The logarithmic and exponential functions

- In the preceding, we have manipulated with functions such as

$$\ln x, \quad e^x, \quad x^\pi$$

  even though we have not defined them formally.

- In particular, we are familiar with the important formula

$$\ln(st) = \ln s + \ln t.$$

Question. Consider the functional equation

$$f(st) = f(t) + f(s), \tag{1}$$

where s and t are independent variables.

It is evident that $f = \ln$ is one solution of this equation.

Are there other functions f obeying this functional equation?

Answer. We first note that (1) evaluated at $s = t = 1$ yields

$$f(1) = 0.$$

Moreover, differentiation of (1) with respect to s leads to

$$t f'(ts) = f'(s)$$

so that at $s = 1$

$$f'(t) = \frac{1}{t} f'(1).$$

If we now demand that $f'(1) = 1$ then f is uniquely determined via integration since $f(1) = 0$.

Conclusion. A function f is uniquely defined by the functional equation

$$f(st) = f(s) + f(t),$$

subject to

$$f'(1) = 1.$$

It is given by (thanks to the FTC1)

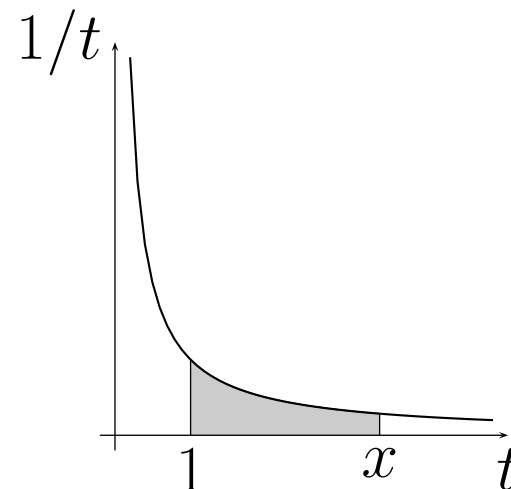$$f(x) = \int_1^x \frac{1}{t}\, dt.$$

The natural logarithm function

Definition. The natural logarithm function

$$\ln : (0, \infty) \to \mathbb{R}$$

is defined by the formula

$$\ln x = \int_1^x \frac{1}{t}\, dt.$$

$\ln x$ is the area of the shaded region.

Theorem. The function $\ln : (0, \infty) \to \mathbb{R}$ has the following properties:

(i) $\ln$ is differentiable on $(0, \infty)$ and

$$\frac{d}{dx} \ln x = \frac{1}{x}.$$

Proof. Apply the first fundamental theorem of calculus to the definition of $\ln$.

(ii) $\ln x > 0 \quad$ for $\quad x > 1,$

$\ln 1 = 0,$

$\ln x < 0 \quad$ for $\quad 0 < x < 1.$

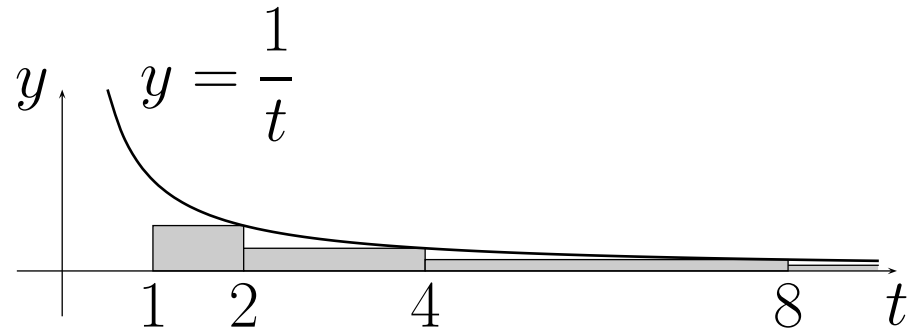**Proof.** This follows from the definition of $\ln$ and the fact that $\frac{1}{t} > 0$ when $t > 0.$

(iii) $\ln x \to -\infty$ as $x \to 0^+$,

$\ln x \to \infty$ as $x \to \infty$.

Proof. The diagram below shows that

$$\int_1^2 \frac{dt}{t} \geq 1 \times \frac{1}{2}, \quad \int_2^4 \frac{dt}{t} \geq 2 \times \frac{1}{4}, \quad \int_4^8 \frac{dt}{t} \geq 4 \times \frac{1}{8}.$$

In general,

$$\int_1^{2^n} \frac{dt}{t} \geq \underbrace{\frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}}_{n \text{ terms}} = \frac{n}{2} \to \infty$$

as $n \to \infty$.

Hence the improper integral

$$\int_1^\infty \frac{1}{t}\,dt$$

'diverges to infinity' and, therefore, $\ln x \to \infty$ as $x \to \infty$.

This argument can be adapted to show that

$$-\int_1^{2^{-n}} \frac{dt}{t} = \int_{2^{-n}}^1 \frac{dt}{t} \to \infty$$

as $n \to \infty$. Hence $\ln x \to -\infty$ as $x \to 0^+$.

(iv) For all $x, y > 0$: $\qquad\qquad \ln(xy) = \ln x + \ln y.$

Proof. Suppose that y is some fixed positive number and that $x > 0$. Then, the chain rule implies that

$$\frac{d}{dx}[\ln(xy)] = \frac{1}{xy}\frac{d}{dx}(xy) = \frac{y}{xy} = \frac{1}{x} = \frac{d}{dx}\ln x.$$

Accordingly,
$$\ln(xy) = \ln(x) + C$$
for some constant C.

Evaluation at $x = 1$ leads to
$$\ln(y) = C$$
and hence
$$\ln(xy) = \ln(x) + \ln(y).$$

(v) For all $x, y > 0$: $\qquad\qquad \ln\left(\dfrac{x}{y}\right) = \ln(x) - \ln(y).$

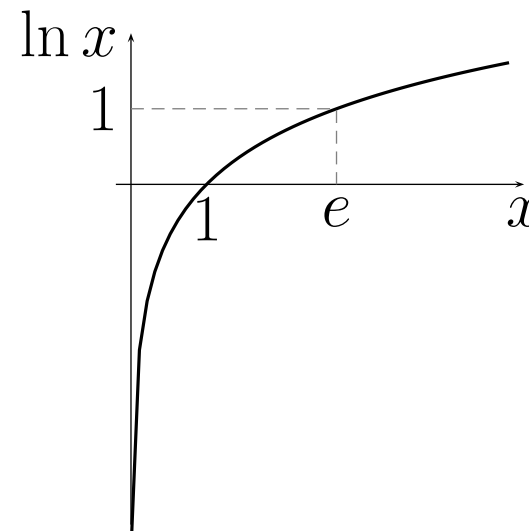(vi) For all $x > 0$ and $r \in \mathbb{Q}$: $\qquad \ln(x^r) = r \ln x.$

Exercise. Prove (v) and (vi) (use technique from the proof of (iv)).

Remark. The above properties imply that

- $\mathrm{Range}(\ln) = \mathbb{R}$, and

- $\ln$ is increasing and hence invertible so that

- $\ln x = 1$ has a unique solution.

Definition. The real number e is defined to be the unique number x satisfying
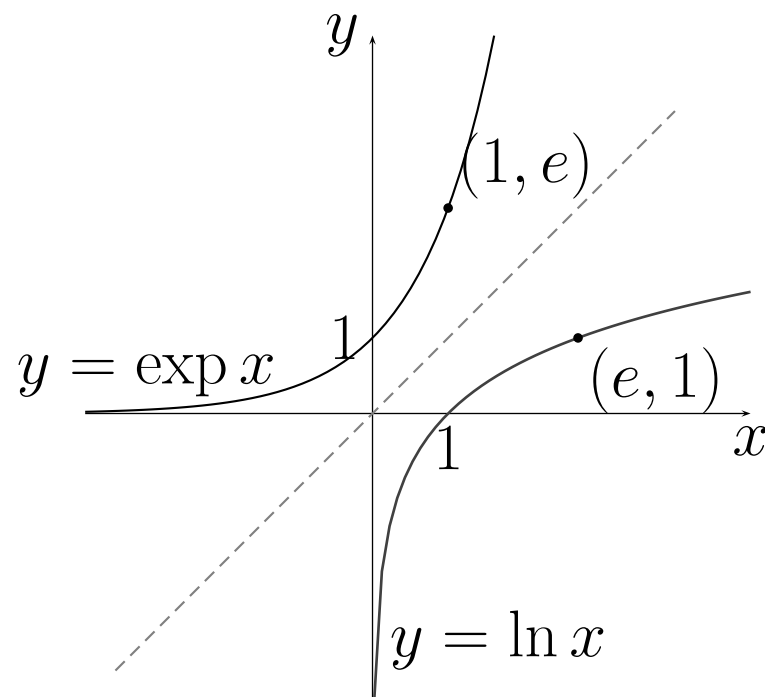
$$\ln x = 1.$$

The exponential function

Definition. The function

$$\exp : \mathbb{R} \to (0, \infty)$$

is defined to be the inverse function of $\ln : (0, \infty) \to \mathbb{R}$.

Remark. For any rational number r, we can evaluate both

$$\exp r \qquad \text{and} \qquad e^r$$

but are these two numbers the same?

Theorem. The function $\exp : \mathbb{R} \to (0, \infty)$ has the following properties:

(i) $\exp(\ln x) = x$ for all $x \in (0, \infty)$,

    $\ln(\exp x) = x$ for all $x \in \mathbb{R}$.

(ii) $\exp(1) = e$ and $\exp(0) = 1$.

(iii) $\exp x \to \infty$ as $x \to \infty$, $\exp x \to 0$ as $x \to -\infty$.

Proof of (i)-(iii) follows from the definition of $\exp$.

(iv) $\exp$ is differentiable on $\mathbb{R}$ with
$$\frac{d}{dx}\exp x = \exp x.$$

Proof. The function $\exp$ is differentiable on $\mathbb{R}$ by virtue of the inverse function theorem and differentiation of
$$\ln(\exp x) = x$$

produces
$$\frac{1}{\exp x}\frac{d}{dx}\exp x = 1$$

(v) For all $x, y \in \mathbb{R}$: $\exp(x + y) = \exp x \, \exp y.$

Proof. For any x and y, we have
$$\exp(x + y) = \exp\left(\ln(\exp x) + \ln(\exp y)\right)$$
$$= \exp\left(\ln(\exp x \, \exp y)\right)$$
$$= \exp x \, \exp y$$

(vi) For all $r \in \mathbb{Q}$ and $x \in \mathbb{R}$: $\quad \exp(rx) = (\exp x)^r$.

Proof. Suppose that $r \in \mathbb{Q}$ and $x \in \mathbb{R}$. Then,

$$\exp(rx) = \exp\left(r \ln \exp x\right)$$
$$= \exp\left(\ln\left((\exp x)^r\right)\right)$$
$$= (\exp x)^r.$$

Remark. In particular, the above theorem implies that

$$\exp(r) = (\exp 1)^r = e^r.$$

for every rational number r.

It is therefore consistent to make the following definition.

Definition. For any $x \notin \mathbb{Q}$, we define the number e^x to be

$$e^x = \exp x.$$

Note. The above definition 'merely' means that e^x is the unique real number R such that

$$\int_1^R \frac{1}{t}\, dt = x$$

for any $x \in \mathbb{R}$ (Uniqueness follows from the properties of the function $\ln$).

By construction, the function

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \exp x = e^x$$

is differentiable (and continuous) and is called the exponential function.

Exponentials and logarithms with other bases

Question. How would one define b^x for $x \notin \mathbb{Q}$ and $b > 0$?

Since, for any rational number r,

$$b^r = \exp\left(\ln(b^r)\right) = \exp\left(r \ln b\right) = e^{r \ln b},$$

the following definition is natural.

Definition. Suppose that $b > 0$ and $x \notin \mathbb{Q}$. Then, the number b^x is defined by

$$b^x = \exp(x \ln b) = e^{x \ln b}.$$

Note. By combining the defintions of b^x for rational x and irrational x, we now obtain a well-defined function

$$f_b : \mathbb{R} \to (0, \infty), \quad f_b(x) = b^x = \exp(x \ln b)$$

for any $b > 0$.

Example. It is seen that

$$f_3(2) = 3^2 = \exp(2\ln 3) = \exp(\ln 3)\,\exp(\ln 3) = 3 \times 3 = 9$$

as one would expect!

Remark. Since f_b is a combination of continuous and differentiable functions, it is also continuous and differentiable with

$$f_b'(x) = (\ln b)\exp(x\ln b) = (\ln b)b^x.$$

Accordingly,

- if $\;\; b > 1 \;\;$ then $\;\; f_b'(x) > 0 \;\;$ for all $\;\; x \in \mathbb{R}$,
- if $\;\; 0 < b < 1 \;\;$ then $\;\; f_b'(x) < 0 \;\;$ for all $\;\; x \in \mathbb{R}$

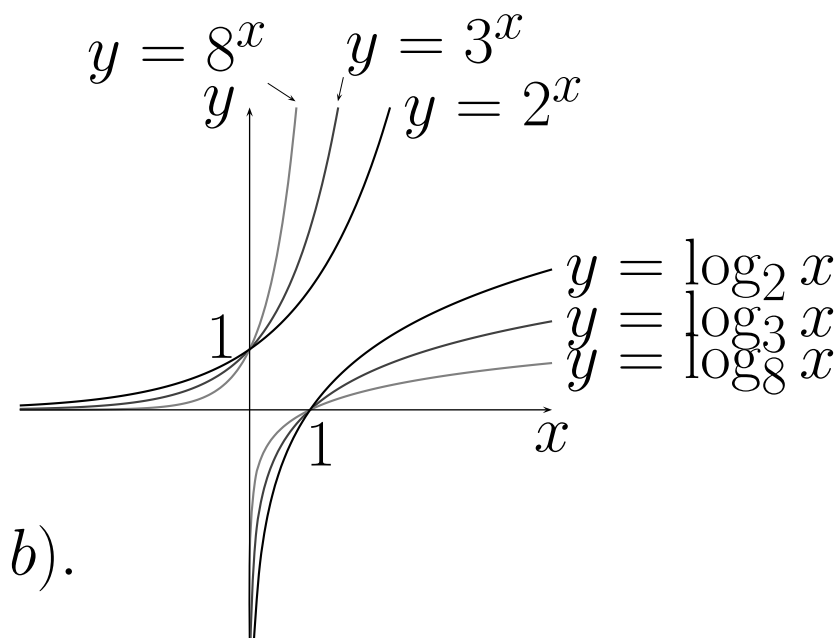so that f_b is invertible for $b \neq 1$ (we had such a theorem).

Definition. Suppose that b is a positive real number with $b \neq 1$. Then, the logarithm function to the base b

$$\log_b : (0, \infty) \to \mathbb{R}$$

is defined to be the inverse of the function

$$f_b : \mathbb{R} \to (0, \infty), \quad f_b(x) = b^x = \exp(x \ln b).$$

In particular, $\log_e x = \ln x$.

$y = 8^x \quad y = 3^x$

$y = 2^x$

$y = \log_2 x$
$y = \log_3 x$
$y = \log_8 x$

Remark. The above definition may be stated as

$$y = b^x \quad \Leftrightarrow \quad x = \log_b y.$$

The following theorem demonstrates that all logarithm functions are just scaled versions of the natural logarithm function.

Theorem. Suppose that b is a positive real number with $b \neq 1$. Then

$$\log_b x = \frac{\ln x}{\ln b}$$

for all $x > 0$.

Proof. Since

$$x = b^{\log_b x} = \exp(\log_b x \ln b),$$

we conclude that

$$\ln x = \log_b x \ln b.$$

Hence, $\log_b$ shares all the properties of $\ln$ such as

$$\frac{d}{dx} \log_b x = \frac{d}{dx} \left(\frac{\ln x}{\ln b} \right) = \frac{1}{x \ln b}$$

or

$$\log_b(x^y) = y \log_b x.$$

Integration and the $\ln$ function

Since

$$\frac{d}{dx}\ln(x) = \frac{1}{x}, \quad x > 0, \quad \text{and} \quad \frac{d}{dx}\ln(-x) = \frac{1}{x}, \quad x < 0,$$

the function $\ln(x)$ is an antiderivative of $1/x$ if $x > 0$ and $\ln(-x)$ is an antiderivative of $1/x$ if $x < 0$. Thus,

$$\int \frac{1}{x}\, dx = \ln|x| + C$$

provided that x is restricted to an interval which does NOT contain 0.

This may be generalised to

$$\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + C$$

provided that f is differentiable and does not vanish on the interval of integration.

Example. On any interval not including zeros of $\cos x$, we have

$$\int \tan x \, dx = -\int \frac{-\sin x}{\cos x} \, dx = -\ln|\cos x| + C.$$

Example. Find the indefinite integral

$$\int \frac{1}{2\sec x + \tan x} \, dx.$$

Rearranging
$$\int \frac{1}{2\sec x + \tan x} \, dx = \int \frac{1}{\frac{2}{\cos x} + \frac{\sin x}{\cos x}} \, dx = \int \frac{\cos x}{2 + \sin x} \, dx.$$
and observing that
$2 + \sin x \neq 0$ for all $x \in \mathbb{R}$
and $(2 + \sin x)' = \cos x$
we have
$\int \frac{1}{2\sec x + \tan x} \, dx = \ln|2 + \sin x| + C.$

Logarithmic differentiation

Logarithms are powerful in that they 'transform' powers into products, products into sums and quotients into differences.

Example. Find the derivative of

$$y = \left(\frac{(3x^2 + 4)(x + 2)}{x^3 + 5x} \right)^{3/5}.$$

The idea is to take $\ln$ of both sides of the equation to obtain

$$\ln y = \frac{3}{5} \ln \left(\frac{(3x^2 + 4)(x + 2)}{x^3 + 5x} \right)$$

$$= \frac{3}{5} \left(\ln(3x^2 + 4) + \ln(x + 2) - \ln(x^3 + 5x) \right).$$

Differentiating both sides with respect to x is relatively easy and leads to

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{5}\left(\frac{6x}{3x^2+4} + \frac{1}{x+2} - \frac{3x^2+5}{x^3+5x}\right).$$

Hence, we obtain

$$\frac{dy}{dx} = \frac{3}{5}\left(\frac{(3x^2+4)(x+2)}{x^3+5x}\right)^{3/5}\left(\frac{6x}{3x^2+4} + \frac{1}{x+2} - \frac{3x^2+5}{x^3+5x}\right).$$

Remark. The above procedure is only valid for intervals on which $y > 0$.

Example. Consider the function

$$f : (0, \pi) \to \mathbb{R}, \quad f(x) = x^{\sin x}.$$

Determine its derivative.

Note first that $y > 0$.
Taking logarithm of $y = x^{\sin x}$, we have
$\ln y = \sin x \ \ln x$.
Differentiating both sides of the equality above, we obtain
$\frac{1}{y}\frac{dy}{dx} = \cos x \ \ln x + \frac{1}{x}\sin x$.
Multiplying by $y = x^{\sin x}$, we have
$\frac{dy}{dx} = x^{\sin x}\left(\cos x \ \ln x + \frac{1}{x}\sin x\right)$.

Indeterminate forms with powers

Consider the limits

$$\lim_{x \to 0^+} x^x \qquad \text{and} \qquad \lim_{x \to \infty} x^{1/x}.$$

The first limit is of the form 0^0 while the second is of the form ∞^0.

Since each limit involves a power, it is natural to first take the logarithm of the limit and then bring l'Hôpital's rule into play.

Example.

Evaluate the limit

$$\lim_{x \to 0^+} x^{2x}.$$

By taking the natural logarithm, we can transform the limit into an indeterminate form of the type $\frac{\infty}{\infty}$:

$$\lim_{x \to 0^+} x^{2x} = \lim_{x \to 0^+} \exp\left(\ln x^{2x} \right) \qquad \text{(since ln and exp are inverses)}$$

$$= \lim_{x \to 0^+} \exp\left(2x \ln x \right)$$

$$= \exp\left(\lim_{x \to 0^+} 2x \ln x \right) \qquad \text{(since exp is continuous)}$$

$$= \exp\left(\lim_{x \to 0^+} \frac{\ln x}{1/(2x)} \right).$$

We can now apply l'Hôpital's rule to the problem. By differentiating the numerator and denominator and then simplifying we obtain

$$\lim_{x \to 0^+} x^{2x} = \exp\left(\lim_{x \to 0^+} \frac{1/x}{-1/(2x^2)} \right)$$

$$= \exp\left(\lim_{x \to 0^+} -2x \right)$$

$$= \exp(0) = 1.$$

The following example is of the indeterminate form "1^∞".

Example. Show that

$$\lim_{x \to \infty} \left(1 + \frac{t}{x}\right)^x = e^t,$$

where t is a constant real parameter.

Since $\ln$ is continuous function, we have
$$\exp \ln(\lim_{x \to \infty} \left(1 + \frac{t}{x}\right)^x) = \exp \lim_{x \to \infty} \ln \left(1 + \frac{t}{x}\right)^x = \exp(\lim_{x \to \infty} x \ln \left(1 + \frac{t}{x}\right)) = \dots$$
dividing by x, we obtain
$$\dots = \exp(\lim_{x \to \infty} \frac{\ln\left(1 + \frac{t}{x}\right)}{\frac{1}{x}}) = \exp("\frac{0}{0}") = \dots$$
using L'Hopital's rule
$$\dots = \exp(\lim_{x \to \infty} \frac{\frac{1}{\left(1 + \frac{t}{x}\right)} \left(-\frac{t}{x^2}\right)}{-\frac{1}{x^2}}) = \exp(\lim_{x \to \infty} \frac{t}{\left(1 + \frac{t}{x}\right)}) = \exp t = e^t.$$