Chapter 3: Complex Numbers

Let's talk about numbers!

First consider the natural numbers, $\mathbb{N} = 0, 1, 2....$

Solve

$$x - 3 = 2, \quad x \in \mathbb{N}.$$

Solution:

$$x = 5.$$

What about

$$x + 3 = 2, \quad x \in \mathbb{N}?$$

No solution!

We can resolve this by considering the integers $\mathbb{Z} = ... - 2, -1, 0, 1, 2...$

$$x + 3 = 2, \quad x \in \mathbb{Z}.$$

has the solution

$$x = -1.$$

Similarly, the equation

$$2 \cdot x = 3, \quad x \in \mathbb{Z}$$

has no solution , but we can introduce a solution by considering $\mathbb{Q}$, the set of **rational numbers** or ratios of integers.

$$2 \cdot x = 3, \quad x \in \mathbb{Q}$$

has the solution

$$x = \frac{3}{2}.$$

Next, we can look at the equation

$$x^2 = 2, \quad x \in \mathbb{Q},$$

which still has no solution! However, we can consider the set $\mathbb{R}$ of all **real numbers**, and

$$x^2 = 2, \quad x \in \mathbb{R},$$

has the solution

$$x = \pm\sqrt{2}.$$

We thus have a hierarchy of types of numbers

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R},$$

and each time we expand our numerical universe, we can solve more types of equations.

What about

$$x^2 + 1 = 0?$$

This still has no solution, even in $\mathbb{R}$.

Enter the complex numbers!

We will introduce a symbol i which plays the role of "$\sqrt{-1}$", so that

$$i^2 = -1.$$

It turns out that this is the end of the road, after this we will be able to solve **any** polynomial equation.

Definition

The set of all numbers of the form $a + bi$ where a, b are real numbers and $i^2 = -1$ is called the set of all **complex numbers** and denoted by $\mathbb{C}$.

We do arithmetic treating i as an ordinary number, except satisfying $i^2 = -1$.

Rules for arithmetic

Let $z = a + ib, w = c + id$ be complex numbers. Then

- $z \pm w = (a \pm c) + i(b \pm d)$
- $zw = (ac - bd) + i(ad + bc)$.
- $\dfrac{z}{w} = \dfrac{a + ib}{c + id} \times \dfrac{c - id}{c - id} = \dfrac{(ac + bd) + i(bc - ad)}{c^2 + d^2}$

We use the notation

$$Re(z) = a, \quad Im(z) = b.$$

Ex: For $z = 2 - 4i, w = 3 + i$, find $z + w, \quad z - w, \quad zw, \quad \dfrac{z}{w}$.

Answer:

$$z + w = (2 + 3) + (-4 + 1)i = 5 - 3i$$

$$z + w = (2 - 3) + (-4 - 1)i = -1 - 5i$$

$$zw = (2 \cdot 3 - (-4) \cdot 1) + (2 \cdot 1 + 3 \cdot (-4))i = 10 - 10i$$

$$\frac{z}{w} = \frac{(2 \cdot 3 + (-4) \cdot 1) + ((-4) \cdot 3 - 2 \cdot 1)i}{3^2 + 1^2} = \frac{2 - 14i}{10}.$$

Ex: Simplify $(1+i)^8$.

Answer:

$$(1+i)^8 = ((1+i)^4)^2 = (((1+i)^2)^2)^2$$

$$= ((2i)^2)^2 = (-4)^2 = 16.$$

A set of numbers is a **field** if it is *closed* under addition, subtraction, multiplication, and division (except by 0), meaning you can do all of these arithmetic operations and remain within the set.

Which of the following are fields?

- $\mathbb{N}$ (not closed under subtraction)
- $\mathbb{Z}$ (not closed under division)
- $\mathbb{Q}$ Yes
- $\mathbb{R}$ Yes
- $\mathbb{C}$ Yes

There are also fields that have different arithmetic operations.

Consider the set of numbers $S = \{0, 1, 2, 3, 4\}$. We will define a new addition $+_S$ and multiplication $\cdot_n$ on S as follows:

$$a +_S b = \text{the remainder of } a + b \text{ when dividing by } 5\ .$$

$$a \cdot_S b = \text{the remainder of } a \cdot b \text{ when dividing by } 5\ .$$

These new operations are called addition and multiplication **modulo** 5, and they are commutative and associative. The set S with these operations is called $\mathbb{Z}_5$.

Is $\mathbb{Z}_5$ a field?

We need to check whether subtraction and division make sense!

Equivalently, whether we can solve all equation of the forms

$$x +_S a = b \quad \text{or, for } a \neq 0, \quad x \cdot_S a = b.$$

Here are the addition and multiplication tables.

$+_S$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\cdot_S$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Subtraction is defined because every column contains each number.

Similarly, division is defined since every column (except the column for 0!) contains each number.

Therefore $\mathbb{Z}_5$ is a field.

Something to think about: does the same thing work for $\mathbb{Z}_4$? $\mathbb{Z}_6$? $\mathbb{Z}_7$? $\mathbb{Z}_{51}$?

A complex number α is called an n-th root of unity if $\alpha^n = 1$.

Ex: Find the third roots of unity.

Answer: Write

$$1 = (x + yi)^3 = x^3 - 3y^2x + (3yx^2 - y^3)i.$$

Equating the real and imaginary parts, we get

$$x^3 - 3y^2x = 1 \text{ and } 3yx^2 - y^3 = 0,$$

so either

$$y = 0 \text{ and } x = 1,$$

or

$$x = -\frac{1}{2} \text{ and } y = \pm\frac{\sqrt{3}}{2}.$$

So the third roots of unity are

$$1, \frac{-1 + \sqrt{3}i}{2}, \frac{-1 - \sqrt{3}i}{2}.$$

We can now solve **any** quadratic equation.

Ex: Solve $5x^2 - 4x + 1 = 0$.

Answer: Using the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{(-4)^2 - 4(5)(1)}}{2(5)}$$

$$= \frac{4 \pm \sqrt{-4}}{10} = \frac{4 \pm \sqrt{4}\sqrt{-1}}{10} = \frac{4 \pm 2i}{10} = \frac{2 \pm i}{5}.$$

Ex: Solve $z^2 - 3z + (3+i) = 0$.

Answer: Again start with the quadratic formula,

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(3+i)}}{2} = \frac{3 \pm \sqrt{-3-4i}}{2}$$

However, we still need to find $\sqrt{-3-4i}$!

Write

$$-3 - 4i = (x + yi)^2 = x^2 - y^2 + 2xyi$$

so we have the equations

$$x^2 - y^2 = -3 \text{ and } 2xy = -4.$$

Combining, we get

$$x^2 - \frac{4}{x^2} = -3$$

or

$$(x^2)^2 + 3x^2 - 4 = 0,$$

which has the solution

$$x^2 = \frac{-3 \pm 5}{2}.$$

Since x is real, we get

$$x^2 = 1,$$

So the two square roots are

$$x = \pm 1, \quad y = \mp 2.$$

Therefore the solution to the orginal quadratic is

$$z = \frac{3 \pm \sqrt{-3 - 4i}}{2} = \frac{3 \pm (1 - 2i)}{2} = 2 - i,\ 1 + i.$$

Theorem (Fundamental Theorem of Algebra)

Every polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where the co-efficients $a_0, a_1, \cdots, a_n$ are complex numbers, with $n > 0$ and $a_n \neq 0$, has a **root** *(a solution to $p(x) = 0$.)*

In fact, $p(x)$ can be factored as

$$a_n(x - c_1)(x - c_2)...(x - c_n).$$

The roots are $c_1, c_2, ..., c_n$. (Some roots may appear multiple times in the factorization).

Unfortunately, just because roots exist doesn't mean they are easy to find!

Let's try to solve cubic equations

$$x^3 + px^2 + qx + r = 0.$$

The method consists of two main ideas.

- The equation

  $$x^3 + px^2 + qx + r = 0$$

  is equivalent to an equation with *no quadratic term*

  $$y^3 + ay + b = 0,$$

  by a change of variable $x = y - c$ for a certain constant c.

- Once there is no quadratic term, we can solve by another change of variables, $y = u + d/u$, for a certain constant d.

Step 1: Getting rid of the quadratic term

Consider the equation

$$x^3 + px^2 + qx + r = 0.$$

First let's let

$$x = y - \frac{p}{3}.$$

Then in terms of y the equation becomes

$$(y - \frac{p}{3})^3 + p(y - \frac{p}{3})^2 + q(y - \frac{p}{3}) + r = 0.$$

The coefficient of y^2 is

$$3(-\frac{p}{3}) + p = 0.$$

Therefore, the equation can be written as

$$y^3 + ay + b = 0,$$

for some numbers a and b.

Ex: Change variables to remove the square term in

$$x^3 - 6x^2 + x + 3.$$

Answer: Here $p = -6$, so let

$$x = y - \frac{p}{3} = y + 2.$$

Then

$$x^3 - 6x^2 + x + 3 = (y+2)^3 - 6(y+2)^2 + (y+2) + 3$$

$$= y^3 - 11y - 11.$$

Step 2: Solving a cubic *without* a quadratic term

Now let's solve the simpler cubic

$$x^3 + px + q = 0.$$

Cardano's method: let $x = u + v$, and expand.

$$0 = (u+v)^3 + p(u+v) + q = u^3 + v^3 + (3uv + p)(u+v) + q.$$

We then let

$$uv = -\frac{p}{3},$$

(which is always possible! Why?)

So we're left with

$$u^3 + v^3 = -q$$

We also have:

$$u^3 v^3 = -\frac{p^3}{27}.$$

Ex: Find the real root of $x^3 + 3x = 1$.

Answer: Here $p = 3$ and $q = -1$.

We solve

$$u^3 + v^3 = -q = 1, \quad u^3 v^3 = -\frac{p^3}{27} = -1,$$

to get

$$u^3 - \frac{1}{u^3} = 1, \text{ or } (u^3)^2 - u^3 - 1 = 0,$$

so

$$u^3 = \frac{1 \pm \sqrt{5}}{2}, \text{ and similarly } v^3 = \frac{1 \pm \sqrt{5}}{2}.$$

Since $u^3 v^3 = -1$, it must be that $u \neq v$. So

$$x = u + v = \sqrt[3]{\frac{1+\sqrt{5}}{2}} + \sqrt[3]{\frac{1-\sqrt{5}}{2}}.$$

How would you come up with this idea if you couldn't figure out to solve a cuubic?

Answer: "Steal" it from someone else!

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Tartaglia and Cardano.

We have to be careful with cube roots when trying Cardano's method.

Ex: Solve $x^3 - 6x + 4 = 0$ (note that $x = 2$ is a root).

Answer: Here $p = -6$ and $q = 4$.

We solve

$$u^3 + v^3 = -4, \quad u^3 v^3 = 8,$$

getting

$$(u^3)^2 + 4u^3 + 8 = 0, \quad \text{so } u^3 = -2 \pm 2i,$$

and similarly

$$v^3 = -2 \pm 2i.$$

What's going on here?

The root 2 may be expressed as $(1 + i) + (1 - i)$

and

$$(1 + i)^3 = -2 + 2i, \quad (1 - i)^3 = -2 - 2i.$$

When solving a quadratic equation

$$az^2 + bz + c$$

with real coefficients, if the discriminant $b^2 - 4ac$ is negative, the roots come in pairs

$$x + iy, \quad x - iy,$$

called conjugates.

This leads to the notion of **complex conjugation**, which is a map taking every complex number to its **conjugate**.

$$z \mapsto \overline{z}$$

$$x + iy \mapsto x - iy$$

Complex conjugation has the following properties:

- $\overline{z \pm w} = \overline{z} \pm \overline{w}$
- $\overline{zw} = \overline{z} \cdot \overline{w}$
- $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$.
- $\overline{z} = z$ if and only if z is real.
- $z + \overline{z} = 2Re(z)$.

Note that it follows that

$$\overline{z^n} = (\overline{z})^n.$$

(Why?)

We can think of the complex numbers as being arranged in the Cartesian plane, with the horizontal axis corresponding to the real part and the vertical axis to the imaginary part.

Then the complex number $z = x + yi$ corresponds to the point (x, y) in the plane.

The **modulus** of a complex number $z = x + yi$, written $|z|$ is the distance of the corresponding point in the plane from the origin; by the Pythagorean theorem this is just $\sqrt{x^2 + y^2}$.

If we identify each complex number $z = x + yi$ with the point (x, y) in the plane, then addition of complex numbers is just the same vector addition we saw in Section 1, and the modulus of a complex number is the length of the corresponding vector.

We can completely specify a point in the plane if in addition to its distance from the origin, we know the angle that the line connecting it with the origin it makes with the positive x-axis.

The **argument** of a non-zero complex number $z = x + yi$, written $\mathrm{Arg}(z)$ is the angle that the line connecting (x, y) to $(0, 0)$ makes with the positive x-axs.

We decree that

$$-\pi < \mathrm{Arg}(z) \leq \pi.$$

If $z = x + yi$ with $y > 0$, we can express $\mathrm{Arg}(z)$ as $\cos^{-1} \frac{x}{|z|}$.

Similarly if $y < 0$, $\mathrm{Arg}(z) = -\cos^{-1} \frac{x}{|z|}$.

Together, the modulus and argument form an alternative geoemtric way of specifying a complex number.

Ex: Find the modulus and argument of $z = -1 + i\sqrt{3}$ and $w = 1 - 2i$.

Answer:

$$|z| = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2, \quad |w| = \sqrt{1^2 + (-2)^2} = 5.$$

$$\mathrm{Arg}(z) = \cos^{-1}\frac{-1}{2} = \frac{2\pi}{3}, \quad \mathrm{Arg}(w) = -\cos^{-1}\frac{1}{5}.$$

The modulus function has the following properties:

- $|zw| = |z||w|$
- $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$, provided $w \neq 0$.
- $|z^n| = |z|^n$
- $|z| = 0 \Leftrightarrow z = 0$.

Any complex number z may be written as

$$z = x + yi = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta).$$

where $r = |z|$ and $\theta = \text{Arg } z$.

This is called the **polar form** of z.

Ex: The polar form of $1 - i$ is:

$$\sqrt{2}(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})).$$

The Cartesian form of $\sqrt{3}(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$ is:

$$\frac{\sqrt{3}}{2} + i\frac{3}{2}.$$

Definition

For any real number θ, let

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

For any two real numbers θ and ϕ, we have

$$e^{i\theta}e^{i\phi} = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$$

$$= (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\cos\theta\sin\phi + \sin\theta\cos\phi)$$

$$= \cos(\theta+\phi) + i\sin(\theta+\phi)$$

$$= e^{i(\theta+\phi)}.$$

This property justifies the exponential notation.

Every complex number can be written in polar form as

$$z = re^{i\theta}, \text{ with } r = |z| \text{ and } \theta = \mathrm{Arg}(z).$$

Ex:

$$z = 1 - i = \sqrt{2}e^{-\frac{\pi i}{4}}.$$

Similarly,

$$-1 = e^{\pi i},$$

This formula relating 1 and e is **Euler's formula**, after the great Swiss mathematician Leonhard Euler.

A nice application of exponential notation is:

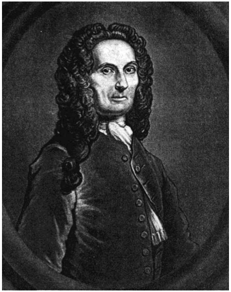
Theorem (De Moivre's theorem)

For any real number θ, and any integer n, we have

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Proof:

$$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta).$$



Ex: Let $z = 1 - i\sqrt{3}$. Find z^{12}.

Answer: We can write

$$z = 1 - i\sqrt{3} = 2\frac{1 - i\sqrt{3}}{2} = 2(\cos(-\frac{\pi}{3}) + i\sin(-\frac{\pi}{3})).$$

Then

$$z^{12} = 2^{12}(\cos(-\frac{\pi}{3}) + i\sin(-\frac{\pi}{3}))^{12}$$

$$= 4096(\cos(12 \cdot (-\frac{\pi}{3})) + i\sin(12 \cdot (-\frac{\pi}{3})))$$

$$= 4096(\cos(-4\pi) + i\sin(-4\pi))$$

$$= 4096.$$

From the polar form, we can see that when you multiply two complex numbers, the moduli get multiplied and the arguments get added:

$$r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}.$$

Therefore

$$\text{Arg}(zw) = \text{Arg}(z) + \text{Arg}(w) \mod 2\pi \ .$$

Similarly, division subtracts the angles

$$\text{Arg}(z/w) = \text{Arg}(z) - \text{Arg}(w) \mod 2\pi \ .$$

It is necessary to make these statement "mod 2π", since we have decreed that $\text{Arg}(z)$ is between $-\pi$ and π.

For example, $\text{Arg}(-1) = \pi$, and

$$\text{Arg}((-1)\cdot(-1)) = \text{Arg}(1) = 0 \neq \text{Arg}(-1) + \text{Arg}(-1) = \pi + \pi = 2\pi.$$

However, the two sides do agree mod 2π.

Arg is therefore sometimes called the *principal argument*.

Suppose $|\alpha| < 1$. Draw a diagram to show that $|\mathrm{Arg}\left(\dfrac{1+\alpha}{1-\alpha}\right)| < \frac{\pi}{2}$.

$|\mathrm{Arg}\left(\dfrac{1+\alpha}{1-\alpha}\right)|$ is the angle between $1+\alpha$ and $1-\alpha$, thought of as vectors in the plane.

The points $1+\alpha$ and $1-\alpha$ lie at opposite ends of the diameter of a circle of radius $|\alpha|$ centered at the point 1. Since $|\alpha| < 1$, the origin lies outside this circle.

But any angle of the form $\angle AOB$, where AB is the diameter of a circle and O is a point outside of the circle, is an acute angle.

(Remember that if AB is the diameter of a circle and O is any other point *on* the circle, then $\angle AOB$ is a right angle; therefore if O lies outside the circle, $\angle AOB$ must be acute.)

Other important properties of the polar form:

- The conjugate of the complex number $z = e^{i\theta}$ is given by $\overline{z} = e^{-i\theta}$.
- $e^{i\theta} = e^{i(\theta+2k\pi)}$ where k is an integer.

We can also write $\cos\theta$ and $\sin\theta$ in terms of the complex exponential as follows:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \text{and} \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Ex: Convert $z = 2e^{i\frac{5\pi}{6}}, w = 3e^{\frac{-\pi i}{3}}$ to Cartesian form.

Answer:

$$z = 2e^{i\frac{5\pi}{6}} = 2(\cos(\frac{5\pi}{6}) + i\sin(\frac{5\pi}{6})) = 2(-\frac{\sqrt{3}}{2} + i\frac{1}{2}) = -\sqrt{3} + i$$

and

$$w = 3e^{\frac{-\pi i}{3}} = 3(\cos(\frac{-\pi}{3}) + i\sin(\frac{-\pi}{3}))$$

$$= 3(\frac{1}{2} - i\frac{\sqrt{3}}{2}) = \frac{3}{2} - i\frac{3\sqrt{3}}{2}.$$

Ex: Evaluate the product $(1+i)(1-i\sqrt{3})$ in two ways to show that $\cos\frac{\pi}{12} = \frac{1+\sqrt{3}}{2\sqrt{2}}$.

Answer: First we expand in Cartesian form:

$$(1+i)(1-i\sqrt{3}) = 1+\sqrt{3}+(1-\sqrt{3})i.$$

Next in polar form:

$$1+i = \sqrt{2}e^{\frac{\pi i}{4}}, \quad (1-i\sqrt{3}) = 2e^{\frac{-\pi i}{3}}.$$

Then

$$(1+i)(1-i\sqrt{3}) = \sqrt{2}e^{\frac{\pi i}{4}} \cdot 2e^{-\frac{\pi i}{3}} = 2\sqrt{2}e^{-\frac{\pi i}{12}}$$

Equating, we get

$$e^{-\frac{\pi i}{12}} = \frac{1+\sqrt{3}+(1-\sqrt{3})i}{2\sqrt{2}},$$

which gives

$$\cos(\frac{\pi}{12}) = \cos(-\frac{\pi}{12}) = \frac{1+\sqrt{3}}{2\sqrt{2}}.$$

Multiplying complex numbers involves stretching and rotation in the complex plane.

The stretching comes from multiplying the moduli, and the rotation from adding the arguments. Multiplying by a complex number of modulus 1 involves only rotation.

In particular, multiplication by i is rotation anti-clockwise about the origin by 90°.

Ex: Find the complex number obtained by rotating $(4+2i)$ anti-clockwise about the origin through $\dfrac{\pi}{2}$.

Answer: We simply multiply by i:

$$i \cdot (4+2i) = -2+4i.$$

More generally, to rotate complex number anticlockwise around 0 through an angle θ, we multiply it by $e^{i\theta}$.

Ex: Rotate $3 - i$ anticlockwise about 0 through an angle of $\dfrac{\pi}{4}$.

Answer: We multiply by $e^{\frac{\pi i}{4}} = \frac{1}{\sqrt{2}}(1 + i)$:

$$\frac{1}{\sqrt{2}}(1+i)(3-i) = \frac{4+2i}{\sqrt{2}}.$$

Yet more important properties of the modulus!

- $z\overline{z} = |z|^2$
- (The Triangle inequality), $|z_1 + z_2| \leq |z_1| + |z_2|$.

The triangle inequality follows from the triangle inequality for vectors in the plane.

Ex: Prove that every root of the polynomial $p(z) = z^4 + z + 3$ lies outside the unit circle in the complex plane.

Answer: Suppse $p(z) = 0$ for some z inside the unit circle.

Then

$$z^4 + z = -3.$$

By the triangle inequality, we have

$$3 = |-3| \leq |z^4| + |z| = |z|^4 + |z|.$$

But since z is inside the unit circle, we know that

$$|z| \leq 1,$$

so the right-hand side is less than or equal to 2, which is impossible.

Finding powers of complex numbers:

Ex: Find $(1-\sqrt{3}i)^{10}$.

Answer: We use polar form:

$$1-\sqrt{3}i = 2e^{-\frac{\pi i}{3}},$$

so

$$(1-\sqrt{3}i)^{10} = (2e^{-\frac{\pi i}{3}})^{10} = 2^{10}(e^{-\frac{\pi i}{3}})^{10} = 1024e^{-\frac{10\pi i}{3}}$$

$$= 1024(\cos\frac{-10\pi}{3} + i\sin\frac{-10\pi}{3}) = 1024(-\frac{1}{2} + i\frac{\sqrt{3}}{2}).$$

Note that $-\frac{10\pi}{3}$ is **not** the principal argument of the answer. To get Arg, we must add an integer multiplie of 2π until we get an angle between $-\pi$ and π. In this case, that would be $\frac{2\pi}{3}$.

Finding roots of complex numbers:

Here we reverse the procedure for finding powers.

Let's say we want to find an nth root of a complex number α. This means that we want to solve the equation

$$z^n = \alpha.$$

We can express α in polar form:

$$\alpha = re^{i\theta}.$$

Then

$$\sqrt[n]{r} \cdot e^{\frac{i\theta}{n}}$$

is a solution, since

$$(\sqrt[n]{r} \cdot e^{\frac{i\theta}{n}})^n = (\sqrt[n]{r})^n (e^{\frac{i\theta}{n}})^n = re^{\frac{in\theta}{n}} = re^{i\theta} = \alpha.$$

However, we can add any integer multiple of $\frac{2\pi}{n}$ to the argument and still have a solution, since when we take the nth power, this amounts to adding an integer multiple of 2π to the argument, which just "takes us around the circle" but doesn't change the answer:

$$(\sqrt[n]{r} \cdot e^{\frac{i(\theta+2\pi k)}{n}})^n = (\sqrt[n]{r})^n (e^{\frac{i(\theta+2\pi k)}{n}})^n = re^{\frac{in(\theta+2\pi k)}{n}}$$
$$= re^{i(\theta+2\pi k)} = re^{i\theta} = \alpha.$$

In fact, there are always exactly n different solutions to

$$z^n = \alpha = re^{i\theta},$$

given by

$$\sqrt[n]{r} \cdot e^{\frac{i(\theta+2k\pi)}{n}}, \;\; k = 0, 1, ..., n-1.$$

These solutions are equally spaced on a circle around the origin, since they all have the same modulus; consecutive solutions are separated by an angle of $\frac{2\pi}{n}$.

Ex: Find the 7^{th} roots of -1.

Answer: First express -1 in polar form:

$$-1 = e^{\pi i}.$$

Then the 7^{th} roots are

$$e^{\frac{i(\pi+2\pi k)}{7}} = e^{\pi i \frac{1+2k}{7}}, \quad k = 0, 1, 2, 3, 4, 5, 6.$$

Ex: Find the 5^{th} roots of $4(1-i)$.

Answer: Again we express $4(1-i)$ in polar form:

$$4(1-i) = 4\sqrt{2}e^{-\frac{\pi i}{4}}.$$

Then the 5^{th} roots are

$$\sqrt[5]{4\sqrt{2}}e^{i(-\frac{\pi}{20}+\frac{2k\pi}{5})} = \sqrt{2}e^{\pi i\frac{8k-1}{20}}, \quad k = 0, 1, 2, 3, 4.$$

Euler's formula gives a dramatic relationship between the exponential and trigonometric functions. We can exploit this to deduce useful relationships and identities in trigonometry.

Ex: Find an expression for $\cos 5\theta$ in terms of sines and cosines.

Answer: We have

$$\cos 5\theta = \mathrm{Re}(e^{5\theta i}) = \mathrm{Re}((e^{\theta i})^5) = \mathrm{Re}((\cos\theta + i\sin\theta)^5).$$

Expanding with the binomial theorem (and ignoring the imaginary terms) , we get

$$\cos 5\theta = \cos^5\theta - \binom{5}{2}\cos^3\theta\sin^2\theta + \binom{5}{4}\cos\theta\sin^4\theta$$

$$= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta.$$

Ex: Express $\sin^5 \theta$ in terms of sine and cosines of multiples of θ.

Answer: We have

$$\sin^5 \theta = \left[\frac{1}{2i}(e^{\theta i} - e^{-\theta i})\right]^5$$

$$= \frac{1}{32i}(\binom{5}{0}e^{5\theta i} - \binom{5}{1}e^{3\theta i} + \binom{5}{2}e^{\theta i} - \binom{5}{2}e^{-\theta i} + \binom{5}{1}e^{-3\theta i} - \binom{5}{0}e^{-5\theta i})$$

$$= \frac{1}{32i}(e^{5\theta i} - 5e^{3\theta i} + 10e^{\theta i} - 10e^{-\theta i} + 5e^{-3\theta i} - e^{-5\theta i})$$

$$= \frac{1}{32i}([e^{5\theta i} - e^{-5\theta i}] + [5e^{-3\theta i} - 5e^{3\theta i}] + [10e^{\theta i} - 10e^{-\theta i}])$$

$$= \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta).$$

Ex: Suppose $0 < \theta < 2\pi$ and n is a positive integer. Show that

$$\mathrm{Re}\left(\frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}\right) = \frac{1}{2} + \frac{\sin(n + \frac{1}{2})\theta}{2 \sin \frac{\theta}{2}}.$$

Answer: We have

$$\frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}} = \frac{e^{-\frac{i\theta}{2}}}{e^{-\frac{i\theta}{2}}} \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}} = \frac{e^{-\frac{i\theta}{2}} - e^{i(n+\frac{1}{2})\theta}}{e^{-\frac{i\theta}{2}} - e^{\frac{i\theta}{2}}}$$

$$= \frac{\cos \frac{\theta}{2} - i \sin \frac{\theta}{2} - (\cos((n + \frac{1}{2})\theta) + i \sin((n + \frac{1}{2})\theta))}{-2i \sin \frac{\theta}{2}}$$

$$= \frac{1}{2} + \frac{\sin((n + \frac{1}{2})\theta)}{2 \sin \frac{\theta}{2}} + i(\ldots).$$

Use the previous result to find a simple formula for

$$1 + \cos\theta + \cos 2\theta + \dots + \cos(n\theta).$$

We use the "telescoping" identity

$$(1 - e^{i\theta})(1 + e^{i\theta} + e^{i2\theta} + \dots + e^{in\theta}) = 1 - e^{i(n+1)\theta}.$$

Then

$$1 + \cos\theta + \cos 2\theta + \dots + \cos(n\theta)$$

$$= \mathrm{Re}(1 + e^{i\theta} + e^{i2\theta} + \dots + e^{in\theta})$$

$$= \mathrm{Re}\left(\frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}\right)$$

$$= \frac{1}{2} + \frac{\sin(n + \frac{1}{2})\theta}{2\sin\frac{\theta}{2}}.$$

We can represent regions in the complex plane by algebraic equations.

For example, the equation $\{z \in \mathbb{C} : |z+1| < 2\}$ is an open disk of radius 2 centered at $z = -1$.

Similarly, $\{z \in \mathbb{C} : 0 \leq \text{Arg}(z) \leq \frac{\pi}{3}\}$ is an infinite wedge of 60° counterclockwise from the x-axis (not including the origin).

Ex: Sketch $\{z \in \mathbf{C} : |z - i + 1| < 2\} \cap \{z \in \mathbf{C} : Re(z) \geq 0\}$

Ex: Sketch $\{z \in \mathbf{C} : |z - 3| < 2\} \cup \{z \in \mathbf{C} : Im(z - 3i) > 0\}$

Remainder Theorem: If $p(x)$ is a polynomial then the remainder r when $p(x)$ is divided by $x - \alpha$ is given by $p(\alpha)$.

Factor Theorem: If $p(\alpha) = 0$ then $(x - \alpha)$ is a factor of $p(x)$.

When factoring, the number field is important!

Ex: $x^2 - 2$ does not factor over the rational numbers, but it does over the real numbers. Similarly, $x^2 + 1$ does not factor over the real numbers but does over the complex numbers.

By the fundamental theorem of algebra, all polynomials completely factor over the complex numbers.

Theorem

Every polynomial complex polynomial of degree $n \geq 1$ has a factorisation into linear factors of the form:

$$p(z) = a(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n)$$

where $\alpha_1, \alpha_2, \cdots, \alpha_n$ are the roots of $p(z)$.

To factor over the real numbers, first factor over the complex numbers.

Ex: Factor $x^4 + 1$ over the complex numbers and hence over the real numbers.

Answer: Over the complex numbers, the roots are the 4^{th} roots of -1:

$$x^4 + 1 = (x - e^{\frac{\pi i}{4}})(x - e^{\frac{3\pi i}{4}})(x - e^{\frac{5\pi i}{4}})(x - e^{\frac{7\pi i}{4}}).$$

These roots are all complex, but we can combine some of them:

$$(x-e^{\frac{\pi i}{4}})(x-e^{\frac{7\pi i}{4}}) = x^2-\sqrt{2}x+1, \quad (x-e^{\frac{3\pi i}{4}})(x-e^{\frac{3\pi i}{4}}) = x^2+\sqrt{2}x+1.$$

Therefore

$$x^4 + 1 = (x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2}x + 1),$$

Ex: Factor $x^6 + 8$ over the complex numbers and hence over the real numbers.

Answer: Over the complex numbers, the roots are the 6^{th} roots of -8:

$$
\begin{aligned}
&x^6 + 8 \\
&\qquad = (x - \sqrt{2}e^{\frac{\pi i}{6}})(x - \sqrt{2}e^{\frac{3\pi i}{6}})(x - \sqrt{2}e^{\frac{5\pi i}{6}}) \\
&\qquad\qquad (x - \sqrt{2}e^{\frac{7\pi i}{6}})(x - \sqrt{2}e^{\frac{9\pi i}{6}})(x - \sqrt{2}e^{\frac{11\pi i}{6}}).
\end{aligned}
$$

Combining conjugate pairs, we get:

$$x^6 + 8 = (x^2 + 2)(x^2 + \sqrt{6}x + 2)(x^2 - \sqrt{6}x + 2).$$

Theorem

Suppose $p(x)$ is a polynomial with real coefficients. If α is a root, then so is $\overline{\alpha}$.

Proof: Suppose α is a root of $p(x)$, so that

$$a_n\alpha^n + ... + a_1\alpha + a_0 = 0,$$

where $a_0, a_1, ..., a_n$ are the coefficients of p. Then we may take the complex conjugate of both sides of the equation to get

$$0 = \overline{0} = \overline{a_n\alpha^n + ... + a_1\alpha + a_0}$$

$$= \overline{a_n}\,\overline{\alpha}^n + ... + \overline{a_1}\,\overline{\alpha} + \overline{a_0}$$

$$= a_n\overline{\alpha}^n + ... + a_1\overline{\alpha} + a_0,$$

where the last step follows since the coefficients are real.

But this means that $\overline{\alpha}$ is a root of $p(x)$ as well.

Theorem

A real polynomial can be factored into a product of real linear and/or real quadratic factors.

Ex: Show that $z = i$ is a root of

$$p(z) = z^4 - 2z^3 + 6z^2 - 2z + 5$$

and hence factor p over $\mathbb{R}$ and $\mathbb{C}$.

Answer: We check that

$$p(i) = i^4 - 2i^3 + 6i^2 - 2i + 5 = 1 + 2i - 6 - 2i + 5 = 0.$$

Then $-i$ is a root as well , and $(z-i)(z+i) = z^2 + 1$ divides $p(z)$.

To find the other quadratic factor, set

$$(z^2 + 1)(az^2 + bz + c) = z^4 - 2z^3 + 6z^2 - 2z + 5,$$

and we get that $a = 1, b = -2, c = 5$, so the other two roots are

$$\frac{2 \pm \sqrt{4 - 20}}{2} = 1 \pm 2i.$$