THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Semester 2 2012

MATH1131
MATHEMATICS 1A

(1) TIME ALLOWED – 2 hours

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

(8) A SHORT TABLE OF INTEGRALS IS SUPPLIED AT THE END OF THE PAPER

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

Use a separate book clearly marked Question 1

1. i) Let $z = 1 + 3i$ and $w = -1 + i$.

   a) Find $z(w - 3i)$ in **Cartesian form.**

   b) Find z/w in **Cartesian form.**

   c) Find $|z + \overline{w}|$.

   ii) Let the set S in the complex plane be defined by

$$S = \left\{ z \in \mathbb{C} \, : \, -\frac{\pi}{4} \leq \operatorname{Arg}(z) \leq \frac{\pi}{4} \text{ and } \operatorname{Re}(z) \leq 2 \right\}.$$

   a) Sketch the set S on a labelled Argand diagram.

   b) By considering your sketch, or otherwise, find the area of the region defined by S.

   iii) Use De Moivre's theorem to prove that $\cos(4\theta) = 1 - 8\sin^2\theta\cos^2\theta$.

   iv) Let $z = -1 - \sqrt{3}i$.

   a) Find $|z|$.

   b) Find $\operatorname{Arg}(z)$.

   c) Use the polar form of z to evaluate z^{18} and express your answer in **Cartesian** form.

   v) A plane Π has parametric vector equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}; \quad \lambda, \mu \in \mathbb{R}.$$

   Find the Cartesian equation for this plane.

vi) Consider the following MAPLE session.

```
> with(LinearAlgebra):
> A:=<<0,1,1>|<1,1,1>|<1,1,0>>;
```

$$A := \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

```
> B:=A^2;
```

$$B := \begin{bmatrix} 2 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix}$$

```
> C:=A^3;
```

$$C := \begin{bmatrix} 3 & 5 & 4 \\ 5 & 7 & 5 \\ 4 & 5 & 3 \end{bmatrix}$$

```
> F:=C-B-3*A;
```

$$F := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Assume that A has an inverse.

Using the above MAPLE session, or otherwise, find the inverse of the matrix A.

Use a separate book clearly marked Question 2

2. i) Let $A = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix}$.

a) State the size of AA^T.

b) Evaluate A^TA or explain why this product does not exist.

ii) Let $M = \begin{pmatrix} 1 & 1 & 0 \\ 1 & \alpha & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

a) Evaluate the determinant of M.

b) Determine the value(s) of α for which M does **not** have an inverse.

c) Find the inverse of M when $\alpha = 1$.

iii) The points A, B, and C in $\mathbb{R}^3$ have position vectors

$$\mathbf{a} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \quad \text{and} \quad \mathbf{c} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

a) Find the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

b) Find a parametric vector equation of the plane passing through A, B, and C.

c) Find the cross product $\overrightarrow{AB} \times \overrightarrow{AC}$.

d) Hence, or otherwise, find the area of the triangle with vertices A, B, and C.

iv) Let ℓ_1 and ℓ_2 be the lines

$$\ell_1 : \quad \mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}; \quad \lambda \in \mathbb{R} \quad \text{and}$$

$$\ell_2 : \quad \mathbf{x} = \begin{pmatrix} -2 \\ 6 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}; \quad \mu \in \mathbb{R}.$$

a) Show that the point B with coordinates $(-1, 4, 3)$ lies on the line ℓ_1.

b) Find the point A at which the lines ℓ_1 and ℓ_2 intersect.

c) Find the projection of the vector $\overrightarrow{AB}$ onto the line ℓ_2.

v) Let A, B, and C be three points on the unit circle in $\mathbb{R}^2$, with position vectors

$$\mathbf{a} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{c} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix},$$

where θ is some real number.

Use vector methods to prove that the triangle with vertices A, B, and C is right-angled.

Use a separate book clearly marked Question 3

3. i) Evaluate each of the following limits:

a) $\displaystyle\lim_{x\to\infty} \frac{3x^3 + \sin(2x)}{4 - x^3}$,

b) $\displaystyle\lim_{x\to 0} \frac{e^{2x} - 1}{\sin(x)}$.

ii) Evaluate each of the following integrals:

a) $\displaystyle I_1 = \int x\cos(x)\,dx$,

b) $\displaystyle I_2 = \int_1^3 \frac{x}{x+1}\,dx$.

iii) a) State the Mean Value Theorem.

b) Find a real number c which satisfies the conclusion of the Mean Value Theorem for the function $f(x) = \ln x$ on the interval $[1, 2]$.

c) By considering a sketch, or otherwise, state which is **larger**:

A: the area under $f(x) = \ln x$ on the interval $[1, 2]$

OR

B: the area under the tangent to $f(x) = \ln x$ at c, (where c is the point determined in (b) above) on the interval $[1, 2]$.

iv) Use the fundamental theorem of calculus to find $\displaystyle\frac{d}{dx}\int_x^{x^2} \cosh(\sqrt{t})\,dt$.

v) A function f defined on the interval $[0, 5]$ has the following three properties:

- f is not continuous at $x = 2$.
- $\displaystyle\lim_{x\to 2} f(x) = 3$.
- For $x \neq 2$, $f'(x) > 0$, $f''(x) < 0$.

Draw a possible sketch of the graph of f.

vi) Show that the improper integral $\displaystyle\int_1^\infty \frac{dx}{x^2 + e^{5x}}$ converges.

Use a separate book clearly marked Question 4

4. i) Consider the polar curve C with equation $r = 2 + \cos 2\theta$.

a) Explain briefly why C is symmetric in the x-axis and in the y-axis.

b) Sketch the graph of C in the x-y plane.

ii) Let $g(x) = \begin{cases} \alpha x + \beta & \text{if } x > 0 \\ e^{2x} & \text{if } x \leq 0. \end{cases}$

a) Find the values of α and β so that g is differentiable at $x = 0$.

b) Assuming that g is differentiable at $x = 0$ determine the equation of the tangent to $y = g(x)$ at $x = 0$.

iii) a) State the definition of $\tanh(y)$ in terms of the exponential function.

b) By considering $x = \tanh(y)$, prove that $\tanh^{-1}(x) = \dfrac{1}{2}\ln\left|\dfrac{1+x}{1-x}\right|$, for $x \neq -1, 1$.

c) Give the exact value of $\tanh^{-1}(2)$.

iv) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 5x + \cos(x) + 1$.

a) Show that f has a inverse function g.

b) Find $g'(2)$.

v) Two functions f and g were entered into MAPLE and the following calculations were performed:

```
> subs(x=0,f);
                    0
> subs(x=0,g);
                    0
> subs(x=0,diff(f,x));
                 -0.6931
> subs(x=0,diff(g,x));
                 -1.099
> subs(x=0,diff(f,x,x));
                 -0.4804
> subs(x=0,diff(g,x,x));
                 -1.208
```

By considering the MAPLE output show that the product fg of the two functions has a stationary point at $x = 0$ and determine its nature. Give reasons for your answer.

BASIC INTEGRALS

$$\int \frac{1}{x}\,dx = \ln|x| + C = \ln|kx|, \qquad C = \ln k$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

$$\int a^x\,dx = \frac{1}{\ln a}a^x + C, \qquad a \neq 1$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$$

$$\int \operatorname{cosec}^2 ax\,dx = -\frac{1}{a}\cot ax + C$$

$$\int \tan ax\,dx = \frac{1}{a}\ln|\sec ax| + C$$

$$\int \cot ax\,dx = \frac{1}{a}\ln|\sin ax| + C$$

$$\int \sec ax\,dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C$$

$$\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C$$

$$\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C$$

$$\int \operatorname{sech}^2 ax\,dx = \frac{1}{a}\tanh ax + C$$

$$\int \operatorname{cosech}^2 ax\,dx = -\frac{1}{a}\coth ax + C$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$\begin{aligned}\int \frac{dx}{a^2 - x^2} &= \frac{1}{a}\tanh^{-1}\frac{x}{a} + C, && |x| < a\\ &= \frac{1}{a}\coth^{-1}\frac{x}{a} + C, && |x| > a > 0\\ &= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C, && x^2 \neq a^2\end{aligned}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\frac{x}{a} + C, \qquad x \geqslant a > 0$$

BASIC INTEGRALS

$$\int x^a\,dx = \frac{1}{a+1}x^{a+1} + C, \qquad a \neq -1$$

$$\int \frac{1}{x}\,dx = \ln|x| + C = \ln|kx|, \qquad C = \ln k$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

$$\int a^x\,dx = \frac{1}{\ln a}a^x + C, \qquad a \neq 1$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$$

$$\int \operatorname{cosec}^2 ax\,dx = -\frac{1}{a}\cot ax + C$$

$$\int \tan ax\,dx = \frac{1}{a}\ln|\sec ax| + C$$

$$\int \cot ax\,dx = \frac{1}{a}\ln|\sin ax| + C$$

$$\begin{aligned}\int \sec ax\,dx &= \frac{1}{a}\ln|\sec ax + \tan ax| + C \\ &= \frac{1}{a}\ln\left|\frac{1+\tan\frac{ax}{2}}{1-\tan\frac{ax}{2}}\right| + C\end{aligned}$$

$$\begin{aligned}\int \operatorname{cosec} ax\,dx &= \frac{1}{a}\ln|\operatorname{cosec} ax - \cot ax| + C \\ &= \frac{1}{a}\ln\left|\tan\frac{ax}{2}\right| + C\end{aligned}$$

$$\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C$$

$$\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C$$

$$\int \operatorname{sech}^2 ax\,dx = \frac{1}{a}\tanh ax + C$$

$$\int \operatorname{cosech}^2 ax\,dx = -\frac{1}{a}\coth ax + C$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$\begin{aligned}\int \frac{dx}{a^2-x^2} &= \frac{1}{a}\tanh^{-1}\frac{x}{a} + C, && |x| < a\\ &= \frac{1}{a}\coth^{-1}\frac{x}{a} + C, && |x| > a > 0\\ &= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C, && x^2 \neq a^2\end{aligned}$$

$$\begin{aligned}\int \frac{dx}{\sqrt{a^2-x^2}} &= \sin^{-1}\frac{x}{a} + C\\ &= -\cos^{-1}\frac{x}{a} + C + \frac{\pi}{2}, && |x| \leqslant a\end{aligned}$$

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2+a^2}} &= \sinh^{-1}\frac{x}{a} + C\\ &= \ln(x+\sqrt{x^2+a^2}) + (C - \ln a)\end{aligned}$$

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2-a^2}} &= \cosh^{-1}\frac{x}{a} + C, && x \geqslant a > 0\\ &= \ln(x+\sqrt{x^2-a^2}) + (C - \ln a)\end{aligned}$$

$$\begin{aligned}\int \frac{dx}{x\sqrt{x^2-a^2}} &= \frac{1}{a}\sec^{-1}\frac{x}{a} + C\\ &= \frac{1}{a}\cos^{-1}\frac{a}{x} + C, && |x| \geqslant a > 0\end{aligned}$$

$$\begin{aligned}\int \frac{dx}{x\sqrt{a^2+x^2}} &= -\frac{1}{a}\sinh^{-1}\frac{a}{x} + C\\ &= -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2+x^2}}{x}\right| + C\end{aligned}$$

$$\begin{aligned}\int \frac{dx}{x\sqrt{a^2-x^2}} &= -\frac{1}{a}\cosh^{-1}\frac{a}{x} + C, && 0 < x \leqslant a\\ &= -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right| + C\end{aligned}$$