions



Jean Baptiste Joseph Fourier (1768-1830)

- Fourier was born in Auxerre! (Bourgogne)

Fourier, Joseph (1768-1830)

French mathematician who discovered that any periodic motion can be written as a superposition of sinusoidal and cosinusoidal vibrations. He developed a mathematical theory of heat in *Théorie Analytique de la Chaleur (Analytic Theory of Heat),* (1822), discussing it in terms of differential equations.

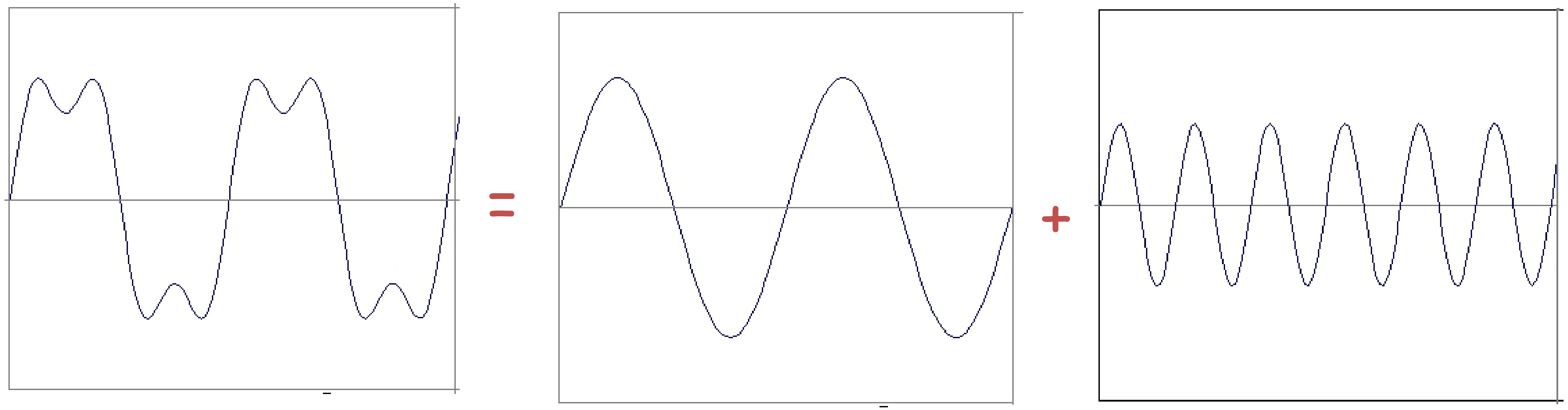
Fourier was a friend and advisor of Napoleon. Fourier believed that his health would be improved by wrapping himself up in blankets, and in this state he tripped down the stairs in his house and killed himself. The paper of Galois which he had taken home to read shortly before his death was never recovered.

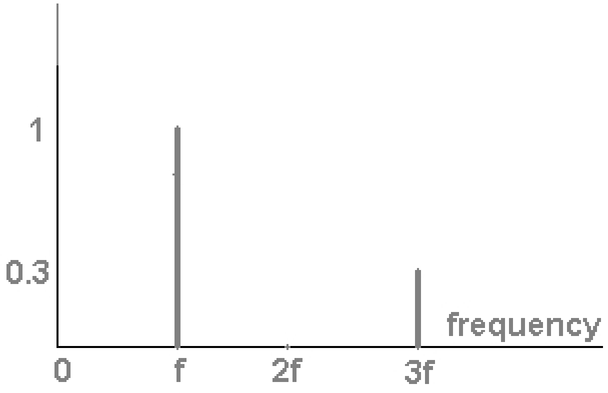
SEE ALSO: Galois

Additional biographies: MacTutor (St. Andrews), Bonn

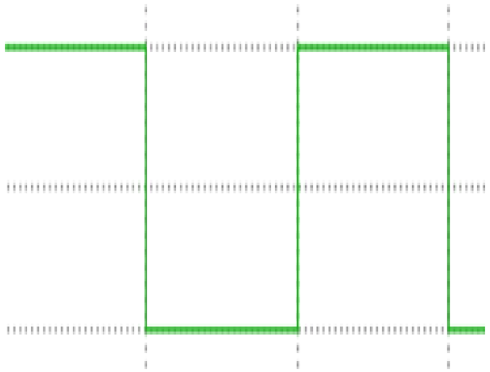
Frequency Spectra

- Example : $g(t) = \sin(2\pi f\, t) + (1/3)\sin(2\pi(3f)\, t)$

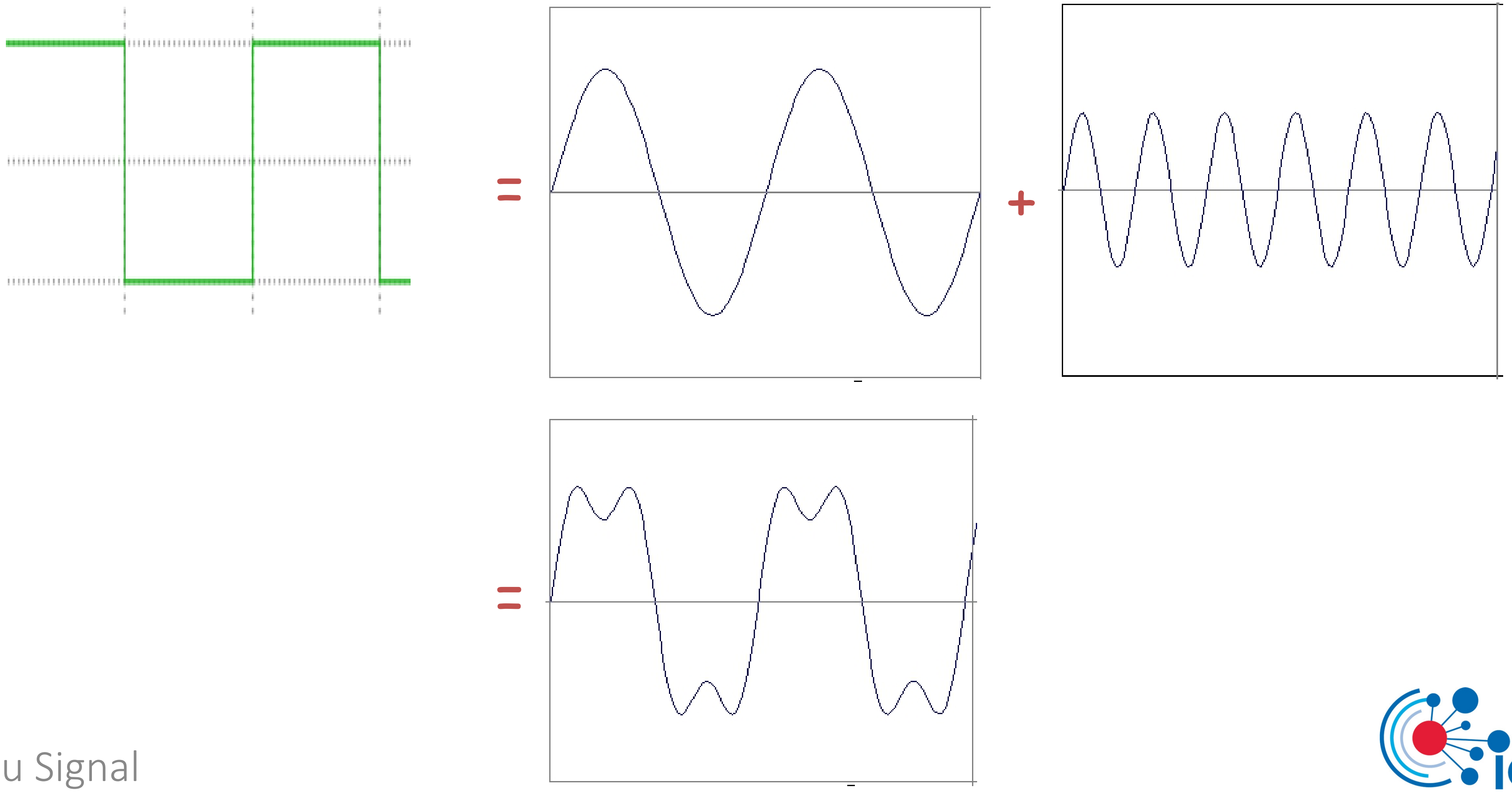




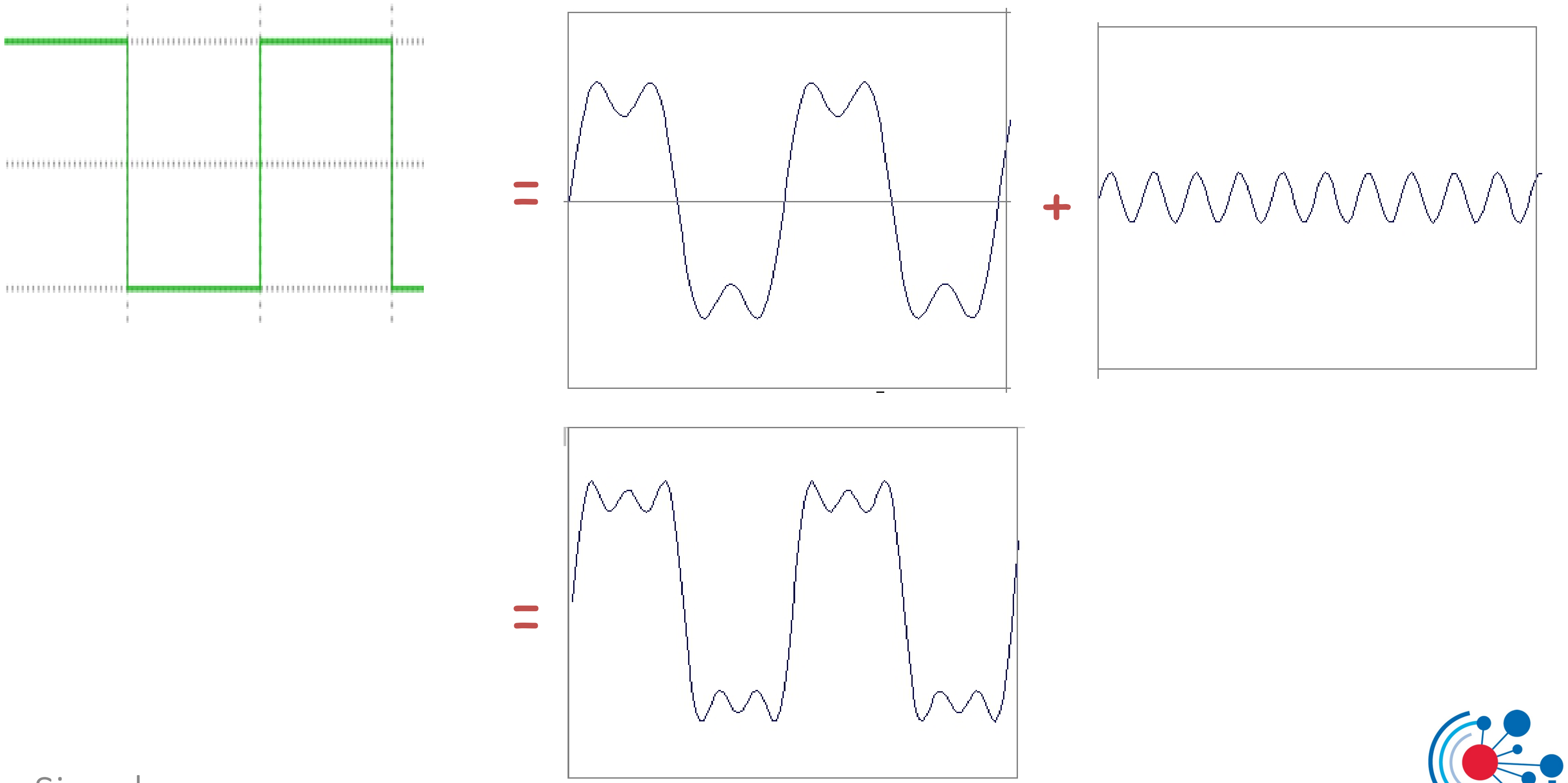
Frequency Spectra



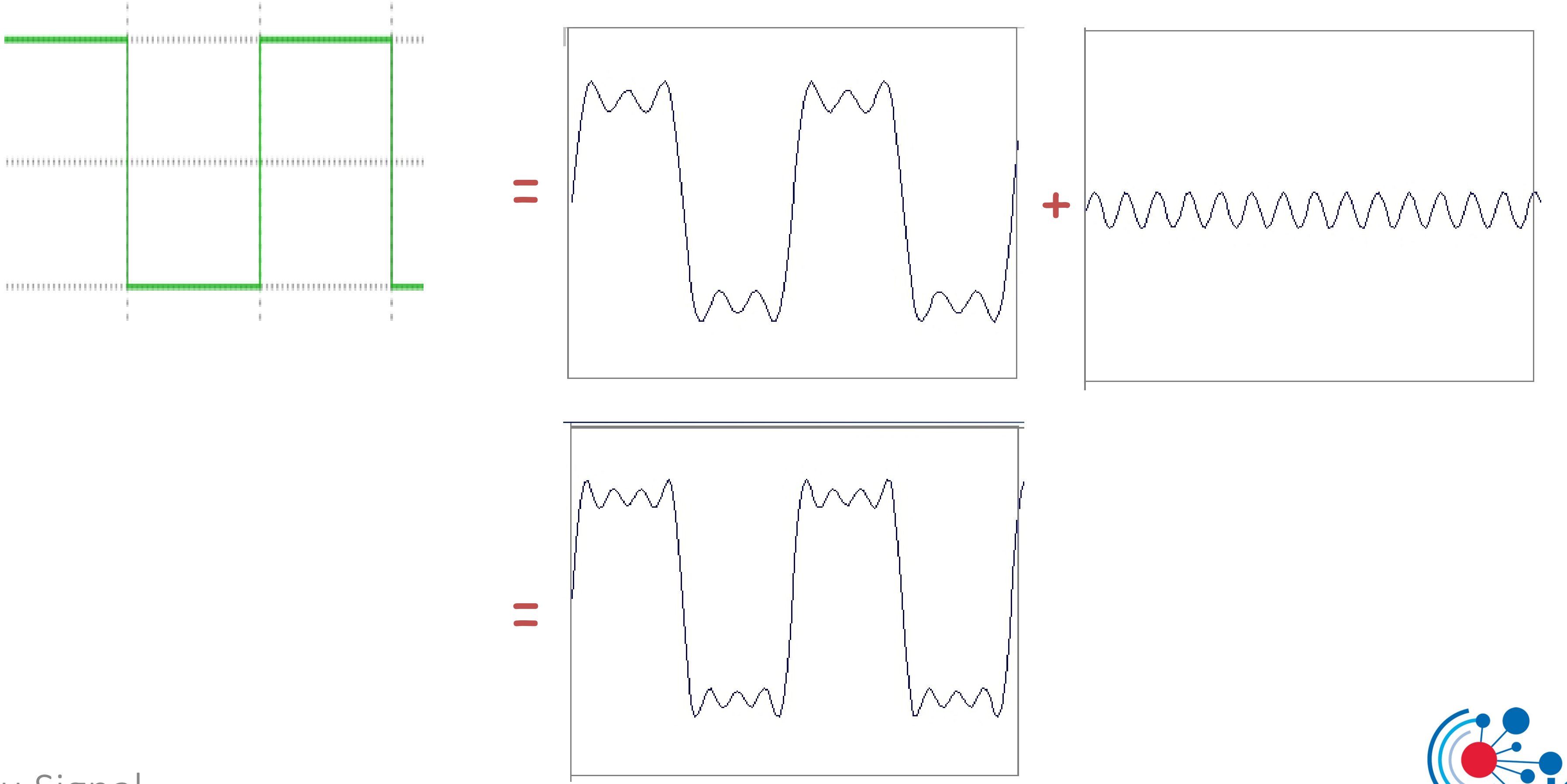
Frequency Spectra



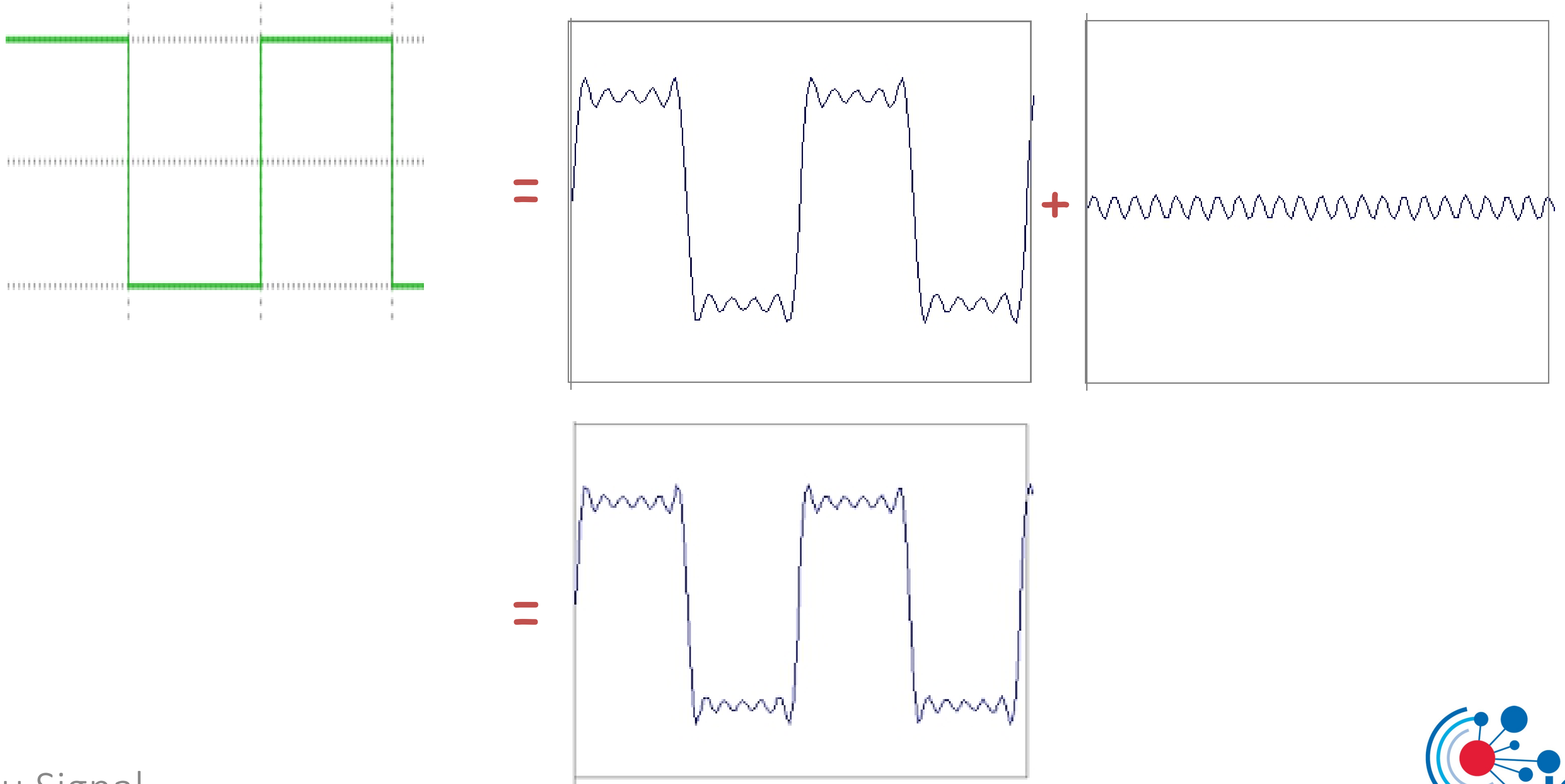
Frequency Spectra



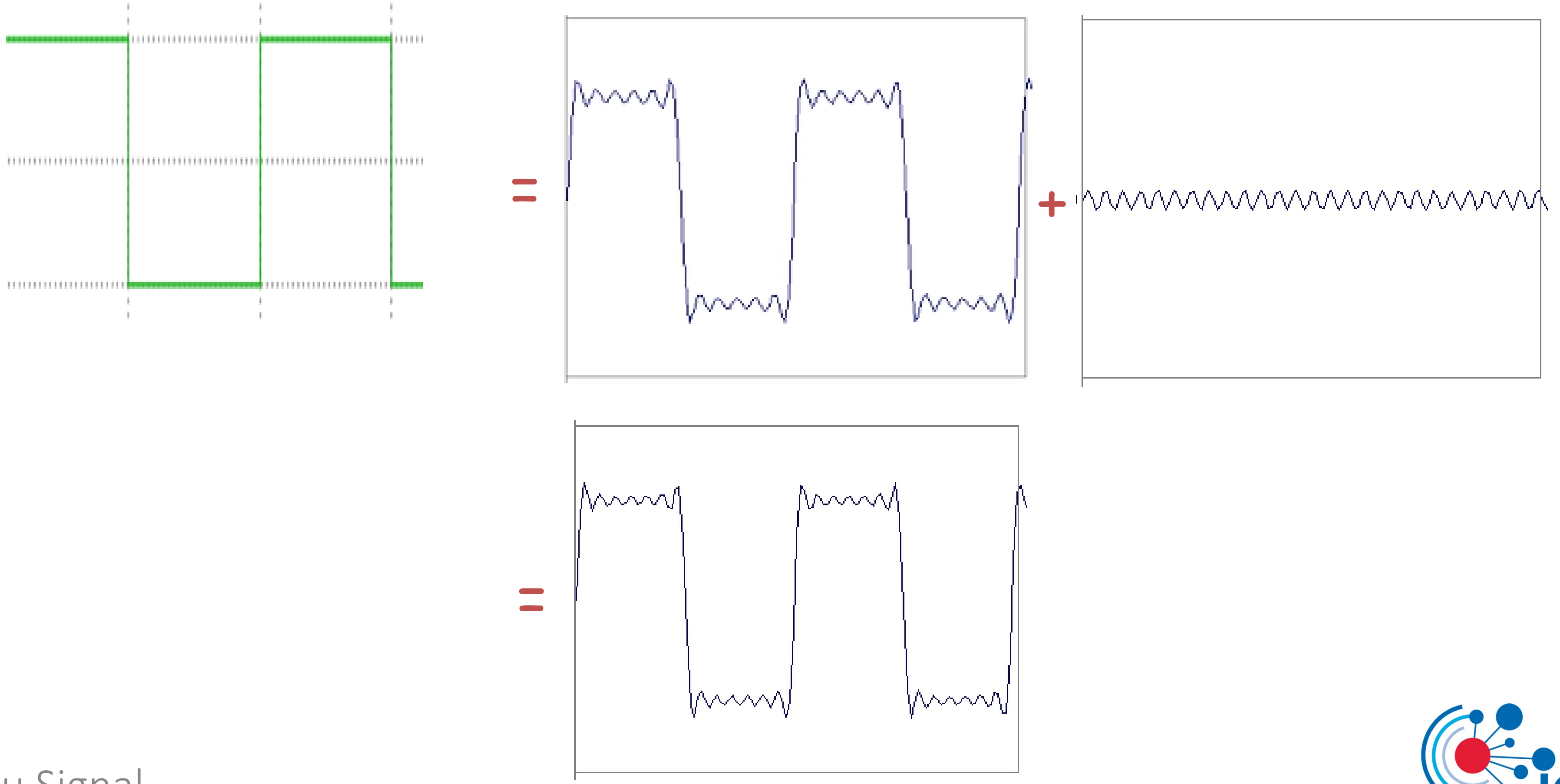
Frequency Spectra



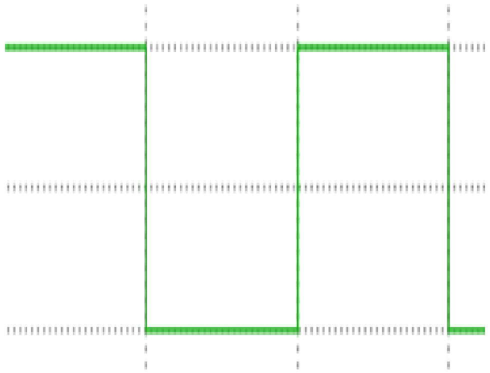
Frequency Spectra



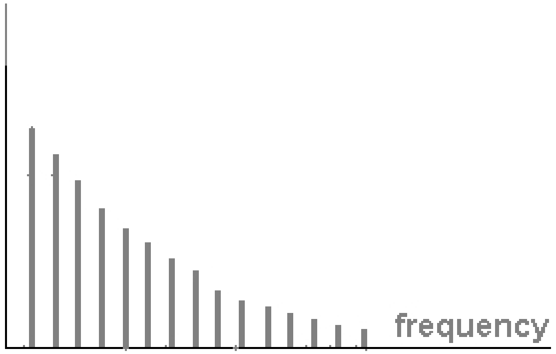
Frequency Spectra



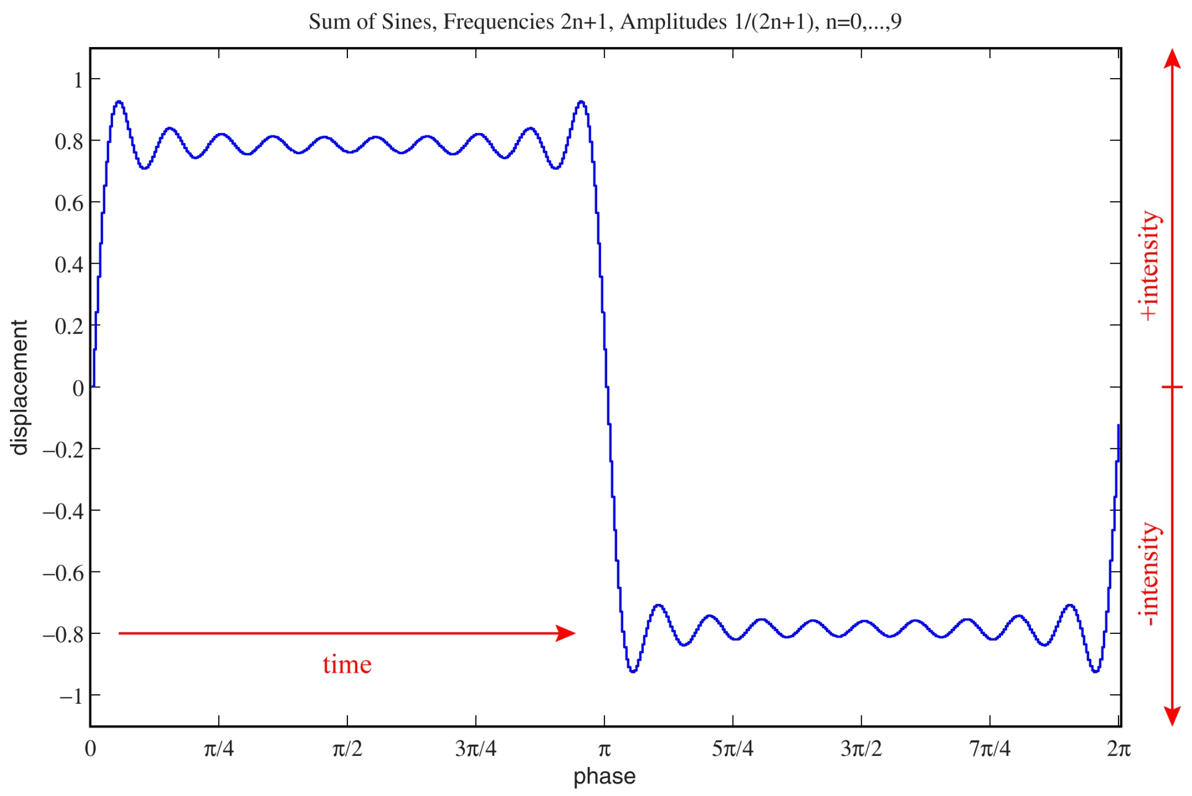
Frequency Spectra



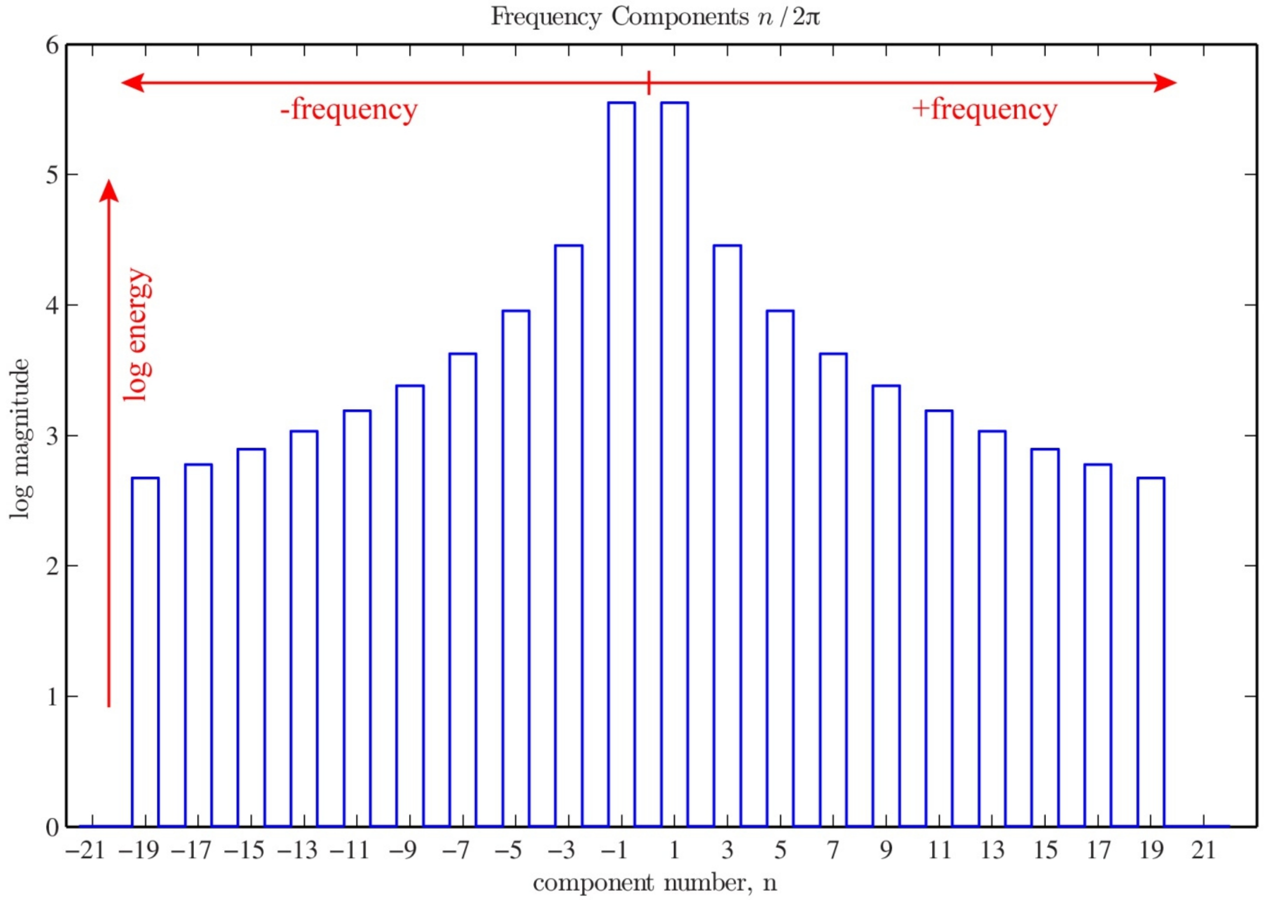
$$= \boxed{A\sum_{k=1}^{\infty}\frac{1}{k}\sin(2\pi kt)}$$



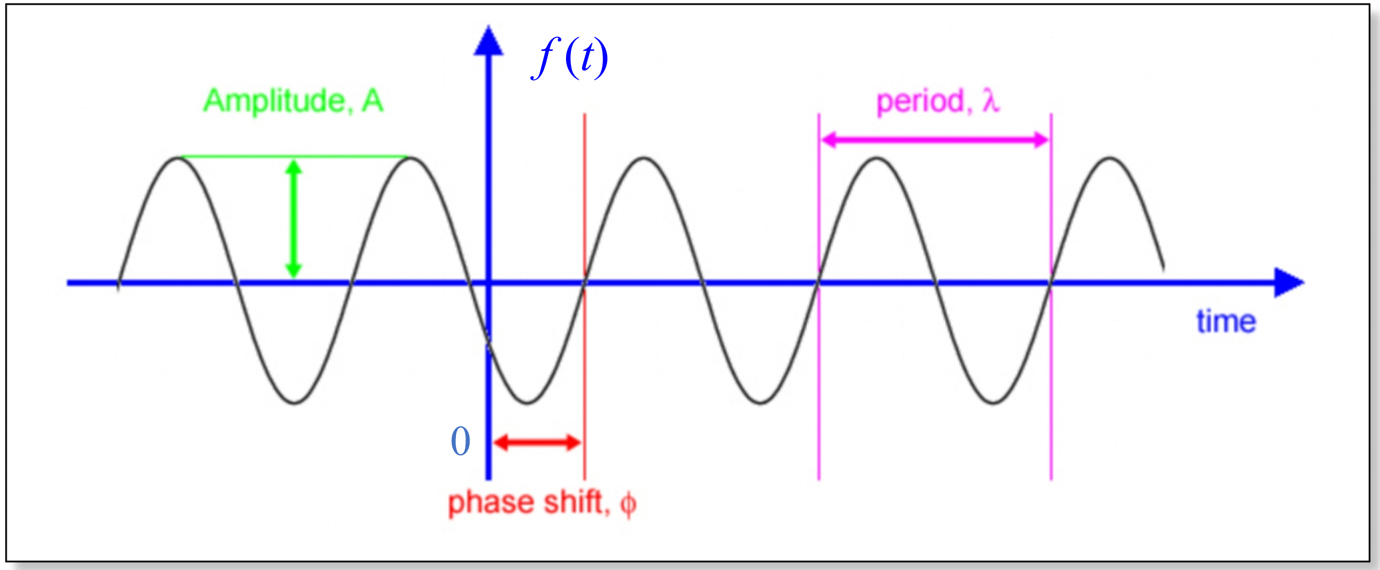
Example: Time-Domain Representation



Example: Frequency-Domain Representation



Anatomy of a Sinusoid



$$f(t) = A\sin\left(\frac{2\pi}{\lambda}t - \phi\right)$$

$1/\lambda$ is the frequency of the sinusoid (Hz).

$2\pi/\lambda$ is the angular frequency (radians/s).

Harmonics Interpretation of Periodic Signals

- Fourier series for periodic signals
  - Basis = set of complex exponentials : $\{ e^{jn\omega t} \}_{n \in [-\infty;+\infty]}$
  - $\omega = 2\pi f = 2\pi/T$, f is the fundamental frequency
  - Signal $s(t)$ can be defined as:

$$s(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n.\cos n\omega t + b_n.\sin n\omega t)$$

  - Or in a more generic form:

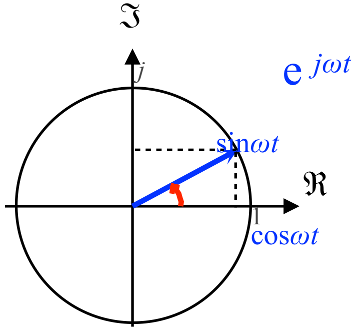
$$s(t) = \sum_{n=-\infty}^{+\infty} c_n.e^{jn\omega t}$$

$$\begin{aligned} c_n &= \frac{a_n - jb_n}{2} \\ &= \frac{d_n}{2}.(\cos\varphi + j.\sin\varphi) \\ |c_n| &= \frac{d_n}{2} \\ \arg[c_n] &= \varphi \end{aligned}$$

Harmonics Interpretation of Periodic Signals

- Orthogonality of the basis

$$
\begin{aligned}
< e^{jn\omega t}, e^{jm\omega t} >_T &= \tfrac{1}{T}\int_T e^{jn\omega t}.e^{-jm\omega t}dt \\
&= \tfrac{1}{T}\int_T e^{j(n-m)\omega t}dt \\
&= \tfrac{1}{T}\tfrac{1}{(n-m)\omega}.\left(e^{j(n-m)2\pi}-1\right) && n \neq m \\
&= \tfrac{1}{T}\tfrac{1}{(n-m)\omega}\times 0 && n \neq m \\
&= 0 && n \neq m \\
&= 1 && n = m
\end{aligned}
$$



- Projection for $\cos \omega t$

$$
\begin{aligned}
< \cos\omega t, e^{j\omega t} >_T &= \frac{1}{T}\int_T \cos\omega t.e^{-j\omega t}dt \\
&= \frac{1}{T}\int_T \frac{e^{j\omega t}+e^{-j\omega t}}{2}.e^{-j\omega t}dt \\
&= \frac{1}{T}\int_T \frac{1+e^{-2j\omega t}}{2}.dt \\
&= \frac{1}{2T}\left\{[t]_0^T + \frac{1}{-2j\omega}\left[e^{-2j\omega t}\right]_0^T dt\right\} \\
&= \frac{1}{2T}\left\{T + \frac{1}{-2j\omega}\left(e^{-2j\omega T}-1\right) dt\right\} \\
&= \frac{1}{2T}.T = \frac{1}{2}
\end{aligned}
$$

$$
< \cos\omega t, e^{jn\omega t} >_T \;=\; 0 \quad \forall n \neq 1
$$

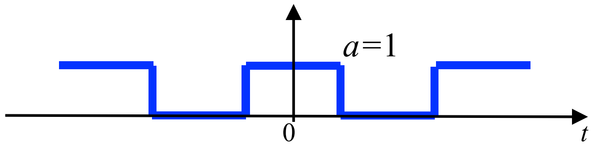
Harmonics Interpretation of Periodic Signals

- Signal is real than basis can be written as $\{\cos n\omega t, \sin n\omega t\}$n $\in [0;+\infty]$:

$$
\begin{aligned}
a_0 &= \frac{2}{T}\int_T s(t)dt \\
&= 2.c_0 \\
b_0 &= 0 \\
a_n &= \frac{2}{T}\int_T s(t)\cos(n\omega t)dt \ , \quad n \geq 1 \\
b_n &= \frac{2}{T}\int_T s(t)\sin(n\omega t)dt \ , \quad n \geq 1
\end{aligned}
$$

a_0 : 2 x mean of the signal

- **(Exercise)** Find the decomposition of "even" wave square signal

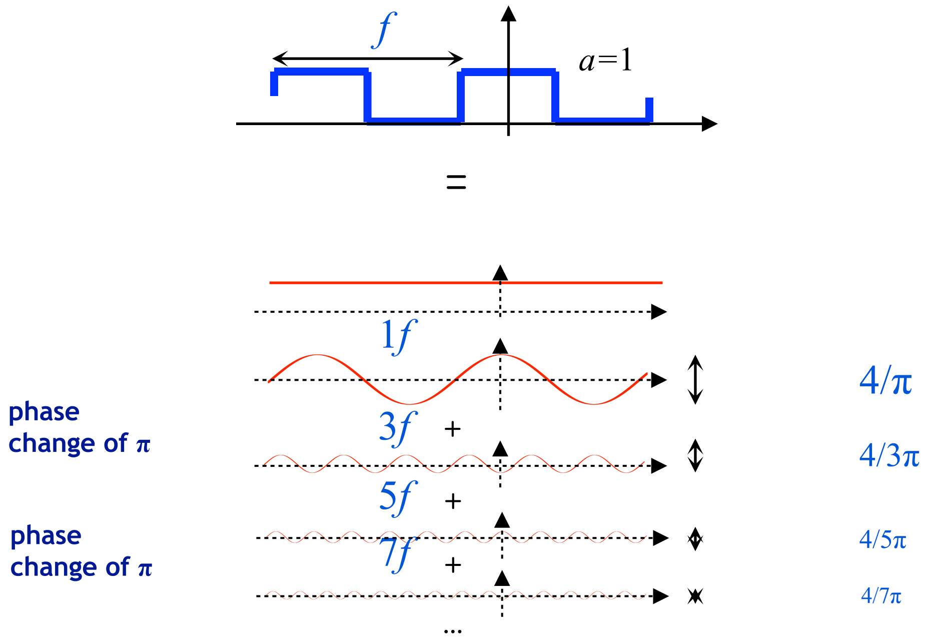


Harmonics Interpretation of Periodic Signals

- Example with “even” squared signal

$$
\begin{aligned}
a_0 &= 2 \times 0.5 \qquad (\bar{s}(t) = 0.5) \\
b_0 &= 0 \\
a_n &= 2a. \times \frac{\sin n\frac{\pi}{2}}{n\frac{\pi}{2}} \;, \quad n \geq 1 \\
b_n &= 0
\end{aligned}
$$

$$
\begin{cases}
a_1 = 2.\frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} = 2 \times 1.\frac{1}{\frac{\pi}{2}} = \frac{4}{\pi} \\
a_2 = 2.\frac{\sin 2.\frac{\pi}{2}}{2.\frac{\pi}{2}} = 0 \\
a_3 = 2.\frac{\sin 3.\frac{\pi}{2}}{3.\frac{\pi}{2}} = -\frac{1}{3}.\frac{4}{\pi} \\
a_4 = 2.\frac{\sin 4.\frac{\pi}{2}}{4.\frac{\pi}{2}} = 0 \\
a_5 = 2.\frac{\sin 5.\frac{\pi}{2}}{5.\frac{\pi}{2}} = \frac{1}{5}.\frac{4}{\pi} \\
\ldots
\end{cases}
$$



Harmonics Interpretation of Periodic Signals

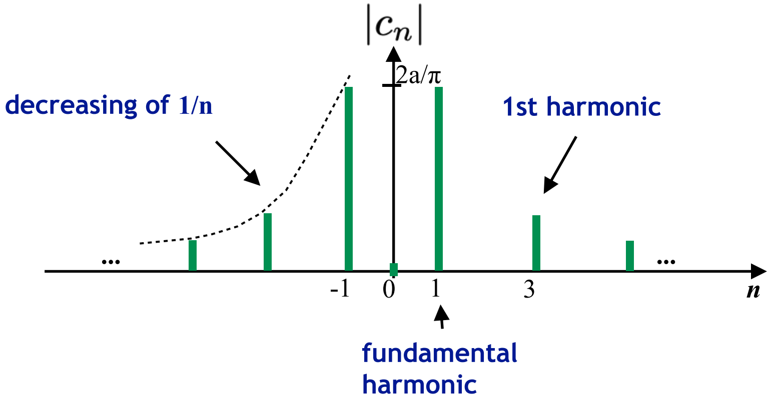
- Projection in one period:

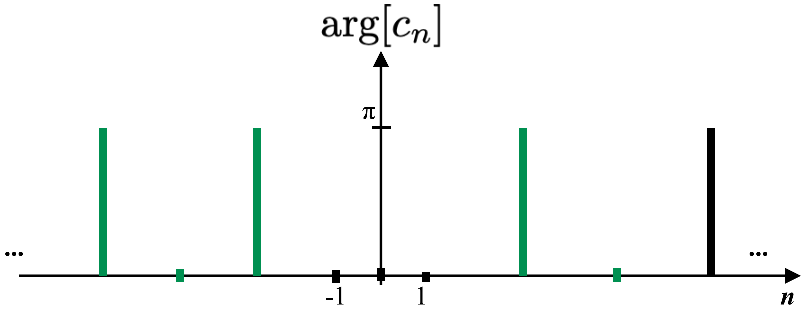
$$
\begin{aligned}
c_n &= \frac{1}{T}\int_T s(t)\overline{e^{jn\omega t}}dt \\
&= \frac{1}{T}\int_T s(t)e^{-jn\omega t}dt \\
&= \frac{1}{T}\int_0^T s(t)e^{-jn\omega t}dt \\
&= \frac{1}{T}\int_a^{a+T} s(t)e^{-jn\omega t}dt, \quad a\in\mathbb{R} \\
&\ldots
\end{aligned}
$$

c_0 : mean of the signal

$$c_n = a.\frac{\sin n\frac{\pi}{2}}{n\frac{\pi}{2}}, \; n \neq 0$$

- Spectra and phase of Fourier coefficients:





Properties of Fourier Series

- Even signal
  - $a_n \neq 0$ in general
  - $b_n = 0$
- Odd signal
  - $a_n = 0$ in general
  - $b_n \neq 0$

- Regularity of the signal and coefficients decrease:
    - Discontinuity of order 0 (square signal, sawtooth wave, ...): decrease of $1/n$
    - Discontinuity of order 1 (triangle wave): decrease of $1/n^2$
    - ...
    - Decrease of coefficients depends on the regularity of the signal

Synthesis of Real Periodic Signals

- In a nutshell:

Fourier series is the decomposition of a periodic signal into a sum of sinusoids.

$$
\begin{aligned}
s(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n . \cos n\omega t + b_n . \sin n\omega t) \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} d_n . \cos (n\omega t + \varphi) \\
&= \sum_{n=-\infty}^{+\infty} c_n . e^{jn\omega t}
\end{aligned}
$$

$$
\begin{aligned}
c_n &= \frac{a_n - jb_n}{2} \\
&= \frac{d_n}{2} . (\cos\varphi + j . \sin\varphi) \\
|c_n| &= \frac{d_n}{2} \\
\arg[c_n] &= \varphi
\end{aligned}
$$

Fourier Transform

- **Fourier transform (FT)** is the decomposition of a *nonperiodic* **signal** into a continuous sum* (integral) of sinusoids
- The spectra can have any frequency in $\mathbb{R}$

- Projection or decomposition (FT):

$$\begin{aligned}\hat{s}(\omega) &= < s(t), e^{j\omega t} > \\ &= \int_{-\infty}^{\infty} s(t)e^{-j\omega t}dt\end{aligned}$$

- Synthesis or reconstruction (FT^{-1}):

$$\begin{aligned}s(t) &= \frac{1}{2\pi} < \hat{s}(\omega), e^{-j\omega t} > \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} s(\omega)e^{j\omega t}d\omega\end{aligned}$$

Fourier Transform

- Fourier transform stores the magnitude and phase at each frequency
  - Magnitude encodes how much signal there is at a particular frequency
  - Phase encodes spatial information (indirectly)
  - For mathematical convenience, this is often notated in terms of real and complex numbers

Amplitude: $A = \pm\sqrt{R(\omega)^2 + I(\omega)^2}$ Phase: $\phi = \tan^{-1}\frac{I(\omega)}{R(\omega)}$

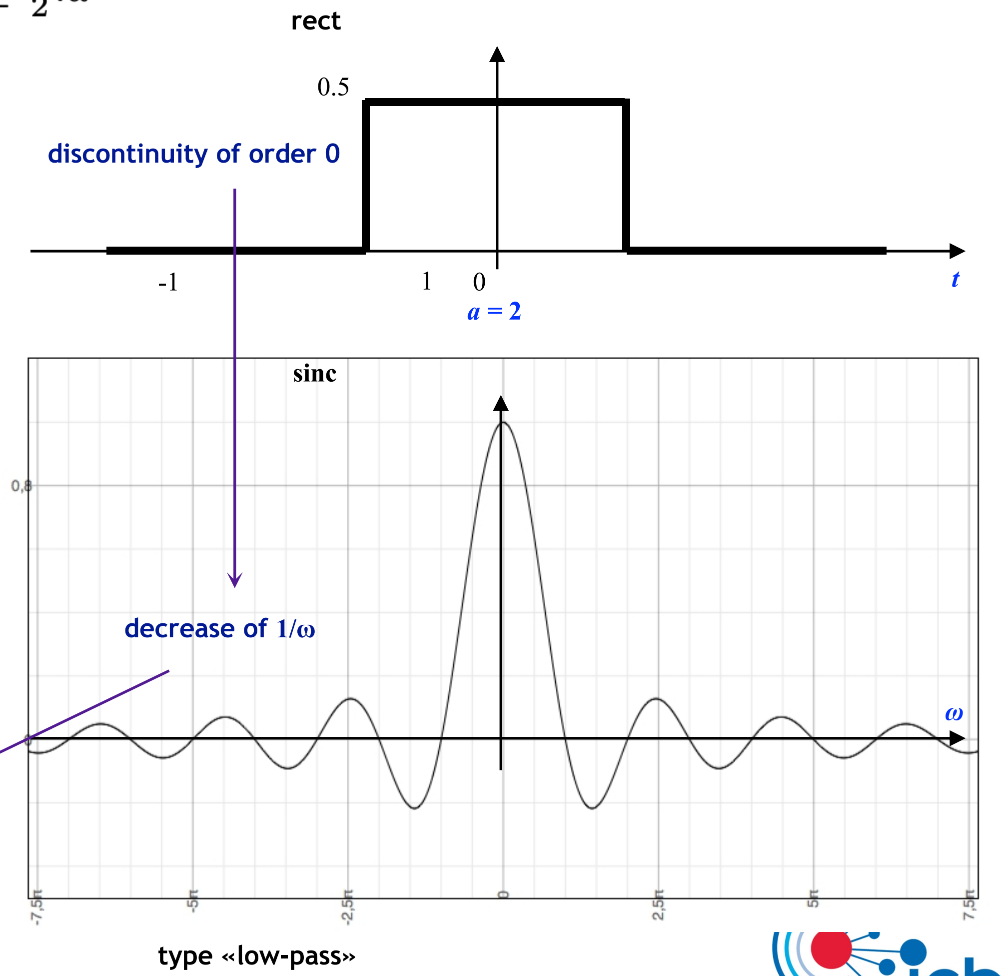
Example

- Rectangular function

$$\Pi_a(t) = \begin{cases} \frac{1}{a} & -\frac{1}{2}.a \leq t \leq \frac{1}{2}.a \\ 0 & \text{sinon} \end{cases}$$

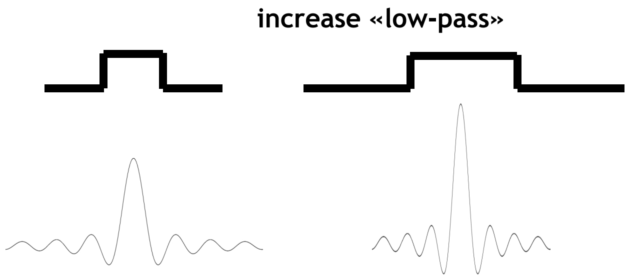
- FT

$$\begin{aligned}
\hat{\Pi}(\omega) &= < \Pi(t), e^{j\omega t} > \\
&= \int_{-\infty}^{\infty} \Pi(t) e^{-j\omega t} dt \\
&= \int_{-\frac{a}{2}}^{\frac{a}{2}} \frac{1}{a}.e^{-j\omega t} dt \\
&= \int_{-\frac{a}{2}}^{\frac{a}{2}} \frac{1}{a}.e^{-j\omega t} dt \\
&= \frac{1}{a}.\frac{1}{-j\omega}[e^{-j\omega}]_{-\frac{a}{2}}^{\frac{a}{2}} dt \\
&= \frac{1}{-\omega a}.\left(e^{-j\omega\frac{a}{2}} - e^{j\omega\frac{a}{2}}\right) dt \\
&= \frac{2}{j\omega a}.\sin\omega\frac{a}{2} \\
&= \frac{\sin\omega\frac{a}{2}}{\omega\frac{a}{2}} \\
&= \text{sinc } \omega\frac{a}{2}
\end{aligned}$$



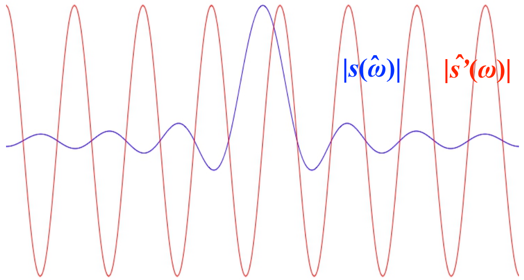
Some FT Properties

- Scaling: $$s(a.t) \longleftrightarrow \frac{1}{|a|}\hat{s}\left(\frac{\omega}{a}\right)$$



- Convolution: $$h * s(t) \longleftrightarrow \hat{h}(\omega).\hat{s}(\omega)$$

- Derivative: $$s'(t) \longleftrightarrow j\omega.\hat{s}(\omega)$$

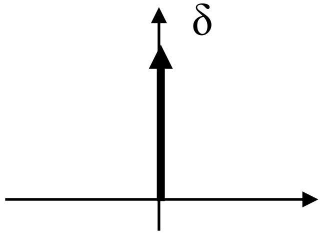


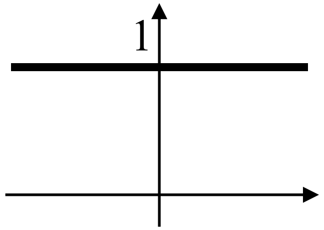
- Energy conservation:

$$\int_{-\infty}^{\infty} |s(t)|^2\, dt = \int_{-\infty}^{\infty} |\hat{s}(\omega)|^2\, d\omega$$

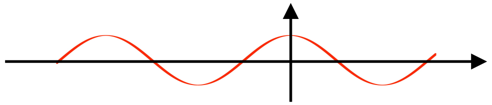
Some FT Pairs

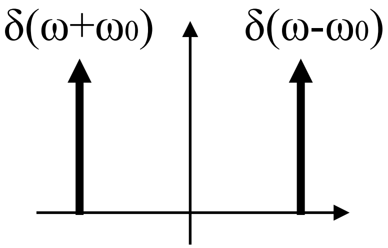
- Impulse





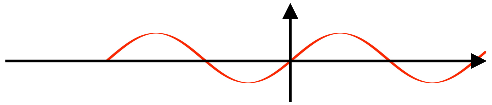
- Cosine

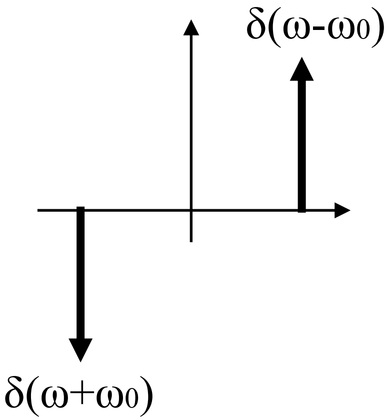




$$\cos \omega t \quad \longleftrightarrow \quad \delta(\omega + \omega_0) + \delta(\omega - \omega_0)$$
$$\sin \omega t \quad \longleftrightarrow \quad \delta(\omega + \omega_0) - \delta(\omega - \omega_0)$$

- Sine





Some FT Pairs

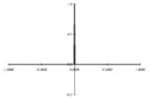
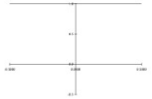
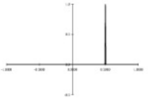
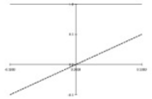
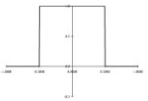
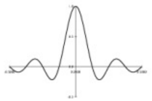
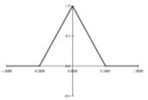
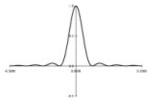
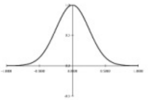
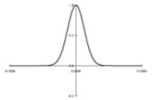
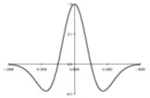
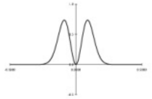
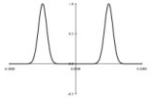
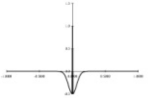
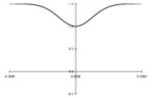
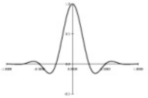
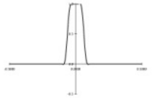
Name		Signal		Transform	
impulse		$\delta(x)$	$\Leftrightarrow$	1	
shifted impulse		$\delta(x-u)$	$\Leftrightarrow$	$e^{-j\omega u}$	
box filter		$\mathrm{box}(x/a)$	$\Leftrightarrow$	$a\mathrm{sinc}(a\omega)$	
tent		$\mathrm{tent}(x/a)$	$\Leftrightarrow$	$a\mathrm{sinc}^2(a\omega)$	
Gaussian		$G(x;\sigma)$	$\Leftrightarrow$	$\frac{\sqrt{2\pi}}{\sigma}G(\omega;\sigma^{-1})$	
Laplacian of Gaussian		$(\frac{x^2}{\sigma^4}-\frac{1}{\sigma^2})G(x;\sigma)$	$\Leftrightarrow$	$-\frac{\sqrt{2\pi}}{\sigma}\omega^2 G(\omega;\sigma^{-1})$	
Gabor		$\cos(\omega_0 x)G(x;\sigma)$	$\Leftrightarrow$	$\frac{\sqrt{2\pi}}{\sigma}G(\omega\pm\omega_0;\sigma^{-1})$	
unsharp mask		$(1+\gamma)\delta(x)$ $-\gamma G(x;\sigma)$	$\Leftrightarrow$	$(1+\gamma)-$ $\frac{\sqrt{2\pi}\gamma}{\sigma}G(\omega;\sigma^{-1})$	
windowed sinc		$\mathrm{rcos}(x/(aW))$ $\mathrm{sinc}(x/a)$	$\Leftrightarrow$	(see Figure 3.29)	

Table: Richard Szeliski, *Computer Vision and Applications*, Springer, 2010, ISBN 978-1-84882-935-0, p.137, http://szeliski.org/Book/.

Fourier Transform Pairs

Function, f(t)	Fourier Transform, F(ω)
Definition of Inverse Fourier Transform $f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{j\omega t}d\omega$	*Definition of Fourier Transform* $F(\omega)=\int_{-\infty}^{\infty}f(t)e^{-j\omega t}dt$
$f(t-t_0)$	$F(\omega)e^{-j\omega t_0}$
$f(t)e^{j\omega_0 t}$	$F(\omega-\omega_0)$
$f(\alpha t)$	$\frac{1}{\lvert\alpha\rvert}F\left(\frac{\omega}{\alpha}\right)$
$F(t)$	$2\pi f(-\omega)$
$\frac{d^n f(t)}{dt^n}$	$(j\omega)^n F(\omega)$
$(-jt)^n f(t)$	$\frac{d^n F(\omega)}{d\omega^n}$
$\int_{-\infty}^{t}f(\tau)d\tau$	$\frac{F(\omega)}{j\omega}+\pi F(0)\delta(\omega)$
$\delta(t)$	1
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
sgn (t)	$\frac{2}{j\omega}$

Function, f(t)	Fourier Transform, F(ω)
$j\frac{1}{\pi t}$	$\mathrm{sgn}(\omega)$
$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
$\sum_{n=-\infty}^{\infty}F_n e^{jn\omega_0 t}$	$2\pi\sum_{n=-\infty}^{\infty}F_n\delta(\omega-n\omega_0)$
$\mathrm{rect}\left(\frac{t}{\tau}\right)$	$\tau\mathrm{Sa}\left(\frac{\omega\tau}{2}\right)$
$\frac{B}{2\pi}\mathrm{Sa}\left(\frac{Bt}{2}\right)$	$\mathrm{rect}\left(\frac{\omega}{B}\right)$
$\mathrm{tri}(t)$	$\mathrm{Sa}^2\left(\frac{\omega}{2}\right)$
$A\cos\left(\frac{\pi t}{2\tau}\right)\mathrm{rect}\left(\frac{t}{2\tau}\right)$	$\frac{A\pi}{\tau}\frac{\cos(\omega\tau)}{\left(\pi/2\tau\right)^2-\omega^2}$
$\cos(\omega_0 t)$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
$\sin(\omega_0 t)$	$\frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
$u(t)\cos(\omega_0 t)$	$\frac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\frac{j\omega}{\omega_0^2-\omega^2}$
$u(t)\sin(\omega_0 t)$	$\frac{\pi}{2j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]+\frac{\omega^2}{\omega_0^2-\omega^2}$
$u(t)e^{-\alpha t}\cos(\omega_0 t)$	$\frac{(\alpha+j\omega)}{\omega_0^2+(\alpha+j\omega)^2}$

Exercise

3.1 Calculate the frequency representation (spectrum) of the rectangular signal $f_1(t)$, with $a = 1$:

$$f_1(t) = \begin{cases} 1/a, & \text{if } t \in [-a/2, a/2] \\ 0, & \text{otherwise} \end{cases}$$

3.2 Calculate the frequency representation (spectrum) of the signal $f_2(t)$, with $a = 1$:

$$f_2(t) = \begin{cases} 1/a, & \text{if } t \in [-a/2, 0] \\ -1/a, & \text{if } t \in [0, a/2] \\ 0, & \text{otherwise} \end{cases}$$

3.3 Plot the two spectra of the two windows. Interpret these spectra by indicating which window corresponds to a "high-pass" and which to a "low-pass".