Ray-Plane Intersection

A plane is defined by the equation: $Ax + By + Cz + D = 0$, or the vector [A B C D].

A, B, and C, define the normal to the plane.

If $A^2 + B^2 + C^2 = 1$ then the unit normal $\mathbf{N} = [A\ B\ C]$. If A, B, and C define a unit normal, then the distance from the origin [0 0 0] to the plane is D.

A ray is defined by a start position and a direction: $p_0 + tv$, $t > 0$

To determine if there is an intersection with the plane, substitute for ray equation into the plane equation and get:
$A(p_{0x} + tv_x) + B(p_{0y} + tv_y) + C(p_{0z} + tv_z) + D = 0$

which yields:
$t = -(Ap_{0x} + B\,p_{0y} + C\,p_{0z} + D) / (Av_x + Bv_y + Cv_z)$
$= -(\mathbf{N} \cdot p_0 + D) / (\mathbf{N} \cdot v)$

First compute $\mathbf{N} \cdot v = V_d$. If $V_d = 0$ (incident angle, q = 90°) then the ray is parallel to the plane and there is no intersection (if ray is in the plane then we ignore it).

If $V_d > 0$ then the normal of the plane is pointing away from the ray. If we use one-sided planar objects then could stop if $V_d > 0$, else continue.

Now compute second dot product $V_0 = -(\mathbf{N} \cdot p_0 + D)$ and compute $t = V_0 / V_d$. If $t < 0$ then the ray intersects plane behind origin, i.e. no intersection of interest, else compute intersection point:
$p = [X_i\ Y_i\ Z_i] = [p_{0x} + tv_x \quad p_{0y} + tv_y \quad p_{0z} + tv_z]$

Now we usually want surface normal for the surface facing the ray, so if $V_d > 0$ (normal facing away) then reverse sign of ray.

Example calculation:
Given a plane [1 0 0 -7] (plane with x = -7)
Ray with $p_0 = [\ 2\ 3\ 4\ ]$, $v = [\ 0.577\ 0.577\ 0.577]$
Compute $V_d = \mathbf{N} \cdot v = 0.577 > 0$, therefore the plane points away from the ray
Next, compute $V_0 = -(\mathbf{N} \cdot p_0 + D) = 5$
$t = 5/0.577 = 8.66 > 0$, therefore the intersection point is not behind the ray, so compute the coordinates of the intersection point:

$X_i = 2 + 0.577 * 8.66 = 7 \quad Y_i = 3 + 0.577 * 8.66 = 8 \quad Z_i = 4 + 0.577 * 8.66 = 9$

Must reverse normal so it is $\mathbf{N} = [-1\ 0\ 0]$

Ray-Sphere Intersection

Consider a simple scene with a unit sphere at origin
$x^2 + y^2 + z^2 = 1$

Let P = (x, y, z) be a point on sphere, then the equation reduces to:
$P \cdot P = 1$

Let ray be represented by a start point p_0 and a direction v, with parameter t representing distance along ray: $p_0 + tv$

Solve equation to find intersection between a ray and unit sphere:
$P \cdot P = 1$
$(p_0 + t\boldsymbol{v}) \cdot (p_0 + t\boldsymbol{v}) = 1$
$p_0^2 + 2tp_0\boldsymbol{v} + t^2\boldsymbol{v}^2 = 1$
$\boldsymbol{v}^2t^2 + 2p_0\boldsymbol{v}t + (p_0^2 - 1) = 0$
$t = [-2p_0\boldsymbol{v} \pm \text{sqrt}((2p_0\boldsymbol{v})^2 - 4\boldsymbol{v}^2 (p_0^2 - 1))] / 2\boldsymbol{v}^2$

Example calculation:
eye point = p_0 = (0, sqrt(2)/2, 3)
direction = v = (0, 0, -1)
find closest intersection point
$p_0\boldsymbol{v}$ = (0, sqrt(2)/2, 3) · (0, 0, -1) = -3
p_0^2 = (0, sqrt(2)/2, 3) · (0, sqrt(2)/2, 3) = 9.5
$\boldsymbol{v}^2$ = (0, 0, -1) · (0, 0, -1) = 1
so $t = [-2(-3) \pm \text{sqrt}((2(-3))^2 - 4(1)((9.5) - 1))] / 2(1)^2 = [6 \pm \text{sqrt}(2)] / 2 = 3 \pm \text{sqrt}(2)/2$

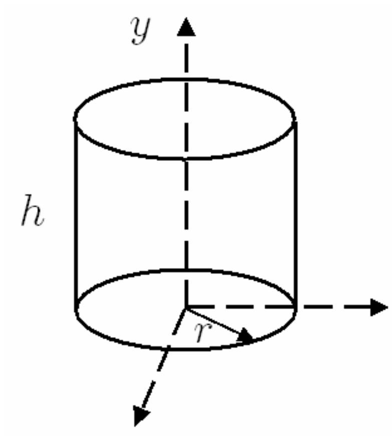
Smallest positive t value is our intersection point
so t = 3 - sqrt(2)/2
so p = (0, sqrt(2)/2, 3) + (3 - sqrt(2)/2)(0, 0, -1) = (0, sqrt(2)/2, sqrt(2)/2)

For colour calculation, we also need the surface normal at the intersection point.
Since this is a unit sphere, **N** = (0, sqrt(2)/2, sqrt(2)/2)

But we need a method for arbitrary spheres (i.e. not unit and not at origin)…

Ray-Cylinder Intersection

Assume we have a cylinder of height *h* and radius *r*.



The equation for the sides of the cylinder is:
$x^2 + z^2 = r^2$ (where $0 <= y <= h$)

We also need to define equations for the top and bottom caps:

Top cap: $y = h$ (where $x^2 + z^2 <= r^2$)
Bottom cap: y = 0 (where $x^2 + z^2 <= r^2$)

Let our ray be represented by $p_0 + tv$

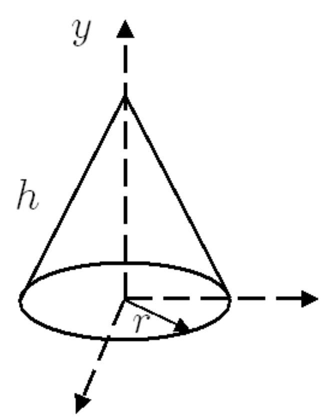
Steps for computing intersections:
1) Intersect ray with the side surface by plugging the ray equation into the cylinder equation: $(p_{0x} + tv_x)^2 + (p_{0z} + tv_z) = r^2$. Solve for *t* (possibly two solutions) and keep those which give an intersection point that satisfies $0 <= y <= h$

2) Intersect ray with the caps (defined as planes). Find the *t* values and keep those which are positive and which give an intersection point which satisfies $x^2 + z^2 <= r^2$

3) From the set of all intersection points (side and caps) choose the smallest positive *t* (if any) as the intersection point along the ray.

Ray-Cone Intersection

Infinite cone with apex at zero:
$x^2 + z^2 = k^2y^2$



Cone with apex on y axis at *h*:

$$x^2 + z^2 = \frac{r^2}{h^2}(y - h)^2 \quad \text{(where } 0 <= y <= h\text{)}$$

Intersections are computed similar to the cylinder: compute intersection with the side and also the intersection with the bottom cap, then choose closest intersection point.

In general

We want to find the intersection between an arbitrary primitive and a ray.

There is a series of transformations M (the combined transformation matrix) that transform a unit primitive (of the correct type) into our arbitrary primitive i.e. M transforms the unit primitive frame into the arbitrary primitive frame.

So M^{-1} transforms our ray into the unit primitive frame, where we can apply the unit primitive equations to solve for an intersection.

Idea:
given ray = p + t**v**
calculate ray' = $M^{-1} \cdot$ ray = $M^{-1} \cdot p + t\, M^{-1} \cdot \mathbf{v} = p' + t\mathbf{v'}$
calculate intersection using ray' with unit primitive, solve for t
substitute t into original ray = p + t**v** to get intersection point