THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Semester 1 2014

MATH1131
MATHEMATICS 1A

(1) TIME ALLOWED – 2 hours

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

(8) A SHORT TABLE OF INTEGRALS IS APPENDED TO THE PAPER

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

Use a separate book clearly marked Question 1

1. i) Let $z = 7 + i$ and $w = 4 + 3i$.

   a) Find $2z - \overline{w}$ in $a + ib$ form.

   b) Find $5(w - i)/z$ in $a + ib$ form.

   c) Find $|zw|$.

   d) Find $\text{Arg}(zw)$.

   e) Hence, or otherwise, show that $\text{Arg}(z) + \text{Arg}(w) = \frac{\pi}{4}$.

   f) Use the polar form of zw to evaluate $(zw)^{40}$.

ii) Consider the following system of equations:

$$\begin{array}{rcrcrcl} x & - & y & - & z & = & 1 \\ x & - & 3y & + & z & = & 1 \\ 2x & - & 3y & - & z & = & 2. \end{array}$$

   a) Write the system in augmented matrix form and reduce it to row echelon form.

   b) Solve the system.

iii) Evaluate the limits

   a)
$$\lim_{x \to \infty} \frac{6x^2 + \sin x}{4x^2 + \cos x};$$

   b)
$$\lim_{x \to 0} \frac{e^{2x} - 2x - 1}{4x^2}.$$

iv) A function g is defined by

$$g(x) = \begin{cases} \dfrac{|x^2 - 16|}{x - 4} & \text{if } x \neq 4 \\ \alpha & \text{if } x = 4. \end{cases}$$

By considering the left and right hand limits at $x = 4$, show that no value of α can make g continuous at the point $x = 4$.

v) Let $f(x) = x^5 + x^3 + x - 2$.

   a) Prove that f has at least one real root in the interval $[0, 2]$, naming any theorems you use.

   b) State, with reasons, the number of real roots of f.

Use a separate book clearly marked Question 2

2. i) Use a substitution to find the integral

$$\int \frac{dx}{x(1+(\log x)^2)}.$$

ii) a) Give the definitions of $\sinh x$ and $\cosh x$ in terms of the exponential function.

b) Use your definitions to prove that $\sinh(2x) = 2\sinh x \cosh x$.

iii) Evaluate the integral

$$\int_0^{\pi/3} x \sin(2x)\, dx.$$

iv) Simplify the matrix expression $(A^TA)^{-1}(A^TA)^T$, where A is an invertible matrix.

v) Consider the three points A, B, C in $\mathbb{R}^3$ with position vectors $\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$, and $\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$ respectively.

a) Find a parametric vector form for the plane Π that passes through points A, B, and C.

b) Calculate the cross product $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC}$, showing your working.

c) Hence, or otherwise, find a **Cartesian** equation for the plane Π.

d) Find the area of the triangle ABC.

e) Find the minimal distance from the point $P \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix}$ to the plane Π.

vi) Consider the following MAPLE session.

```
> with(LinearAlgebra):
> A := <<0,1,1>|<1,-1,-1>|<-2,1,0>>;
```

$$A := \begin{bmatrix} 0 & 1 & -2 \\ 1 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

```
> B := A^2;
```

$$B := \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & -3 \\ -1 & 2 & -3 \end{bmatrix}$$

```
> C:=A^3;
```

$$C := \begin{bmatrix} 2 & -3 & 3 \\ -2 & 2 & 1 \\ -1 & 0 & 4 \end{bmatrix}$$

```
> F := 2A + B + C;
```

$$F := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Using the above MAPLE session, or otherwise, find the 3×3 matrix which is the inverse of the matrix A.

Use a separate book clearly marked Question 3

3. i) A block of wood is subject to 3 vector forces:
$\mathbf{F}_1 = 1$, in the direction West, $\mathbf{F}_2 = 1$ in the direction South, $\mathbf{F}_3 = 2$ in the direction North-East, each measured in Newtons.
Let $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$ be the resultant force on the block.

a) On a scale diagram draw the 3 forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ and the resultant force $\mathbf{F}$.

b) Find the exact value of $|\mathbf{F}|$ and the direction of $\mathbf{F}$.

ii) Let $M = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 3 \\ -1 & 2 & 1 \end{pmatrix}$ and $N = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$.

a) Evaluate the determinant of M.

b) Write down the inverse of N.

iii) Let $p(z) = z^4 + 2z^2 - 3$.

a) Show that $p(1) = p(-1) = 0$.

b) Factor $p(z)$ into two real quadratic polynomials $q(z)$ and $r(z)$.

c) Find the roots of $p(z)$.

d) Factor $p(z)$ into four complex linear polynomials.

iv) Consider the line ℓ and the plane Π given by the following equations:

$$\ell : \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \lambda \in \mathbb{R},$$

$$\Pi : 6x + 8y - 9z = 0.$$

Determine the point of intersection of the line ℓ and the plane Π.

v) a) Use De Moivre's Theorem to prove that $4\cos^3\theta = \cos 3\theta + 3\cos\theta$.

b) Deduce that $2\cos\frac{\pi}{9}$ is a root of the polynomial $q(z) = z^3 - 3z - 1$.

vi) Let $\mathbf{u} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ \beta \end{pmatrix}$ be two vectors in $\mathbb{R}^3$.

a) Find the value of β so that the vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

b) For the value $\beta = 0$, find the projection, $\text{proj}_{\mathbf{v}}\,\mathbf{u}$, of $\mathbf{u}$ onto $\mathbf{v}$.

c) Find the value of β so that the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\frac{\pi}{4}$.

Use a separate book clearly marked Question 4

4. i) Use the Fundamental Theorem of Calculus to find
$$\frac{d}{dx}\int_{x^2}^{x^3} \cos\left(\frac{1}{t}\right)\, dt.$$

ii) Let $f(x) = \dfrac{175x^2 - 350x + 10}{x^2 - 2x + 2}$.
Consider the following MAPLE session.

```
> f:=(175*x^2-350*x+10)/(x^2-2*x+2);
```
$$\frac{175\,x^2 - 350\,x + 10}{x^2 - 2\,x + 2}$$
```
> subs(x=0.0,f);
```
$$5.0$$
```
> subs(x=4.0,f);
```
$$141.0$$
```
> solve(f=0,x);
```
$$1 + \sqrt{1155}/35,\ 1 - \sqrt{1155}/35$$
```
> evalf(%);
```
$$1.971008312,\ 0.0289916875$$
```
> fdash:=diff(f,x);
```
$$\frac{350\,x - 350}{x^2 - 2\,x + 2} - \frac{(175\,x^2 - 350\,x + 10)\,(2\,x - 2)}{(x^2 - 2\,x + 2)^2}$$
```
> solve(fdash=0,x);
```
$$1$$
```
> subs(x=1.0,f);
```
$$-165.0$$

a) Use the information in the MAPLE output to give a rough sketch of $f(x) = \dfrac{175x^2 - 350x + 10}{x^2 - 2x + 2}$, for $0 \le x \le 4$.

b) Hence, or otherwise, find the maximum and minimum values of $f(x) = \left|\dfrac{175x^2 - 350x + 10}{x^2 - 2x + 2}\right|$ over the closed interval $[0, 4]$.

iii) Determine, with reasons, whether the improper integral
$$K = \int_0^{\infty} \frac{dx}{e^{2x} + \cos^2 x}$$
converges or diverges.

iv) Sketch in the xy-plane, the graph of the polar curve given by $r = 1 - \cos\theta$. (You are NOT required to find the slope of the curve.)

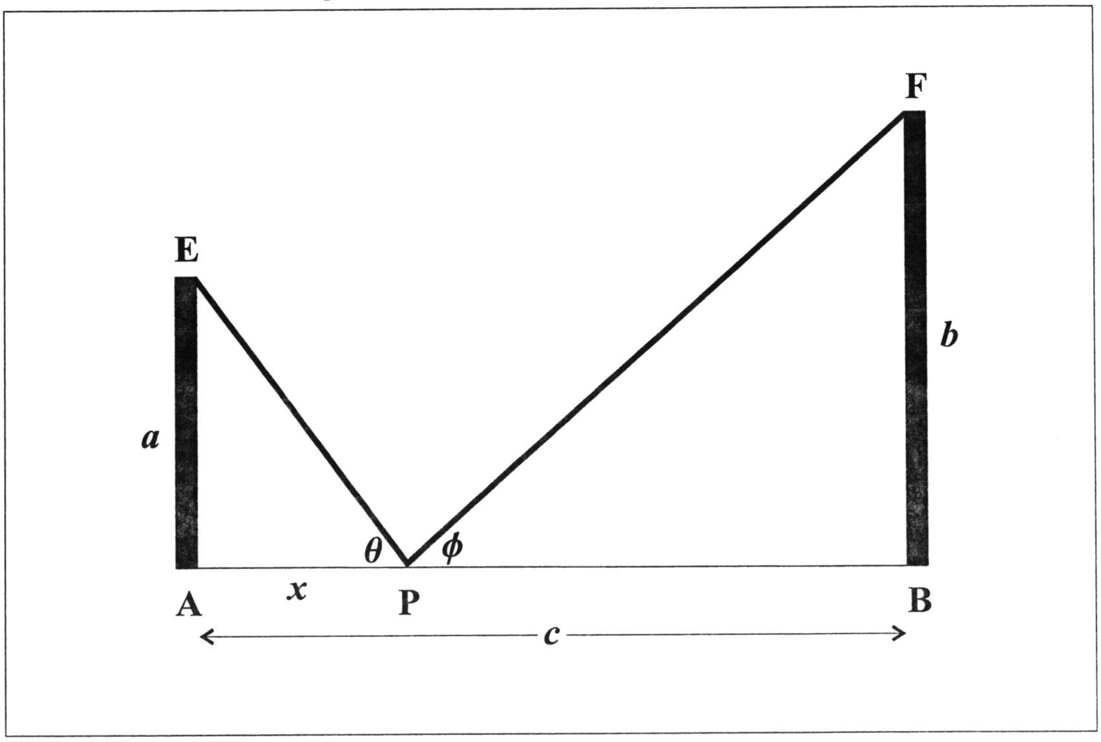
v) a) State carefully the Mean Value Theorem.

b) Suppose $-1 < x < y < 1$. By applying the Mean Value Theorem to the function $f(t) = \sin^{-1} t$ on the interval $[x, y]$, prove that

$$\sin^{-1} y - \sin^{-1} x \geq y - x.$$

vi) Two poles A and B, of heights a metres and b metres respectively, are c metres apart on the horizontal ground. A single tight rope runs from the top of pole A to the point P on the ground between A and B and then to the top of pole B.

Assume that the distance from A to P is x, and that the angles θ and ϕ are as shown in the diagram.



a) Explain why the length L of the rope is given by

$$L = \sqrt{a^2 + x^2} + \sqrt{b^2 + (c - x)^2}.$$

b) Prove that $\cos\theta = \cos\phi$ when $\dfrac{dL}{dx} = 0$.

c) Assuming that $\cos\theta = \cos\phi$ minimizes L, using similar triangles, or otherwise, find the value of x that minimizes L.

BASIC INTEGRALS

$$\int \frac{1}{x}\,dx = \ln|x| + C = \ln|kx|, \qquad C = \ln k$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

$$\int a^x\,dx = \frac{1}{\ln a}a^x + C, \qquad a \neq 1$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$$

$$\int \operatorname{cosec}^2 ax\,dx = -\frac{1}{a}\cot ax + C$$

$$\int \tan ax\,dx = \frac{1}{a}\ln|\sec ax| + C$$

$$\int \cot ax\,dx = \frac{1}{a}\ln|\sin ax| + C$$

$$\int \sec ax\,dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C$$

$$\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C$$

$$\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C$$

$$\int \operatorname{sech}^2 ax\,dx = \frac{1}{a}\tanh ax + C$$

$$\int \operatorname{cosech}^2 ax\,dx = -\frac{1}{a}\coth ax + C$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$\begin{aligned}\int \frac{dx}{a^2 - x^2} &= \frac{1}{a}\tanh^{-1}\frac{x}{a} + C, && |x| < a\\ &= \tfrac{1}{a}\coth^{-1}\tfrac{x}{a} + C, && |x| > a > 0\\ &= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C, && x^2 \neq a^2\end{aligned}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\frac{x}{a} + C, \qquad x \geqslant a > 0$$