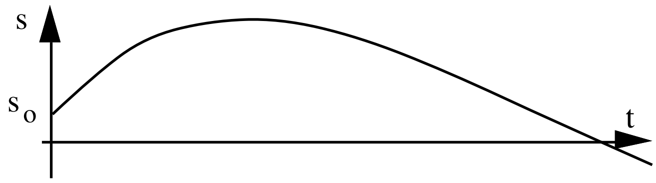
Kinematics - study of motion

PHYSICS 1A UNSW
Joe Wolfe



See HR&W chapers 1-4
Physclips Chs 2&3 and support pages
P2P chapters 1-4

Is this straightforward, or are there subtleties?
Measure lengths to get (relative) positions
Measure durations to get (relative) times
How?
At first, we use rulers and repeated cycles for space and time. First we count ratios, then indirect methods
But what *are* time and space? Here are some subtleties:

What happens when we move the clocks and rulers around? Does this change them?
If we use 'identical' ones? How do we calibrate them?
Can you (always) give an object a position or an event a time? Can you do so more than once?
Can you make continuous measurements? What is a point? A particle?

Kinematics We shall go very quickly over some bits as they are assumed knowledge*

Motion with constant acceleration

$a_y = \text{constant}$ $\qquad a_y = \frac{dv_y}{dt}$ the ***definition of acceleration***, so, by integrating,

$v_y = \int a_y \, dt$ $\qquad$ *subscript y for y direction*

$= a_y t + \text{const}$ $\qquad$ *what is the constant? Use the initial state:* at $t = 0$, $v_y = v_{y0}$

$v_y = v_{y0} + a_y t$ $\qquad$ (i) $\qquad$ *common convention: use subscript 0 for t = 0*

$y = \int v_y \, dt$ $\qquad$ *again, using the definition* $v_y = \frac{dy}{dt}$. *Substitute (i) gives:*

$= \int v_{y0} + a_y t$ $\qquad$ *and integration gives*

$= v_{y0}t + \frac{1}{2} a_y t^2 + \text{constant}$ $\qquad$ *again, use the initial state at t = 0*

$y = y_0 + v_{y0}t + \frac{1}{2} a_y t^2.$ $\qquad$ (ii) $\qquad$ *Don't forget the* y_0 *term*

Definition of constant acceleration and rearrange: $t = (v - v_0)/a_y$

ii) gives $y - y_0 = v_{y0}t + \frac{1}{2} a_y t^2$ $\qquad$ substitute from $t = (v_y - v_{y0})/ a_y$

$a_y(y - y_0) = v_{y0}(v_y - v_{y0}) + \frac{1}{2} (v_y - v_{y0})^2$

$2a_y(y - y_0) = v_y^2 - v_{y0}^2$ $\qquad$ (iii)

You may also remember these as $v = u + at$ (i) $\qquad \Delta s = ut + \frac{1}{2} at^2$ (ii) $\qquad v^2 - u^2 = 2as$ (iii)

However, we shall be looking at motion in both x and y directions so you will need subscripts

* *What if this is not clear to you? See* HR&W chs 1-4, Physclips Chs 2&3, P2P chs 1-3

See also Calculus, a supporting page in Physclips www.animations.physics.unsw.edu.au

Example. Joe runs at constant speed 6 ms^{-1} towards a stationary bus. When he is 30 m from it, the bus accelerates away at 2 m.s^{-2}. Can he overtake it?

Note: problem with 2 objects. Need subscripts

Translate the words … *into maths*

" Joe runs at constant speed 6 ms^{-1}" –> $v_J = 6$ ms^{-1}, $a_J = 0$

" stationary bus " & " accelerates away at 2 m.s^{-2}" –> $v_{b0} = 0$, $a_b = 2$ m.s^{-2}

" When he is 30 m from it " –> $x_{b0} - x_{J0} = 30$ m

" Can he overtake it?" –> Does $x_J = x_b$ at any t?

$$x_b = x_{b0} + v_{b0}t + \frac{1}{2} a_b t^2 \qquad \text{(i)}$$

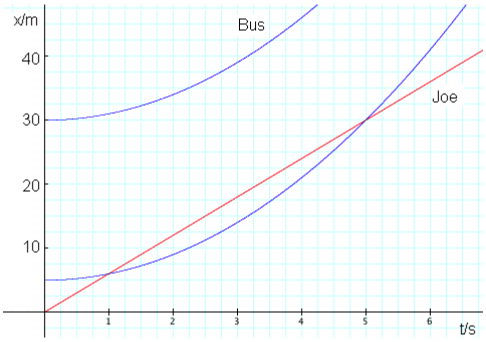
(the pale terms are zero, but I thought about them!)

$$x_J = x_{J0} + v_{J0}t + \frac{1}{2} a_J t^2 \qquad \text{(ii)}$$

it's easiest to remember the general form and lose terms

Eliminate x_{Jo} by choice of origin, x_{bo} = 30 m

Draw a diagram or two. Which of the following graphs is correct?



In kinematics, displacement time graphs are useful!

What does overtake mean? How to write it mathematically?

$x_b = x_J$ substitute from (i) and (ii) gives:

$$\frac{1}{2} a_b t^2 - v_{Jo}t + x_{bo} = 0$$

Recall quadratic: solutions are $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ so

$$t = \frac{v_{Jo} \pm \sqrt{v^2_{Jo} - 2a_b x_{bo}}}{a_b}$$

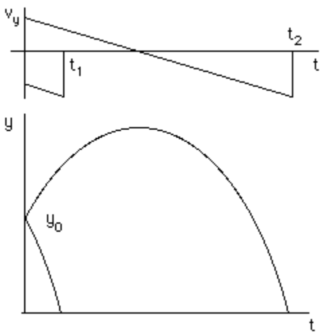
are the units correct?

Put in numbers $\sqrt{\ } = \sqrt{6^2 - 2*2*30}\ \text{ms}^{-1} \rightarrow \sqrt{\text{-ve}}$ $\therefore$ no (real) solutions, $\therefore$ no overtaking.

what would the imaginary solutions mean?

Example. Ball 1 thrown vertically up at 5 ms^{-1} from 20 m above ground. Simultaneously, ball 2 thrown vertically down at 5 ms^{-1} from 20 m above ground. What are their speeds when they hit the ground, and what is the time interval between collisions?

We are given: $v_{o1} = 5$ ms^{-1} $v_{o2} = -5$ ms^{-1} $y_o = 20$ m

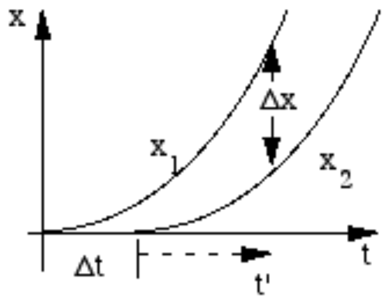


Translate the question: When y = 0, t = ? Tactics: (ii) to get t_1 and t_2. Use (i) to get v_{y1} and v_{y2}.

Solve it yourself: solving simultaneous equations is Assumed Knowledge

How much do you lose if you miss the starter's gun?

Two runners at different times (Δt apart). During the (constant) acceleration phase, when are they a distance Δx apart ?



$x_1 = ½at^2$ ½

$x_2 = ½a(t')^2$

$t' = t - \Delta t$

$x_1 - x_2 = ½at^2 - ½a(t - \Delta t)^2$ *Solve this for t:* $x_1 - x_2 = (a\Delta t)t - ½a(\Delta t)^2$

$t = (x_1 - x_2 + ½a(\Delta t)^2)/a\Delta t$

A trickier but real example. Over a marked kilometre, on a flat track with no wind, solar car ***sUNSWift*** slows from v_0 = 70 k.p.h. to v = 50 k.p.h. Assume that $a = -kv^2$ (turbulent drag only). What is k?

Given x (= 1 km), v_0, v, need k. Must relate these. Must eliminate a. Must introduce x.

$a = -kv^2$ and by definition $a = \frac{dv}{dt}$ so

$\frac{dv}{dt} = -kv^2$ But this has t. Can eliminate dt using dx = vdt. Multiply both sides by dt:

$dv = -kv.(vdt) = -kv.dx$ Separate variables (i.e. get v on one side, x on the other):

so $\frac{dv}{v} = -k.dx$ *Finally v and x! Nearly there.*

Remember that $\frac{d(\ln y)}{dy} = \frac{1}{y}$ *so* $\frac{dv}{v} = d(\ln v)$

Physclips has an introduction to calculus

$d(\ln v) = -k\,dx$ *finally it looks easy: just integrate both sides!*

$\ln v = -\int k\,dx = -kx + \text{const}$ *What constant?Once again, use initial conditions*

at $x = 0$, $v = v_0$

$\ln v_0 = -kx + \text{const} = -k*0 + \text{const}$ *subtract these and remember ln (a/b) = ln a – ln b*

$\ln v - \ln v_0 = \ln\frac{v}{v_0} = -kx$

so $k = -\ln(v/v_0)/x$ (It's also interesting that $v = v_0 e^{-kx}$)

Physclips does this and more in detail

www.animations.physics.unsw.edu.au/jw/car-physics.htm

https://www.youtube.com/watch?v=3VnEGnRh28U

Vectors

have direction and magnitude

e.g. displacement, velocity, acceleration, force, spin, electric field are all vectors

displacement = 2 m towards door; wind velocity is 3.7 ms^{-1} at 31° E. of N., acceleration is – 9.8 ms^{-2} up, Force is 4 N in +ve x direction etc

(*cf* Scalars: mass, length, heat, temperature..., which also have magnitude, but no direction)

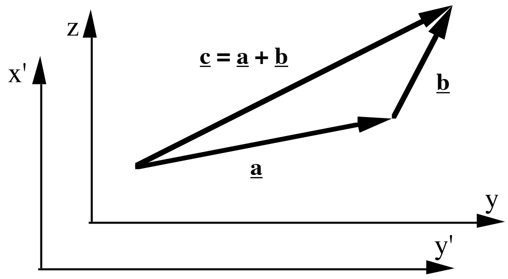
Notation: **a** in most texts

$\underset{\sim}{a}$ when hand writing

<u>**a**</u> often in my notes, to combine the above

$\vec{a}$ in Halliday, Resnick and Walker

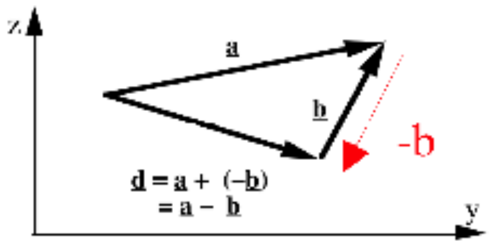
Addition



put them head to tail to add.

Does <u>**a**</u> + <u>**b**</u> = <u>**b**</u> + <u>**a**</u> *?*

Subtraction



Think: (<u>**a**</u> – <u>**b**</u>) *is what I have to add* <u>**b**</u> *to in order to get* <u>**a**</u>.

to subtract, rewrite the equation:

<u>**a**</u> – <u>**b**</u> = <u>**a**</u> + (– <u>**b**</u>)

head to head to subtract vectors

or <u>**a**</u> – <u>**b**</u> = <u>**d**</u> –> <u>**a**</u> = <u>**d**</u> + <u>**b**</u>

See Vectors, a supporting page in Physclips

Consider the displacement 2 m North

Consider adding 2 m North and 2 m East

What sort of quantity is 2 m North ?

What does 2 m mean? What does North mean?

Should/could I write 2 m <u>**North**</u> ?

How big is North?

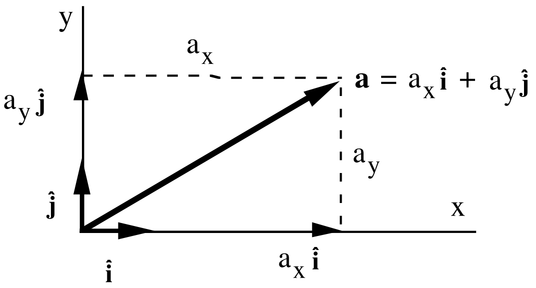
magnitude direction

(2 m) distance (in the direction of North)

We have a word for "in the direction of North" (It's just "North").

Wouldn't it be useful to have a shorthand for "in the positive x direction"? So let's introduce

Vector components and unit vectors



$a_x = a \cos \theta$, $a_y = a \sin \theta$

a_x is the **component** of **a** in the x direction – it is a *scalar*

$\mathbf{\hat{i}}$ is **unit vector**: magnitude of 1 in x direction. $\mathbf{\hat{i}}$ means "in the positive x direction"

$\mathbf{a} = a_x \mathbf{\hat{i}} + a_y \mathbf{\hat{j}}$ *i.e. means a_x in the positive x direction plus a_y in the positive y direction*

$a = \sqrt{a_x^2 + a_y^2}$ $\theta = \tan^{-1} \frac{a_y}{a_x}$ *so we can convert back to magnitude and direction*

Addition by components

$\underline{\mathbf{c}} = \underline{\mathbf{a}} + \underline{\mathbf{b}} = (a_x \mathbf{\hat{i}} + a_y \mathbf{\hat{j}}) + (b_x \mathbf{\hat{i}} + b_y \mathbf{\hat{j}})$

$c_x \mathbf{\hat{i}} + c_y \underline{\mathbf{j}} = (a_x + b_x) \mathbf{\hat{i}} + (a_y + b_y) \mathbf{\hat{j}}$

$c_x = a_x + b_x$ and $c_y = a_y + b_y$ *so we have two scalar equations from one vector one*

vector eqn in n dimensions –> n independent algebraic eqns.

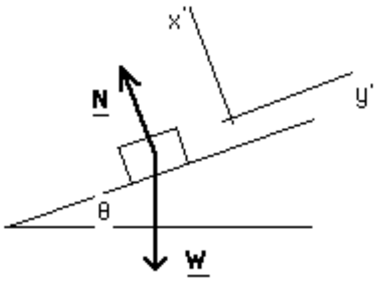
important case: $0 = 0 \mathbf{\hat{i}} + 0 \mathbf{\hat{j}}$

$\therefore$ if $\underline{\mathbf{a}} + \underline{\mathbf{b}} = 0$, (e.g. mechanical equilibrium) *more on this later in Newton's laws.*

$a_x + b_x = 0$ ***and*** $a_y + b_y = 0$

Resolving vectors is often useful. *more on this later in Newton's laws*

Choose convenient axes:



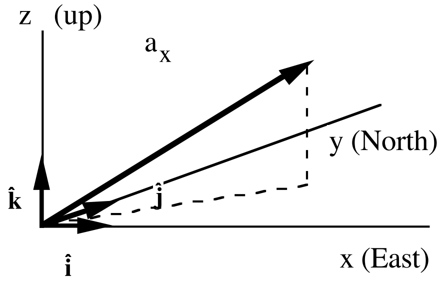
Component of $\underline{\mathbf{W}}$ in direction of plane = - Wsinθ

Total force in y' direction = - Wsinθ $\left(= m \frac{d^2y'}{dt^2}\right)$

Newton's second law requires that components in normal direction add to zero:

$N – W\cos\theta = 0$

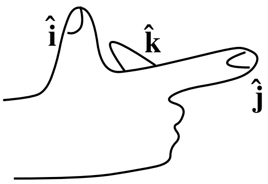
In three dimensions:



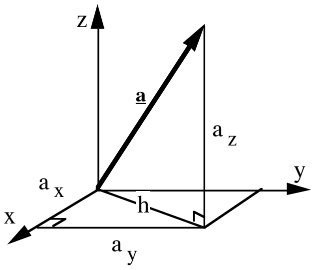
$\underline{\mathbf{r}} \;=\; r_x\,\hat{\mathbf{i}} + r_y\,\hat{\mathbf{j}} + r_z\,\hat{\mathbf{k}}$ (*sometimes just* **i, j, k**)

right hand convention:

$\hat{\mathbf{i}}$	$\hat{\mathbf{j}}$	$\hat{\mathbf{k}}$	in dir^n of
thumb	index	middle	fingers of right hand



Pythagoras' theorem in three dimensions



What is magnitude of $\underline{\mathbf{a}}$?

Hypotenuse h:

$h^2 = a_x{}^2 + a_y{}^2.$

Now look at triangle h,a_z,a:

$a^2 = h^2 + a_z{}^2$

$= a_x{}^2 + a_y{}^2 + a_z{}^2.$

$a = \sqrt{a_x{}^2 + a_y{}^2 + a_z{}^2}$

Not in our syllabus, but, for your interest, in four dimensions:

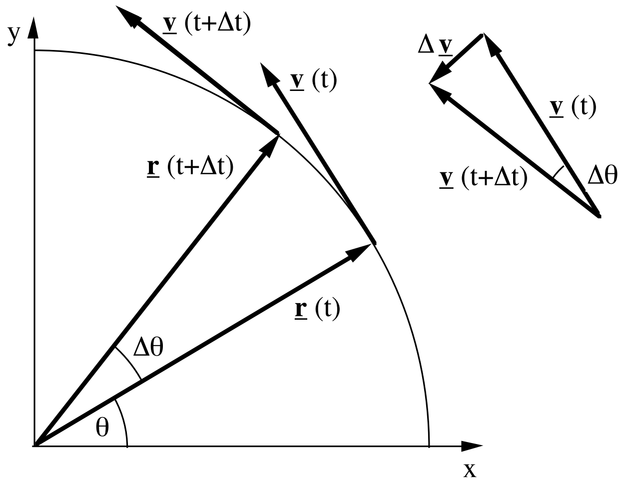
we write $j = \sqrt{-1}$ and use c as the natural unit of velocity.

Two events at $(e_{x1},e_{y1},e_{z1},jct_1)$ and $(e_{x2},e_{y2},e_{z2},jct_2)$

are separated by $\sqrt{(e_{x2}-e_{x1})^2+(e_{y2}-e_{y1})^2+(e_{z2}-e_{z1})^2-(ct_2-ct_1)^2}$

Uniform circular motion

Write $\theta = \omega t$. where ω = constant ω is the **angular velocity**



As Δt and Δθ –> 0, triangles are very narrow so $\Delta\underline{\mathbf{v}}$ –> right angles to $\underline{\mathbf{v}}$

$\therefore \quad \underline{\mathbf{a}} = \lim_{\Delta t \to 0}\left(\frac{\Delta \mathbf{v}}{\Delta t}\right)$ From the triangle, we see it is parallel to $-\underline{\mathbf{r}}$, i.e. towards centre. We call it centripetal acceleration (and centripetal acceleration = –1*radial acceleration). We need magnitudes, so

$\Delta s = r\Delta\theta$ *definition of angle*

$v = \frac{ds}{dt}$ *defininition of speed*

$= r\frac{d\theta}{dt} = r\omega$ Use arc ≅ straight line of triangle, here for $\Delta\underline{\mathbf{v}}$:

$|\Delta\underline{\mathbf{v}}| \cong |v\Delta\theta|$ (*Interesting: notice that* $|\Delta\underline{\mathbf{v}}| \neq \Delta|\underline{\mathbf{v}}|$)

$\lim_{\Delta t \to 0} |d\underline{\mathbf{v}}| = |v d\theta|$

$|\underline{\mathbf{a}}| = \frac{|d\underline{\mathbf{v}}|}{dt}$

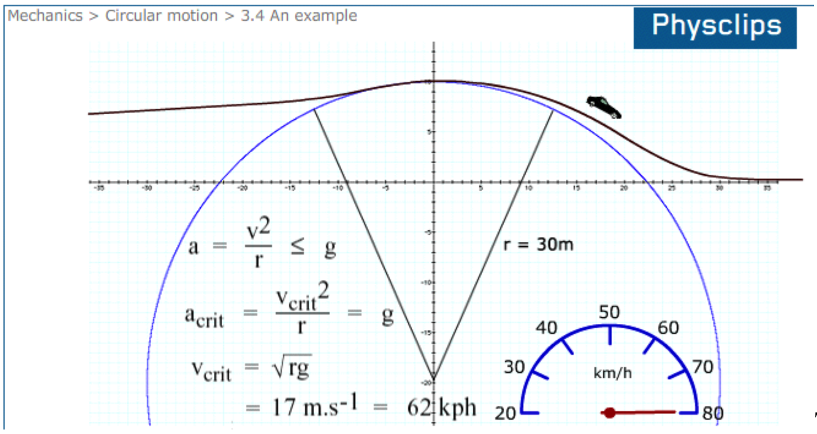
$= v\frac{d\theta}{dt} = v\omega$

$a = \frac{v^2}{r} = \omega^2 r$ but $\underline{\mathbf{a}}$ // $-\underline{\mathbf{r}}$ *(// means 'is parallel to'.)*

so $\underline{\mathbf{a}} = -\omega^2\underline{\mathbf{r}}$

Example. Car travelling at v goes over hill with vertical radius r = 30 m (>> height of car) at summit. It doesn't slow down. How high must v be for the car to lose contact with the ground at the summit?

The only force pulling the car down is gravity, so the downwards acceleration cannot be greater than g. So the centripetal acceleration must be less than or equal to g (or equal to g for the critical value).

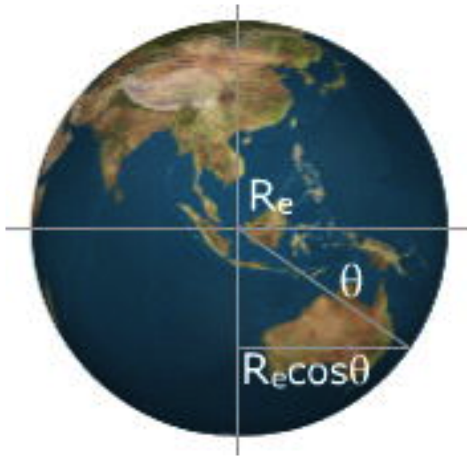


This example on Physclips

Example

What is the acceleration of this theatre?

Due to rotation of the Earth:



$$a_{rot} = \omega^2 r$$

$$= \left(\frac{2\pi}{T}\right)^2 r$$

(T_{Earth} ~ 24hrs, $r_{Sydney\ to\ Earth's\ axis}$ ~ 5300 km)

$$= \left(\frac{2\pi}{23.9*3600\ s}\right)^2 (5.3\ 10^6\ m)$$

$$= 28\ mm.s^{-2}$$

Due to Earth's orbit around the sun:

$$a_{orb} = \left(\frac{2\pi}{T}\right)^2 r \qquad (T_{orbit}, r_{orbit})$$

$$= \left(\frac{2\pi}{365.24*24*3600\ s}\right)^2 (1.5\ 10^{11}\ m)$$

$$= 6\ mm.s^{-2}$$ *So, even though the earth moves rapidly, we cannot feel this motion: it's a tiny acceleration*

Due to Sun's orbit around the centre of the galaxy:

$r \sim 2\ 10^{20}$ m, $T \sim 10^{16}$ s

$a_{galactic} \sim 0.1\ nm.s^{-2}$

Example

$$\mathbf{r} = (A.\sin\omega t)\,\hat{\mathbf{i}} + (A.\cos\omega t)\,\hat{\mathbf{j}} + (Bt)\,\hat{\mathbf{k}}$$

where A and B are constants.

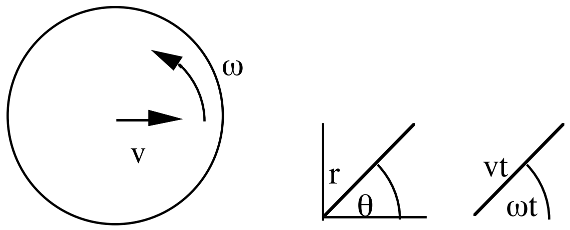
What shape is **r** ? What is **a** ?

$$\mathbf{a} = \frac{d^2}{dt^2}\mathbf{r} = -(A\omega^2.\sin\omega t)\,\hat{\mathbf{i}} - (A\omega^2.\cos\omega t)\,\hat{\mathbf{j}}$$

so **a** is in xy plane and

$$a = \sqrt{a_x^2 + a_y^2} = ... = A\omega^2 = \text{constant}$$ (but direction is always changing)

Example A cockroach on a turntable crawls in the radial direction (initially the x direction) at speed v. The turntable rotates at ω anticlockwise. Describe his path in **i**, **j** coordinates.



We can specify his position with the coordinates (r,θ), where r = vt, and θ = ωt.

$x = r\cos\theta,\ \ y = r\sin\theta$

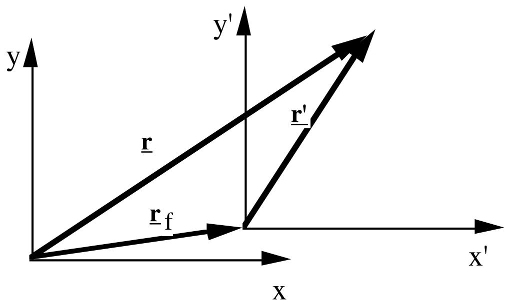
path is $\qquad \underline{\mathbf{r}}(t) = vt\cos\omega t\,\underline{\mathbf{i}} + vt\sin\omega t\,\underline{\mathbf{j}}$

What is this shape?

Relative velocities

(Gallilean/Newtonian relativity *watch for hidden assumptions*)

origin of frame (x',y') is at $\underline{\mathbf{r}}_f$ and moves with $\underline{\mathbf{v}}_f$ with respect to the (x,y) frame. No rotation.



Subraction of vectors (see above): Directly from the geometry, we write

$\underline{\mathbf{r}}' = \underline{\mathbf{r}} - \underline{\mathbf{r}}_f$ and, by definition, $\underline{\mathbf{v}}' = \frac{d}{dt}\underline{\mathbf{r}}'$

So we differentiate both sides to give

$$\frac{d}{dt}\underline{\mathbf{r}}' = \frac{d}{dt}\underline{\mathbf{r}} - \frac{d}{dt}\underline{\mathbf{r}}_f$$

$$\underline{\mathbf{v}}' = \underline{\mathbf{v}} - \underline{\mathbf{v}}_f$$

(*Tricky? See P2P chapter 2*)

It's important to understand this.

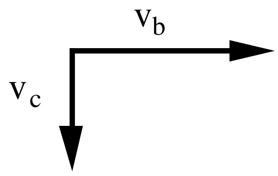
Think of this example:

velocity of wind over the ground = velocity of wind with respect to me + my velocity over the ground

or

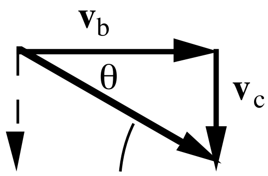
velocity of boat over the ground = velocity of river over ground plus velocity of boat over river

Example A boat heads East at 8 km.hr^{-1}. The current flows South at 6 km.hr^{-1}. What is the boat's velocity relative to the earth?



$\mathbf{v} = \mathbf{v}_{boat} + \mathbf{v}_{current}$

To add the vectors, draw them head-to-tail.



$\mathbf{v} = \mathbf{v}_b + \mathbf{v}_c$

magnitude: $v = \sqrt{v_b^2 + v_c^2}$

$= \sqrt{(8 \text{ km.hr}^{-1})^2 + (6 \text{ km.hr}^{-1})^2}$

$= \sqrt{(8^2 + 6^2)\ (\text{km.hr}^{-1})^2}$

$= \sqrt{(64 + 36)}\ \sqrt{(\text{km.hr}^{-1})^2}$

$= 10 \text{ km.hr}^{-1}$

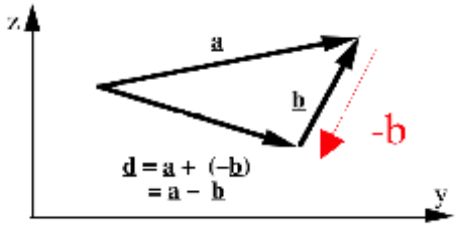
direction: $\theta = \tan^{-1} \dfrac{6 \text{ km.hr}^{-1}}{8 \text{ km.hr}^{-1}} = \tan^{-1} 0.75$

$= 37°$

Answer: 10 km.hr^{-1} at 40° South of East

Reminder about vector subtraction

Draw the vectors head to head to subtract them.



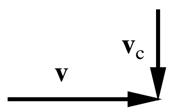
to subtract, rewrite the equation:

$\underline{\mathbf{a}} - \underline{\mathbf{b}} = \underline{\mathbf{a}} + (-\underline{\mathbf{b}})$

i.e. we can also rearrange subtraction so that it becomes addition

Example A sailor wants to travel East at

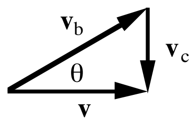
8 km.hr^{-1}. The current flows South at 6 km.hr^{-1}. What direction must she head, and what speed should she make relative to the water?



$\mathbf{v} = \mathbf{v}_{boat} + \mathbf{v}_{current}$

$\mathbf{v}_{boat} = \mathbf{v} - \mathbf{v}_{current}$

To subtract the vectors, draw them head-to-head.



magnitude: $v_b = \sqrt{v^2 + v_c{}^2}$

$= \sqrt{(8 \text{ km.hr}^{-1})^2 + (6 \text{ km.hr}^{-1})^2}$

$= 10 \text{ km.hr}^{-1}$

direction: $\theta = \tan^{-1}\frac{v_c}{v} = \tan^{-1}\frac{6 \text{ km.hr}^{-1}}{8 \text{ km.hr}^{-1}}$

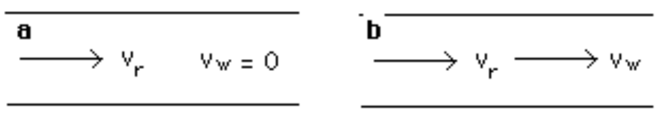
$= 37°$

She must head 40° North of East and travel at 10 km.hr^{-1} with respect to the water.

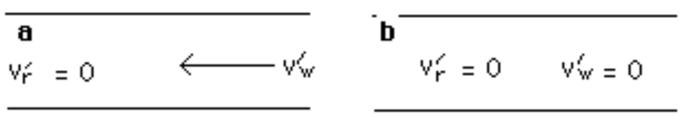
Puzzle River flows East at 10 km/hr. Sailing boat travels East down the river.

Can the boat travel faster

a) with no wind?
b) with 10 km/hr wind from West?



Consider motion *relative to the water*.

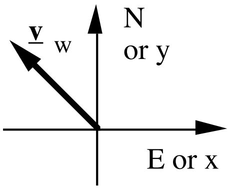


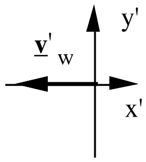
b) $v'_w = 0$ while a) 10 km/hr headwind.

Not in syllabus. See Physclips Physics of Sailing if interested.

A SE wind blows at 30 km/hr. If you are travelling North, how fast must you travel before the wind is coming (i) exactly from your right? (ii) From 30° E of N?

From the ground: From your frame:





or

Relative motion: (*add vel wrt me to my vel to get vel wrt ground*)

$\mathbf{v}_W = \mathbf{v}_{you} + \mathbf{v}'_W$ so $\mathbf{v}'_W = \mathbf{v}_W - \mathbf{v}_{you}$

Either) Do it algebraically. Given:

$\mathbf{v}_W = -\, v_W \cos 45°\, \mathbf{i} + v_W \sin 45°\, \mathbf{j}$

$\mathbf{v}_{you} = 0\, \mathbf{i} + v_{you}\, \mathbf{j}$

$\mathbf{v}'_W = v'_{Wx}\, \mathbf{i} + 0\, \mathbf{j}$

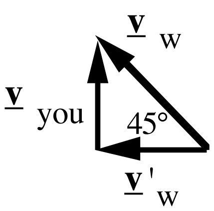
y direction: $v_W \sin 45° = v_{you} + 0$

$v_{you} = v_W \sin 45° = 21$ km/hr

x direction: $-\, v_W \cos 45° = 0 + v'_{Wx}$

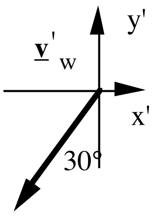
$v'_{Wx} = -\, v_W \cos 45° = 21$ km/hr from E

Or) Do it with vector diagrams:



Same answers, but directly.

Second case: $\mathbf{v}'_W$ is 30° E of N



x direction: $v_W \cos 45° = v'_W \cos 60°$.

y direction: $v_{you} = v_W \sin 45° + v'_W \sin 60°$

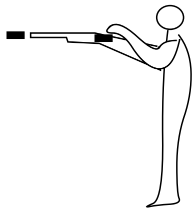
2 equations, 2 unknowns, first gives

$$v'_W = \frac{v_W \cos 45°}{\cos 60°} = 42 \text{ km/hr}$$

Substitute this in second equation:

$v_{you} = 60$ km/hr

Projectiles



Question. A man fires a gun horizontally. At the same time he drops a bullet. Which hits the ground first? ***Explain your reasoning.***

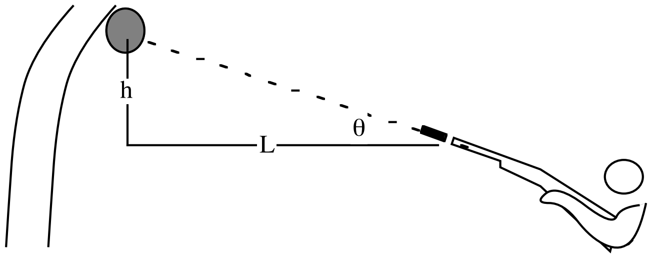
(To simplify, let's say it happens on the moon.)

F = m**a**. So a force in y direction does not cause an acceleration in x direction.

Independence of x and y motion.

(Careful: in air, there are other forces)

Question. A man shoots at a coconut. At the instant that he fires, the coconut falls. What happens?



obvious method (c for coconut, b for bullet). *From the triangle:*

$h = L \tan\theta$ *Now write down the heights of both*

$y_c = h - \frac{1}{2}gt^2 \qquad y_b = v_{yo}t - \frac{1}{2}gt^2$ *Will they miss? What is the difference between the heights?*

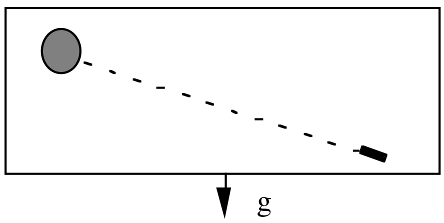
$y_c - y_b = h - v_{yo}t = h - v_o \sin\theta . t$ *The bullet has no horizontal acceleration, so*

$L = t\, v_o \cos\theta$ which gives $T = \dfrac{L}{v_o \cos\theta}$ *substitute gives*

$y_c - y_b = h - \dfrac{v_o \sin\theta . L}{v_o \cos\theta} = h - L\tan\theta$ *and from the geometry above*

$= 0$

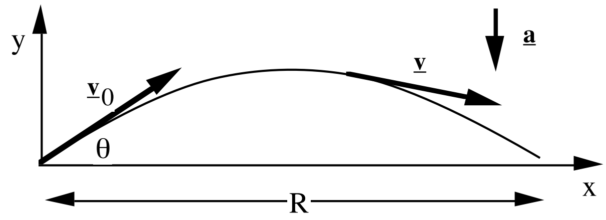
Alternatively: consider a frame of reference falling with g.



The non-vegetarian version of this problem is very old one, called The Monkey and the Hunter. Physclips and P2P have film clips, using a toy monkey. No real monkeys were etc.

Projectiles

Without air, $a_y = -g =$ constant. $a_x = 0$ (Galileo: independence of horizontal & vertical motion)



Strategy: kinematics eqn (ii) gives y(t), x(t):

(ii) –> $y = y_o + v_{yo}t - \frac{1}{2} gt^2.$ $x = x_o + v_x t.$

Our strategy: *Eliminate t gives y(x) Important: y(x) is the trajectory or path it follows.*

For range, it hits the ground at x = R, so y(R) = 0

Rearrange to get R = R(θ)

Find maximum in R(θ)

(ii) –> $y = y_o + v_{yo}t - \frac{1}{2} gt^2.$ *motion with constant acceleration*

(ii) –> $x = x_o + v_x t.$ *motion with no acceleration*

Choose axes so that $x_o = y_o = 0$ and use (ii) to eliminate t (i.e. substitute $t = x/v_x$):

$$y = v_{yo}\left(\frac{x}{v_x}\right) - \frac{1}{2} g \left(\frac{x}{v_x}\right)^2 \qquad (*)$$

Now we have parabolic curve in x, not t.

y = 0 when x = 0 (launching) or when x = R (landing)

$$(*) \;\to\; v_{yo}\left(\frac{R}{v_x}\right) = \frac{1}{2} g\left(\frac{R}{v_x}\right)^2 \qquad (**)$$

$v_{yo} = v_o \sin\theta, \quad v_x = v_o \cos\theta \quad \to \quad R = R(v_o,\theta)$

How to find the maximum of a function? Set $\frac{\partial R}{\partial \theta} = 0$ to obtain θ for maximum range.

We had $v_{yo}\left(\frac{R}{v_x}\right) = \frac{1}{2} g\left(\frac{R}{v_x}\right)^2$ (**) solve (**) for R:

so $R = \dfrac{2v_x v_{yo}}{g}$ (*check units*) *substitute gives:*

$$= \frac{2v_0 \sin\theta . v_0 \cos\theta}{g}$$

sin θ cos θ = … standard trigonometic identity

$$= \frac{v_0^2 \sin 2\theta}{g}$$

$$\frac{\partial R}{\partial \theta} = \frac{2v_0^2 \cos 2\theta}{g}$$

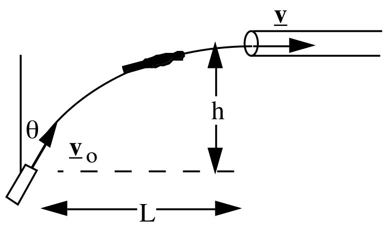
So maximum R when $2\theta = 90°$ i.e $\theta = 45°$

Example The human cannon of Circus Oz has a muzzle velocity v_o. For a new trick, they will fire the human canonball into a horizontal teflon tube at height h above the canon mouth. To avoid damage to the canonball, he must arrive with purely horizontal velocity. Calculate the position of the canon and its angle to the vertical.

i) *draw a diagram*

ii) *put in symbols for quantities*

iii) *translate question*



given h, v_o and final $v_y = 0$

Find L and θ Think about this: chose θ to get highest point correct, then choose L so as to hit the target.

Relate h, v_y and v_{yo}. v_{yo} depends on θ.

During flight, the acceleration is –g upwards. The desired v_y is zero, so (*think* $v^2 = u^2 + 2a\Delta s$)

$$0 = v_y^2 = v_{yo}^2 + 2a_y(\Delta y) = v_o^2 \cos^2\theta - 2gh$$

$$\therefore \quad v_o^2 \cos^2\theta = 2gh$$

$$\therefore \quad \cos\theta = \frac{\sqrt{2gh}}{v_o}$$

$$\theta = \cos^{-1}\left(\frac{\sqrt{2gh}}{v_o}\right)$$

Find time of flight. $v_x t = L$.

$$0 = v_y = v_{yo} + a_y t$$

$$\therefore \quad t = \frac{v_o \cos\theta}{g}$$

$$L = v_o\, t \sin\theta.$$

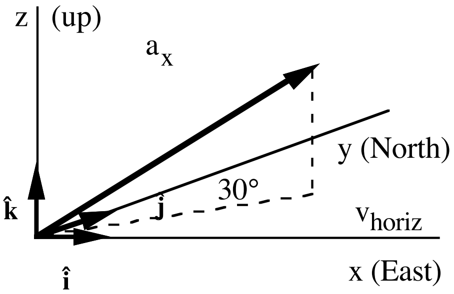
$$= v_o \sin\theta \, \frac{v_o \cos\theta}{g}$$

$$\left(\textit{<optional>} = \frac{v_o^2}{2g} \sin 2\theta \right)$$

$$\text{where } \theta = \cos^{-1}\left(\frac{\sqrt{2gh}}{v_o}\right)$$ *simplify optional*

Projectile in 3D

Example. Throw object from origin at 20 ms^{-1}, 30° East and at 45° to horizontal. Take x = East, y = North, describe its motion as a function of t in $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$ notation.



$v_{zo} = v_o \sin 45° = 14\ ms^{-1}$

$v_{horiz} = v_o \cos 45° = 14\ ms^{-1}$ (= const)

$v_x = v_{horiz} \cos 60° = 7\ ms^{-1}$ (= const)

$v_y = v_{horiz} \cos 30° = 12\ ms^{-1}$ (= const)

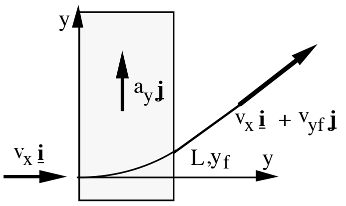
$$\underline{\mathbf{r}} = r_x\,\hat{\mathbf{i}} + r_y\,\hat{\mathbf{j}} + r_z\,\hat{\mathbf{k}}$$

$$= (7\ ms^{-1}).t\,\hat{\mathbf{i}} + (12\ ms^{-1}).t\,\hat{\mathbf{j}}$$

$$+ \left((14\ ms^{-1})t - \tfrac{1}{2}(9.8\ ms^{-2})t^2\right)\hat{\mathbf{k}}$$

But normally we should just make it a 2D problem.

Example. Electron enters a uniform electric field at (x,y,t)=(0,0,0), with velocity $v_x\mathbf{i}$. The field extends from x = 0 to x = L, but is zero for x < 0 and x > L. In the field, the electron is accelerated at $a_y\mathbf{j}$. Write an equation for the position of the electron for x > L. Hint: divide the problem into parts



Outer parts with linear motion, middle one with projectile motion

First region

$x \le 0$ $(t \le 0)$ $\qquad \mathbf{v} = v_x\mathbf{i} = \text{const}$ *easy*

Second region

$0 \le x \le L$ $(0 \le t \le L/v_x)$ *constant acceleration in the y direction. We need final position & velocity*

$$x = x_0 + v_{x0}t + \frac{1}{2}a_xt^2 \qquad v_x = v_{x0} + a_xt$$

grey terms are zero here.

$$y = y_0 + v_{y0}t + \frac{1}{2}a_yt^2 \qquad v_y = v_{y0} + a_yt$$

$$y_f = \frac{1}{2}a_yt_f^2 = \frac{1}{2}a_y\left(\frac{L}{v_x}\right)^2 \qquad v_{yf} = a_y\left(\frac{L}{v_x}\right)$$

Third region

$x \ge L \qquad x = v_xt$ (*still no acceleration in x direction*)

$$y = y_f + v_{yf}\left(t - \left(\frac{L}{v_x}\right)\right)$$

now all we need to do is to substitute for v_{yf}:

position is $v_xt\,\mathbf{i} + \left(\frac{1}{2}a_y\left(\frac{L}{v_x}\right)^2 + a_y\left(\frac{L}{v_x}\right)\left(t - \left(\frac{L}{v_x}\right)\right)\right)\mathbf{j}$