THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Semester 2 2014

MATH1131
MATHEMATICS 1A

(1) TIME ALLOWED – 2 hours

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

(8) A SHORT TABLE OF INTEGRALS IS SUPPLIED AT THE END OF THE PAPER

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

Use a SEPARATE book clearly marked Question 1

1. i) For each of the following, find the limit, if it exists:

a) $\displaystyle\lim_{x\to\infty} \frac{6x^2 - 2x + \sin x}{2x^2 + x + \cos x}$.

b) $\displaystyle\lim_{x\to 1} \left|\frac{x^2 - 5x + 4}{x - 1}\right|$,

c) $\displaystyle\lim_{x\to 3} \frac{e^{x^2} - e^9}{\sin(x-3)}$,

ii) Does the following improper integral converge? If so, find its value. If not, show that it diverges.

$$\int_e^\infty \frac{dx}{x + \ln x}.$$

iii) Use logarithmic differentiation to find the derivative $\dfrac{dy}{dx}$ of

$$y = 2^{\sin x}.$$

iv) Sketch the polar curve

$$r = 3 + 3\sin\theta.$$

You should show any lines of symmetry, and indicate any points where the curve intersects the x and y axes.

v) a) State the Intermediate Value Theorem.

b) Show that the equation

$$e^x = x + 2$$

has a solution in the interval $[0, 2]$.

c) Find the equation of the tangent to $f(x) = e^x$ at $x = 0$ and hence, or otherwise, write down the values of a for which the equation

$$e^x = x + a$$

has at least one solution.

Use a SEPARATE book clearly marked Question 2

2. i) Find

$$\int_0^{\cosh^{-1}(2)} \cosh^3 x \sinh x \, dx.$$

Express your answer as simply as possible.

ii) Find the integral

$$\int x \sin(2x)\, dx.$$

iii) Use the First Fundamental Theorem of Calculus to find

$$\frac{d}{dx}\left[\int_x^{x^2} \ln(1+t^2)\, dt\right].$$

iv) A spherical balloon of radius r is being inflated at a rate of 5cm^3/sec. At what rate is the radius of the sphere increasing when $r = 3$ cm?

v) Consider the function

$$f(x) = x - \frac{1}{x}$$

defined on the interval $(1, \infty)$.

a) Show that f is an increasing function.

b) Let g be the inverse function of f. What is the domain of g?

c) Find $g(\frac{3}{2})$ and $g'(\frac{3}{2})$.

vi) a) State the Mean Value Theorem, including any assumptions required.

b) By applying the Mean Value Theorem to $f(t) = \tan^{-1} t$ on the interval $[0, x], x \geq 0$, prove that

$$\tan^{-1} x \leq x,$$

for $x \geq 0$.

Use a SEPARATE book clearly marked Question 3

3. i) Write down a parametric vector equation for the line in $\mathbb{R}^3$ passing through the points $A = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

ii) A plane Π in $\mathbb{R}^3$ has a parametric vector equation given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \lambda, \mu \in \mathbb{R}.$$

a) Write down a point on the plane.

b) Find a vector $\mathbf{v}$ parallel to the plane which is **not** a scalar multiple of $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

c) Evaluate $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

d) Hence, or otherwise, find the Cartesian equation for the plane Π.

iii) Let $z = 1 + i$.

a) Find $|z|$.

b) Find the argument of z.

c) Hence, or otherwise, show that $1 + i$ is a twentieth root of $-(2^{10})$.

iv) Let S be the region in the Argand plane defined by

$$S = \left\{ x \in \mathbb{C} : 0 \leq \operatorname{Arg}(z - (1 + i)) \leq \frac{\pi}{2} \right\}.$$

Sketch the region S carefully, labelling any relevant information.

v) Given that $z^5 - 1 = (z - 1)(z^4 + z^3 + z^2 + z + 1)$ find, in polar form, all solutions to the equation $z^4 + z^3 + z^2 + z + 1 = 0$.

vi) Suppose that w and z are non-zero complex numbers such that

$$|w - z| = |w + z|.$$

Prove that $\dfrac{w}{z}$ is purely imaginary, that is, has real part zero.

Use a SEPARATE book clearly marked Question 4

4. i) Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 3 & 6 & 11 \end{pmatrix}$.

a) Calculate $\det(A)$.

b) Is A an invertible matrix? Give a reason for your answer.

ii) Let $C = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & -4 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 1 & 5 \\ 0 & 4 & 6 \end{pmatrix}$

a) Evaluate CD^T.

b) Without calculating the matrix, explain why the product $C^T D$ exists and write down the size of this matrix.

iii) The current ages of Xena, Yenny and Zac are x, y and z years respectively. You are given that 48 years ago Zac's age was triple that of Yenny's age at that time.

a) Explain why $3y - z = 96$.

b) It is also known that the sum of their three current ages is 200 and the currently Xena's age is the sum of Yenna's and Zac's ages.
Set up a system of 3 equations in the three unknowns x, y and z.

c) By reducing your system to echelon form and back substituting, find the current ages of the Xena, Yenny and Zac.

iv) The following MAPLE output carries out some matrix operations on two 4×4 matrices A and B.

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> with(LinearAlgebra):

> A.B;
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$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{bmatrix}$$

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> B.A;
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$$\begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

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> A^2;
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$$\begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

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> B^2;
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$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Use the MAPLE Session above to write down the matrix $(A - B)^2$.

v) A system of three equations in three unknowns has been reduced to the following echelon form:

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & (3-\beta) & 6 & 4 \\ 0 & 0 & (\beta-1)(\beta-2) & (\beta-1) \end{array}\right).$$

For which value(s) of β will the system have no solution? Give reasons for your answer.

vi) a) Suppose $\mathbf{v}$ and $\mathbf{n}$ are non-zero vectors in $\mathbb{R}^3$. The projection of $\mathbf{v}$ onto $\mathbf{n}$ is given by

$$\text{proj}_{\mathbf{n}}\mathbf{v} = \left(\frac{\mathbf{v}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n}.$$

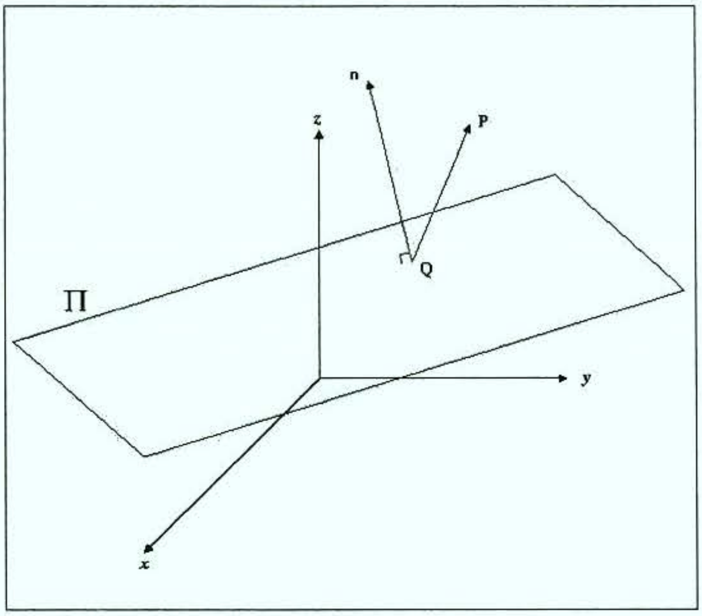
Show that

$$|\text{proj}_{\mathbf{n}}\mathbf{v}| = \frac{|\mathbf{v}\cdot\mathbf{n}|}{|\mathbf{n}|}.$$

b) Suppose that Π is a plane in $\mathbb{R}^3$ with Cartesian equation $Ax + By + Cz = D$.

The vector $\mathbf{n}$ is normal to Π, the point $Q = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ lies on Π and $P = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ is a point in space not on the plane.

Copy the diagram below into your exam booklet and indicate clearly on your diagram the vector $\text{proj}_{\mathbf{n}}\, \overrightarrow{QP}$.



c) Using parts (a) and (b), prove that the shortest distance d between the point P and the plane Π is given by

$$d = \frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}}.$$

BASIC INTEGRALS

$$\int \frac{1}{x}\,dx = \ln|x| + C = \ln|kx|, \qquad C = \ln k$$
$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$
$$\int a^x\,dx = \frac{1}{\ln a}a^x + C, \qquad a \neq 1$$
$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$
$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$$
$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$$
$$\int \operatorname{cosec}^2 ax\,dx = -\frac{1}{a}\cot ax + C$$
$$\int \tan ax\,dx = \frac{1}{a}\ln|\sec ax| + C$$
$$\int \cot ax\,dx = \frac{1}{a}\ln|\sin ax| + C$$
$$\int \sec ax\,dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C$$
$$\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C$$
$$\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C$$
$$\int \operatorname{sech}^2 ax\,dx = \frac{1}{a}\tanh ax + C$$
$$\int \operatorname{cosech}^2 ax\,dx = -\frac{1}{a}\coth ax + C$$
$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$
$$\begin{aligned}\int \frac{dx}{a^2-x^2} &= \frac{1}{a}\tanh^{-1}\frac{x}{a} + C, \qquad |x| < a\\ &= \frac{1}{a}\coth^{-1}\frac{x}{a} + C, \qquad |x| > a > 0\\ &= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C, \qquad x^2 \neq a^2\end{aligned}$$
$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$$
$$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\frac{x}{a} + C$$
$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} + C, \qquad x \geqslant a > 0$$