Introduction

In 1932, Gödel published a brief note demonstrating that the intuitionistic propositional calculus (IPC) cannot be characterized by finite-valued semantics. Although the argument he employed was brilliant, it was presented in an extremely concise manner (merely two paragraphs!), making it challenging to comprehend. In the subsequent discussion, I will endeavor to present it in a more detailed fashion, addressing any gaps with additional or alternative definitions and arguments. Following Gödel's lead, we will denote the intuitionistic propositional calculus as $\mathbf{H}$.

Preliminaries

> (**Language**) We will use $\mathcal{L}$ to denote the set of every wff induced by the unary connective $\neg$ and the binary connectives $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ in a countable infinite set of propositional letters $\{p_0, p_1, \ldots\}$.

> (**Matrix**) A matrix for $\mathcal{L}$ is a tuple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ where $\langle \mathcal{V}, \mathcal{O} \rangle$ is an algebra for $\mathcal{L}$ and $\mathcal{D} \subset \mathcal{V}$ is the set of designated values.

> (**Valuation function**) Given a matrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, a valuation in $\mathcal{M}$ is a function $v : \mathcal{L} \rightarrow \mathcal{V}$ such that, for every n-ary connective $\circ$ and every $\alpha_1, \ldots, \alpha_n \in \mathcal{L}$,
>
> $$v(\circ(\alpha_1, \ldots, \alpha_n)) = \tilde{\circ}(v(\alpha_1), \ldots, v(\alpha_n))$$
>
> where $\tilde{\circ} \in \mathcal{O}$ is the algebraic counterpart of the connective.

> (**Valid formula**) We say that a formula $\varphi \in \mathcal{L}$ is valid in a matrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ if, and only if, for every valuation function v, $v(\varphi) \in \mathcal{D}$. We will denote this as $\mathcal{M} \Vdash \varphi$.

The strategy

Suppose there is a matrix for $\mathcal{L}$ which validates those, and only those, formulas that are provable in $\mathbf{H}$. More formally, we are interested in a finite-valued matrix $\mathcal{M}$ such that,

$$\forall \varphi \in \mathcal{L}, \mathcal{M} \Vdash \varphi \iff \mathbf{H} \vdash \varphi$$

The right-to-left direction ("those") is trivial: just take the usual matrix for classical logic. We are interested in the left-to-right ("only those") direction, which is equivalent to:

$$\neg \exists \varphi \in \mathcal{L}, \mathcal{M} \Vdash \varphi \wedge \mathbf{H} \nvdash \varphi$$

Therefore, if we find a formula φ valid in $\mathcal{M}$ and show that this formula is not provable in $\mathbf{H}$, we are forced to reject $\mathcal{M}$ as an adequate characterization of $\mathbf{H}$. The strategy of Gödel's argument will be to show that, given any finite-valued matrix strong enough to validate every formula provable in $\mathbf{H}$, this matrix will necessarily validate also a formula not provable in $\mathbf{H}$, implying that no finite-valued matrix can satisfy only the formulas provable in intuitionistic propositional calculus.

The formula F_n

Let $n > 1$ be a natural number. Consider the following family of formulas:

$$F_n = \bigvee_{1 \leq i < k \leq n} (p_i \leftrightarrow p_k)$$

where p_i and p_k are propositional letters. Here are some examples.

n	F_n
2	$p_1 \leftrightarrow p_2$
3	$(p_1 \leftrightarrow p_2) \vee (p_1 \leftrightarrow p_3) \vee (p_2 \leftrightarrow p_3)$
4	$(p_1 \leftrightarrow p_2) \vee (p_1 \leftrightarrow p_3) \vee (p_1 \leftrightarrow p_4) \vee (p_2 \leftrightarrow p_3) \vee (p_2 \leftrightarrow p_4) \vee (p_3 \leftrightarrow p_4)$

Lemma 1 Let $n \geq 1$. For every n-valued matrix $\mathcal{M}$ such that $\mathcal{M} \Vdash (p \leftrightarrow p) \vee q$, we have that $\mathcal{M} \Vdash F_{n+1}$.

Proof Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where $\mathcal{V} = \{1, 2, \ldots, n\}$, be a matrix. By definition, a valuation in $\mathcal{M}$ is a function $v : \mathcal{L} \rightarrow \mathcal{V}$. Assume that $\mathcal{D} \subset V$ is the set of designated values. We want to show that, for every v, $v(F_{n+1}) \in \mathcal{D}$. Let v' be a valuation function. Now, note that F_{n+1} have $n+1$ propositional letters and, by hypothesis, we have only n truth-values. By the **pigeonhole principle**, we are forced to conclude that there are at least two propositional letters, say p_i and p_j, such that $v'(p_i) = v'(p_j)$ (**A**). Now, let us rearrange the expression in the following way: $F_{n+1} = (p_i \leftrightarrow p_j) \vee \varphi$ and let $\beta = (p \leftrightarrow$

> $p) \vee q$. By hypothesis, for every valuation v in $\mathcal{M}$, $v(\beta) \in \mathcal{D}$. In particular, $v'(\beta) \in \mathcal{D}$. By uniform substitution, $v'(\beta[p_i/p, \varphi/q]) \in \mathcal{D}$ (**B**). Now, by definition of valuation function,
>
> $$v'(F_{n+1}) = \tilde{\vee}(\tilde{\leftrightarrow}(v'(p_i), v'(p_j)), v'(\varphi))$$
>
> Using (**A**) and (**B**), we conclude that $v'(F_{n+1}) \in \mathcal{D}$, as desired.

As a consequence of this lemma, we can conclude, for instance, that in the usual 2-valued matrix for classical logic, the formula $(p \leftrightarrow q) \vee (p \leftrightarrow r) \vee (q \leftrightarrow r)$ is valid. More generally, every F_n with $n \geq 3$ is true in classical logic. As a consequence of the next lemma, we will conclude that the same is not the case on IPC.

F_n is not intuitionistically valid

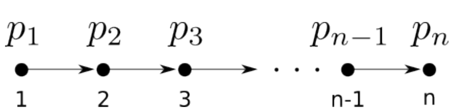
The algebra S_n

The idea is that, given any index $n > 1$, we can construct a Heyting algebra, called S_n, on which there is a valuation, called the *canonical valuation* h, where $h(F_n) \notin \mathcal{D}$.

> (S_n) Given some $n > 1$, define the universe of the algebra as $\{1, 2, \ldots, n\}$, which are ordered in the usual way, and set the designated value as 1. Now, define the operators as:
>
> $$a \vee b = min(a, b)$$
> $$a \wedge b = max(a, b)$$
> $$a \rightarrow b = 1 \text{ for } a \geq b \text{ and } a \rightarrow b = b \text{ for } a < b$$
> $$\neg a = n \text{ for } a \neq n \text{ and } \neg n = 1$$

> (**Remark**) To reconstruct the argument using the same algebra as provided by Gödel, we will subtract $\leftrightarrow$ from our language and redefine $\varphi \leftrightarrow \psi$ as a notation for $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$.

> (**Canonical valuation**) Let h be a valuation function in S_n, such that, for each propositional letter p_i, $h(p_i) = i$, as illustrated by the following image.



Recursive F_n

We will give an alternative definition of F_n, which we will call as recursive F_n (or rF_n) that will be useful for the next lemma.

> (rF_n) Given $n > 1$, let rF_n be the formula defined as follows:
>
> $$rF_2 = p_1 \leftrightarrow p_2$$
> $$rF_{n+1} = rF_n \vee \bigvee_{1 \leq i < n} (p_i \leftrightarrow p_{n+1})$$

It is not difficulty to see that:

> **Proposition** For every n, $F_n = rF_n$.

Now, using this "new" definition of F_n, we can prove the following lemma by induction.

> **Lemma 2** For every $n > 1$, $h(rF_n) = 2$.

> **Proof** Induction on n. (**Base case**) By definition of h, $h(p_1) = 1$ and $h(p_2) = 2$. Therefore, $h(p_1 \leftrightarrow p_2) = max(h(p_1 \to p_2), h(p_2 \to p_1)) = max(2, 1) = 2$. (**Inductive step**) Assume that $h(rF_n) = 2$. By definition of rF_{n+1}, $h(rF_{n+1}) = min(h(rF_n), h(\bigvee_{1 \leq i < n}(p_i \leftrightarrow p_{n+1}))) = min(2, h(\bigvee_{1 \leq i < n}(p_i \leftrightarrow p_{n+1})))$ (**A**). Note that, for every i, $i < n$. Therefore, $h(p_i) < h(p_{n+1})$. By definition of h, $h(p_i \leftrightarrow p_{n+1}) = max(h(p_i \to p_{n+1}), h(p_{n+1} \to p_i)) = max(n + 1, 1) = n + 1$. Therefore, $h(\bigvee_{1 \leq i < n}(p_i \leftrightarrow p_{n+1})) = n + 1$. Using (**A**), we get that $h(rF_{n+1}) = min(2, n + 1) = 2$.

From this lemma, we can conclude that, for every $n > 1$, $h(rF_n) \neq 1$. Therefore, $h(rF_n) \notin \mathcal{D}$. Using the Proposition above with Lemma 2, we have that:

> **Corollary** For every $n > 1$, $S_n \not\Vdash F_n$.

And then, using the completeness of the Heyting algebra S_n with respect to the intuitionistic propositional calculus, we conclude that:

> **Corollary** For every $n > 1$, $\mathbf{H} \not\vdash F_n$.

> (**Remark**) Is worth noting that, as another interesting consequence of the pigeonhole principle, we can conclude that, for every $n > 1$, $S_n \Vdash F_{n+1}$.