Self Balancing Robot Derivations

Michael Rencheck

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1 Problem Set Up

To solve the Euler-Lagrange equations, it is best to first define the frames and state vector that describe the problem and identify the transformations needed:

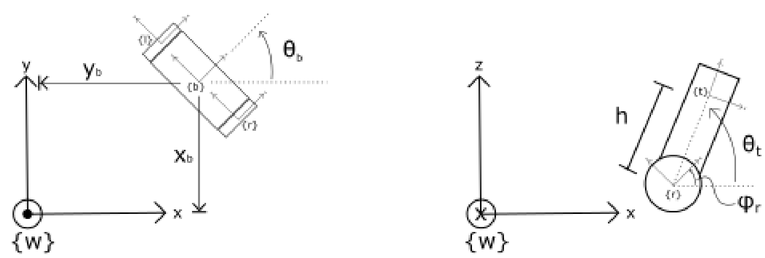

Figure 1: The frames used for the derivation

Frames:

w: world frame
b: robot base frame - centered between the wheels on the axis of rotation
r: right wheel frame - centered on the wheel
l: left wheel frame - centered on the wheel
t: center of mass frame - located line with the base frame

Parameters:

x_b: robot x-position
y_b: robot y-position
θ_b: robot heading angle
θ_t: robot body angle
ϕ_r: right wheel angle
ϕ_l: left wheel angle
r: wheel radius
h_t: distance from rotation point to the center of mass
D: distance between the two wheels

State Vector:

$$q = \begin{bmatrix} x_b \\ y_b \\ \theta_b \\ \phi_r \\ \phi_l \\ \theta_t \end{bmatrix}$$

$$\dot{q} = \begin{bmatrix} \dot{x_b} \\ \dot{y_b} \\ \dot{\theta_b} \\ \dot{\phi_r} \\ \dot{\phi_l} \\ \dot{\theta_t} \end{bmatrix}$$

Transforms:

$$T_{wb} = \begin{bmatrix} cos(\theta_b) & -sin(\theta_b) & 0 & x_b \\ sin(\theta_b) & cos(\theta_b) & 0 & y_b \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{br} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -D/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} cos(\phi_r) & 0 & sin(\phi_r) & 0 \\ 0 & 1 & 0 & 0 \\ -sin(\phi_r) & 0 & cos(\phi_r) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{bl} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & D/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} cos(\phi_l) & 0 & sin(\phi_l) & 0 \\ 0 & 1 & 0 & 0 \\ -sin(\phi_l) & 0 & cos(\phi_l) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{bt} = \begin{bmatrix} 1 & 0 & 0 & hcos(\theta_t) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & hsin(\theta_t) \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} cos(\pi/2 - \theta_t) & 0 & sin(\pi/2 - \theta_t) & 0 \\ 0 & 1 & 0 & 0 \\ -sin(\pi/2 - \theta_t) & 0 & cos(\pi/2 - \theta_t) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

After defining the transforms, we need to get the transform for every link wrt the world frame.

$$T_{wr} = T_{wb} * T_{br}$$

$$T_{wl} = T_{wb} * T_{bl}$$

$$T_{wt} = T_{wb} * T_{bt}$$

Now assemble the Lagrangian:

$$L = KE_{tot} - PE_{tot}$$

Find the kinetic energy of each link:

$$KE_{tot} = KE_r + KE_l + KE_t$$

$$KE_r = \frac{1}{2} * \mathcal{V}_r^T * G_{wheel} * \mathcal{V}_r$$

$$KE_l = \frac{1}{2} * \mathcal{V}_l^T * G_{wheel} * \mathcal{V}_l$$

$$KE_t = \frac{1}{2} * \mathcal{V}_t^T * G_{body} * \mathcal{V}_t$$

Where:
$\mathcal{V}$: is a 6Vector twist
G: is a spatial inertia matrix
m: is mass of the respective body
I: is the moment of inertia about a principle axis

$$\mathcal{V}_x = T_{wx}^{-1} * \dot{T}_{wx}$$

$$G_x = \begin{bmatrix} m_x & 0 & 0 & 0 & 0 & 0 \\ 0 & m_x & 0 & 0 & 0 & 0 \\ 0 & 0 & m_x & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{xx} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{yy} & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{zz} \end{bmatrix}$$

Find the Potential Energy of each link:

$$PE_{tot} = PE_r + PE_l + PE_t$$

$$PE_r = 0$$

$$PE_l = 0$$

$$PE_t = (m_t g h) sin(\theta_t)$$

Define the constraint equations[1]:

$$\lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

$$\nabla\phi = \begin{bmatrix} 1 & 0 & 0 & r * cos(\theta_b) & r * cos(\theta_b) & 0 \\ 0 & 1 & 0 & r * sin(\theta_b) & r * sin(\theta_b) & 0 \end{bmatrix}$$

Last define the external force vector:

[1]For more details, see Robotic Manipulation, Murry, Li, Sastry pg. 272

$$F_{ext} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \tau_{motor} \\ \tau_{motor} \\ 0 \end{bmatrix}$$

Finally solve the systems of equations for $\ddot{q}$ and λ:

$$\frac{dL}{d\dot{q}} - \frac{dL}{dq} = F_{ext} + \lambda \nabla \phi$$

$$\ddot{\phi} = \frac{d}{dt}(\nabla \phi * \dot{q})$$

See the included txt for the raw latex code of the solutions. This output was generated using the included python code.

A couple notation changes in the txt file: subscript w: wheel subscript b: body t variables: θ p variables: ϕ