Winner Pays Bid Auction Minimizes Variance

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1) English Auction

art and antiques (e.g. Sotheby's), sales of agricultural commodities, estate sales and Australian real estate, ...

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2) First Price Sealed Bid / Pay-As-Bid / Winner-Pays-Bid

used by governments to sell oil leases and for government procurement in general.

Used by many modern Ads platforms (e.g. Google Display Ads)

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3) Dutch Auction. Tulip sales

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4) Sealed Bid Second Price

Google Ads

ebay

stamp dealers used it for mail-based auctions as early as 1893 (before Vickrey 1961 paper).

Primary mechanism in Internet Advertising, eBay Auctions

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4) All Pay Auctions

Raffles, Lobbying, Patents Races, Crowdsourcing contest design (bids are either money or effort).

More generally: many auctions have a loser-pay component: bidders off-shore oil auctions must estimate not just the amount of oil available, but the costs of getting that oil to their refineries

Revenue Equivalence Theorem

However, they all must have the same revenue in IPV (independent private value).

Examples two bidders with independent U[0,1] values. We allocate to the highest bidders. If $\mathcal{A}_i(\boldsymbol{b})$ is the allocation function, consider three payment rules:

$$\mathcal{P}_i^{\mathsf{SP}}(\boldsymbol{b}) = \mathcal{A}_i(\boldsymbol{b}) \max_{j \neq i} b_j \qquad \mathcal{P}_i^{\mathsf{FP}}(\boldsymbol{b}) = \mathcal{A}_i(\boldsymbol{b}) b_i \qquad \mathcal{P}_i^{\mathsf{AP}}(\boldsymbol{b}) = b_i$$

which lead to different bidding strategies for the buyers:

$$b_i^{\mathsf{SP}}(v_i) = v_i \qquad b_i^{\mathsf{FP}}(v_i) = v_i/2 \qquad b_i^{\mathsf{AP}}(v_i) = v_i^2/2$$

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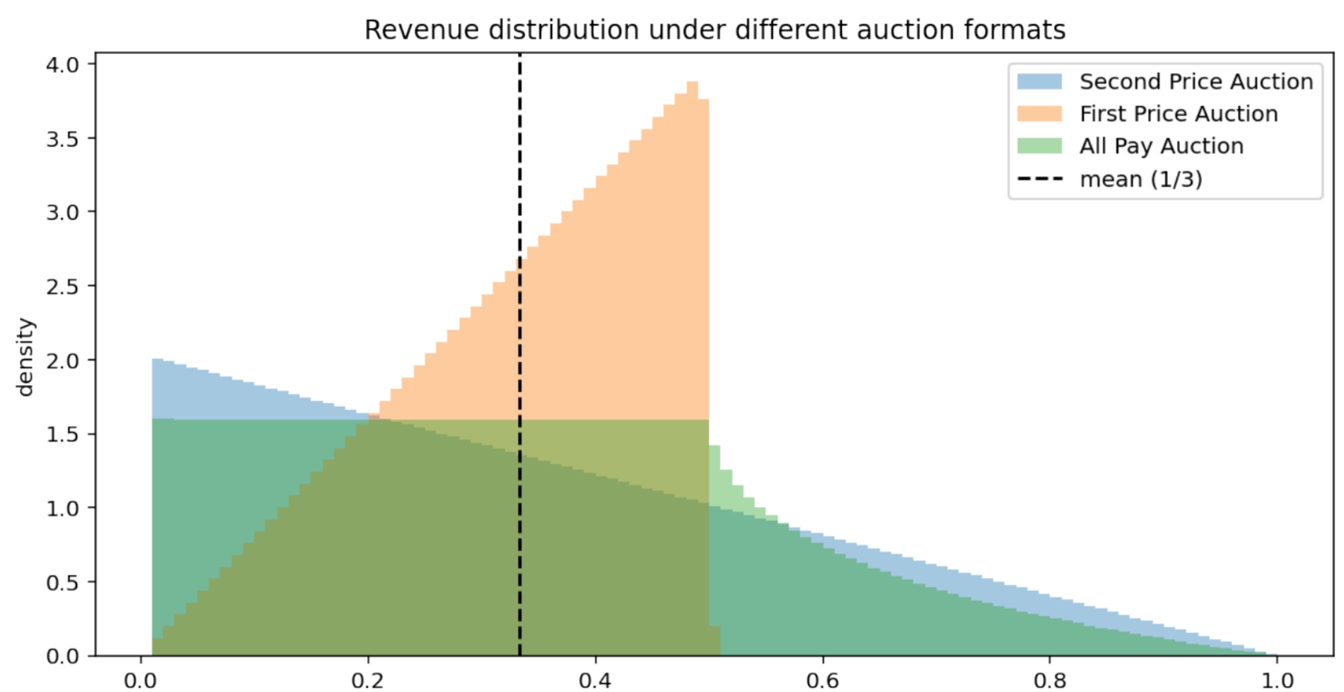
and the expected revenue is 1/3 in all cases. However, the distribution of revenue is different.



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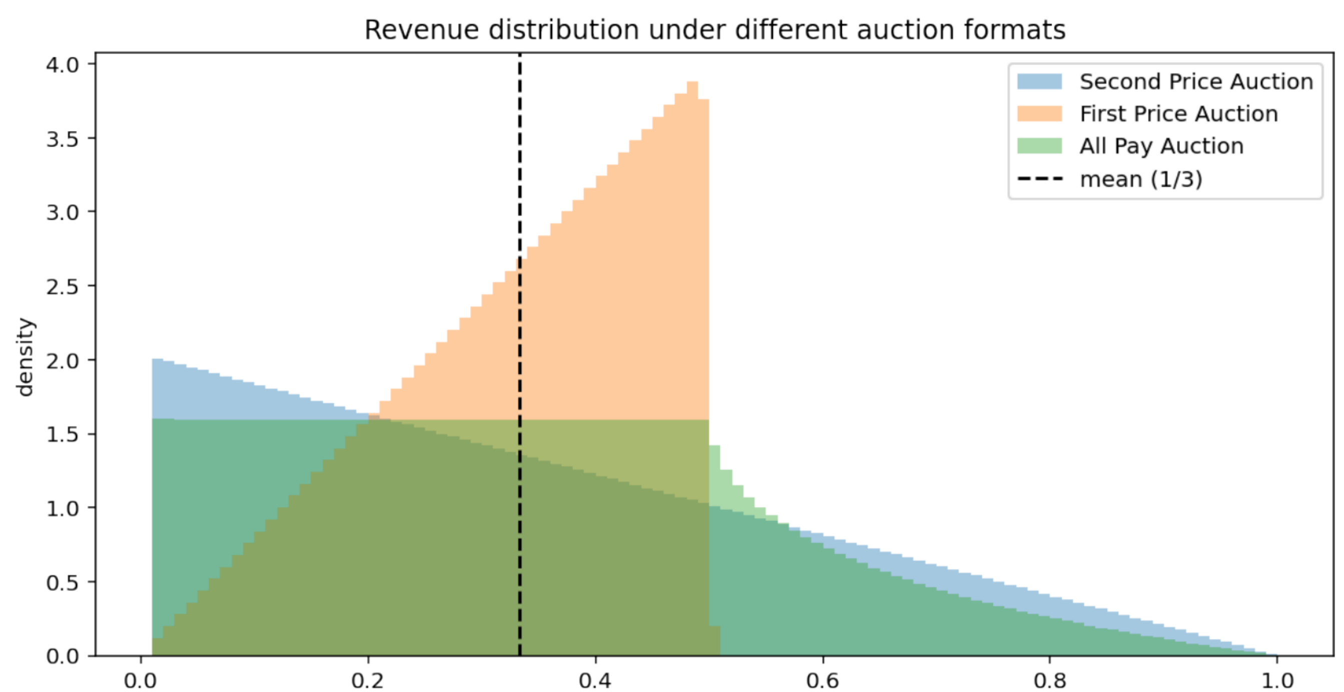
Main Question: Which of them minimize auction variance (and other risk measures)?



Standard Auction Setting

- n bidders each with valuation v_i drawn from independent distributions F_i
- social choice function $A_i(v)$ in $\{0,1\}$ with $\sum_i A_i(v) \leq 1$
- Our goal is to optimize over payment rules $P_i(v)$ that implement that social choice rule. By the revelation principle, we can focus on Bayesian incentive compatible rules (BIC), i.e., reporting your value is an optimal strategy.

$$\mathbb{E}[v_i A_i(\boldsymbol{v}) - P_i(\boldsymbol{v}) \mid v_i] \geq \mathbb{E}[v_i A_i(v_i', \boldsymbol{v}_{-i}) - P_i(v_i', \boldsymbol{v}_{-i}) \mid v_i], \forall i, v_i, v_i'$$

- Examples of BIC implementation of the auction formats discussed:

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- We will consider two types of IR constraints:
  - Ex-post IR: $v_i A_i(v) - P_i(v) \geq 0$ (e.g. SP, FP, ...)
  - Interim IR: $\mathbb{E}[v_i A_i(v) - P_i(v) \mid v_i] \geq 0$ (e.g. All-Pay and other loser-pay)

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- Interim allocation: $x_i(v_i) = \mathbb{E}[A_i(v) \mid v_i]$
- Myerson's integral: $z_i(v_i) := v_i x_i(v_i) - \int_0^{v_i} x_i(u) du$

Risk Minimization

- We want to understand the revenue random variables $R = \sum_i P_i(v)$
- Revenue Equivalence Theorem: $\mathbb{E}[P_i(v) \mid v_i] = z_i(v_i)$ for all BIC + interim IR and as a consequence $\mathbb{E}[R]$ is the same across all auction formats

- Revenue Variance: $\mathsf{Var}(R) = \mathbb{E}[(R - \mathbb{E}R)^2] = \mathbb{E}[R^2] - \mathbb{E}[R]^2$
  - Since $\mathbb{E}[R]^2$ is constant we want to find the auction minimizing the second moment of the revenue $\mathbb{E}[R^2]$
- More generally, we want to minimize $\mathbb{E}[f(R)]$ for a convex function f.

- **Main Question (more formally)**: Given an social choice rule $A_i(v)$ and independent type distributions F_i, and a risk measure f, what are the payment rules (subject to BIC and interim-IR) that maximize the risk $\mathbb{E}[f(R)]$.

Winner Pays Bid Auction

- Given an implementable social choice rule $A_i(v)$ and it corresponding interim allocation $x_i(v)$ and Myerson integral $z_i(v)$, we define the associated WPB payment rule as:

$$P_i^{\mathsf{WPB}}(\boldsymbol{v}) = b_i^{\mathsf{WPB}}(v_i) A_i(\boldsymbol{v}), \text{ where } b_i^{\mathsf{WPB}}(v_i) := \frac{z_i(v_i)}{x_i(v_i)}$$

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- For ex-post IR, WPB minimizes any convex risk measure.
- For interim IR + symmetric settings, WPB minimizes any convex risk measure.
- For interim IR + asymmetric, there exist auctions with lower variance/risk. We will characterize the optimal variance.

Prior Work: Ex-post IR Risk Minimization

- For the efficient allocation and ex-post IR, Waehrer, Harstad and Rothkopf (1998) showed that first price auction minimize any risk measure.

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- Slight generalization: for any allocation rule and ex-post IR, the winner pays bid auction minimizes any risk measure.

- One property of ex-post IR that is useful, at most one term of $\sum_i P_i(v_i)$ is non-zero, and hence:

$$(\textstyle\sum_i P_i(v_i))^2 = \sum_i P_i(v_i)^2$$

  for ex-post IR, the variance of revenue is the sum of payment variances.

- For all-pay auctions:

$$(\textstyle\sum_i P_i(v_i))^2 = \sum_i P_i(v_i)^2 + \sum_{i \neq j} P_i(v) P_j(v)$$

Interim IR Risk Minimization

- **Theorem**: The interim IR auction that minimize $\mathbb{E}[\sum P_i(v)^2]$ is the all-pay.

- ... but that doesn't account for cross terms $P_i(v)P_j(v)$. Once we do it, WPB is again the optimal.
- Symmetric environment: for the talk efficient allocation + IID valuations. (More generally we can have reserves, entry-fees,... as long as they preserve symmetry)

- **Main Theorem**: Among all interim IR and BIC payment rules for symmetric environments, WPB minimizes $\mathbb{E}[f(R)]$ for any convex risk measure f.

- Proof is based on a new revenue decomposition theorem.

Revenue Decomposition

- Let $P_i(v)$ be any (BIC+IR) payment rule and $P_i^{WPB}(v)$ be the WPB payment rule. We want to compare $\mathbb{E}[f(\sum_i P_i(v))]$ and $\mathbb{E}[f(\sum_i P_i^{\mathsf{WPB}}(v))]$

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- Using convexity we have:

$$f(\textstyle\sum_j P_j(\boldsymbol{v})) \geq f(\sum_j P_j^{\mathsf{WPB}}(\boldsymbol{v})) + f'(\sum_j P_j^{\mathsf{WPB}}(\boldsymbol{v})) \cdot (\sum_j P_j(\boldsymbol{v}) - \sum_j P_j^{\mathsf{WPB}}(\boldsymbol{v}))$$

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We want to show that this term is non-negative in expectation

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$$f'(\textstyle\sum_j P_j^{\mathsf{WPB}}(\boldsymbol{v})) = \lambda_i(v_i) + \mu_i(\boldsymbol{v})$$

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Non-negative (because f is convex) and is zero when the bidder pays: $P_i^{\mathsf{WPB}}(v)\mu_i(v) = 0$ which is a complementarity condition.

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- Let $P_i(v)$ be any (BIC+IR) payment rule and $P_i^{WPB}(v)$ be the WPB payment rule. We want to compare $\mathbb{E}[f(\sum_i P_i(v))]$ and $\mathbb{E}[f(\sum_i P_i^{\mathsf{WPB}}(v))]$

- Re-write the risk difference as:

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$\square$

Asymmetric Settings

- Example: two bidders with independent distributions $F_1(v_1) = v_1$ and $F_2(v_2) = v_2^2$. and the efficient allocation. The second moment of WPB is 0.188.

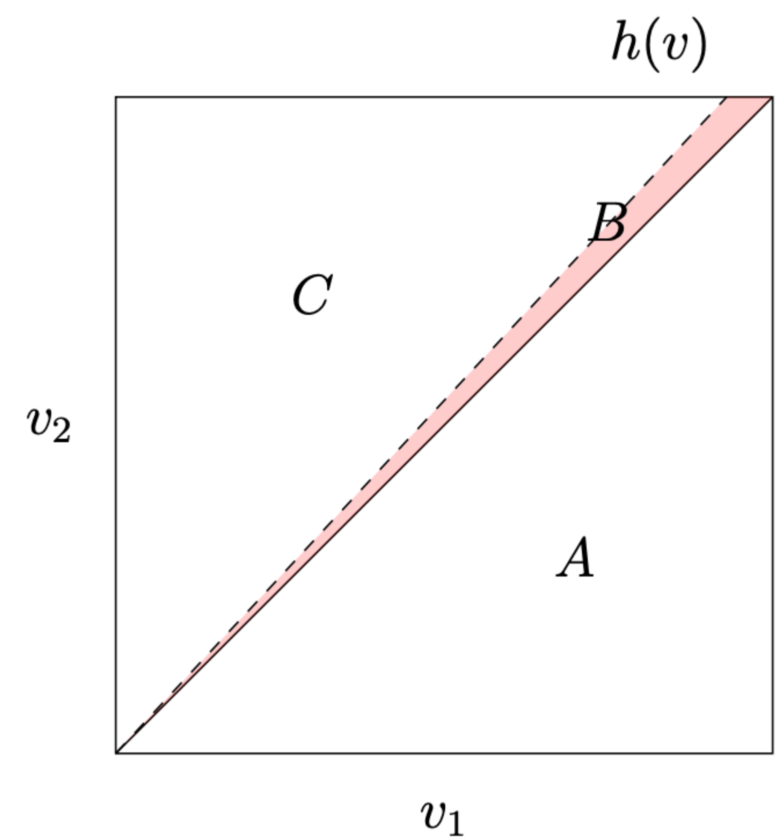
- Let's construct a payment rule with strictly smaller variance:

$$P_1^*(v_1, v_2) = \pi_1(v_1) \cdot \mathbf{1}\{h(v_1) \geq v_2\} \qquad P_2^*(v_1, v_2) = \pi_2(v_2) \cdot \mathbf{1}\{v_2 > h(v_1)\}$$

$$h(u) = u\left(\frac{4}{3}\right)^{1/4} \qquad \pi_1(v_1) = \begin{cases} v_1/\sqrt{3}, & \text{for } v_1 \leq (3/4)^{1/4} \\ 2v_1^3/3, & \text{for } v_1 > (3/4)^{1/4} \end{cases} \qquad \pi_2(v_2) = \frac{v_2}{2}\left(\frac{4}{3}\right)^{1/4}$$

- We can check it is BIC and interim-IR (but not ex-post IR). The second moment is 0.186.

- Region B = bidder 2 is allocated but bidder 1 pays



Asymmetric Settings

- Why? if we increase the region where bidder 1 pays, we can spread our their payments more and decrease their variance, moving variance from bidder 1 to bidder 2.

- Turns out we can do it more generally:

  Theorem: for any allocation A(v), if there exist scores $\pi_i(v_i)$ such that the payment function is such that only the player with highest score pays $\pi_i(v_i)$ and the payment rule implements A(v) then this payment rule minimizes risk.

- **Theorem*** (under some technical conditions): For any allocation function, we can construct scores as above.

Conclusion

Optimization perspective: fix the allocation rule and optimize payment rules.

Auction perspective: variance reduction is an important explanation for the prevalence of the Winner Pays Bid payment rules.

English Auction:

- Higher revenue in interdependent settings (Milgrom-Weber)

First Price Sealed Bid:

- Collusion-resistent in theory (Milgrom) but collusion observed in practice (famous milk cartels)
- Transparency (Bergemann and Horner)
- Credibility (Akbarpour, Li)
- Risk-minimizing for the auctioneer