THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Semester 2 2013

MATH1131
MATHEMATICS 1A

(1) TIME ALLOWED – 2 hours

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

(8) A SHORT TABLE OF INTEGRALS IS SUPPLIED AT THE END OF THE PAPER

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

Use a separate book clearly marked Question 1

1. i) Let $z = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i$ and $w = 2 + 2i$.

a) Find zw in **Cartesian form.**

b) Find $|z|$ and $\operatorname{Arg}(z)$.

c) Find z^{2014} in **Cartesian form.**

ii) a) Prove that

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

b) Use the formula above to prove that

$$\sin^4\theta = \frac{1}{8}\left(\cos 4\theta - 4\cos 2\theta + 3\right).$$

c) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sin^4\theta \, d\theta$.

iii) Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 5 \\ 1 & 4 \end{pmatrix}$.

a) Explain why AB does not exist.

b) Evaluate $A^T B$.

c) Find the inverse of B.

d) Write down the inverse of B^{-1}.

iv) A farmer intends to raise chickens, ducks and pigs on his $5000\,\mathrm{m}^2$ farm. The farmer will provide each chicken with an area of $10\,\mathrm{m}^2$, each duck an area of $10\,\mathrm{m}^2$, and each pig an area of $50\,\mathrm{m}^2$. The cost of food is \$0.30 per week for a chicken, \$0.40 per week for a duck and \$3.00 per week for a pig. He will spend a total of \$180.00 per week feeding the animals that occupy the $5000\,\mathrm{m}^2$ available. Let x be the number of chickens, y the number of ducks and z the number of pigs on the farm.

a) Write down a system of two equations in x, y and z relating the numbers of chickens, ducks and pigs to the space and cost requirements of this farm.

b) Solve this system of equations to find all possible solutions.

c) The farmer wishes to minimize the total number of animals on his farm. How many pigs should he raise?

Use a separate book clearly marked Question 2

2. i) Sketch the following region on the Argand diagram:

$$S = \left\{ z \in \mathbb{C} \ : \ |z-i| > 1 \quad \text{and} \quad 0 \leq \operatorname{Arg}(z-i) \leq \frac{\pi}{2} \right\}$$

ii) A plane has parametric vector equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}; \quad \lambda, \mu \in \mathbb{R}.$$

Write down three (non-zero) vectors which are parallel to the plane.

iii) Evaluate the determinant

$$\begin{vmatrix} 1 & 4 & 1 \\ 2 & 1 & 7 \\ 0 & 0 & 2 \end{vmatrix}.$$

iv) The points P and Q in $\mathbb{R}^3$ have position vectors $\mathbf{p} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix}$ respectively. Let ℓ be the line through P and Q, and let Π be the plane with **Cartesian** equation $x + y + 2z = 52$.

a) Find the vector $\overrightarrow{PQ}$.

b) Find a parametric vector equation for the line l.

c) Determine the point of intersection A of the line l and the plane Π.

d) Hence find a point B, (different from P), on the line l whose perpendicular distance from the plane Π is equal to the perpendicular distance of P from Π.

v) Suppose that A is a 3×3 matrix, and I is the 3×3 identity matrix.

a) Expand out and simplify the matrix product

$$(A - I)(A^3 + A^2 + A + I).$$

Now consider the MAPLE output below, which gives information regarding the powers of a 3×3 matrix A.

```
> with(LinearAlgebra):
> A^2;
```

$$\begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

```
> A^3;
```

$$\begin{bmatrix} 0 & 0 & -1 \\ -1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

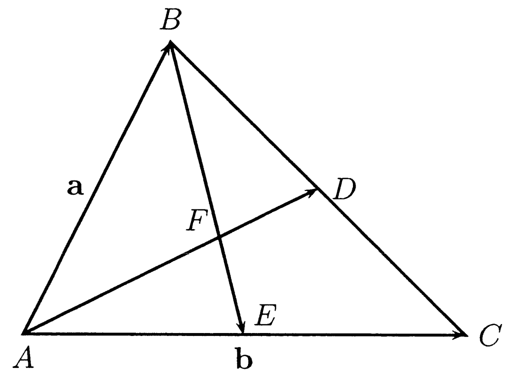
```
> A^4;
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b) You are given that $(A - I)$ is invertible.
Explain why $A^3 + A^2 + A + I = \mathbf{0}$, where $\mathbf{0}$ is the 3×3 zero matrix.

c) Hence find explicitly the matrix A in the MAPLE session above.

vi) Consider the triangle ABC as shown, with $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{AC} = \mathbf{b}$. Let E be the midpoint of AC and D be the midpoint of BC. The lines AD and BE meet at the point F.



a) Write down the area of the triangle ABC in terms of $\mathbf{a}$ and $\mathbf{b}$.

b) Show that $\overrightarrow{BE} = \frac{1}{2}\mathbf{b} - \mathbf{a}$, and find the vector $\overrightarrow{AD}$.

c) Given that $BF = \frac{2}{3}BE$ and $AF = \frac{2}{3}AD$, find, in terms of $\mathbf{a}$ and $\mathbf{b}$, the area of triangle ABF.

Use a separate book clearly marked Question 3

3. i) For each of the following, find the limit, if it exists:

a) $\displaystyle\lim_{x\to 0} \frac{\cos(2x)-1}{x^2}$,

b) $\displaystyle\lim_{x\to\infty} \frac{\sin x + x^2}{3x^2 - x\cos x}$,

c) $\displaystyle\lim_{x\to\infty} \left(\sqrt{x^2+4x} - x\right)$.

ii) Find the following integrals:

a) $\displaystyle I_1 = \int_0^{\pi/2} \cos^2 x \sin x \, dx.$

b) $\displaystyle I_2 = \int x e^{3x} \, dx.$

iii) A ladder of length 13 metres is placed against a vertical wall. The bottom of the ladder is x metres horizontally from the base of the wall and the top of the ladder is y metres vertically from the base of the wall.

a) Write down a relationship connecting x and y.

b) The bottom of the ladder is being pulled away from the wall at a rate of 0.25 m/sec. At what rate is the top of the ladder moving when the base of the ladder is 12 metres horizontally from the base of the wall?

iv) a) State carefully the Mean Value Theorem.

b) By applying the Mean Value theorem to the function $g(t) = 1 + \sin t - e^t$, prove that for $x > 0$,

$$1 + \sin x < e^x$$

v) Show that the following improper integral converges:

$$\int_0^{\infty} \frac{dx}{4+x^2} \, dx.$$

State its value.

Use a separate book clearly marked Question 4

4. i) Prove that $\displaystyle\int_0^1 x\cosh x\,dx = \frac{e-1}{e}$.

ii) Let $f(x) = x^7 + 2x + e^x$.

a) Explain why f has an inverse function g with domain $\mathbb{R}$.

b) Find $g'(1)$.

iii) Consider the following MAPLE session:

```
> p:=x->x^5-4*x^4+3*x^3-2*x^2+x+3;
```

$$x \to x^5 - 4\,x^4 + 3\,x^3 - 2\,x^2 + x + 3$$

```
> dp:=diff(p(x),x);
```

$$5\,x^4 - 16\,x^3 + 9\,x^2 - 4\,x + 1$$

```
> sols:=solve(dp=0,x);
```

$$\frac{3}{2} + \frac{\sqrt{5}}{2}, \frac{3}{2} - \frac{\sqrt{5}}{2}, \frac{1}{10} + \frac{1}{10} I\sqrt{19}, \frac{1}{10} - \frac{1}{10} I\sqrt{19}$$

```
> x1:=sols[1];
```

$$\frac{3}{2} + \frac{\sqrt{5}}{2}$$

```
> x2:=sols[2];
```

$$\frac{3}{2} - \frac{\sqrt{5}}{2}$$

```
> evalf(p(x1));
```

$$-19.18033986$$

```
> evalf(p(x2));
```

$$3.180339888$$

Let $p(x) = x^5 - 4x^4 + 3x^3 - 2x^2 + x + 3$. By using the above MAPLE session, and drawing a rough sketch, determine the number of real roots of p.

iv) Without using calculus, sketch the polar curve given by $r = 2 + \sin\theta$.

v) Let $f(x) = \dfrac{5x + |x|}{x - 3}$.

a) What is the domain of f?

b) Find $\lim\limits_{x\to\infty} f(x)$? Find $\lim\limits_{x\to-\infty} f(x)$?

c) By considering the cases $x > 0$ and $x < 0$ separately, write a formula for $f'(x)$.

d) Is f differentiable at 0? Give reasons for your answer.

e) Sketch the graph of f carefully, showing all asymptotes.

BASIC INTEGRALS

$$\int \frac{1}{x}\, dx = \ln|x| + C = \ln|kx|, \qquad C = \ln k$$

$$\int e^{ax}\, dx = \frac{1}{a}e^{ax} + C$$

$$\int a^x\, dx = \frac{1}{\ln a}a^x + C, \qquad a \neq 1$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax + C$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax + C$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax + C$$

$$\int \operatorname{cosec}^2 ax\, dx = -\frac{1}{a}\cot ax + C$$

$$\int \tan ax\, dx = \frac{1}{a}\ln|\sec ax| + C$$

$$\int \cot ax\, dx = \frac{1}{a}\ln|\sin ax| + C$$

$$\int \sec ax\, dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C$$

$$\int \sinh ax\, dx = \frac{1}{a}\cosh ax + C$$

$$\int \cosh ax\, dx = \frac{1}{a}\sinh ax + C$$

$$\int \operatorname{sech}^2 ax\, dx = \frac{1}{a}\tanh ax + C$$

$$\int \operatorname{cosech}^2 ax\, dx = -\frac{1}{a}\coth ax + C$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$\begin{aligned}\int \frac{dx}{a^2 - x^2} &= \frac{1}{a}\tanh^{-1}\frac{x}{a} + C, \qquad |x| < a \\ &= \frac{1}{a}\coth^{-1}\frac{x}{a} + C, \qquad |x| > a > 0 \\ &= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C, \qquad x^2 \neq a^2\end{aligned}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\frac{x}{a} + C, \qquad x \geqslant a > 0$$