THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Semester 2 2011

MATH1131
MATHEMATICS 1A

(1) TIME ALLOWED – 2 hours

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

(8) A SHORT TABLE OF INTEGRALS WILL BE SUPPLIED

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

Use a separate book clearly marked Question 1

1. i) Let $z = 3 + 2i$ and $w = -4 + 4i$.

a) Find zw in **Cartesian form**.

b) Find z/w in **Cartesian form**.

c) Write down $\operatorname{Arg}(w)$.

ii) Let the set S in the complex plane be defined by

$$S = \{z \in \mathbb{C} : |z - (1 + i)| = 1\}.$$

a) Sketch the set S on a labelled Argand diagram.

b) By considering your sketch, or otherwise, find the maximum value of $|z|$ for $z \in S$.

iii) Use De Moivre's Theorem to express $\sin(3\theta)$ as a sum of powers of $\sin(\theta)$.

iv) Define the function $f(\theta)$ by $f(\theta) = \sin\theta + i\cos\theta$.

a) Show that $|f(\theta)| = 1$.

b) Prove that $f(\theta) = e^{i(\pi/2-\theta)}$.

c) Hence, or otherwise, evaluate $[f(\frac{\pi}{6})]^{99}$ in **Cartesian form**.

v) A plane Π has parametric vector equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}; \quad \lambda, \mu \in \mathbb{R}.$$

Find three (non-parallel) vectors that are parallel to the plane Π.

vi) Consider the following MAPLE session.

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> with(LinearAlgebra):
> A:=<<1,2,-1>|<0,0,1>|<1,1,0>>;
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$$A := \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

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> B:=A^2;
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$$B := \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$$

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> C:=A^3;
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$$C := \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

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> F:=C-B;
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$$F := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Without carrying out any row reduction, use the above Maple session to find the inverse of the matrix A^2.

Use a separate book clearly marked Question 2

2. i) Let $A = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 6 & 4 \end{pmatrix}$.

a) State the size of B^T.

b) Evaluate each of the following matrix products, or explain why the product does not exist.

α) AB;

β) BA.

ii) For what value(s) of α does the matrix $M = \begin{pmatrix} \alpha & 8 \\ 2 & \alpha \end{pmatrix}$ **not** have an inverse.

iii) Let $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

a) Find the cross product $\mathbf{u} \times \mathbf{v}$.

b) Hence find the **Cartesian** equation of the plane parallel to $\mathbf{u}$ and $\mathbf{v}$ and passing through the point $\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$.

iv) a) Solve the following system of linear equations expressing your answer in parametric vector form.

$$\begin{aligned} x + y + 2z &= 5 \\ x + 4y - z &= 8 \end{aligned}$$

b) Hence, or otherwise, solve the non-linear system of equations

$$\begin{aligned} x + y + 2z &= 5 \\ x + 4y - z &= 8 \\ x^2 + y^2 + z^2 &= 6. \end{aligned}$$

v) Let l_1 and l_2 be the lines

$$\ell_1 : \ \mathbf{x} = \begin{pmatrix} 6 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}; \quad t \in \mathbb{R} \quad \text{and}$$

$$\ell_2 : \ \mathbf{x} = \begin{pmatrix} 0 \\ 1 \\ 12 \end{pmatrix} + s \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}; \quad s \in \mathbb{R}.$$

a) Show that l_1 and l_2 intersect at the point $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

b) Prove that l_1 is perpendicular to l_2.

c) Find points A, B, C and D on the lines l_1 and l_2 such that $ABCD$ is a square of side length equal to $\sqrt{2}$.

Use a separate book clearly marked Question 3

3. i) Evaluate each limit showing all necessary working:

a)
$$\lim_{x\to\infty} \sqrt{x^2+3x} - (x+1);$$

b)
$$\lim_{x\to 0} \frac{e^{2x}-1}{\sin 3x}.$$

ii) Let $f(x) = x^{11} + x^6 + x - 1$.

a) Use the Intermediate Value Theorem to show that $f(x)$ has at least one root in $[-1, 1]$.

b) Determine, with reasons, the number of real roots of $f(x)$.

iii) Find $\dfrac{dy}{dx}$ for each of:

a) $y = 3^{\cos 2x}$;

b) $3y^3 + 6x^2y = x^4 + y^4$.

iv) Let $f : (0,\infty) \to \mathbb{R}$ be defined by $f(x) = \sqrt{x^3+x+1}$.

a) Find the range of f.

b) Show that f has an inverse $g = f^{-1}$ defined on the range of f. (Do NOT try to find g.)

c) Find $g'(\sqrt{3})$.

v) Consider the polar curve

$$r = 1 + \cos(\frac{\theta}{2}) \quad \text{for} \quad 0 \le \theta \le 2\pi.$$

By evaluating at suitable values of θ, sketch this curve.
(Do NOT attempt to find $\frac{dy}{dx}$.)

Use a separate book clearly marked Question 4

4. i) Evaluate the following integrals:

a) $\displaystyle\int x\cos(x^2)\,dx$;

b) $\displaystyle\int x^2\cos(x)\,dx$.

ii) Find

$$\frac{d}{dx}\int_1^{\sqrt{x}}\frac{\sin(t^2)}{t^2}\,dt,\qquad \text{for } x>0.$$

(Do NOT try to evaluate the integral.)

iii) Does the improper integral

$$\int_1^{\infty}\frac{\sin(t^2)}{t^2}\,dt$$

converge or diverge? Give reasons for your answer.

iv) a) Use the definitions of $\sinh x$ and $\cosh x$ to prove that

$$\cosh^2 x-\sinh^2 x=1.$$

b) Find the **second** derivative of $y=\sinh^2 x$. Express your answer in terms of cosh only.

c) Find $\dfrac{d}{dx}\sinh^{-1}(\sqrt{x})$ for $x>0$, and hence evaluate

$$\int\frac{dx}{\sqrt{x^2+x}}.$$

v) Suppose the function f is continuous on $[a,b]$ where $a<b$.

$$\text{Let}\quad g(x)=\int_a^x f(t)\,dt-\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)\cdot(x-a).$$

a) Evaluate $g(a)$ and $g(b)$.

b) Find $g'(x)$.

c) Hence apply the Mean Value Theorem to show that there exists a real number c, between a and b, such that

$$f(c)=\frac{1}{b-a}\int_a^b f(t)\,dt.$$