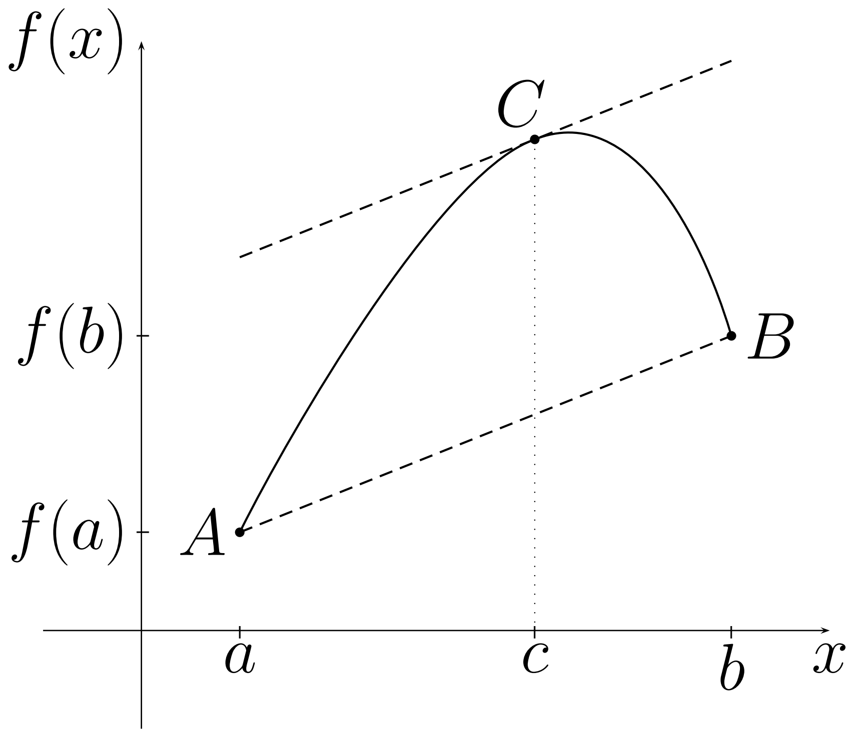
The mean value theorem

This section is about an odd looking result that stands at the foundations of much of calculus, the Mean Value Theorem (MVT).



The mean value theorem. Suppose that a function f is continuous on $[a, b]$ and differentiable on (a, b). Then, there exists at least one real number c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Briefly speaking the theorem says:

> If you travel from point A to point B, then at some stage during the journey you must have been doing exactly your average speed.

Example. A car enters a tunnel at a speed of 30km/h and after 1 minute leaves the tunnel at a speed of 40 km/h. The length of the tunnel is 1 km. Did the driver break the speed limit of 50 km/h?

Let $f(t)$ be a distance (km) from the entry of the tunnel at the moment t (h).
That is $f(0) = 0$ and $f(\frac{1}{60}) = 1$.
The derivative of f at t is a speed of the car at the moment t.
By MVT there is a $c \in (0, \frac{1}{60})$ such that
$\frac{f(\frac{1}{60})-f(0)}{\frac{1}{60}-0} = f'(c)$.
or $f'(c) = 60$.
So there is a moment $c \in (0, \frac{1}{60})$ such that the speed is 60 km/h.

Remark. In the above theorem, it is required that f is continuous on the closed interval but differentiable only on the open interval!

Example. Find counterexamples which demonstrate that the continuity and differentiability requirements must be met.

Consider the function

$$f(x) = \begin{cases} x, & 1 < x \leq 2 \\ 0, & x = 1. \end{cases}$$

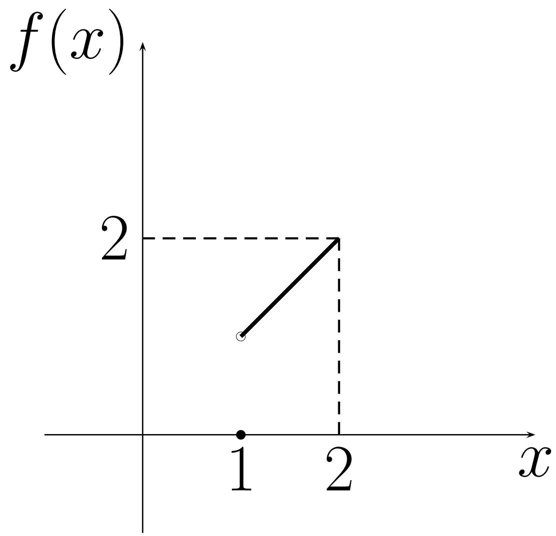
We have that

$f'(x) = 1$, for every $1 < x < 2$.

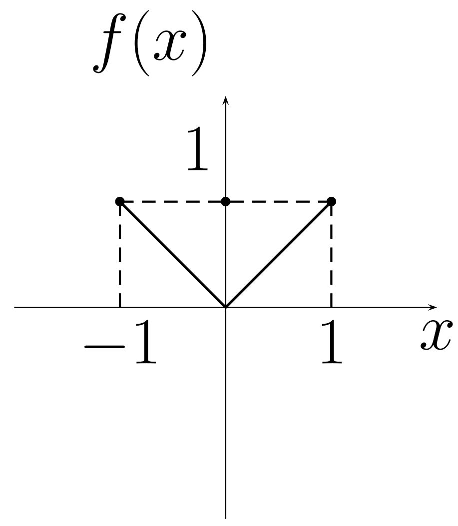
$\frac{f(2)-f(1)}{2-1} = \frac{2-0}{1} = 2.$

there is no $c \in (1, 2)$ such that $f'(c) = 2$.

So MVT does not work for discontinuous functions.



Consider the function
$f(x) = |x|$, for $-1 \leq x \leq 1$.
We have that
$f'(x) = 1$, for every $0 < x < 1$.
$f'(x) = -1$, for every $-1 < x < 0$.
The derivative $f'(0)$ does not exist.
$\frac{f(1)-f(-1)}{1-(-1)} = \frac{0}{2} = 0$.
there is no $c \in (-1, 1)$ such that $f'(c) = 0$.



So MVT does not work for non-differentiable functions.

Example. Apply the mean value theorem to the function f defined by $f(x) = x^5$ and $a = -1$, $b = 4$. Find the value(s) of c which satisfy the conclusion of the theorem (MVT).

Observe that $f'(x) = 5x^4$.

$\frac{f(4)-f(-1)}{4-(-1)} = f'(c)$

$\frac{4^5-(-1)^5}{5} = 5c^4$

$\frac{1025}{5} = 5c^4$.

$c = \pm\sqrt[4]{41} \sim \pm 2.53$

Since $c \in (-1, 4)$, we take $c = 2.53$

Applications of the mean value theorem (MVT).

Before we prove the theorem, it is worth asking why this is interesting.

1. MVT is behind important facts that you will have just looked at intuitively, for example if $f'(x) \geq 0$ on (a, b) then f is 'increasing' on (a, b).

2. MVT is used in the proof of the Fundamental Theorem of Calculus: if $F' = f$ then $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$.

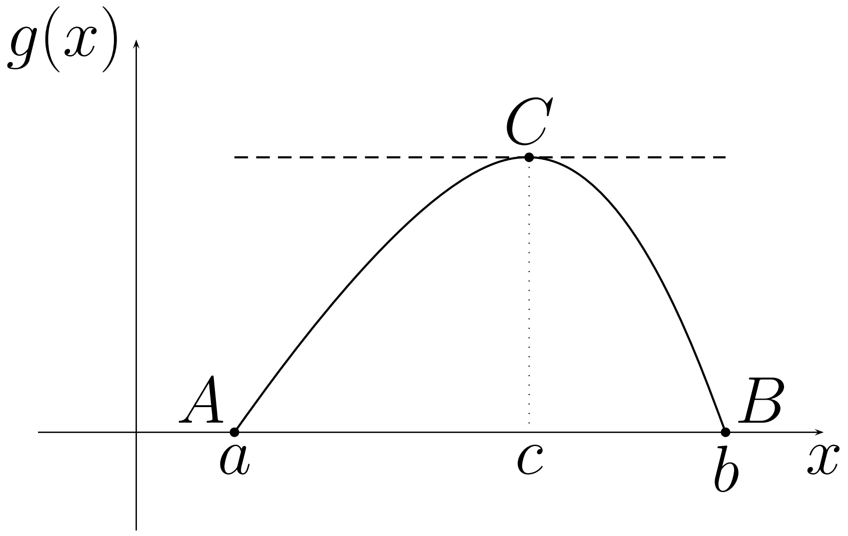
3. MVT allows you to prove inequalities like
$$x - \frac{x^3}{3} \leq \sin x \leq x \text{ for } 0 \leq x \leq 1.$$

4. MVT is used to prove L'Hôpital's Rule which is used for
$$\lim_{x \to 0} \frac{\tan x - \sin x}{x^3}.$$

5. Counting zeros, generalized max-min tests, …

Proof of the mean value theorem

If $f(a) = f(b) = 0$ then the mean value theorem reduces to Rolle's theorem.



Rolle's theorem. Suppose that a function g is continuous on $[a, b]$ and differentiable on (a, b). If, in addition,

$$g(a) = g(b) = 0$$

then there exists a c in (a, b) such that $g'(c) = 0$.

Proof of Rolle's theorem.

Case 1: Suppose that

$$g(x) = 0, \quad \text{for all} \quad x \in [a, b].$$

Then, $g'(c) = 0$ for every c in (a, b).

Case 2: Suppose that there exists a point d in (a, b) such that

$$g(d) > 0.$$

By the max-min theorem, g attains a maximum value at some point c in $[a, b]$.

Moreover,

$$g(c) \geq g(d) > g(a) = g(b) = 0$$

so that c must lie in (a, b).

Since g is differentiable on (a, b), we know from previous lectures that

$$g'(c) = 0.$$

Case 3: Suppose that

$$g(x) \leq 0$$

for all x in $[a, b]$ and that g is not constant on $[a, b]$.

Then, g attains a minimum at a point c in (a, b) and hence $g'(c) = 0$.

In order to prove the mean value theorem, we merely 'deform' the graph of f in such a way that Rolle's theorem applies:

Proof of the mean value theorem.
Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

We consider the function g defined by

$$g(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}(x - a) + f(a)\right].$$

The function g is continuous on $[a, b]$, differentiable on (a, b) (find its derivative) and $g(a) = g(b) = 0$.

Using Rolle's theorem, there exists a c in (a, b) such that $g'(c) = 0$, that is, such that

$$f'(c) - \left[\frac{f(b) - f(a)}{b - a}\right] = 0.$$

Hence,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

as required.

Proving inequalities using the mean value theorem

An important application of the mean value theorem is in proving inequalities.

General 'philosophy.' Apply the mean value theorem to an appropriate function f and find a lower or upper bound for $f'(c)$ on (a, b).

Example. Show that

$$e^x > 1 + x \quad \text{for all} \quad x > 0.$$

Hint: Fix $x > 0$ and make the identification $[a, b] = [0, x]$ for this fixed $x > 0$.

Suppose that $x > 0$ and consider the closed interval $[0, x]$.
We define a function $f : [0, x] \to \mathbb{R}$ by $f(t) = e^t$.
Now, f is continuous on $[0, x]$ and differentiable on $(0, x)$
so that MVT implies that $\frac{f(x)-f(0)}{x-0} = f'(c)$ for some $c \in (0, x)$.
Thus, $\frac{e^x-1}{x} = e^c$ for some c between 0 and x.
A lower bound of $f'(c)$ on $(0, x)$ is given by $f'(c) = e^c > 1$.
We therefore conclude that $\frac{e^x-1}{x} > 1$ or, equivalently, $e^x > x + 1$.

Example. It is known that any polynomial 'grows faster than' the logarithm. For, instance, show that

$$\ln x < x - 1 \quad \text{for all} \quad x > 1.$$

Suppose that $x > 1$ and consider the closed interval $[1, x]$.
We define a function $f : [1, x] \to \mathbb{R}$ by $f(t) = \ln t$.
Now, f is continuous on $[1, x]$ and differentiable on $(1, x)$
so that MVT implies that $\frac{f(x)-f(1)}{x-1} = f'(c)$ for some c in $(1, x)$.
Thus, $\frac{\ln x}{x-1} = \frac{1}{c}$ for some c between 1 and x.
An upper bound of $f'(c)$ on $(1, x)$ is given by
$f'(c) = \frac{1}{c} < 1$, since $c > 1$.
We therefore conclude that $\frac{\ln x}{x-1} < 1$ or, equivalently, $\ln x < x - 1$.

Error bounds

A second application of the mean value theorem is in calculating error bounds.

Question: How much bigger than $\frac{1}{2}$ can $\sin 31^\circ$ be?

Let $f(x) = \sin x$, and choose the interval $[a, b]$ with $a = \frac{\pi}{6}$ and $b = \frac{31\pi}{180}$ as above. (b is just 31° in radians and a is such that $\sin a = \frac{1}{2}$.)

Then

$$\frac{f(b) - f(a)}{b - a} = \frac{\sin b - \frac{1}{2}}{\pi/180}.$$

Applying MVT to f, there is some number $c \in (a, b)$ such that

$f'(c) = \cos c = \dfrac{\sin b - \frac{1}{2}}{\pi/180}$. In other words, $\sin b = \frac{1}{2} + \frac{\pi\ cosc}{180}$.

In fact $\frac{1}{\sqrt{2}} < \cos c < \frac{\sqrt{3}}{2}$ as $\frac{\pi}{6} < c < \frac{\pi}{4}$.

Thus, $$0.5123 < \frac{1}{2} + \frac{\pi}{180\sqrt{2}} < \sin b < \frac{1}{2} + \frac{\pi\sqrt{3}}{360} < 0.5152$$

The sign of a derivative

Definition. Let a function f be defined on an interval I. We say that

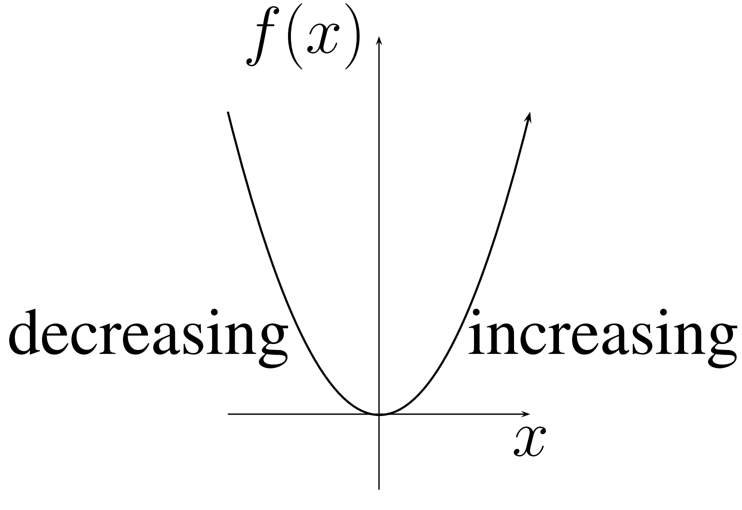
- f is increasing on I if for every two points x_1 and x_2 in I,

$$x_1 < x_2 \quad \text{implies that} \quad f(x_1) < f(x_2).$$

- f is decreasing on I if for every two points x_1 and x_2 in I,

$$x_1 < x_2 \quad \text{implies that} \quad f(x_1) > f(x_2).$$

Example. The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is increasing on $[0, 10)$ and decreasing on $[-5, 0]$.



Theorem. Let f be continuous on $[a, b]$ and differentiable on (a, b).

- If $f'(x) > 0$ for all x in (a, b) then f is increasing on $[a, b]$.
- If $f'(x) < 0$ for all x in (a, b) then f is decreasing on $[a, b]$.
- If $f'(x) = 0$ for all x in (a, b) then f is constant on $[a, b]$.

Proof. Suppose that $f'(x) > 0$ for all x in (a, b) and choose two points x_1 and x_2 in $[a, b]$ such that $x_1 < x_2$.

Since f is differentiable on (a, b), it is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). Hence, by the mean value theorem,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

for some c in (x_1, x_2).

Accordingly, $$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) > 0.$$

The remaining two statements are proven in a similar manner.

The above theorem may be directly used to classify stationary points.

Example. Find and classify all stationary points of the function $f : \mathbb{R} \to \mathbb{R}$ whose derivative is given by

$$f'(x) = (x-4)(x-1)(x+5)^2.$$

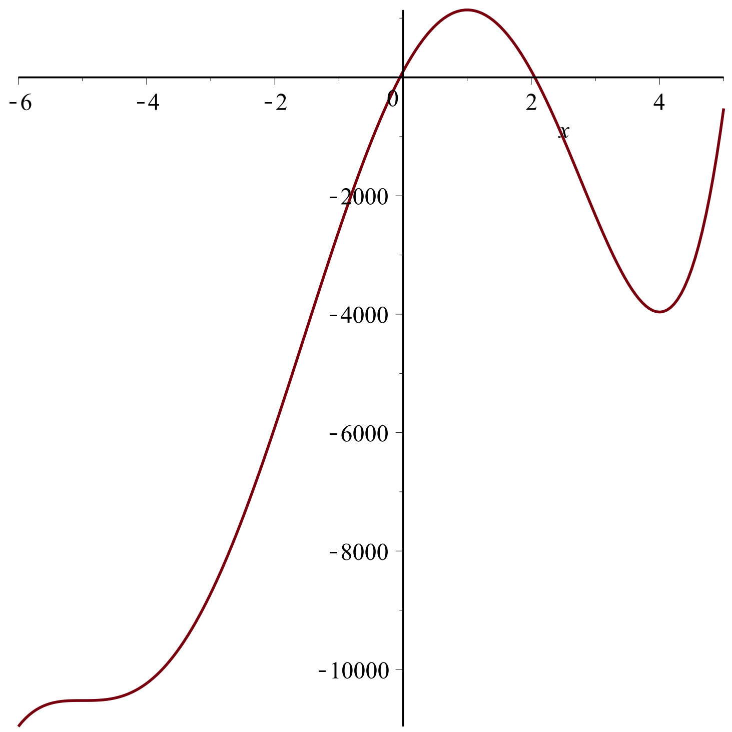
Stationary points: Set $f'(x) = 0$.
Result: $x = 4$, $x = 1$ and $x = -5$.
Classification: Investigate $f'(x)$ in a 'small' neighbourhood of any stationary point.

	-5^-	-5	-5^+	1^-	1	1^+	4^-	4	4^+
$x-4$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	0	$+$
$x-1$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$
$(x+5)^2$	$+$	0	$+$	$+$	$+$	$+$	$+$	$+$	$+$
$f'(x)$	$+$	0	$+$	$+$	0	$-$	$-$	0	$+$
Gradient	$\nearrow$		$\nearrow$	$\nearrow$		$\searrow$	$\searrow$		$\nearrow$

Result:

- $x = 4$: local minimum point
- $x = 1$: local maximum point
- $x = -5$: horizontal point of inflexion

Look at the graph: >plot($x^5/5 + 5x^4/4 - 7x^3 - 85x^2/2 + 100x$, $x = -6..5$)



The second derivative and applications

Another (potential) method for classifying the stationary points of a function f involves the second derivative of f, which is denoted by

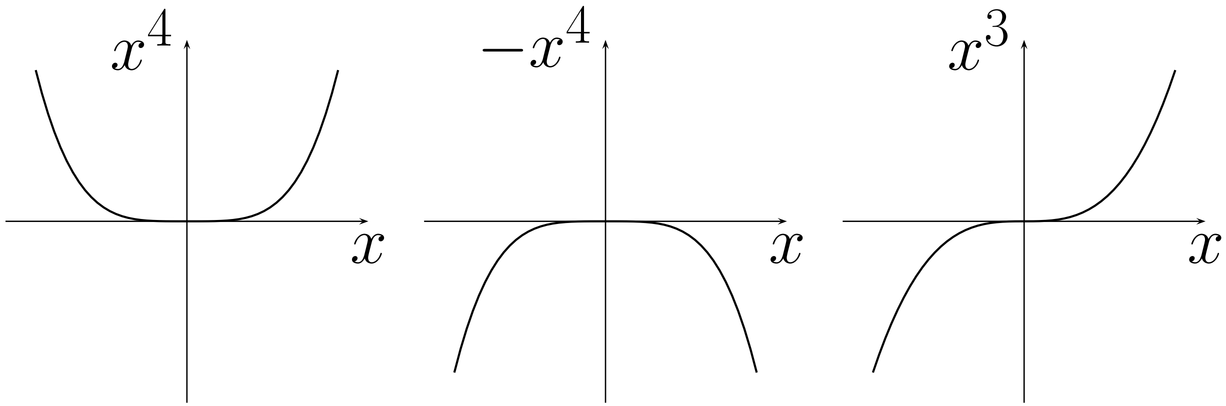
$$f'' \quad \text{or} \quad \frac{d^2y}{dx^2}, \quad \text{or} \quad y''$$

if we set $y = f(x)$.

Theorem (The second derivative test). Suppose that a function f is twice differentiable on (a, b) and that $c \in (a, b)$.

- If $f'(c) = 0$ and $f''(c) > 0$ then c is a local minimum point of f.
- If $f'(c) = 0$ and $f''(c) < 0$ then c is a local maximum point of f.

Remark. If $f'(c) = f''(c) = 0$, no conclusion may be drawn!



- If $f(x) = x^4$ then $f'(0) = f''(0) = 0$ and there is a local minimum at 0.
- If $f(x) = -x^4$ then $f'(0) = f''(0) = 0$ and there is a local maximum at 0.
- If $f(x) = x^3$ then $f'(0) = f''(0) = 0$ and there is a horizontal point of inflexion at 0.

Hence if $f'(c) = f''(c) = 0$ then it is best to classify the stationary point c by examining the sign of the derivative on either side of c!

Exercise. Find and classify the stationary points of the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = x^3 - 6x^2 + 9x - 5.$$

$f'(x) = 3x^2 - 12x + 9 = 0$, whenever, $x^2 - 4x + 3 = 0$.
So $x_1 = 1$, $x_2 = 3$ are stationary points.
$f''(x) = 6x - 12$
$f''(1) < 0$, therefore, $x = 1$ is a local maximum.
$f''(3) > 0$, therefore, $x = 3$ is a local minimum.

Critical points, maxima and minima

Question. How does one find global maxima or minima?

Definition. Suppose that f is defined on $[a, b]$. We say that a point c in $[a, b]$ is a critical point for f on $[a, b]$ if c satisfies one of the following properties:

- c is an endpoint a or b of the interval $[a, b]$,
- f is not differentiable at c,
- f is differentiable at c and $f'(c) = 0$.

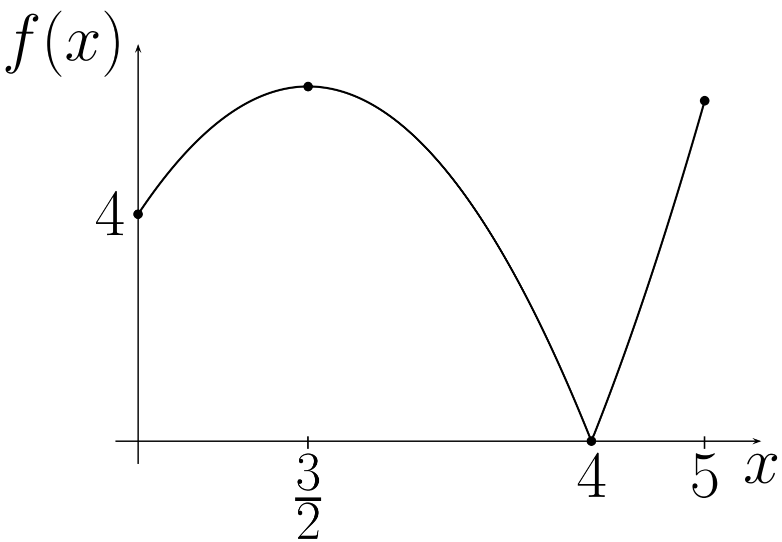
Theorem. Suppose that f is continuous on $[a, b]$. Then, f has a global maximum and global minimum on $[a, b]$. Moreover, the global maximum point and the global minimum point are both critical points for f on $[a, b]$.

Example. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is given by the rule

$$f(x) = |x^2 - 3x - 4|.$$

Find the global maximum and global minimum values of f on the interval $[0, 5]$ (cf. previous graph).

From the graph it is clear that the global minimum is at $x = 4$.
The global minimum value is $f(4) = 0$.
The global maximum point is $x = 3/2$.
The global maximum value is $f(3/2) = 6.25$
(Checking $f(5) = 6$ we see that $3/2$ is indeed maximum)



Counting zeros

Strategy. Using the information where the function is increasing and decreasing, about all maxima and minima, we may roughly sketch the graph and see how many times the graph intersects Ox.

Example. Determine the number of (real) zeros of

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = x^4 - x^3 - 3x^2 - 8x - 5$$

and give an approximate location for each zero.

Differentiating $f'(x) = (x-2)(4x^2 + 5x + 4)$ so that $f'(2) = 0$ and $f'(x) < 0$ on $(-\infty, 2)$ and $f'(x) > 0$ on $(2, \infty)$.

So f is decreasing on $(-\infty, 2)$ and therefore can not have more than one zero on this interval.

$f(-1) = 2$ and $f(0) = -5$, hence, the intermediate value theorem implies that f has zero on $(-1, 0)$.

f is increasing on $(2, \infty)$ and therefore can not have more than one zero on this interval.

$f(3) = -2$ and $f(4) = 107$, hence, the intermediate value theorem implies that f has zero on $(3, 4)$.

Result: f has two real zeros, one in the interval $(-1, 0)$ and one in the interval $(3, 4)$.

Antiderivatives

Remark. The velocity of a particle is the time-derivative of its position. Accordingly, the position of a particle may be regarded as an antiderivative of its velocity.

Definition. Suppose that f is continuous on an open interval I. A function F is said to be an antiderivative (or a primitive) of f on I if

$$F'(x) = f(x) \quad \text{for all} \quad x \in I.$$

The process of finding an antiderivative of a function is called antidifferentiation.

Remark. It is evident that if F is and antiderivative of f then G defined by

$$G(x) = F(x) + C,$$

where C is an arbitrary constant, is also an antiderivative of f.

Theorem. Suppose that f is a continuous function on an open interval I and that F and G are two antiderivatives of f on I. Then, there exists a real constant C such that

$$G(x) = F(x) + C$$

for all x in I.

Proof. Let H denote the function given by

$$H(x) = G(x) - F(x)$$

for all x in I. Then, H is differentiable on I and

$$\begin{aligned} H'(x) &= G'(x) - F'(x) \\ &= f(x) - f(x) \\ &= 0 \end{aligned}$$

for all x in I. Hence, there exists a constant C such that $H(x) = C$ for all x in I (prove it, use MVT) so that

$$G(x) = F(x) + C, \quad \text{for all } \; x \in I.$$

Some well-known antiderivatives are given below.

Function	Antiderivative
x^r, where r is rational and $r \neq -1$	$\frac{1}{r+1}x^{r+1} + C$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$
e^{ax}	$\frac{1}{a}e^{ax} + C$
$\dfrac{f'(x)}{f(x)}$	$\ln\|f(x)\| + C$

L'Hôpital's rule

Question. What is the limit of the 'indeterminate expression'

$$\lim_{x\to 0}\frac{x\sin x}{1-\cos x}\,?$$

Theorem (l'Hôpital's rule). Suppose that f and g are both differentiable functions in a neighbourhood of some $a\in\mathbb{R}$ and that either one of the two following conditions hold:

- $f(x)\to 0$ and $g(x)\to 0$ as $x\to a$
- $f(x)\to\infty$ and $g(x)\to\infty$ as $x\to a$.

If

$$\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

exists then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

Remark. The theorem also holds for

- limits as $x \to \infty$ or $x \to -\infty$
- one-sided limits (as $x \to a^+$ or $x \to a^-$).

L'Hôpital's rule is proved using the mean value theorem!

Example. Determine the limit

$$\lim_{x\to 0} \frac{1 - \cos x}{x}.$$

$$\boxed{\lim_{x\to 0} \tfrac{1-\cos x}{x} = \lim_{x\to 0} \tfrac{\sin x}{1} = 0.}$$

Remark. l'Hôpital's rule may be applied iteratively.

Example.

$$\lim_{x\to 0} \frac{x - \sin x}{x^2} = \frac{1}{2} \lim_{x\to 0} \frac{1 - \cos x}{x} = \text{... see above.}$$

Remark. It is important that the limit exists after a finite number of applications of l'Hôpital's rule!

Exercise. What is the limit

$$\lim_{x\to\infty} \frac{2x - \sin x}{3x + \sin x}$$

and why can l'Hôpital's rule not be applied?

$$\lim_{x\to\infty} \frac{2x - \sin x}{3x + \sin x} = \lim_{x\to\infty} \frac{2 - \frac{\sin x}{x}}{3 + \frac{\sin x}{x}} = \frac{2}{3}.$$

We can not apply l'Hôpital's rule, since the limit

$$\lim_{x\to\infty} \frac{2 - \cos x}{3 + \cos x}$$

does not exist (prove this).

Example. Find $\lim_{x \to 0^+} x \ln x$.

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x} = \left(\frac{\infty}{\infty}\right) =$$

$$= \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} -x = 0.$$

The use of L'Hôpital's rule was justified as the final limit exists.

Example. Find $\lim_{x \to 0} \frac{\tan x - \sin x}{x^3}$.

($\frac{0}{0}$ form)

$$\lim_{x\to 0} \frac{\tan x - \sin x}{x^3} = \lim_{x\to 0} \frac{\sec^2 x - \cos x}{3x^2} = \text{(still } \tfrac{0}{0}\text{)}$$

$$= \lim_{x\to 0} \frac{2\sec^2 x \tan x + \sin x}{6x} = \text{(still } \tfrac{0}{0}\text{)}$$

Differentiating again and simplifying

$$= \lim_{x\to 0} \frac{\cos^5 x - 4\cos^2 x + 6}{6\cos^4 x} = \frac{1}{2}.$$

The use of L'Hôpital's rule was justified as the final limit exists.