Chapter 11

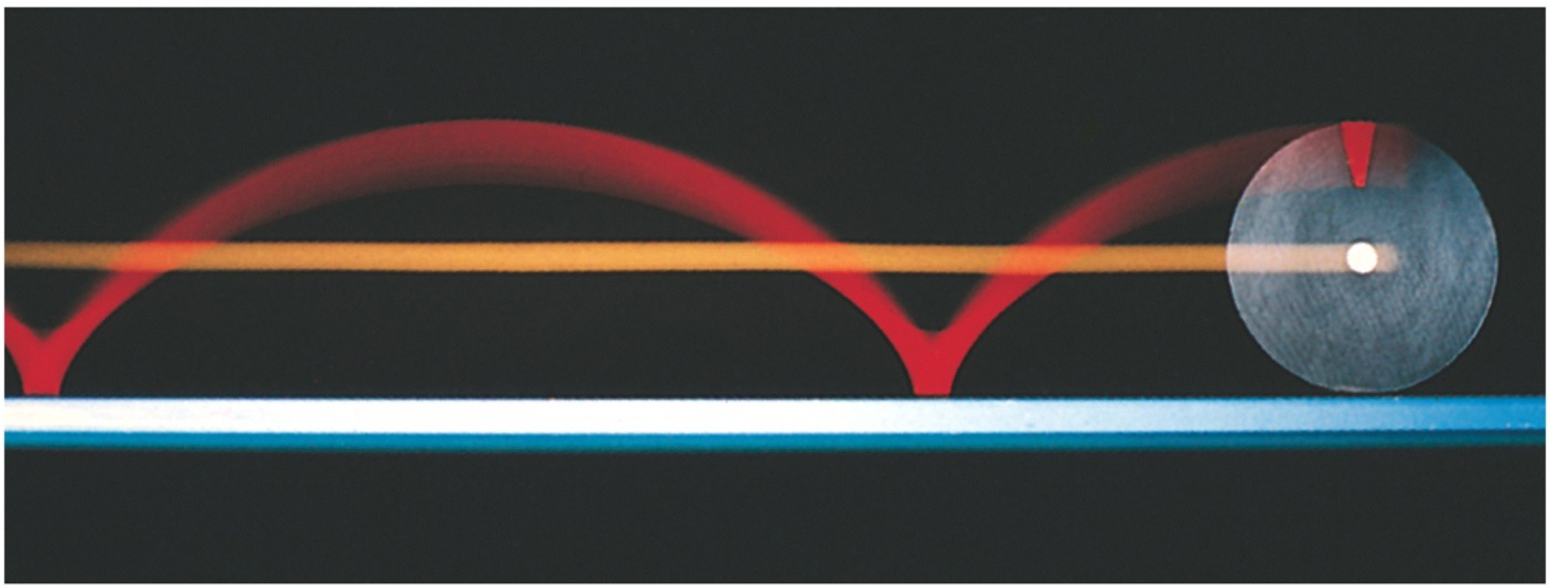
Rolling, Torque, and Angular Momentum

11-1 Rolling as Translation and Rotation Combined

Learning Objectives

11.01 Identify that smooth rolling can be considered as a combination of pure translation and pure rotation.

11.02 Apply the relationship between the center-of-mass speed and the angular speed of a body in smooth rolling.



Richard Megna/Fundamental Photographs

Figure 11-2

11-1 Rolling as Translation and Rotation Combined

- We consider only objects that roll smoothly (no slip)
- The center of mass (com) of the object moves in a straight line parallel to the surface
- The object rotates around the com as it moves
- The rotational motion is defined by:

$$s = \theta R,$$ Eq. (11-1)

$$v_{com} = \omega R$$ Eq. (11-2)

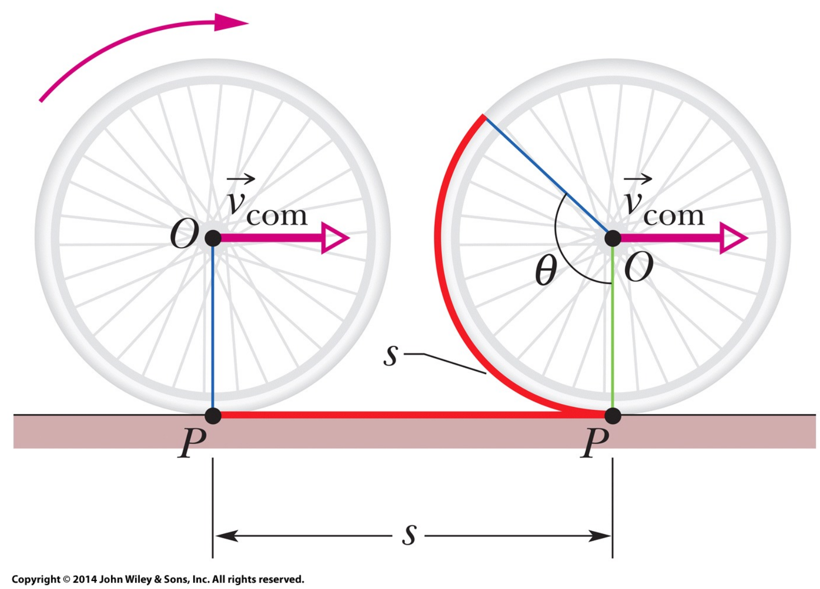


Figure 11-3

11-1 Rolling as Translation and Rotation Combined

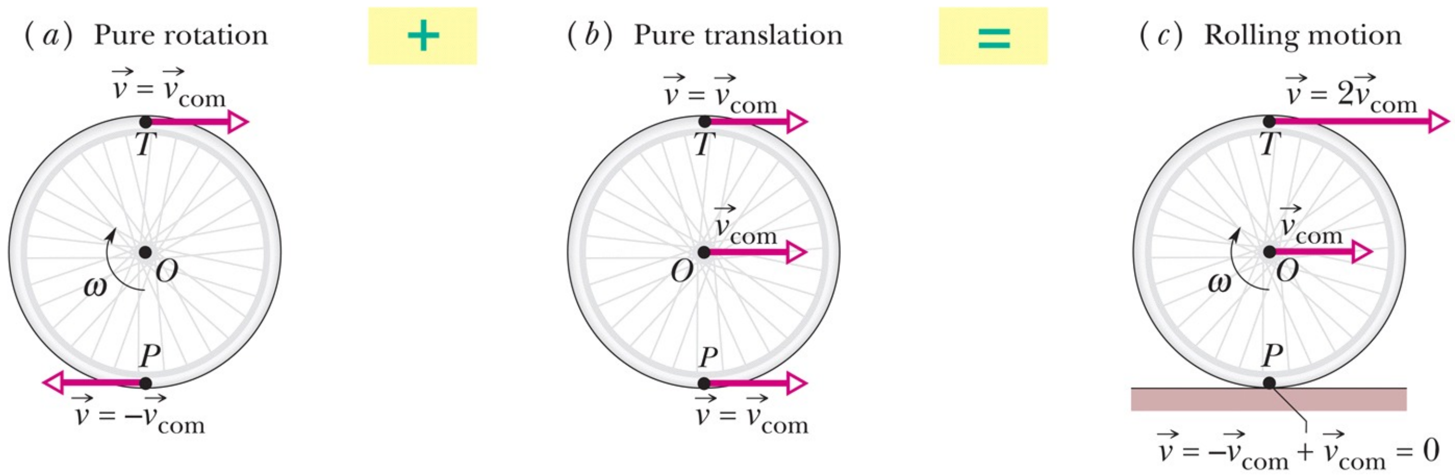


Figure 11-4

- The figure shows how the velocities of translation and rotation combine at different points on the wheel

Checkpoint 1

The rear wheel on a clown's bicycle has twice the radius of the front wheel. (a) When the bicycle is moving, is the linear speed at the very top of the rear wheel greater than, less than, or the same as that of the very top of the front wheel? (b) Is the angular speed of the rear wheel greater than, less than, or the same as that of the front wheel?

Answer: (a) the same (b) less than

11-2 Forces and Kinetic Energy of Rolling

Learning Objectives

11.03 Calculate the kinetic energy of a body in smooth rolling as the sum of the translational kinetic energy of the center of mass and the rotational kinetic energy around the center of mass.

11.04 Apply the relationship between the work done on a smoothly rolling object and its kinetic energy change.

11.05 For smooth rolling (and thus no sliding), conserve mechanical energy to relate initial energy values to the values at a later point.

11.06 Draw a free-body diagram of an accelerating body that is smoothly rolling on a horizontal surface or up or down on a ramp.

11-2 Forces and Kinetic Energy of Rolling

11.07 Apply the relationship between the center-of-mass acceleration and the angular acceleration.

11.08 For smooth rolling up or down a ramp, apply the relationship between the object's acceleration, its rotational inertia, and the angle of the ramp.

11-2 Forces and Kinetic Energy of Rolling

- Combine translational and rotational kinetic energy:

$$K = \tfrac{1}{2} I_{\text{com}} \omega^2 + \tfrac{1}{2} M v_{\text{com}}^2.$$

Eq. (11-5)

A rolling object has two types of kinetic energy: a rotational kinetic energy ($\tfrac{1}{2} I_{\text{com}} \omega^2$) due to its rotation about its center of mass and a translational kinetic energy ($\tfrac{1}{2} M v_{\text{com}}^2$) due to translation of its center of mass.

- If a wheel accelerates, its angular speed changes
- A force must act to prevent slip

$$a_{\text{com}} = \alpha R$$

Eq. (11-6)

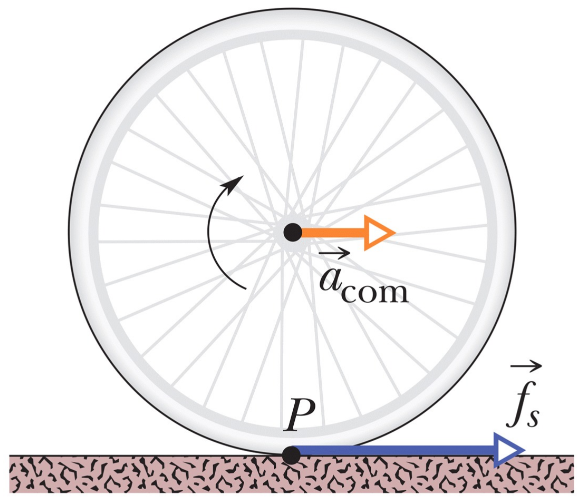


Figure 11-7

11-2 Forces and Kinetic Energy of Rolling

- If slip occurs, then the motion is *not* smooth rolling!
- For smooth rolling down a ramp:
  1. The gravitational force is vertically down
  2. The normal force is perpendicular to the ramp
  3. The force of friction points up the slope

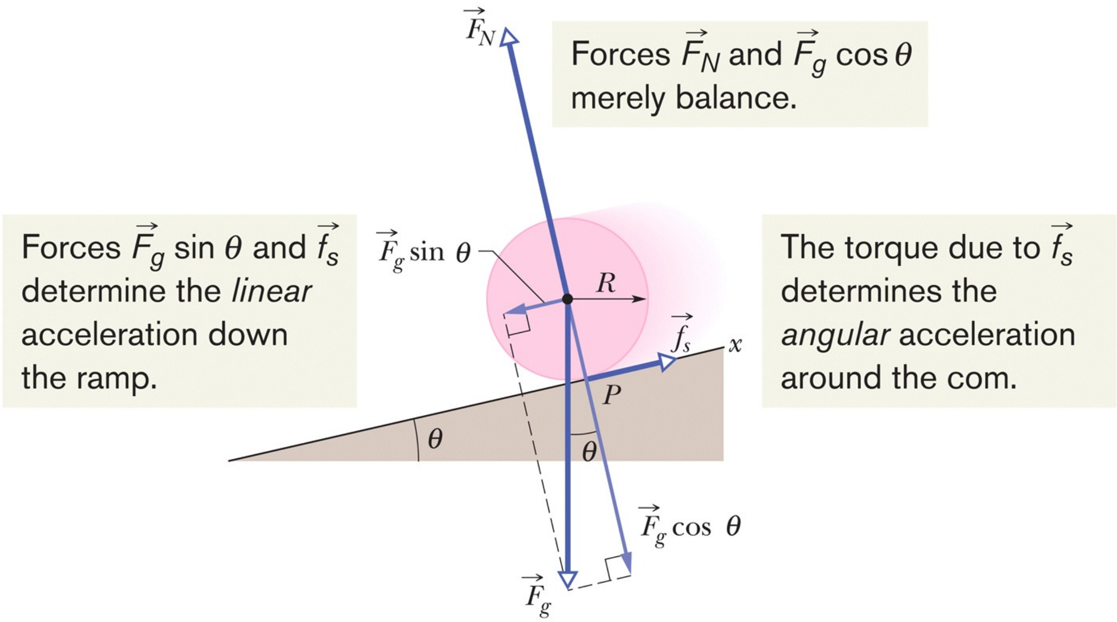


Figure 11-8

11-2 Forces and Kinetic Energy of Rolling

- We can use this equation to find the acceleration of such a body

$$a_{\text{com},x} = -\frac{g \sin \theta}{1 + I_{\text{com}}/MR^2}.$$ **Eq. (11-10)**

- Note that the frictional force produces the rotation
- Without friction, the object will simply slide

Checkpoint 2

Disks *A* and *B* are identical and roll across a floor with equal speeds. Then disk *A* rolls up an incline, reaching a maximum height *h*, and disk *B* moves up an incline that is identical except that it is frictionless. Is the maximum height reached by disk *B* greater than, less than, or equal to *h*?

Answer: The maximum height reached by *B* is less than that reached by *A*. For *A*, all the kinetic energy becomes potential energy at *h*. Since the ramp is frictionless for B, all of the rotational *K* stays rotational, and only the translational kinetic energy becomes potential energy at its maximum height.

11-3 The Yo-Yo

Learning Objectives

11.09 Draw a free-body diagram of a yo-yo moving up or down its string.

11.10 Identify that a yo-yo is effectively an object that rolls smoothly up or down a ramp with an incline angle of 90°.

11.11 For a yo-yo moving up or down its string, apply the relationship between the yo-yo's acceleration and its rotational inertia.

11.12 Determine the tension in a yo-yo's string as the yo-yo moves up or down the string.

11-3 The Yo-Yo

- As a yo-yo moves down a string, it loses potential energy *mgh* but gains rotational and translational kinetic energy
- To find the linear acceleration of a yo-yo accelerating down its string:
  1. Rolls down a “ramp” of angle 90°
  2. Rolls on an axle instead of its outer surface
  3. Slowed by tension *T* rather than friction

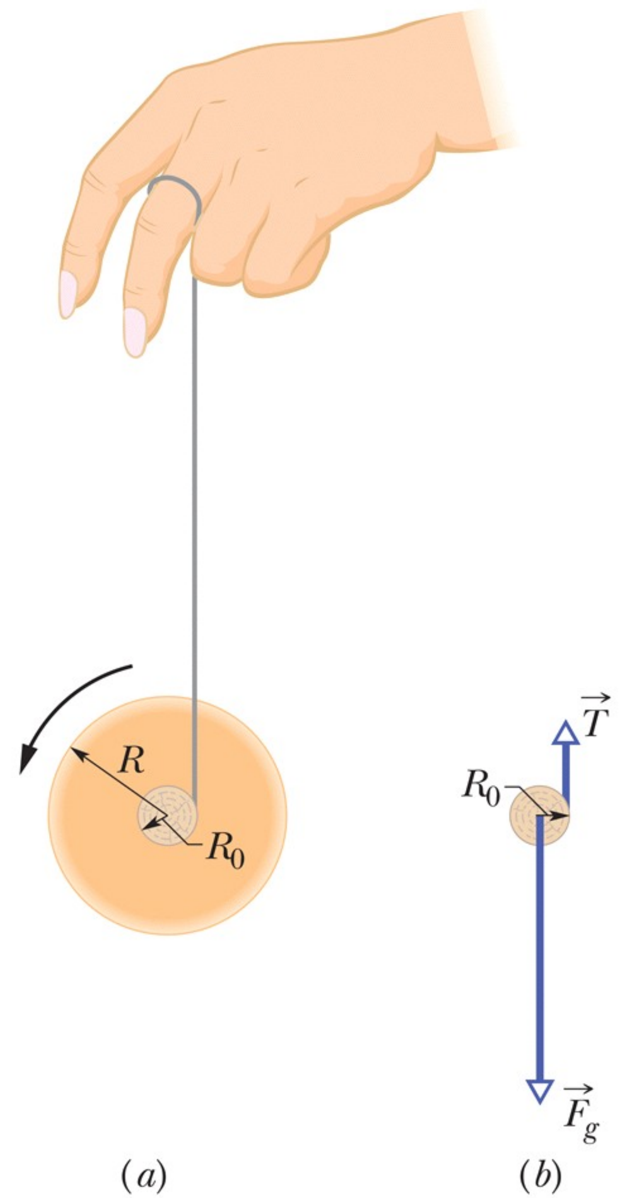


Figure 11-9

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11-3 The Yo-Yo

- Replacing the values in 11-10 leads us to:

$$a_{\text{com}} = -\frac{g}{1 + I_{\text{com}}/MR_0^2}, \quad \textbf{Eq. (11-13)}$$

Example Calculate the acceleration of the yo-yo

- M = 150 grams, R_0 = 3 mm, I_{com} = $Mr^2/2$ = 3E-5 kg m^2
- Therefore a_{com} = -9.8 m/s^2 / (1 + 3E-5 / (0.15 * 0.003^2))
  = - .4 m/s^2

11-4 Torque Revisited

Learning Objectives

11.13 Identify that torque is a vector quantity.

11.14 Identify that the point about which a torque is calculated must always be specified.

11.15 Calculate the torque due to a force on a particle by taking the cross product of the particle's position vector and the force vector, in either unit-vector notation or magnitude-angle notation.

11.16 Use the right-hand rule for cross products to find the direction of a torque vector.

11-4 Torque Revisited

- Previously, torque was defined only for a rotating body and a fixed axis
- Now we redefine it for an individual particle that moves along any path relative to a fixed point
- The path need not be a circle; torque is now a vector
- Direction determined with right-hand-rule

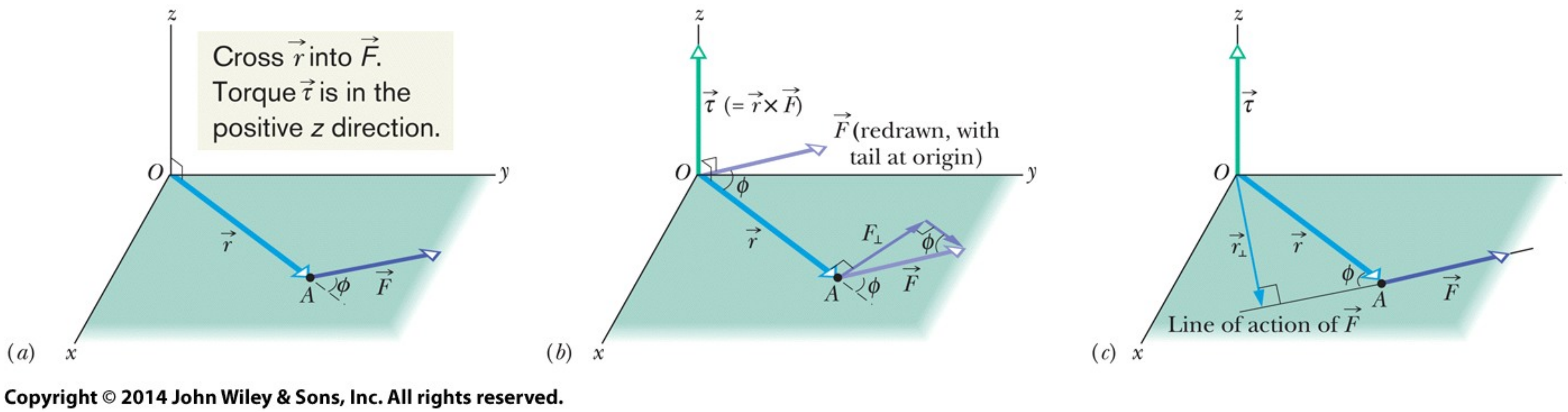


Figure 11-10

11-4 Torque Revisited

- The general equation for torque is:

$$\vec{\tau} = \vec{r} \times \vec{F}$$ Eq. (11-14)

- We can also write the magnitude as:

$$\tau = rF \sin \phi,$$ Eq. (11-15)

- Or, using the perpendicular component of force or the moment arm of *F*:

$$\tau = rF_{\perp},$$ Eq. (11-16)

$$\tau = r_{\perp}F,$$ Eq. (11-17)

11-4 Torque Revisited

Checkpoint 3

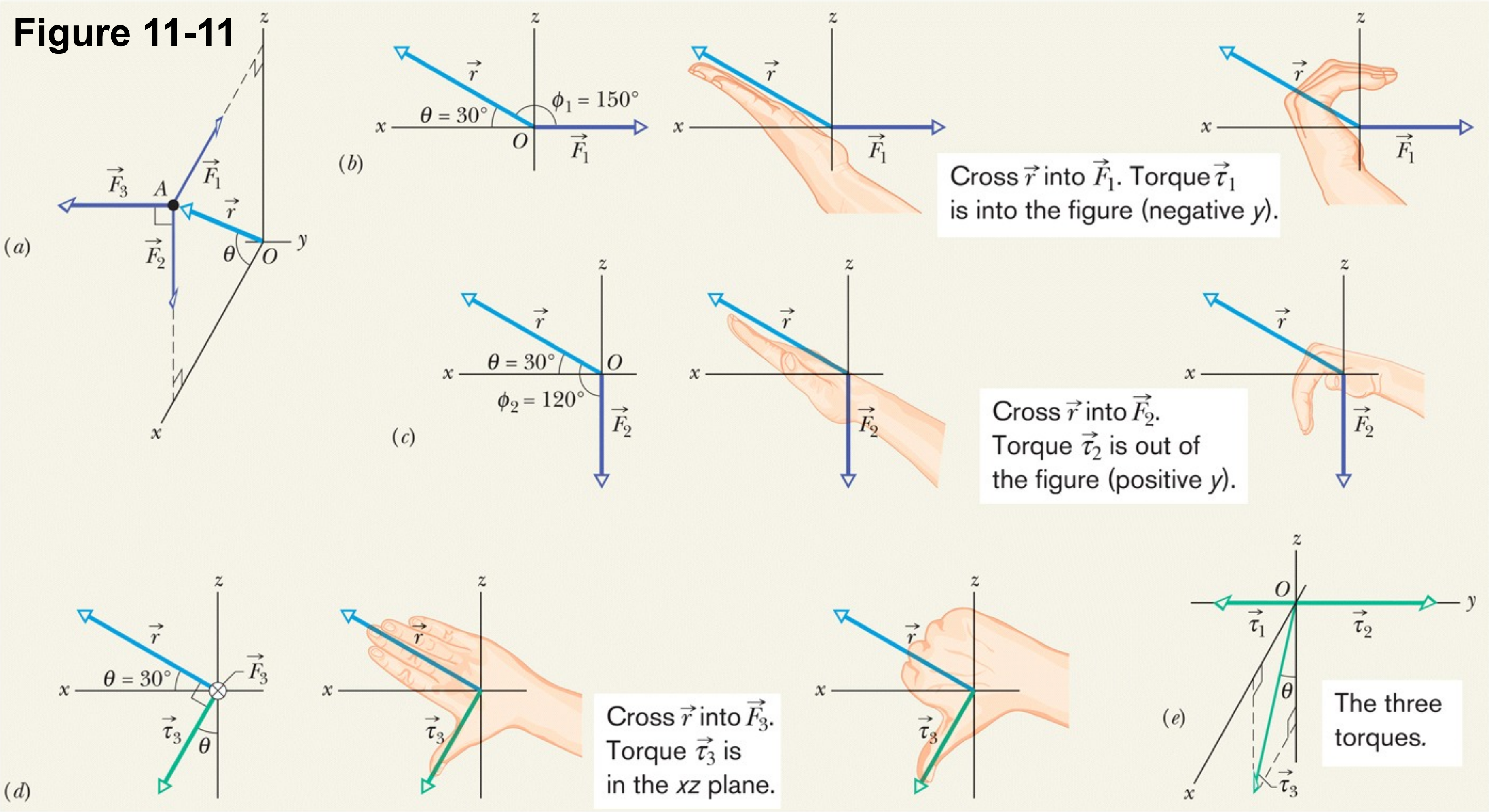
The position vector $\vec{r}$ of a particle points along the positive direction of a z axis. If the torque on the particle is (a) zero, (b) in the negative direction of x, and (c) in the negative direction of y, in what direction is the force causing the torque?

Answer: (a) along the z direction (b) along the $+y$ direction (c) along the $+x$ direction

11-4 Torque Revisited

Example Calculating net torque:

Figure 11-11



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11-5 Angular Momentum

Learning Objectives

11.17 Identify that angular momentum is a vector quantity.

11.18 Identify that the fixed point about which an angular momentum is calculated must always be specified.

11.19 Calculate the angular momentum of a particle by taking the cross product of the particle's position vector and its momentum vector, in either unit-vector notation or magnitude-angle notation.

11.20 Use the right-hand rule for cross products to find the direction of an angular momentum vector.

11-5 Angular Momentum

- Here we investigate the angular counterpart to linear momentum
- We write:

$$\vec{\ell} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$$ **Eq. (11-18)**

- Note that the particle need not rotate around *O* to have angular momentum around it
- The unit of angular momentum is kg m^2/s, or J s

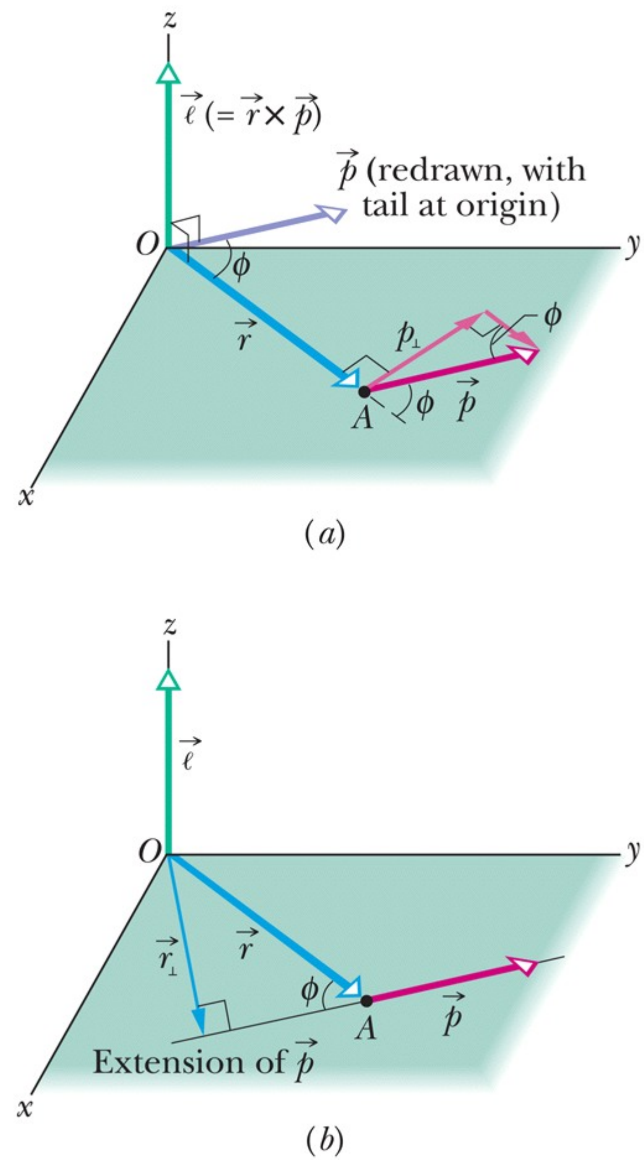

Figure 11-12

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11-5 Angular Momentum

- To find the direction of angular momentum, use the right-hand rule to relate *r* and *v* to the result
- To find the magnitude, use the equation for the magnitude of a cross product:

$$\ell = rmv \sin \phi,$$ Eq. (11-19)

- Which can also be written as:

$$\ell = rp_\perp = rmv_\perp,$$ Eq. (11-20)

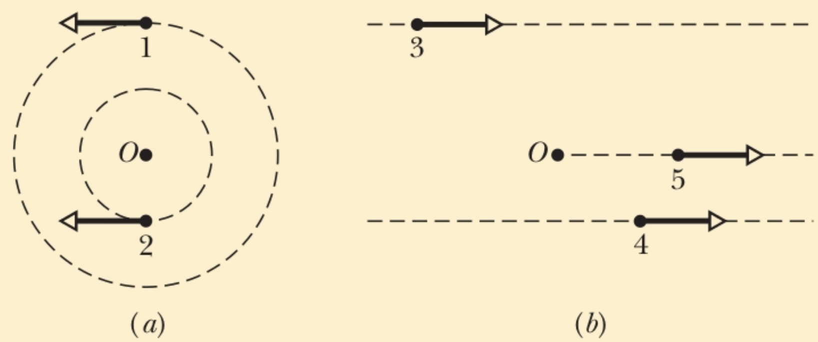
$$\ell = r_\perp p = r_\perp mv,$$ Eq. (11-21)

11-5 Angular Momentum

- Angular momentum has meaning only with respect to a specified origin
- It is always perpendicular to the plane formed by the position and linear momentum vectors

Checkpoint 4

In part *a* of the figure, particles 1 and 2 move around point O in circles with radii 2 m and 4 m. In part *b*, particles 3 and 4 travel along straight lines at perpendicular distances of 4 m and 2 m from point O. Particle 5 moves directly away from O. All five particles have the same mass and the same constant speed. (a) Rank the particles according to the magnitudes of their angular momentum about point O, greatest first. (b) Which particles have negative angular momentum about point O?



Answer: (a) 1 & 3, 2 & 4, 5

(b) 2 and 3 (assuming counterclockwise is positive)

11-6 Newton's Second Law in Angular Form

Learning Objectives

11.21 Apply Newton's second law in angular form to relate the torque acting on a particle to the resulting rate of change of the particle's angular momentum, all relative to a specified point.

11-6 Newton's Second Law in Angular Form

- We rewrite Newton's second law as:

$$\vec{\tau}_{\text{net}} = \frac{d\vec{\ell}}{dt} \quad \text{(single particle).}$$

Eq. (11-23)

- The torque and the angular momentum must be defined with respect to the same point (usually the origin)

> The (vector) sum of all the torques acting on a particle is equal to the time rate of change of the angular momentum of that particle.

- Note the similarity to the linear form:

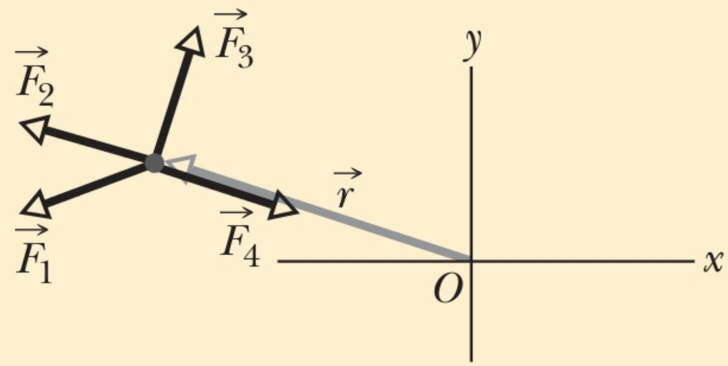
$$\vec{F}_{\text{net}} = \frac{d\vec{p}}{dt} \quad \text{(single particle)}$$

Eq. (11-22)

11-6 Newton's Second Law in Angular Form

Checkpoint 5

The figure shows the position vector $\vec{r}$ of a particle at a certain instant, and four choices for the direction of a force that is to accelerate the particle. All four choices lie in the xy plane. (a) Rank the choices according to the magnitude of the time rate of change ($d\vec{\ell}/dt$) they produce in the angular momentum of the particle about point O, greatest first. (b) Which choice results in a negative rate of change about O?



Answer: (a) F_3, F_1, F_2 & F_4 (b) F_3 *(assuming counterclockwise is positive)*

11-7 Angular Momentum of a Rigid Body

Learning Objectives

11.22 For a system of particles, apply Newton's second law in angular form to relate the net torque acting on the system to the rate of the resulting change in the system's angular momentum.

11.23 Apply the relationship between the angular momentum of a rigid body rotating around a fixed axis and the body's rotational inertia and angular speed around that axis.

11.24 If two rigid bodies rotate about the same axis, calculate their total angular momentum.

11-7 Angular Momentum of a Rigid Body

- We sum the angular momenta of the particles to find the angular momentum of a system of particles:

$$\vec{L} = \vec{\ell}_1 + \vec{\ell}_2 + \vec{\ell}_3 + \cdots + \vec{\ell}_n = \sum_{i=1}^{n} \vec{\ell}_i.$$

Eq. (11-26)

- The rate of change of the net angular momentum is:

$$\frac{d\vec{L}}{dt} = \sum_{i=1}^{n} \vec{\tau}_{\text{net},i}.$$

Eq. (11-28)

- In other words, the net torque is defined by this change:

$$\vec{\tau}_{\text{net}} = \frac{d\vec{L}}{dt} \qquad \text{(system of particles)},$$

Eq. (11-29)

The net external torque $\vec{\tau}_{\text{net}}$ acting on a system of particles is equal to the time rate of change of the system's total angular momentum $\vec{L}$.

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11-7 Angular Momentum of a Rigid Body

- Note that the torque and angular momentum must be measured relative to the same origin
- If the center of mass is accelerating, then that origin *must* be the center of mass
- We can find the angular momentum of a rigid body through summation:

$$L_z = \sum_{i=1}^{n} \ell_{iz} = \sum_{i=1}^{n} \Delta m_i \, v_i r_{\perp i} = \sum_{i=1}^{n} \Delta m_i (\omega r_{\perp i}) r_{\perp i}$$

$$= \omega \left(\sum_{i=1}^{n} \Delta m_i \, r_{\perp i}^2 \right). \qquad \textbf{Eq. (11-30)}$$

- The sum is the rotational inertia *I* of the body

11-7 Angular Momentum of a Rigid Body

- Therefore this simplifies to:

$$L = I\omega \qquad \text{(rigid body, fixed axis).}$$

Eq. (11-31)

Table 11-1

Table 11-1 More Corresponding Variables and Relations for Translational and Rotational Motion[a]

Translational		Rotational	
Force	$\vec{F}$	Torque	$\vec{\tau}\ (= \vec{r} \times \vec{F})$
Linear momentum	$\vec{p}$	Angular momentum	$\vec{\ell}\ (= \vec{r} \times \vec{p})$
Linear momentum[b]	$\vec{P}\ (= \Sigma \vec{p}_i)$	Angular momentum[b]	$\vec{L}\ (= \Sigma \vec{\ell}_i)$
Linear momentum[b]	$\vec{P} = M\vec{v}_{com}$	Angular momentum[c]	$L = I\omega$
Newton's second law[b]	$\vec{F}_{net} = \dfrac{d\vec{P}}{dt}$	Newton's second law[b]	$\vec{\tau}_{net} = \dfrac{d\vec{L}}{dt}$
Conservation law[d]	$\vec{P}$ = a constant	Conservation law[d]	$\vec{L}$ = a constant

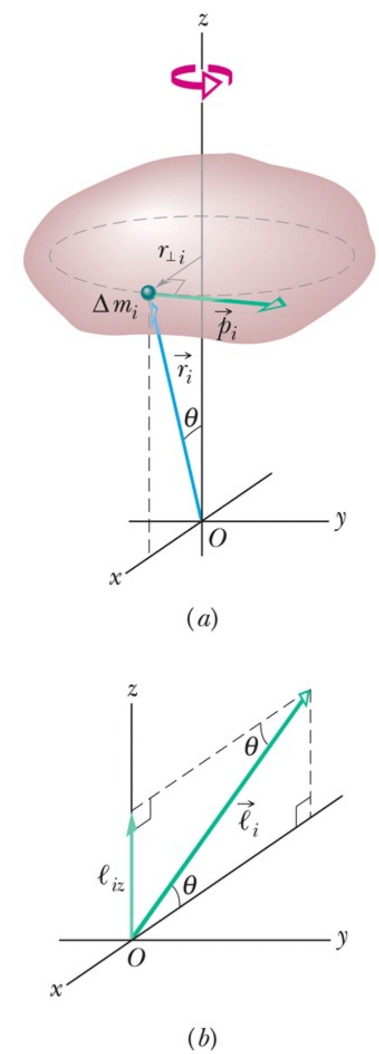
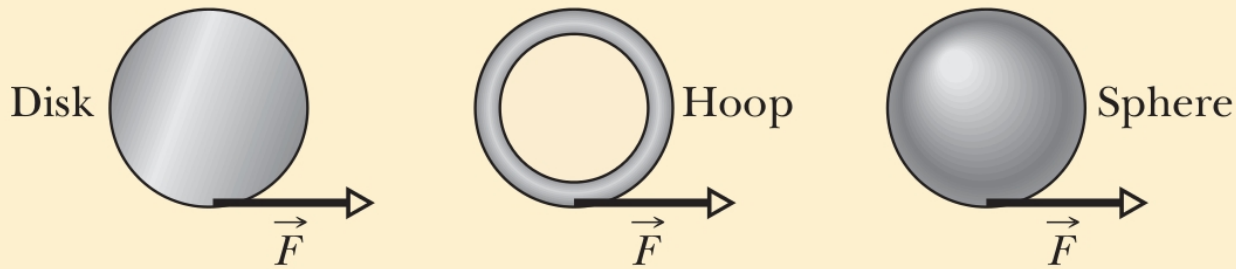


Figure 11-15

11-7 Angular Momentum of a Rigid Body

Checkpoint 6

In the figure, a disk, a hoop, and a solid sphere are made to spin about fixed central axes (like a top) by means of strings wrapped around them, with the strings producing the same constant tangential force $\vec{F}$ on all three objects. The three objects have the same mass and radius, and they are initially stationary. Rank the objects according to (a) their angular momentum about their central axes and (b) their angular speed, greatest first, when the strings have been pulled for a certain time t.



Answer: (a) All angular momenta will be the same, because the torque is the same in each case (b) sphere, disk, hoop

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11-8 Conservation of Angular Momentum

Learning Objectives

11.25 When no external net torque acts on a system along a specified axis, apply the conservation of angular momentum to relate the initial angular momentum value along *that axis* to the value at a later instant.

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11-8 Conservation of Angular Momentum

- Since we have a new version of Newton's second law, we also have a new conservation law:

$$\vec{L} = \text{a constant} \quad \text{(isolated system).}$$ Eq. (11-32)

- The **law of conservation of angular momentum** states that, for an isolated system,

(*net initial angular momentum*) = (*net final angular momentum*)

$$\vec{L}_i = \vec{L}_f \quad \text{(isolated system).}$$ Eq. (11-33)

11-8 Conservation of Angular Momentum

If the net external torque acting on a system is zero, the angular momentum $\vec{L}$ of the system remains constant, no matter what changes take place within the system.

- Since these are vector equations, they are equivalent to the three corresponding scalar equations
- This means we can separate axes and write:

If the component of the net *external* torque on a system along a certain axis is zero, then the component of the angular momentum of the system along that axis cannot change, no matter what changes take place within the system.

- If the distribution of mass changes with no external torque, we have:

$$I_i\omega_i = I_f\omega_f. \quad \textbf{Eq. (11-34)}$$

11-8 Conservation of Angular Momentum

Example Angular momentum conservation

- A student spinning on a stool: rotation speeds up when arms are brought in, slows down when arms are extended
- A springboard diver: rotational speed is controlled by tucking her arms and legs in, which reduces rotational inertia and increases rotational speed
- A long jumper: the angular momentum caused by the torque during the initial jump can be transferred to the rotation of the arms, by windmilling them, keeping the jumper upright

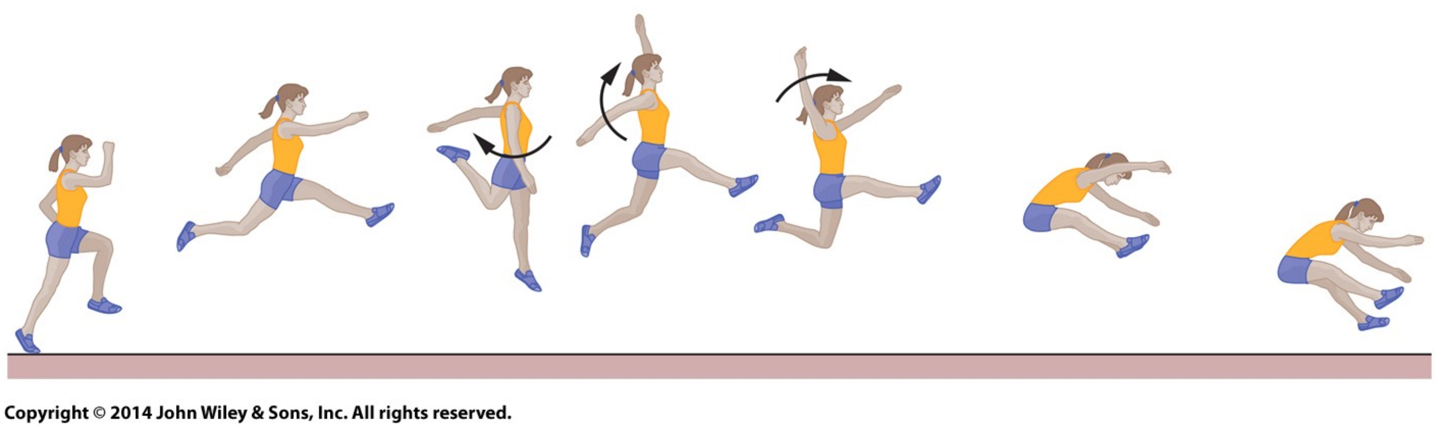

Figure 11-18

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11-8 Conservation of Angular Momentum

Checkpoint 7

A rhinoceros beetle rides the rim of a small disk that rotates like a merry-go-round. If the beetle crawls toward the center of the disk, do the following (each relative to the central axis) increase, decrease, or remain the same for the beetle–disk system: (a) rotational inertia, (b) angular momentum, and (c) angular speed?

Answer: (a) decreases (b) remains the same (c) increases

11-9 Precession of a Gyroscope

Learning Objectives

11.26 Identify that the gravitational force acting on a spinning gyroscope causes the spin angular momentum vector (and thus the gyroscope) to rotate about the vertical axis in a motion called precession.

11.27 Calculate the precession rate of a gyroscope.

11.28 Identify that a gyroscope's precession rate is independent of the gyroscope's mass.

11-9 Precession of a Gyroscope

- A nonspinning gyroscope, as attached in 11-22 (a), falls
- A spinning gyroscope (b) instead rotates around a vertical axis
- This rotation is called **precession**

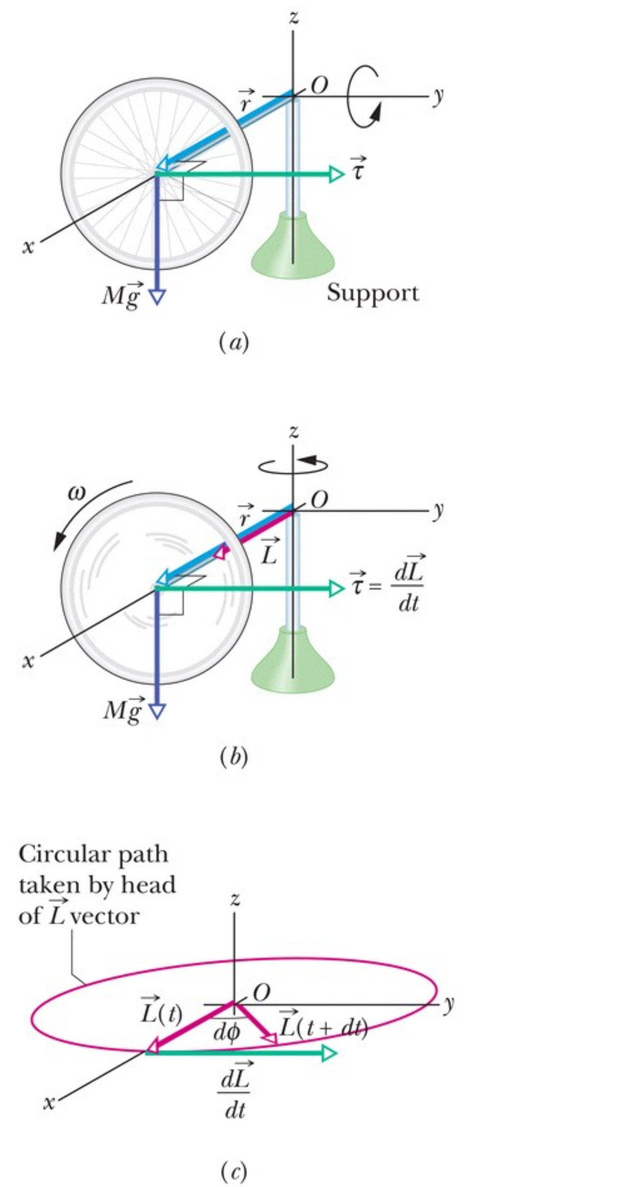

Figure 11-22

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11-9 Precession of a Gyroscope

- The angular momentum of a (*rapidly spinning*) gyroscope is:

$$L = I\omega, \qquad \textbf{Eq. (11-43)}$$

- The torque can only change the direction of *L*, not its magnitude, because of (11-43)

$$d\vec{L} = \vec{\tau}\,dt. \qquad \textbf{Eq. (11-44)}$$

- The only way its direction can change along the direction of the torque without its magnitude changing is if it rotates around the central axis
- Therefore it precesses instead of toppling over

11-9 Precession of a Gyroscope

- The **precession rate** is given by:

$$\Omega = \frac{Mgr}{I\omega}$$

Eq. (11-46)

- True for a sufficiently rapid spin rate
- Independent of mass, (*I* is proportional to *M*) but does depend on *g*
- Valid for a gyroscope at an angle to the horizontal as well (a top for instance)

11 Summary

Rolling Bodies

$$v_{com} = \omega R$$ Eq. (11-2)

$$K = \tfrac{1}{2} I_{com} \omega^2 + \tfrac{1}{2} M v_{com}^2.$$ Eq. (11-5)

$$a_{com} = \alpha R$$ Eq. (11-6)

Torque as a Vector

- Direction given by the right-hand rule

$$\vec{\tau} = \vec{r} \times \vec{F}$$ Eq. (11-14)

Angular Momentum of a Particle

$$\vec{\ell} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$$

Eq. (11-18)

Newton's Second Law in Angular Form

$$\vec{\tau}_{net} = \frac{d\vec{\ell}}{dt}$$ Eq. (11-23)

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11 Summary

Angular Momentum of a System of Particles

$$\vec{L} = \vec{\ell}_1 + \vec{\ell}_2 + \vec{\ell}_3 + \cdots + \vec{\ell}_n = \sum_{i=1}^{n} \vec{\ell}_i.$$

Eq. (11-26)

$$\vec{\tau}_{\text{net}} = \frac{d\vec{L}}{dt}$$

Eq. (11-29)

Angular Momentum of a Rigid Body

$$L = I\omega$$

Eq. (11-31)

Conservation of Angular Momentum

$$\vec{L} = \text{a constant}$$

Eq. (11-32)

$$\vec{L}_i = \vec{L}_f$$

Eq. (11-33)

Precession of a Gyroscope

$$\Omega = \frac{Mgr}{I\omega}$$

Eq. (11-46)