Theoretical and practical investigations on fuzzy relational systems

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Summary

Fuzzy relational systems (FRSs) have been used in many applications in different areas during the last decades, for example:

- Biomathematics [Barros et al., 2017];
- Engineering [Pedrycz and Gomide, 2007] [Ross, 2005];
- Control systems [Moura and Sussner, 2018];

which emphasizes the importance of studying FRSs.

Introduction - FRSs

The fuzzy relational system proposed by Mamdani and Assilian in 1975 [Mamdani and Assilian, 1975] became popular due to his ease of application in practical problems and intuitive graphic interpretation.

Nevertheless, as demonstrated by the works of Martin Stepnicka *et al*, there are other kinds relational systems suitable to be applied in practical problems besides having a different logic interpretation than the Mamdani-Assilian system.

Introduction - Moser-Navara axioms

The axioms proposed by Moser and Navara in 2002 [Moser and Navara, 2002] give a mathematical foundation to evaluate the coherence of an inference system.

Later, this work was extended by Martin Stepnicka *et al.* to evaluate other inference systems besides the one proposed by Mamdani-Assilian.

Introduction - Applications

Finally, we will use the the Wang-Mendel algorithm [Wang and Mendel, 1992] for learning fuzzy rules to analyze the behaviour of the different studied FRSs in some benchmark regression problems found on the internet.

Summary

Concepts - Crisp binary relation

A crisp binary relation can be seen as the relationship (or the absence of it) between elements of two sets. In other words, it is a relation $R(x, y)$ with $x \in X$ and $y \in Y$ is a map $R : X \times Y \to \{0, 1\}$.

We will use the following definition for a crisp binary relation over fuzzy sets:

Definition 2.1

Given two fuzzy sets $A \in \mathcal{F}(X)$ and $B \in \mathcal{F}(Y)$ and a binary operator $\sim: [0, 1] \times [0, 1] \to \{0, 1\}$, we define a crisp binary relation $R_{A\sim B}$ as the set of ordered pairs (x, y) so that

$$(x, y) \in R_{A\sim B} \Leftrightarrow A(x) \sim B(y). \tag{1}$$

Concepts - Subsets based on crisp binary relation I

Using this definition of crisp binary relation, we will introduce a notation for selecting subsets of the space X.

Definition 2.2

Let $A \in \mathcal{F}(X)$, $B \in \mathcal{F}(Y)$, a binary operator $\sim: [0,1] \times [0,1] \to \{0,1\}$ and $y \in Y$ be arbitrary. The set $X_{A\sim B(y)}$ is given by

$$X_{A\sim B(y)} = \{x \in X | (x,y) \in R_{A\sim B}\}. \tag{2}$$

Concepts - Example of subset based on crisp binary relation I

Example 2.3

Let $A \in \mathcal{F}(X)$, $B \in \mathcal{F}(Y)$, and $y \in Y$ be arbitrary. The set $X_{A \geq B(y)} \subseteq X$ is given by

$$X_{A \geq B(y)} = \{x \in X | A(x) \geq B(y)\}. \quad (3)$$

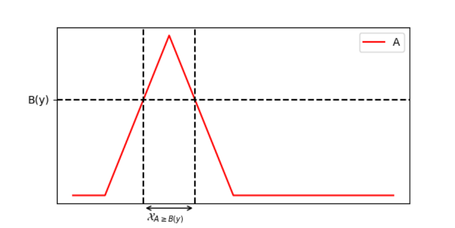


Figure: Subset $X_{A \geq B(y)}$ of X.

Concepts - Subsets based on binary relation II

Now considering two fuzzy sets defined in the same domain, that is $Y = X$, we define the following notation:

Definition 2.4

Let $A, B \in \mathcal{F}(X)$ and a binary operator $\sim : [0,1] \times [0,1] \to \{0,1\}$. The set $X_{A\sim B}$ is given by

$$X_{A\sim B} = \{x \in X | (x, x) \in R_{A\sim B}\}. \tag{4}$$

Concepts - Example of subset based on binary relation II

Example 2.5

Let $A, B \in \mathcal{F}(X)$. The set $X_{A<B} \subseteq X$ is given by

$$X_{A<B} = \{x \in X | A(x) < B(x)\}. \quad (5)$$

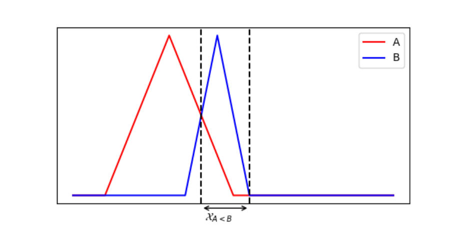


Figure: Subset $X_{A<B}$ of X.

Concepts - Residuated lattices

Since we will deal with residual implications in this work, it is important to introduce the concept of a residuated lattice. This algebraic structure that has some properties that will be very useful for developing fuzzy relational systems later.

Concepts - Partially ordered set

Definition 2.6

[Birkhoff, 1940] A partially ordered set U is a set in which a binary relation $x \leq y$ is defined, which satisfies for all $x, y, z \in U$ the following conditions:

- Reflexive: $x \leq x$
- Antisymmetric: If $x \leq y$ and $y \leq x$, then $x = y$
- Transitive: If $x \leq y$ and $y \leq z$, then $x \leq z$

If $x \leq y$ it is said that x "is less than" or "is contained in" y. On the other way, it can also be read as y "contains" x.

Concepts - Lattice

The *sup* between two elements x and y from a partially ordered set P, denoted by $x \vee y$, is the is the smallest element of P that is greater than or equal to both x and y. [Birkhoff, 1940].

Similarly, the *inf* between two elements x and y from a partially ordered set P, denoted by $x \wedge y$, is the greatest element of P that is smaller than or equal to both x and y [Birkhoff, 1940].

Definition 2.7

[Birkhoff, 1940] A lattice is a partially ordered set P in which any two of its elements have a *sup* $(x \vee y)$ and an inf $(x \wedge y)$.

A lattice P is called complete if $\bigwedge X$ and $\bigvee X$ are defined in P for any subset $X \subseteq P$ [Gratzer, 1971].

Concepts - Residuated lattice

Definition 2.8

[Perfilieva, 2005] A residuated lattice is an algebra $\mathcal{L} = \langle L, \vee, \wedge, *, \rightarrow, 0, 1\rangle$ with four binary operations and two constants such that:

- $\langle L, \vee, \wedge, 0, 1\rangle$ is a lattice with the ordering $\leq$ defined using the operations $\vee$, $\wedge$ as usual and 0, 1 are the smallest and largest elements, respectively;
- $\langle L, *, 1\rangle$ is a commutative monoid, that is, $*$ is a commutative and associative operation with the identity $x * 1 = x$;
- the $\rightarrow$ is a residuation operation with respect to $*$, that is

$$x * y \leq z \quad \Leftrightarrow \quad x \leq y \rightarrow z. \tag{6}$$

If a residuated lattice $\mathcal{L}$ is complete, then we speak of a complete residuated lattice [Sussner, 2015] [Belohlávek, 2012].

A well-known example of a complete residuated lattice is the algebra $\mathcal{L}_* = \langle [0,1], \vee, \wedge, *, \rightarrow, 0, 1 \rangle$, where $*$ is a left-continuous t-norm and $\rightarrow$ is its adjoint residual implication [Holčapek et al., 2022] [Belohlávek, 2012].

Concepts - Complete residuated lattice

Proposition 1

Let $\mathcal{L}_*$ be a complete residuated lattice and $a, b, c \in L$. If $a > b$ then we have that

$$a \to (b * c) = (a \to b) * c. \tag{7}$$

Proposition 2

Let $\mathcal{L}_*$ be a complete residuated lattice and $a, b, c \in L$. If $b > c$ then we have that

$$a * c \leq a * (b \to c) \leq a \wedge (b \to c). \tag{8}$$

Summary

FRS - Modus Ponens

In general, inference is a process to obtain new information using existing knowledge [Lee, 2004]. One type of deductive procedure to perform inferences is the classical *modus ponens* [Nguyen et al., 2018]. The general form of an inference following the *modus ponens* procedure is as follows:

$$\begin{array}{c} \textit{premise}\text{: If } x \text{ is } A \text{ then } y \text{ is } B \\ \textit{fact}\text{: } x \text{ is } A \\ \textit{conclusion}\text{: } y \text{ is } B \end{array}$$

FRS - Generalized Modus Ponens

Approximate or fuzzy reasoning refers to processes by which imprecise conclusions are inferred from imprecise premises [Nguyen et al., 2018]. To this end, one can use a generalized version of the *modus ponens* procedure:

$$\begin{gathered}\textit{premise}\text{: If } x \text{ is } A \text{ then } y \text{ is } B\\ \textit{fact}\text{: } x \text{ is } A'\\ \textit{conclusion}\text{: } y \text{ is } B'\end{gathered}$$

It is possible to observe that, even in a scenario with uncertainties, it is still possible to reach a conclusion in light of the facts. Thus, modeling the generalized *modus ponens* using the concepts of fuzzy set theory we obtain a fuzzy inference mechanism.

FRS - Rule base

A fuzzy rule base can be represented using the concepts of fuzzy relations and sets seen in the previous section. In this work we will use two approaches to this modeling [Pedrycz and Gomide, 2007]:

Definition 3.1

Given a finite set of rules of the form

$$\textit{If } x \textit{ is } A_i \textit{ then } y \textit{ is } B_i \quad i = 1, \ldots, n,$$

we define a fuzzy rule base as follows:

1. Conjunctive rule base: $\check{R}(x, y) = \bigvee_{i=1}^{n} A_i(x) * B_i(y)$.
2. Implicative rule base: $\hat{R}(x, y) = \bigwedge_{i=1}^{n} A_i(x) \to B_i(y)$.

FRS - Conjunctive rule base

Dubois and Prade made an important study [Dubois and Prade, 1996] about the semantics of a conjunctive rule base, that we will quote here:

> *"It seems that fuzzy rules modeled in this way are not seen as restrictions, but rather pieces of information. So, the aggregation by the maximum expresses the accumulation of information."*

It is interesting to note that this approach does not directly correspond to the IF-THEN rule model, so the notation most suitable for a rule base modeled in this way would be the following:

$$x \textit{ is } A_i \textit{ and } y \textit{ is } B_i \quad i = 1, \ldots, n$$

FRS - Implicative rule base

Dubois and Prade also studied the semantics of an implicative rule base [Dubois and Prade, 1996]. Quoting the authors again:

> *"In this view, each piece of information (fuzzy rule) is seen as a restriction. This view naturally leads to a conjunctive way of aggregating the individual pieces of information since the more information, the more constraints and fewer possible values that satisfies them."*

FRS - Rule bases example

Example 3.2

Suppose a set of rules like:

$$\textit{If } x \textit{ is } A_i \textit{ then } y \textit{ is } B_i \qquad i = 1, 2, 3$$

where

$$A_1(x) = triang(x, 1, 3, 5) \qquad A_2(x) = triang(x, 3, 5, 7) \qquad A_3(x) = triang(x, 5, 7, 9)$$

$$B_1(y) = triang(y, 1, 2, 3) \qquad B_2(y) = triang(y, 3, 4, 5) \qquad B_3(y) = triang(y, 5, 6, 7)$$

Also consider the following two types of rule bases (conjunctive and implicative) from Definition 3.1:

$$\check{R}(x, y) = \bigvee_{i=1}^{3} A_i(x) * B_i(y); \qquad \hat{R}(x, y) = \bigwedge_{i=1}^{3} A_i(x) \to B_i(y).$$

FRS - Rule bases example

Example

Taking the Minimum t-norm (Conjunctive rule base) and the Gödel implication (Implicative rule base), we have the following visualizations of these fuzzy relations in a three-dimensional space:

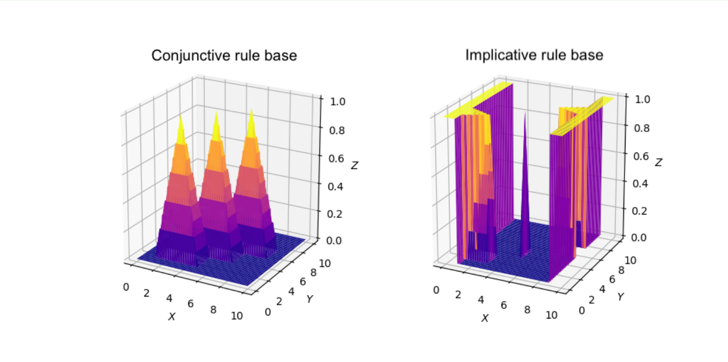


Figure: Representation of the Conjunctive and Implicative fuzzy rule bases.

FRS - Rule bases example

Example

This graphical example clarifies the meaning of ”snippets of information” (conjunctive rule base) and ”constraints” (implicative rule base) cited in the work of D. Dubois and H. Prade.

FRS - Fuzzy relational systems

Given a fuzzy rule base represented by a fuzzy relation $R \in \mathcal{F}(X \times Y)$, and an input represented by a fuzzy set $A' \in \mathcal{F}(X)$, we can obtain an output $B' \in \mathcal{F}(Y)$ from the composition of fuzzy relations [Pedrycz and Gomide, 2007]:

1. Sup-t composition: $B'(y) = A'(x) \circ R(x,y) = \bigvee_{x \in X} A'(x) * R(x,y)$.
2. Inf-I composition: $B'(y) = A'(x) \lhd R(x,y) = \bigwedge_{x \in X} A'(x) \to R(x,y)$.

The Sup-t composition described here is known in the literature as the Compositional Rule of Inference or CRI [Zadeh, 1973]; the Inf-I composition is also known as the Bandler-Kohout Subproduct or BKS [Bandler and Kohout, 1980]. Therefore, we will refer to them as CRI and BKS compositions respectively.

FRS - Combinations

Thus, we have 4 types of fuzzy relational systems formed by the combinations of compositions and rule bases [Stepnicka and Mandal, 2018] [Stepnicka, 2016] [Stepnicka and Mandal, 2015]:

1. Conjunctive rules and CRI: $B'(y) = A'(x) \circ \check{R}(x,y)$
2. Implicative rules and CRI: $B'(y) = A'(x) \circ \hat{R}(x,y)$
3. Conjunctive rules and BKS: $B'(y) = A'(x) \lhd \check{R}(x,y)$
4. Implicative rules and BKS: $B'(y) = A'(x) \lhd \hat{R}(x,y)$

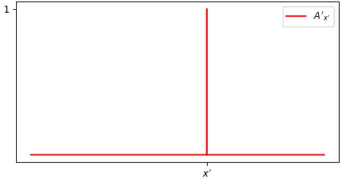
FRS - Characterization of the outputs

To facilitate the understanding of the study of the FRSs, we will divide our analysis in three cases with increasing levels of complexity.

FRS - Crisp input

In the first case, let us consider a crisp input $A'_{x'}(x)$ and a finite rule base $R = (A_i, B_i)$ for $i = 1, \ldots, n$, where $A_i \in \mathcal{F}(X)$ e $B_i \in \mathcal{F}(Y)$.

$$A'_{x'}(x) = crisp(x, x') = \begin{cases} 1, & \text{if } x = x', \\ 0, & \text{otherwise,} \end{cases}$$



FRS - Crisp input

Proposition 3

Given a crisp input $A'_{x'}(x)$, the combinations of CRI composition with conjunctive rules and BKS composition with conjunctive rules are equal, which means

$$A'_{x'}(x) \circ \check{R}(x,y) = A'_{x'}(x) \lhd \check{R}(x,y) = \bigvee_{i=1}^{n} (A_i(x') * B_i(y)). \tag{9}$$

FRS - Crisp input

Proposition 4

Given a crisp input $A'_{x'}(x)$, the combinations of CRI composition with implicative rules and BKS composition with implicative rules are equal, which means

$$A'_{x'}(x) \circ \hat{R}(x, y) = A'_{x'}(x) \triangleleft \hat{R}(x, y) = \bigwedge_{i=1}^{n} (A_i(x') \to B_i(y)). \tag{10}$$

FRS - Crisp input

Example 3.3

Consider the sets $X, Y = [0, 10] \subset \mathcal{R}$, an input $A'_{3.5}(x) = crisp(x, 3.5)$ and the following rule base:

Antecedents	**Consequents**
$A_1 = triang(x, 1, 3, 5)$	$B_1 = triang(y, 1, 2, 3)$
$A_2 = triang(x, 3, 5, 7)$	$B_2 = triang(y, 3, 4, 5)$
$A_3 = triang(x, 5, 7, 9)$	$B_3 = triang(y, 5, 6, 7)$

Also consider the minimum t-norm $\wedge$ and its adjoint Gödel implication $\rightarrow_M$.

Example

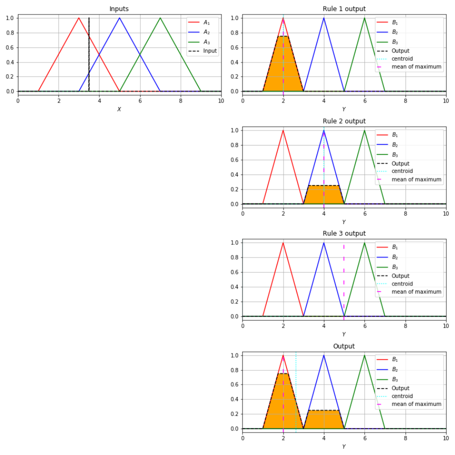


Figure: Output for conjunctive rules

FRS - Crisp input

Example

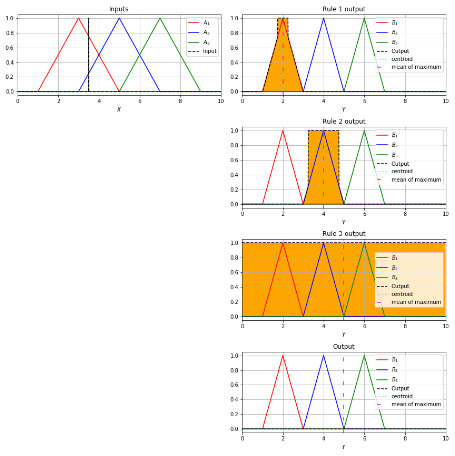


Figure: Output for implicative rules

FRS - Crisp input

Let us look at another similar example, but with consequents that have a non-empty intersection.

Example 3.4

Consider the sets $X, Y = [0, 10] \subset \mathcal{R}$, an input $A'_{3.5}(x) = crisp(x, 3.5)$ and the following rule base:

Antecedents	**Consequents**
$A_1 = triang(x, 1, 3, 5)$	$B_1 = triang(y, 1, 3, 5)$
$A_2 = triang(x, 3, 5, 7)$	$B_2 = triang(y, 3.5, 5.5, 7.5)$
$A_3 = triang(x, 5, 7, 9)$	$B_3 = triang(y, 7, 8, 9)$

Also consider the minimum t-norm $\wedge$ and its adjoint Gödel implication $\rightarrow_M$.

FRS - Crisp input

Example

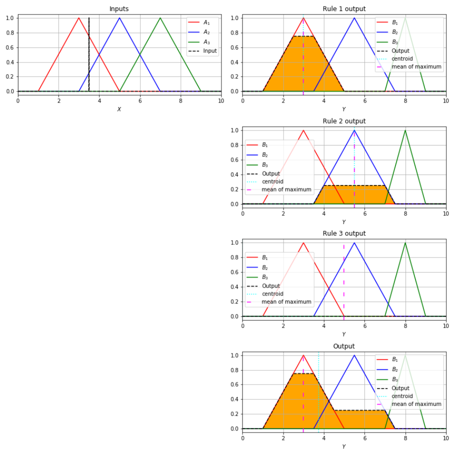


Figure: Output for conjunctive rules

FRS - Crisp input

Example

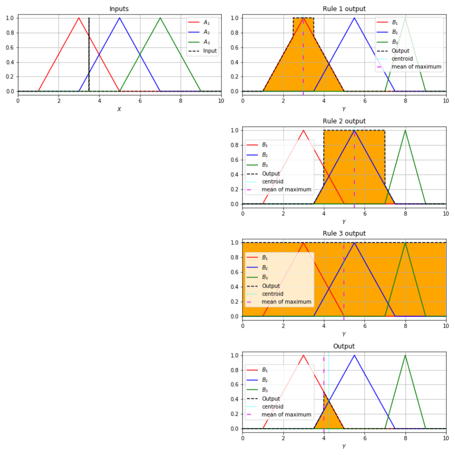


Figure: Output for implicative rules

Now let us look at a slightly more complex case, where the input A' is a normal fuzzy set and the rule base has only a single rule $R = (A_1, B_1)$.

For the combination of CRI composition with a single conjunctive rule, applying the associative property of the t-norm we have that

$$\begin{aligned} B'(y) &= \bigvee_{x \in X} A'(x) * (A_1(x) * B_1(y)) \\ &= \bigvee_{x \in X} (A'(x) * A_1(x)) * B_1(y) \\ &= \alpha * B_1(y), \end{aligned} \tag{11}$$

where $\alpha = \bigvee_{x \in X} (A'(x) * A_1(x))$ is defined as the "degree of activation" of the rule. The value of α is fixed and independent of y.

FRS - Fuzzy input and one rule (BKS + conj)

In the combination of the BKS composition with a single conjunctive rule, we have that $B'(y) = \bigwedge_{x \in X} A'(x) \to (A_1(x) * B_1(y))$. Using Definition 2.4 to partition the domain X as in Example 2.5 leads to

$$B'(y) = \bigwedge_{x \in X_{A' \leq A_1}} A'(x) \to (A_1(x) * B_1(y)) \wedge \bigwedge_{x \in X_{A' > A_1}} A'(x) \to (A_1(x) * B_1(y)). \tag{12}$$

FRS - Fuzzy input and one rule (BKS + conj)

As we are working with normal fuzzy sets, it is guaranteed that $X_{A' \leq A_1} \neq \emptyset$. However, the same cannot be said about the set $X_{A' > A_1}$.

Proposition 5

Let $A', A_1 \in \mathcal{F}(X)$ be normal and $b \in [0, 1]$. If $X_{A' > A_1} \neq \emptyset$, then

$$\bigwedge_{x \in X} A'(x) \to (A_1(x) * b) = \bigwedge_{x \in X_{A' > A_1}} A'(x) \to (A_1(x) * b). \tag{13}$$

FRS - Fuzzy input and one rule (BKS + conj)

Corollary 1

Let $A', A_1 \in \mathcal{F}(X)$ be normal and $b \in [0,1]$. If $X_{A'>A_1} \neq \emptyset$, then

$$\bigwedge_{x \in X} A'(x) \to (A_1(x) * b) = \beta * b, \tag{14}$$

where $\beta = \bigwedge_{x \in X_{A'>A_1}} (A'(x) \to A_1(x))$.

If $X_{A'>A_1} = \emptyset$, then $A' \subseteq A_1$ and we obtain the following result:

Corollary 2

Let $A', A_1 \in \mathcal{F}(X)$ be normal and $b \in [0,1]$. If $X_{A'>A_1} = \emptyset$, then

$$\bigwedge_{x \in X} A'(x) \to (A_1(x) * b) = b. \tag{15}$$

Therefore, using the results from Proposition 5, Corollary 1 and Corollary 2, we can conclude that

$$B'(y) = \bigwedge_{x \in X} A'(x) \to (A_1(x) * B_1(y)) = \beta * B_1(y), \tag{16}$$

since if $X'_{A' > A_1} = \emptyset$ we would have $\beta = \bigwedge \emptyset = 1$.

FRS - Fuzzy input and one rule (CRI + imp)

For the combination of CRI composition with a single implicative rule, we have that $B'(y) = \bigvee_{x \in X} A'(x) * (A_1(x) \to B_1(y))$.

As we have seen before, this case depends on the t-norm and implication chosen but we can develop for some specific cases.

Corollary 3

Let $A', A_1 \in \mathcal{F}(X)$ be normal and $b \in [0,1]$. Considering the minimum t-norm $\wedge$ with the Gödel implication $\to_M$, the product t-norm $\cdot$ with the Goguen implication $\to_P$ and the Lukasiewicz t-norm $*_L$ with the Lukasiewicz implication $\to_L$, we obtain the following identities:

- $\bigvee_{x \in X} A'(x) \wedge (A_1(x) \to_M b) = \bigvee_{x \in X_{A_1 \leq b}} A'(x) \vee (\bigvee_{x \in X_{A_1 > b}} A'(x) \wedge b)$;
- $\bigvee_{x \in X} A'(x) \cdot (A_1(x) \to_P b) = \bigvee_{x \in X_{A_1 \leq b}} A'(x) \vee (\bigvee_{x \in X_{A_1 > b}} A'(x) \cdot \frac{b}{A_1(x)})$;
- $\bigvee_{x \in X} A'(x) *_L (A_1(x) \to_L b) = \bigvee_{x \in X_{A_1 \leq b}} A'(x) \vee \bigvee_{x \in X_{A_1 > b}} [0 \vee (A'(x) + b - A_1(x))]$.

FRS - Fuzzy input and one rule (BKS + imp)

Finally, for the combination of BKS composition with a single implicative rule, we have that $B'(y) = \bigwedge_{x \in X} A'(x) \to (A_1(x) \to B_1(y))$. Using the fact that $x \to (y \to z) = (x * y) \to z$, we can obtain

$$B'(y) = \bigvee_{x \in X} (A'(x) * A_1(x)) \to B_1(y) = \alpha \to B_1(y), \tag{17}$$

where $\alpha = \bigvee_{x \in X} (A'(x) * A_1(x))$.

FRS - Fuzzy input and one rule example

Example 3.5

Consider the sets $X, Y = [0, 10] \subset R$, an input $A'(x) = triang(x, 2, 3, 4)$ and the following rule base with just a single rule:

Antecedents	**Consequents**
$A_1 = triang(x, 1, 4, 7)$	$B_1 = triang(y, 1, 3, 5)$

Also, consider the minimum t-norm $\wedge$ and the Gödel implication $\rightarrow_M$. Doing the calculations we obtain the following values of α and β:

$$\alpha = \bigvee_{x \in X} (A'(x) \wedge A_1(x)) = 0.75,$$

$$\beta = \bigwedge_{x \in X} (A'(x) \rightarrow_M A_1(x)) = 0.5.$$

FRS - Fuzzy input and one rule example

Example

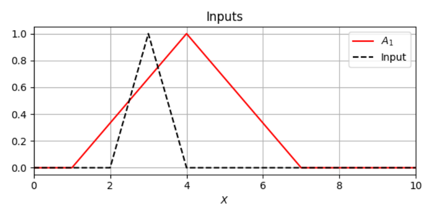
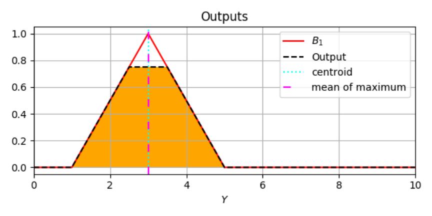


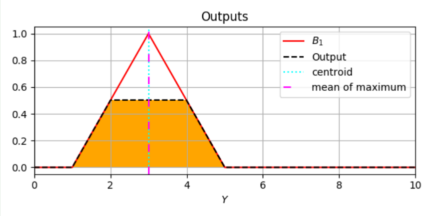
Figure: Input fuzzy set and antecedent

FRS - Fuzzy input and one rule example

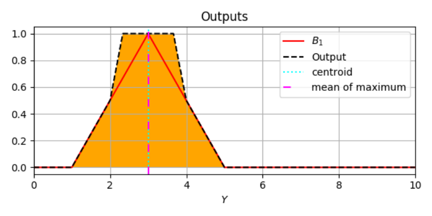
Example



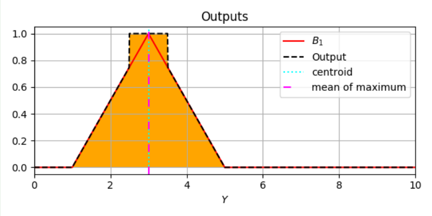
(a) CRI + conjunctive rules



(b) BKS + conjunctive rules



(c) CRI + implicative rules



(d) BKS + implicative rules

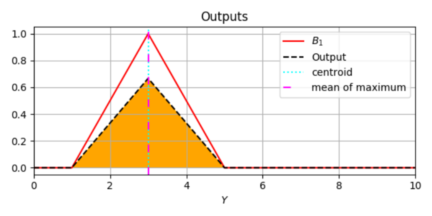
Example 3.6

Considering the same rule base and input as before, take now the product t-norm $\cdot$ and the associated Goguen implication $\to_P$. Calculating the values of α and β we have

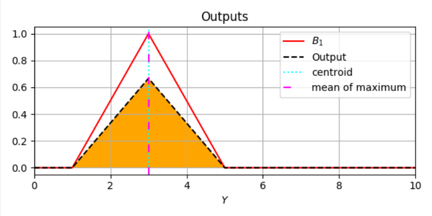
$$\alpha = \bigvee_{x \in X} (A'(x) \cdot A_1(x)) \approx 0.67,$$
$$\beta = \bigwedge_{x \in X} (A'(x) \to_P A_1(x)) \approx 0.67.$$

FRS - Fuzzy input and one rule example

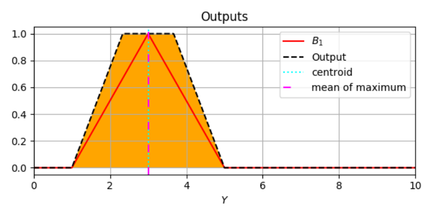
Example



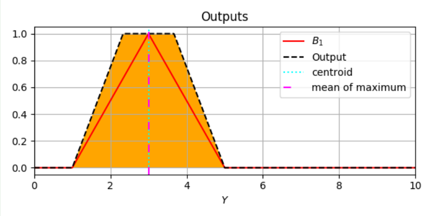
(a) CRI + conjunctive rules



(b) BKS + conjunctive rules



(c) CRI + implicative rules



(d) BKS + implicative rules

FRS - Fuzzy input and one rule example

Example 3.7

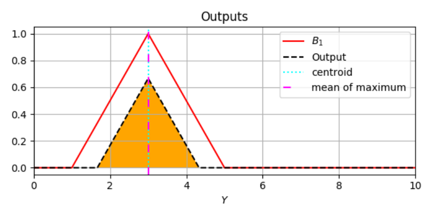
Finally, consider the Lukasiewicz t-norm $*_L$ and Lukasiewicz implication $\rightarrow_L$. Again, calculating the values of α and β we have

$$\alpha = \bigvee_{x \in X} (A'(x) *_L A_1(x)) \approx 0.67,$$

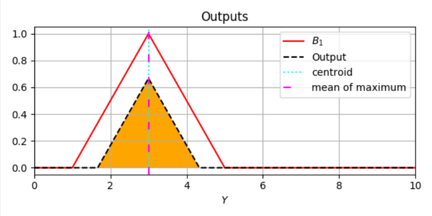
$$\beta = \bigwedge_{x \in X} (A'(x) \rightarrow_L A_1(x)) \approx 0.67.$$

FRS - Fuzzy input and one rule example

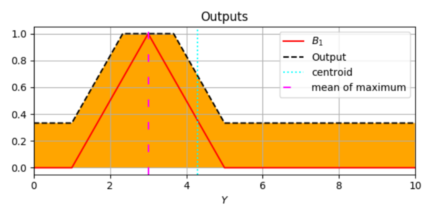
Example



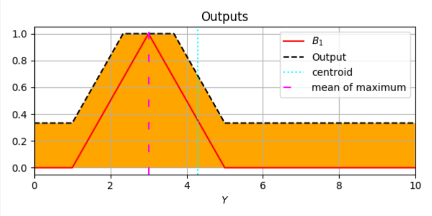
(a) CRI + conjunctive rules



(b) BKS + conjunctive rules



(c) CRI + implicative rules



(d) BKS + implicative rules

FRS - Fuzzy input and multiple rules

To conclude this study, let us look at the general case of a fuzzy input A' and a fuzzy rule base with a finite number of rules $R = (A_i, B_i)$ for $i = 1, \ldots, n$.

FRS - Fuzzy input and multiple rules (CRI + conj)

For the combination of CRI composition with conjunctive rules, we have that

$$B'(y) = \bigvee_{x \in X} [A'(x) * \bigvee_{i=1}^{n} (A_i(x) * B_i(y))]. \tag{18}$$

Proposition 6

P

Let $A', A_i \in \mathcal{F}(X)$ and $B_i \in \mathcal{F}(Y)$. Then

$$B'(y) = \bigvee_{x \in X} [A'(x) * \bigvee_{i=1}^{n} (A_i(x) * B_i(y))] = \bigvee_{i=1}^{n} \alpha_i * B_i(y). \tag{19}$$

where $\alpha_i = \bigvee_{x \in X} (A'(x) * A_i(x))$.

For the combination of the BKS composition with conjunctive rules, we have that

$$B'(y) = \bigwedge_{x \in X} [A'(x) \to \bigvee_{i=1}^{n} (A_i(x) * B_i(y))]. \tag{20}$$

FRS - Fuzzy input and multiple rules (BKS + conj)

Proposition 7

P

Let $A', A_i \in \mathcal{F}(X)$ be normal, $B_i \in \mathcal{F}(Y)$ and $y \in Y$. Let $X' = Supp(A')$ and $X'_l \subseteq X'$ be such that $\bigcup_{l=1}^{m} X'_l = X'$, and there exists $i_l \in I = \{i \in [1, n] | Supp(A') \cap Supp(A_i) \neq \emptyset\}$ for some $l = 1, ..., m$, such that $\bigvee_{i \in I}(A_i(x) * B_i(y)) = A_{i_l}(x) * B_{i_l}(y)$ for every $x \in X'_l$. The following equation holds true:

$$B'(y) = \bigwedge_{x \in X'} [A'(x) \to \bigvee_{i=1}^{n}(A_i(x) * B_i(y))] = \bigvee_{l=1}^{m}(\beta_{i_l} * B_{i_l}(y)), \tag{21}$$

where $\beta_{i_l} = \bigwedge_{x \in X'_{l_{A' > A_{i_l}}}} [A'(x) \to A_{i_l}(x)]$.

For the combination of CRI composition with implicative rules, we have that

$$B'(y) = \bigvee_{x \in X} [A'(x) * \bigwedge_{i=1}^{n} (A_i(x) \to B_i(y))]. \tag{22}$$

FRS - Fuzzy input and multiple rules (CRI + imp)

Proposition 8

P

Let $A', A_i \in \mathcal{F}(X)$ be normal, $B_i \in \mathcal{F}(Y)$ and an arbitrary $y \in Y$. Let $X' = Supp(A')$ and $X'_l \subseteq X'$ be such that $\bigcup_{l=1}^{m} X'_l = X'$, and there exists $i_l \in I = \{i \in [1, n] | Supp(A') \cap Supp(A_i) \neq \emptyset\}$ for some $l = 1, ..., m$, such that $\bigwedge_{i \in I} (A_i(x) \to B_i(y)) = A_{i_l}(x) \to B_{i_l}(y)$ for every $x \in X'_l$. The following equation holds true:

$$\begin{aligned} B'(y) &= \bigvee_{x \in X} [A'(x) * \bigwedge_{i=1}^{n} (A_i(x) \to B_i(y))] \\ &= \bigvee_{l=1}^{m} \{ \bigvee_{x \in X'_{l_{A_{i_l} \leq B_{i_l}(y)}}} A'(x) \vee \bigvee_{x \in X'_{l_{A_{i_l} > B_{i_l}(y)}}} [A'(x) * (A_{i_l}(x) \to B_{i_l}(y))] \} \end{aligned} \tag{23}$$

FRS - Fuzzy input and multiple rules (CRI + imp)

Corollary 4

P Considering the conditions from Proposition 8 and the minimum t-norm $\wedge$ with the Gödel implication $\rightarrow_M$, the product t-norm $\cdot$ with the Goguen implication $\rightarrow_P$ and the Lukasiewicz t-norm $*_L$ with the Lukasiewicz implication $\rightarrow_L$, we obtain the following identities:

- $$\bigvee_{x \in X} [A'(x) \wedge \bigwedge_{i=1}^{n} (A_i(x) \rightarrow_M B_i(y))] = \bigvee_{l=1}^{m} [\bigvee_{x \in X'_{{}^{l}A_{i_l} \leq B_{i_l}(y)}} A'(x) \vee \bigvee_{x \in X'_{{}^{l}A_{i_l} > B_{i_l}(y)}} (A'(x) \wedge B_{i_l}(y))];$$
- $$\bigvee_{x \in X} [A'(x) \cdot \bigwedge_{i=1}^{n} (A_i(x) \rightarrow_P B_i(y))] = \bigvee_{l=1}^{m} [\bigvee_{x \in X'_{{}^{l}A_{i_l} \leq B_{i_l}(y)}} A'(x) \vee \bigvee_{x \in X'_{{}^{l}A_{i_l} > B_{i_l}(y)}} (A'(x) \cdot \frac{B_{i_l}(y)}{A_{i_l}(x)})];$$
- $$\bigvee_{x \in X} [A'(x) *_L \bigwedge_{i=1}^{n} (A_i(x) \rightarrow_L B_i(y))] =$$
  $$\bigvee_{l=1}^{m} \{ \bigvee_{x \in X'_{{}^{l}A_{i_l} \leq B_{i_l}(y)}} A'(x) \vee \bigvee_{x \in X'_{{}^{l}A_{i_l} > B_{i_l}(y)}} [0 \vee (A'(x) + B_{i_l}(y) - A_{i_l}(x))] \}.$$

FRS - Fuzzy input and multiple rules (BKS + imp)

Finally, for the combination of BKS composition with implicative rules, we have that

$$B'(y) = \bigwedge_{x \in X} [A'(x) \to \bigwedge_{i=1}^{n} (A_i(x) \to B_i(y))]. \tag{24}$$

Proposition 9

Let $A', A_i \in \mathcal{F}(X)$ and $B_i \in \mathcal{F}(y)$. Defining $\alpha_i = \bigvee_{x \in X} (A'(x) * A_i(x))$, then we have

$$\bigwedge_{x \in X} [A'(x) \to \bigwedge_{i=1}^{n} (A_i(x) \to B_i(y))] = \bigwedge_{i=1}^{n} \alpha_i \to B_i(y). \tag{25}$$

FRS - Fuzzy input and multiple rules example

Example 3.8

Consider the sets $X, Y = [0, 10] \subset R$, an input $A'(x) = triang(x, 2.5, 3.5, 4.5)$ and the following rule base:

Antecedents	**Consequents**
$A_1 = triang(x, 1, 3, 5)$	$B_1 = triang(y, 1, 3, 5)$
$A_2 = triang(x, 3, 5, 7)$	$B_2 = triang(y, 3.5, 5.5, 7.5)$
$A_3 = triang(x, 5, 7, 9)$	$B_3 = triang(y, 7, 8, 9)$

Also, consider the minimum t-norm $\wedge$ and the Gödel implication $\rightarrow_M$.

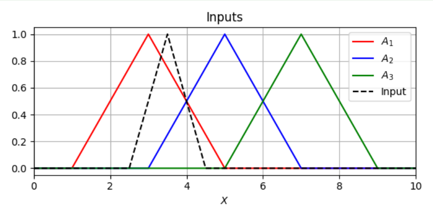
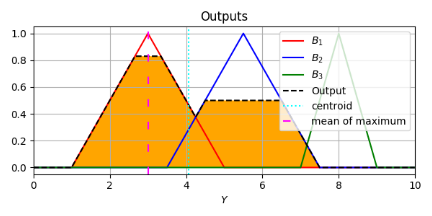
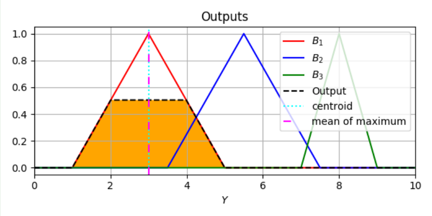


Figure: Input fuzzy set and antecedents

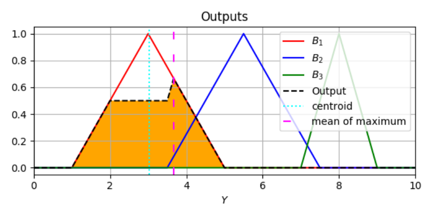
Example



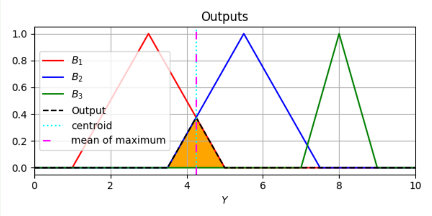
(a) CRI + conjunctive rules



(b) BKS + conjunctive rules



(c) CRI + implicative rules



(d) BKS + implicative rules

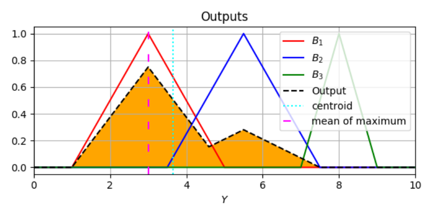
FRS - Fuzzy input and multiple rules example

Example 3.9

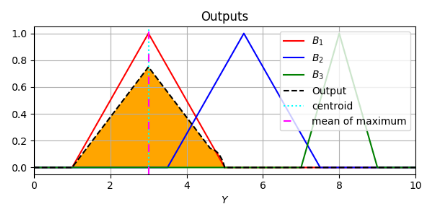
Now, consider the product t-norm $\cdot$ and its associated Goguen implication $\rightarrow_P$ with the same input $A'(x) = triang(x, 2.5, 3.5, 4.5)$ and the rule base as before.

FRS - Fuzzy input and multiple rules example

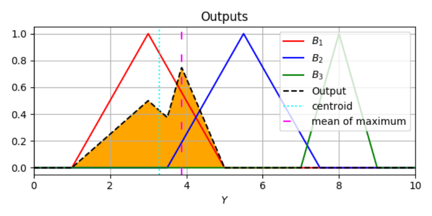
Example



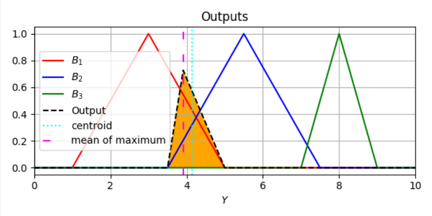
(a) CRI + conjunctive rules



(b) BKS + conjunctive rules



(c) CRI + implicative rules



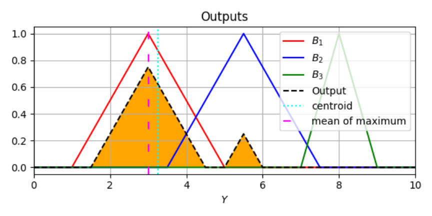
(d) BKS + implicative rules

FRS - Fuzzy input and multiple rules example

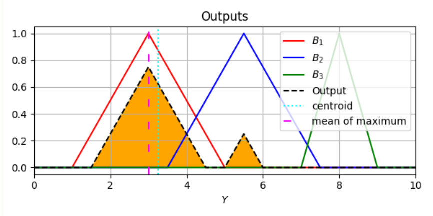
Example 3.10

Finally, consider the Lukasiewicz t-norm $*_L$ and the Lukasiewicz implication $\to_L$. Also, consider the same input fuzzy set $A'(x) = triang(x, 2.5, 3.5, 4.5)$ as before and the same rule.

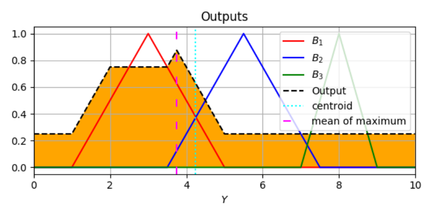
Example



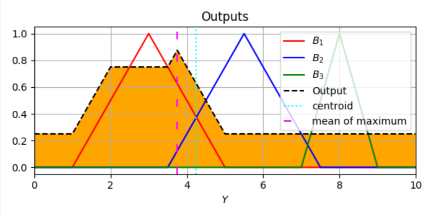
(a) CRI + conjunctive rules



(b) BKS + conjunctive rules



(c) CRI + implicative rules



(d) BKS + implicative rules

Example 3.11

Consider the sets $X, Y = [0, 10] \subset R$, an input $A'(x) = triang(x, 0, 0.5, 1)$ and the same rule base as before:

Antecedents	**Consequents**
$A_1 = triang(x, 1, 3, 5)$	$B_1 = triang(y, 1, 3, 5)$
$A_2 = triang(x, 3, 5, 7)$	$B_2 = triang(y, 3.5, 5.5, 7.5)$
$A_3 = triang(x, 5, 7, 9)$	$B_3 = triang(y, 7, 8, 9)$

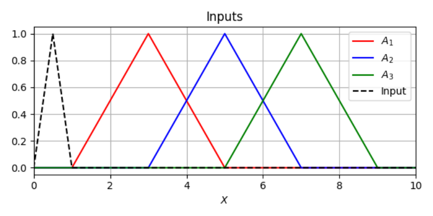
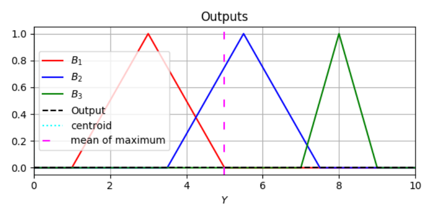


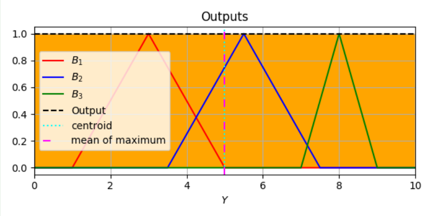
Figure: Input fuzzy set and antecedents

FRS - Fuzzy input and multiple rules example

Example



(a) Conjunctive rules



(b) Implicative rules

FRS - Fuzzy input and multiple rules example

Example 3.12

Consider the sets $X, Y = [0, 10] \subset R$, an input $A'(x) = triang(x, 4, 5, 6)$ and the same rule base as before:

Antecedents	**Consequents**
$A_1 = triang(x, 1, 3, 5)$	$B_1 = triang(y, 1, 3, 5)$
$A_2 = triang(x, 3, 5, 7)$	$B_2 = triang(y, 3.5, 5.5, 7.5)$
$A_3 = triang(x, 5, 7, 9)$	$B_3 = triang(y, 7, 8, 9)$

Also, consider the minimum t-norm $\wedge$ and the Gödel implication $\rightarrow_M$.

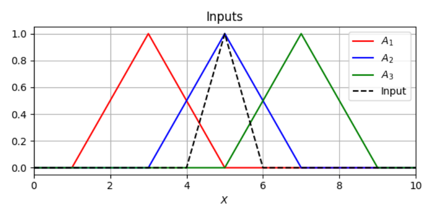
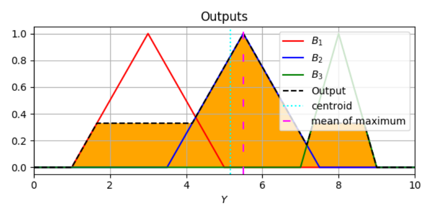


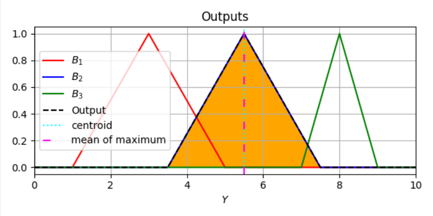
Figure: Input fuzzy set and antecedents

FRS - Fuzzy input and multiple rules example

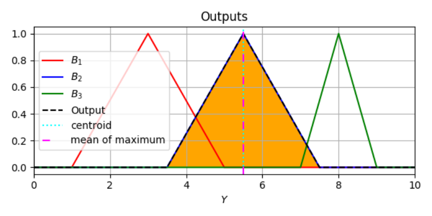
Example



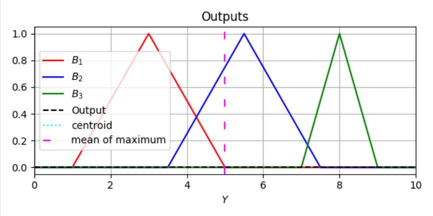
(a) CRI + Conjunctive rules



(b) BKS + Conjunctive rules



(c) CRI + implicative rules



(d) BKS + implicative rules

Summary

Moser-Navara axioms

To ensure the robustness and coherence of fuzzy relational systems, B. Moser and M. Navara [Moser and Navara, 2002] proposed three axioms that these systems should obey primarily.

Moser-Navara axioms

Conceptually the axioms are as follows:

1. Interpolation: Given an input A' equals to and antecedent A_i, the output of the system should be equal to the corresponding consequent B_i.
2. Significance of the generated outputs: For all normal input A' $(\exists x \in X | A'(x) = 1)$, the system produces a non-trivial output, which means, $\exists y \in Y | B'(y) \neq 0$ in case of conjunctive rules and $\exists y \in Y | B'(y) \neq 1$ in case of implicative rules.
3. Robustness: Given any input, the corresponding output must be contained in the union of the consequents of the activated rules $(A' \cap A_i \neq \emptyset)$ in case of conjunctive rules; or must contain the intersection of consequents of activated rules in case of implicative rules.

Moser-Navara axioms - Conjunctive rules

Formal definition of the axioms considering an inference system @ (CRI $\circ$ or BKS $\lhd$) and conjunctive rules [Stepnicka and Mandal, 2015] [Stepnicka, 2016] [Stepnicka and Mandal, 2018]:

A_C1 For all $i \in 1, \ldots, n$

$$A_i @ \check{R} = B_i;$$

A_C2 For each normal input $A' \in \mathcal{F}(X)$ there exists an index i such that

$$A' @ \check{R} \not\subseteq B_i;$$

A_C3 The output $A' @ \check{R}$ belongs to the union of consequents B_i of activated rules, which means,

$$A' @ \check{R} \subseteq \bigcup_{i \in F} B_i$$

where

$F = \{i | Supp(A_i) \cap Supp(A') \neq \emptyset\}, (B_i \cup B_j)(y) = B_i(y) \vee B_j(y).$

Moser-Navara axioms - Implicative rules

And analogously for implicative rules:

A_I1 For all $i \in 1, \ldots, n$
$$A_i @ \hat{R} = B_i;$$

A_I2 For each normal input $A' \in \mathcal{F}(X)$ there exists an index i such that
$$A' @ \hat{R} \not\supseteq B_i;$$

A_I3 The output $A' @ \hat{R}$ contains the intersection of consequents B_i of activated rules, which means,
$$A' @ \hat{R} \supseteq \bigcap_{i \in F} B_i$$
where
$F = \{i | Supp(A_i) \cap Supp(A') \neq \emptyset\}, (B_i \cap B_j)(y) = B_i(y) \wedge B_j(y).$

Moser-Navara axioms - CRI + conjunctive rules problem

Proposition 10

[Moser and Navara, 2002] Be $*$ a t-norm without zero divisors. Be the fuzzy sets $A_i, i = 1, \ldots, n$ continuous and normal; and $B_i, i = 1, \ldots, n$ with mutually distinct supports. So the combination of the conjunctive rules model $\check{R}$ and the CRI composition $\circ$ do not satisfy axioms 1 and 2 simultaneously.

Example 4.1

Take the following fuzzy rule base:

	Antecedents	**Consequents**
R_1:	$A_1(x) = triang(x, 0, 2, 4)$	$B_1(y) = triang(y, 1, 2, 3)$
R_2:	$A_2(x) = triang(x, 2, 4, 6)$	$B_2(y) = triang(y, 4, 5, 6)$
R_3:	$A_3(x) = triang(x, 4, 6, 8)$	$B_3(y) = triang(y, 7, 8, 9)$

Table: Fuzzy rule base.

Suppose an input $A'(x) = A_2(x) = triang(x, 2, 4, 5)$, then the output $B'(y)$ of the combination CRI + conjunctive rules is given Figure 20.

Example

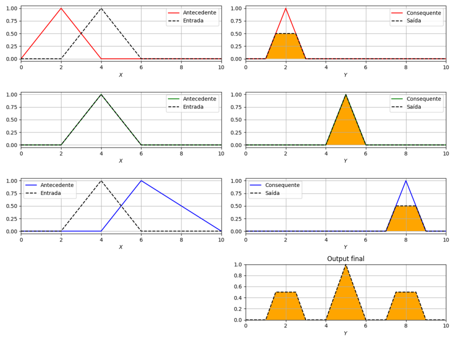


Figure: Example of violation of the axioms.

Given an input $A'(x) = A_2(x)$ we have an output $B'(y) \neq B_2(y)$.

Moser-Navara axioms - BKS + implicative rules problem

Proposition 11

[Stepnicka and Mandal, 2015] Let $*$ be a left-continuous t-norm and without zero divisors. Let $A_i, i = 1, \dots, n$ be continuous and normal and let $B_i, i = 1, \dots, n$ fuzzy sets with mutually different supports. So the model of implicative rules $\hat{R}$ (with the residual implication derived from the t-norm $*$) and the BKS composition $\triangleleft$ do not satisfy axioms 1 and 2 simultaneously.

Example 4.2

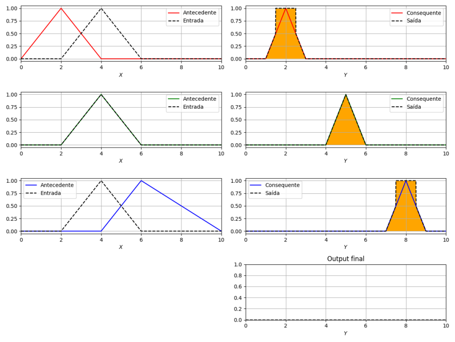


Figure: Example of violation of the axioms.

Given an input $A'(x) = A_2(x)$ we have an output $B'(y) \neq B_2(y)$.

Summary

Applications

Now we will we will apply the relational systems presented before to a series of reference problems, available in the repository *Knowledge Extraction Based on Evolutionary Learning* [Alcala-Fdez et al., 2011].

To learn the fuzzy rule bases from the data, we will use the Wang-Mendel algorithm [Wang and Mendel, 1992] whose steps will be explained next. The Python implementation of the algorithm is available at `https://github.com/renatolm/wang-mendel`.

Applications - Wang-Mendel algorithm - Step 1

1. Partitioning the attributes and outputs spaces in fuzzy sets:
   1. Let the attributes be $x_i, i \in [1, n]$, and the output y.
   2. Taking their domain intervals $X_i = [x_i^-, x_i^+]$ and $Y = [y^-, y^+]$.
   3. Each of these intervals is divided in N fuzzy regions (N can be different for each attribute).

   Obs: In this work will be used triangular membership functions for each fuzzy region, in such a way that one of the vertices is on the center of the fuzzy region with unitary value and the other two be on the center of the neighbor regions.

Applications - Wang-Mendel algorithm - Step 2

2. For each data in the training set, it is defined a new rule and its strength is calculated:
    1. First the membership degrees of the inputs and the output are calculated in each of the fuzzy regions.
    2. Then a fuzzy rule is composed by the fuzzy regions with the highest membership degrees for each variable.
    3. Finally, a strength is associated for the rule, given by the product of the membership degrees of each variable.

Applications - Wang-Mendel algorithm - Step 3

3. Cleaning of the rules:
   1. For identical rules, the one with higher strength is maintained e and the duplicates are removed.
   2. For equal rules with different consequents, the one with higher strength is maintained e and the inconsistents are removed.

Applications - Defuzzification and error

After this training phase (learning of fuzzy rules), the combinations seen before are applied on the test set, and the centroid and mean of maxima (MOM) defuzzification methods are used to get the results [Klir and Yuan, 1995][Pedrycz and Gomide, 2007][Barros et al., 2017].

In the end we calculated the Root Mean Square Error (RMSE) of the predictions.

The Table 2 contains basic information about the datasets: the number of samples, the quantities of numerical attributes and the number of rules learned by the Wang-Mendel algorithm.

	Number of samples	**Numerical attributes**	**# of rules learned**
Diabetes	43	2	15
Ele-1	495	2	13
Plastic	1650	2	15
Quake	2178	3	53
Laser	993	4	58
Ele-2	1056	4	65
AutoMPG6	392	5	115
MachineCPU	209	6	35
Dee	365	6	177
AutoMPG8	392	7	161

Table: Description of the datasets

Applications - Regression problems

In the fuzzy relational systems, the t-norm of the minimum $\wedge$ and its adjunct implication $\rightarrow_M$ (Gödel's implication) were used. Each of the domain ranges X_i and Y was divided into $N = 5$ fuzzy regions.

The numbers written in *italic text* are the best results in each experiment configuration (i.e. in each row of each table), and the numbers written in **bold text** are the best results for each dataset independent of the defuzzification method (i.e. best for each row in each pair of tables). In this first simulation, crisp inputs were considered for the relational systems.

Applications - Crisp input results

	Conjunctive Rules + CRI	Conjunctive Rules + BKS	Implicative Rules + CRI	Implicative Rules + BKS
Diabetes	***0.47***	***0.47***	0.52	0.52
Ele-1	***715.7***	***715.7***	1167.3	1167.3
Plastic	***1.76***	***1.76***	2.37	2.37
Quake	***0.21***	***0.21***	0.28	0.28
Laser	***10.8***	***10.8***	28.4	28.4
Ele-2	***267.3***	***267.3***	644.9	644.9
AutoMPG6	***4.1***	***4.1***	6.6	6.6
MachineCPU	153.4	153.4	*119.8*	*119.8*
Dee	***0.56***	***0.56***	0.73	0.73
AutoMPG8	***4.2***	***4.2***	7.1	7.1

Table: Results using centroid defuzzification

Applications - Crisp input results

	Conjunctive Rules + CRI	Conjunctive Rules + BKS	Implicative Rules + CRI	Implicative Rules + BKS
Diabetes	*0.74*	*0.74*	**0.74**	**0.74**
Ele-1	*1141.02*	*1141.02*	1183.04	1183.04
Plastic	*2.68*	*2.68*	3.25	3.25
Quake	*0.30*	*0.30*	0.39	0.39
Laser	*29.95*	*29.95*	43.35	43.35
Ele-2	638.77	638.77	*509.38*	*509.38*
AutoMPG6	5.56	5.56	*4.71*	*4.71*
MachineCPU	120.47	120.47	***109.47***	***109.47***
Dee	0.71	0.71	*0.64*	*0.64*
AutoMPG8	4.91	4.91	*4.25*	*4.25*

Table: Results using MOM defuzzification

Applications - Crisp input results

As already highlighted before, the results of combinations with conjunctive rules are the same, considering a crisp input. Analogously for relational systems with implicative rules.

Regarding the accuracy of the results, it can be noted that in most problems the relational systems with conjunctive rules obtained a better performance using the centroid defuzzification. The MOM defuzzification seems to be more adequate for FRSs that use implicative rules, specially in problems with higher number of input variables.

Applications - Fuzzy input results

Now we will see the results from a simulation considering a fuzzy input.

For the fuzzification of the inputs, triangular membership functions centered on the point in question x_i and with vertices at $x_i - r$ and $x_i + r$ were considered, where r is 5% of the domain interval of that variable . That is, $r = 0.05(x_i^+ - x_i^-)$ and $A_i'(x) = triang(x, x_i - r, x_i, x_i + r)$.

Applications - Fuzzy input results

	Conjunctive Rules + CRI	Conjunctive Rules + BKS	Implicative Rules + CRI	Implicative Rules + BKS
Diabetes	0.69	*0.67*	1.78	1.77
Ele-1	***727.4***	975.8	1206.5	1760.6
Plastic	2.23	***2.08***	4.8	5.6
Quake	***0.28***	0.48	0.74	0.83
Laser	19.4	***15.8***	70.3	78.8
Ele-2	***367.4***	488.1	808.7	1584.9
AutoMPG6	***3.7***	6.3	15.0	22.4
MachineCPU	186.7	187.3	*137.9*	156.1
Dee	*0.64*	0.68	1.71	2.27
AutoMPG8	*5.5*	5.6	7.8	11.2

Table: Results considering a fuzzy input using centroid defuzzification

Applications - Fuzzy input results

	Conjunctive Rules + CRI	Conjunctive Rules + BKS	Implicative Rules + CRI	Implicative Rules + BKS
Diabetes	***0.64***	***0.64***	0.79	0.81
Ele-1	1142.38	1112.94	*1057.77*	1698.88
Plastic	*2.73*	*2.73*	3.11	3.35
Quake	*0.33*	*0.33*	0.37	0.39
Laser	26.46	*25.45*	34.04	46.29
Ele-2	663.31	640.56	*404.56*	782.84
AutoMPG6	4.7	5.02	*4.28*	5.46
MachineCPU	***108.17***	146.79	119.12	145.04
Dee	0.6	0.65	***0.55***	0.60
AutoMPG8	4.78	4.4	***3.22***	3.59

Table: Results considering a fuzzy input using MOM defuzzification

Applications - Fuzzy input results

As expected, the results were a little worse than in previous experiments due to the uncertainty considered in the inputs. It is also possible to conclude that performance is more related to the choice of fuzzy rules representation (conjunctive or implicative rules) and defuzzification method, than to the composition (CRI or BKS).

It is interesting to note that for the problems with a higher number of variables (specifically the last four), the results obtained using a fuzzy input were the best overall. This could indicate that the datasets are not large enough to learn an exhaustive rule base, and in this case an uncertainty in the inputs is beneficial for the performance.

Applications - Results

In general, the relational systems using implicative rules had worst performance. This can be explained by the way that the Wang-Mendel algorithm works: it learns conjunctive rules by acumulating information, and not excluding possibilities, which aligns with the interpretation made by Dubois et al. [Dubois et al., 1999].

Summary

Final considerations

In this work, a bibliographic review of three important works related to fuzzy relational systems was made:

1. the study of combinations of fuzzy compositions and conjunctive/implicative fuzzy rules by Martin Stepnicka et al., as well as Moser-Navara axioms;
2. the interpretation of fuzzy rules proposed by Didier Dubois et al.;
3. the Wang-Mendel algorithm for learning fuzzy rules.

Final considerations

We saw that there are suitable alternatives to the combination of CRI composition with conjunctive rules (Mandani-Assilian relational system), which can be as good or even more adequate depending on the situation.

The applications highlighted the hypothesis raised by Didier Dubois et al. [Dubois et al., 1999] that conjunctive rules are more suitable for modeling the knowledge acquired through data (observations), while implicative rules would be more suitable for modeling an expert's knowledge (constraints).

On this subject, there is a fertile field for the elaboration of new, more extensive studies, including the combination of these two types of rules in a single fuzzy rule base, in order to complement each other.

Final considerations

Finally, the presentation of the Moser-Navara axioms and the most recent results obtained by Martin Stepnicka et al. was made in order to add more theoretical foundation to the discussed subject. However, despite violating the axioms, the combination of CRI composition with conjunctive rules performs well in real problems.

This is directly related to the way that the rules are generated by the Wang-Mendel algorithm and the chosen defuzzification method. Once again, there is much room for further work exploring the possible negative implications that violating these axioms can have.

Summary

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Summary

8 Proofs and sketches

Proposition 1 back.

Sketch of the proof

Using the definition of residual implication

$$a \to (b * c) = \bigvee\{z \in [0,1] | a * z \leq b * c\} = \bigvee Z = z' \tag{26}$$

and also

$$a \to b = \bigvee\{y \in [0,1] | a * y \leq b\} = \bigvee Y = y'. \tag{27}$$

Let it be a $y \in Y$, since $a > b \Rightarrow a > b * c$, by the definitions above we have

$$y * c \leq z' \Rightarrow Y * c \subseteq Z \Rightarrow (a \to b) * c \leq a \to (b * c). \tag{28}$$

where $Y * c = y * c, \forall y \in Y$.

Proposition 1 back.

Sketch of the proof

As $*$ is left-continuous and increasing, $z' = a \to (b * c)$ satisfies $a * z' = b * c$.
In the same way, $y' = a \to b$ satisfies $a * y' = b$.
Since $*$ is continuous, then $\exists y_z$ such that $z' = y_z * c$. So, we can write:

$$z' = y_z * c \leq y' * c \Leftrightarrow a \to (b * c) \leq (a \to b) * c. \tag{29}$$

Proposition 2 back.

Proof.

Using the fact that the minimum is the greatest t-norm we have

$$a * (b \to c) \leq a \wedge (b \to c). \tag{30}$$

On the other hand, using the fact that the Godel implication is the smallest residual implication we have

$$a * (b \to c) \geq a * c. \tag{31}$$

$\square$

Proposition 5 back.

Proof

Since $X = X_{A' \leq A_1} \cup X_{A' > A_1}$, we have

$$\bigwedge_{x \in X} A'(x) \to (A_1(x) * b) = \bigwedge_{x \in X_{A' \leq A_1}} A'(x) \to (A_1(x) * b) \wedge \bigwedge_{x \in X_{A' > A_1}} A'(x) \to (A_1(x) * b). \tag{32}$$

Proposition 5 back.

Proof

On the one hand, using the fact that the implication is decreasing in the first argument, we have that

$$\bigwedge_{x \in X_{A' \leq A_1}} A'(x) \to (A_1(x) * b) \geq \bigwedge_{x \in X_{A' \leq A_1}} A_1(x) \to (A_1(x) * b). \tag{33}$$

Property 3 from Lemma **??** ($b \leq a \to (a * b)$) implies that

$$\bigwedge_{x \in X_{A' \leq A_1}} A_1(x) \to (A_1(x) * b) \geq b. \tag{34}$$

Proposition 5 back.

Proof.

On the other hand, from the definition of residual implication we have that

$$A'(x) \to (A_1(x) * b) = \bigvee \{z \in [0,1] : A'(x) * z \leq A_1(x) * b\}. \quad (35)$$

Let $z' = A'(x) \to (A_1(x) * b)$. For any $x \in X_{A' > A_1}$, we have that $A'(x) > A_1(x)$. By the monotonicity property of the t-norm, we have

$$\begin{aligned} & A'(x) * z' \leq A_1(x) * b \\ \Rightarrow & \bigwedge_{x \in X_{A' > A_1}} A'(x) \to (A_1(x) * b) \leq b \\ \Rightarrow & \bigwedge_{x \in X} A'(x) \to (A_1(x) * b) = \bigwedge_{x \in X_{A' > A_1}} A'(x) \to (A_1(x) * b). \end{aligned} \quad (36)$$

□

Corollary 1 back.

Proof.

If $0 \leq d_i < c_i \leq 1$ such that $i \in I$ for some arbitrary index set I and $e \in [0,1]$, then Proposition 1 implies that $c_i \to (d_i * e) = (c_i \to d_i) * e$. Taking the infimum over the set I we have

$$\bigwedge_{i \in I} [c_i \to (d_i * e)] = \bigwedge_{i \in I} [(c_i \to d_i) * e] = \bigwedge_{i \in I} [c_i \to d_i] * e \tag{37}$$

$$\Rightarrow \bigwedge_{x \in X_{A' > A_1}} A'(x) \to (A_1(x) * b) = \bigwedge_{x \in X_{A' > A_1}} (A'(x) \to A_1(x)) * b = \beta * b, \tag{38}$$

where $\beta = \bigwedge_{x \in X_{A' > A_1}} (A'(x) \to A_1(x))$. A brief glance at Proposition 5 suffices to conclude that the claim of Corollary 1 is satisfied. $\square$

Corollary 2 back.

Proof

Since $X_{A'>A_1} = \emptyset$, we have that

$$\bigwedge_{x \in X} A'(x) \to (A_1(x) * b) = \bigwedge_{x \in X_{A' \leq A_1}} A'(x) \to (A_1(x) * b). \quad (39)$$

As we have seen in the demonstration of Proposition 5, using the fact that the implication is decreasing in the first argument and Property 3 from Lemma **??**, we have

$$\bigwedge_{x \in X_{A' \leq A_1}} A'(x) \to (A_1(x) * b) \geq \bigwedge_{x \in X_{A' \leq A_1}} A_1(x) \to (A_1(x) * b) \geq b. \quad (40)$$

Corollary 2 back.

Proof.

Now, since A' and A_1 are normal fuzzy sets, there exists a $x' \in X$ such that $A'(x') = 1$ and therefore $A_1(x') = 1$. In view of Proposition **??**, this leads to

$$A'(x') \to (A_1(x') * b) = 1 \to (1 * b) = b. \tag{41}$$

Thus, $\bigwedge_{x \in X} A'(x) \to (A_1(x) * b) = b$.

□

Proposition ?? back.

Proof.

Property 3 of Lemma ?? states that $a * (a \to b) \leq b$. Therefore,

$$\bigvee_{x \in X_{A' \leq A_1}} A'(x) * (A_1(x) \to b) \leq \bigvee_{x \in X_{A' \leq A_1}} A_1(x) * (A_1(x) \to b) \leq b. \quad (42)$$

On the other hand, consider $x^* \in X$ such that $A_1(x^*) < 1 = A'(x^*)$:

$$\bigvee_{x \in X_{A' > A_1}} A'(x) * (A_1(x) \to b) \geq 1 * (A_1(x^*) \to b) = A_1(x) \to b \geq b. \quad (43)$$

Using a combination of Equations 42 and 43 we can conclude the following:

$$\bigvee_{x \in X} A'(x) * (A_1(x) \to b) = \bigvee_{x \in X_{A' > A_1}} A'(x) * (A_1(x) \to b). \quad (44)$$

□

Corollary **??** back.

Proof.

Since $X_{A'>A_1} = \emptyset$, we have that

$$\bigvee_{x \in X} A'(x) * (A_1(x) \to b) = \bigvee_{x \in X_{A' \leq A_1}} A'(x) * (A_1(x) \to b). \tag{45}$$

Property 3 of Lemma **??** states that $a * (a \to b) \leq b$. Therefore,

$$\bigvee_{x \in X_{A' \leq A_1}} A'(x) * (A_1(x) \to b) \leq \bigvee_{x \in X_{A' \leq A_1}} A_1(x) * (A_1(x) \to b) \leq b. \tag{46}$$

Since A' and A_1 are normal fuzzy sets, there exists $x' \in X$ such that $A'(x') = A_1(x') = 1$. This leads to

$$A'(x') * (A_1(x') \to b) = 1 * (1 \to b) = b. \tag{47}$$

Thus, $\bigvee_{x \in X} A'(x) * (A_1(x) \to b) = b$. □

Corollary **??** back.

Proof.

Property 7 of Lemma **??** states that $A_1(x) \to b = 1$ is equivalent to $A_1(x) \leq b$. Therefore,

$$\bigvee_{x \in X_{A_1 \leq b}} A'(x) * (A_1(x) \to b) = \bigvee_{x \in X_{A_1 \leq b}} A'(x) * 1 = \bigvee_{x \in X_{A_1 \leq b}} A'(x). \quad (48)$$

$\square$

Corollary 3 back.

Proof.

Since $X = X_{A_1 \leq b} \cup X_{A_1 > b}$, we have

$$\bigvee_{x \in X} A'(x) * (A_1(x) \to b) = \bigvee_{x \in X_{A_1 \leq b}} A'(x) * (A_1(x) \to b) \vee \bigvee_{x \in X_{A_1 > b}} A'(x) * (A_1(x) \to b).$$

From Corollary **??**, we have

$$\bigvee_{x \in X} A'(x) * (A_1(x) \to b) = \bigvee_{x \in X_{A_1 \leq b}} A'(x) \vee \bigvee_{x \in X_{A_1 > b}} A'(x) * (A_1(x) \to b) \quad (49)$$

Substituting the aforementioned pairs of t-norms and adjoint implications, we derive the equations of Corollary 3. □

Proposition 6 back.

Proof.

This is a straightforward application of the previous results as

$$\bigvee_{x\in X}[A'(x)*\bigvee_{i=1}^{n}(A_i(x)*B_i(y))]=\bigvee_{i=1}^{n}\bigvee_{x\in X}[A'(x)*(A_i(x)*B_i(y))], \quad (50)$$

by Property 1 of Proposition **??**. In addition, Equation 11 implies that

$$\bigvee_{i=1}^{n}\bigvee_{x\in X}[A'(x)*(A_i(x)*B_i(y))]=\bigvee_{i=1}^{n}\alpha_i*B_i(y). \quad (51)$$

□

Corollary 4 back.

Proof.

Substituting the aforementioned pairs of t-norms and adjoint implications on Proposition 8, we derive the equations of Corollary 4. □

Proposition 9 back.

Proof.

Again, this is a straightforward application of the previous results as

$$\bigwedge_{x\in X}[A'(x) \to \bigwedge_{i=1}^{n}(A_i(x) \to B_i(y))] = \bigwedge_{i=1}^{n}\bigwedge_{x\in X}[A'(x) \to (A_i(x) \to B_i(y))], \tag{52}$$

using the property 2 of Proposition **??**. Also

$$\bigwedge_{i=1}^{n}\bigwedge_{x\in X}[A'(x) \to (A_i(x) \to B_i(y))] = \bigwedge_{i=1}^{n}\alpha_i \to B_i(y), \tag{53}$$

using the result of Equation 17. □

Proposition 7 back.

Sketch of the proof

Let $y \in Y$ be arbitrary. Since $X = X' \cup X \setminus X'$, Equation 20 can be rewritten as follows:

$$B'(y) = \bigwedge_{x \in X'} [A'(x) \to \bigvee_{i=1}^{n} (A_i(x) * B_i(y))] \wedge \bigwedge_{x \in X \setminus X'} [A'(x) \to \bigvee_{i=1}^{n} (A_i(x) * B_i(y))] \tag{54}$$

$$= \bigwedge_{x \in X'} [A'(x) \to \bigvee_{i=1}^{n} (A_i(x) * B_i(y))]. \tag{55}$$

Using the definitions of i_l and X'_l, as well as the fact that $\bigcup_{l=1}^{m} X'_l = X'$, we can partition the set X' into subsets as follows:

$$\bigwedge_{x \in X'} [A'(x) \to \bigvee_{i=1}^{n} (A_i(x) * B_i(y))] = \bigwedge_{x \in X'_1} [A'(x) \to (A_{i_1}(x) * B_{i_1}(y))]$$

$$\cdots$$

$$\vee \bigwedge_{x \in X'_m} [A'(x) \to (A_{i_m}(x) * B_{i_m}(y))]. \tag{56}$$

Proposition 7 back.

Sketch of the proof

Note that for each of these partitions X'_l we obtain the single fuzzy rule case studied before. So, we can write

$$\bigwedge_{x \in X'_l} [A'(x) \to (A_{i_l}(x) * B_{i_l}(y))] = \beta_{i_l} * B_{i_l}(y). \tag{57}$$

And finally, aggregating all the partitions we have

$$\bigwedge_{x \in X'} [A'(x) \to \bigvee_{i=1}^{n} (A_i(x) * B_i(y))] = \bigvee_{l=1}^{m} (\beta_{i_l} * B_{i_l}(y)), \tag{58}$$

where $\beta_{i_l} = \bigwedge_{x \in X'_{l_{A' > A_{i_l}}}} (A'(x) \to A_{i_l}(x))$.

Proposition 8 back.

Sketch of the proof

Let $y \in Y$ be arbitrary. Since $X = X' \cup X \setminus X'$, Equation 22 can be rewritten as follows:

$$B'(y) = \bigvee_{x \in X'} [A'(x) * \bigwedge_{i=1}^{n} (A_i(x) \to B_i(y))] \vee \bigvee_{x \in X \setminus X'} [A'(x) * \bigwedge_{i=1}^{n} (A_i(x) \to B_i(y))] \tag{59}$$

$$= \bigvee_{x \in X'} [A'(x) * \bigwedge_{i=1}^{n} (A_i(x) \to B_i(y))]. \tag{60}$$

Using the definitions of i_l and X'_l, as well as the fact that $\bigcup_{l=1}^{m} X'_l = X'$, we can partition the set X' into subsets as follows:

$$\bigvee_{x \in X'} [A'(x) * \bigwedge_{i=1}^{n} (A_i(x) \to B_i(y))] = \bigvee_{x \in X'_1} [A'(x) * (A_{i_1}(x) \to B_{i_1}(y))]$$

$$\cdots$$

$$\vee \bigvee_{x \in X'_m} [A'(x) * (A_{i_m}(x) \to B_{i_m}(y))] \tag{61}$$

FRS - Fuzzy input and multiple rules (CRI + imp)

Proposition 8 back.

Sketch of the proof

Note that for each of these partitions X_l' we obtain the single fuzzy rule case studied before. So, we can write

$$\bigvee_{x \in X_l'} [A'(x) * (A_{i_l}(x) \to B_{i_l}(y))] = \bigvee_{x \in X'_{l_{A_{i_l} \leq B_{i_l}(y)}}} A'(x) \vee \bigvee_{x \in X'_{l_{A_{i_l} > B_{i_l}(y)}}} [A'(x) * (A_{i_l}(x) \to B_{i_l}(y))]. \quad (62)$$

And finally, aggregating all the partitions we have

$$B'(y) = \bigvee_{l=1}^{m} \{ \bigvee_{x \in X'_{l_{A_{i_l} \leq B_{i_l}(y)}}} A'(x) \vee \bigvee_{x \in X'_{l_{A_{i_l} > B_{i_l}(y)}}} [A'(x) * (A_{i_l}(x) \to B_{i_l}(y))] \} \quad (63)$$