Inverse functions

We often think of a function as a rule which takes in an input and assigns to it an output.

Usually we have a nice formula or recipe which tells us how to calculate the output for a given input.

Many hard and interesting problems go the other way: You know the output and you want to work out what the input must have been.

Examples:

(a) Find all the x such that $f(x) = x^3 - 3x^2 + x - 4 = 0$. This is much harder than finding $f(x)$ for a given input!

(b) In a scanning device like a CT-machine, the output (what is picked up by the detectors) is a function of what is inside the object being scanned.

The challenge is to reconstruct the 'input data' from the output information.

More abstractly, the problem is

Given a function $f : A \to B$, if we set $y = f(x)$, under what circumstances is it possible to express x as a function of y, that is, to find a function $g : B \to A$ such that $x = g(y)$?

The first things to worry about are:

1. Is it true that: for any $y \in B$ there is $x \in A$ such that $y = f(x)$?
2. If so, is this x unique?

In answering these questions it is vital that one considers not just the formula for f, but also what the domain of f is.

Standard example. Consider the rule

$$y = x^2.$$

Whether any function defined by this rule is invertible depends on the domain:

- $f_1 : [0, \infty) \to \mathbb{R}, \qquad y = f_1(x) = x^2$

  If we take into account that $\text{Range}(f_1) = [0, \infty)$ then the inverse function is given by

  $$g_1 : [0, \infty) \to [0, \infty), \qquad x = g_1(y) = \sqrt{y}.$$

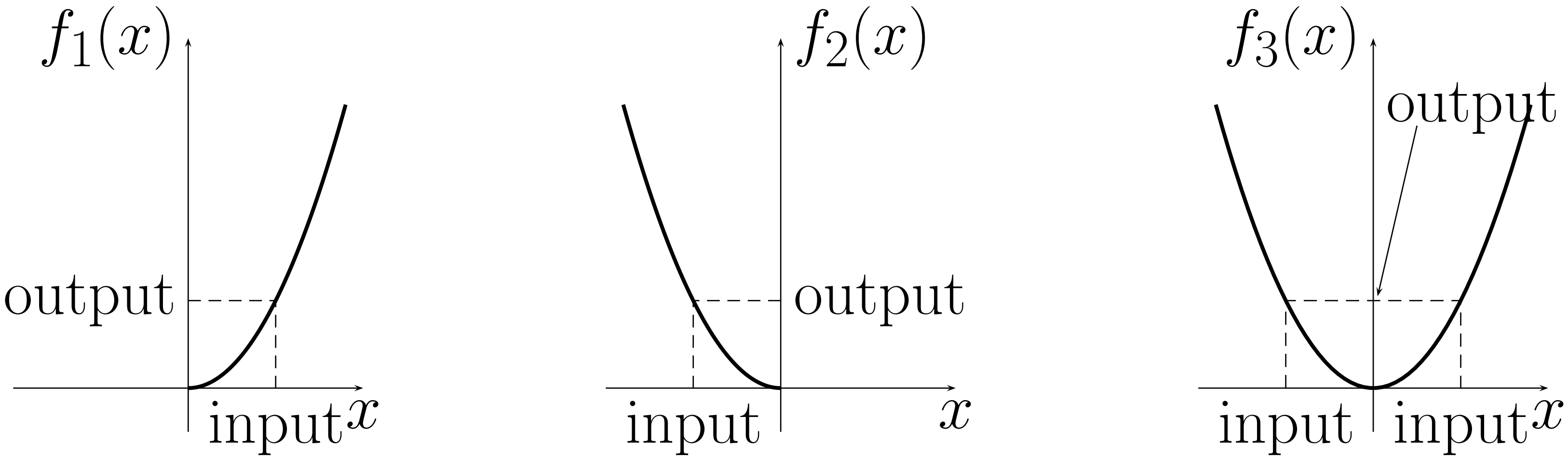
- $f_2 : (-\infty, 0] \to \mathbb{R}, \qquad y = f_2(x) = x^2$

If we take into account that, again, $\text{Range}(f_2) = [0, \infty)$ then the inverse function is given by

$$g_2 : [0, \infty) \to (-\infty, 0], \qquad x = g_2(y) = -\sqrt{y}.$$

- $f_3 : \mathbb{R} \to \mathbb{R}, \qquad y = f_3(x) = x^2$

  The latter is not invertible since for any 'output' $y \neq 0$ there exist two 'inputs' $x = \sqrt{y}$ and $x = -\sqrt{y}$.



Remark. It is evident that it might be possible to construct an invertible function by restricting the domain of a given function.

Conclusion. The main criterion for invertibility is the existence of a one-to-one correspondence between ‘inputs’ and ‘outputs’.

One-to-one functions

Idea. A function is one-to-one if every 'output' corresponds to a unique 'input'.

Definition. A function f is said to be one-to-one

$$\text{if} \quad f(x_1) = f(x_2) \quad \text{implies that} \quad x_1 = x_2$$

for all $x_1, x_2 \in \mathrm{Dom}(f)$.

Terminology. One-to-one functions are also called injective functions.

Remark. An injective function is equivalently characterised by

$$f(x_1) \neq f(x_2) \quad \text{for all} \quad x_1 \neq x_2$$

in the domain of f.

Example. Show that the function

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = x^3 + x + 1$$

is one-to-one.

Solution. Assume that $f(x_1) = f(x_2)$ for some x_1, $x_2 \in \mathbb{R}$. We have that

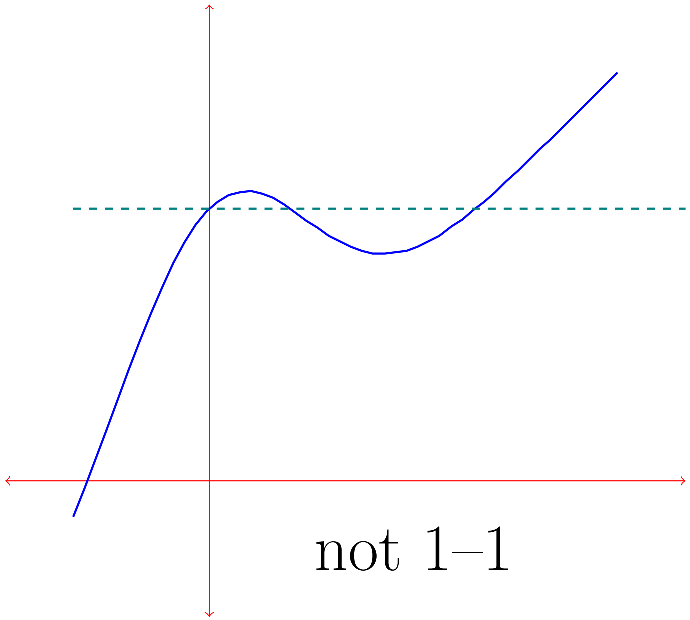
$$x_1^3 + x_1 + 1 = x_2^3 + x_2 + 1 \quad \Leftrightarrow \quad x_1^3 - x_2^3 + x_1 - x_2 = 0$$
$$(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2 + 1) = 0.$$

Suppose x_1, x_2 has the same sign, then $x_1^2 + x_1x_2 + x_2^2 + 1 > 0$ for all $x_1, x_2 \in \mathbb{R}$. If x_1, x_2 has different signs, we can write

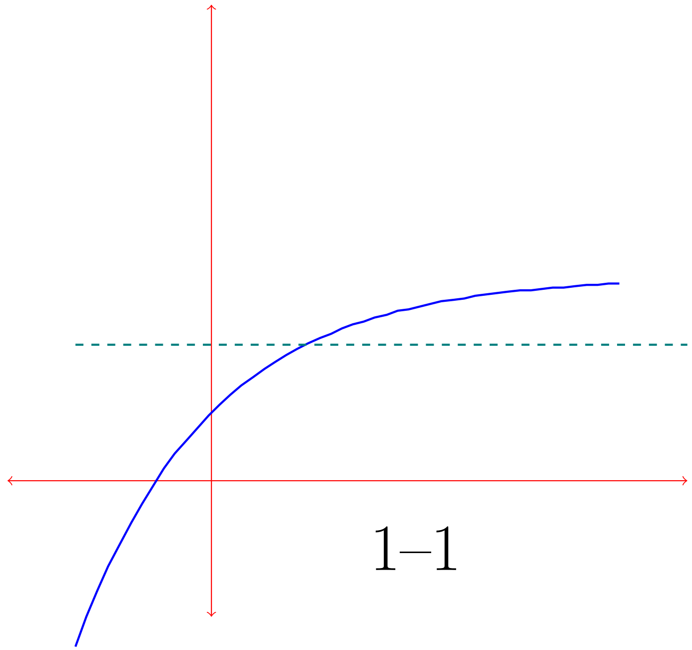
$$x_1^2 + x_1x_2 + x_2^2 + 1 = (x_1 + x_2)^2 - x_1x_2 + 1 > 0.$$

Thus, we have that $x_1^2 + x_1x_2 + x_2^2 + 1$ never equals zero. Therefore, we have $x_1 - x_2 = 0$, or, equivalently, $x_1 = x_2$. Hence, the function f is one-to-one.

If $f : A \to \mathbb{R}$ where $A \subseteq \mathbb{R}$, then you can easily identify one-to-one functions by looking at the graph of f.



not 1–1



1–1

f is one-to-one if each horizontal line cuts the graph **at most once**.

The horizontal line test. Suppose that f is a real-valued function defined on some subset of $\mathbb{R}$. Then, f is one-to-one if and only if every horizontal line in the Cartesian plane intersects the graph of f at most once.

Calculus provides an easy tool for checking whether a function is one-to-one.

Proposition. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). If $f'(x) \neq 0$ for all $x \in (a, b)$ then f is one-to-one.

Proof. Suppose that for some $x_1 < x_2$ we have $f(x_1) = f(x_2)$. Then Rolle's theorem implies that there exists some $c \in (x_1, x_2)$ such that $f'(c) = 0$. But this is impossible so we must have had $x_1 = x_2$.

More generally, if f is either strictly increasing or strictly decreasing on some interval, then f is one-to-one.

This includes cases like:

1. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ which is strictly increasing, but where $f'(x)$ is sometimes zero.
2. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 3x - |x|$ which is strictly increasing, but not differentiable.

Example. Is the function $f : (-1, 1) \to \mathbb{R}$ defined by

$$f(x) = 3 + 2\tan\left(\frac{\pi}{2}x\right)$$

one-to-one?

Solution. The function $\tan\left(\frac{\pi}{2}x\right)$ is continuous and differentiable at every point of the interval $(-1, 1)$. Therefore, the function f is also continuous and differentiable on $(-1, 1)$.

We have

$$f'(x) = \frac{\pi}{\cos^2\left(\frac{\pi}{2}x\right)} \neq 0,$$

for every $x \in (-1, 1)$. Hence, by the proposition above, the function f is one-to-one.

Remark. Not every function whose derivative is only positive (or only negative) on its domain is one-to-one. For example,

$$\frac{d}{dx} \tan x = \sec^2 x \geq 1$$

but tan is not one-to-one on its maximal domain!

Inverse functions

Theorem. Suppose that f is a one-to-one function. Then, there exists a unique function g satisfying

$$g(f(x)) = x \quad \text{for all} \quad x \in \mathrm{Dom}(f)$$

and

$$f(g(y)) = y \quad \text{for all} \quad y \in \mathrm{Range}(f).$$

Moreover,

$$\mathrm{Dom}(g) = \mathrm{Range}(f), \qquad \mathrm{Range}(g) = \mathrm{Dom}(f)$$

and g is one-to-one.

Proof. Set $D = \mathrm{Dom}(f)$ and $R = \mathrm{Range}(f)$ and define the function

$$g : R \to D$$

by choosing as $g(y)$ the unique $x \in D$ for which $y = f(x)$.

It is then left as an exercise to show that g has the properties listed above.

The theorem allows us to define the term inverse function.

Definition. Suppose that f is a one-to-one function. Then the inverse function of f is the unique function g given by the above theorem. The inverse function for f is often denoted by f^{-1}.

Remark. If f^{-1} denotes the inverse function of a one-to-one function f then the relations in the above theorem may be expressed as

$$f^{-1}(f(x)) = x \quad \text{for all} \quad x \in \mathrm{Dom}(f)$$

and

$$f(f^{-1}(y)) = y \quad \text{for all} \quad y \in \mathrm{Range}(f)$$

so that f may also be interpreted as the inverse of the function f^{-1}.

Note. $f^{-1}(y)$ does not mean $1/f(y)$!

Remark. Since f^{-1} is a function just like any other function, we regard it as a function

$$x \mapsto f^{-1}(x)$$

so that we can graph f^{-1} in the usual manner.

Example. Determine f^{-1}, where

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = 4 - \frac{1}{3}x^3.$$

Set

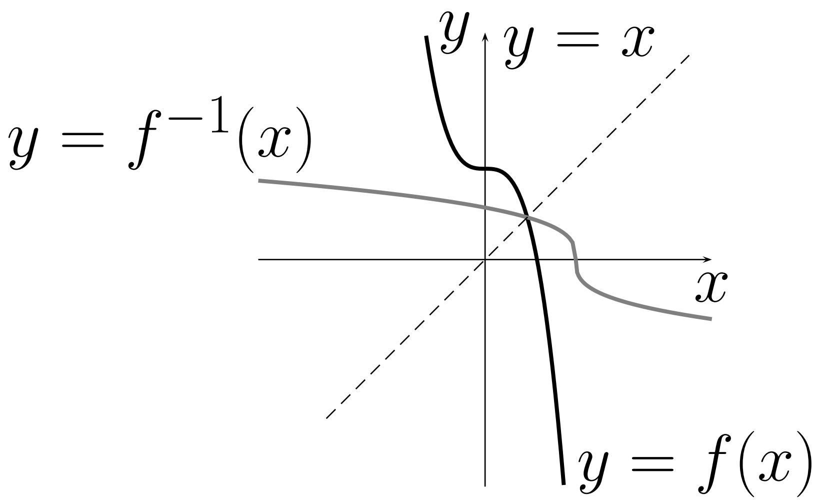
$$y = 4 - \frac{1}{3}x^3$$

so that

$$x = \sqrt[3]{12 - 3y}.$$

Hence, (interchanging x and y),

$$f^{-1} : \mathbb{R} \to \mathbb{R}, \qquad f^{-1}(x) = \sqrt[3]{12 - 3x}.$$



The inverse function theorem

Question. If the derivative of an invertible function exists, under what circumstances is the inverse function also differentiable?

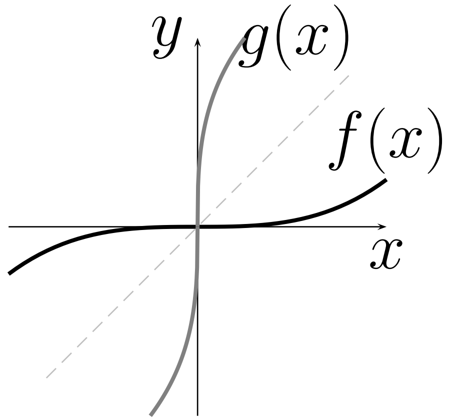
Subtlety. Consider the function

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = x^3.$$

Its inverse is given by

$$g : \mathbb{R} \to \mathbb{R}, \qquad g(x) = \sqrt[3]{x}$$

but g is not differentiable at $x = 0$!



The inverse function theorem. Suppose that I is an open interval, $f : I \to \mathbb{R}$ is differentiable and

$$f'(x) \neq 0$$

for all x in I. Then,

- f is one-to-one and has an inverse function

$$g : \mathrm{Range}(f) \to \mathrm{Dom}(f)$$

- g is differentiable at all points in $\mathrm{Range}(f)$
- The derivative of g is given by

$$g'(y) = \frac{1}{f'(g(y))}$$

  for all $y \in \mathrm{Range}(f)$.

Proof.

- Since $f'(x) \neq 0$ on I, f is one-to-one (mean value theorem!).
- g is differentiable ... too hard! (actually not, but we will skip the proof)
- Differentiation of

$$f(g(y)) = y$$

with respect to y yields

$$f'(g(y)) \times g'(y) = 1.$$

Remark. Once again, we usually write the derivative of the inverse function g as

$$g'(x) = \frac{1}{f'(g(x))}$$

for $x \in \text{Range}(f)$.

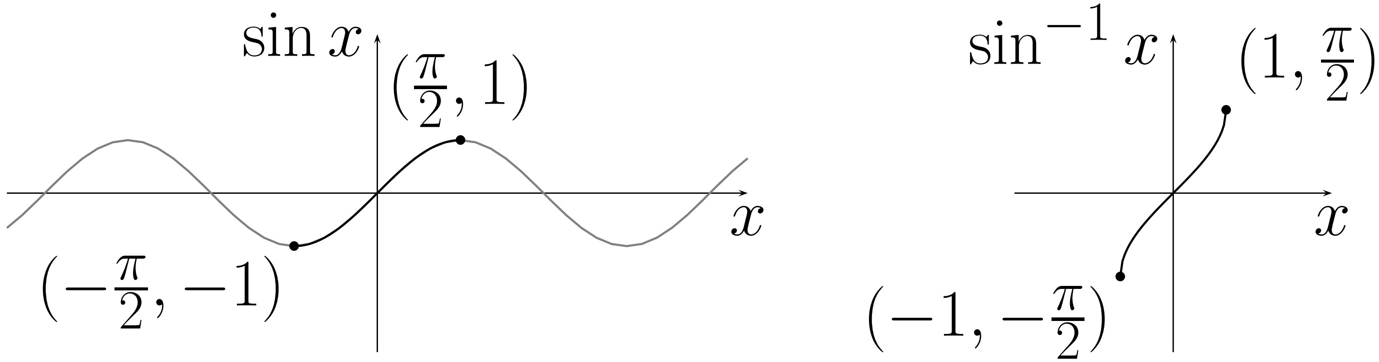
Applications to the trigonometric functions

The inverse sine function. We consider the restricted sine function

$$\sin : [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \to [-1, 1].$$

This function is one-to-one and therefore has an inverse

$$\sin^{-1} : [-1, 1] \to [-\tfrac{\pi}{2}, \tfrac{\pi}{2}].$$



On $(-1, 1)$, according to the inverse function theorem, the derivative of $\sin^{-1}$ is given by

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\cos(\sin^{-1} x)}.$$

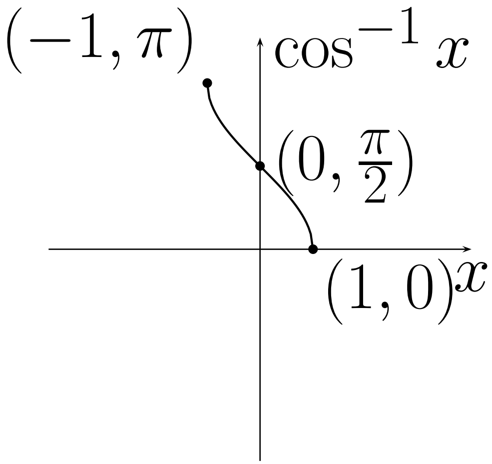
Since cos is positive on $(-\frac{\pi}{2}, \frac{\pi}{2})$, we conclude that

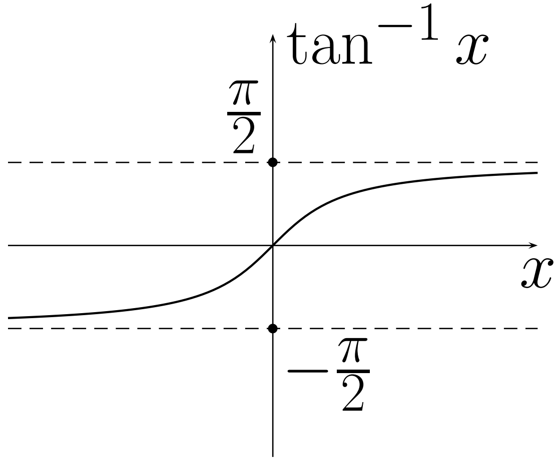
$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - \sin^2(\sin^{-1} x)}} = \frac{1}{\sqrt{1 - x^2}}.$$

Note. $\frac{d}{dx}(\sin^{-1} x) \to \infty$ as $x \to \pm 1$.

Table of inverse trigonometric functions.

Function	Domain	Range	Derivative
sin	$[-\frac{\pi}{2}, \frac{\pi}{2}]$	$[-1, 1]$	$\frac{d}{dx}(\sin x) = \cos x$
$\sin^{-1}$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$	$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$
cos	$[0, \pi]$	$[-1, 1]$	$\frac{d}{dx}(\cos x) = -\sin x$
$\cos^{-1}$	$[-1, 1]$	$[0, \pi]$	$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1 - x^2}}$
tan	$(-\frac{\pi}{2}, \frac{\pi}{2})$	$(-\infty, \infty)$	$\frac{d}{dx}(\tan x) = \sec^2 x$
$\tan^{-1}$	$(-\infty, \infty)$	$(-\frac{\pi}{2}, \frac{\pi}{2})$	$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}$





Remark. Even though

$$\sin(\sin^{-1} x) = x$$

for $x \in [-1, 1]$, in general,

$$\sin^{-1}(\sin x) \neq x$$

unless $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Example. Determine

(a)

$$\cos\left(2\sin^{-1}\frac{3}{5}\right).$$

Solution.

We have

$$\begin{aligned}\cos\left(2\sin^{-1}\frac{3}{5}\right) &= 1-2\sin^2\left(\sin^{-1}\frac{3}{5}\right)\\ &= 1-2\left(\frac{3}{5}\right)^2 = \frac{7}{25}.\end{aligned}$$

(b)

$$\sin^{-1}\left(\sin\frac{5\pi}{6}\right).$$

Solution. Since $\frac{5\pi}{6}$ does not belong to the interval $[-\frac{\pi}{2},\frac{\pi}{2}]$ we do not have $\sin^{-1}\left(\sin\frac{5\pi}{6}\right)=\frac{5\pi}{6}$.

We have

$$\begin{aligned}\sin^{-1}\left(\sin\frac{5\pi}{6}\right) &= \sin^{-1}\left(\sin(\pi-\frac{\pi}{6})\right)\\ &= \sin^{-1}\left(\sin(\frac{\pi}{6})\right)=\frac{\pi}{6}.\end{aligned}$$