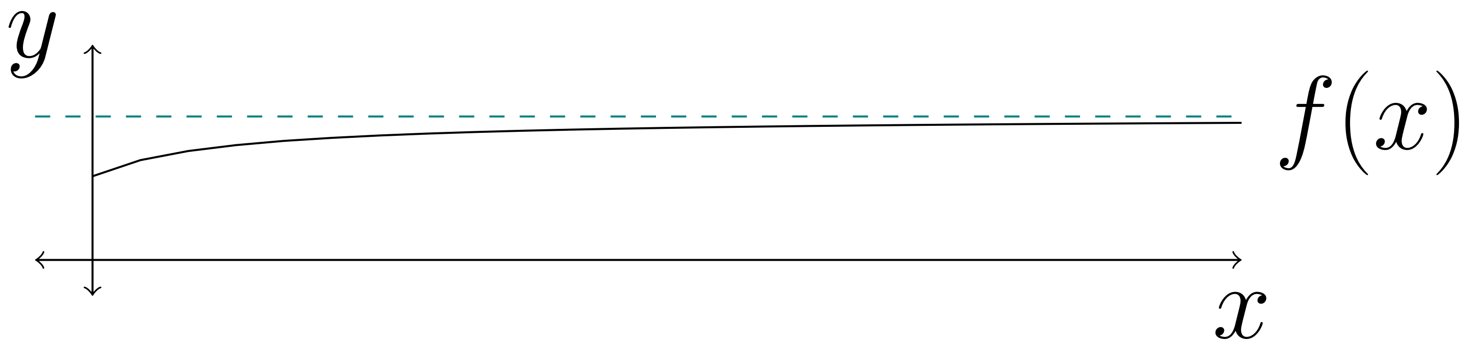
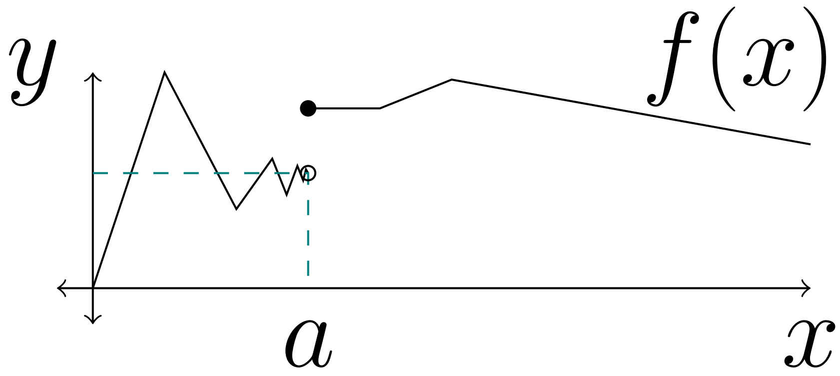
Limits

Limit is the fundamental concept in calculus. There two main types of limits.

Limits at ∞. *What is the long term behaviour of the function f?*

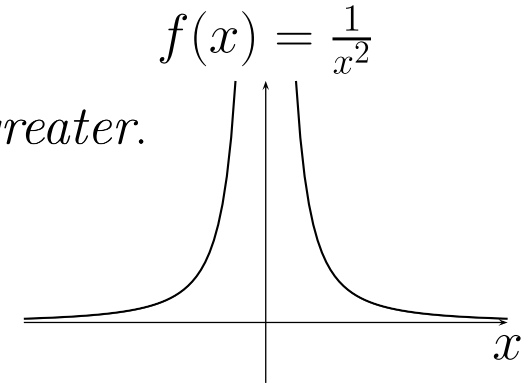


Limits at a point. *What is the local behaviour of f for x near some point $a \in \mathbb{R}$?*



Limits of functions at infinity

Rough definition.



- We say that $f(x)$ has **limit** L **as** x **goes to** ∞ if $f(x)$ gets *closer and closer* to L as x gets *greater and greater*. In this case, we write

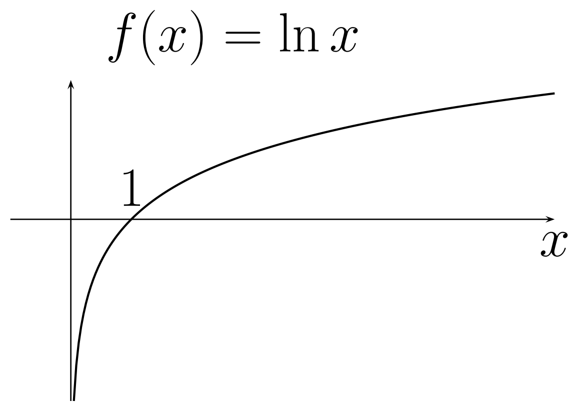
$$\lim_{x \to \infty} f(x) = L,$$

or

$$f(x) \to L \quad \text{as} \quad x \to \infty.$$

- If $f(x)$ gets 'arbitrarily large' (that is, 'approaches' ∞) as x tends to ∞, then we say also that the limit does not exist and we write

$$f(x) \to \infty \quad \text{as} \quad x \to \infty.$$



Example. Why do we believe that

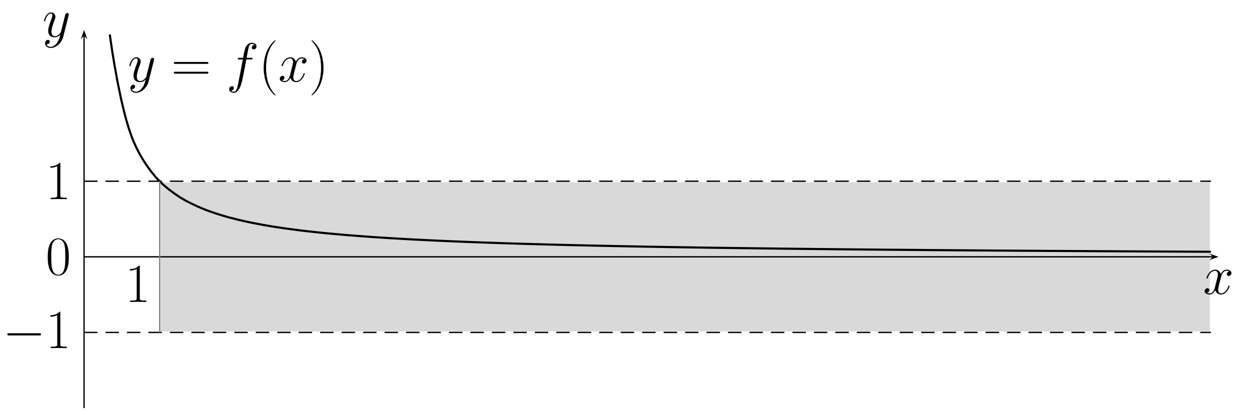
$$\lim_{x\to\infty} f(x) = 0 \quad \text{for} \quad f(x) = \frac{1}{x}\,?$$

Consider the distance between $f(x)$ and 0 denoted by

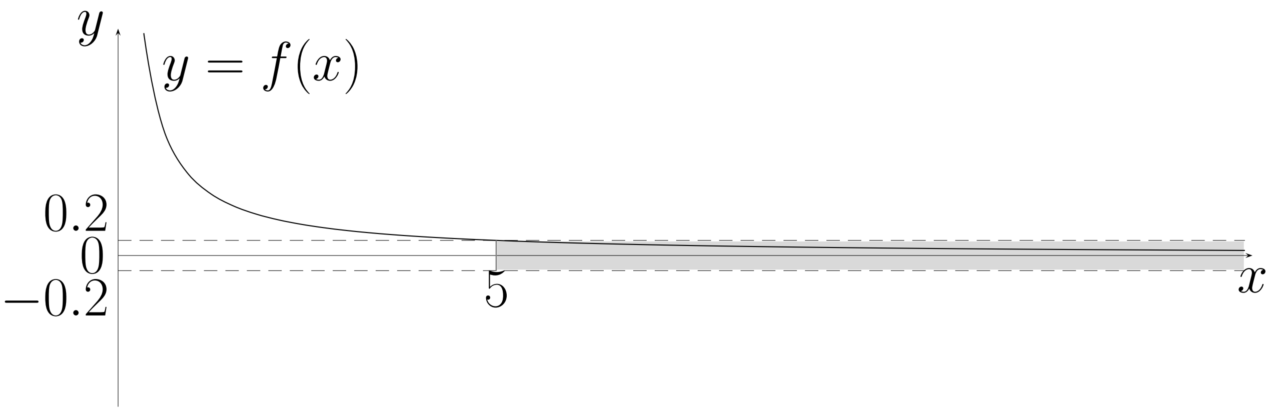
$$\text{error}(x) = |f(x) - 0|.$$

Facts.

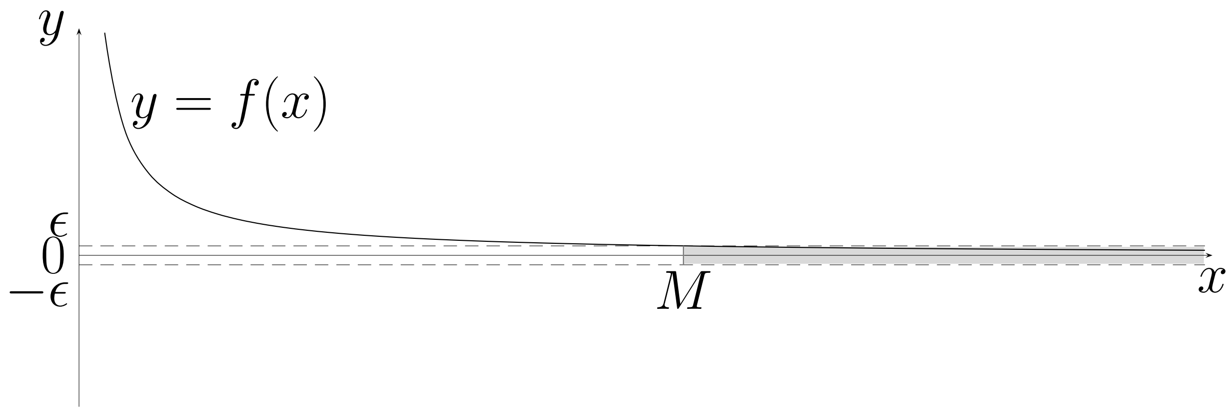
- $\text{error}(x) < 1$ whenever $x > 1$.



- $\text{error}(x) < 0.2$ whenever $x > 5$.



- $\text{error}(x) < 0.1$ whenever $x > 10$.
- $\text{error}(x) < 0.01$ whenever $x > 100$.
- $\text{error}(x) < 0.0001$ whenever $x > 10000$.
- Set $\epsilon = 1/M$. Then, $\text{error}(x) < \epsilon$ whenever $x > M$.

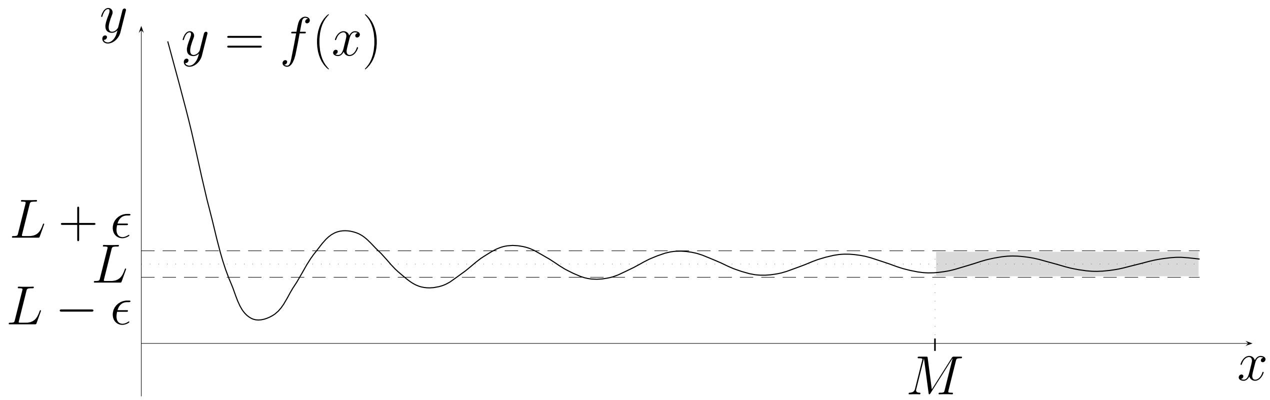


Definition. Let f be a function defined on some interval (b, ∞) and let L be a real number. We say that

$$\lim_{x \to \infty} f(x) = L$$

if

$$\text{for every } \epsilon > 0, \text{ there exists a real number } M \text{ such that}$$
$$\text{if } x > M \quad \text{then} \quad |f(x) - L| < \epsilon.$$



To show that $\lim_{x \to \infty} f(x) = L$ using the definition **you** need to give a recipe for finding an M that works for different ϵ.

$$\textbf{If } x > M_\epsilon \textbf{ then } L - \epsilon < f(x) < L + \epsilon\textbf{.}$$

Proving that $\lim_{x \to \infty} f(x) = L$ using the limit definition

Example. Prove that
$$\lim_{x \to \infty} \frac{2x+3}{x+5} = 2.$$

Proof. We consider the distance (we called it "error" earlier)

$$\begin{aligned}
|f(x) - L| &= \left| \frac{2x+3}{x+5} - 2 \right| \\
&= \left| \frac{2x+3-2x-10}{x+5} \right| \\
&= \left| \frac{-7}{x+5} \right| \\
&= \frac{7}{x+5} && \text{for } x > -5 \\
&< \frac{7}{x} && \text{[to make algebra simpler later on]}
\end{aligned}$$

In summary,

$$|f(x) - L| < \frac{7}{x}.$$

This inequality gives an upper bound for the distance between $f(x)$ and L! Accordingly,

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad \frac{7}{x} < \epsilon.$$

The latter condition is equivalent to

$$x > \frac{7}{\epsilon}$$

and hence if we set

$$M = \frac{7}{\epsilon}$$

then

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad x > M,$$

In the preceding example it was easy to find an upper bound for $|f(x) - L|$. For most problems it is not even possible exactly solve $|f(x) - L| < \epsilon$, and when it is, it usually gives a really messy formula for M.

General strategy. Given ϵ, we need to find a number M such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad x > M.$$

The number M can be found by following the procedure below.

1. Find a **good upper bound** for $|f(x) - L|$.
2. Find a **simple condition** on x such that this upper bound is less than ϵ.
3. Use this condition to state an appropriate value for M (in terms of ϵ).

Remarks.

- In general, M depends on ϵ but it is not uniquely defined.
- The definition of the limit does NOT require to specify M for a given ϵ! It requires to show (**to prove**) that such an M exists!!!
- The definition of the limit does not tell you what the limit is.
- The definition may be used to prove theorems which allow you to justify methods of finding limits.
- Applying the definition to verify an educated guess for a limit is usually the last resort.
- Make use of the theorems unless you are specifically asked to apply the definition.

Basic rules for limits

'Elementary' rules.

- If f is a constant function, that is $f(x) = c$ for all x, then

$$\lim_{x\to\infty} f(x) = c.$$

- If $f(x) \to \infty$ as $x \to \infty$ then

$$\lim_{x\to\infty} \frac{1}{f(x)} = 0.$$

These are intuitively obvious and give limits such as

$$\lim_{x\to\infty} \frac{2}{x^2} = 0, \qquad \lim_{x\to\infty} e^{-x} = \lim_{x\to\infty} \frac{1}{e^x} = 0.$$

Exercise. Prove these elementary rules.

Theorem. Suppose that

$$\lim_{x\to\infty} f(x) = a, \qquad \lim_{x\to\infty} g(x) = b$$

for some functions f and g. Then

- $\displaystyle\lim_{x\to\infty} [f(x) + g(x)] = a + b$
- $\displaystyle\lim_{x\to\infty} [f(x) - g(x)] = a - b$
- $\displaystyle\lim_{x\to\infty} [f(x)g(x)] = ab$
- $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{a}{b}$ provided that $b \neq 0$.

Proof of the limit of a sum of two functions. Suppose that

$$\lim_{x \to \infty} f(x) = L_1, \qquad \lim_{x \to \infty} g(x) = L_2$$

and set

$$\epsilon_1 = \epsilon_2 = \frac{\epsilon}{2}$$

for any given $\epsilon > 0$.

By definition of the limit, there exist numbers M_1 and M_2 such that

$$|f(x) - L_1| < \epsilon_1 \quad \text{whenever} \quad x > M_1$$

and

$$|g(x) - L_2| < \epsilon_2 \quad \text{whenever} \quad x > M_2.$$

Hence, by the triangle inequality (that is, $|a + b| \leq |a| + |b|, a, b \in \mathbb{R}$) we have

$$\begin{aligned} \big|[f(x) + g(x)] - (L_1 + L_2)\big| &\leq |f(x) - L_1| + |g(x) - L_2| \\ &< \epsilon_1 + \epsilon_2 \\ &= \epsilon \end{aligned}$$

whenever both $x > M_1$ and $x > M_2$.

If we set $M = \max\{M_1, M_2\}$ then

$$\Big|[f(x) + g(x)] - (L_1 + L_2)\Big| < \epsilon \quad \text{whenever} \quad x > M$$

so that

$$\lim_{x\to\infty} [f(x) + g(x)] = L_1 + L_2.$$

The other rules may be proven in a similar manner.

Example. Determine the limit of

$$f(x) = \frac{3 + \dfrac{1}{3x^3}}{5 - e^{-x}}$$

as $x \to \infty$.

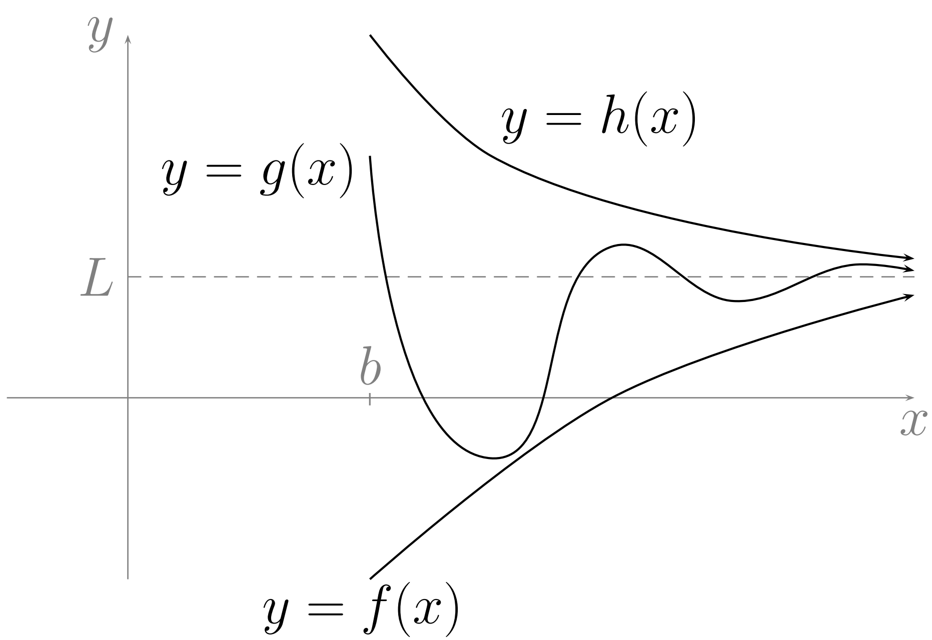
Solution.

$$\begin{aligned}
\lim_{x\to\infty} \frac{3 + \frac{1}{3x^3}}{5 - e^{-x}} &= \frac{\lim\limits_{x\to\infty} (3 + \dfrac{1}{3x^3})}{\lim\limits_{x\to\infty} (5 - e^{-x})} && \text{(rule (iv))} \\
&= \frac{\lim\limits_{x\to\infty} 3 + \dfrac{1}{3} \lim\limits_{x\to\infty} \dfrac{1}{x^3}}{\lim\limits_{x\to\infty} 5 - \lim\limits_{x\to\infty} e^{-x}} && \text{(rules (i) and (ii))} \\
&= \frac{3 + 0}{5 - 0} \\
&= \frac{3}{5}.
\end{aligned}$$

$\square$

The pinching theorem

Idea. Assume that two functions f and h have the same limit as $x \to \infty$ and the graph of a function g lies between the graphs of f and h (if x is large enough). Then, g has the same limit as f and h.



The pinching theorem. Suppose that f, g and h are three functions such that

$$f(x) \leq g(x) \leq h(x)$$

on an interval (b, ∞) for some $b \in \mathbb{R}$ and

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} h(x) = L.$$

Then

$$\lim_{x \to \infty} g(x) = L.$$

Proof of the pinching Theorem. See page 37 of the Calculus Notes.

Example. Determine the limit of

$$g(x) = \frac{\cos x}{x}$$

as $x \to \infty$.

Solution. We begin with the basic inequality

$$-1 \leq \cos x \leq 1,$$

which is valid for every real number x. Since $x \to \infty$, we may assume that $x > 0$ and then we have

$$-\frac{1}{x} \leq \frac{\cos x}{x} \leq \frac{1}{x}.$$

Now

$$\lim_{x \to \infty} -\frac{1}{x} = \lim_{x \to \infty} \frac{1}{x} = 0,$$

and so

$$\lim_{x \to \infty} \frac{\cos x}{x} = 0$$

by the pinching theorem. $\square$

Limits of the form $f(x)/g(x)$

Suppose that we want to calculate a limit of the form

$$\lim_{x\to\infty} \frac{f(x)}{g(x)},$$

where both $f(x)$ and $g(x)$ tend to infinity as $x \to \infty$.

Problem. We cannot apply the preceding rules since f and g do not have limits.

Idea. Divide both f and g by the leading term, that is the fastest growing term appearing in the denominator g (if it exists).

Example. Find the following limit (if it exists).

$$\lim_{x\to\infty} \frac{6x^3 - 4\sin x}{\cos 3x + 5x - x^3}$$

Solution. The leading term in this example is x^3, therefore, we divide both numerator and denominator by x^3. We have

$$\lim_{x\to\infty} \frac{6x^3 - 4\sin x}{\cos 3x + 5x - x^3} = \lim_{x\to\infty} \frac{6 - 4\frac{\sin x}{x^3}}{\frac{\cos 3x}{x^3} + \frac{5}{x^2} - 1}$$

$$= \frac{6 - 4\displaystyle\lim_{x\to\infty} \frac{\sin x}{x^3}}{\displaystyle\lim_{x\to\infty} \frac{\cos 3x}{x^3} + \lim_{x\to\infty} \frac{5}{x^2} - 1}.$$

Exercise. Prove that

$$\lim_{x\to\infty} \frac{\sin x}{x^3} = 0, \quad \lim_{x\to\infty} \frac{\cos 3x}{x^3} = 0.$$

Since also $\lim\limits_{x\to\infty} \dfrac{5}{x^2} = 0$, we obtain that

$$\lim_{x\to\infty} \frac{6x^3 - 4\sin x}{\cos 3x + 5x - x^3} = \frac{6}{-1} = -6.$$

Example. Find the following limit (if it exists).

$$\lim_{x\to\infty} \frac{x^2 + 5x}{\sqrt{x^4+1} - 4x}$$

Solution. The leading term here is x^2, and therefore,

$$\lim_{x\to\infty} \frac{x^2 + 5x}{\sqrt{x^4+1} - 4x} = \lim_{x\to\infty} \frac{1 + \frac{5}{x}}{\sqrt{\frac{x^4+1}{x^4}} - \frac{4}{x}} = \lim_{x\to\infty} \frac{1 + \frac{5}{x}}{\sqrt{1 + \frac{1}{x^4}} - \frac{4}{x}} = 1$$

Limits of the form $\sqrt{f(x)} - \sqrt{g(x)}$

Idea. We divide and multiply by the factor $\sqrt{f(x)} + \sqrt{g(x)}$. Then we arrive at limits of the previous type.

Sometimes the **lower order** terms **may** determine the limit!

Example. Determine the limit of

$$f(x) = \sqrt{x^2 + 2x} - \sqrt{x^2 - 1}$$

as $x \to \infty$.

Solution. We have

$$\begin{aligned}
f(x) &= \sqrt{x^2+2x} - \sqrt{x^2-1} \\
&= \frac{\left(\sqrt{x^2+2x} - \sqrt{x^2-1}\right)\left(\sqrt{x^2+2x} + \sqrt{x^2-1}\right)}{\sqrt{x^2+2x} + \sqrt{x^2-1}} \\
&= \frac{x^2+2x-x^2+1}{\sqrt{x^2+2x} + \sqrt{x^2-1}} \\
&= \frac{2x+1}{\sqrt{x^2+2x} + \sqrt{x^2-1}} \\
&= \frac{2+\frac{1}{x}}{\sqrt{1+\frac{2}{x}} + \sqrt{1-\frac{1}{x^2}}}.
\end{aligned}$$

Hence, the limit $\lim\limits_{x\to\infty} f(x)$ exists and

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2+\frac{1}{x}}{\sqrt{1+\frac{2}{x}} + \sqrt{1-\frac{1}{x^2}}} = 1.$$

Exercise. Does

$$\lim_{x\to\infty} \sqrt{x^4 - x^3} - \sqrt{x^4 + 1}$$

exist?

Indeterminate forms

The following limits have the form "$\frac{\infty}{\infty}$" but each displays a very different limiting behaviour as $x \to \infty$:

- $\dfrac{x^2}{x} \to \infty$
- $\dfrac{x}{x^2} \to 0$
- $\dfrac{2x^2}{x^2} \to 2$

Since we cannot determine in advance what kind of limiting behaviour something of the form "$\frac{\infty}{\infty}$" has, we say that "$\frac{\infty}{\infty}$" is an indeterminate form.

Other types of indeterminate forms are

- "$\frac{0}{0}$"
- "$\infty - \infty$"
- "$0 \times \infty$"

and appear in various applications. We will come back to them later.

Continuity

Question. How would you define continuity?

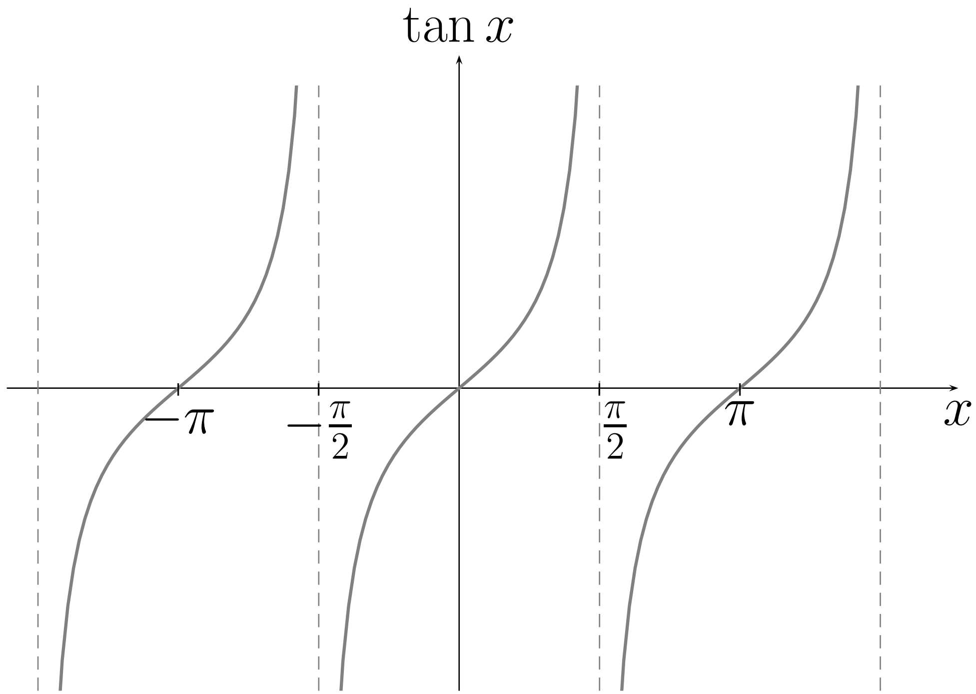
'Intuitive' (incorrect) answer. The function is continuous if 'its graph can be drawn without lifting the pencil off the page'.

Rigorous (correct) answer. Via limits at a point!

‘Counterexample’. Consider the function

$$\tan : A \to \mathbb{R}$$

with $A = \mathrm{Dom}(f) = \{x \in \mathbb{R} : x \neq \frac{\pi}{2} + n\pi,\ n \in \mathbb{Z}\}$.



The function tan is continuous on its domain! The break in the graph is merely due to the ‘missing’ points in the domain A.

To avoid such errors, we need much more logically robust definitions!

The first thing to discuss continuity at a point x_0:

Definition: Suppose that $f : (a, b) \to \mathbb{R}$ and that $x_0 \in (a, b)$. We say that f is **continuous at** x_0 if $\lim_{x \to x_0} f(x) = f(x_0)$.

Of course, this just moves the problem into deciding what $\lim_{x \to x_0} f(x)$ means!

Informally: $\lim_{x \to x_0} f(x) = L$ means that the closer and closer x gets to x_0, the closer and closer $f(x)$ gets to L.

Again the main challenge is to be more precise about expression **'closer and closer'**.

Definition. Suppose that $f(x)$ is defined for all x near x_0, although not necessarily at x_0. We say that $\lim_{x \to x_0} f(x) = L$ if, **for all** $\epsilon > 0$**, there exists** $\delta > 0$ **such that if** $x_0 - \delta < x < x_0 + \delta$ **and** $x \neq x_0$ **then** $|f(x) - L| < \epsilon$**.**

Note the '$x \neq x_0$'. Even if $f(x_0)$ is defined, we don't care what its value is!

Limit Theorems

Theorem. Suppose that f and g are defined in an interval containing x_0 (but not necessarily at x_0) and that $\lim\limits_{x \to x_0} f(x)$ and $\lim\limits_{x \to x_0} g(x)$ both exist. Then

1. $\lim\limits_{x \to x_0} (f+g)(x) = \lim\limits_{x \to x_0} f(x) + \lim\limits_{x \to x_0} g(x)$,
2. $\lim\limits_{x \to x_0} (f-g)(x) = \lim\limits_{x \to x_0} f(x) - \lim\limits_{x \to x_0} g(x)$,
3. $\lim\limits_{x \to x_0} (fg)(x) = \left(\lim\limits_{x \to x_0} f(x)\right) \cdot \left(\lim\limits_{x \to x_0} g(x)\right)$,
4. $\lim\limits_{x \to x_0} (f/g)(x) = \dfrac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)}$, as long as $\lim\limits_{x \to x_0} g(x) \neq 0$.

Exercise. Adapt our earlier proof for $\lim_{x \to \infty}$ to prove (1).

Polynomials

To use the limit theorems you need to know **some** simple limits!

Exercise. Let $f(x) = c$ (a constant).
Suppose $x_0 \in \mathbb{R}$. Use the definition to prove that $\lim_{x \to x_0} f(x) = c$.

Exercise. Let $g(x) = x$.
Use the definition to prove that $\lim_{x \to x_0} g(x) = x_0$.

Observation: Every polynomial $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ is made up from combining 'f' and 'g' from the exercises above a finite number of times, so by the theorem on the last slide $\lim_{x \to x_0} p(x) = p(x_0)$.

For example, $\lim_{x \to x_0} (x^2 + 3) = \left(\lim_{x \to x_0} x \right)^2 + \lim_{x \to x_0} 3 = x_0^2 + 3$.

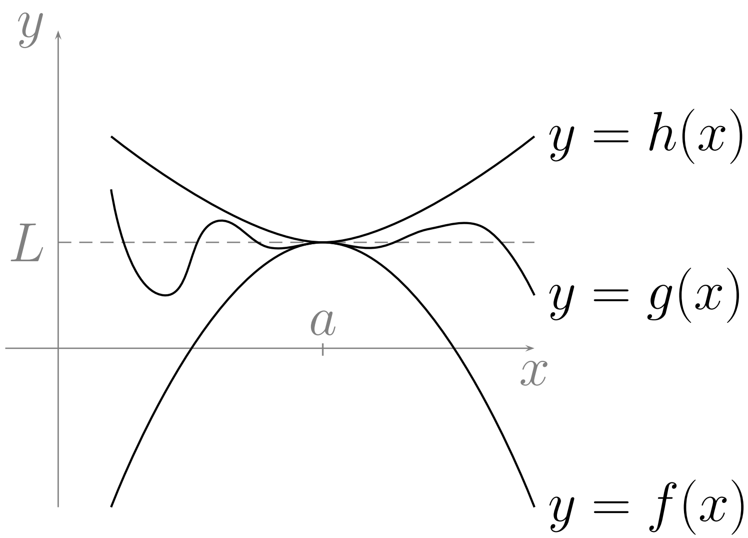
Exercise. Show that $\lim_{x\to 0} \sin(1/x)$ doesn't exist, by showing that, *no matter what L you thought might be the limit,* that even for a big value of ϵ like $\frac{1}{2}$, you could never find a value of δ small enough so that $|\sin(1/x) - L| < \frac{1}{2}$ whenever $|x - 0| < \delta$ and $x \neq 0$.

Hint. Take $x_n = \frac{1}{2\pi n}$ and $y_n = \frac{1}{2\pi n + \pi/2}$, $n \in \mathbb{N}$.

The Pinching Theorem for limit at a point

Theorem. Suppose that f, g, h are defined on an interval I containing x_0 (except possibly at x_0), and that $f(x) \leq g(x) \leq h(x) \qquad x \in I,\ x \neq x_0$.

If $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = L$, then $\lim_{x \to x_0} g(x)$ exists and equals L too.



Example. Find $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$.

Solution. Let $f(x) = -x^2$ and $h(x) = x^2$. Then for all $x \neq 0$,

$$f(x) \leq x^2 \sin\left(\frac{1}{x}\right) \leq h(x).$$

Also $\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x) = 0$ so by the Pinching Theorem $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$ exists and equals 0 too.

Left-hand, right-hand and two-sided limits

You should be comfortable with high school limits like

$$\begin{aligned}
\lim_{x\to 1}\frac{x^3-1}{x^2-1} &= \lim_{x\to 1}\frac{(x-1)(x^2+x+1)}{(x-1)(x+1)} \\
&= \lim_{x\to 1}\frac{x^2+x+1}{x+1} \textbf{(not a } \frac{0}{0} \textbf{ form)} \\
&= \frac{\lim_{x\to 1} x^2+x+1}{\lim_{x\to 1} x+1} \\
&= \frac{3}{2}.
\end{aligned}$$

But what about $\lim_{x\to 1}\dfrac{|x^3-1|}{x^2-1}$?

Let $f(x) = \dfrac{|x^3 - 1|}{x^2 - 1}$.

If $x > 1$ then $f(x) = \dfrac{x^3 - 1}{x^2 - 1} = \dfrac{x^2 + x + 1}{x + 1} \approx \dfrac{3}{2}$ for x near 1.

If $x < 1$ then $x^3 - 1$ is negative so $f(x) = -\dfrac{x^3 - 1}{x^2 - 1} = -\dfrac{x^2 + x + 1}{x + 1} \approx -\dfrac{3}{2}$ for x near 1.

In this case $\lim_{x \to 1} f(x)$ does not exist. The value of $f(x)$ does not get closer and closer to a single number as x approaches closer and closer to 1.

On the other hand, if you only sneak up on 1 from the right, $f(x)$ gets closer and closer to $\dfrac{3}{2}$.

We say that f has a **right hand limit** at 1 and write $\lim_{x \to 1^+} f(x) = \dfrac{3}{2}$.

Similarly, this f also has a **left hand limit** at 1: $\lim_{x \to 1^-} f(x) = -\dfrac{3}{2}$.

Definition. Let f be a function defined at least on an interval $(a-p, a)$ with $p > 0$. We say that L is **left hand limit** of the function f at a point a, if for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\text{if } a-\delta < x < a, \text{ then } |f(x) - L| < \epsilon.$$

Exercise. Write an $\epsilon - \delta$-definition of the right-hand limit.

The standard limit theorems all have one sided versions, eg

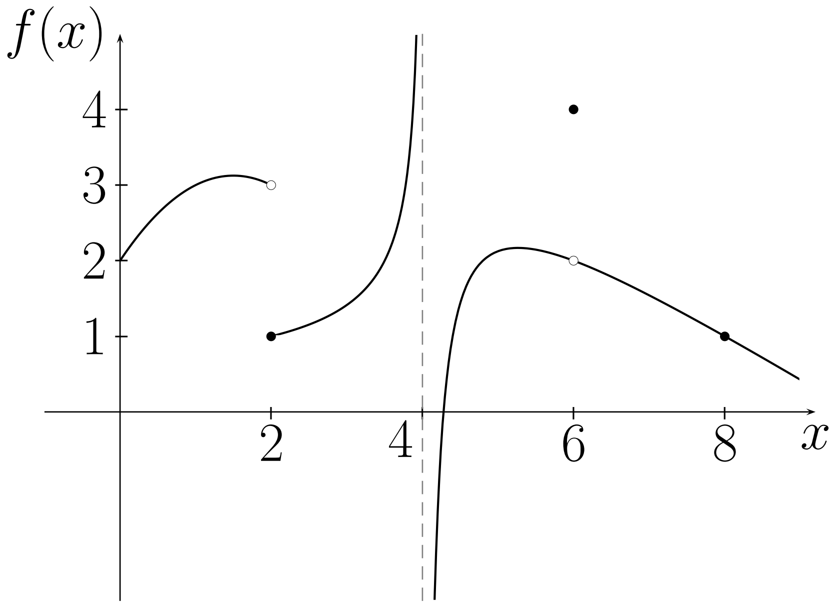
$$\lim_{x \to x_0^+} (f+g)(x) = \lim_{x \to x_0^+} f(x) + \lim_{x \to x_0^+} g(x)$$

if the right hand side limits exist.

Of course, for some functions, one of these limits may exist but not the other!

Theorem. Let f be defined on an open interval containing x_0. Then $\lim_{x \to x_0} f(x)$ exists **if and only if** $\lim_{x \to x_0^+} f(x)$ and $\lim_{x \to x_0^-} f(x)$ both exist, and are equal.

Example. Consider the function f whose graph is shown below.



With reference to this graph, we will discuss the behaviour of $f(x)$ when x is near the points 2, 4, 6 and 8.

- For $a = 2$:

$$\lim_{x \to 2^-} f(x) = 3, \qquad \lim_{x \to 2^+} f(x) = 1$$

  The two-sided limit does not exist.

- For $a = 4$:

$$f(x) \to \pm\infty \quad \text{as} \quad x \to 4^{\mp}$$

  No limit exists.

- For $a = 6$:

$$\lim_{x \to 6^-} f(x) = \lim_{x \to 6^+} f(x) = 2, \qquad f(6) = 4$$

  The two-sided limit exists but does not coincide with the value of f at $a = 6$.

- For $a = 8$:

$$\lim_{x \to 8^-} f(x) = \lim_{x \to 8^+} f(x) = f(8) = 1$$

  The two-sided limit exists and coincides with the value of f at $a = 8$.

Question. What is so special about the above function at $x = 8$?

Limits and continuous functions

Definition. Let f be defined on some open interval containing the point a. We say that f is continuous at a if

$$\lim_{x \to a} f(x) = f(a);$$

otherwise we say that f is discontinuous at a.

If f is continuous at every point of its domain, we simply say that f is continuous.

Previous example. The function f is continuous everywhere except at $x = 2$ and $x = 6$.

Note that $x = 4$ is not part of the domain of f and hence asking whether or not f is continuous at $x = 4$ does not make any sense.

Remark. Continuity is a deep property for a function to have. Contrary to the impression that many students form from their study of functions at school, most functions are *not* continuous. Two interesting functions with discontinuities are mentioned below.

- The function $f : \mathbb{R} \to \mathbb{R}$, defined by
$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}, \end{cases}$$
is discontinuous at every point in its domain.

- The function $f : \mathbb{R} \to \mathbb{R}$, defined by
$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ -x & \text{if } x \notin \mathbb{Q}, \end{cases}$$
is continuous at 0 but discontinuous everywhere else.