Elec4A - Traitement du Signal

Frequential Signal Analysis

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UFR Sciences & Techniques - IEM, 2026

Agenda

- **Fourier Decomposition**
  - 1D Fourier Decomposition
    - Fourier Series (periodic signals)
    - Fourier Transform
  - Sampling and Aliasing
  - 2D Signals and Discrete Fourier Transform

Acknowledgments

- The course slides are based on materials generously made publicly available by many other people and lectures:

  Derek Hoiem (Illinois), James Hays (Georgia Tech), Andrew Zisserman (Oxford), Alexei Efros (UC Berkeley), Fabrice Meriaudeau (uB), Olivier Laligant (uB), Steve Seitz (UW)

- I might not have credits on every slide (which is bad, sorry).

Part I :

Decomposition of signals - Fourier!

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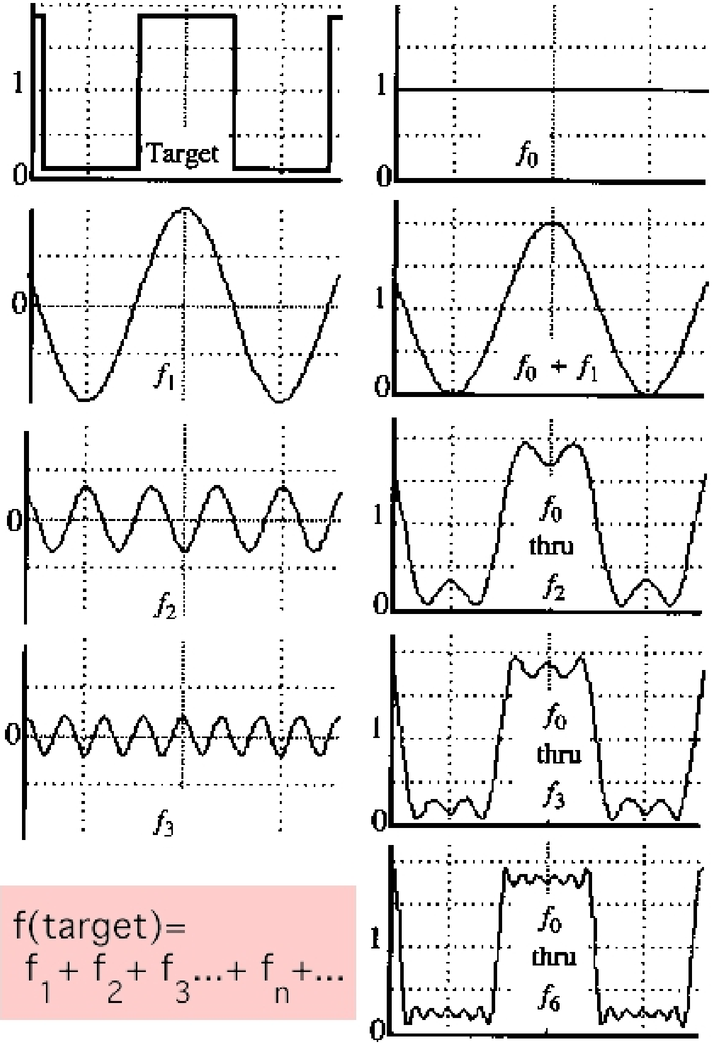
Decomposition of signals - Fourier!

A Sum of Sines

Building block of periodic functions:

$$A\sin(\omega x + \phi)$$

Add enough of them to get any signal *g(x)* you want!



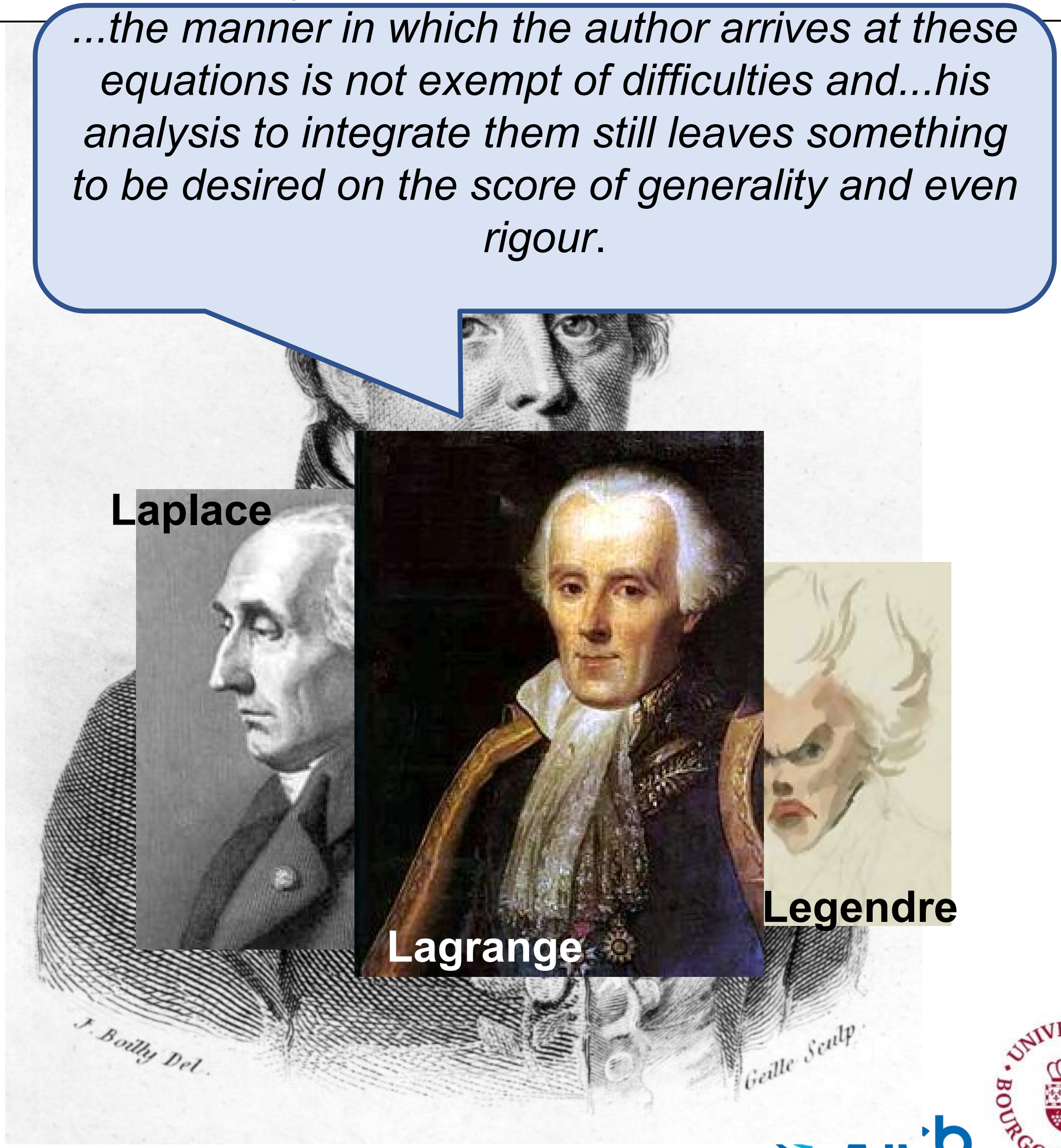
f(target)=
$f_1 + f_2 + f_3 ... + f_n + ...$

Jean Baptiste Joseph Fourier (1768-1830)

had crazy idea (1807):

> ***Any*** *univariate function can be rewritten as a weighted sum of sines and cosines of different frequencies.*

- Don't believe it?
  - Neither did Lagrange, Laplace, Poisson and other big wigs
  - Not translated into English until 1878!
- But it's (mostly) true!
  - called Fourier Series
  - there are some subtle restrictions



Jean Baptiste Joseph Fourier (1768-1830)

- Fourier was born in Auxerre! (Bourgogne)

Fourier, Joseph (1768-1830)

French mathematician who discovered that any periodic motion can be written as a superposition of sinusoidal and cosinusoidal vibrations. He developed a mathematical theory of heat in *Théorie Analytique de la Chaleur (Analytic Theory of Heat),* (1822), discussing it in terms of differential equations.

Fourier was a friend and advisor of Napoleon. Fourier believed that his health would be improved by wrapping himself up in blankets, and in this state he tripped down the stairs in his house and killed himself. The paper of Galois which he had taken home to read shortly before his death was never recovered.

SEE ALSO: Galois

Additional biographies: MacTutor (St. Andrews), Bonn

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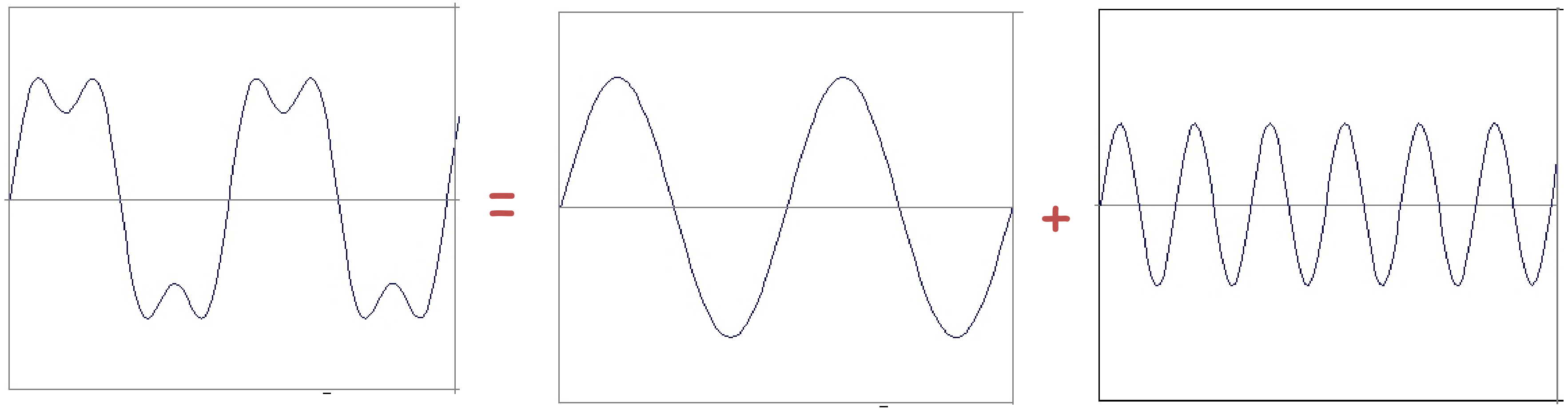
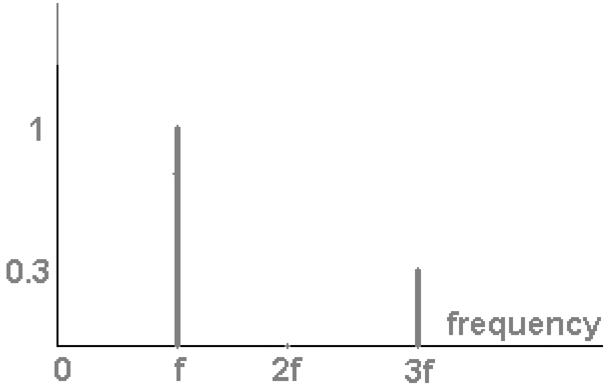
Frequency Spectra

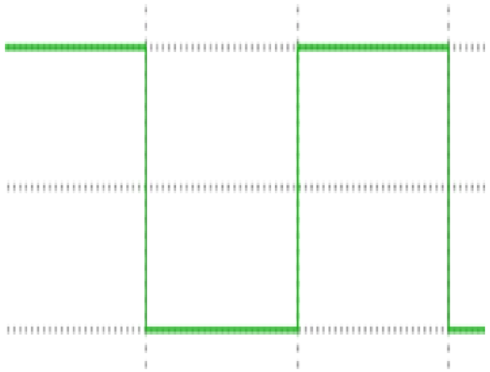
- Example : $g(t) = \sin(2\pi f\, t) + (1/3)\sin(2\pi(3f)\, t)$



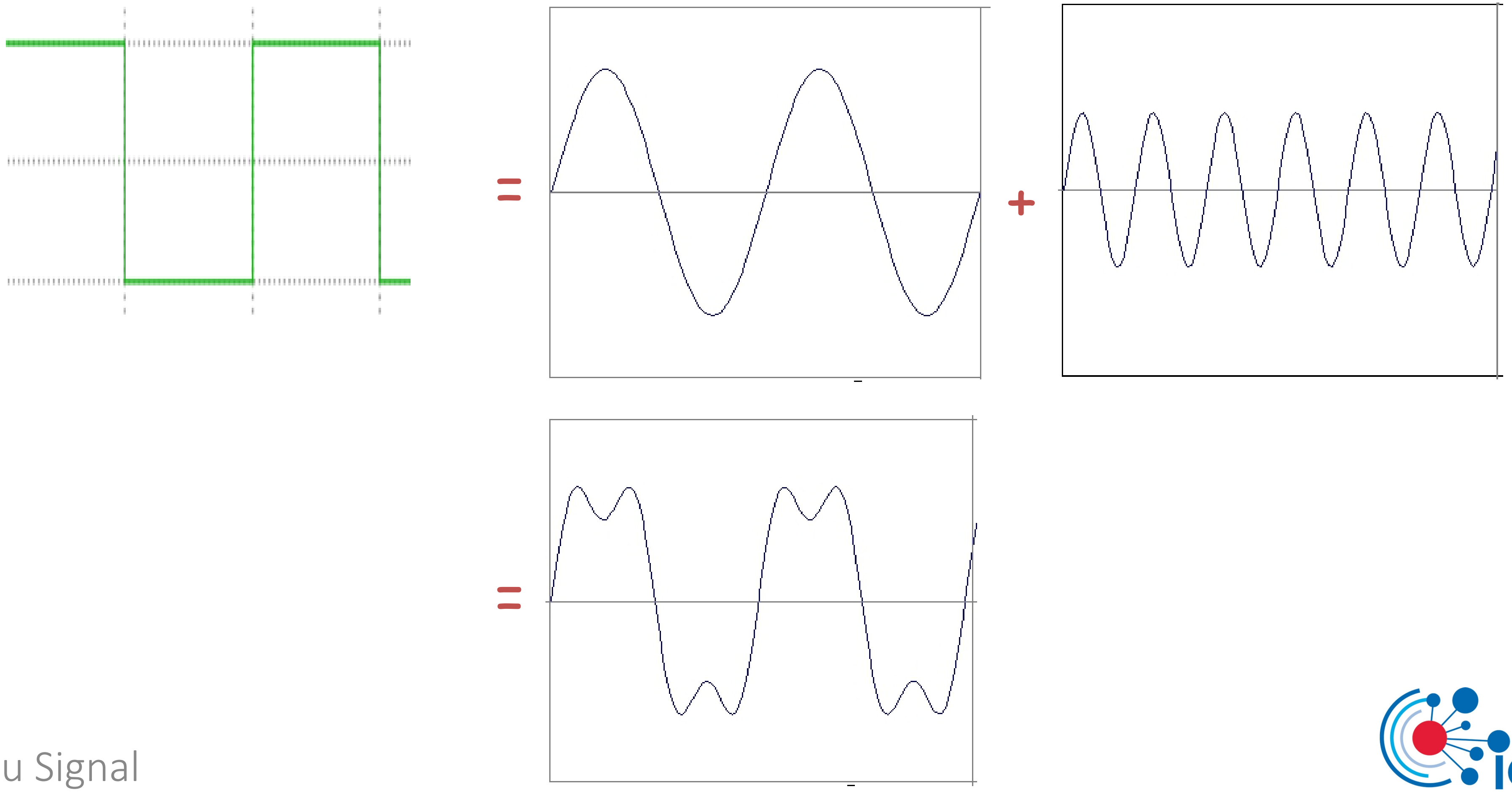


slides: Efros

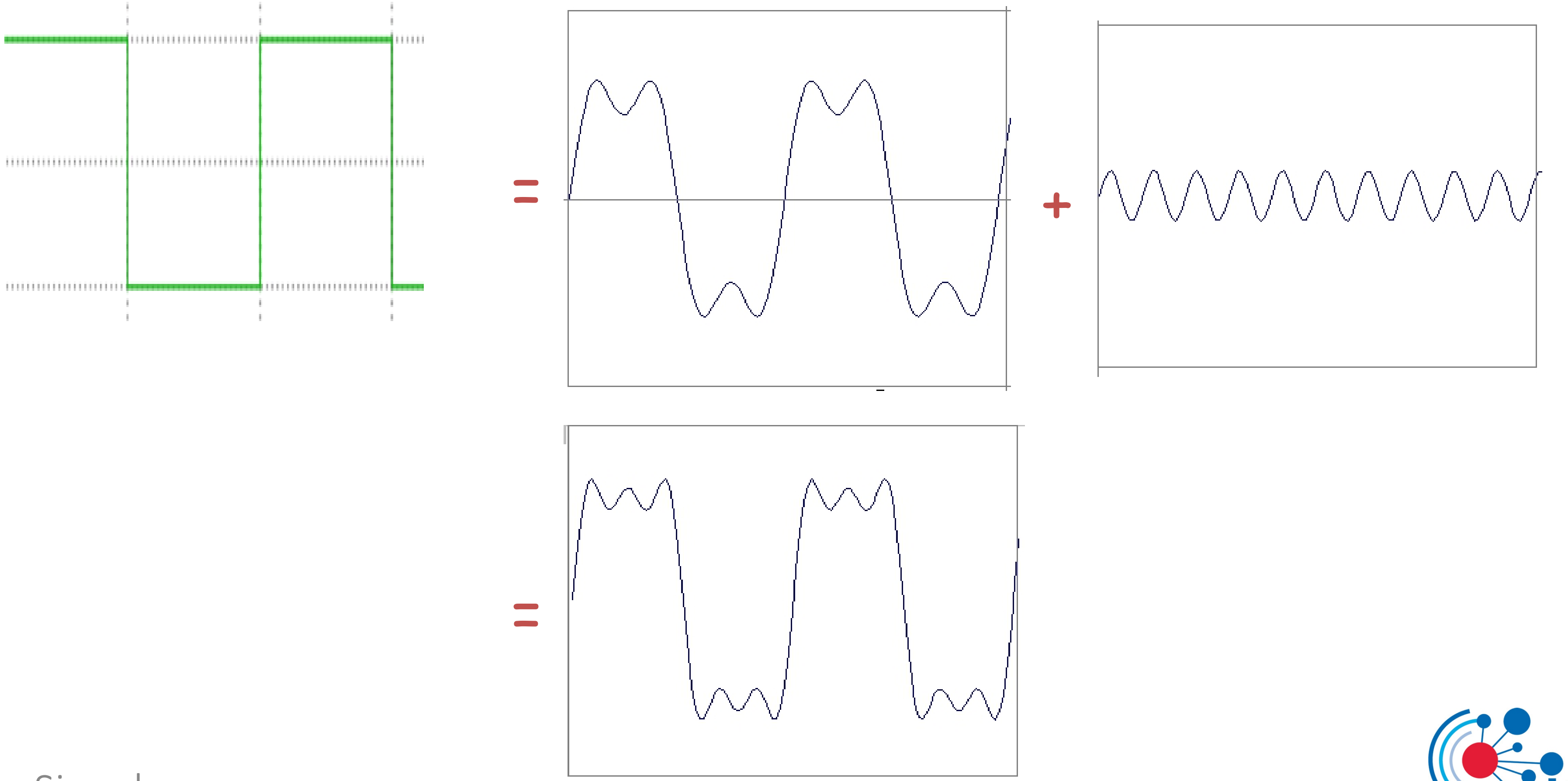
Frequency Spectra



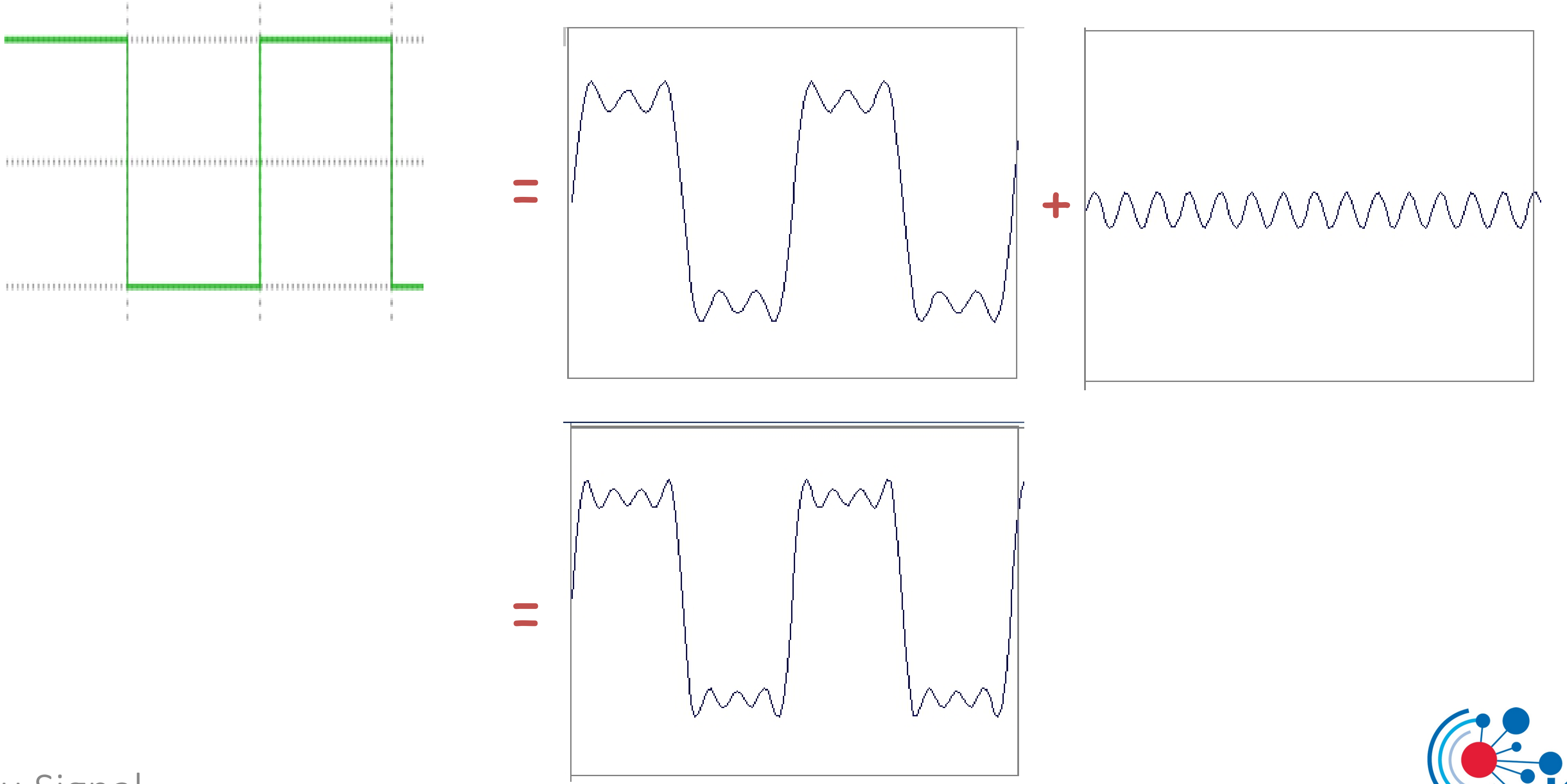
Frequency Spectra



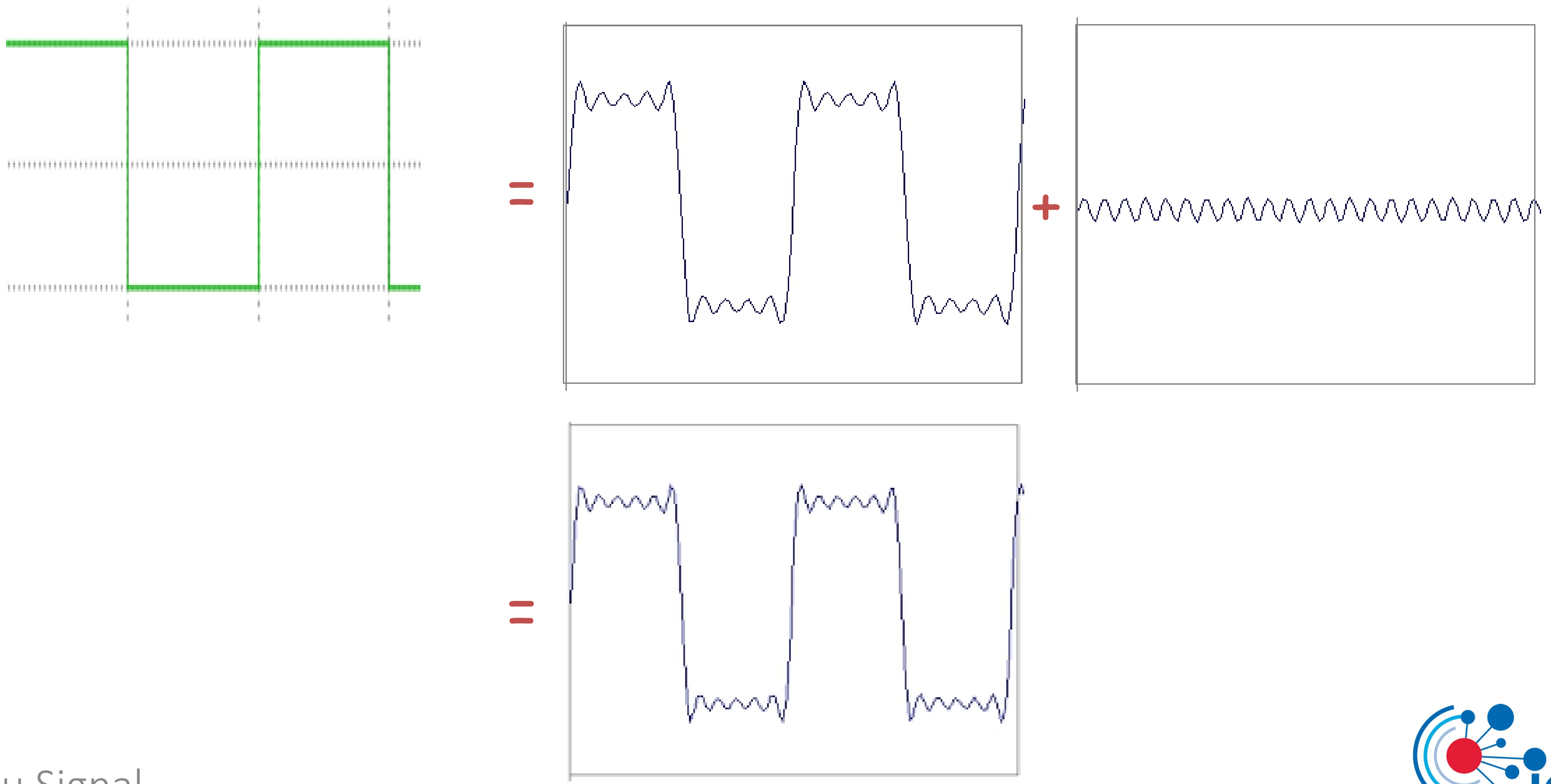
Frequency Spectra



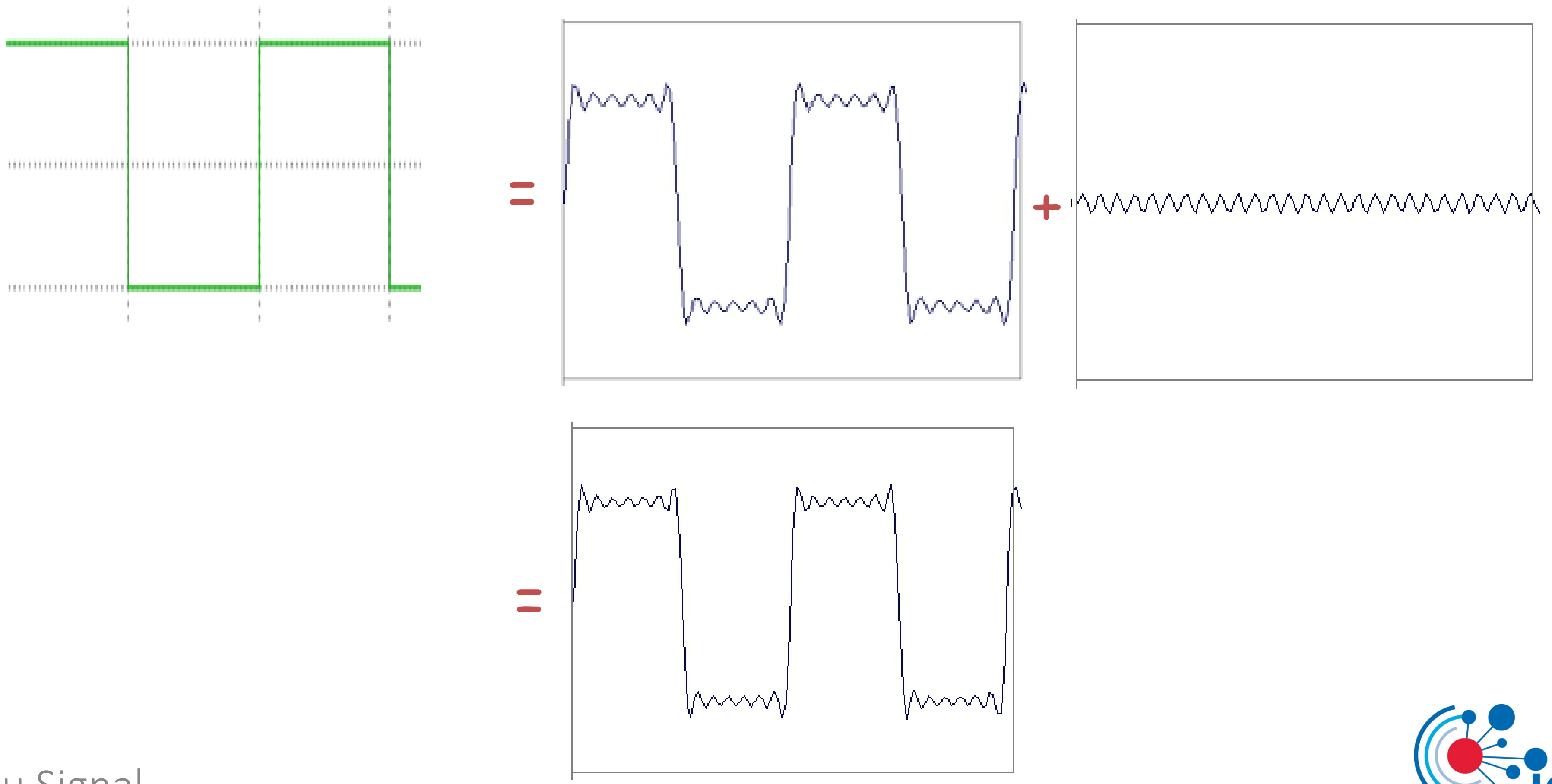
Frequency Spectra



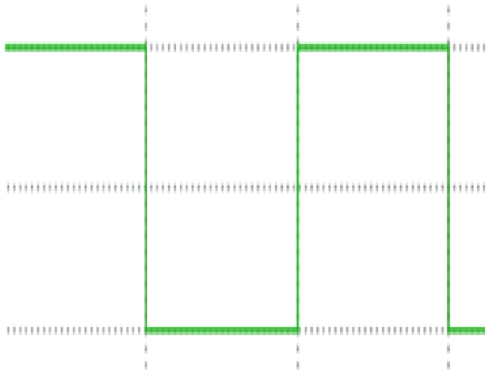
Frequency Spectra



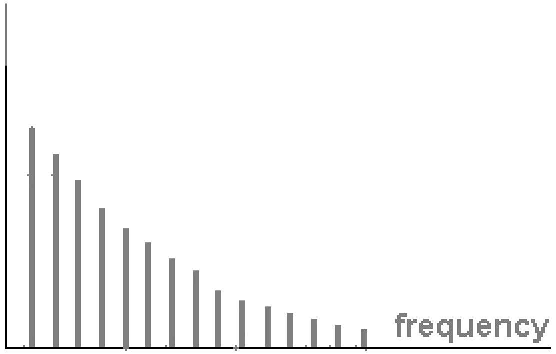
Frequency Spectra



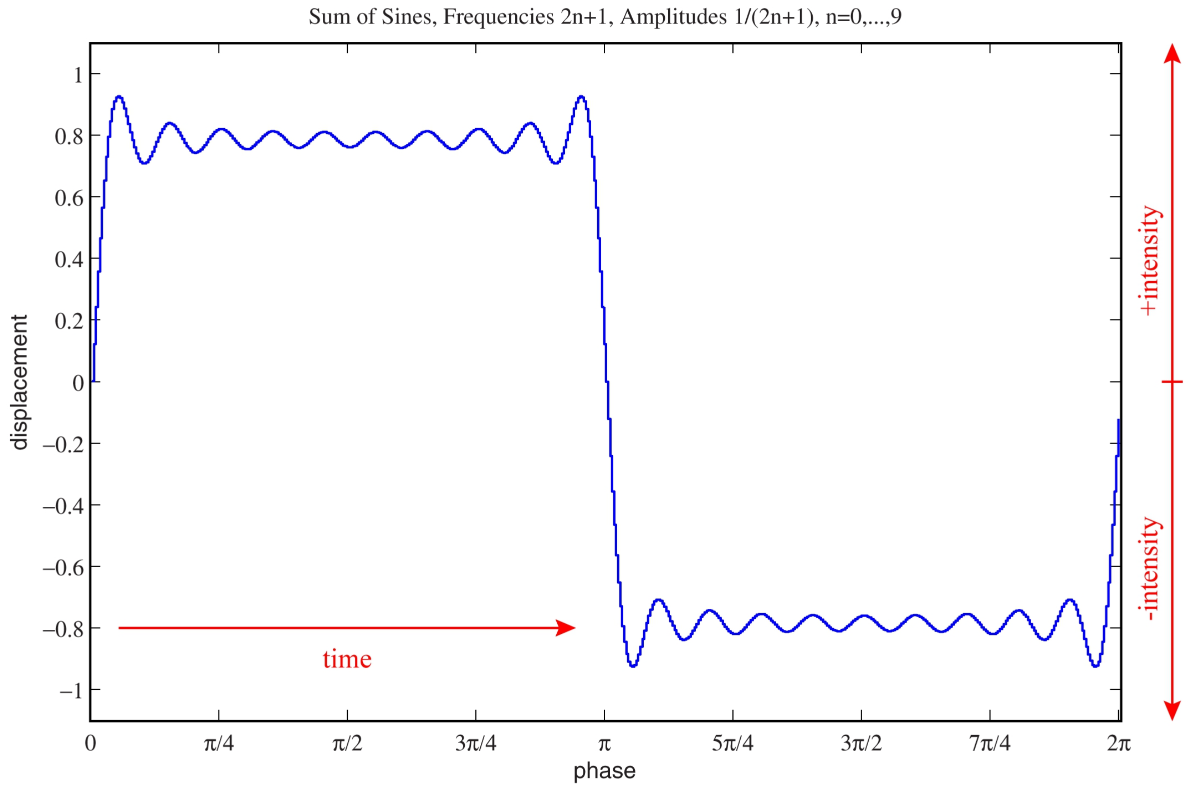
Frequency Spectra



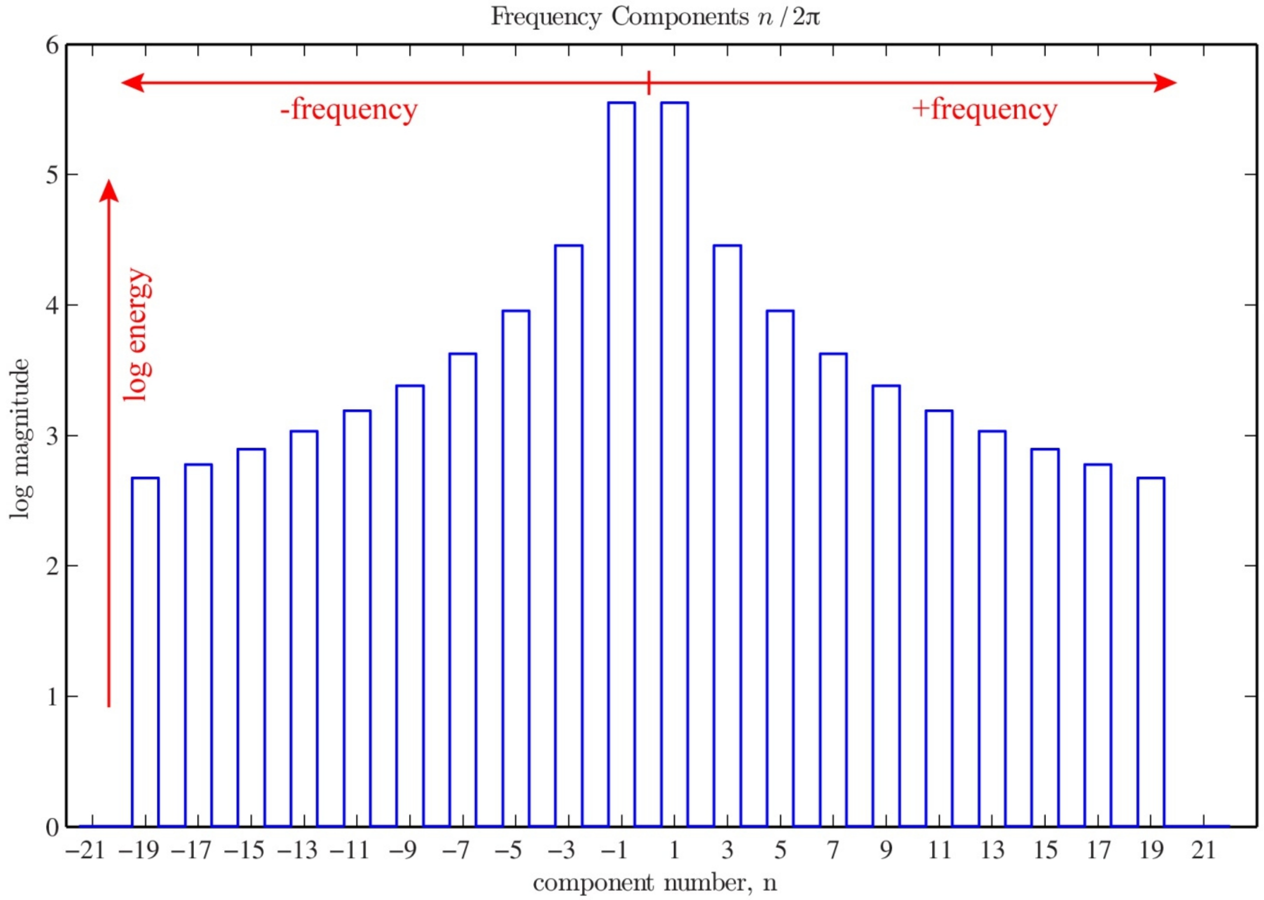
$$= \boxed{A\sum_{k=1}^{\infty}\frac{1}{k}\sin(2\pi kt)}$$



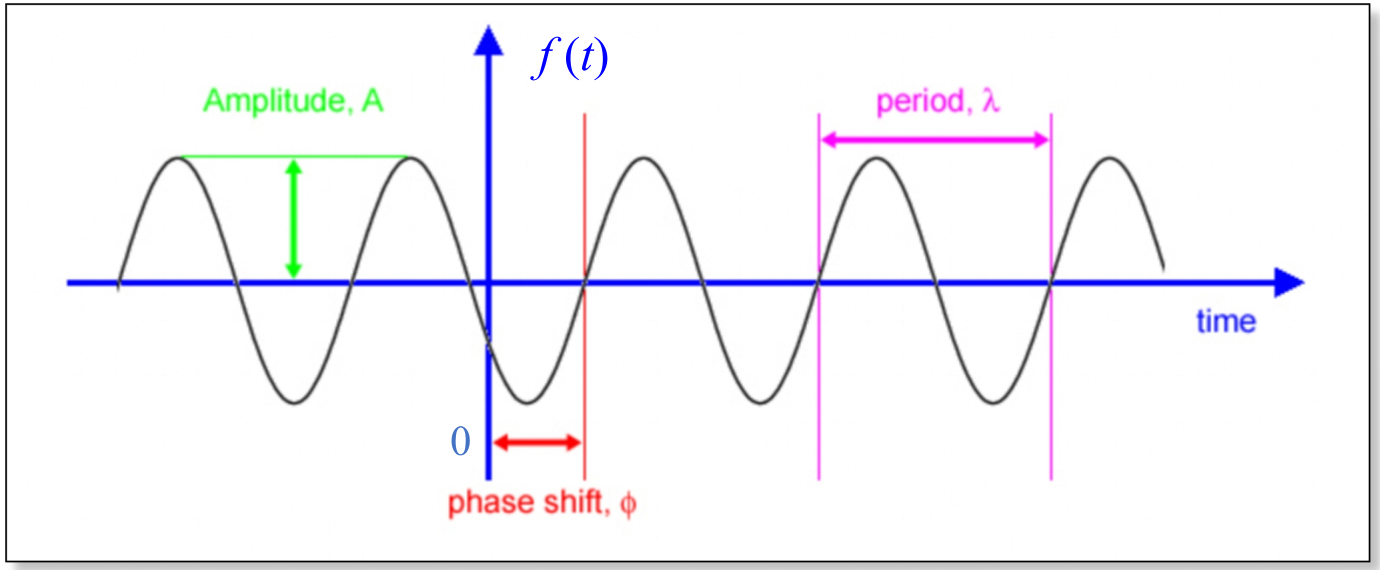
Example: Time-Domain Representation



Example: Frequency-Domain Representation



Anatomy of a Sinusoid



$$f(t) = A\sin\left(\frac{2\pi}{\lambda}t - \phi\right)$$

$1/\lambda$ is the frequency of the sinusoid (Hz).
$2\pi/\lambda$ is the angular frequency (radians/s).

Harmonics Interpretation of Periodic Signals

- Fourier series for periodic signals

  - Basis = set of complex exponentials : $\{ e^{jn\omega t} \}_{n \in [-\infty;+\infty]}$

  - $\omega = 2\pi f = 2\pi/T$, f is the fundamental frequency

  - Signal $s(t)$ can be defined as:

$$s(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n . \cos n\omega t + b_n . \sin n\omega t)$$

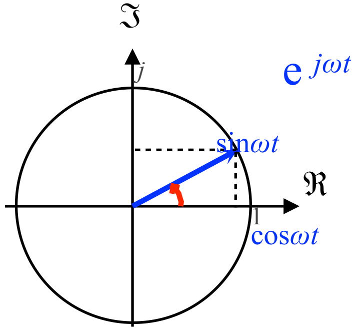
  - Or in a more generic form:

$$s(t) = \sum_{n=-\infty}^{+\infty} c_n . e^{jn\omega t}$$

$$\begin{aligned} c_n &= \frac{a_n - jb_n}{2} \\ &= \frac{d_n}{2} . (\cos\varphi + j . \sin\varphi) \\ |c_n| &= \frac{d_n}{2} \\ \arg[c_n] &= \varphi \end{aligned}$$

- Orthogonality of the basis

$$
\begin{aligned}
< e^{jn\omega t}, e^{jm\omega t} >_T &= \tfrac{1}{T}\int_T e^{jn\omega t}.e^{-jm\omega t}dt \\
&= \tfrac{1}{T}\int_T e^{j(n-m)\omega t}dt \\
&= \tfrac{1}{T}\tfrac{1}{(n-m)\omega}.\left(e^{j(n-m)2\pi}-1\right) && n \neq m \\
&= \tfrac{1}{T}\tfrac{1}{(n-m)\omega}\times 0 && n \neq m \\
&= 0 && n \neq m \\
&= 1 && n = m
\end{aligned}
$$



- Projection for $\cos \omega t$

$$
\begin{aligned}
< \cos\omega t, e^{j\omega t} >_T &= \frac{1}{T}\int_T \cos\omega t.e^{-j\omega t}dt \\
&= \frac{1}{T}\int_T \frac{e^{j\omega t}+e^{-j\omega t}}{2}.e^{-j\omega t}dt \\
&= \frac{1}{T}\int_T \frac{1+e^{-2j\omega t}}{2}.dt \\
&= \frac{1}{2T}\left\{[t]_0^T + \frac{1}{-2j\omega}\left[e^{-2j\omega t}\right]_0^T dt\right\} \\
&= \frac{1}{2T}\left\{T + \frac{1}{-2j\omega}\left(e^{-2j\omega T}-1\right) dt\right\} \\
&= \frac{1}{2T}.T = \frac{1}{2}
\end{aligned}
$$

$$
< \cos\omega t, e^{jn\omega t} >_T \;=\; 0 \quad \forall n \neq 1
$$

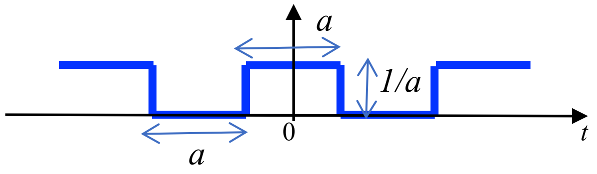
Harmonics Interpretation of Periodic Signals

- Signal is real than basis can be written as $\{\cos n\omega t, \sin n\omega t\}$n $\in [0;+\infty]$:

$$
\begin{aligned}
a_0 &= \frac{2}{T}\int_T s(t)dt \\
&= 2.c_0 \\
b_0 &= 0 \\
a_n &= \frac{2}{T}\int_T s(t)\cos(n\omega t)dt \ , \quad n \geq 1 \\
b_n &= \frac{2}{T}\int_T s(t)\sin(n\omega t)dt \ , \quad n \geq 1
\end{aligned}
$$

a_0 : 2 x mean of the signal

- **(Exercise)** Find the decomposition of "even" wave square signal

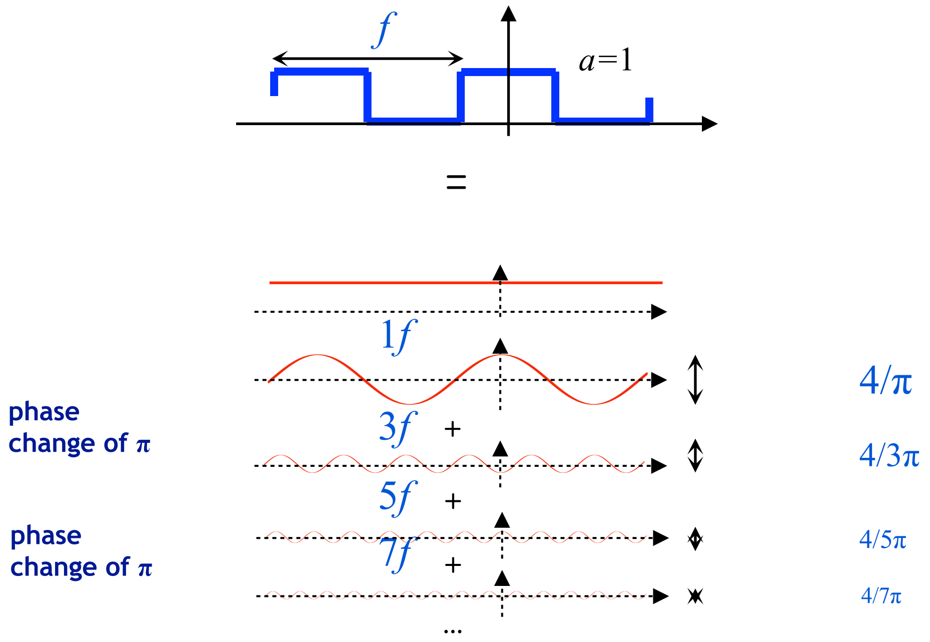


Harmonics Interpretation of Periodic Signals

- Example with “even” squared signal

$$
\begin{aligned}
a_0 &= 2 \times 0.5 \qquad (\bar{s}(t) = 0.5) \\
b_0 &= 0 \\
a_n &= 2a. \times \frac{\sin n\frac{\pi}{2}}{n\frac{\pi}{2}} \;,\quad n \geq 1 \\
b_n &= 0
\end{aligned}
$$

$$
\begin{cases}
a_1 = 2.\frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} = 2 \times 1.\frac{1}{\frac{\pi}{2}} = \frac{4}{\pi} \\
a_2 = 2.\frac{\sin 2.\frac{\pi}{2}}{2.\frac{\pi}{2}} = 0 \\
a_3 = 2.\frac{\sin 3.\frac{\pi}{2}}{3.\frac{\pi}{2}} = -\frac{1}{3}.\frac{4}{\pi} \\
a_4 = 2.\frac{\sin 4.\frac{\pi}{2}}{4.\frac{\pi}{2}} = 0 \\
a_5 = 2.\frac{\sin 5.\frac{\pi}{2}}{5.\frac{\pi}{2}} = \frac{1}{5}.\frac{4}{\pi} \\
\ldots
\end{cases}
$$



Harmonics Interpretation of Periodic Signals

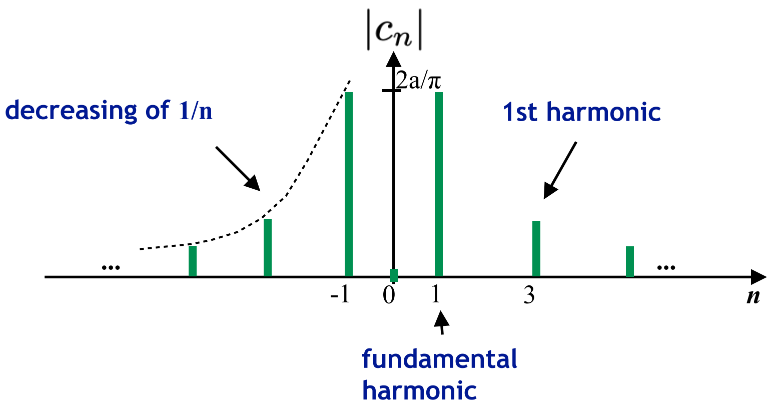
- Projection in one period:

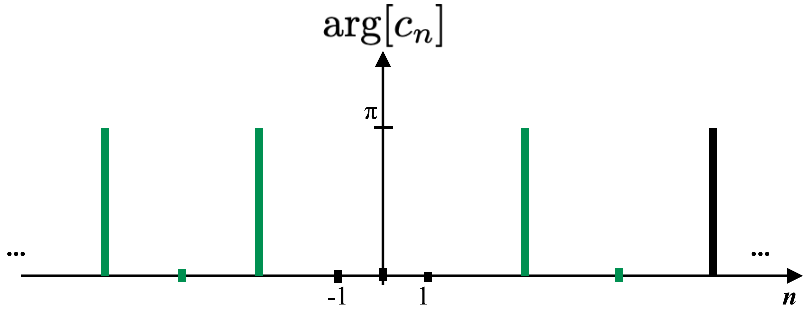
$$
\begin{aligned}
c_n &= \frac{1}{T}\int_T s(t)\overline{e^{jn\omega t}}dt \\
&= \frac{1}{T}\int_T s(t)e^{-jn\omega t}dt \\
&= \frac{1}{T}\int_0^T s(t)e^{-jn\omega t}dt \\
&= \frac{1}{T}\int_a^{a+T} s(t)e^{-jn\omega t}dt, \quad a \in \mathbb{R} \\
&\dots
\end{aligned}
$$

c_0 : mean of the signal

$$
c_n = a.\frac{\sin n\frac{\pi}{2}}{n\frac{\pi}{2}}, \; n \neq 0
$$

- Spectra and phase of Fourier coefficients:





Properties of Fourier Series

- Even signal
  - $a_n \neq 0$ in general
  - $b_n = 0$
- Odd signal
  - $a_n = 0$ in general
  - $b_n \neq 0$

- Regularity of the signal and coefficients decrease:

  - Discontinuity of order 0 (square signal, sawtooth wave, ...): decrease of $1/n$
  - Discontinuity of order 1 (triangle wave): decrease of $1/n^2$
  - ...
  - Decrease of coefficients depends on the regularity of the signal

Summary for Periodic Signals

- In a nutshell:

Fourier series is the decomposition of a periodic signal into a sum of sinusoids.

$$
\begin{aligned}
s(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n . \cos n\omega t + b_n . \sin n\omega t) \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} d_n . \cos (n\omega t + \varphi) \\
&= \sum_{n=-\infty}^{+\infty} c_n . e^{jn\omega t}
\end{aligned}
$$

$$
\begin{aligned}
c_n &= \frac{a_n - jb_n}{2} \\
&= \frac{d_n}{2} . (\cos\varphi + j . \sin\varphi) \\
|c_n| &= \frac{d_n}{2} \\
\arg[c_n] &= \varphi
\end{aligned}
$$

Fourier Transform

- **Fourier transform (FT)** is the decomposition of a ***nonperiodic*** **signal** into a continuous sum* (integral) of sinusoids
- The spectra can have any frequency in $\mathbb{R}$
- Projection or decomposition (FT):

$$\begin{aligned}\hat{s}(\omega) &= < s(t), e^{j\omega t} > \\ &= \int_{-\infty}^{\infty} s(t) e^{-j\omega t} dt\end{aligned}$$

- Synthesis or reconstruction (FT^{-1}):

$$\begin{aligned}s(t) &= \frac{1}{2\pi} < \hat{s}(\omega), e^{-j\omega t} > \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} s(\omega) e^{j\omega t} d\omega\end{aligned}$$

Fourier Transform

- Fourier transform stores the magnitude and phase at each frequency
  - Magnitude encodes how much signal there is at a particular frequency
  - Phase encodes spatial information (indirectly)
  - For mathematical convenience, this is often notated in terms of real and complex numbers

Amplitude: $$A = \pm\sqrt{R(\omega)^2 + I(\omega)^2}$$

Phase: $$\phi = \tan^{-1}\frac{I(\omega)}{R(\omega)}$$

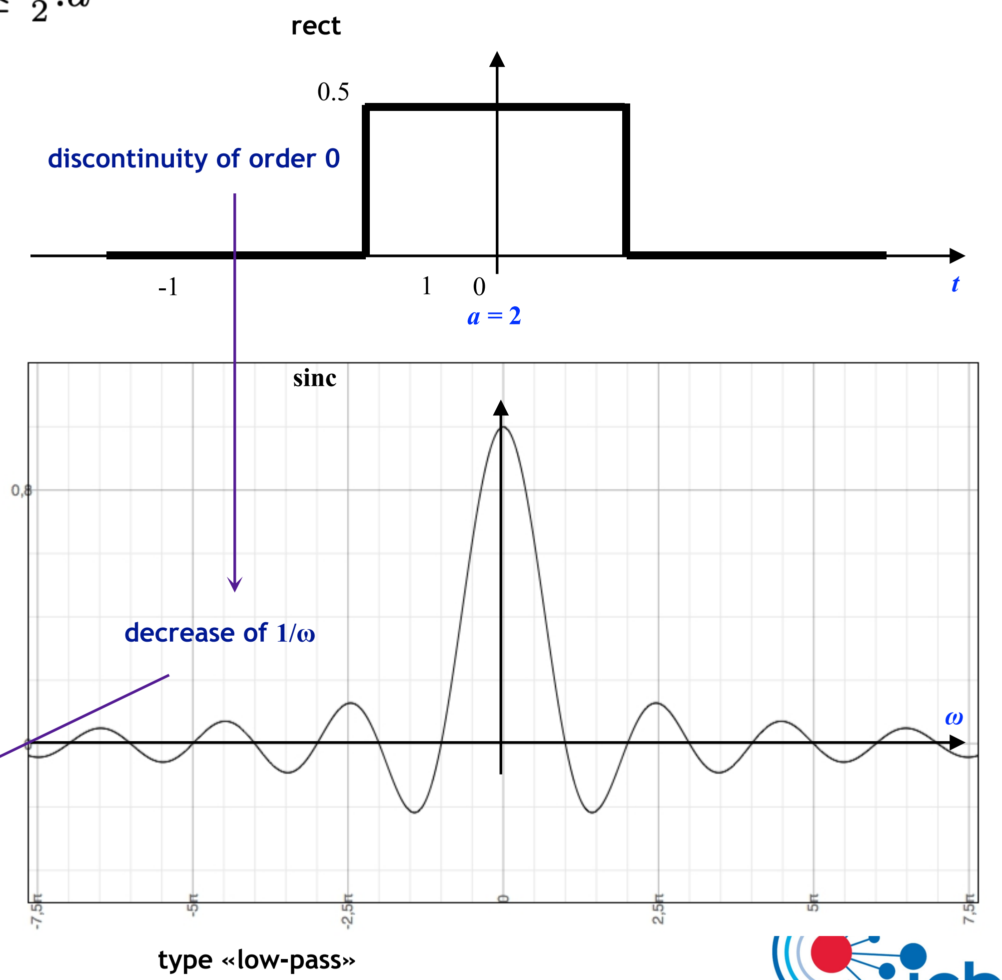
Example

- Rectangular function

$$\Pi_a(t) \quad = \quad \begin{cases} \frac{1}{a} & -\frac{1}{2}.a \leq t \leq \frac{1}{2}.a \\ 0 & \text{sinon} \end{cases}$$

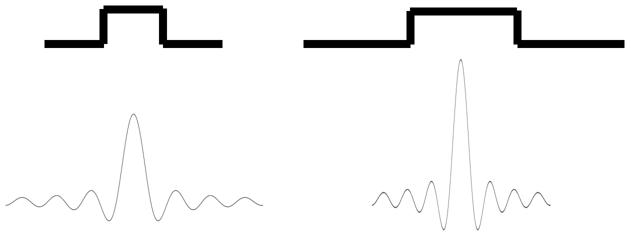
- FT

$$\begin{aligned}
\hat{\Pi}(\omega) &= <\Pi(t), e^{j\omega t}> \\
&= \int_{-\infty}^{\infty} \Pi(t) e^{-j\omega t} dt \\
&= \int_{-\frac{a}{2}}^{\frac{a}{2}} \frac{1}{a}.e^{-j\omega t} dt \\
&= \int_{-\frac{a}{2}}^{\frac{a}{2}} \frac{1}{a}.e^{-j\omega t} dt \\
&= \frac{1}{a}.\frac{1}{-j\omega}[e^{-j\omega}]_{-\frac{a}{2}}^{\frac{a}{2}} dt \\
&= \frac{1}{-\omega a}.\left(e^{-j\omega \frac{a}{2}} - e^{j\omega \frac{a}{2}}\right) dt \\
&= \frac{2}{j\omega a}.\sin \omega \frac{a}{2} \\
&= \frac{\sin \omega \frac{a}{2}}{\omega \frac{a}{2}} \\
&= \text{sinc } \omega \frac{a}{2}
\end{aligned}$$



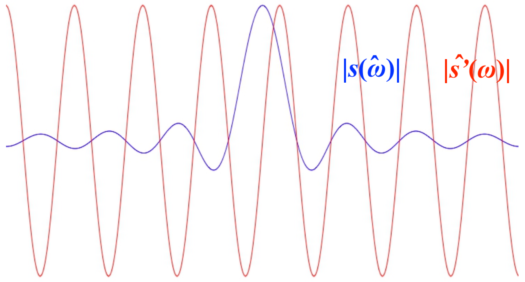
Some FT Properties

- Scaling: $$s(a.t) \longleftrightarrow \frac{1}{|a|}\hat{s}\left(\frac{\omega}{a}\right)$$



- Convolution: $$h * s(t) \longleftrightarrow \hat{h}(\omega).\hat{s}(\omega)$$

- Derivative: $$s'(t) \longleftrightarrow j\omega.\hat{s}(\omega)$$

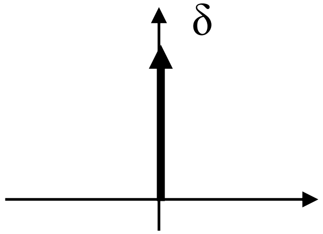


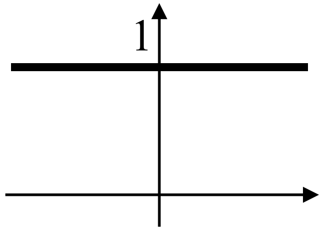
- Energy conservation:

$$\int_{-\infty}^{\infty} |s(t)|^2 \, dt = \int_{-\infty}^{\infty} |\hat{s}(\omega)|^2 \, d\omega$$

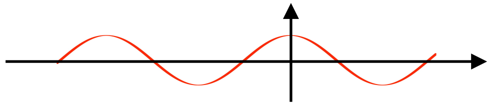
Some FT Pairs

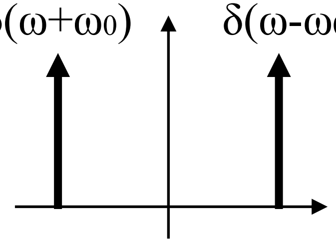
- Impulse





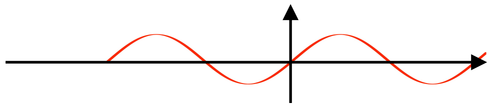
- Cosine

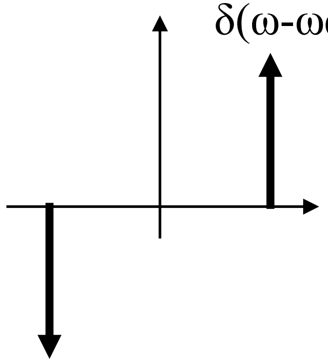




$$\cos \omega t \quad \longleftrightarrow \quad \delta(\omega+\omega_0)+\delta(\omega-\omega_0)$$
$$\sin \omega t \quad \longleftrightarrow \quad \delta(\omega+\omega_0)-\delta(\omega-\omega_0)$$

- Sine





Some FT Pairs

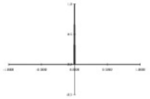
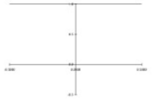
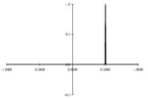
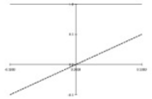
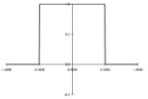
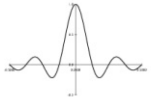
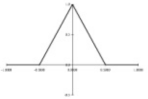
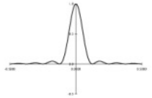
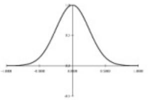
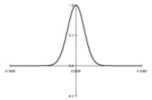
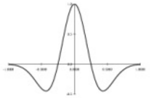
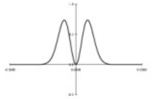
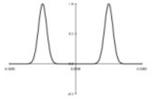
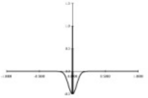
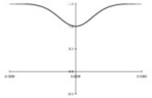
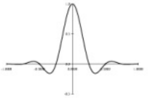
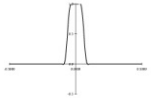
Name		Signal		Transform	
impulse		$\delta(x)$	$\Leftrightarrow$	1	
shifted impulse		$\delta(x-u)$	$\Leftrightarrow$	$e^{-j\omega u}$	
box filter		$\mathrm{box}(x/a)$	$\Leftrightarrow$	$a\mathrm{sinc}(a\omega)$	
tent		$\mathrm{tent}(x/a)$	$\Leftrightarrow$	$a\mathrm{sinc}^2(a\omega)$	
Gaussian		$G(x;\sigma)$	$\Leftrightarrow$	$\frac{\sqrt{2\pi}}{\sigma}G(\omega;\sigma^{-1})$	
Laplacian of Gaussian		$(\frac{x^2}{\sigma^4}-\frac{1}{\sigma^2})G(x;\sigma)$	$\Leftrightarrow$	$-\frac{\sqrt{2\pi}}{\sigma}\omega^2 G(\omega;\sigma^{-1})$	
Gabor		$\cos(\omega_0 x)G(x;\sigma)$	$\Leftrightarrow$	$\frac{\sqrt{2\pi}}{\sigma}G(\omega\pm\omega_0;\sigma^{-1})$	
unsharp mask		$(1+\gamma)\delta(x)$ $-\gamma G(x;\sigma)$	$\Leftrightarrow$	$(1+\gamma)-$ $\frac{\sqrt{2\pi}\gamma}{\sigma}G(\omega;\sigma^{-1})$	
windowed sinc		$\mathrm{rcos}(x/(aW))$ $\mathrm{sinc}(x/a)$	$\Leftrightarrow$	(see Figure 3.29)	

Table: Richard Szeliski, *Computer Vision and Applications*, Springer, 2010, ISBN 978-1-84882-935-0, p.137, http://szeliski.org/Book/.

Fourier Transform Pairs

Function, f(t)	Fourier Transform, F(ω)
Definition of Inverse Fourier Transform $f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{j\omega t}d\omega$	*Definition of Fourier Transform* $F(\omega)=\int_{-\infty}^{\infty}f(t)e^{-j\omega t}dt$
$f(t-t_0)$	$F(\omega)e^{-j\omega t_0}$
$f(t)e^{j\omega_0 t}$	$F(\omega-\omega_0)$
$f(\alpha t)$	$\frac{1}{\lvert\alpha\rvert}F\left(\frac{\omega}{\alpha}\right)$
$F(t)$	$2\pi f(-\omega)$
$\frac{d^n f(t)}{dt^n}$	$(j\omega)^n F(\omega)$
$(-jt)^n f(t)$	$\frac{d^n F(\omega)}{d\omega^n}$
$\int_{-\infty}^{t}f(\tau)d\tau$	$\frac{F(\omega)}{j\omega}+\pi F(0)\delta(\omega)$
$\delta(t)$	1
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
sgn (t)	$\frac{2}{j\omega}$

Function, f(t)	Fourier Transform, F(ω)
$j\frac{1}{\pi t}$	$\mathrm{sgn}(\omega)$
$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
$\sum_{n=-\infty}^{\infty}F_n e^{jn\omega_0 t}$	$2\pi\sum_{n=-\infty}^{\infty}F_n\delta(\omega-n\omega_0)$
$\mathrm{rect}\left(\frac{t}{\tau}\right)$	$\tau\mathrm{Sa}\left(\frac{\omega\tau}{2}\right)$
$\frac{B}{2\pi}\mathrm{Sa}\left(\frac{Bt}{2}\right)$	$\mathrm{rect}\left(\frac{\omega}{B}\right)$
$\mathrm{tri}(t)$	$\mathrm{Sa}^2\left(\frac{\omega}{2}\right)$
$A\cos\left(\frac{\pi t}{2\tau}\right)\mathrm{rect}\left(\frac{t}{2\tau}\right)$	$\frac{A\pi}{\tau}\frac{\cos(\omega\tau)}{\left(\pi/2\tau\right)^2-\omega^2}$
$\cos(\omega_0 t)$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
$\sin(\omega_0 t)$	$\frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
$u(t)\cos(\omega_0 t)$	$\frac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\frac{j\omega}{\omega_0^2-\omega^2}$
$u(t)\sin(\omega_0 t)$	$\frac{\pi}{2j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]+\frac{\omega^2}{\omega_0^2-\omega^2}$
$u(t)e^{-\alpha t}\cos(\omega_0 t)$	$\frac{(\alpha+j\omega)}{\omega_0^2+(\alpha+j\omega)^2}$

Exercise

3.1 Calculate the frequency representation (spectrum) of the rectangular signal $f_1(t)$, with $a = 1$:

$$f_1(t) = \begin{cases} 1/a, & \text{if } t \in [-a/2, a/2] \\ 0, & \text{otherwise} \end{cases}$$

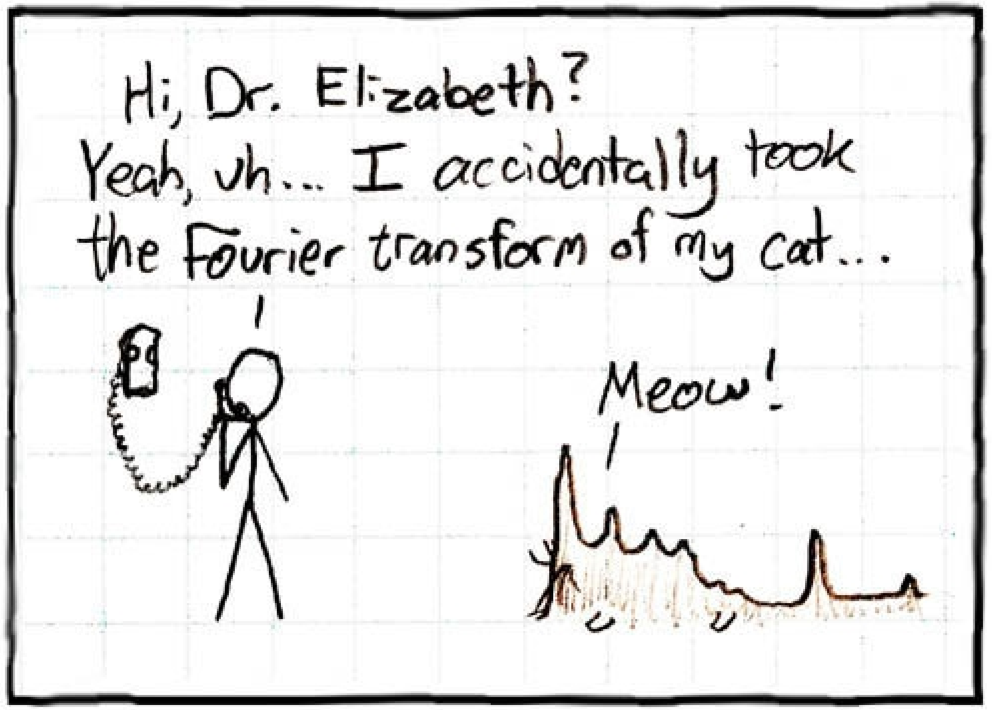
3.2 Calculate the frequency representation (spectrum) of the signal $f_2(t)$, with $a = 1$:

$$f_2(t) = \begin{cases} 1/a, & \text{if } t \in [-a/2, 0] \\ -1/a, & \text{if } t \in [0, a/2] \\ 0, & \text{otherwise} \end{cases}$$

3.3 Plot the two spectra of the two windows. Interpret these spectra by indicating which window corresponds to a "high-pass" and which to a "low-pass".

Other signals

- We can also think of all kinds of other signals the same way



xkcd.com

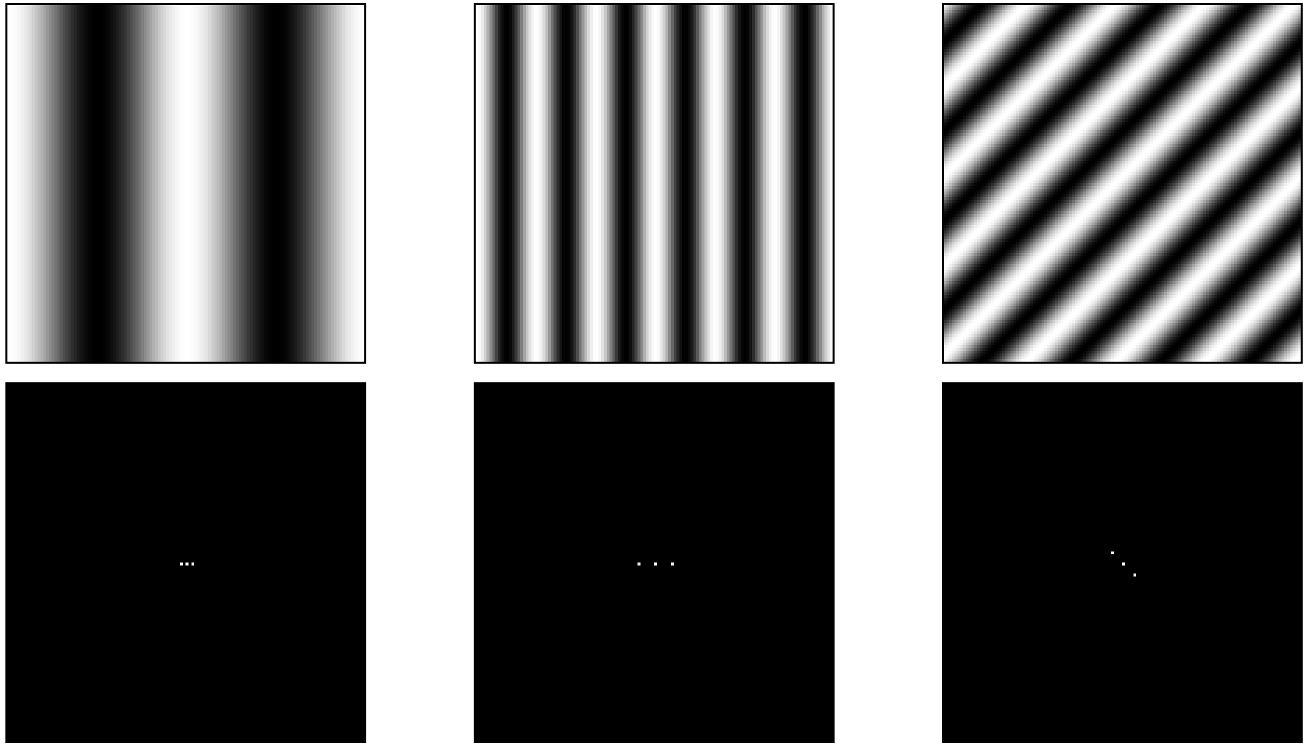
Dates

- Controle continu - CC : 03/03/2026

Fourier Analysis in Images

Intensity Image

Fourier Image



http://sharp.bu.edu/~slehar/fourier/fourier.html#filtering

2D Fourier Transform

Let $\mathbf{I}(r,c)$ be a single-band (intensity) digital image with R rows and C columns. Then, $\mathbf{I}(r,c)$ has Fourier representation

$$\mathbf{I}(r,c)=\frac{1}{RC}\sum_{u=0}^{R-1}\sum_{v=0}^{C-1}\mathbf{I}(v,u)e^{+i2\pi\left(\frac{vr}{R}+\frac{uc}{C}\right)},$$

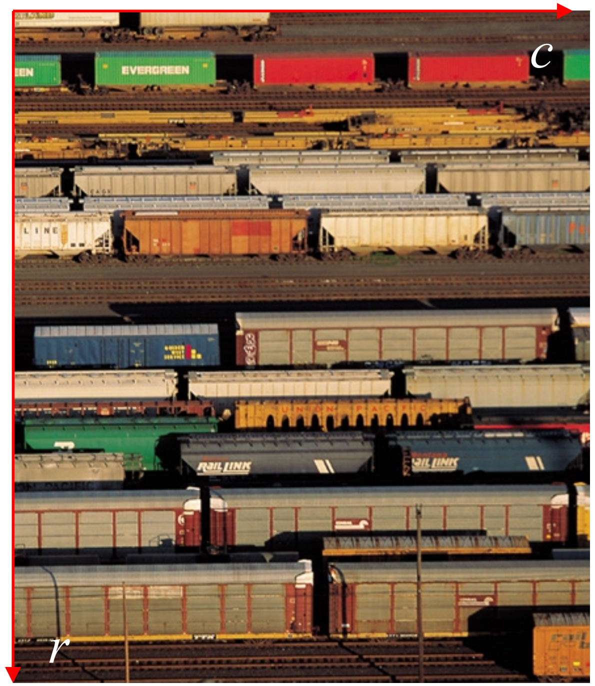
these complex exponentials are 2D sinusoids.

where

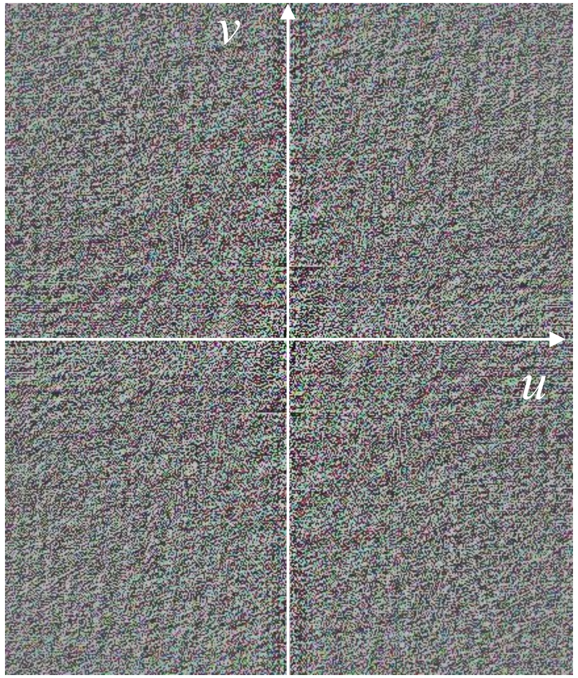
$$\mathbf{I}(v,u)=\sum_{r=0}^{R-1}\sum_{c=0}^{C-1}\mathbf{I}(r,c)e^{-i2\pi\left(\frac{vr}{R}+\frac{uc}{C}\right)}$$

are the R x C Fourier coefficients.

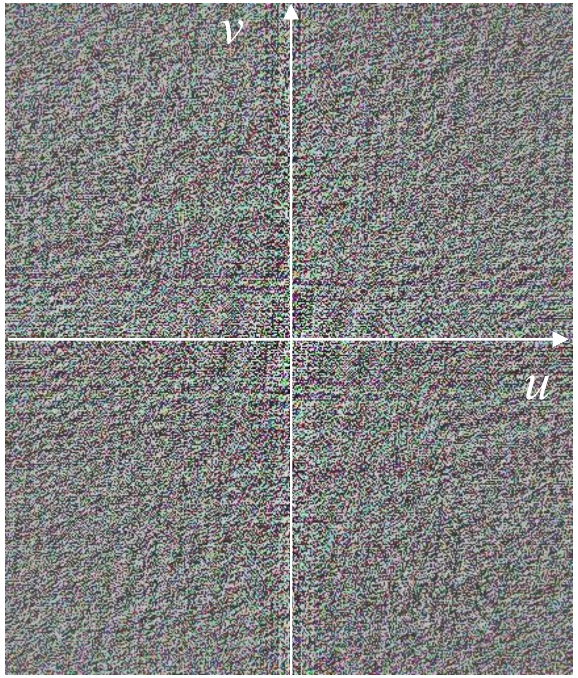
Fourier Transform of Images



$\mathbf{I}$



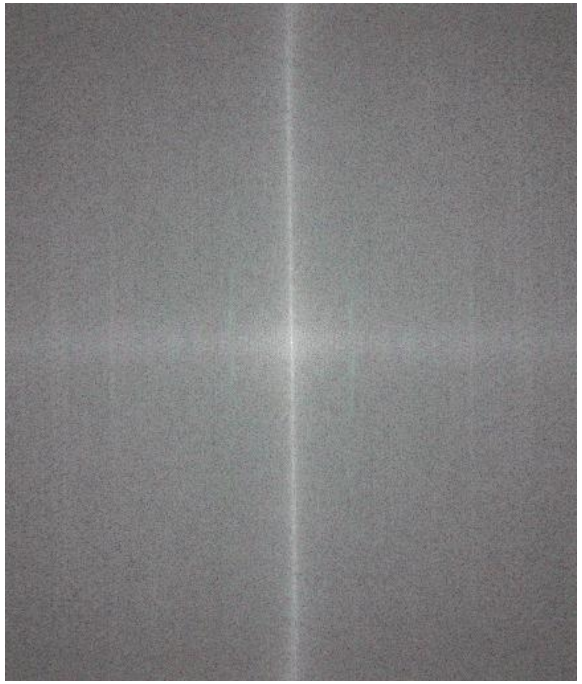
$\mathrm{Re}[\mathcal{F}\{\mathbf{I}\}]$



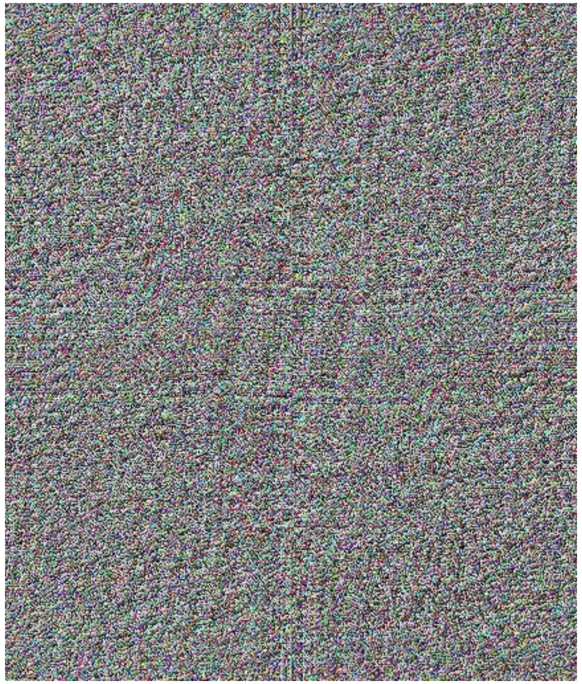
$\mathrm{Im}[\mathcal{F}\{\mathbf{I}\}]$

FT of an Image (Magnitude + Phase)

$\mathbf{I}$



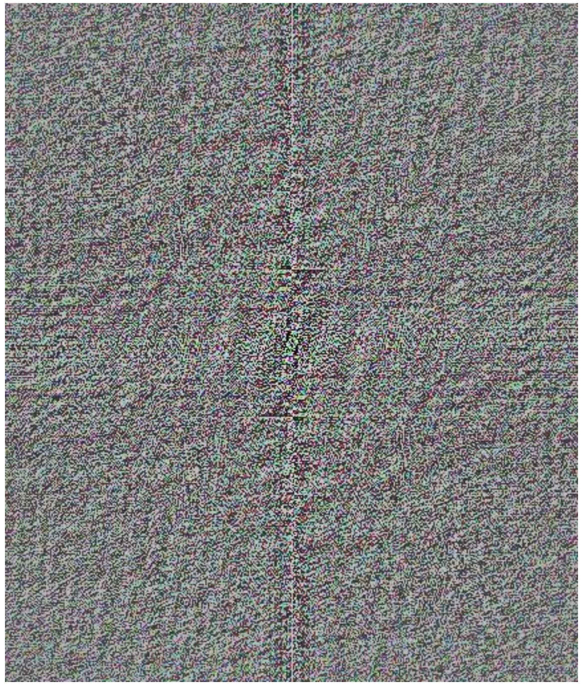
$\log\{|\mathcal{F}\{\mathbf{I}\}|^2+1\}$



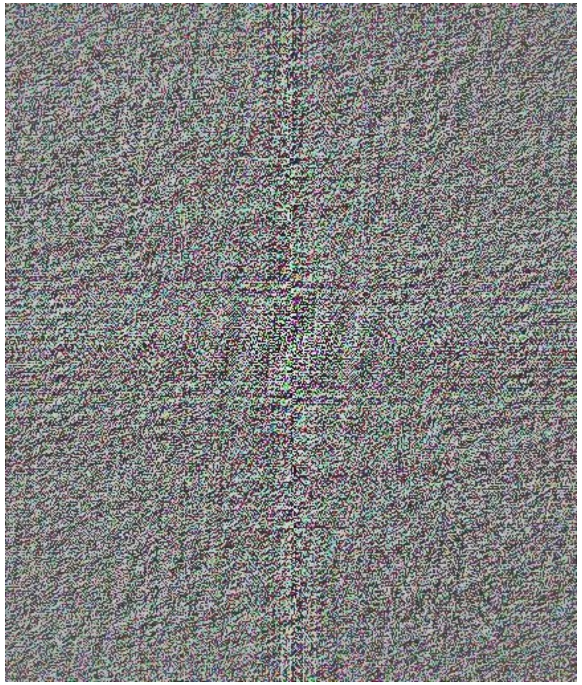
$\angle[\mathcal{F}\{\mathbf{I}\}]$

FT of an Image (Real + Imaginary)

$\mathbf{I}$

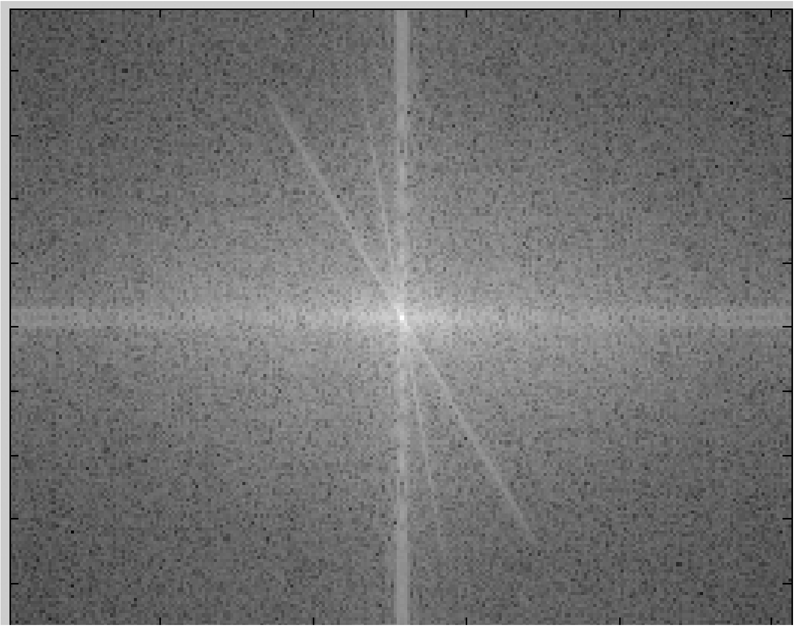


$\mathrm{Re}[\mathcal{F}\{\mathbf{I}\}]$

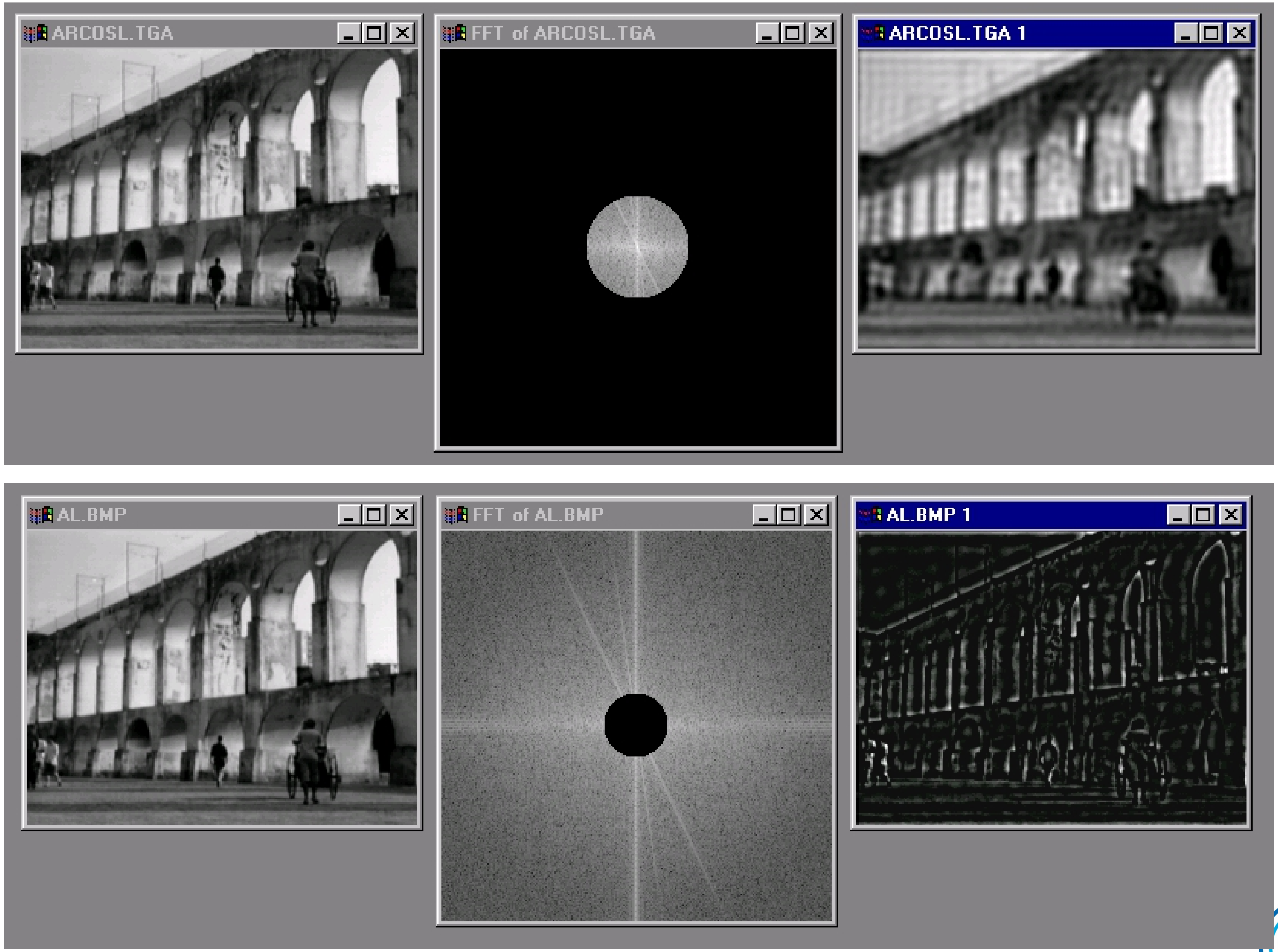


$\mathrm{Im}[\mathcal{F}\{\mathbf{I}\}]$

Man-made Scene

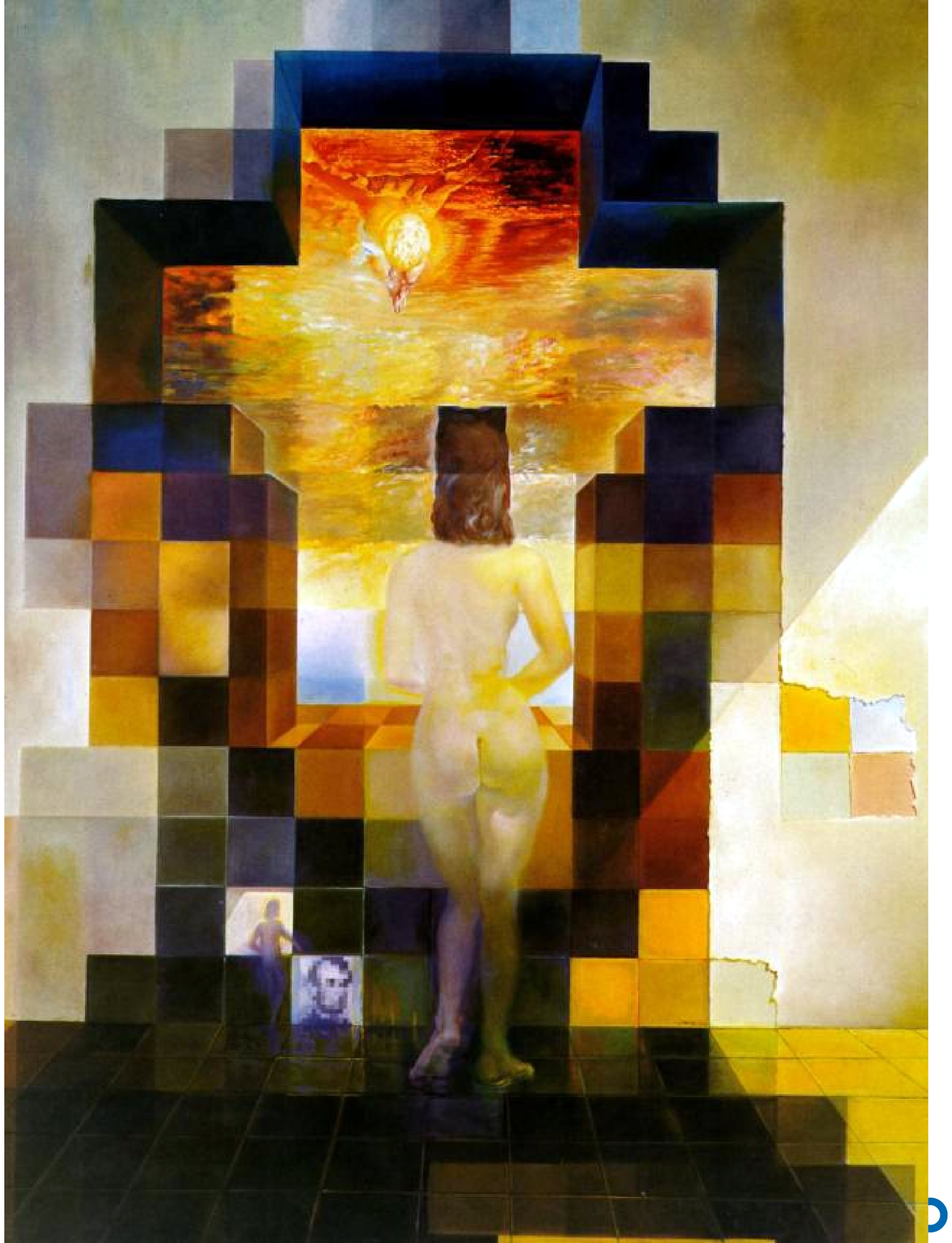


Low and High Pass filtering

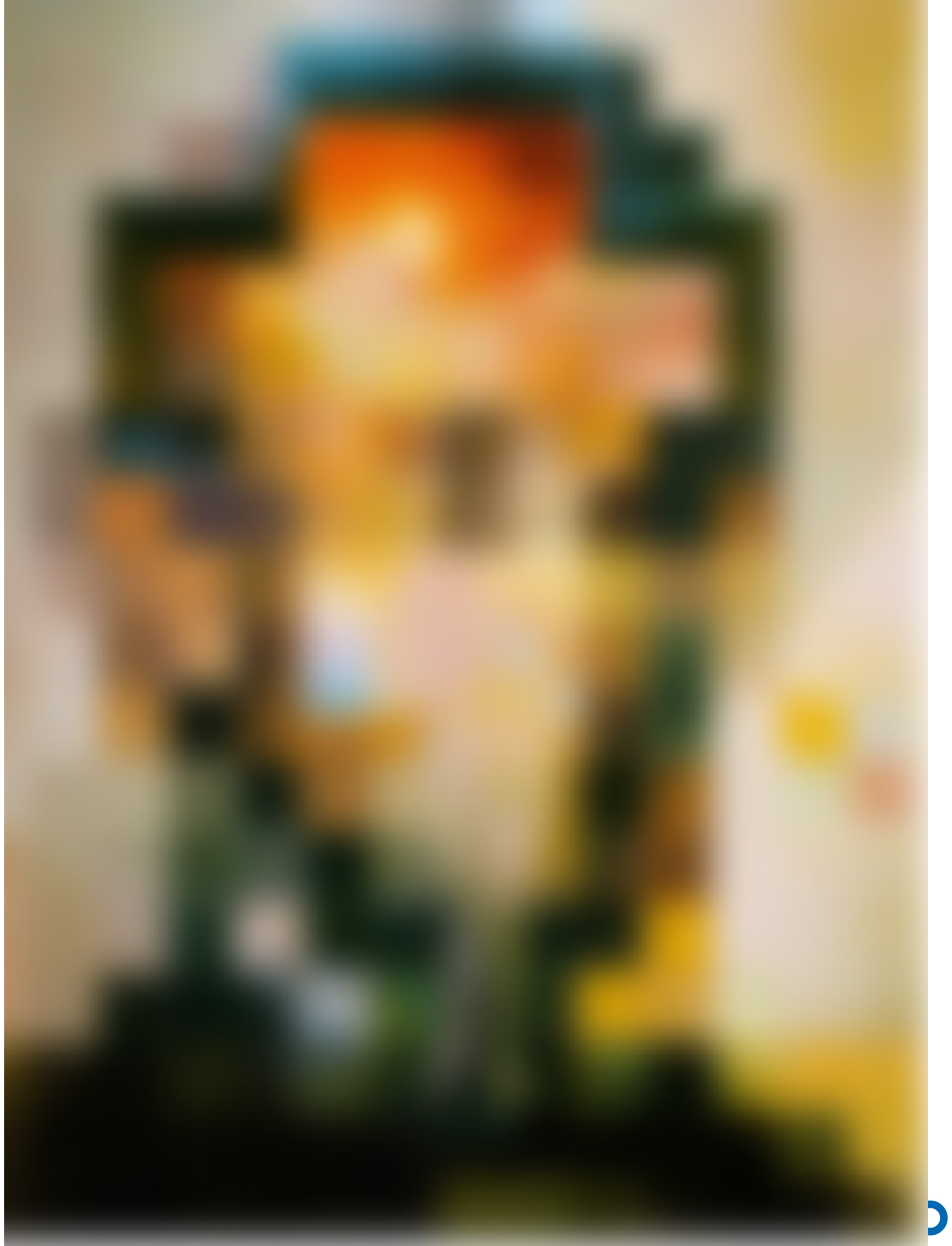


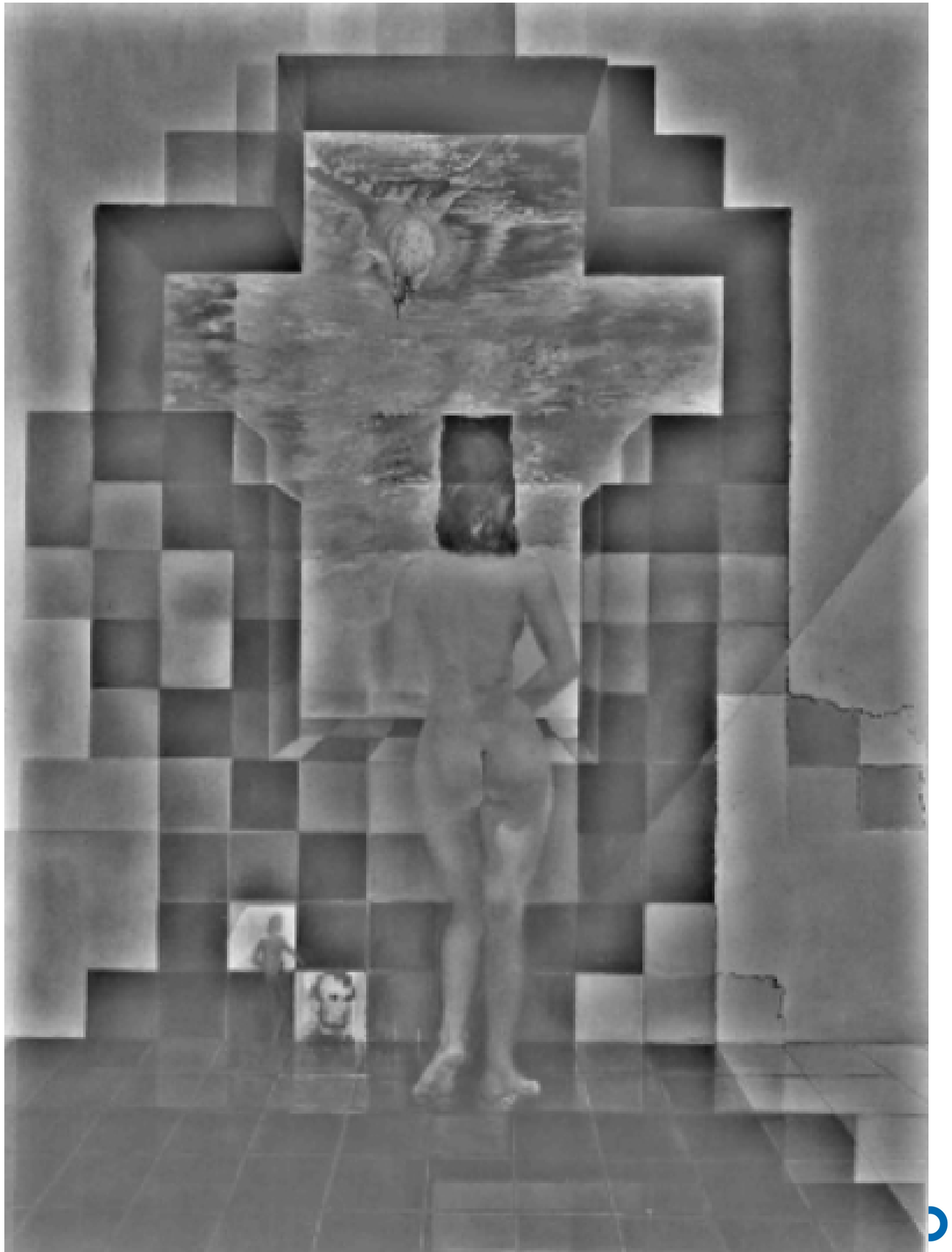
Can Change Spectrum, Then Reconstruct

Salvador Dali invented Hybrid Images?



Salvador Dali
“Gala Contemplating the Mediterranean Sea, which at 20 meters becomes the portrait of Abraham Lincoln”, 1976





Visual Human Perception & Frequency

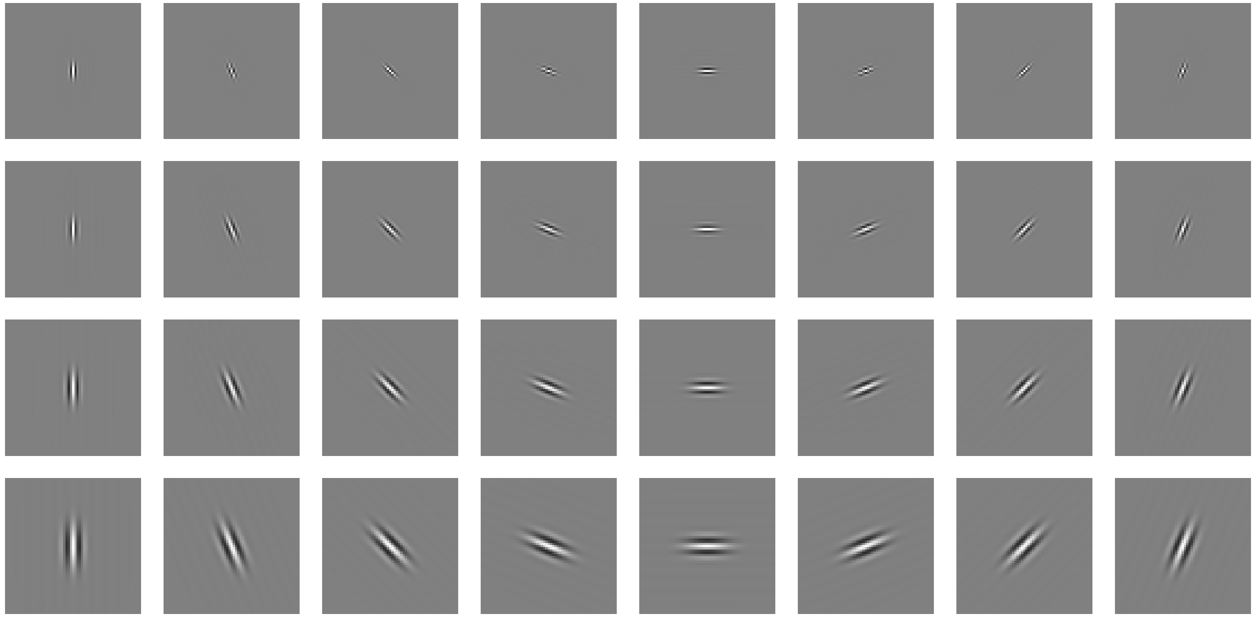
Why do we get different, distance-dependent interpretations of hybrid images?

?

Slide: Hoiem

Visual Human Perception & Frequency

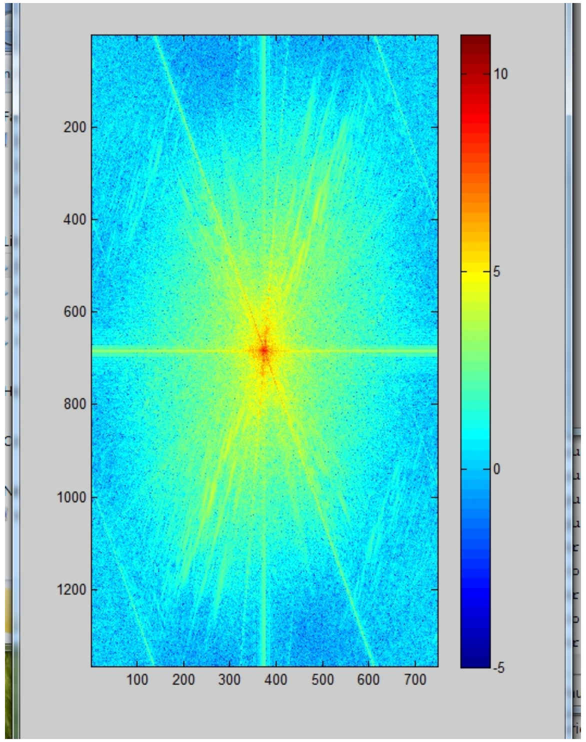
- Early processing in humans filters for various orientations and scales of frequency
- Perceptual cues in the mid-high frequencies dominate perception
- When we see an image from far away, we are effectively subsampling it



Early Visual Processing: Multi-scale edge and blob filters

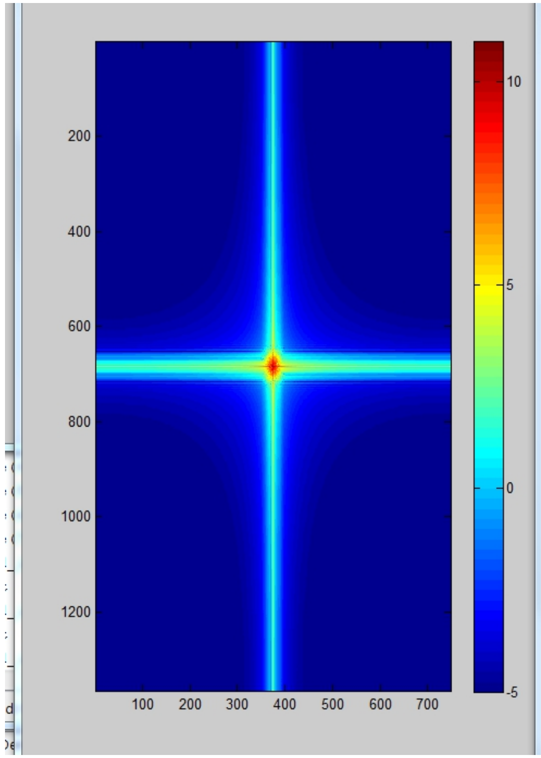
Hybrid Image in FFT

Hybrid Image



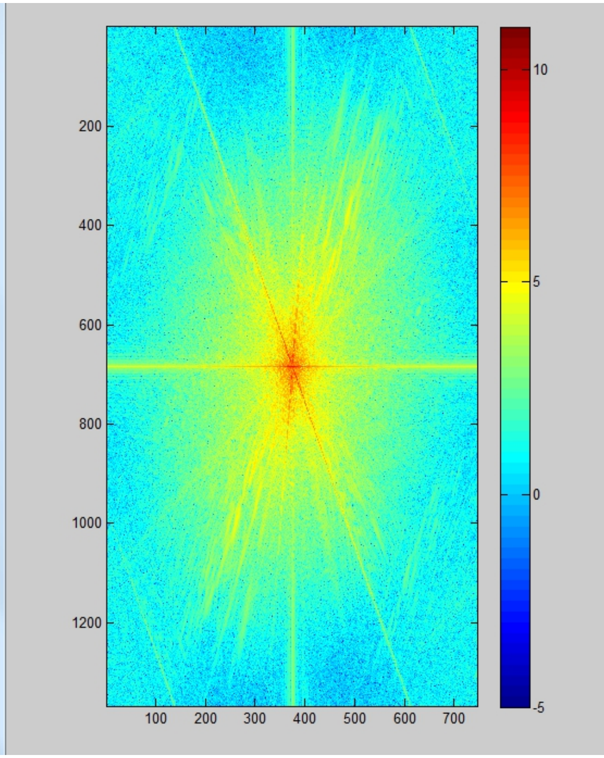
=

Low-passed Image

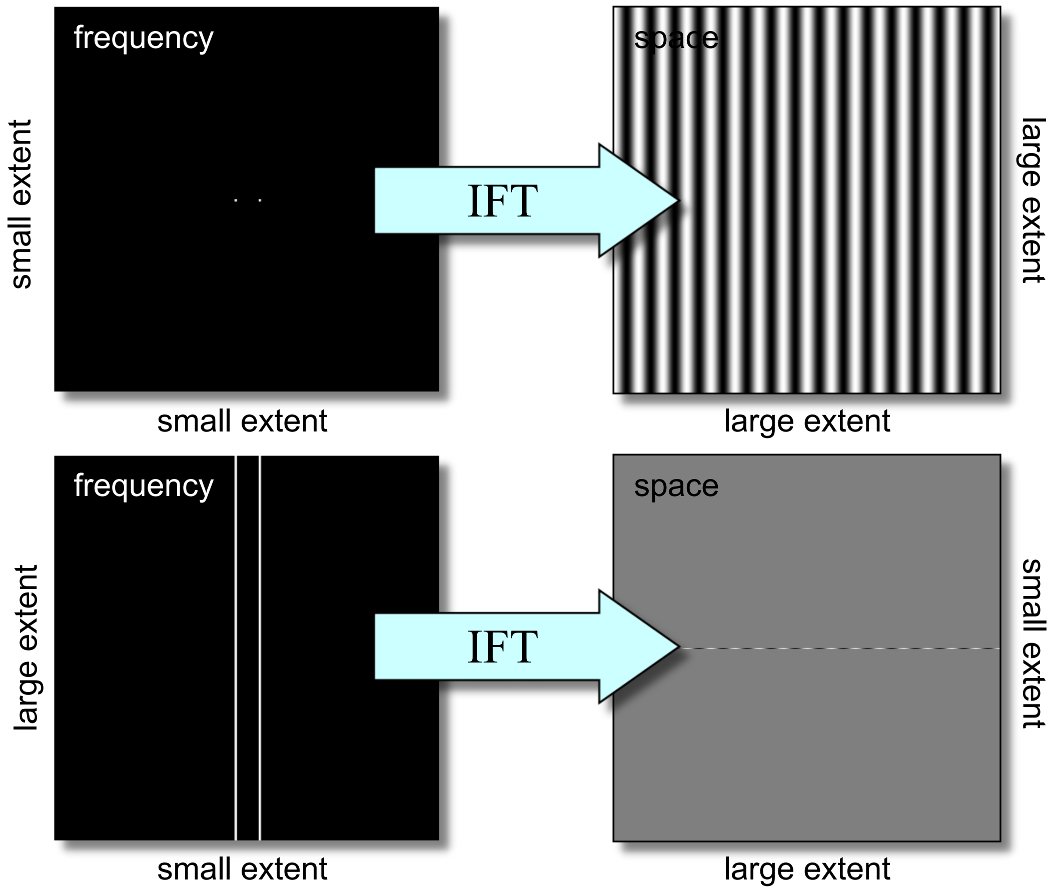


+

High-passed Image



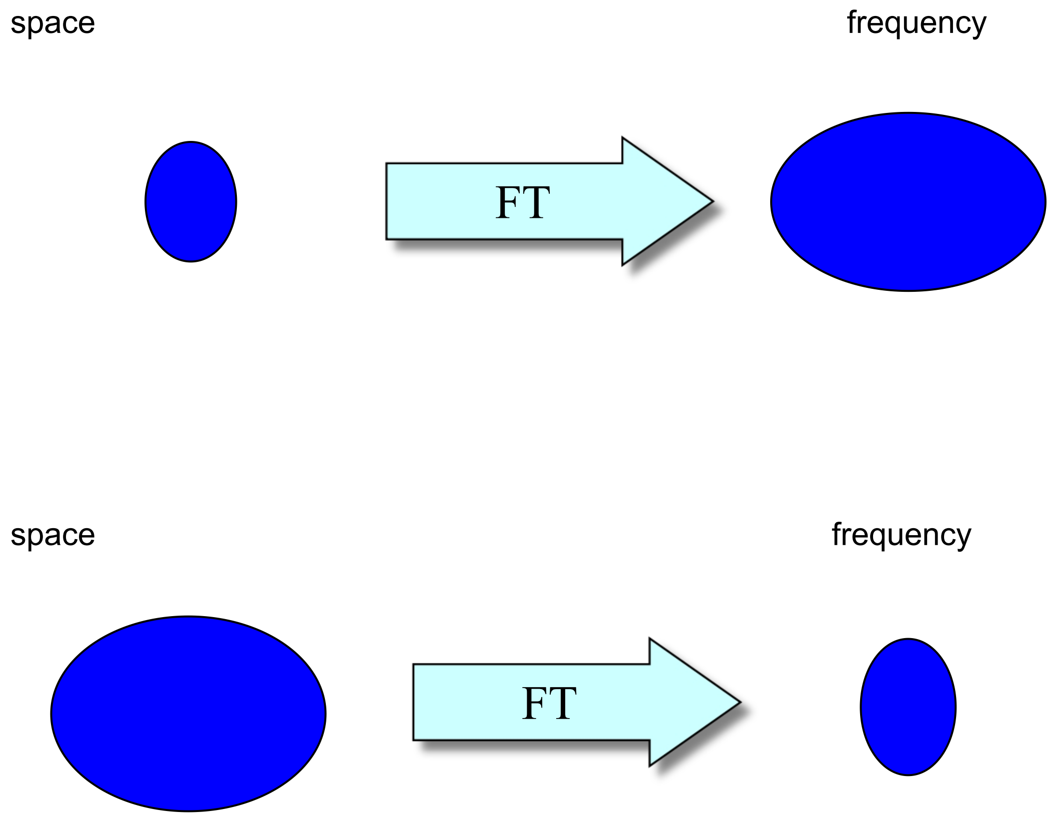
Spatial and Spectra Relations



Recall: a symmetric pair of impulses in the frequency domain becomes a sinusoid in the spatial domain.

A symmetric pair of lines in the frequency domain becomes a sinusoidal line in the spatial domain.

Spatial and Spectra Relations



If $\Delta x \, \Delta y$ is the extent of the object in space and if $\Delta u \, \Delta v$ is its extent in frequency then,

$$\Delta x \, \Delta y \cdot \Delta u \, \Delta v \geq \frac{1}{16\pi^2}$$

A small object in space has a large frequency extent and vice-versa.

Power Spectrum

For display, the log of the power spectrum is often used.

The power spectrum of a signal is the square of the magnitude of its Fourier Transform.

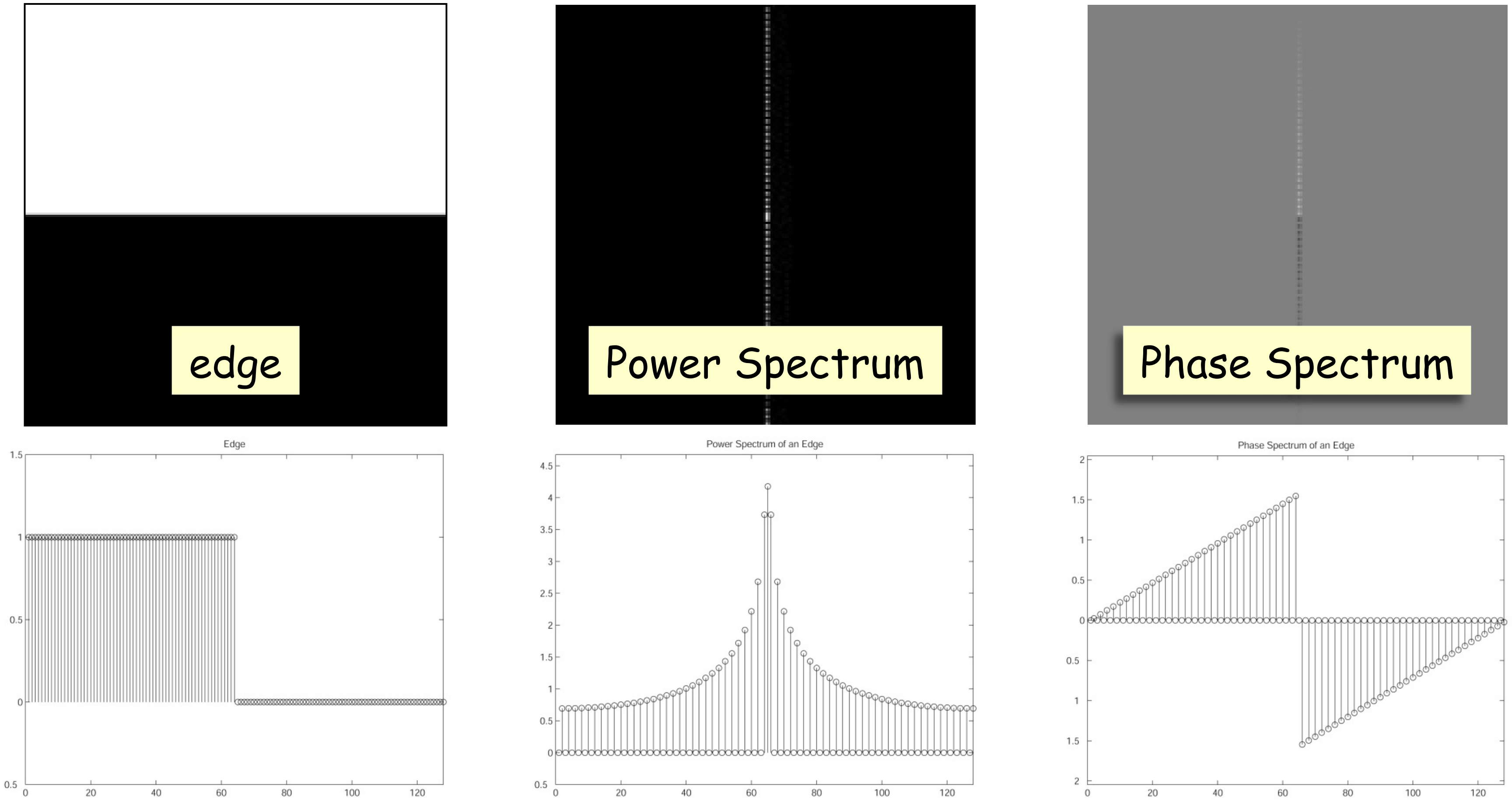
$$\begin{aligned}\left|\mathbf{I}(u,v)\right|^2 &= \mathbf{I}(u,v)\,\mathbf{I}^*(u,v)\\ &= \left[\operatorname{Re}\mathbf{I}(u,v) + i\operatorname{Im}\mathbf{I}(u,v)\right]\left[\operatorname{Re}\mathbf{I}(u,v) - i\operatorname{Im}\mathbf{I}(u,v)\right]\\ &= \left[\operatorname{Re}\mathbf{I}(u,v)\right]^2 + \left[\operatorname{Im}\mathbf{I}(u,v)\right]^2.\end{aligned}$$

At each location (u,v) it indicates the squared intensity of the frequency component with period $\lambda = 1\big/\sqrt{u^2+v^2}$ and orientation

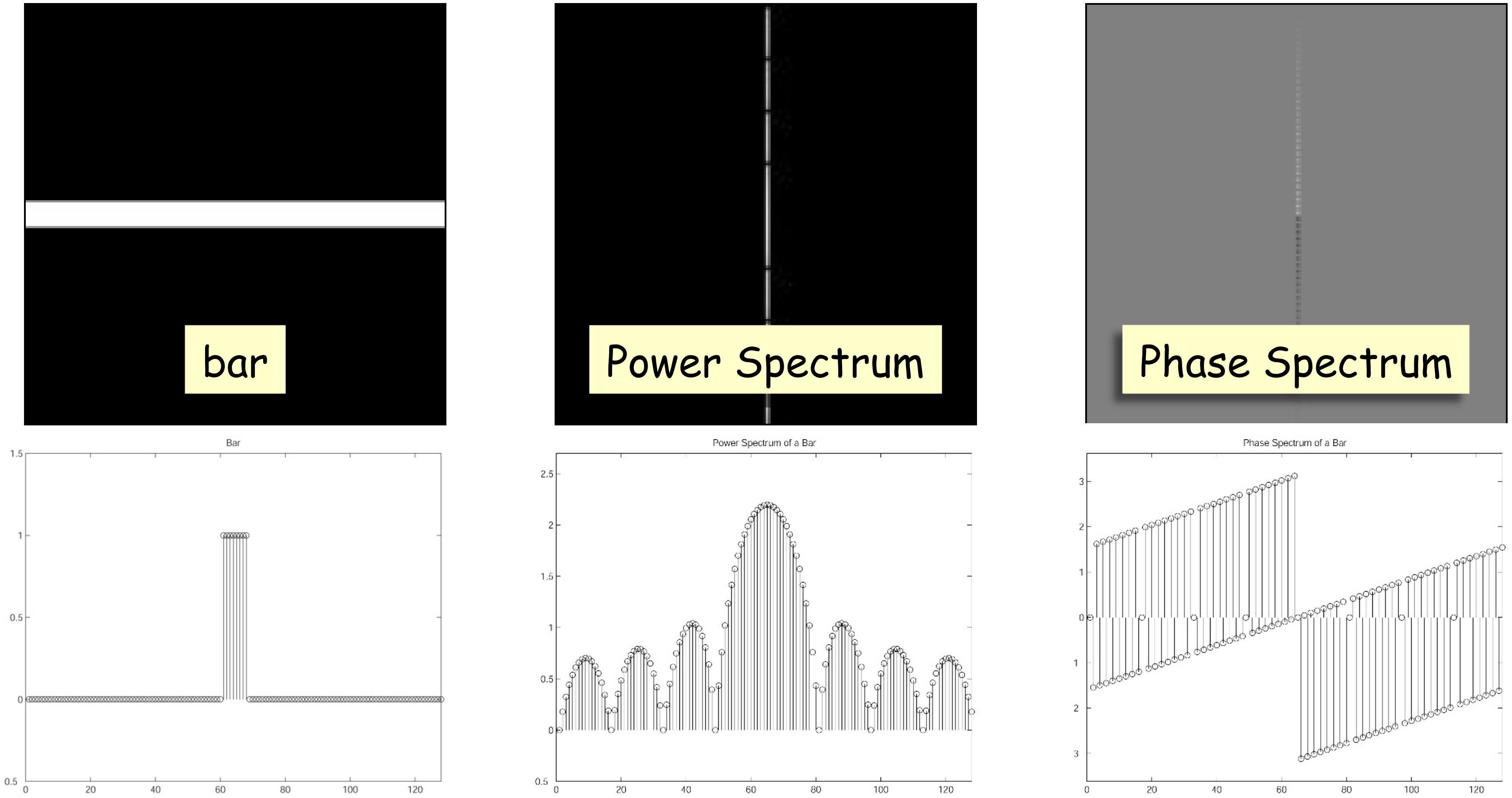
$$\theta_{\mathrm{wf}} = \tan^{-1}\left(\frac{\omega_v}{\omega_u}\right) = \tan^{-1}\left(\frac{vC}{uR}\right).$$

```
For display in Matlab:
PS = fftshift(2*log(abs(fft2(I))+1));
```

Fourier Transform of Edges



Fourier Transform Bar



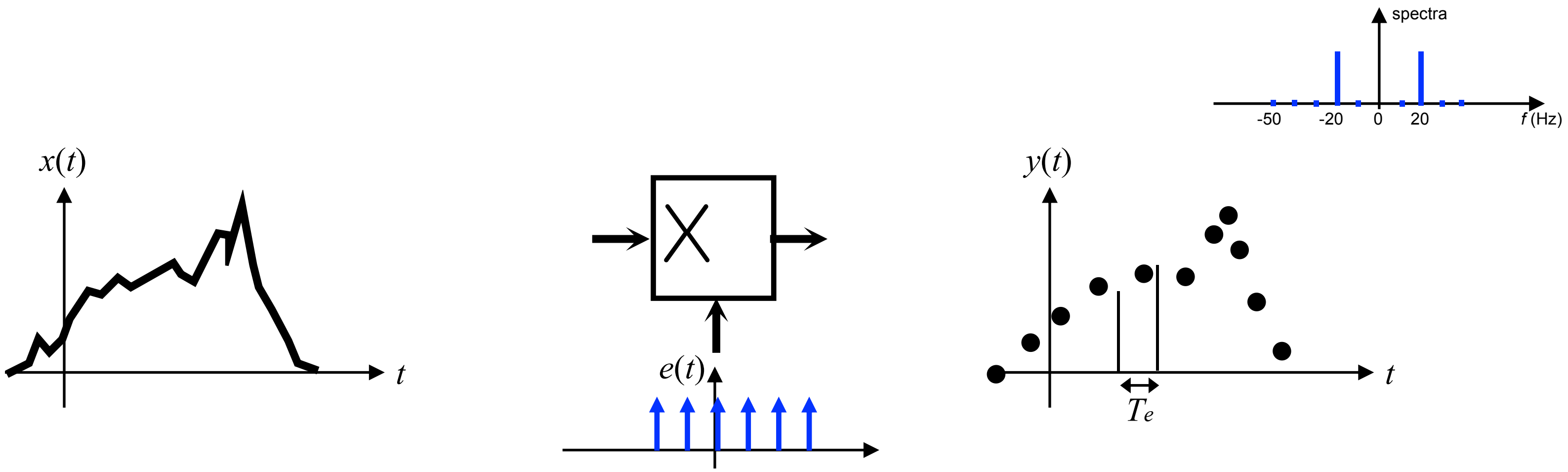
2D Fourier Transform Properties

$af(r,c)+bg(r,c) \Leftrightarrow aF(v,u)+bG(v,u)$	Linearity
$f(r-r_0,c-c_0) \Leftrightarrow e^{-j2\pi(vr_0+uc_0)}F(v,u)$	Shifting
$e^{j2\pi(rv_0+cu_0)}f(r,c) \Leftrightarrow F(v-v_0,u-u_0)$	Modulation
$f(r,c)*g(r,c) \Leftrightarrow F(v,u)G(v,u)$	Convolution
$f(r,c)g(r,c) \Leftrightarrow F(v,u)*G(v,u)$	Multiplication
$f(r,c)=f(r)f(c) \Leftrightarrow F(v,u)=F(v)F(u)$	Separability
$\sum_{r=1}^{R}\sum_{c=1}^{C}\left\|f(r,c)\right\|^2 = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\left\|F(v,u)\right\|^2 dvdu$	Parseval Thm.

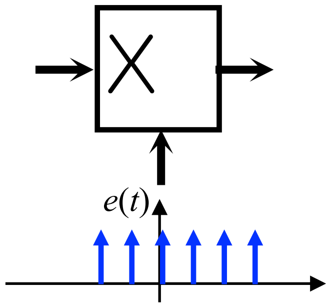
Discrete Signals

Sampling & Discrete Signals

- Recall on sampling continuous signal



- Relation to Shannon Theorem
  - For the signal $x(t)$ sampled with frequency f_e



$$
\begin{aligned}
y(t) &= x(t).e(t) \\
&= x(t) \sum_{k=-\infty}^{\infty} \delta(t-kT_e) \\
&= \sum_{k=-\infty}^{\infty} x(t).\delta(t-kT_e) \\
&= \sum_{k=-\infty}^{\infty} x(kT_e).\delta(t-kT_e)
\end{aligned}
$$

  - So the FT

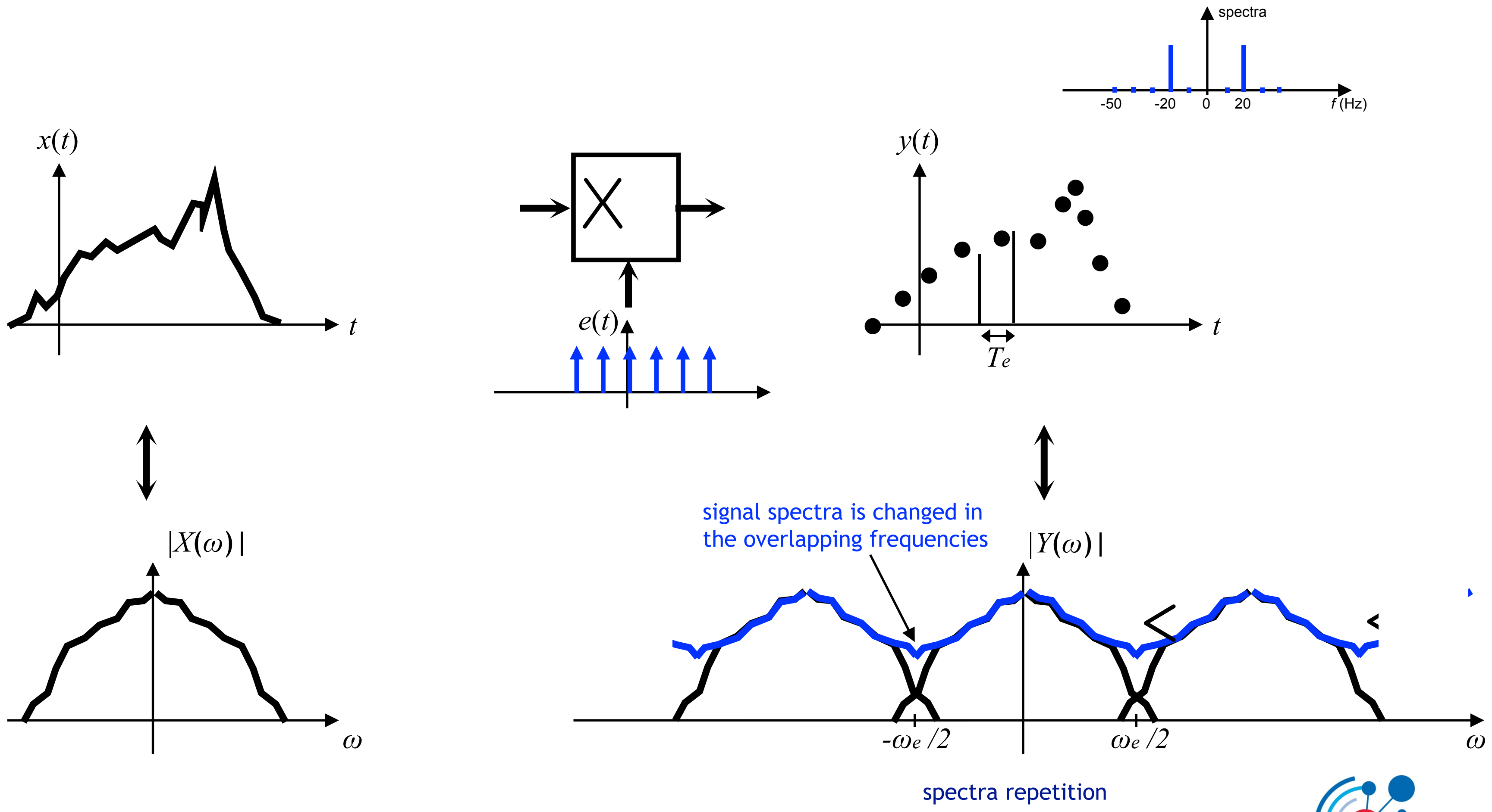
$$
\begin{aligned}
Y(\omega) &= \frac{1}{2\pi} X(\omega) * \omega_e \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_e) \\
&= \frac{1}{T_e} \sum_{n=-\infty}^{\infty} X(\omega) * \delta(\omega - n\omega_e) \\
&= \frac{1}{T_e} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_e)
\end{aligned}
$$

transformations

$$
\begin{aligned}
\sum_{k=-\infty}^{\infty} \delta(t-kT_e) &\longleftrightarrow \omega_e \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_e) \\
x(t).y(t) &\longleftrightarrow \frac{1}{2\pi} X(\omega) * Y(\omega) \\
x(t) * y(t) &\longleftrightarrow X(\omega).Y(\omega)
\end{aligned}
$$

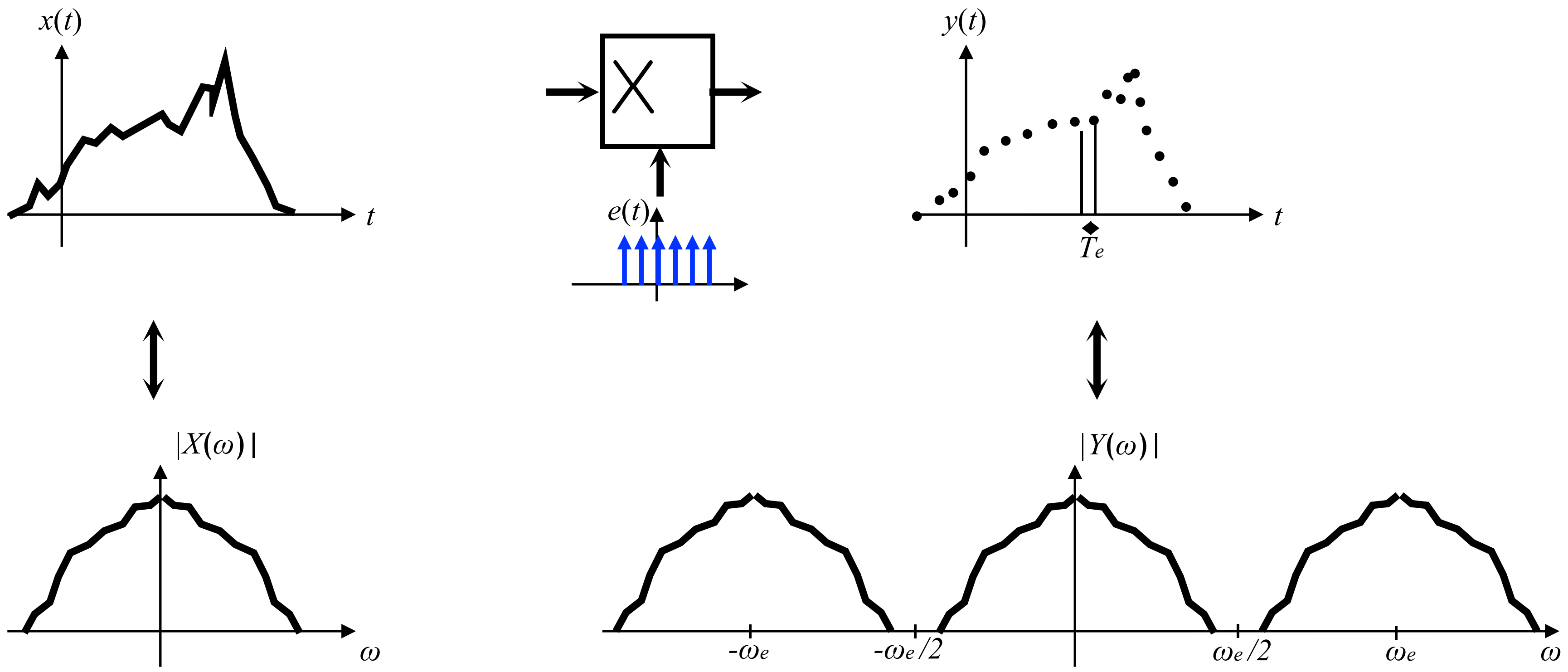
Sampling & Discrete Signals

- If frequency of sampling is smaller than twice the max frequency in the spectra



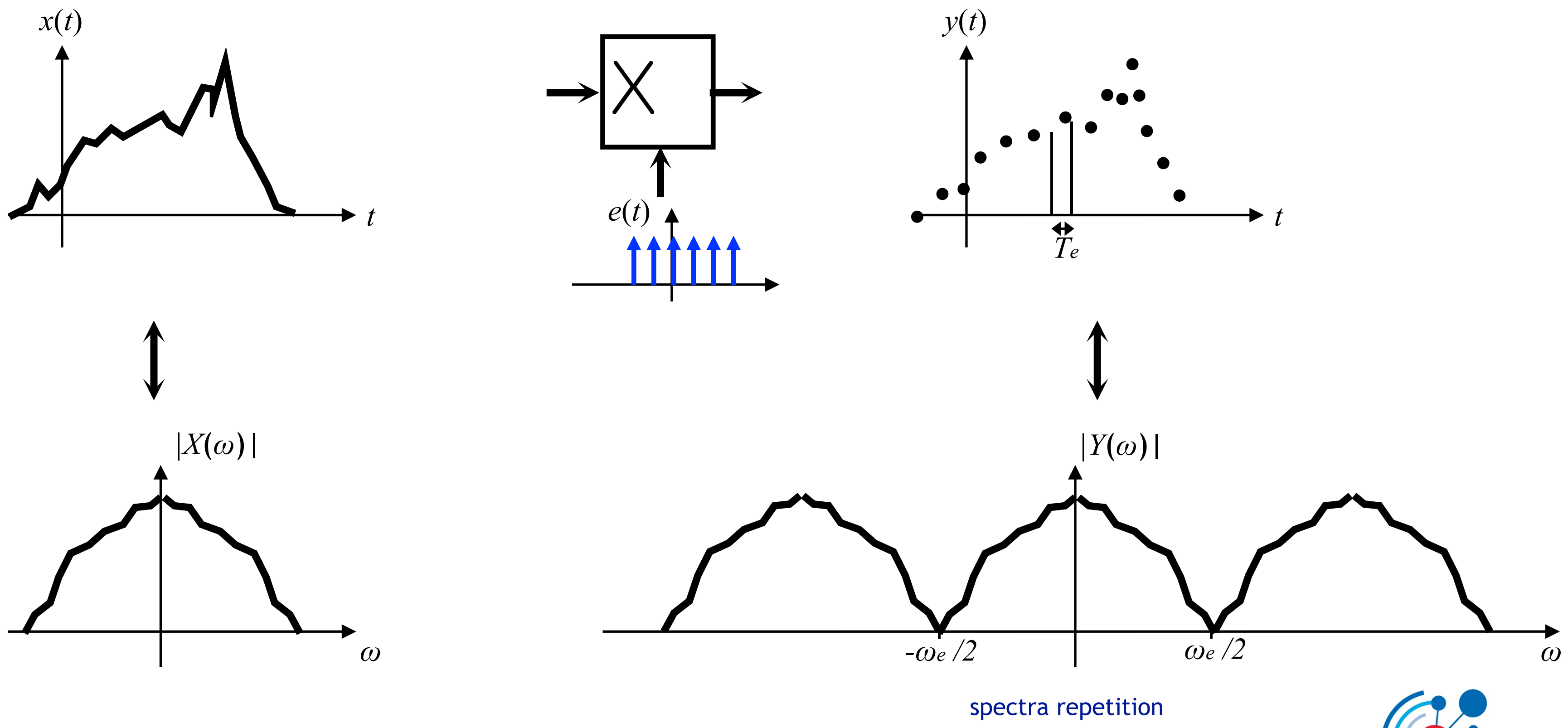
Sampling & Discrete Signals

- Sampling makes the frequency components periodic!



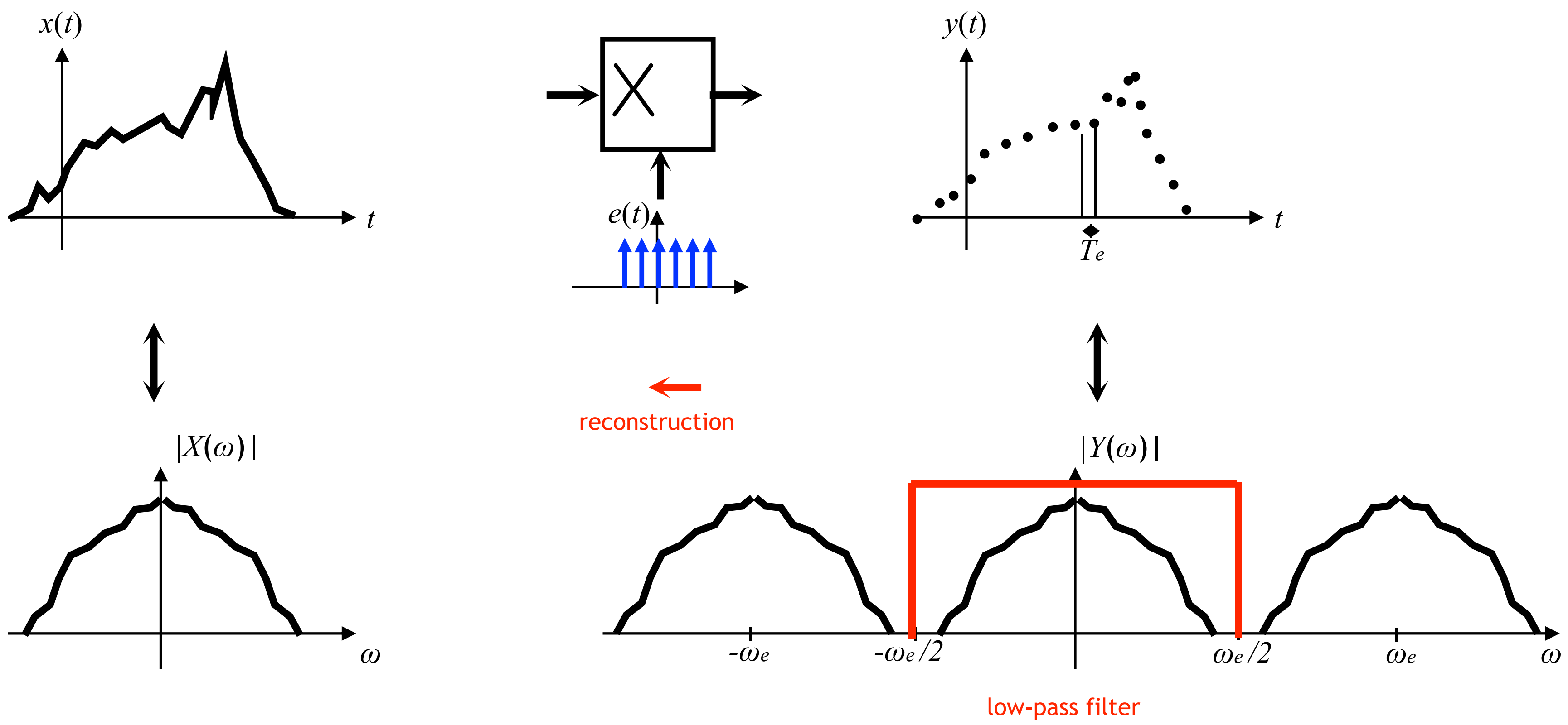
Shannon-Nyquist Condition

- No loss of information when sampling if $f_{max} < f_e/2$



Signal Reconstruction

- Reconstruction of the signal if cropping the frequencies to $f_{\max} = f_e/2$



Discrete-Time Fourier Transform - DTFT

- DTFT is directly linked with the sampling of the signal

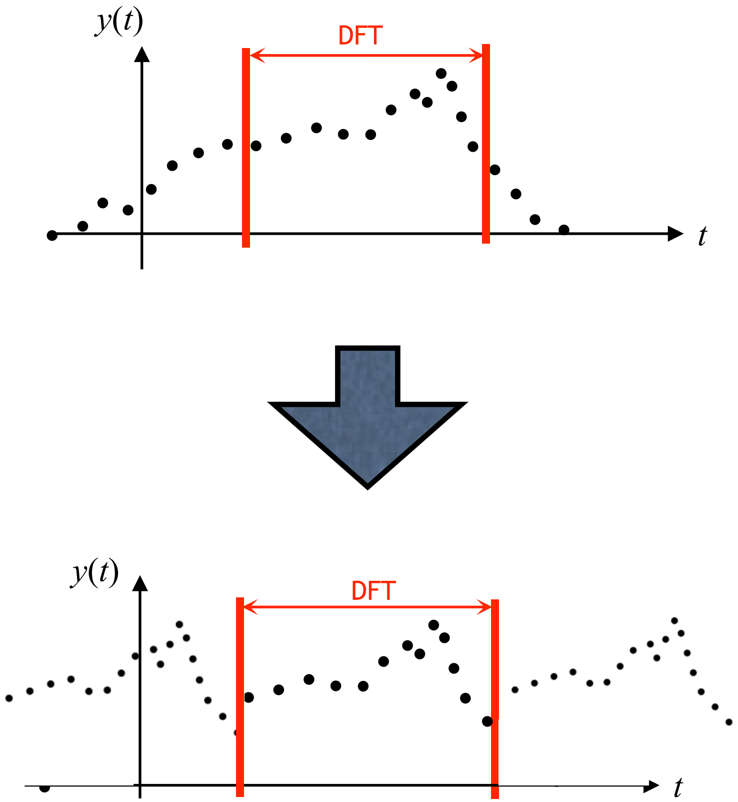
$$S(\omega) \quad = \quad \sum_{k=-\infty}^{\infty} s[k]e^{-j\omega kT_e}$$

- Problems:
  - Infinite sum
  - S is continuous but s is discrete

- Thus this definition is not adapted to discrete signals in practice...

DTFT to DFT

We limit the sum to N points ⇒ we therefore apply a window to the signal and make it periodic



This signal, made periodic by the DFT, often exhibits discontinuities.
These discontinuities generate high-frequency spectra.

DTFT to DFT

- Starting from N points in the spatial domain, we arrive at N points in the frequency domain ⇒ the frequency step is therefore Δf = fe / N, since
$f \in [-fe/2; fe/2]$

$$\begin{aligned} S(n) &= \sum_{k=0}^{N-1} s[k] e^{-j2\pi.n.\Delta f.kT_e} \\ &= \sum_{k=0}^{N-1} s[k] e^{-j2\pi.n.\frac{fe}{N}f.kT_e} \\ &= \sum_{k=0}^{N-1} s[k] e^{-j2\pi.\frac{n.k}{N}} \end{aligned}$$

$n \in [-N/2; N/2\text{-}1]$

- DTF inverse :

$$x[k] = \frac{1}{N} \sum_{n=-N/2}^{N/2-1} X(n) e^{j2\pi.\frac{n.k}{N}}$$

Computing the Fourier Transform

$$H(\omega) = \mathcal{F}\{h(x)\} = Ae^{j\phi}$$

Continuous

$$H(\omega) = \int_{-\infty}^{\infty} h(x)e^{-j\omega x}dx$$

Discrete

$$H(k) = \frac{1}{N}\sum_{x=0}^{N-1} h(x)e^{-j\frac{2\pi kx}{N}}$$ *k = -N/2..N/2*

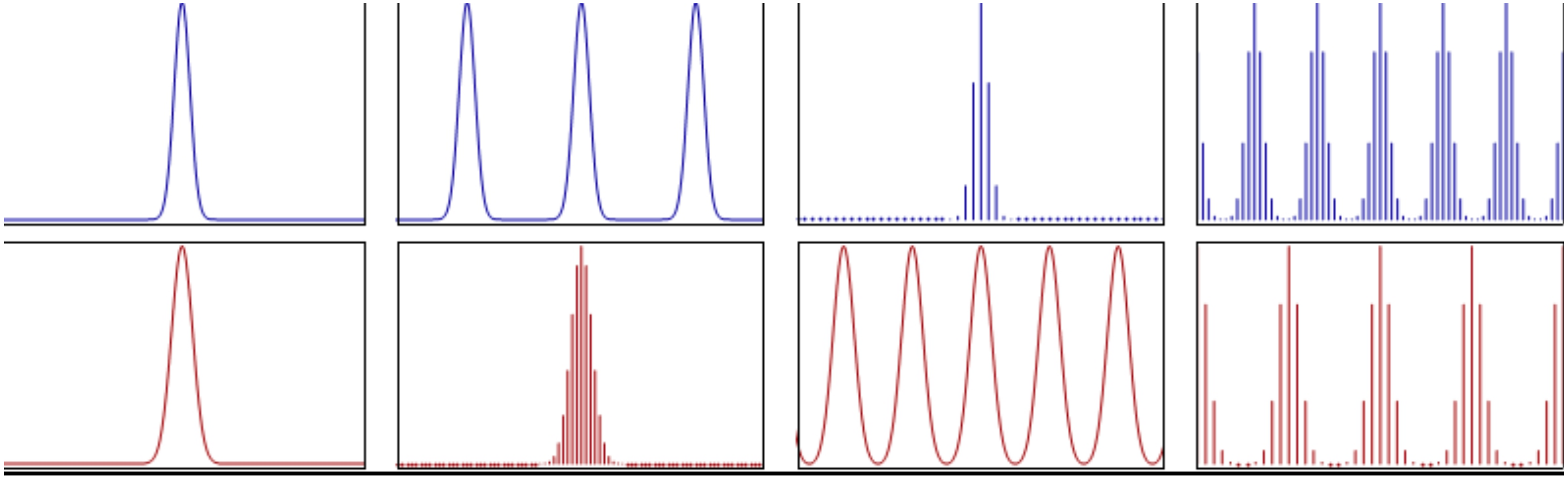
Fast Fourier Transform (FFT): N.logN

Sampling & Frequency Overview

non-periodic | periodic | discrete | DFT (FFT)

signal

frequency components



Sampling & Frequency Overview

signal continu apériodique	$\overset{\text{TF}}{\leftrightarrow}$	continu apériodique
signal continu périodique	$\overset{\text{série de F}}{\leftrightarrow}$	discret apériodique
signal discret apériodique	$\overset{\text{TF(TD)}}{\leftrightarrow}$	continu périodique
signal discret périodique	$\overset{\text{TFD}}{\leftrightarrow}$	discret périodique