Clinching Auctions with Online Supply

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Abstract

Auctions for perishable goods such as internet ad inventory need to make real-time allocation and pricing decisions as the supply of the good arrives in an online manner, without knowing the entire supply in advance. These allocation and pricing decisions get complicated when buyers have some global constraints. In this work, we consider a multi-unit model where buyers have global *budget* constraints, and the supply arrives in an online manner. Our main contribution is to show that for this setting there is an individually-rational, incentive-compatible and Pareto-optimal auction that allocates these units and calculates prices on the fly, without knowledge of the total supply. We do so by showing that the Adaptive Clinching Auction satisfies a *supply-monotonicity* property.

We also analyze and discuss, using examples, how the insights gained by the allocation and payment rule can be applied to design better ad allocation heuristics in practice. Finally, while our main technical result concerns multi-unit supply, we propose a formal model of online supply that captures scenarios beyond multi-unit supply and has applications to sponsored search. We conjecture that our results for multi-unit auctions can be extended to these more general models.

1 Introduction

The problem of selling advertisement on the web is essentially an online problem - the supply (pageviews) arrives dynamically and decisions on how to allocate ads to pageviews and price these ads need to be taken instantaneously, without full knowledge of the future supply. What makes these decisions complex is the fact that buyers have budget constraints, which ties the allocation and pricing decisions across different time steps. Another complicating feature of the online advertisement markets is that buyers are strategic and can misreport their values to their own advantage.

These observations have sparkled a fruitful line of research in two different directions. First is that of designing online algorithms where one assumes that the supply is coming online, but makes a simplifying assumption that buyers are non-strategic. This line of research has led to novel tools and techniques in the design of online algorithms (see for example [14, 5, 8, 1]). The second line of research considers the design of incentive-compatible mechanisms assuming that buyers are strategic, but makes an assumption that the supply is known beforehand. Handling budget constraints using truthful mechanisms is non-trivial since standard VCG-like techniques fail when the player utilities are not quasi-linear. In a seminal work, Dobzinski, Lavi and Nisan [9] showed that one can adapt Ausubel's clinching auction [2] to achieve Pareto-optimal outcomes for the case of multi-unit supply. In settings with budget constraints, the goal of maximizing social welfare is unattainable and efficiency is achieved through Pareto-optimal outcomes. In fact, if budgets are sufficiently large, Pareto-optimal outcomes are exactly the ones that maximize social welfare [11].

From a practical standpoint, it is important to understand what can be done when both the above scenarios are present at the same time. Motivated by this, we study the following question in this paper: *Can one design efficient incentive-compatible mechanisms for the case when agents have budget constraints and the supply arrives online?*

A closely related question was studied by Babaioff, Blumrosen and Roth [3], who asked weather it was possible to obtain efficient incentive-compatible mechanisms with online supply, but instead of budget constraints, they considered capacity constraints, i.e., each agent wants at most k items (capacity) rather than having at most B dollars to spend (budget). They showed that no such mechanism can be efficient and proved lower bounds on the efficiency that could be achieved.

Such lower bounds seem to offer a grim perspective on what can be done with budget constraints, since typically, budget constraints are less well-behaved than capacity constraints. On the contrary, and somewhat surprisingly, we show that for budget constraints it is possible to obtain incentive compatible and Pareto-optimal auctions that allocate and charge for items as they arrive, by showing that the Adaptive Clinching Auction in [9] for multi-unit supply can be implemented in an online manner. More formally, we show that the clinching auction for the multi-unit supply case satisfies the following *supply-monotonicity* property: Given the allocation and payments obtained by running the auction for initial supply s, one can obtain the allocation and payments for any other supply $s' \geq s$ by *augmenting* to the auction outcome for supply s. In other words, it is possible to find an allocation for the extra $s' - s$ items and extra (non-negative) payments such that when added to clinching auction outcome for the supply s, we obtain the clinching auction outcome for supply s'. Moreover, we show that each agent's utility is also monotone with respect to the supply, i.e., agents do not have incentive to leave the auction prematurely.

From a technical perspective, proving the above result requires a deeper understanding of the structure of the clinching auction, which in general is difficult to analyze because it is described using a differential ascending price procedure rather than a one-shot outcome like VCG. In order to do so, we study the description of the clinching auction given by Bhattacharya, Conitzer, Munagala and Xia [4] by means of a differential equation. At its heart, the proof of the supply monotonicity is a coupling argument. We analyze two parallel differential procedures whose limits correspond to

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the outcome of the clinching auction with the same values and budgets but different initial supplies. We prove that either one stays ahead of the other or they meet and from this point on they evolve identically (for carefully chosen concepts of 'stay ahead' and 'meet'). We identify many different invariants in the differential description of the auction, and use tools from real analysis to show that these invariants hold.

Towards better heuristics for ad-allocation. One of the main goals of this research program is to provide insights for the design of better heuristics to deal with budget-constrained agents in real ad auctions. Most heuristics in practice are based on bid-throttling or bid-lowering. Bid-throttling probabilistically removes a player from the auction based on her spent budget (throttling). Bid-lowering runs a standard second price auction with modified bids. While sound from an algorithmic perspective, bid-throttling and bid-lowering are not integrated with the underlying auction from the perspective of incentives. We believe clinching auctions provide better insights into designing heuristics that are more robust to strategic behavior. Towards this goal, we analyze the online allocation rule obtained from the clinching auction and provide a qualitative description of how allocation and payments evolve when new items arrive, and show how this description is significantly different from the bid-throttling or bid-lowering heuristics that are applied in practice. In clinching, the fact that an agent got many items for an expensive price in the beginning (once the items were scarce) gives him an advantage over items in the future if/once they become abundant, i.e., this agent will have the possibility of acquiring these items for a lower price than the other agents.

Online supply beyond Multi Unit Auctions. Since the groundbreaking work of Dobzinski, Lavi and Nisan [9], clinching auctions have been extended beyond multi-unit auctions in the offline-supply setting: first to matching markets by Fiat, Leonardi, Saia and Sankowski [10], then to sponsored search setting by Colini-Baldeschi, Henzinger, Leonardi and Starnberger [6] and to general polymatroids by Goel, Mirrokni and Paes Leme [11]. It is natural to ask if such offline-auctions have online supply counterparts.

We leave this question as the main open problem in this paper. However, before one formulates this question, we need to define what we mean by online supply in such settings. One contribution of this paper is to define a *model of online supply* for allocation constraints beyond multi-units. Using our definition of online supply for generic constraints and the definition of AdWords Polytope in [11], we can capture for example: (1) sponsored search with multiple slots, where for each pageview we need to decide on an allocation of agents to slots and (ii) matching constraints: each arriving pageview can only be allocated to a subset of advertisers.

Related Work. The study of auctions with online supply was initiated in Mahdian and Saberi [12] who study multi-unit auctions with the objective of maximizing revenue. They provide a constant competitive auction with the optimal offline single-price revenue. Devanur and Hartline [7] study this problem in both the Bayesian and prior-free model. In the Bayesian model, they argue that there is no separation between the online and offline problem. This discussion is then extended to the prior-free setting. The results in [7] assume that the payments can be deferred until all supply is realized, while allocation needs to be done online.

Our work is more closely related to the work by Babaioff, Blumrosen and Roth [3], which study the online supply model with the goal of maximizing social welfare. Unlike previous work, they insist (as we also do) that payments are charged in an online manner. This is a desirable property from a practical standpoint, since it allows players to monitor their spend in real-time. Their results are mainly negative: they prove lower bounds on the approximability of social welfare in setting

where the supply is online. Efficiency is only recovered when stochastic information on the supply is available.

We should also note that there is a long line of research at the intersection of online algorithms and mechanism design, mostly dealing with agents arriving and departing in an online manner. We refer to Parkes [16] for a survey.

Another stream of related works comes from the literature on mechanism design with budget constrained agent. This line of inquire was initiated in Dobzinski et al [9], who proposed a mechanism based on Ausubel's celebrated clinching framework [2]. The authors propose a mechanism for multi-unit auctions and indivisible goods and a mechanism for 2 players and divisible goods called the *Adaptive Clinching Auction*. The mechanism is extended to n players by Bhattacharya et al [4] by means of a differential ascending process whose limit is the allocation and payments of the Adaptive Clinching Auction. The authors also show how to use randomization to enable the auction to handle private budgets. Many subsequent papers deal with extending the clinching auctions to more general environments beyond multi-units: Fiat el al [10], Colini-Baldeschi et al [6] and Goel et al [11].

2   Preliminary Definitions

An auction is defined by a set of n *players* equipped with utility functions and an *environment*, which specifies the set of feasible allocations. Formally, we consider a divisible[1] good g such that an allocation of this good will be represented by a vector $x = (x_1, \ldots, x_n)$, meaning that player i got x_i units of the good. Player i has a set of private types Θ_i and his utility function will depend on his type θ_i, the amount x_i he is allocated and the amount of money π_i he is charged for it. We will represent it by a function $u_i(\theta_i, x_i, \pi_i)$. Moreover, we will consider a set $P \subseteq \mathbb{R}_+^n$ that specifies the set of feasible allocations. We will call such set the *environment*.

Definition 2.1 *An auction for this setting consists of two mappings: the allocation $x : \times_i \Theta_i \to P$ and the payment $\pi : \times_i \Theta_i \to \mathbb{R}_+^n$. The auction is said to be individually rational if $u_i(\theta_i, x_i(\theta), \pi_i(\theta)) \geq 0$ for all type vectors $\theta = (\theta_1, \ldots, \theta_n)$. The auction is said to be incentive-compatible (a.k.a. truthful) if no player can improve his utility by misreporting his type, i.e.:*

$$u_i(\theta_i, x_i(\theta_i, \theta_{-i}), \pi_i(\theta_i, \theta_{-i})) \geq u_i(\theta_i, x_i(\theta_i', \theta_{-i}), \pi_i(\theta_i', \theta_{-i})), \qquad \forall \theta_i', \theta_i \in \Theta_i, \forall i$$

In this paper, we will be particularly interested in agents with budget-constrained utility functions. For this setting $\Theta_i = \mathbb{R}_+$ representing the value v_i the agent has for one unit of the good. There is a *public* budget B_i such that: $u_i(v_i, x_i, \pi_i) = v_i \cdot x_i - \pi_i$ if $\pi_i \leq B_i$ and $-\infty$ otherwise. Incentive compatibility for this setting means that the agents don't have incentives to misreport their value. For this setting, we are interested in auctions producing Pareto-optimal outcomes:

Definition 2.2 *Given an auction with environment P and agents equipped with budget-constrained utility functions, B_i being the public budgets, we say that an outcome $(x, \pi), x \in P, \pi \leq B$ is* Pareto optimal *if there is no alternative outcome $x' \in P, \pi' \leq B$ such that $v_i x_i' - \pi_i' \geq v_i x_i - \pi_i$ for all i, $\sum_i \pi_i' \geq \sum_i \pi_i$ and at least one of those inequalities is strict.*

For the remainder of this paper, we will assume that budgets are public for simplicity. For multi-unit auctions (i.e. $P = \{x; \sum_i x_i \leq s\}$), our results extend to private budgets by applying the budget elicitation schemes in Bhattacharya el al [4].

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[1]our decision of considering divisible goods is motivated by our application. In sponsored search, the number of items (pageviews) arriving at each time is enormous, making fractional allocations essentially feasible.

3 Online Supply Model

We consider auctions where the feasibility set is not known in advance to the auctioneer. For each time $t \in \{0, \ldots, T\}$, we associate an environment $P_t \subseteq \mathbb{R}^n_+$, which keeps track of the allocations done in times $t' = 1..t$ [2]. In each step, the mechanism needs to output an allocation vector $x^t = (x^t_1, \ldots, x^t_n) \in P_t$ and a payment vector $\pi^t = (\pi^t_1, \ldots, \pi^t_n) \geq 0$ by augmenting x^{t-1} and π^{t-1}. Given a set of desirable properties, we would like to maintain them for all t. To make the problem tractable, have to restrict the set of possible histories $\{P_t\}_{t \geq 0}$. We do so by defining a partial ordering $\preceq$ on the set of feasibility constraints such that if $t \leq s$ then $P_t \preceq P_s$.

Our main goal is to design auctions where the auctioneer can allocate and charge payments *'on the fly'*. The auctioneer will face a set of environments $P_1 \preceq P_2 \preceq \ldots \preceq P_t$ and at time t, he needs to allocate $x^t \in P_t$ and charge π^t, maintaining a set of desirable properties. He doesn't know if P_t will be the final outcome, or if some new environment $P_{t+1} \succeq P_t$ will arrive, in which case he will need to augment $x^t \in P_t$ to an allocation x^{t+1} in P_{t+1}. It is crucial that his decision at time t doesn't depend the knowledge about P_{t+1}.

Definition 3.1 (Online Supply Model) *Consider a family of feasibility allocation constraints indexed by $\mathcal{F}$, i.e, for each $f \in \mathcal{F}$ associate a set of feasible allocation vectors $P^f \subseteq \mathbb{R}^n_+$ (a set P^f is often called environment). Also, consider a partial order $\preceq$ defined over $\mathcal{F}$ such that if $f \preceq f'$ then $P^f \subseteq P^{f'}$. An auction for environment P^f consists of functions $x^f : \Theta = \times_i \Theta_i \to P^f$ and $\pi^f : \Theta \to \mathbb{R}^n_+$.*

*An **auction in the strong online supply model** for $(\mathcal{F}, \preceq)$ is a family of auctions such that $x^f \leq x^{f'}$ and $\pi^f \leq \pi^{f'}$ whenever $f \preceq f'$. Moreover, we say that the auction satisfies a certain property if it satisfies this property for each f (e.g. the auction is incentive compatible if for each $f \in \mathcal{F}$, (x^f, π^f) is an incentive compatible auction).*

*An **auction in the weak online supply model** for $(\mathcal{F}, \preceq)$ is essentially the same, except that we drop the requirement of $\pi^f \leq \pi^{f'}$. The intuition is that we are required to allocate goods online, but are allowed to charge payments only in the end.*

The main idea behind the definition is that if at some point the auctioneer runs the auction (x^f, π^f) for some environment P^f and at a later time some more goods arrive perhaps with new constraints such that the environment is augmented to $P^{f'}$ with $f' \succeq f$, then the auctioneer can run $(x^{f'}, \pi^{f'})$ and augment the allocation of player i by $x^{f'}_i(v) - x^f_i(v)$ goods and charge him more $\pi^{f'}_i(v) - \pi^f_i(v)$.

Example 3.2 (Multi-unit auctions) *Let $\Delta_s = \{x \in \mathbb{R}^n_+; \sum_i x_i \leq s\}$ and define $\mathcal{F}^{MU} = \{\Delta_s; s \geq 0\}$ and let $\Delta_s \preceq^{MU} \Delta_t$ iff $s \leq t$. Let the value of player i for one unit of the good, v_i, lies in $\Theta_i = \mathbb{R}_+$. Now $u_i = v_i x_i - \pi_i$. Thus, we are in a simple multi-unit auction setting. In this setting, VCG is incentive compatible, individually rational and efficient (in the sense that it has those three properties once run for each Δ_s) auction in the strong online model for $(\mathcal{F}^{MU}, \preceq^{MU})$.*

Example 3.3 (Multi-unit auctions with capacities) *Curiously, if players have capacity constraints, i.e., their utilities are $u_i = v_i \min\{x_i, C_i\} - \pi_i$, then the VCG allocations for $(\mathcal{F}^{MU}, \preceq^{MU})$ are still monotone in the supply, but the payments are not. For example, consider two agents with values $v_1 = 1, v_2 = 2$ and capacities $C_1 = C_2 = 1$. With supply 1, one item is allocated to player 2 and he is charged 1. With supply 2, both players get one unit of the item, but the VCG*

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[2]We want to stress the fact that P_t *doesn't* represent the set of allocations allowed *at* time t, but the set of allocations allowed *until* time t. The set of new possible allocations in time t is the difference between P_t and P_{t-1}

prices are zero. Therefore, there is no incentive compatible, individually rational and efficient in the strong online model. Babaioff, Blumrosen and Roth [3] strengthen this result showing that no $\Omega(\log\log n)$-approximately efficient auction exists in the strong online model.

Example 3.4 (Polymatroidal auctions) *Now, let $\mathcal{F}^{PM}$ be the set of all polymatroidal domains and consider the naive-partial-order $\preccurlyeq^N$ to be such that $f \preccurlyeq f'$ iff $P^f \subseteq P^{f'}$. The VCG is not even online in the weak sense for $(\mathcal{F}^{PM}, \preccurlyeq^N)$. Consider the following example:*

$$P^f = \{x \in \mathbb{R}^2_+; x_1 \le 2, x_2 \le 2, x_1 + x_2 \le 3\} \quad and \quad P^{f'} = \{x \in \mathbb{R}^2_+; x_1 + x_2 \le 4\}$$

then clearly $f \preccurlyeq f'$ but if $v_1 > v_2$. $x^f(v) = (2, 1)$ but $x^{f'} = (4, 0)$ violating the monotonicity property. But now, let's define a different partial order $\preccurlyeq^{PM}$ such that $f \preccurlyeq^{PM} f'$ if there is a polymatroid P' such that $P^{f'} = P^f + P'$ where the sum is the Minkowski sum. In Lemma A.1 (Appendix A) we show that VCG is an auction in the strong online model for $(\mathcal{F}^{PM}, \preccurlyeq^{PM})$.

One interesting property of incentive-compatible auctions in the online supply model is that utilities are monotone with the supply. If bidders have the option of leaving in each timestep collecting their current allocations for their current payment, they still (weakly) prefer to stay until the end of the auction. This is formally stated and proved in Lemma A.2 in Appendix A.

4   Clinching Auctions and Supply Monotonicity

Our main theorem states that the Adaptive Clinching Auction (defined in Dobzinski, Lavi and Nisan [9] and Bhattacharya el al [4]) is an incentive-compatible auction in the strong online supply model for budget constrained agents in the multi-unit setting. Formally:

Theorem 4.1 *Given n agents with public budgets B_i and single-dimensional types $v_i \in \mathbb{R}_+$ such that their utility is given by $u_i = v_i x_i - \pi_i$ if $\pi_i \le B_i$ and $u_i = -\infty$ otherwise, the Adaptive Clinching Auction is an auction in the strong online supply model for $(\mathcal{F}^{MU}, \preccurlyeq^{MU})$. In other words, if $x(v, B, s)$ and $\pi(v, B, s)$ is the outcome of the auction for valuation profile v, budgets B and supply s, then if $s \le s'$, then: $x(v, B, s) \le x(v, B, s')$ and $\pi(v, B, s) \le \pi(v, B, s')$.*

Notice that this is in sharp contrast with what happens in Example 3.3 where getting a Pareto optimal auction in the strong online model is not possible, not even in an approximate way [3]. This is somewhat surprising, since capacity constraints on the allocations are usually more nicely-behaved compared to budget constraints.

Before proving the result, we review the Adaptive Clinching Auction [4, 9], presenting it in a way which will be more convenient for the proof.

4.1   Adaptive Clinching Auction

The clinching auction takes as input the valuation profile v, the budget profile B and the initial supply s, then it runs a procedure based on the ascending price framework to determine final allocation and payments. There is a price clock p, and for each price, the auction mantains[4] $x_i(p)$ denoting the current allocation of player i and $B_i(p)$, which is the current remaining budget of player i. Initially, $x_i(0) = 0$ and $B_i(0) = B_i$, their initial budget. For each p, the auction defines

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[3]for that setting, since there are no budgets, Pareto optimality boils down to efficiency.

[4]note that here we prefer to index the ascending process by the price itself rather then an external variable, like in Bhattacharya el al [4].

the values of the right-derivatives $\partial_p x_i(p)$ and $\partial_p B_i(p)$ and described its behavior in the points in which it is discontinuous. Notice we will use $\partial_p f(p)$ to denote the right-derivative of f at p.

For simplicity, we define the auction and prove our results for valuation profiles v such that $v_i \neq v_j$ for each $i \neq j$ (we call it a profile in generic form) This is mainly a technical assumption to avoid over-complicating the statement and the proof. In Appendix C, we extend this for any valuation profile v. For the definition and its subsequent discussion, we will use the following implicitly defined notation:

- remnant supply: $S(p) = s - \sum_i x_i(p)$
- active players: $A(p) = \{i; v_i > p\}$
- clinching players: $C(p) = \{i \in A(p); S(p) = \sum_{j \in A(p)\setminus i} \frac{B_j(p)}{p}\}$
- maximum remaining budget: $B_*(p) = \max_{i \in A(p)} B_i(p)$
- for any function f, let $f(\bar{p}-) = \lim_{p \uparrow \bar{p}} f(p)$ and $f(\bar{p}+) = \lim_{p \downarrow \bar{p}} f(p)$

Definition 4.2 (Adaptive Clinching Auction) *Given as input a valuation vector v in generic form, a budget vector B and initial supply s, consider the functions $x_i(p), B_i(p)$ such that:*

(i) $x_i(0) = 0$ and $B_i(0) = B_i$.

(ii) $\partial_p x_i(p) = \frac{S(p)}{p}$ and $\partial_p B_i(p) = -S(p)$ if $i \in C(p)$ and $\partial_p x_i(p) = \partial_p B_i(p) = 0$ otherwise.

(iii) the functions x_i and B_i are right-continuous at all points p, i.e., $x_i(p) = x_i(p+)$ and $B_i(p) = B_i(p+)$ for all p and it is left-continuous at all points $p \notin \{v_1, \ldots, v_n\}$, i.e., $x_i(p-) = x_i(p)$ and $B_i(p-) = B_i(p)$ for all $p \notin \{v_1, \ldots, v_n\}$

(iv) for $p = v_i$, let $\delta_j = \left[S(v_i-) - \sum_{k \in A(v_i)\setminus j} \frac{B_k(v_i-)}{v_i} \right]^+$. For $j \in A(v_i)$, let $x_j(v_i) = x_j(v_i-) + \delta_j$ and $B_j(v_i) = B_j(v_i-) - v_i \delta_j$ and for $j \notin A(v_i)$, $x_j(v_i) = x_j(v_i-)$ and $B_j(v_i) = B_j(v_i-)$.

The existence and uniqueness of those functions follow from elementary real analysis. The outcome associated with v, B, s is $x_i = \lim_{p \to \infty} x_i(p)$ and $\pi_i = B_i(0) - \lim_{p \to \infty} B_i(p)$. Notice that this is well defined since x and B are constant for $p > \max_i v_i$.

The verb *clinch* means acquiring goods that are underdemanded at the current price. So clinching a δ_i amount at price p means receiving δ_i amount of the good and paying $\delta_i p$ for it. When we refer to a player clinching some amount, either we refer to the infinitesimal clinching happening in item (ii) or the player clinching positive units in (iv).

The reader familiar with the clinching auction for indivisible goods will notice that the definition above is nothing more than the limit as $\epsilon \to 0$ of this auction run with $\frac{1}{\epsilon}s$ indivisible goods and valuations ϵv_i per unit. This auction satisfies all the desirable properties for multi-unit auctions with budgets:

Theorem 4.3 (Bhattacharya et al [4]) *The Adaptive Clinching Auction in Definition 4.2 is incentive-compatible, individually-rational, budget-feasible and produces Pareto-optimal outcomes.*

As one can possibly guess, it is possible to solve the differential equation in each interval between two adjacent values of v_i and give an explicit description of the clinching auction. We do that in Appendix D. Nevertheless, we mostly prove our results using the differential form in Definition 4.2 which is more insightful than the explicit version.
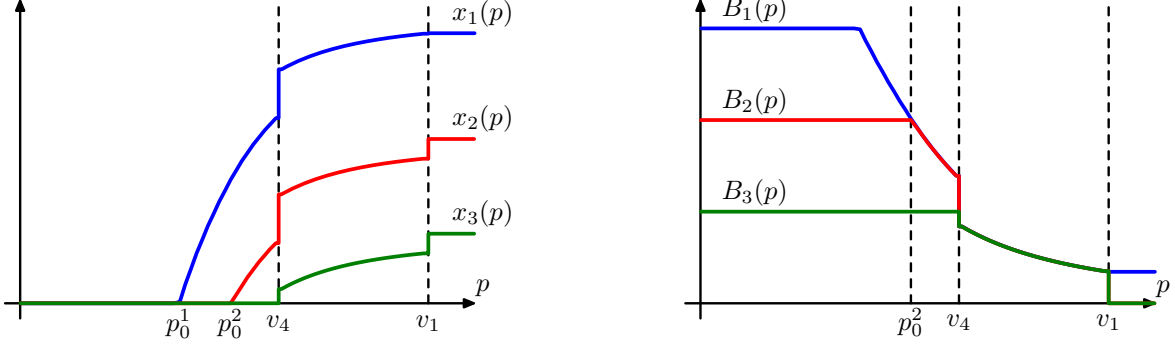
Figure 1: The functions $x(p)$ and $B(p)$ for an auction in Example 4.4

Example 4.4 *At this point, it is instructive to consider an example of the auction. Consider an auction between $n = 4$ players with valuations $v = [9, 10, 11, 5.7]$ and $B = [3, 2, 1, .5]$. The functions $x_i(p), B_i(p)$ are depicted in Figure 1. For $p < p_0^1 = 3.5$, the clinching set $C(p)$ is empty. At this price $S(p_0^1) = 1 = \frac{2+1+0.5}{3.5} = \sum_{j \neq 1} \frac{B_j}{p}$, so player 1 alone begins clinching.*

Since he is clinching alone for a while, $x_1(p) = s - S(p)$. Now by derivating this expression we get that $\frac{S(p)}{p} = \partial_p x_1(p) = -\partial_p S(p)$. Solving for the supply with the condition that $S(p_0^1) = 1$, we get: $S(p) = \frac{sp_0^1}{p}$ and $x_1(p) = s - \frac{sp_0^1}{p}$. This continues while no other player enters the clinching set. The supply function $S(p)$ is illustrated in the first part of Figure 2.

Notice that for this period, the budget of 1 is being spent while the budgets of the other agents are intact. Eventually, the budget of 1 meets the budget of 2, and at this point, those two players are indistinguishable from the perspective of the differential procedure as long as both are active. Therefore, both start spending their budget at the same rate and acquiring goods at the same rate $\partial_p x_1(p) = \partial_p x_2(p) = S(p)$. Since from this point on $S(p) = s - x_1(p) - x_2(p)$, then $\frac{S(p)}{p} = \partial_p x_1(p) = -\frac{1}{2}\partial_p S(p)$. Solving again for the supply and the boundary condition $S(p_0^2) = \frac{sp_0^1}{p_0^2}$ we get: $S(p) = \frac{sp_0^1 p_0^2}{p^2}$. Using this, one can calculate $x_1(p)$ and $x_2(p)$. Both continue clinching at the same rate until the price reaches $p = v_4$, where player 4 exits the active set prompting the agents to clinch a positive amount according to (iv).

Their allocation $x_i(p)$ and budgets $B_i(p)$ are discontinuous at this point, but continue to follow the differential procedure after this point, having their budgets all equal (not coincidentally, as we will see in Lemma 4.11), until price $p = v_1$ is reached and player 1 exits the active set. At this point, all the remaining active players clinch a positive amount according to (iv) that exhausts the supply. Therefore, allocations and budgets are constant from this point on.

One important tool in analyzing this auction is the concept of the *wishful allocation*. We define a $\Psi_i(p)$ as a function of $x_i(p)$ and $B_i(p)$ which is continuous even at the points where $x_i(p)$ and $B_i(p)$ are not. It is carefully set up so that the discontinuities from both functions cancel out. Intuitively, it represents a sum of what the player acquired already at the current price $x_i(p)$ with the maximum amount he would like to acquire at this price, which is $\frac{B_i(p)}{p}$.

Definition 4.5 (Wishful allocation) *The wishful allocation is defined as $\Psi_i(p) = x_i(p) + \frac{B_i(p)}{p}$.*

Lemma 4.6 *The wishful allocation is continuous and right-differentiable for all $p \geq 0$. Moreover, its right-derivative is given by: $\partial_p \Psi_i(p) = -\frac{B_i(p)}{p^2}$.*
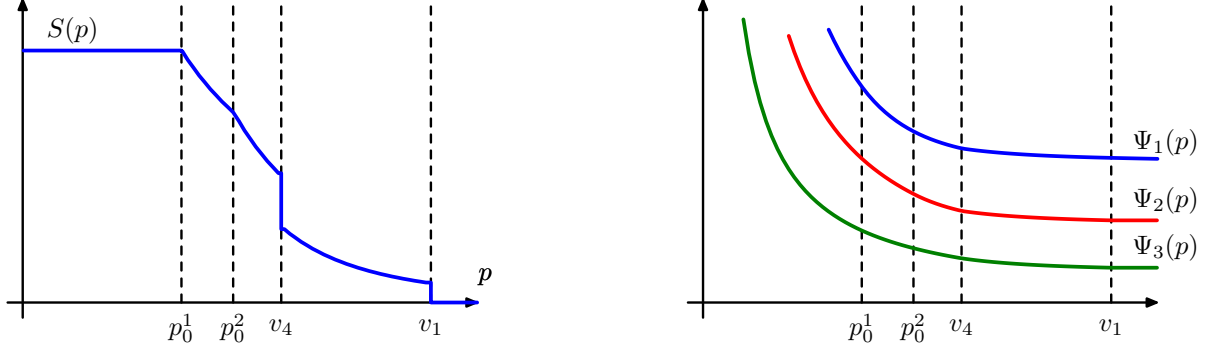
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Figure 2: Supply $S(p)$ and wishful allocation $\Psi(p)$ for an auction in Example 4.4

The proof is elementary and can be found in Appendix A. Since $\Psi_i(p) \geq x_i(p)$ and is a monontone non-increasing function converging to the final allocation as $p \to \infty$, it constantly gives us an upper bound of the final allocation.

Now, we study some other properties of the above auction, which will be useful in the proof of our main theorem. First we prove a Meta Lemma that sets the basic structure for most of our proofs. The lemma is based on elementary facts of real analysis. Its proof, as well as other missing proofs of this section can be found in Appendix A.

Meta-Lemma 4.7 *Given a property Λ that depends on p, if we want to prove for all $p \geq p_0$, it is enough to prove the following facts:*
(a) it holds for $p = p_0$.
(b) if Λ holds for p, then there is some $\epsilon_p > 0$ such that Λ holds for $[p, p + \epsilon_p)$
(c) if Λ holds for all p' such that $p_0 \leq p' < p$, then Λ also holds for p.

For most properties Λ that we want to prove about the Adaptive Clinching Auction, part (a) is easy to show, part (b) requires using the right-continuity of the function and the value of the right-derivatives given in item (ii) of Definition 4.2 and part (c) is usually proved using continuity for $p \notin \{v_1, \ldots, v_n\}$ and using part (iv) of Definition 4.2.

The first two lemmas (whose proof is based on the Meta-Lemma) state that once a player start acquiring goods (i.e. $\partial_p x_i(p) > 0$), he continues to do so for all the prices until p becomes equal to his value v_i. All the proofs can be found in Appendix A.

Lemma 4.8 *Once a player i enters the clinching set, then he is in the clinching set until he becomes inactive, i.e., if $i \in C(p)$ for some p, then $i \in C(p')$ for all $p' \in [p, v_i)$.*

Lemma 4.9 *For each price p and each active player i, $S(p) \leq \sum_{j \in A(p) \setminus i} \frac{B_j(p)}{p}$.*

Corollary 4.10 *If at price $p = v_j$, player $i \in A(v_j)$ acquires any positive amount of the good $\delta_i > 0$, then he enters in the clinching set (if he wasn't previously), i.e., $i \in C(v_j)$.*

A crucial observation for our proof is that the evolution of the profile of remaining budget profiles follows a very structured format. At any given price, the remaining budget of an agent is either his original budget or the maximum budget among all agents. It is instructive to observe that in the second part of Figure 1.

Lemma 4.11 *For each price p, if $C(p) \neq \emptyset$, then $C(p) = \{i \in A(p); B_i(p) = B_*(p)\}$.*

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Corollary 4.12 *For each $i \in A(p)$, $B_i(p) = \min\{B_i(0), B_*(p)\}$.*

4.2 Supply Monotonicity

Now we are ready to prove Theorem 4.1 which is our main result. For that we fix a budget profile B and a valuation profile v in generic form (i.e. $v_i \neq v_j$ for $i \neq j$, which is not needed for the proof and is mainly intended to simplify the exposition; See Appendix C). Now, we consider two executions of the adaptive clinching auction. One with initial supply s^{b} which we call the *base auction* and one with initial supply $s^{\text{a}} \geq s^{\text{b}}$ which we call the *augmented auction*. Running the base and augmented auction with the same valuations and budgets we get functions $x^{\text{b}}(p), B^{\text{b}}(p)$ and $x^{\text{a}}(p), B^{\text{a}}(p)$. From this point on, we use superscripts b and a to refer to the base and augmented auctions respectively. For the set of active players at a given price, we omit the superscript, since $A^{\text{b}}(p) = A^{\text{a}}(p)$ for all p.

   As the first step toward the proof of Theorem 4.1, we prove that the payments are monotone with the supply, that the final payment of each agent in the augmented auction is higher than in the base auction:

Proposition 4.13 (Payment Monotonicity) *Given the base and augmented auction as defined above, then for all $p \geq 0$ and all agents i, $B_i^{\text{b}}(p) \geq B_i^{\text{a}}(p)$.*

   The full proof of this proposition is delicate and involves keeping track of many invariants as the allocation and budget profiles evolve with prices. We delay the proof to Appendix B. Here we present a warm-up to the proof that deals with the case where all values v_i are very large. This special case will highlight the core of the proof which is essentially a coupling argument. Also, we keep the discussion here informal and delay the formal arguments to Appendix B.

Warm Up: If valuations are large compared to budgets, $C^{\text{a}}(p) = C^{\text{b}}(p) = [n]$ for some $p < \min_i v_i$ and once $\bar{p} = \min_i v_i$ is reached, some player i becomes inactive and all the other players $j \neq i$, clinch their entire demand $\delta_j = \frac{B_j(\bar{p}-)}{\bar{p}}$.

   This case is nice because it allows us to ignore part (iv) of Definition 4.2 and simply analyze the continuous function defined in $[0, \bar{p})$ by part (ii) of the definition. We start by defining $p_0^{\text{a}} = \min\{p; C^{\text{a}}(p) \neq \emptyset\}$ and $p_0^{\text{b}} = \min\{p; C^{\text{b}}(p) \neq \emptyset\}$. Since the supply is larger in the augmented auction, $p_0^{\text{a}} < p_0^{\text{b}}$. Then we can divide the analysis in three intervals: in the interval $[0, p_0^{\text{a}})$ where no player is clinching, so budgets are constant, i.e., equal to the initial budget. In the interval $[p_0^{\text{a}}, p_0^{\text{b}})$ no player is clinching in the base auction but some are clinching in the augmented auction, so clearly the remaining budgets are larger in the base auction. For the remaining interval $[p_0^{\text{b}}, \infty)$ we can use Corollary 4.12 to see that all we need to show is that since both clinching sets are non-empty we just need to prove that $B_*^{\text{a}}(p) \leq B_*^{\text{b}}(p)$ for all $p \in [p_0^{\text{b}}, \infty)$. This is true for $p = p_0^{\text{b}}$ by continuity of B_*^{a} and B_*^{b} (since $p < \min_i v_i$). Now we argue that if $B_*^{\text{a}}(p) \leq B_*^{\text{b}}(p)$ for some $p \geq p_0^{\text{b}}$, then $B_*^{\text{a}}(p') \leq B_*^{\text{b}}(p')$ for $p' \in [p, p + \epsilon)$ for some $\epsilon > 0$. Then we invoke the Meta-Lemma to extend this to all $p \geq p_0^{\text{b}}$. Now, in order to prove that we analyze two cases: if $B_*^{\text{a}}(p) < B_*^{\text{b}}(p)$ then by continuity of B_*^{a} and B_*^{b}, there exists some $\epsilon > 0$ such that $B_*^{\text{a}}(p') < B_*^{\text{b}}(p')$ for $p' \in [p, p + \epsilon)$. Now, if $B_*^{\text{a}}(p) = B_*^{\text{b}}(p)$, then by Corollary 4.12, $B_i^{\text{a}}(p) = B_i^{\text{b}}(p)$ for all agents i and moreover, the remnant supply is the same $S^{\text{a}}(p) = S^{\text{b}}(p)$, since $S(p) = \sum_{i \in A(p)} \frac{B_i(p)}{p} - \frac{B_*(p)}{p}$. Notice that the evolution of budgets in $p' \geq p$ depends only on $B(p)$ and $S(p)$ and since those are equal for the augmented and for the base auction, then $B_i^{\text{a}}(p') = B_i^{\text{b}}(p')$ for all $p' \geq p$. We say that at this point, the auctions get *fully coupled*. This completes the discussion. The heart of the proof is to show that the maximum budget of the base auction stays higher then the one in the augmented auction. If eventually they

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meet, then the two auctions become *fully coupled*, in the sense that they evolve in the same way from this price on. ∎

Now, we want to establish allocation monotonicity, i.e., that $x_i^{\mathsf{a}}(p) \geq x_i^{\mathsf{b}}(p)$ for all $p \geq 0$. We will prove a stronger claim, that the *wishful allocation* Ψ_i is monotone in the supply, i.e., $\Psi_i^{\mathsf{a}}(p) \geq \Psi_i^{\mathsf{b}}(p)$ for all $p \geq 0$.

Proposition 4.14 (Allocation Monotonicity) *For all $p \geq 0$ and all agents i, the following invariant holds:* $\Psi_i^{\mathsf{b}}(p) \leq \Psi_i^{\mathsf{a}}(p)$.

Proof : This proof follows from combining Proposition 4.13 and Lemma 4.6. For small values of p, $\Psi_i^{\mathsf{b}}(p) \leq \Psi_i^{\mathsf{a}}(p)$ is definitely true, since both are equal to $\frac{B_i}{p}$. Now, if it is true for some small p, then it is true for any $p' \geq p$, since:

$$\Psi_i^{\mathsf{a}}(p') = \Psi_i^{\mathsf{a}}(p) - \int_p^{p'} \frac{B_i^{\mathsf{a}}(\rho)}{\rho^2} d\rho \geq \Psi_i^{\mathsf{b}}(p) - \int_p^{p'} \frac{B_i^{\mathsf{b}}(\rho)}{\rho^2} d\rho = \Psi_i^{\mathsf{b}}(p').$$

∎

The proof of our main theorem follows immediately from Propositions 4.13 and 4.14.

Proof of Theorem 4.1 : For the allocation monotonicity, Proposition 4.14 implies that $x_i^{\mathsf{a}}(p) + \frac{B_i^{\mathsf{a}}(p)}{p} \geq x_i^{\mathsf{b}}(p) + \frac{B_i^{\mathsf{b}}(p)}{p}$. Since $B_i^{\mathsf{a}}(p) \leq B_i^{\mathsf{b}}(p)$, then clearly: $x_i^{\mathsf{a}}(p) \geq x_i^{\mathsf{b}}(p)$, taking $p \to \infty$ we get that for each player i, the final allocation in the augmented auction and in the base auction are such that $x_i^{\mathsf{a}} \geq x_i^{\mathsf{b}}$.

The monotonicity of the payment function follows directly from Proposition 4.13. The remaining budget in the end is larger in the base auction then in the augmented auction for each agent. So, the final payments are such that $\pi_i^{\mathsf{a}} \geq \pi_i^{\mathsf{b}}$.

∎

4.3   Qualitative Description of the Adaptive Clinching Auction

Once we established that the Adaptive Clinching Auction is an auction in the online supply model, we have a feasible online allocation and price rule in our hands, i.e., a rule that tells how to allocate and charge for an ϵ amount of the good when it arrives after s supply has already been allocated. At this point, it is worth studying qualitative behavior of such an allocation rule, and develop more insights that can guide us in the design of heuristic to apply in real-world ad auctions.

We do this analysis in details for 2 players in Appendix E. We observe that for the first units of supply, the clinching auction behaves like VCG: allocating to the highest bidder and charging the second highest bid. After that, depending on the relation between v_1 and v_2, two distinct behaviors can happen: either at a certain point the high value player gets his budget depleted, and the auction starts allocating new arriving units to the low valued player (still charging for those items) and then at a certain further point, it starts splitting the goods among them (charging only the player with non-depleted budget). Or alternatively, the auction continues to allocate to the high-valued player but start charging him a discounted version of VCG. Then when his budget gets depleted, the auction starts splitting goods among the players (charging only the player with non-depleted budget). We refer to Appendix E for a detailed discussion of the intuition behind this online rule.

References

[1] G. Aggarwal, G. Goel, C. Karande, and A. Mehta. Online vertex-weighted bipartite matching and single-bid budgeted allocations. In *SODA*, pages 1253–1264, 2011.

[2] L. M. Ausubel. An efficient ascending-bid auction for multiple objects. *American Economic Review*, 94, 1997.

[3] M. Babaioff, L. Blumrosen, and A. Roth. Auctions with online supply. In *ACM Conference on Electronic Commerce*, pages 13–22, 2010.

[4] S. Bhattacharya, V. Conitzer, K. Munagala, and L. Xia. Incentive compatible budget elicitation in multi-unit auctions. In *SODA*, pages 554–572, 2010.

[5] N. Buchbinder, K. Jain, and J. Naor. Online primal-dual algorithms for maximizing ad-auctions revenue. In *ESA*, pages 253–264, 2007.

[6] R. Colini-Baldeschi, M. Henzinger, S. Leonardi, and M. Starnberger. On multiple round sponsored search auctions with budgets. *AdAuctions Workshop*, 2012.

[7] N. R. Devanur and J. D. Hartline. Limited and online supply and the bayesian foundations of prior-free mechanism design. In *ACM EC*, pages 41–50, 2009.

[8] N. R. Devenur and T. P. Hayes. The adwords problem: online keyword matching with budgeted bidders under random permutations. In *ACM Conference on Electronic Commerce*, pages 71–78, 2009.

[9] S. Dobzinski, R. Lavi, and N. Nisan. Multi-unit auctions with budget limits. *Games and Economic Behavior*, 74(2):486–503, 2012.

[10] A. Fiat, S. Leonardi, J. Saia, and P. Sankowski. Single valued combinatorial auctions with budgets. In *ACM Conference on Electronic Commerce*, pages 223–232, 2011.

[11] G. Goel, V. S. Mirrokni, and R. Paes Leme. Polyhedral clinching auctions and the adwords polytope. In *STOC*, pages 107–122, 2012.

[12] M. Mahdian and A. Saberi. Multi-unit auctions with unknown supply. In *Proceedings of the 7th ACM conference on Electronic commerce*, EC '06, pages 243–249, New York, NY, USA, 2006. ACM.

[13] C. J. H. McDiarmid. Rado's theorem for polymatroids. *Mathematical Proceedings of the Cambridge Philosophical Society*, 78(02):263–281, 1975.

[14] A. Mehta, A. Saberi, U. V. Vazirani, and V. V. Vazirani. Adwords and generalized online matching. *J. ACM*, 54(5), 2007.

[15] R. Myerson. Optimal auction design. *Mathematics of Operations Research*, 6(1):58–73, 1981.

[16] D. C. Parkes. Online mechanisms. In N. Nisan, T. Roughgarden, E. Tardos, and V. Vazirani, editors, *Algorithmic Game Theory*, chapter 16, pages 411–439. Cambridge University Press, 2007.

A   Missing Proofs in Sections 3 and 4

Lemma A.1 *VCG is an auction in the strong online model for $(\mathcal{F}^{PM}, \preccurlyeq^{PM})$.*

Proof :   Assume that $P^f, P^{f'}, P'$ are defined respectively by the monotone submodular functions f, f', g. If $P^{f'} = P^f + P'$, then by McDiarmid's Theorem [13], $f' = f + g$.

Now, let's remind how VCG allocated for this setting. If the polymatroid is defined by f, VCG begins by sorting the players by their value (and breaking ties lexicographically). So, we can assume $v_1 \geq \ldots \geq v_n$. Then it chooses the outcome:

$$x_i = f([i]) - f([i-1])$$

$$\pi_i = v_{i+1} \cdot (f([i+1] \setminus i) - f([i-1]) - x_{i+1}) + \sum_{j > i+1} v_j \cdot (f([j] \setminus i) - f([j-1] \setminus i) - x_j)$$

where $[i]$ is an abbreviation for $\{1, \ldots, i\}$. Now, once we do this for f, f' we notice that the allocation and payments for f' are simply the sum of the allocation and payments for f and g, hence they are monotone along $\preccurlyeq^{PM}$. ∎

Lemma A.2 (Utility monotonicity) *Consider a setting where agents have single-parameter valuations $\Theta_i = \mathbb{R}_+$ and quasilinear utilities $u_i = v_i x_i - \pi_i$. Given a truthful auction in the weak online supply model and $f \preccurlyeq f'$, then the utility of agent i increased with the supply, i.e.: $u^{f'} = v_i x^{f'} - \pi_i^{f'} \geq v_i x^f - \pi_i^f = u^f$*

Proof :   The proof follows directly from Myerson's characterization [15] of payments in quasi-linear settings: $u^{f'} = v_i x_i^{f'} - \pi_i^{f'} = \int_0^{v_i} x_i^{f'}(u)du \geq \int_0^{v_i} x^f(u)du = v_i x^f - \pi_i^f = u^f$. ∎

Proof of Lemma 4.6 :   The function $\Psi_i(p)$ is clearly continuous for $p \notin \{v_1, \ldots, v_n\}$ and right-continuous everywhere. Now, we claim that it is also left-continuous at v_j, i.e., $\Psi(v_j-) = \Psi(v_j)$. This fact is almost immediate:

$$\Psi_i(v_j) = x_i(v_j) + \frac{B_i(v_j)}{v_j} = [x_i(v_j-) + \delta_i] + \frac{[B_i(v_j-) - \delta_i v_j]}{v_j} = x_i(v_j-) + \frac{B_i(v_j-)}{v_j} = \Psi_i(v_j-)$$

Calculating its derivative is also easy:

$$\partial_p \Psi_i(p) = \partial_p \left[x_i(p) + \frac{B_i(p)}{p} \right] = \partial_p x_i(p) + \frac{\partial_p B_i(p)}{p} - \frac{B_i(p)}{p^2} = -\frac{B_i(p)}{p^2}$$

since $\partial_p x_i(p) = \frac{S(p)}{p} = -\frac{\partial_p B_i(p)}{p}$. ∎

Proof of the Meta Lemma 4.7 :   Let $F = \{p \geq p_0; \Lambda \text{ doesn't hold for } p\}$. We want to show that if the properties (a),(b),(c) above hold, then $F = \emptyset$. Assume for contradiction that (a),(b),(c) hold but $F \neq \emptyset$. Let $\bar{p} = \inf F$, i.e., the smallest $\bar{p}$ such that for all $\epsilon > 0$, $[\bar{p}, \bar{p} + \delta) \cap F \neq \emptyset$ for all $\delta > 0$.

Now, there are two possibilities:

(1) either $\bar{p} \notin F$, in this case we can invoke (b) to see that there should be an $\epsilon > 0$ such that $[\bar{p}, \bar{p} + \epsilon) \cap F = \emptyset$ which contradicts the fact that $\bar{p} = \inf F$.

(2) or $\bar{p} \in F$. By (a), we know $\bar{p} > p_0$. Then we can use that by the definition of inf, Λ holds for all $p < \bar{p}$, so we can invoke (c) to show that Λ should hold for $\bar{p}$. And again we arrive in a contradiction. ∎

Proof of Lemma 4.8 : The proof is based on the Meta Lemma. Part (a) is trivial.

For part (b), there is $\epsilon > 0$ such that in $[p, p+\epsilon)$ the active set is the same as $A(p)$. We will show that if $i \in C(p)$, then $i \in C(p')$ for all $p' \in [p, p+\epsilon)$, or in other words: $S(p') = \sum_{j \in A(p)\setminus i} \frac{B_j}{p'}$. This equality holds for p. Now, we will simply show that the derivative of the is the same in $[p, p+\epsilon)$, i.e.: $\partial_p S(p) = \partial_p \sum_{j \in A(p)\setminus i} \frac{B_j(p)}{p}$.

$$\partial_p \sum_{j \in A(p)\setminus i} \frac{B_j(p)}{p} = \frac{p[\sum_{j \in A(p)\setminus i} \partial_p B_j(p)] - \sum_{j \in A(p)\setminus i} B_j(p)}{p^2} = \frac{1}{p} \sum_{j \in C(p)\setminus i} \partial_p B_j(p) - \frac{1}{p} S(p) =$$

$$= - \sum_{j \in C(p)\setminus i} \frac{1}{p} S(p) - \frac{1}{p} S(p) = - \sum_{j \in C(p)} \partial_p x_j(p) = - \sum_{j \in A(p)} \partial_p x_j(p) = \partial_p S(p)$$

For part (c), it is trivial for $p \notin \{v_1, \ldots, v_n\}$ by left-continuity: if $S(p') = \sum_{j \in A(p')\setminus i} \frac{B_j}{p'}$ for $p' < p$ and the functions involved are left-continuous, then it holds for p. Now, for $p = v_j$, if $S(v_j-) = \sum_{k \in A(v_j-)\setminus i} \frac{B_k(v_j-)}{v_j}$, then for δ_k as defined in (iii) of Definition 4.2 we have:

$$S(v_j) = S(v_j-) - \sum_{k \in A(v_j)} \delta_k = \left[\sum_{k \in A(v_j)\setminus i} \frac{B_k(v_j-) - \delta_k v_j}{v_j} \right] + \frac{B_j(v_j-)}{v_j} - \delta_i = \sum_{k \in A(v_j)\setminus i} \frac{B_k(v_j)}{v_j}$$

since:

$$\delta_i = \left[S(v_j-) - \sum_{k \in A(v_j)\setminus i} \frac{B_k(v_j-)}{v_j} \right]^+ = \left[\sum_{k \in A(v_j-)\setminus i} \frac{B_k(v_j-)}{v_j} - \sum_{k \in A(v_j)\setminus i} \frac{B_k(v_j-)}{v_j} \right]^+ = \frac{B_j(v_j-)}{v_j}$$

∎

Proof of Lemma 4.9 : Again we prove it using the Meta Lemma. (a) is trivial, for (b) if $S(p) < \sum_{j \in A(p)\setminus i} \frac{B_j(p)}{p}$, then by right-continuity the strict inequality continues to hold in some region $[p, p+\epsilon)$. If $S(p) = \sum_{j \in A(p)\setminus i} \frac{B_j(p)}{p}$ we can do the same analysis as in Lemma 4.8. For (c) it is again trivial for $p \notin \{v_1, \ldots, v_n\}$ by left-continuity and for $p = v_j$ we use the fact that comes directly from the proof of the previous lemma:

$$S(v_j) - \sum_{k \in A(v_j)\setminus i} \frac{B_k(v_j)}{v_j} = \left[S(v_j-) - \sum_{k \in A(v_j-)\setminus i} \frac{B_k(v_j-)}{v_j} \right] + \frac{B_j(v_j-)}{v_j} - \delta_i \leq 0 \qquad (1)$$

by the definition of δ_i, since:

$$\delta_i = \left[S(v_j-) - \sum_{k \in A(v_j)\setminus i} \frac{B_k(v_j-)}{v_j} \right]^+ \geq \left[S(v_j-) - \sum_{k \in A(v_j-)\setminus i} \frac{B_k(v_j-)}{v_j} \right] + \frac{B_j(v_j-)}{v_j}$$

∎

Proof of Corollary 4.10 : If $\delta_i > 0$, then, $\delta_i = S(v_j-) - \sum_{k \in A(v_j)\setminus i} \frac{B_k(v_j-)}{v_j}$. Substituting that in equation (1) we get that $S(v_j) = \sum_{k \in A(v_j)\setminus i} \frac{B_k(v_j)}{v_j}$ and therefore $i \in C(v_j)$. ∎

Proof of Lemma 4.11 : It is easy to see that all bidders in the clinching set have the same remaining budget, since if $i, i' \in C(p)$, then $\sum_{j \in A(p)\setminus i} \frac{B_j(p)}{p} = S(p) = \sum_{j \in A(p)\setminus i'} \frac{B_j(p)}{p}$ and therefore $B_i(p) = B_{i'}(p)$. Also, clearly, all players with the same budget will be in the clinching set. The fact that the players clinching have the largest budget follows directly from Lemma 4.9. ∎

B Proof of Proposition 4.13

Proof of Proposition 4.13 :

The first part of the proof consists of showing that clinching starts first in the augmented auction. Then we divide the prices in three intervals: in the first where no clinching happens in both auctions, in the second where clinching happens only in the augmented auction and the third in which clinching happens in both auctions. Then we prove the claim in each of the intervals.

First part of the proof: Clinching starts earlier in the augmented auction

Let $p_0^{\mathsf{b}} = \min\{p; C^{\mathsf{b}}(p) \neq \emptyset\}$ and $p_0^{\mathsf{a}} = \min\{p; C^{\mathsf{a}}(p) \neq \emptyset\}$. We claim that $p_0^{\mathsf{a}} \leq p_0^{\mathsf{b}}$. In order to see that, assume the contrary: $p_0^{\mathsf{b}} < p_0^{\mathsf{a}}$. At p_0^{b}, there is one agent i such that $S^{\mathsf{b}}(p_0^{\mathsf{b}}) = \sum_{k \in A(p_0^{\mathsf{b}}) \setminus i} \frac{B_k^{\mathsf{b}}(p_0^{\mathsf{b}})}{p_0^{\mathsf{b}}}$. If $p_0^{\mathsf{b}} \notin \{v_1, \ldots, v_n\}$, then by Corollary 4.10, no budget was spent in neither of the auctions at this price and no goods were acquired, so $S^{\mathsf{b}}(p_0^{\mathsf{b}}) = s^{\mathsf{b}}$, $S^{\mathsf{a}}(p_0^{\mathsf{b}}) = s^{\mathsf{a}}$, $B_k^{\mathsf{b}}(p_0^{\mathsf{b}}) = B_k^{\mathsf{b}}(0)$ and $B_k^{\mathsf{a}}(p_0^{\mathsf{b}}) = B_k^{\mathsf{a}}(0)$. This implies that at this point $S^{\mathsf{a}}(p_0^{\mathsf{b}}) > S^{\mathsf{b}}(p_0^{\mathsf{b}}) = \sum_{k \in A(p_0^{\mathsf{b}}) \setminus i} \frac{B_k^{\mathsf{a}}(p_0^{\mathsf{b}})}{p_0^{\mathsf{b}}}$, which contradicts Lemma 4.9 for the augmented auction. Now, the case left to analyze is the one where $p_0^{\mathsf{b}} = v_j$ for some $j \neq i$ and i entered the clinching set after acquiring a positive amount of good $\delta_i^{\mathsf{b}} > 0$ at price v_j. Then: $\delta_i^{\mathsf{b}} = s^{\mathsf{b}} - \sum_{k \in A(v_j) \setminus i} \frac{B_k(0)}{v_j} > 0$. But in this case $\delta_i^{\mathsf{a}} > 0$, contradicting that $p_0^{\mathsf{a}} > p_0^{\mathsf{b}}$.

Second part of the proof: Proof for the first interval $[0, p_0^{\mathsf{a}})$.

For the p in the interval $[0, p_0^{\mathsf{a}})$, no clinching occurs, so $B_i^{\mathsf{a}}(p) = B_i^{\mathsf{a}}(0) = B_i^{\mathsf{b}}(0) = B_i^{\mathsf{b}}(p)$.

Third part of the proof: Proof for the second interval $[p_0^{\mathsf{a}}, p_0^{\mathsf{b}})$.

In the interval $[p_0^{\mathsf{a}}, p_0^{\mathsf{b}})$, some players are acquiring goods in the augmented auction but no player is neither acquiring goods nor paying anything in the base auction, so: $B_i^{\mathsf{a}}(p) \leq B_i^{\mathsf{a}}(0) = B_i^{\mathsf{b}}(0) = B_i^{\mathsf{b}}(p)$.

Fourth part of the proof: Proof for the third interval $[p_0^{\mathsf{b}}, \infty)$.

In this interval, both players are clinching. Now, we use the Meta Lemma to show that for all $p \geq p_0^{\mathsf{b}}$, the property $B_i^{\mathsf{b}}(p) \geq B_i^{\mathsf{a}}(p)$ for all i holds.

For part (a) of the Meta Lemma, we need to show that $B_i^{\mathsf{b}}(p_0^{\mathsf{b}}) \geq B_i^{\mathsf{a}}(p_0^{\mathsf{b}})$. If $p_0^{\mathsf{b}} \notin \{v_1, \ldots, v_n\}$ this follows directly from continuity and the third part of the proof. If $p_0^{\mathsf{b}} = v_j$ for some j, then by the previous cases we know that $B_i^{\mathsf{a}}(v_j-) \leq B_i^{\mathsf{b}}(v_j-)$. We have that $B_i^{\mathsf{a}}(v_j) = B_i^{\mathsf{a}}(v_j-) - \delta_i^{\mathsf{a}} v_j$ and $B_i^{\mathsf{b}}(v_j) = B_i^{\mathsf{b}}(v_j-) - \delta_i^{\mathsf{b}} v_j$. Now we analyze the clinched amounts δ_i^{a} and δ_i^{b}. If $p_0^{\mathsf{a}} = p_0^{\mathsf{b}}$, it is straightforward to see that $\delta_i^{\mathsf{a}} \geq \delta_i^{\mathsf{b}}$ and therefore $B_i^{\mathsf{a}}(v_j) \leq B_i^{\mathsf{b}}(v_j)$. So, let's focus on the case where $p_0^{\mathsf{a}} < p_0^{\mathsf{b}}$. For this case:

$$\delta_i^{\mathsf{a}} = \left[S^{\mathsf{a}}(v_j-) - \sum_{k \in A(v_j) \setminus i} \frac{B_k^{\mathsf{a}}(v_j-)}{v_j} \right]^+ = \left[\sum_{k \in A(v_j-)} \frac{B_k^{\mathsf{a}}(v_j-)}{v_j} - \frac{B_*^{\mathsf{a}}(v_j-)}{v_j} - \sum_{k \in A(v_j) \setminus i} \frac{B_k^{\mathsf{a}}(v_j-)}{v_j} \right]^+ =$$

$$= \left[\frac{B_i^{\mathsf{a}}(v_j-)}{v_j} + \frac{B_j^{\mathsf{a}}(v_j-)}{v_j} - \frac{B_*^{\mathsf{a}}(v_j-)}{v_j} \right]^+ = \frac{1}{v_j} \left[\min\{B_i(0), B_*^{\mathsf{a}}(v_j-)\} + \min\{B_j(0), B_*^{\mathsf{a}}(v_j-)\} - B_*^{\mathsf{a}}(v_j-) \right]^+$$

where the last step is an invocation of Corollary 4.12. For the base auction we have essentially the same, except that $S^{\mathsf{b}}(v_j-) \leq \sum_{k \in A(v_j-)} \frac{B_k^{\mathsf{b}}(v_j-)}{v_j} - \frac{B_*^{\mathsf{b}}(v_j-)}{v_j}$ holds as an inequality rather than

equality, so we get:

$$\delta_i^{\mathsf{b}} \leq \left[\frac{B_i^{\mathsf{b}}(v_j-)}{v_j} + \frac{B_j^{\mathsf{b}}(v_j-)}{v_j} - \frac{B_*^{\mathsf{b}}(v_j-)}{v_j} \right]^+ = \frac{1}{v_j} \left[\min\{B_i(0), B_*^{\mathsf{b}}(v_j-)\} + \min\{B_j(0), B_*^{\mathsf{b}}(v_j-)\} - B_*^{\mathsf{b}}(v_j-) \right]^+$$

In order to prove that $B_i^{\mathsf{a}}(v_j) \leq B_i^{\mathsf{b}}(v_j)$, we study two cases:

- Case A: $B_*^{\mathsf{a}}(v_j-) \leq B_j^{\mathsf{a}}(0)$, i.e. $B_j^{\mathsf{a}}(v_j-) = B_*^{\mathsf{a}}(v_j-)$. In this case, $\delta_i^{\mathsf{a}} = \frac{B_i^{\mathsf{a}}(v_j-)}{v_j}$ and therefore $B_j^{\mathsf{a}}(v_j) = 0$, so, it is trivial that $B_i^{\mathsf{a}}(v_j) = 0 \leq B_i^{\mathsf{b}}(v_j)$.

- Case B: $B_*^{\mathsf{a}}(v_j-) > B_j^{\mathsf{a}}(0)$. Now, consider the function $\Phi(\beta) = [\min\{\beta, \mu\} + \min\{\beta, \gamma\} - \beta]^+$ for $\beta \geq \min\{\mu, \gamma\}$. This function is monotone non-increasing in such range. Now, take $\mu = B_i(0), \gamma = B_j(0)$ and use that $B_*^{\mathsf{a}}(v_j-) \leq B_*^{\mathsf{b}}(v_j-)$ to conclude that $\delta_i^{\mathsf{a}} = \Phi(B_*^{\mathsf{a}}(v_j-)) \geq \Phi(B_*^{\mathsf{b}}(v_j-)) \geq \delta_i^{\mathsf{b}}$. This implies $B_i^{\mathsf{a}}(v_j) \leq B_i^{\mathsf{b}}(v_j)$.

This finishes the proof of part (a) of the Meta Lemma.

Now, for part (b) of the Meta-Lemma, consider two cases:

- $B_*^{\mathsf{a}}(p) < B_*^{\mathsf{b}}(p)$, then by right-continuity of the budget function, there is some $\epsilon > 0$ such $B_*^{\mathsf{a}}(p') < B_*^{\mathsf{b}}(p')$ for any $p' \in [p, p + \epsilon)$.

- $B_*^{\mathsf{a}}(p) = B_*^{\mathsf{b}}(p)$, therefore, $B_i^{\mathsf{a}}(p) = B_i^{\mathsf{b}}(p)$ for all $i \in A(p)$, moreover, $S^{\mathsf{a}}(p) = S^{\mathsf{b}}(p)$, since

$$S^{\mathsf{a}}(p) = \sum_{i \in A(p)} \frac{B_i^{\mathsf{a}}(p)}{p} - \frac{B_*^{\mathsf{a}}(p)}{p} = \sum_{i \in A(p)} \frac{B_i^{\mathsf{b}}(p)}{p} - \frac{B_*^{\mathsf{b}}(p)}{p} = S^{\mathsf{b}}(p)$$

  Since the behavior of the function $B(\cdot)$ for $p' \geq p$ just depends on $S(p)$ and $B(p)$, for all $p' \geq p$, then for all $p' \geq p$, $B^{\mathsf{a}}(p) = B^{\mathsf{b}}(p)$. In other words, when $B_*^{\mathsf{b}}(p)$ and $B_*^{\mathsf{a}}(p)$ meet, then the auctions become *fully coupled*.

Part (c) of the Meta-Lemma is essentially the same argument made in item (a). This part is trivial for $p \notin \{v_1, \ldots, v_n\}$ by continuity of $B(p)$. For $p = v_j$ we use that $B_*^{\mathsf{a}}(v_j-) \leq B_*^{\mathsf{b}}(v_j-)$ and study δ_i^{a} and δ_i^{b}. As in (c) we get:

$$\delta_i^{\mathsf{a}} = \frac{1}{v_j} \left[\min\{B_i(0), B_*^{\mathsf{a}}(v_j-)\} + \min\{B_j(0), B_*^{\mathsf{a}}(v_j-)\} - B_*^{\mathsf{a}}(v_j-) \right]^+$$

$$\delta_i^{\mathsf{b}} = \frac{1}{v_j} \left[\min\{B_i(0), B_*^{\mathsf{b}}(v_j-)\} + \min\{B_j(0), B_*^{\mathsf{b}}(v_j-)\} - B_*^{\mathsf{b}}(v_j-) \right]^+$$

Now, by analyzing cases A and B as in part (a) of the Meta-Lemma, we conclude that $B_i^{\mathsf{a}}(v_j) = B_i^{\mathsf{a}}(v_j-) - v_j \delta_i^{\mathsf{a}} \leq B_i^{\mathsf{b}}(v_j-) - v_j \delta_i^{\mathsf{b}} = B_i^{\mathsf{b}}(v_j)$ as desired. ∎

C   Adaptive Clinching Auction with Repeated Values

In this section, we modify the auction given in Definition 4.2 to account for the possibility of valuation profiles having repeated values, i.e., two agents i, i' with $v_i = v_{i'}$. The modification is quite simple:

Definition C.1 (Adaptive Clinching Auction revisited) *Given any valuation vector v, budget vector B and initial supply s, consider functions $x_i(p), B_i(p)$ satisfying (i),(ii) and (iii) in Definition 4.2 and also:*

(iv') for $p = v_j-$, Let $\tilde{A} = A(v_j-)$, $\tilde{x}_i = x_i(v_j-)$, $\tilde{B}_i = B_i(v_j-)$ and $\tilde{S} = S(v_j-)$. Now, run the following procedure on $(\tilde{A}, \tilde{x}, \tilde{B}, \tilde{S})$:

  $\circ$ *while there is $\tilde{j} \in \tilde{A}$ with $v_{\tilde{j}} = v_j$*

   $\circ$ *let $\tilde{j}$ be the lexicographic first of such elements*
   $\circ$ *remove $\tilde{j}$ from $\tilde{A}$*
   $\circ$ *define $\delta_k = \tilde{S} - \sum_{k \in \tilde{A}} \frac{\tilde{B}_k}{v_j}$ for each $k \in \tilde{A}$*
   $\circ$ *update $\tilde{x}_k = \tilde{x}_k + \delta_k$, $\tilde{B}_k = \tilde{B}_k - \delta_k v_j$ for each $k \in \tilde{A}$*
   $\circ$ *update $\tilde{S} = \tilde{S} - \sum_{k \in \tilde{A}} \delta_k$.*

 and then set $x_i(v_j) = \tilde{x}_i$, $B_i(v_j) = \tilde{B}_i$.

Notice this is essentially repeating (iv) in Definition 4.2 for as many times as elements with the same value v_j. Notice that all the proofs in Section 4 carry out naturally for this setting, simply by repeating the same argument done for (iv) multiple times, showing that the invariants analyzed are true after each while iteration.

D   Algorithmic Form of the Adaptive Clinching Auction

We presented the Adaptive Clinching Auction in Definition 4.2 as the limit as $p \to \infty$ of a differential procedure following Bhattacharya el al [4]. Here we present the same auction in an algorithmic format, i.e., an $\tilde{O}(n)$ steps procedure to compute (x, π) from (v, B, s). The idea is quite simple: given a price p and the values of $B(p), x(p)$, we solve the differential equation in item (i) of Definition 4.2 and using it, we compute the next point $\bar{p}$ where either a player leaves the active set, or a player enters the clinching set. Given that, we compute $B(\bar{p}-), x(\bar{p}-)$. Then we obtain the values of $B(\bar{p}), x(\bar{p})$ either by the procedure in (iv) if a player leaves the active set on $\bar{p}$ or simply by taking $B(\bar{p}) = B(\bar{p}-)$ and $x(\bar{p}) = x(\bar{p}-)$ otherwise.

Lemma D.1 *Consider the functions $x(p)$ and $B(p)$ obtained in the Adaptive Clinching Auction. If for prices $p' \in [p, \bar{p})$, the clinching and active set are the same, i.e., $C(p') = C(p)$ and $A(p') = A(p)$, then given $k = |C(p)|$, the players i in the clinching set are such that:*

- *if $k = 1$, $S(p') = \frac{pS(p)}{p'}$, $x_i(p') = x_i(p) + [S(p') - S(p)]$ and $B_i(p') = B_i(p) + pS(p)[\log p - \log p']$.*

- *if $k > 1$, $S(p') = \frac{p^k S(p)}{(p')^k}$, $x_i(p') = x_i(p) + \frac{1}{k}[S(p') - S(p)]$ and $B_i(p') = B_i(p) + \frac{p^k S(p)}{k-1}\left[\frac{1}{p'^{k-1}} - \frac{1}{p^{k-1}}\right]$*

Proof :   The proof is straightforward. For the case of $k = 1$, we follow the discussion in Example 4.2: let i be the only player in $C(p)$, then $S(p') + x_i(p')$ is constant in this range, since all that is subtracted from the supply is added to the allocation of player 1, therefore:

$$\partial S(p') = -\partial_p x_i(p') = -\frac{S(p')}{p'} \Rightarrow S(p') = \frac{\alpha}{p'}.$$

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using the boundary condition $S(p) = \frac{\alpha}{p}$, we get the value of $\alpha = pS(p)$. Now, clearly $x(p') = x(p) + [S(p') - S(p)]$, since player i is the only one clinching. For his budget:

$$B_i(p') - B_i(p) = \int_p^{p'} \partial_p B_i(\rho)d\rho = \int_p^{p'} -S(\rho)d\rho = \int_p^{p'} -\frac{pS(p)}{\rho}d\rho = pS(p)[\log p - \log p']$$

For $k > 1$, $S(p') + \sum_{i \in C(p')} x_i(p')$ is constant and therefore:

$$\partial S(p') = - \sum_{i \in C(p')} \partial_p x_i(p') = -k\frac{S(p')}{p'} \Rightarrow S(p') = \frac{\alpha}{(p')^k}.$$

Using the boundary condition $S(p) = \frac{\alpha}{p^k}$, we get the value of $\alpha = p^k S(p)$. We use the observation in Lemma 4.11 that players in the clinching set have the same budget, and therefore the auction treats them equally from this point on as long as they remain in the active set, i.e., they will get allocated and charged at the same rate. Therefore: $x(p') = x(p) + \frac{1}{k}[S(p') - S(p)]$. For the budgets:

$$B_i(p') - B_i(p) = \int_p^{p'} -S(\rho)d\rho = \int_p^{p'} -\frac{pS(p)}{\rho^k}d\rho = \frac{p^k S(p)}{k-1}\left[\frac{1}{(p')^{k-1}} - \frac{1}{p^{k-1}}\right]$$

∎

Theorem D.2 (Algorithmic Form) *It is possible to compute the allocation and payments of the Adaptive Clinching Auction in $\tilde{O}(n)$ time.*

Proof :   Using the lemma above, we just need to compute x and B for the points where one of the following events happen: (a) one leaves the active set and (b) one player enters the clinching set. Clearly there are at most n events of type (a) and by Lemma 4.8 also at most n events of type (b).

The algorithm starts at price $p = 0$ and at each time computes the next event. For example, at price $p = 0$, the next event of type (a) occurs in $p = \min_i v_i$. The next event of type (b) occurs at price $p = \frac{1}{s}[\sum_A B_i - \max_A B_i]$ if no event of type (a) happens before. First we compute which one occurs first. Let $\bar{p}$ be such a price. Then, computing $B(\bar{p}-), x(\bar{p}-)$ is trivial, since no clinching happened so far, so at that price: $B(\bar{p}-) = B$ (initial budgets) and $x(\bar{p}-) = 0$. Now, if $\bar{p}$ is an event of type (a), then use step (iv) in Definition 4.2 to compute $x(\bar{p}), B(\bar{p})$. If not, simply take $B(\bar{p}) = B(\bar{p}-)$ and $x(\bar{p}) = x(\bar{p}-)$.

From this point on, at each considered price p, the clinching set will be non-empty, so we know the format of $x(p')$ and $B(p')$ for $p' \in [p, p + \epsilon)$. If the next event that happens is of type (a), it happens at $\min\{v_j; v_j > p\}$, if it is of type (b), it happens at $\min\{p'; B_*(p') = \max_{i' \in A(p')\backslash C(p')} B_{i'}\}$, where the expression for $B(p')$ is given in the previous lemma. For example, if $|C(p)| = 1$, then this happens at:

$$p' = \exp\left[\frac{1}{pS(p)}(B_*(p) - \max_{i' \in A(p)\backslash C(p)} B_{i'}(p)) + \log p\right]$$

and if $|C(p)| = k > 1$, it happens at:

$$p' = \frac{p^k S(p)}{(k-1)(B_*(p) - \max_{i' \in A(p)\backslash C(p)} B_{i'}(p)) + pS(p)}$$

Those expressions are easily obtained by taking $B_i(p')$ as calculated in the previous lemma and calculating for which p' it becomes equal to $\max_{i' \in A(p)\backslash C(p)} B_i'(p)$. Now, we simply need to find out which of those events happen first. Let it be $\bar{p}$, then we compute $B(\bar{p}-), x(\bar{p}-)$ using the previous lemma and then update to $B(\bar{p}), x(\bar{p})$ as described above. ∎

E Details of the Qualitative Description

In Theorem 4.1, we showed that the Adaptive Clinching Auction is an auction in the online supply model. It is natural to ask how allocation and payment qualitatively evolve as supply arrives. In order words, how does it allocate and charge for an ϵ amount of the good when it arrives after s supply has already been allocated. We perform in this section a qualitative analysis for two players based on the explicit formula of the Adaptive Clinching Auction for $n = 2$ players derived in Dobzinski et al [9] (which can alternatively be obtained from Theorem D.2). This is reproduced in Figure 3.

As we will see, the clinching auction has a very natural behavior for the first few units of supply that arrive - it simply allocates them using VCG. At a certain point, when budget constraints start to kick-in, the allocation and payments evolve in a quite non-expected way. The main goal of this section is to highlight this point.

Depending on the relation between v_1 and v_2, two distinct behaviors can happen: either at a certain point, the high-value player gets his budget depleted, and the auction starts allocating new arriving units to the low-value player (still charging for those items) and then at a certain further point, it starts splitting the goods among them (charging only the player with non-depleted budget). Or alternatively, the auction continues to allocate to the high-value player, but start charging him a discounted version of VCG. Then when his budget gets depleted, the auction starts splitting goods among the players (charging only the player with non-depleted budget).

$v_2 \geq v_1$ and $s \cdot v_1 \leq B_2$	$x = (0, s)$ $p = (0, sv_1)$
$v_2 \geq v_1$ and $B_2 \leq s \cdot v_1 \leq B_1'$	$x = \left(s - \frac{B_2}{v_1}, \frac{B_2}{v_1} \right)$ $p = (B_2(\log(sv_1) - \log B_2), B_2)$
$v_2 \geq v_1$ and $B_1' \leq s \cdot v_1$	$x = \left(s - \frac{sB_2}{2B_1'}[1 + (\frac{B_1'}{sv_1})^2], \frac{sB_2}{2B_1'}[1 + (\frac{B_1'}{sv_1})^2] \right)$ $p = \left(B_2(1 - \frac{B_1'}{sv_1}) + B_2(\log B_1' - \log B_2), B_2 \right)$
$v_2 < v_1$ and $s \cdot v_2 \leq B_2$	$x = (s, 0)$ $p = (sv_2, 0)$
$v_2 < v_1$ and $B_2 \leq s \cdot v_2 \leq B_1'$	$x = (s, 0)$ $p = (B_2 + B_2(\log(sv_2) - \log B_2), 0)$
$v_2 < v_1$ and $B_1' \leq s \cdot v_2$	$x = \left(s + \frac{sB_2}{2B_1'}[-1 + (\frac{B_1'}{sv_2})^2], \frac{sB_2}{2B_1'}[1 - (\frac{B_1'}{sv_2})^2] \right)$ $p = \left(B_1, B_2 - \frac{B_1'B_2}{sv_2} \right)$

Figure 3: Explicit formula of the Adaptive Clinching Auction for $n = 2$.
Valuations v_1, v_2, budgets $B_1 \geq B_2$, initial supply s and $B_1' = \exp(\frac{B_1}{B_2} - 1 + \log B_2)$

Now, for some fixed pair of valuations v_1, v_2 and budgets $B_1 \geq B_2$, we study how the allocation and payments evolve with the available supply s. As one can easily see, if supply is sufficiently small $s \leq \frac{\min_i B_i}{\min_i v_1}$, the auction is essentially VCG, as the good arrives, it allocates it to the highest-value player and she pays per unit the value of the other player. The behavior afterwards depends on the relation between v_1 and v_2.

Case $\mathbf{v_2 \geq v_1}$: As goods arrive, we allocate to the higher-value player charging him v_1 per unit. This is essentially VCG. This is possible until $s = \frac{B_2}{v_1}$, when the budget of 2 is depleted. As more goods arrive, we allocate them entirely to player 1 charging them at a rate proportionally to $\frac{B_2}{s}$, i.e., the fraction between player 2's original budget and supply that has arrived so far. We
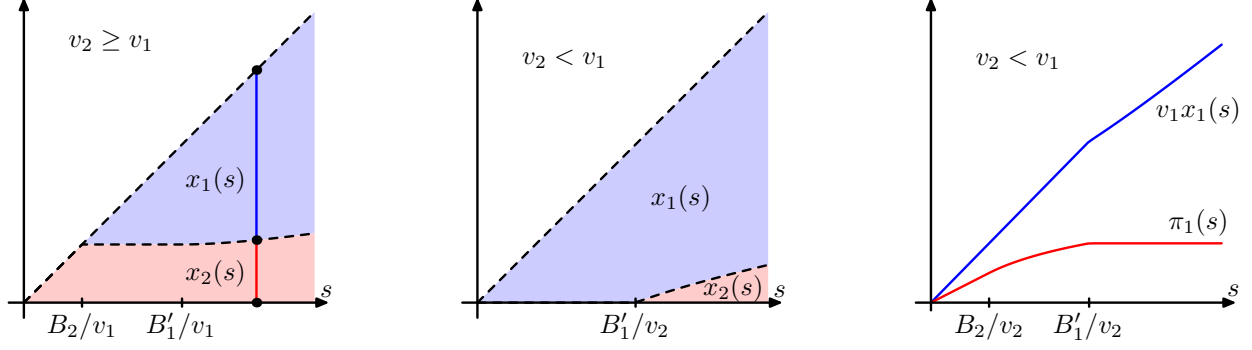
Figure 4: Evolution of allocations x_1 and x_2 as supply arrives in the cases $v_2 \geq v_1$ (first plot) and $v_2 < v_1$ (second plot). The third plot represents the evolution of value $v_1 x_1(s)$ and payment $\pi_1(s)$ of player 1 as more supply arrives in the case $v_2 < v_1$.

continue doing that until $s = \frac{1}{v_1} B_1' = \frac{1}{v_1} \exp(\frac{B_1}{B_2} - 1 + \log B_2)$. At this point, the remaining budget of player 1 is the same as the original budget of player 1. From this point on, each amount of the good that arrives is split among 1 and 2 at a rate $\partial_s x = (1 - \frac{B_2}{2B_1'} + \frac{B_2 B_1'}{2v_1^2 s^2}, \frac{B_2}{2B_1'} - \frac{B_2 B_1'}{2v_1^2 s^2})$. Player 2 is clearly not charged (since his budget is already depleted) and player 1 is charged at a rate $\frac{B_1' B_2}{s^2 v_1}$ per arriving unit of s. (The allocation is depicted in the first part of Figure 4.)

Case $\mathbf{v_2 < v_1}$: Again we start allocating like VCG, i.e., allocating the goods to player 1 and charging him v_2 for each unit of the good. We do that up to supply $s = \frac{B_2}{v_2}$. From this point on, we continue allocating incoming goods to player 1, but we charge him at a cheaper rate than v_2, precisely, we charge him at a rate $\frac{B_2}{s}$. We do so until the budget of 1 is depleted, which happens at $s = \frac{B_1'}{v_2}$. From this point on, we split the arriving goods between players 1 and 2 at a rate $\partial_s x = (1 - \frac{B_2}{2B_1'} - \frac{B_2 B_1'}{2v_1^2 s^2}, \frac{B_2}{2B_1'} + \frac{B_2 B_1'}{2v_1^2 s^2})$. Naturally, we cannot charge player 1, because his budget is already depleted, but we charge player 2 at a rate $\frac{B_1' B_2}{s^2 v_2}$ per arriving unit of s. (The allocation is depicted in the second and third part of Figure 4.)

Relation to bid throttling. One important remark is that the auction is *not* a special case of a bid-throttling scheme, since an agent is allocated items even after his budget is completely depleted and even if the other agent still has budget left. Intuitively, the fact that an agent got many items for an expensive price in the beginning (once the items were scarce) gives him an advantage over items in the future if/once they become abundant, i.e., this agent will have the possibility of acquiring these items for a lower price compared to other agents.

The final goal of this research direction is to provide better simple heuristics to deal with budgets for real-world ad auctions. We believe that the qualitative analysis above hints to new heuristics to manage budget constrained agents. We illustrate the effectiveness of such heuristic in the following scenario: consider a set of advertisers competing for ad slots on queries for a highly volatile query, say 'sunglasses'. If weather is rainy, there are very few queries and budget constraints do not kick in, but if weather is sunny, there is a high volume of queries and we would like to split the queries to advertisers according to their budgets. Imagine now that the weather is completely unpredictable. If the day starts rainy, very few queries arrive in the morning. It is unclear is the weather is changing in the afternoon. If the search engine knew the weather in the afternoon, it could run second-price auctions on modified bids to ensure that the high value players still have sufficient budget left in the afternoon if it is sunny and would run second price on real bids if they knew the weather would

still be rainy. The heuristic proposed by clinching on the other hand, is completely agnostic to that matter. It will allocate using VCG in the beginning. If the weather becomes sunny, it will give items for cheaper for the high-valued players to compensate for the more expensive items acquired in the beginning.