Elec4A - Traitement du Signal

Lecture 2:
Short Recall of Probability Distributions and Error Propagation

Renato Martins, ICB UMR CNRS - Univ. Bourgogne
UFR Sciences & Techniques, 2025

3-1 Introduction

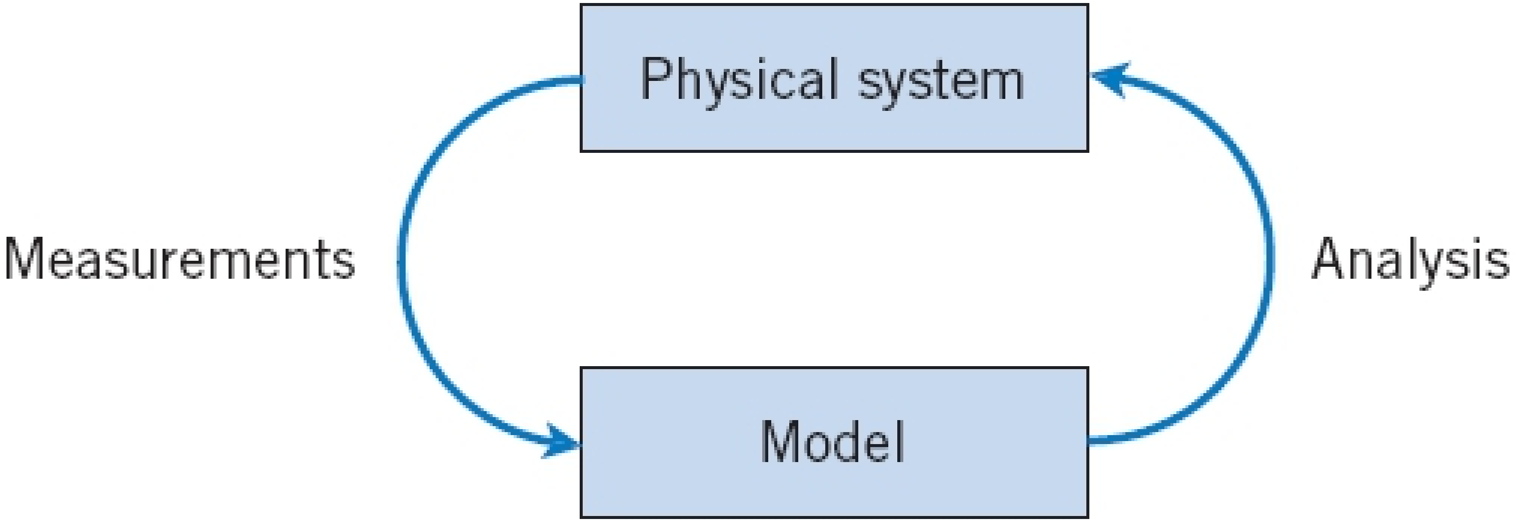


Figure 3-1 Continuous iteration between model and physical system.

3-1 Introduction

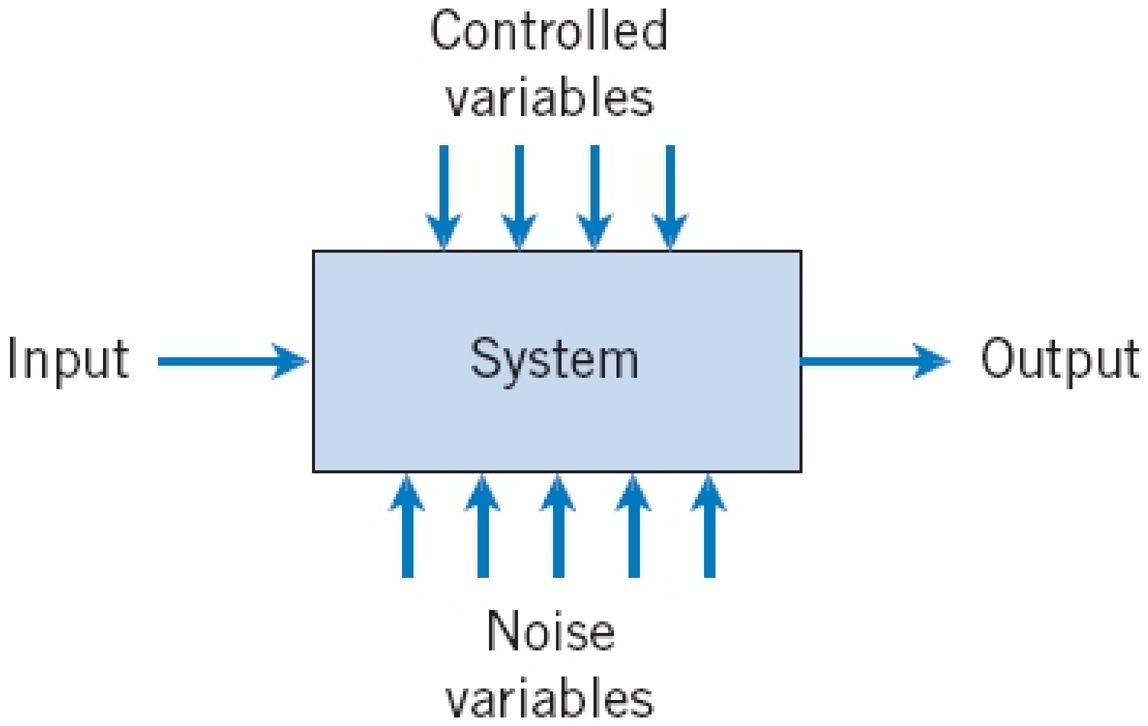
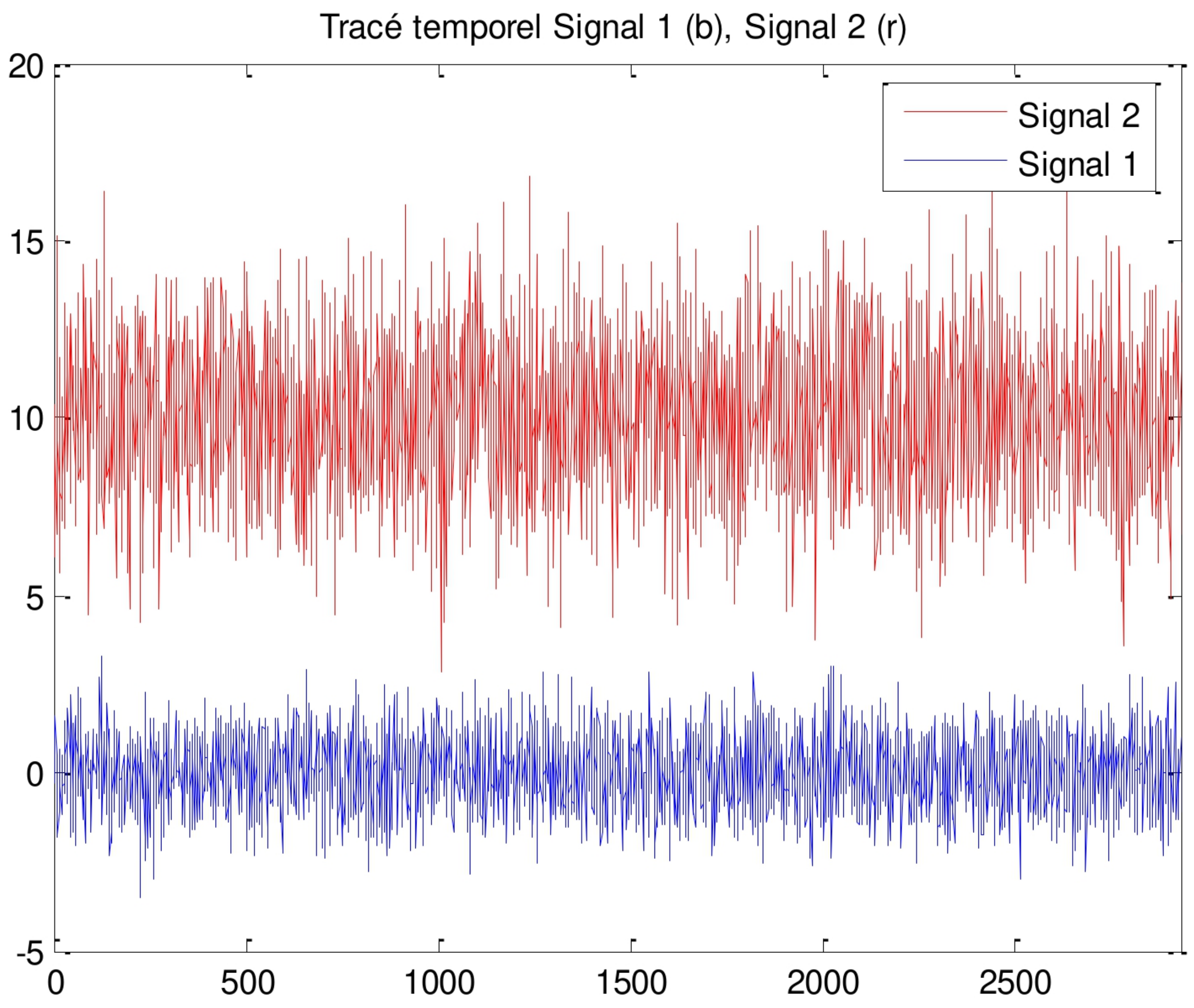


Figure 3-2 Noise variables affect the transformation of inputs to outputs.

3-1 Introduction



3-1 Introduction

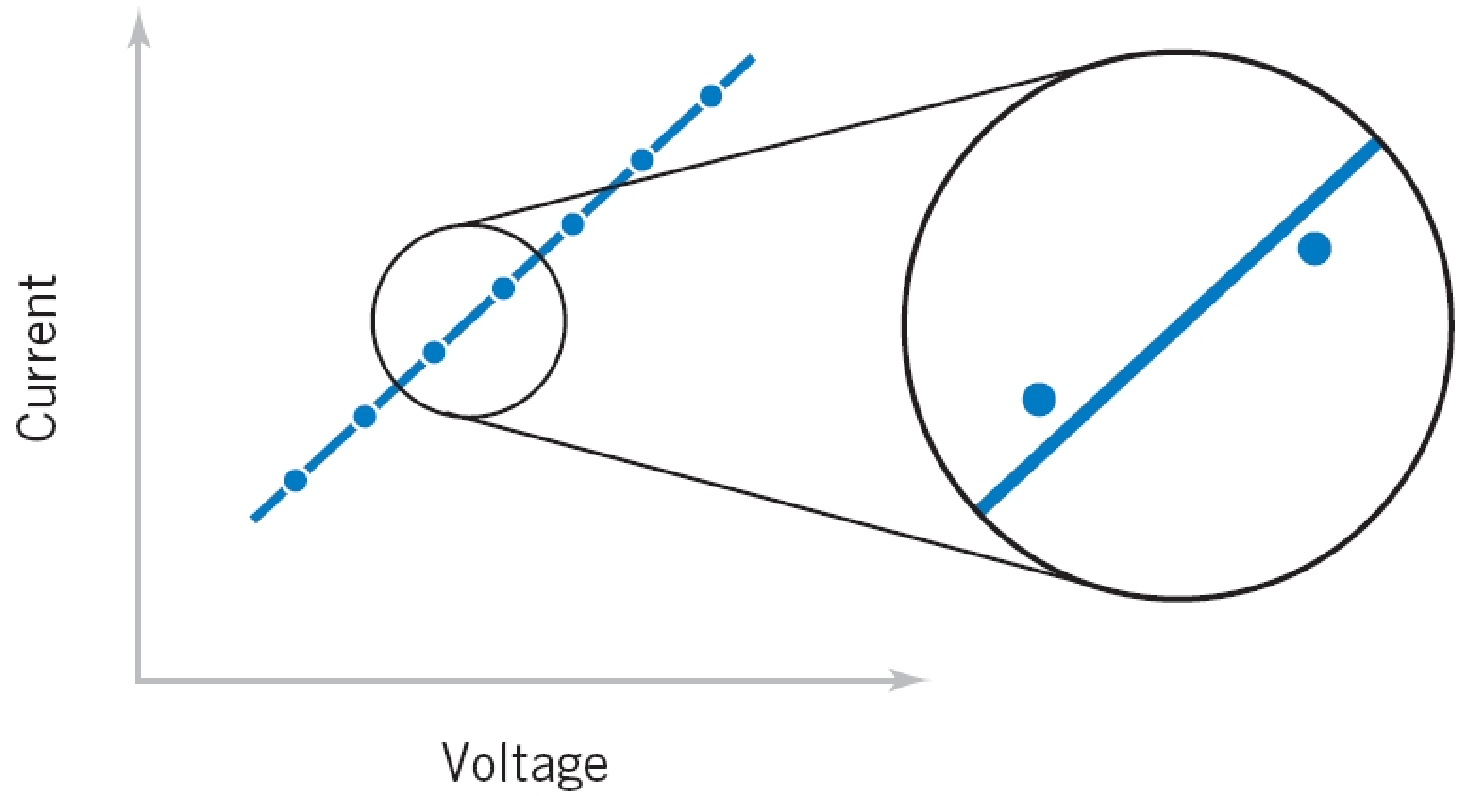


Figure 3-3 A closer examination of the system identifies deviations from the model.

3-2 Random Variables

- In an experiment, a measurement is usually denoted by a variable such as X.

- In a random experiment, a variable whose measured value can change (from one replicate of the experiment to another) is referred to as a **random variable**.

3-2 Random Variables

A **random variable** is a numerical variable whose measured value can change from one replicate of the experiment to another.

A **discrete** random variable is a random variable with a finite (or countably infinite) set of real numbers for its range.
A **continuous** random variable is a random variable with an interval (either finite or infinite) of real numbers for its range.

Examples of **continuous** random variables:
electrical current, length, pressure, temperature, time, voltage, weight

Examples of **discrete** random variables:
number of scratches on a surface, proportion of defective parts among 1,000 tested, number of transmitted bits received in error

3-3 Probability

- Used to quantify likelihood or chance
- Used to represent risk or uncertainty in engineering applications
- Can be interpreted as our **degree of belief** or **relative frequency**

3-3 Probability

- Probability statements describe the likelihood that particular values occur.

- The likelihood is quantified by assigning a number from the interval [0, 1] to the set of values (or a percentage from 0 to 100%).

- Higher numbers indicate that the set of values is more likely.

3-3 Probability

- A probability is usually expressed in terms of a random variable.
- For the part length example, X denotes the part length and the probability statement can be written in either of the following forms

$$P(X \in [10.8, 11.2]) = 0.25 \qquad \text{or} \qquad P(10.8 \leq X \leq 11.2) = 0.25$$

- Both equations state that the probability that the random variable X assumes a value in [10.8, 11.2] is 0.25.

3-3 Probability

Complement of an Event

- Given a set E, the complement of E is the set of elements that are not in E. The **complement** is denoted as E'.

Mutually Exclusive Events

- The sets E_1 , E_2 ,...,E_k are **mutually exclusive** if the intersection of any pair is empty. That is, each element is in one and only one of the sets E_1 , E_2 ,...,E_k .

3-3 Probability

Probability Properties

1. $P(X \in R) = 1$, where R is the set of real numbers.
2. $0 \leq P(X \in E) \leq 1$ for any set E. (3-1)
3. If $E_1, E_2, \ldots, E_k$ are mutually exclusive sets,
   $P(X \in E_1 \cup E_2 \cup \ldots \cup E_k) = P(X \in E_1) + \cdots + P(X \in E_k)$.

3-4 Continuous Random Variables

3-4.1 Probability Density Function

- The **probability distribution** or simply **distribution** of a random variable X is a description of the set of the probabilities associated with the possible values for X.

The **probability density function** (or pdf) $f(x)$ of a continuous random variable is used to determine probabilities from areas as follows:

$$P(a < X < b) = \int_a^b f(x)\,dx \qquad (3\text{-}2)$$

The properties of the pdf are

(1) $f(x) \geq 0$

(2) $\int_{-\infty}^{\infty} f(x) = 1$

3-4 Continuous Random Variables

3-4.1 Probability Density Function

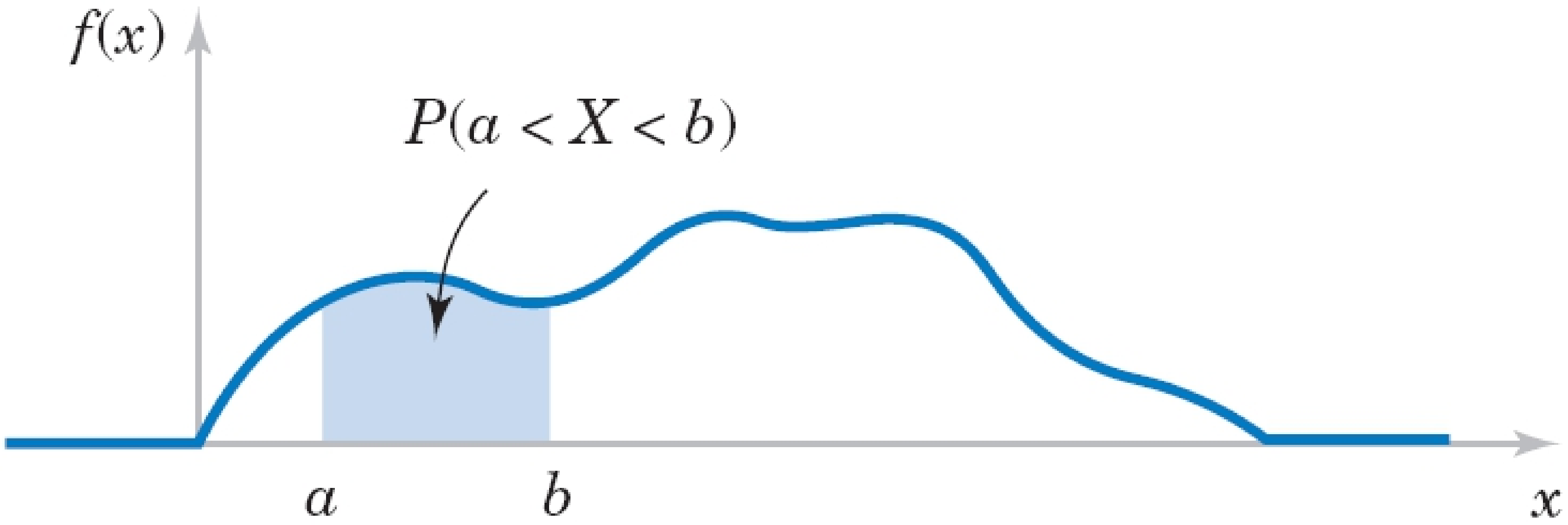


Figure 3-6 Probability determined from the area under $f(x)$.

3-4 Continuous Random Variables

3-4.1 Probability Density Function

If X is a continuous random variable, for any x_1 and x_2,

$$P(x_1 \leq X \leq x_2) = P(x_1 < X \leq x_2) = P(x_1 \leq X < x_2) = P(x_1 < X < x_2)$$

3-4 Continuous Random Variables

3-4.2 Cumulative Distribution Function

The **cumulative distribution function** (or cdf) of a continuous random variable X with probability density function $f(x)$ is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)\, du$$

for $-\infty < x < \infty$.

3-4 Continuous Random Variables

3-4.3 Mean and Variance

Suppose X is a continuous random variable with pdf $f(x)$. The **mean** or **expected value** of X, denoted as μ or $E(X)$, is

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx \tag{3-3}$$

The **variance** of X, denoted as $V(X)$ or σ^2, is

$$\sigma^2 = V(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\, dx = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2$$

The **standard deviation** of X is σ.

3-5 Important Continuous Distributions

3-5.1 Normal Distribution

Undoubtedly, the most widely used model for the distribution of a random variable is a **normal distribution**.

- Central limit theorem
- Gaussian distribution

3-5 Important Continuous Distributions

3-5.1 Normal Distribution

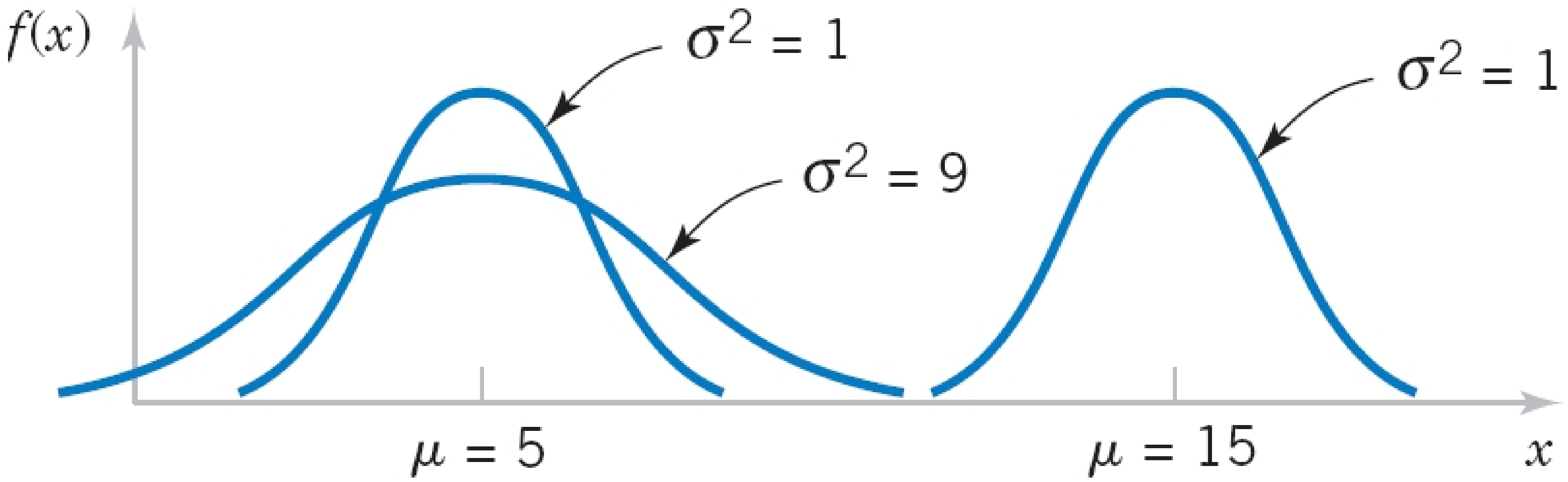


Figure 3-11 Normal probability density functions for selected values of the parameters μ and σ^2.

3-5 Important Continuous Distributions

3-5.1 Normal Distribution

A random variable X with probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \quad \text{for} \quad -\infty < x < \infty \tag{3-4}$$

has a **normal distribution** (and is called a **normal random variable**) with parameters μ and σ, where $-\infty < \mu < \infty$, and $\sigma > 0$. Also,

$$E(X) = \mu \quad \text{and} \quad V(X) = \sigma^2$$

The mean and variance of the normal distribution are derived at the end of this section.

3-5 Important Continuous Distributions

3-5.1 Normal Distribution

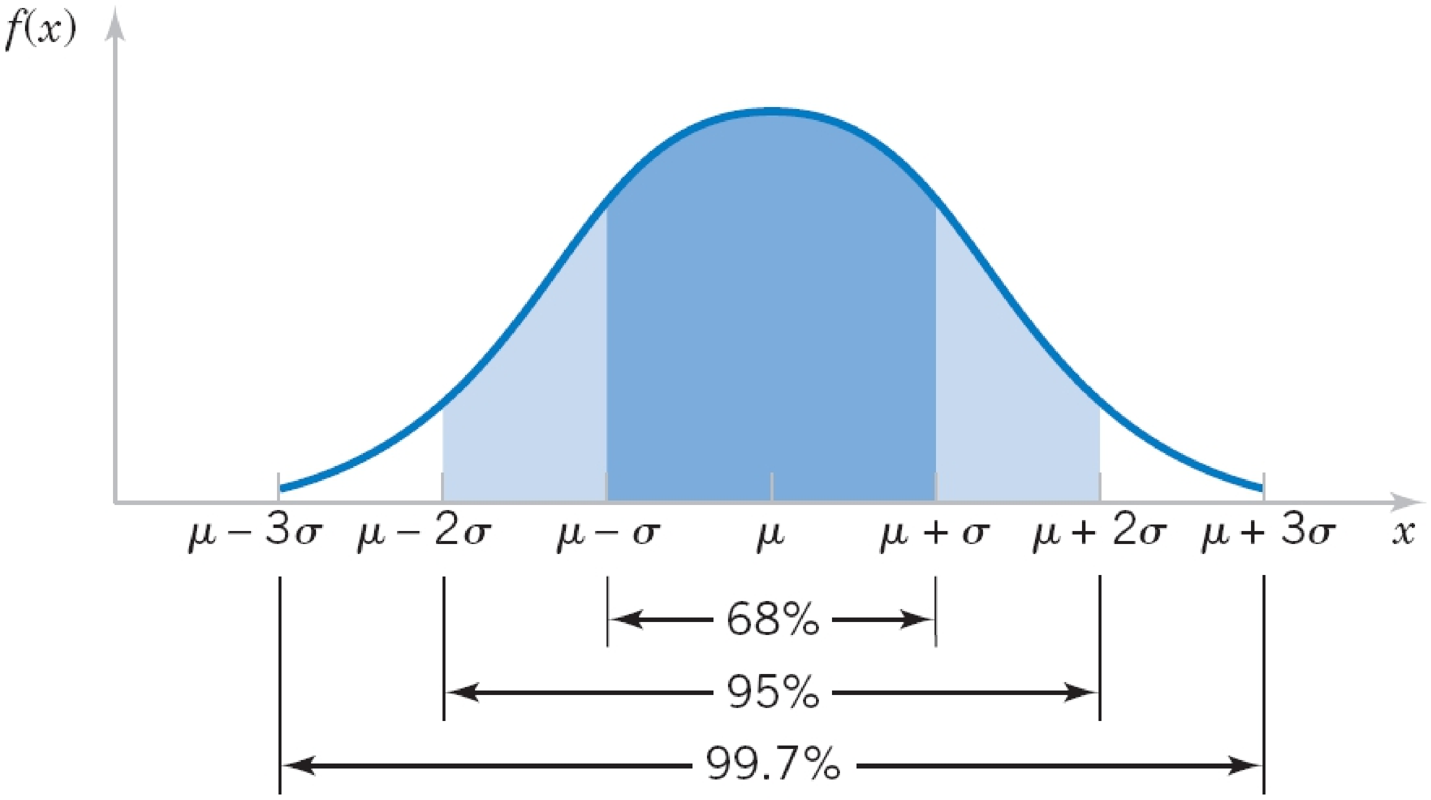


Figure 3-13 Probabilities associated with a normal distribution.

3-5 Important Continuous Distributions

3-5.1 Normal Distribution

A normal random variable with $\mu = 0$ and $\sigma^2 = 1$ is called a **standard normal** random variable. A standard normal random variable is denoted as Z.

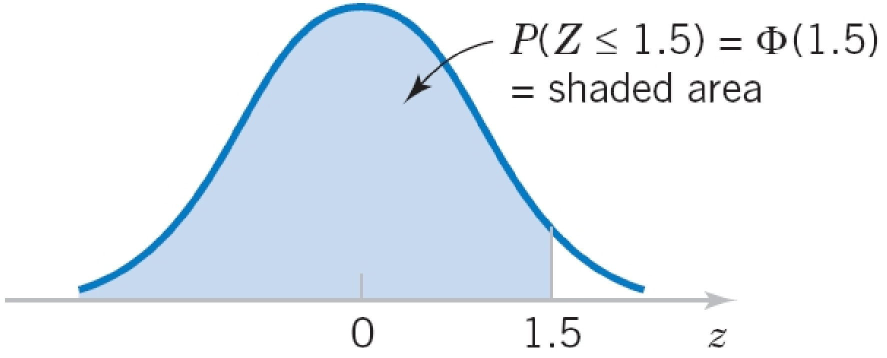
The function

$$\Phi(z) = P(Z \leq z)$$

is used to denote a probability from Appendix A Table I. It is the **cumulative distribution function** of a standard normal random variable. A table (or computer software) is required because the probability can't be determined by elementary methods.

3-5 Important Continuous Distributions

3-5.1 Normal Distribution



z	0.00	0.01	0.02	0.03
0	0.50000	0.50399	0.50398	0.51197
⋮		⋮		
1.5	0.93319	0.93448	0.93574	0.93699

Figure 3-14 Standard normal probability density function.

3-5 Important Continuous Distributions

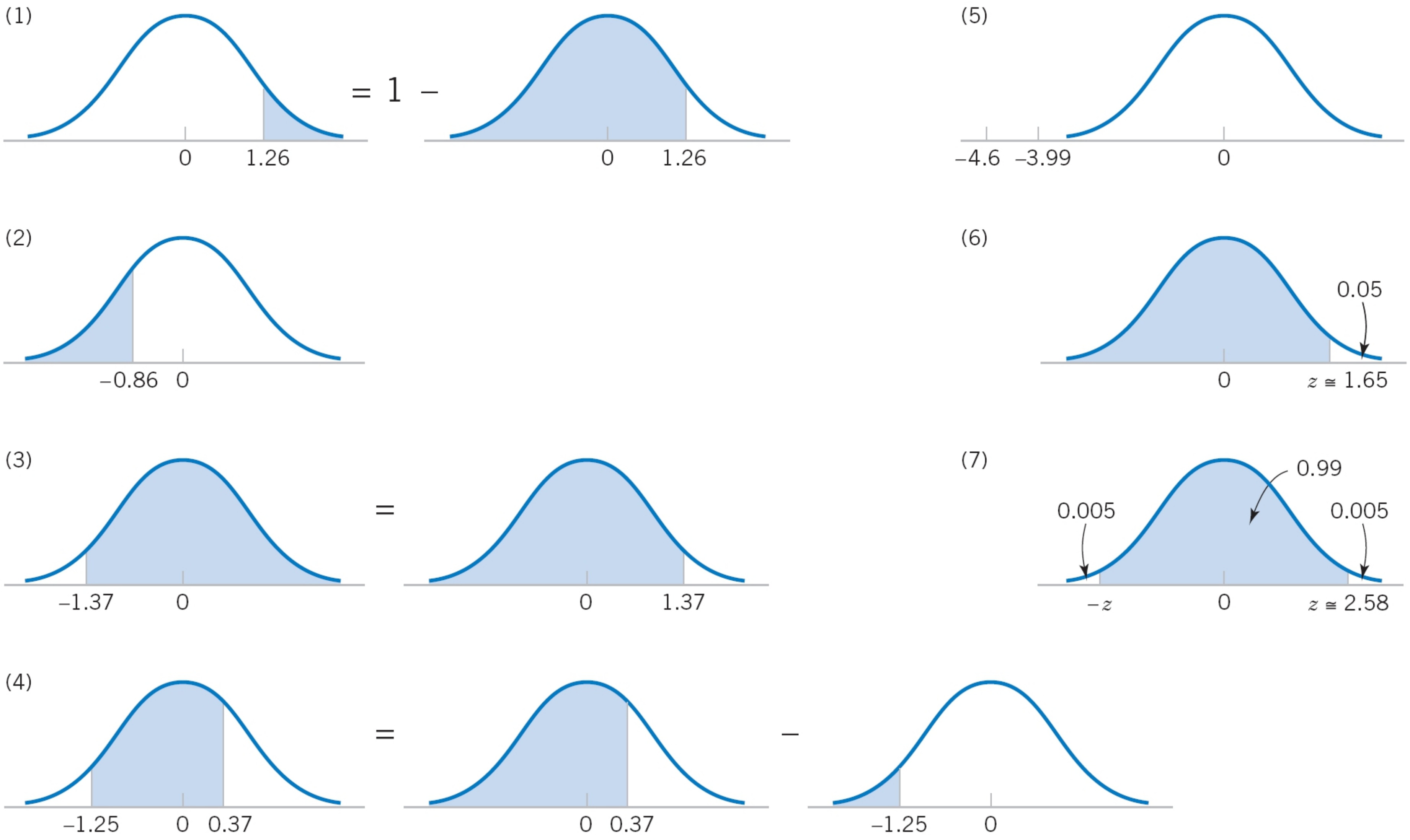


Figure 3-15 Graphical displays for Example 3-8.

3-5 Important Continuous Distributions

3-5.1 Normal Distribution

If X is a normal random variable with $E(X) = \mu$ and $V(X) = \sigma^2$, the random variable

$$Z = \frac{X - \mu}{\sigma}$$

is a normal random variable with $E(Z) = 0$ and $V(Z) = 1$. That is, Z is a **standard normal** random variable.

3-5 Important Continuous Distributions

3-5.1 Normal Distribution

Suppose X is a normal random variable with mean μ and variance σ^2. Then,

$$P(X \leq x) = P\left(\frac{X - \mu}{\sigma} \leq \frac{x - \mu}{\sigma}\right) = P(Z \leq z) \tag{3-5}$$

where

Z is a **standard normal** random variable, and

$z = (x - \mu)/\sigma$ is the ***z*-value** obtained by **standardizing** x.

The probability is obtained by entering **Appendix A Table I** with $z = (x - \mu)/\sigma$.

3-13 Random Samples, Statistics, and The Central Limit Theorem

Independent random variables $X_1, X_2, \ldots, X_n$ with the same distribution are called a **random sample.**

A **statistic** is a function of the random variables in a random sample.

The probability distribution of a statistic is called its **sampling distribution.**

3-13 Random Samples, Statistics, and The Central Limit Theorem

Central Limit Theorem

If $X_1, X_2, \ldots, X_n$ is a random sample of size n taken from a population with mean μ and variance σ^2, and if $\overline{X}$ is the sample mean, the limiting form of the distribution of

$$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \tag{3-39}$$

as $n \rightarrow \infty$, is the standard normal distribution.

3-13 Random Samples, Statistics, and The Central Limit Theorem

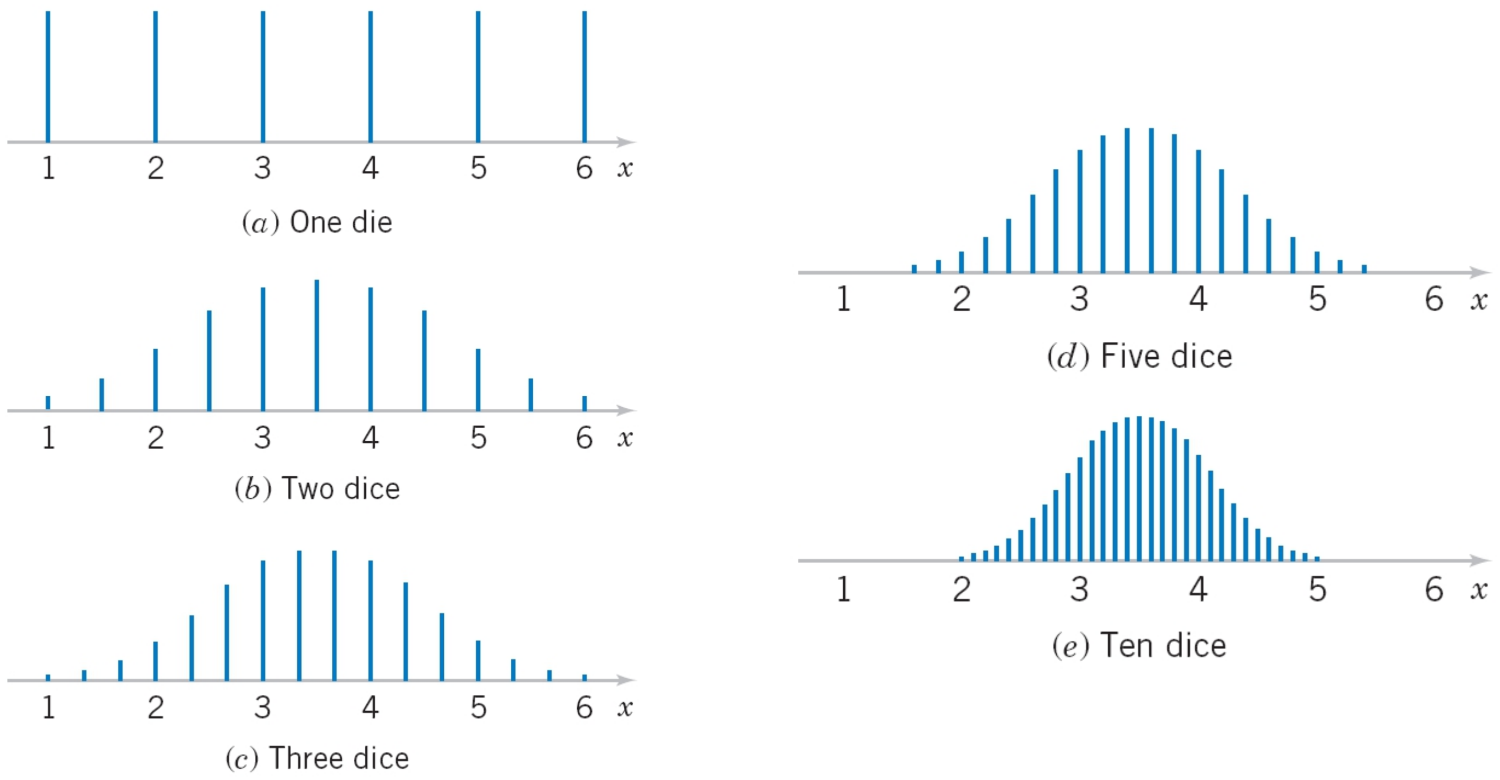


Figure 3-42 Distributions of average scores from throwing dice. [Adapted with permission from Box, Hunter, and Hunter (1978).]

3-7 Discrete Random Variables

3-7.1 Probability Mass Function

For a discrete random variable X with possible values $x_1, x_2, \ldots, x_n$, the **probability mass function** (or pmf) is

$$f(x_i) = P(X = x_i) \tag{3-13}$$

3-7 Discrete Random Variables

3-7.2 Cumulative Distribution Function

The **cumulative distribution function** of a discrete random variable X is

$$F(x) = P(X \le x) = \sum_{x_i \le x} f(x_i)$$

3-7 Discrete Random Variables

3-7.2 Cumulative Distribution Function

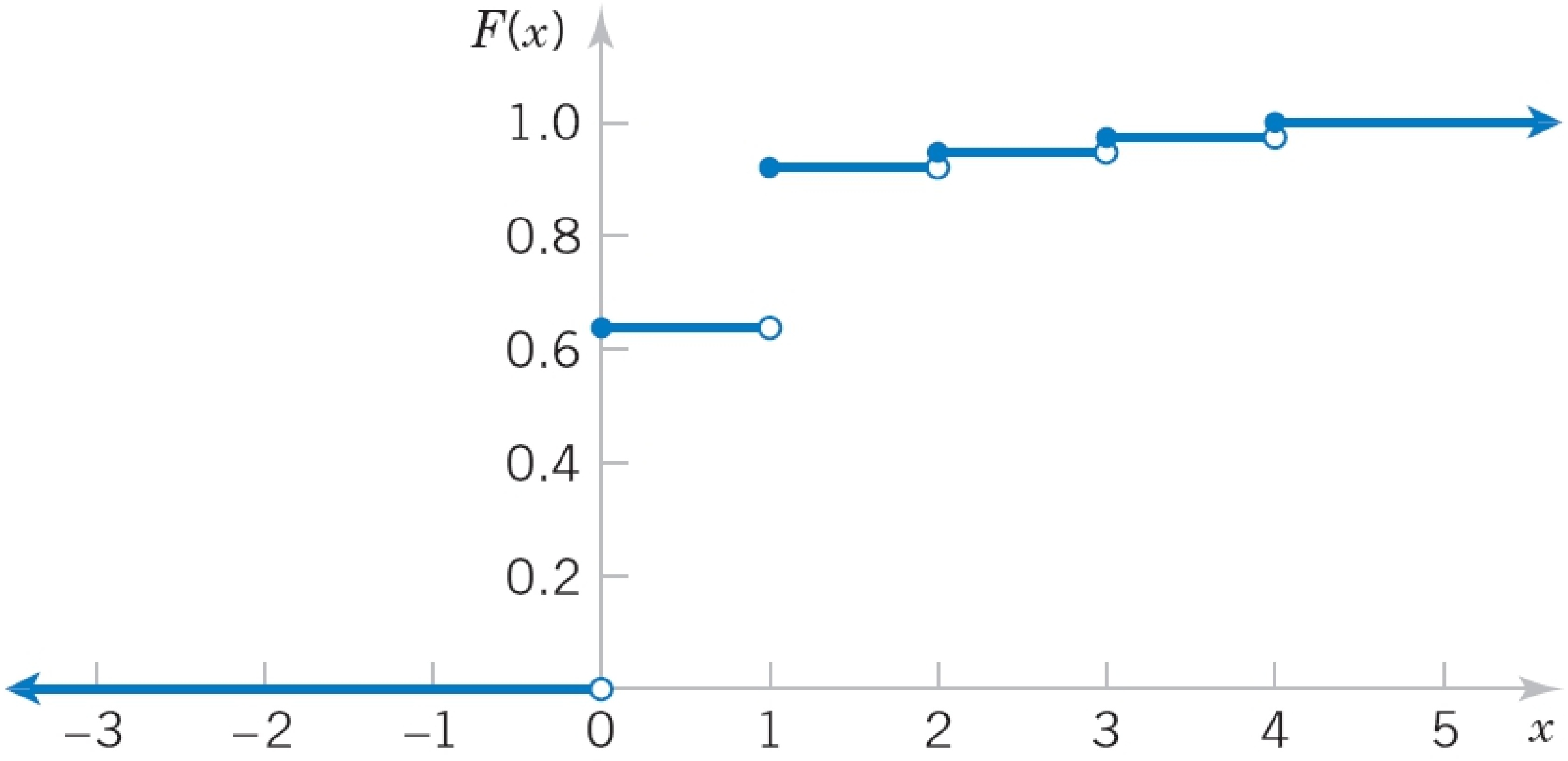


Figure 3-30 Cumulative distribution function for x in Example 3-19.

3-7 Discrete Random Variables

3-7.3 Mean and Variance

Let the possible values of the random variable X be denoted as $x_1, x_2, \ldots, x_n$. The pmf of X is $f(x)$, so $f(x_i) = P(X = x_i)$.

The **mean** or **expected value** of the discrete random variable X, denoted as μ or $E(X)$, is

$$\mu = E(X) = \sum_{i=1}^{n} x_i f(x_i) \tag{3-14}$$

The **variance** of X, denoted as σ^2 or $V(X)$, is

$$\sigma^2 = V(X) = E(X - \mu)^2 = \sum_{i=1}^{n} (x_i - \mu)^2 f(x_i) = \sum_{i=1}^{n} x_i^2 f(x_i) - \mu^2$$

The **standard deviation** of X is σ.

3-8 Binomial Distribution

- A trial with only two possible outcomes is used so frequently as a building block of a random experiment that it is called a **Bernoulli trial**.

- It is usually assumed that the trials that constitute the random experiment are **independent**. This implies that the outcome from one trial has no effect on the outcome to be obtained from any other trial.

- Furthermore, it is often reasonable to assume that the **probability of a success on each trial is constant**.

3-8 Binomial Distribution

- Consider the following random experiments and random variables.
  - Flip a coin 10 times. Let X = the number of heads obtained.
  - Of all bits transmitted through a digital transmission channel, 10% are received in error. Let X = the number of bits in error in the next 4 bits transmitted.

  Do they meet the following criteria:

1. Does the experiment consist of **Bernoulli trials**?
2. **Are the trials that constitute the random experiment are independent?**
3. **Is probability of a success on each trial is constant?**

3-8 Binomial Distribution

A random experiment consisting of n repeated trials such that

1. the trials are independent,
2. each trial results in only two possible outcomes, labeled as *success* and *failure*, and
3. the probability of a success on each trial, denoted as p, remains constant

is called a *binomial experiment.*

The random variable X that equals the number of trials that result in a *success* has a **binomial distribution** with parameters p and n where $0 < p < 1$ and $n = \{1, 2, 3, \ldots\}$. The pmf of X is

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \ldots, n \qquad (3\text{-}15)$$

3-8 Binomial Distribution

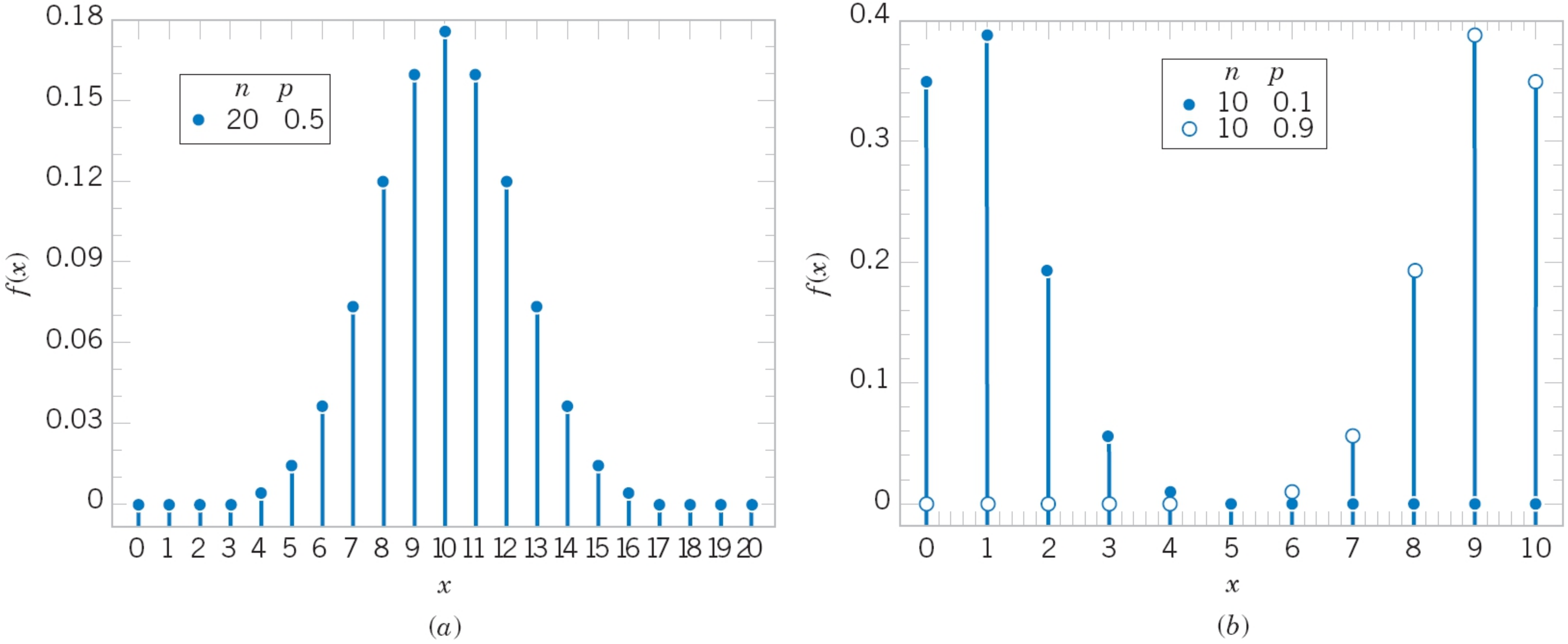


Figure 3-31 Binomial distribution for selected values of n and p.

3-8 Binomial Distribution

If X is a binomial random variable with parameters p and n,

$$\mu = E(X) = np \quad \text{and} \quad \sigma^2 = V(X) = np(1 - p) \qquad (3\text{-}16)$$

EXAMPLE 3-25

Bit Transmission Errors: Binomial Mean and Variance

For the number of transmitted bits received in error in Example 3-18, $n = 4$ and $p = 0.1$ so

$$E(X) = 4(0.1) = 0.4$$

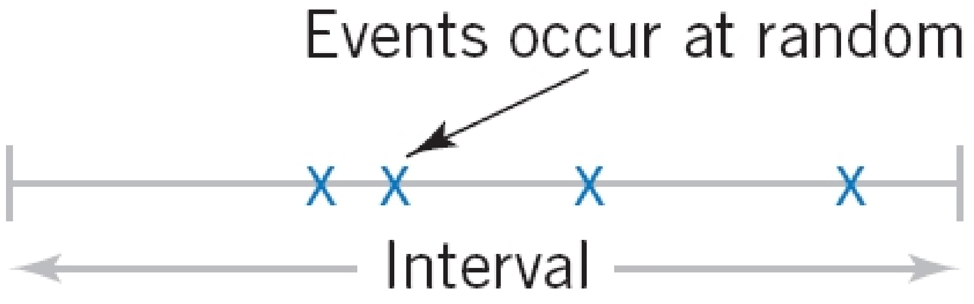
The variance of the number of defective bits is

$$V(X) = 4(0.1)(0.9) = 0.36$$

These results match those that were calculated directly from the probabilities in Example 3-20. ■

3-9 Poisson Process

Figure 3-32 In a Poission process, events occur at random in an interval.



3-9 Poisson Process

3-9.1 Poisson Distribution

Assume that events occur at random throughout the interval. If the interval can be partitioned into subintervals of small enough length such that

1. The probability of more than one event in a subinterval is zero,
2. The probability of one event in a subinterval is the same for all subintervals and proportional to the length of the subinterval, and
3. The event occurrence in each subinterval is independent of other subintervals, the random experiment is called a *Poisson process*.

If the mean number of events in the interval is $\lambda > 0$, the random variable X that equals the number of events in the interval has a **Poisson distribution** with parameter λ, and the pmf of X is

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \qquad x = 0, 1, 2, \ldots \tag{3-17}$$

The mean and variance of X are

$$E(X) = \lambda \qquad \text{and} \qquad V(X) = \lambda \tag{3-18}$$

3-9 Poisson Process

3-9.1 Poisson Distribution

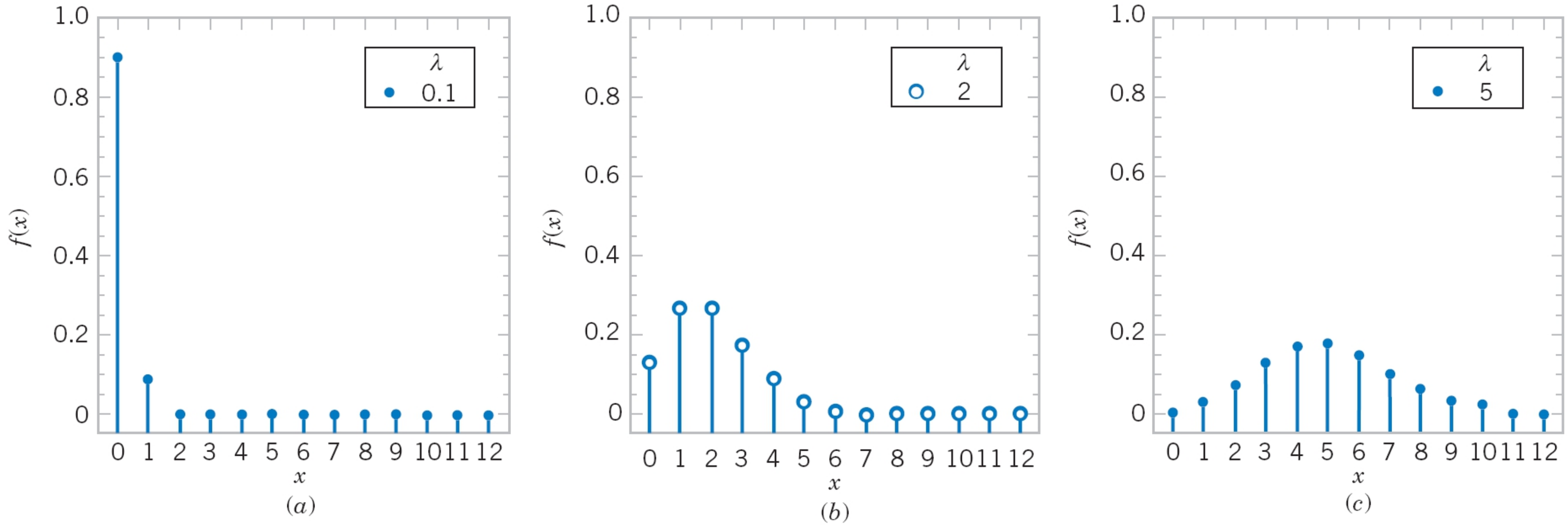


Figure 3-33 Poisson distribution for selected values of the parameter λ.

3-9 Poisson Process

3-9.2 Exponential Distribution

• The discussion of the Poisson distribution defined a random variable to be the number of flaws along a length of copper wire. The distance between flaws is another random variable that is often of interest.

• Let the random variable X denote the *length* from any starting point on the wire until a flaw is detected.

• As you might expect, the distribution of X can be obtained from knowledge of the distribution of the number of flaws. The key to the relationship is the following concept:

> The distance to the first flaw exceeds 3 millimeters if and only if there are no flaws within a length of 3 millimeters—simple, but sufficient for an analysis of the distribution of X.

3-9 Poisson Process

3-9.2 Exponential Distribution

The random variable X that equals the distance between successive events of a Poisson process with mean $\lambda > 0$ has an **exponential distribution** with parameter λ. The pdf of X is

$$f(x) = \lambda e^{-\lambda x}, \qquad \text{for } 0 \leq x < \infty \tag{3-19}$$

The mean and variance of X are

$$E(X) = \frac{1}{\lambda} \quad \text{and} \quad V(X) = \frac{1}{\lambda^2} \tag{3-20}$$

3-9 Poisson Process

3-9.2 Exponential Distribution

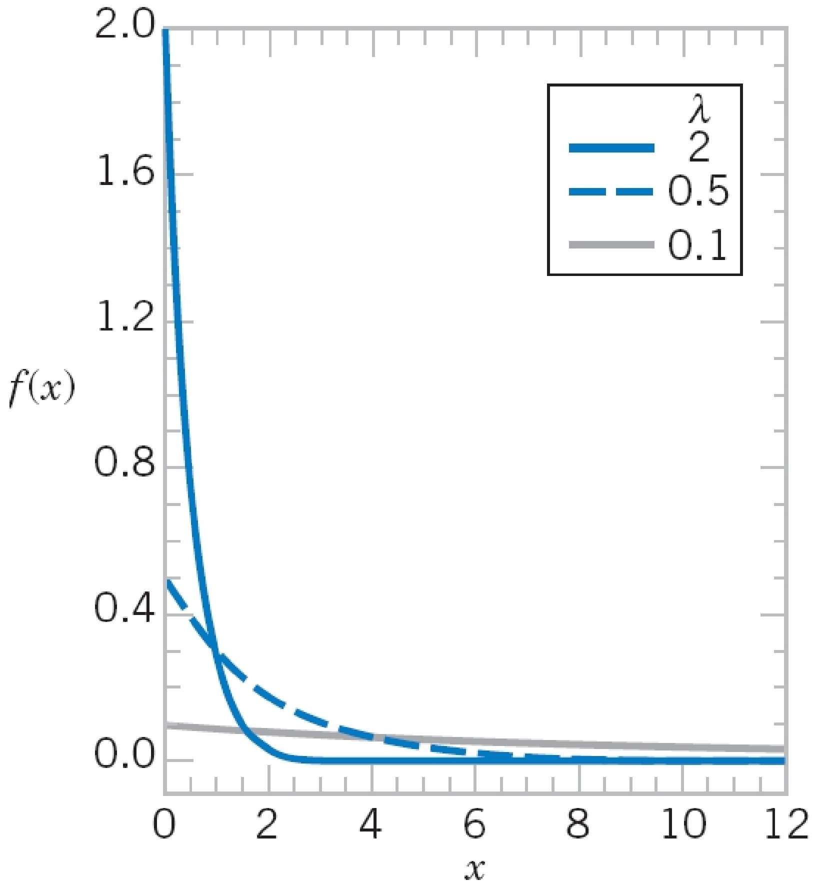


Figure 3-34 Probability density function of an exponential random variable for selected values of λ.

3-9 Poisson Process

3-9.2 Exponential Distribution

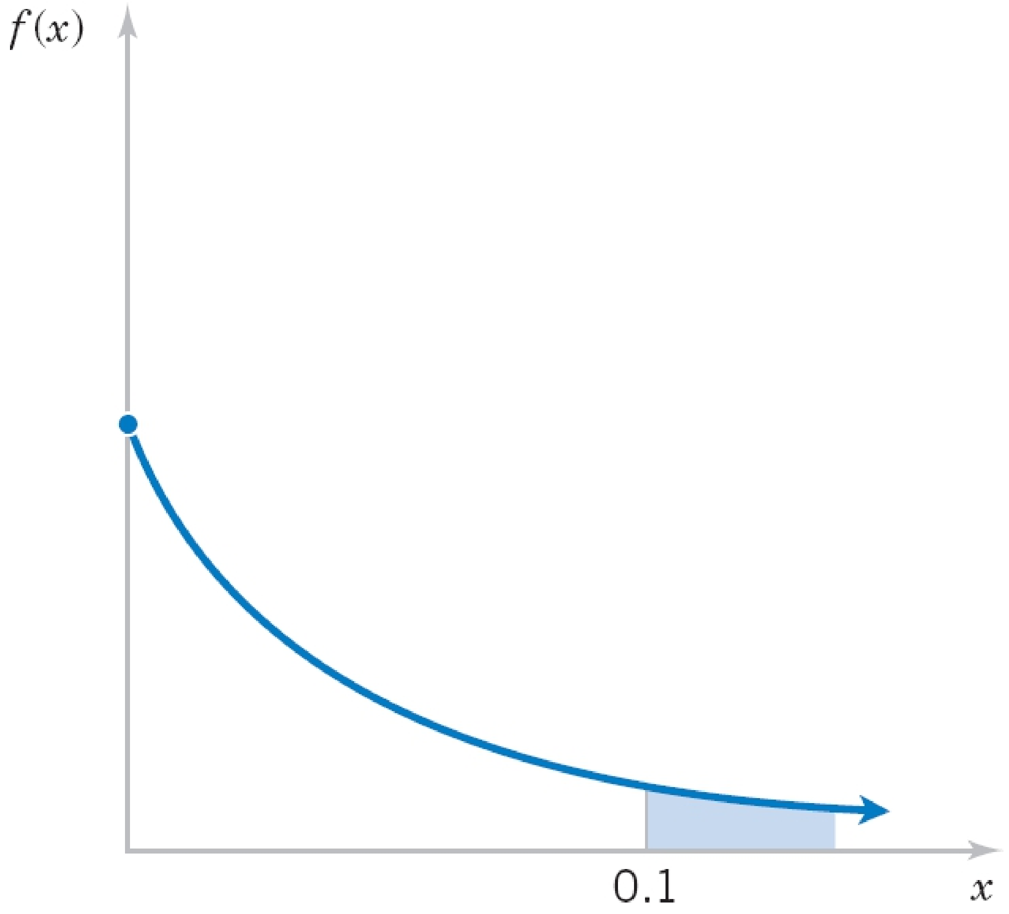


Figure 3-35 Probability for the exponential distribution in Example 3-30.

3-9 Poisson Process

3-9.2 Exponential Distribution

- The exponential distribution is often used in reliability studies as the model for the time until failure of a device.

- For example, the lifetime of a semiconductor chip might be modeled as an exponential random variable with a mean of 40,000 hours. The **lack of memory property** of the exponential distribution implies that the *device does not wear out*. The lifetime of a device with failures caused by random shocks might be appropriately modeled as an exponential random variable.

- However, the lifetime of a device that suffers slow mechanical wear, such as bearing wear, is better modeled by a distribution that does not lack memory.

3-10 Normal Approximation to the Binomial and Poisson Distributions

Normal Approximation to the Binomial

If X is a binomial random variable,

$$Z = \frac{X - np}{\sqrt{np(1-p)}} \qquad (3\text{-}21)$$

is approximately a standard normal random variable. Consequently, probabilities computed from Z can be used to approximate probabilities for X.

3-10 Normal Approximation to the Binomial and Poisson Distributions

Normal Approximation to the Binomial

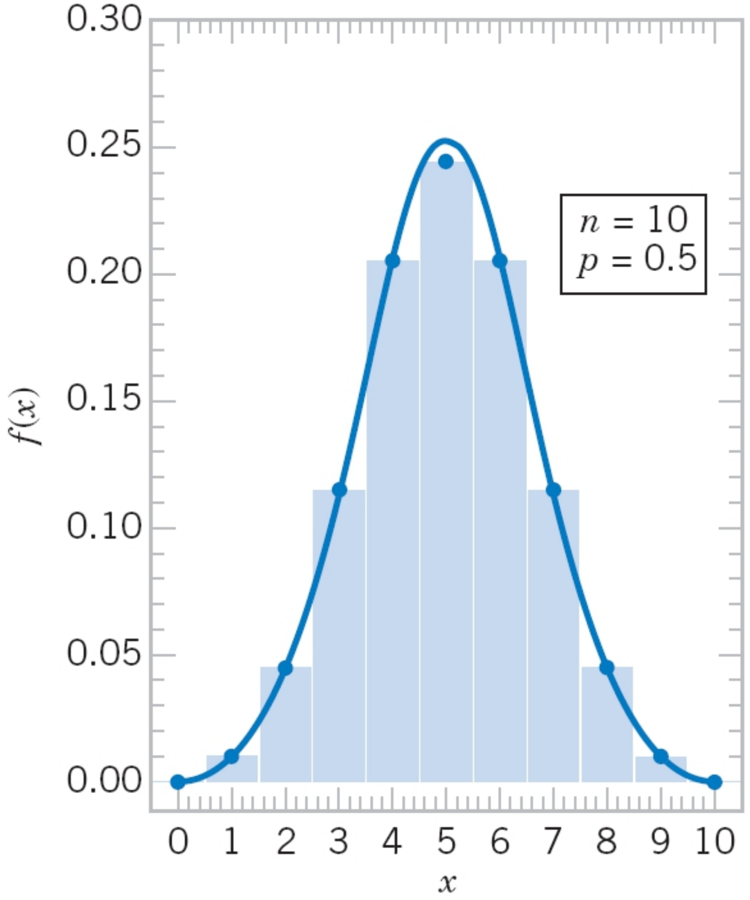


Figure 3-36 Normal approximation to the binomial distribution.

3-10 Normal Approximation to the Binomial and Poisson Distributions

Normal Approximation to the Poisson

If X is a Poisson random variable with $E(X) = \lambda$ and $V(X) = \lambda$,

$$Z = \frac{X - \lambda}{\sqrt{\lambda}} \tag{3-22}$$

is approximately a standard normal random variable.

3-11 More Than One Random Variable and Independence

3-11.1 Joint Distributions

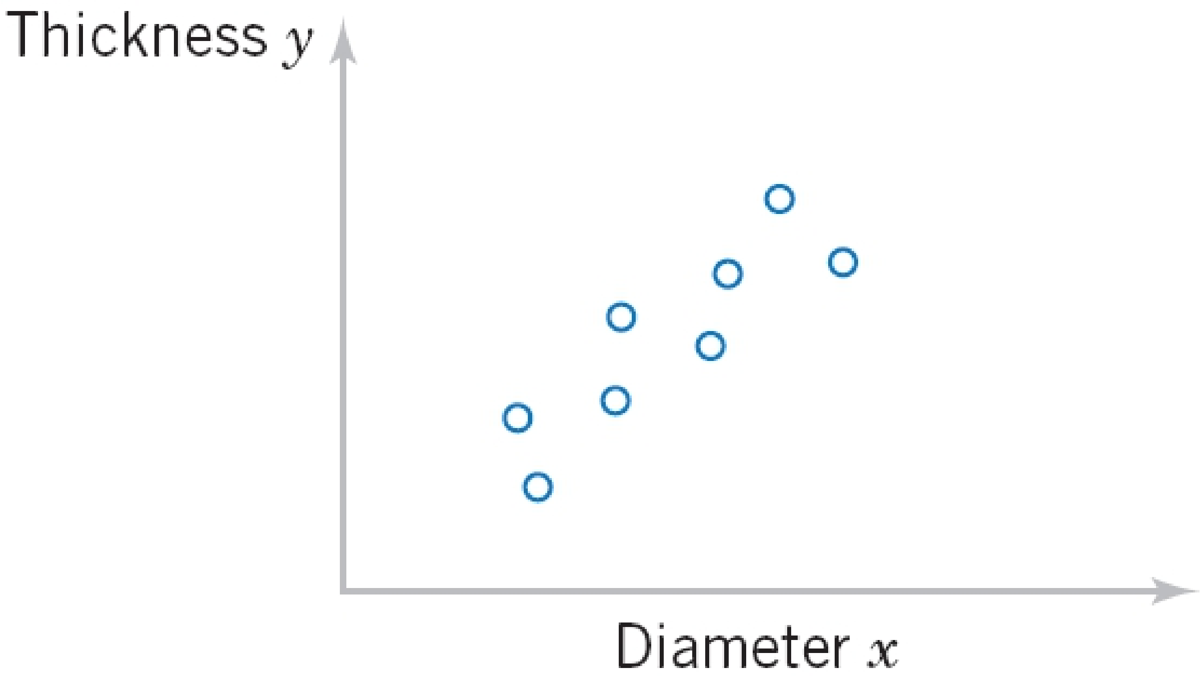


Figure 3-38 Scatter diagram of diameter and thickness measurements.

3-11 More Than One Random Variable and Independence

3-11.1 Joint Distributions

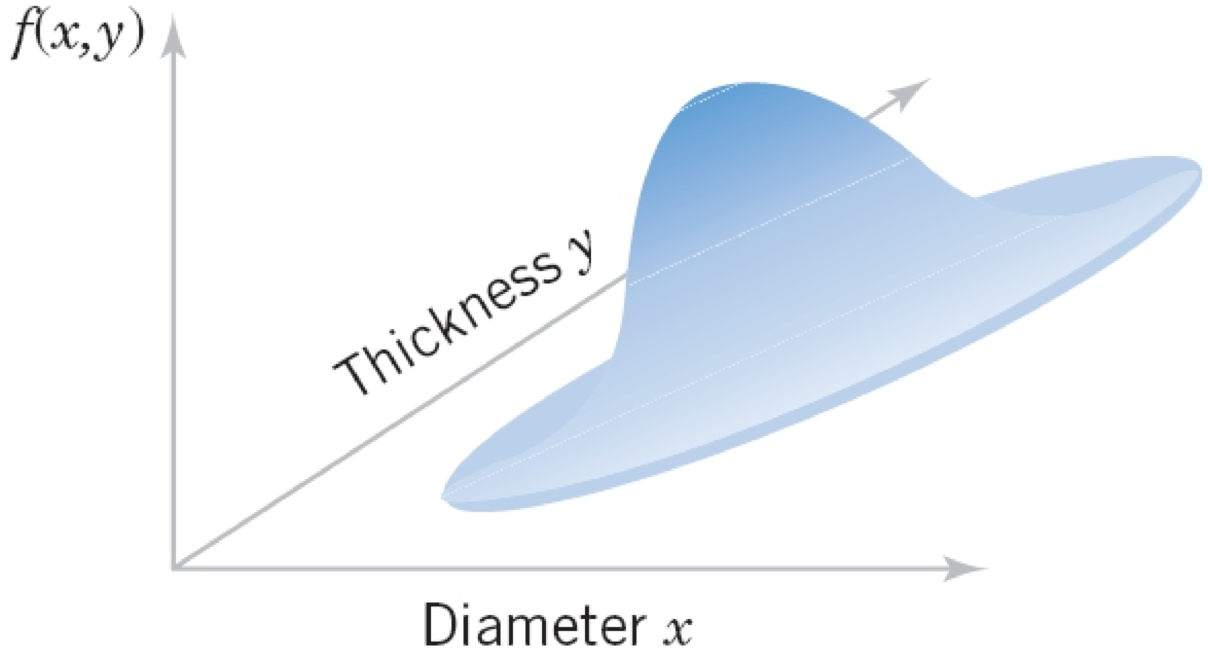


Figure 3-39 Joint probability density function of x and y.

3-11 More Than One Random Variable and Independence

3-11.1 Joint Distributions

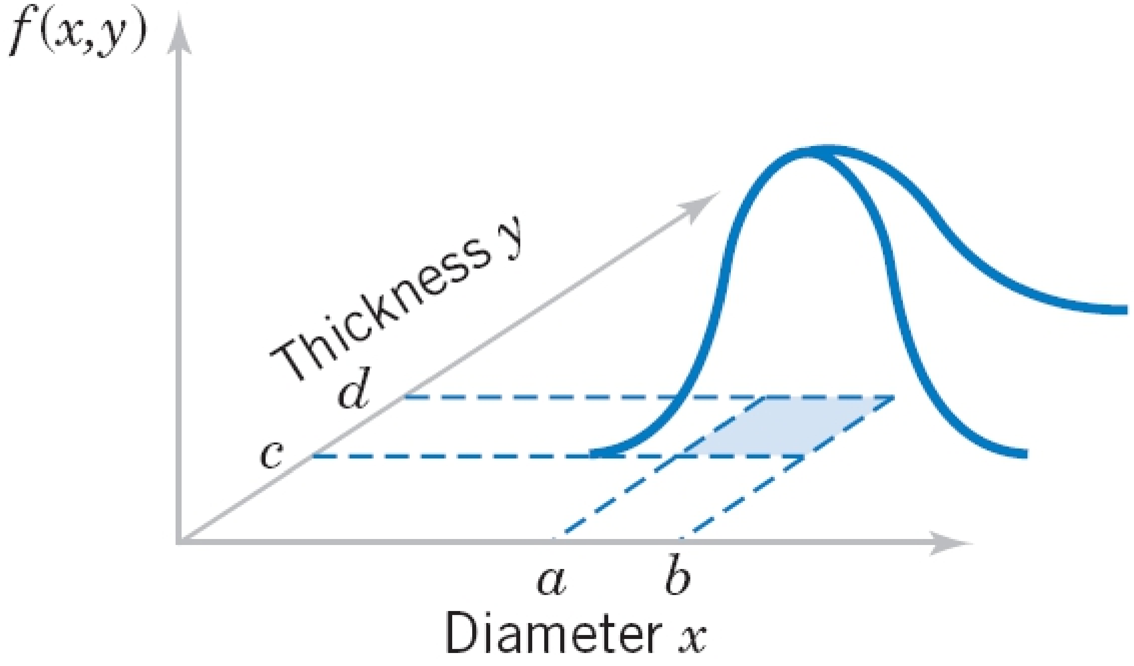


Figure 3-40 Probability of a region is the volume enclosed by $f(x, y)$ over the region.

3-11 More Than One Random Variable and Independence

3-11.1 Joint Distributions

$$P(a < X < b, c < Y < d) = \int_a^b \int_c^d f(x, y)\, dy\, dx$$

3-11 More Than One Random Variable and Independence

3-11.2 Independence

The random variables $X_1, X_2, \ldots, X_n$ are **independent** if

$$P(X_1 \in E_1, X_2 \in E_2, \ldots, X_n \in E_n) = P(X_1 \in E_1)P(X_2 \in E_2)\cdots P(X_n \in E_n)$$

for *any* sets $E_1, E_2, \ldots, E_n$.

3-12 Functions of Random Variables

$$Y = X + c$$

$$E(Y) = E(X) + c = \mu + c \tag{3-23}$$

$$V(Y) = V(X) + 0 = \sigma^2 \tag{3-24}$$

$$Y = cX$$

$$E(Y) = E(cX) = cE(X) = c\mu \tag{3-25}$$

$$V(Y) = V(cX) = c^2V(X) = c^2\sigma^2 \tag{3-26}$$

3-12 Functions of Random Variables

3-12.1 Linear Combinations of Independent Random Variables

The mean and variance of the linear function of **independent** random variables are

$$Y = c_0 + c_1X_1 + c_2X_2 + \cdots + c_nX_n$$

$$E(Y) = c_0 + c_1\mu_1 + c_2\mu_2 + \cdots + c_n\mu_n \qquad (3\text{-}27)$$

and

$$V(Y) = c_1^2\sigma_1^2 + c_2^2\sigma_2^2 + \cdots + c_n^2\sigma_n^2 \qquad (3\text{-}28)$$

3-12 Functions of Random Variables

3-12.1 Linear Combinations of Independent Random Variables

Let $X_1, X_2, \ldots, X_n$ be independent, normally distributed random variables with means $E(X_i) = \mu_i$, $i = 1, 2, \ldots, n$ and variances $V(X_i) = \sigma_i^2$, $i = 1, 2, \ldots, n$. Then the linear function

$$Y = c_0 + c_1X_1 + c_2X_2 + \cdots + c_nX_n$$

is normally distributed with mean

$$E(Y) = c_0 + c_1\mu_1 + c_2\mu_2 + \cdots + c_n\mu_n$$

and variance

$$V(Y) = c_1^2\sigma_1^2 + c_2^2\sigma_2^2 + \cdots + c_n^2\sigma_n^2$$

3-12 Functions of Random Variables

3-12.2 What If the Random Variables Are Not Independent?

The correlation between two random variables X_1 and X_2 is

$$\rho_{X_1X_2} = \frac{E(X_1X_2) - \mu_1\mu_2}{\sqrt{\sigma_1^2\sigma_2^2}} = \frac{\text{Cov}(X_1, X_2)}{\sqrt{\sigma_1^2\sigma_2^2}} \qquad (3\text{-}29)$$

with $-1 \leq \rho_{X_1X_2} \leq +1$, and $\rho_{X_1X_2}$ is usually called the **correlation coefficient.**

3-12 Functions of Random Variables

3-12.2 What If the Random Variables Are Not Independent?

Let $X_1, X_2, \ldots, X_n$ be random variables with means $E(X_i) = \mu_i$ and variances $V(X_i) = \sigma_i^2$, $i = 1, 2, \ldots, n$, and covariances $\text{Cov}(X_1, X_2)$, $i, j = 1, 2, \ldots, n$ with $i < j$. Then the mean of the linear combination

$$Y = c_0 + c_1X_1 + c_2X_2 + \cdots + c_nX_n$$

is

$$E(Y) = c_0 + c_1\mu_1 + c_2\mu_2 + \cdots + c_n\mu_n \tag{3-30}$$

and the variance is

$$V(Y) = c_1^2\sigma_1^2 + c_2^2\sigma_2^2 + \cdots + c_n^2\sigma_n^2 + 2\sum_{i<j}\sum c_ic_j \,\text{Cov}(X_i, X_j) \tag{3-31}$$

3-12 Functions of Random Variables

3-12.3 What If the Function Is Nonlinear?

If X has mean μ_X and variance σ_X^2, the approximate mean and variance of Y can be computed using the following result:

$$E(Y) = \mu_Y \simeq h(\mu_X) \tag{3-32}$$

$$V(Y) = \sigma_Y^2 \simeq \left(\frac{dh}{dX}\right)^2 \sigma_X^2 \tag{3-33}$$

where the derivative dh/dX is evaluated at μ_X.

3-12 Functions of Random Variables

3-12.3 What If the Function Is Nonlinear?

Let

$$Y = h(X_1, X_2, \ldots, X_n)$$

for independent random variables X_i, $i = 1,2,\ldots, n$, each with mean μ_i and variance σ_i^2, the approximate mean and variance of Y are

$$E(Y) = \mu_Y \simeq h(\mu_1, \mu_2, \ldots, \mu_n) \tag{3-37}$$

$$V(Y) = \sigma_Y^2 \simeq \sum_{i=1}^{n} \left(\frac{\partial h}{\partial X_i}\right)^2 \sigma_i^2 \tag{3-38}$$

where the partial derivatives $\partial h / \partial X_i$ are evaluated at $\mu_1, \mu_2, \ldots, \mu_n$.