## The logarithmic and exponential functions

- In the preceding, we have manipulated with functions such as

$$\ln x, \quad e^x, \quad x^\pi$$

even though we have not defined them formally.

- In particular, we are familiar with the important formula

$$\ln(st) = \ln s + \ln t.$$

**Question.** Consider the functional equation

$$f(st) = f(t) + f(s), \tag{1}$$

where $s$ and $t$ are independent variables.
It is evident that $f = \ln$ is one solution of this equation.
Are there other functions $f$ obeying this functional equation?

**Answer.** We first note that (1) evaluated at $s = t = 1$ yields

$$f(1) = 0.$$

Moreover, differentiation of (1) with respect to $s$ leads to

$$tf'(ts) = f'(s)$$

so that at $s = 1$

$$f'(t) = \frac{1}{t} f'(1).$$

If we now demand that $f'(1) = 1$ then $f$ is uniquely determined via integration since $f(1) = 0$.

**Conclusion.** A function $f$ is uniquely defined by the functional equation

$$f(st) = f(s) + f(t),$$

subject to

$$f'(1) = 1.$$

It is given by (thanks to the FTC1)

$$f(x) = \int_1^x \frac{1}{t}\,dt.$$

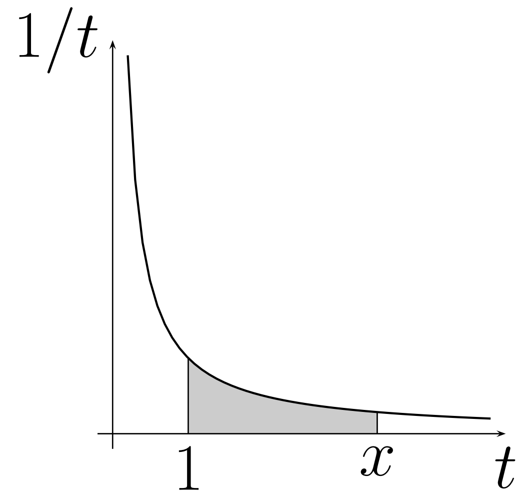
# The natural logarithm function

**Definition.** The natural logarithm function

$$\ln : (0, \infty) \to \mathbb{R}$$

is defined by the formula

$$\ln x = \int_1^x \frac{1}{t}\, dt.$$




$\ln x$ is the area of the shaded region.

**Theorem.** The function $\ln : (0, \infty) \to \mathbb{R}$ has the following properties:

(i) $\ln$ is differentiable on $(0, \infty)$ and

$$\frac{d}{dx} \ln x = \frac{1}{x}.$$

**Proof.** Apply the first fundamental theorem of calculus to the definition of $\ln$.

(ii) $\ln x > 0 \quad \text{for} \quad x > 1,$

$\ln 1 = 0,$

$\ln x < 0 \quad \text{for} \quad 0 < x < 1.$

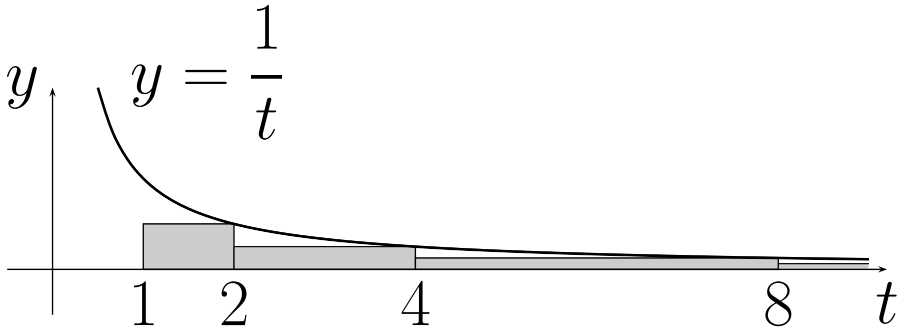
**Proof.** This follows from the definition of $\ln$ and the fact that $\frac{1}{t} > 0$ when $t > 0$.

(iii) $\ln x \to -\infty \quad$ as $\quad x \to 0^+$,

$\ln x \to \infty \quad$ as $\quad x \to \infty$.

**Proof.** The diagram below shows that

$$\int_1^2 \frac{dt}{t} \geq 1 \times \frac{1}{2}, \quad \int_2^4 \frac{dt}{t} \geq 2 \times \frac{1}{4}, \quad \int_4^8 \frac{dt}{t} \geq 4 \times \frac{1}{8}.$$




In general,

$$\int_1^{2^n} \frac{dt}{t} \geq \underbrace{\frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}}_{n \text{ terms}} = \frac{n}{2} \to \infty$$

as $n \to \infty$.

Hence the improper integral

$$\int_1^\infty \frac{1}{t}\,dt$$

'diverges to infinity' and, therefore, $\ln x \to \infty$ as $x \to \infty$.

This argument can be adapted to show that

$$-\int_1^{2^{-n}} \frac{dt}{t} = \int_{2^{-n}}^1 \frac{dt}{t} \to \infty$$

as $n \to \infty$. Hence $\ln x \to -\infty$ as $x \to 0^+$.

(iv) For all $x, y > 0$: $\qquad \ln(xy) = \ln x + \ln y.$

**Proof.** Suppose that $y$ is some fixed positive number and that $x > 0$. Then, the chain rule implies that

$$\frac{d}{dx}[\ln(xy)] = \frac{1}{xy}\frac{d}{dx}(xy) = \frac{y}{xy} = \frac{1}{x} = \frac{d}{dx}\ln x.$$

Accordingly,

$$\ln(xy) = \ln(x) + C$$

for some constant $C$.

Evaluation at $x = 1$ leads to

$$\ln(y) = C$$

and hence

$$\ln(xy) = \ln(x) + \ln(y).$$

(v) For all $x, y > 0$: $$\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y).$$

(vi) For all $x > 0$ and $r \in \mathbb{Q}$: $$\ln(x^r) = r \ln x.$$

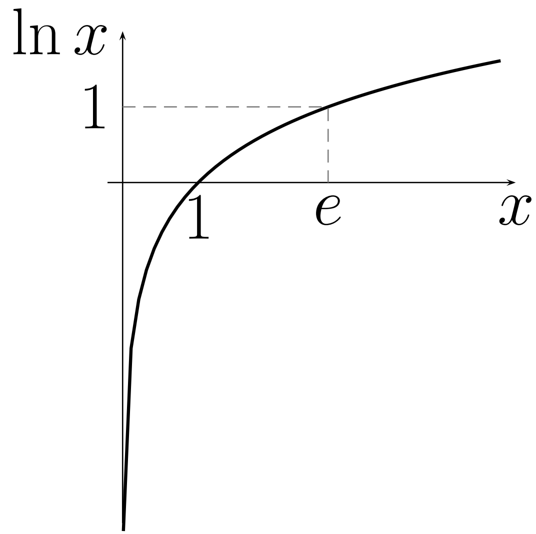
**Exercise.** Prove (v) and (vi) (use technique from the proof of (iv)).

**Remark.** The above properties imply that

- $\mathrm{Range}(\ln) = \mathbb{R}$, and
- $\ln$ is increasing and hence invertible so that
- $\ln x = 1$ has a unique solution.

**Definition.** The real number $e$ is defined to be the unique number $x$ satisfying

$$\ln x = 1.$$

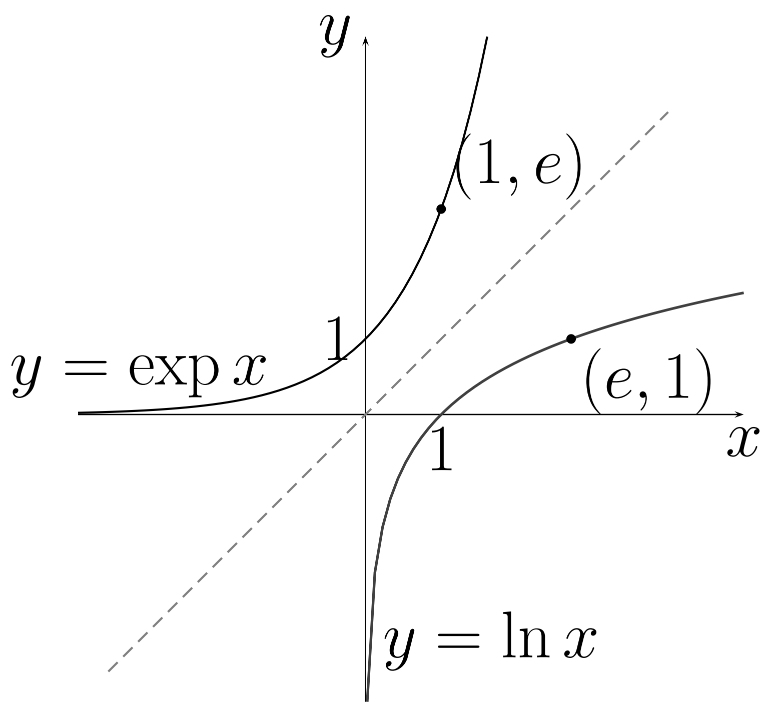


## The exponential function

**Definition.** The function

$$\exp : \mathbb{R} \to (0, \infty)$$

is defined to be the inverse function of $\ln : (0, \infty) \to \mathbb{R}$.



**Remark.** For any rational number $r$, we can evaluate both

$$\exp r \qquad \text{and} \qquad e^r$$

but are these two numbers the same?

**Theorem.** The function $\exp : \mathbb{R} \to (0, \infty)$ has the following properties:

(i) $\exp(\ln x) = x \quad$ for all $\quad x \in (0, \infty)$,
$\ln(\exp x) = x \quad$ for all $\quad x \in \mathbb{R}$.

(ii) $\exp(1) = e$ and $\exp(0) = 1$.

(iii) $\exp x \to \infty \quad$ as $\quad x \to \infty, \quad \exp x \to 0 \quad$ as $\quad x \to -\infty$.

**Proof** of (i)-(iii) follows from the definition of $\exp$.

(iv) $\exp$ is differentiable on $\mathbb{R}$ with

$$\frac{d}{dx}\exp x = \exp x.$$

**Proof.** The function $\exp$ is differentiable on $\mathbb{R}$ by virtue of the inverse function theorem and differentiation of

$$\ln(\exp x) = x$$

produces

$$\frac{1}{\exp x}\frac{d}{dx}\exp x = 1$$

(v) For all $x, y \in \mathbb{R}$: $\quad \exp(x+y) = \exp x\ \exp y.$

**Proof.** For any $x$ and $y$, we have

$$\begin{aligned}\exp(x+y) &= \exp\Big(\ln(\exp x) + \ln(\exp y)\Big)\\ &= \exp\Big(\ln(\exp x\ \exp y)\Big)\\ &= \exp x\ \exp y\end{aligned}$$

(vi) For all $r \in \mathbb{Q}$ and $x \in \mathbb{R}$: $\quad \exp(rx) = (\exp x)^r$.

**Proof.** Suppose that $r \in \mathbb{Q}$ and $x \in \mathbb{R}$. Then,

$$\begin{aligned}
\exp(rx) &= \exp\Big(r \ln \exp x\Big) \\
&= \exp\Big(\ln\big((\exp x)^r\big)\Big) \\
&= (\exp x)^r.
\end{aligned}$$

**Remark.** In particular, the above theorem implies that

$$\exp(r) = (\exp 1)^r = e^r.$$

for every rational number $r$.

It is therefore consistent to make the following definition.

**Definition.** For any $x \notin \mathbb{Q}$, we define the number $e^x$ to be

$$e^x = \exp x.$$

**Note.** The above definition 'merely' means that $e^x$ is the unique real number $R$ such that

$$\int_1^R \frac{1}{t}\, dt = x$$

for any $x \in \mathbb{R}$ (Uniqueness follows from the properties of the function $\ln$).

By construction, the function

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \exp x = e^x$$

is differentiable (and continuous) and is called the exponential function.

**Exponentials and logarithms with other bases**

**Question.** How would one define $b^x$ for $x \notin \mathbb{Q}$ and $b > 0$?

Since, for any rational number $r$,

$$b^r = \exp\big(\ln(b^r)\big) = \exp\big(r \ln b\big) = e^{r \ln b},$$

the following definition is natural.

**Definition.** Suppose that $b > 0$ and $x \notin \mathbb{Q}$. Then, the number $b^x$ is defined by

$$b^x = \exp(x \ln b) = e^{x \ln b}.$$

**Note.** By combining the defintions of $b^x$ for rational $x$ and irrational $x$, we now obtain a well-defined function

$$f_b : \mathbb{R} \to (0, \infty), \quad f_b(x) = b^x = \exp(x \ln b)$$

for any $b > 0$.

**Example.** It is seen that

$$f_3(2) = 3^2 = \exp(2\ln 3) = \exp(\ln 3)\ \exp(\ln 3) = 3 \times 3 = 9$$

as one would expect!

**Remark.** Since $f_b$ is a combination of continuous and differentiable functions, it is also continuous and differentiable with

$$f_b'(x) = (\ln b)\exp(x \ln b) = (\ln b)b^x.$$

Accordingly,

- if $\quad b > 1 \quad$ then $\quad f_b'(x) > 0 \quad$ for all $\quad x \in \mathbb{R}$,
- if $\quad 0 < b < 1 \quad$ then $\quad f_b'(x) < 0 \quad$ for all $\quad x \in \mathbb{R}$

so that $f_b$ is invertible for $b \neq 1$ (we had such a theorem).

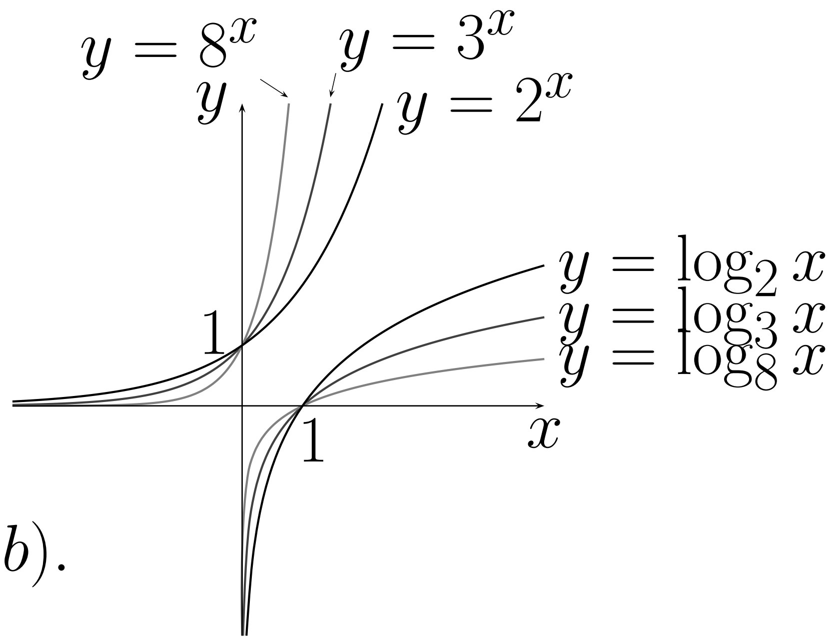
**Definition.** Suppose that $b$ is a positive real number with $b \neq 1$. Then, the logarithm function to the base $b$

$$\log_b : (0, \infty) \to \mathbb{R}$$

is defined to be the inverse of the function

$$f_b : \mathbb{R} \to (0, \infty), \qquad f_b(x) = b^x = \exp(x \ln b).$$

In particular, $\log_e x = \ln x$.




**Remark.** The above definition may be stated as

$$y = b^x \quad \Leftrightarrow \quad x = \log_b y.$$

The following theorem demonstrates that all logarithm functions are just scaled versions of the natural logarithm function.

**Theorem.** Suppose that $b$ is a positive real number with $b \neq 1$. Then

$$\log_b x = \frac{\ln x}{\ln b}$$

for all $x > 0$.

**Proof.** Since

$$x = b^{\log_b x} = \exp(\log_b x \ln b),$$

we conclude that

$$\ln x = \log_b x \ln b.$$

Hence, $\log_b$ shares all the properties of $\ln$ such as

$$\frac{d}{dx} \log_b x = \frac{d}{dx} \left( \frac{\ln x}{\ln b} \right) = \frac{1}{x \ln b}$$

or

$$\log_b(x^y) = y \log_b x.$$

## Integration and the $\ln$ function

Since

$$\frac{d}{dx}\ln(x) = \frac{1}{x}, \quad x > 0, \quad \text{and} \quad \frac{d}{dx}\ln(-x) = \frac{1}{x}, \quad x < 0,$$

the function $\ln(x)$ is an antiderivative of $1/x$ if $x > 0$ and $\ln(-x)$ is an antiderivative of $1/x$ if $x < 0$. Thus,

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

provided that $x$ is restricted to an interval which does NOT contain $0$.

This may be generalised to

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

provided that $f$ is differentiable and does not vanish on the interval of integration.

**Example.** On any interval not including zeros of $\cos x$, we have

$$\int \tan x \, dx = -\int \frac{-\sin x}{\cos x} \, dx = -\ln|\cos x| + C.$$

**Example.** Find the indefinite integral

$$\int \frac{1}{2\sec x + \tan x} \, dx.$$

**Logarithmic differentiation**

Logarithms are powerful in that they 'transform' powers into products, products into sums and quotients into differences.

**Example.** Find the derivative of

$$y = \left( \frac{(3x^2+4)(x+2)}{x^3+5x} \right)^{3/5}.$$

The idea is to take $\ln$ of both sides of the equation to obtain

$$\begin{aligned} \ln y &= \frac{3}{5} \ln \left( \frac{(3x^2+4)(x+2)}{x^3+5x} \right) \\ &= \frac{3}{5} \Big( \ln(3x^2+4) + \ln(x+2) - \ln(x^3+5x) \Big). \end{aligned}$$

Differentiating both sides with respect to $x$ is relatively easy and leads to

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{5}\left(\frac{6x}{3x^2+4} + \frac{1}{x+2} - \frac{3x^2+5}{x^3+5x}\right).$$

Hence, we obtain

$$\frac{dy}{dx} = \frac{3}{5}\left(\frac{(3x^2+4)(x+2)}{x^3+5x}\right)^{3/5}\left(\frac{6x}{3x^2+4} + \frac{1}{x+2} - \frac{3x^2+5}{x^3+5x}\right).$$

**Remark.** The above procedure is *only* valid for intervals on which $y > 0$.

**Example.** Consider the function

$$f : (0, \pi) \to \mathbb{R}, \qquad f(x) = x^{\sin x}.$$

Determine its derivative.

**Indeterminate forms with powers**

Consider the limits

$$\lim_{x\to 0^+} x^x \qquad \text{and} \qquad \lim_{x\to\infty} x^{1/x}.$$

The first limit is of the form $0^0$ while the second is of the form $\infty^0$.

Since each limit involves a power, it is natural to first take the logarithm of the limit and then bring l'Hôpital's rule into play.

**Example.**

Evaluate the limit

$$\lim_{x\to 0^+} x^{2x}.$$

By taking the natural logarithm, we can transform the limit into an indeterminate form of the type $\frac{\infty}{\infty}$:

$$\begin{aligned}
\lim_{x\to 0^+} x^{2x} &= \lim_{x\to 0^+} \exp\left(\ln x^{2x}\right) && \text{(since } \ln \text{ and } \exp \text{ are inverses)} \\
&= \lim_{x\to 0^+} \exp\left(2x \ln x\right) \\
&= \exp\left(\lim_{x\to 0^+} 2x \ln x\right) && \text{(since } \exp \text{ is continuous)} \\
&= \exp\left(\lim_{x\to 0^+} \frac{\ln x}{1/(2x)}\right).
\end{aligned}$$

We can now apply l'Hôpital's rule to the problem. By differentiating the numerator and denominator and then simplifying we obtain

$$\begin{aligned}
\lim_{x\to 0^+} x^{2x} &= \exp\left(\lim_{x\to 0^+} \frac{1/x}{-1/(2x^2)}\right) \\
&= \exp\left(\lim_{x\to 0^+} -2x\right) \\
&= \exp(0) = 1.
\end{aligned}$$

The following example is of the indeterminate form "$1^\infty$".

**Example.** Show that

$$\lim_{x\to\infty}\left(1+\frac{t}{x}\right)^x = e^t,$$

where $t$ is a constant real parameter.