THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Semester 1 2011

MATH1131
MATHEMATICS 1A

(1) TIME ALLOWED – 2 hours

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

(8) A SHORT TABLE OF INTEGRALS WILL BE SUPPLIED

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

Use a separate book clearly marked Question 1

1. i) Let $z = -1 - i$.

a) Find $|z|$.

b) Find $\text{Arg}(z)$.

c) Use the polar form of z to evaluate z^{102} and then express your answer in **Cartesian form**.

ii) a) Simplify $(2 + 4i)^2$.

b) Hence, or otherwise, solve the quadratic equation $z^2 - 4z + (7 - 4i) = 0$.

iii) Sketch the following region on the Argand diagram

$$S = \{z \in \mathbb{C} : 0 \leq \text{Arg}(z - i) \leq \frac{\pi}{4}\}.$$

iv) Evaluate the limit

$$\lim_{x \to \infty} \frac{1}{x - \sqrt{x^2 - 6x - 4}} .$$

v) Evaluate the improper integral

$$\int_1^\infty x^{-5/4}\, dx .$$

vi) A curve in the plane is defined implicitly by the equation

$$x^2 - 3xy^2 + 11 = 0 .$$

a) Show that the curve has slope at the point (x, y) given by

$$\frac{dy}{dx} = \frac{2x - 3y^2}{6xy} .$$

b) Find the equation of the tangent to the curve at the point $(1, 2)$.

c) Write a Maple command to plot the curve in the region $1 \leq x \leq 4$ and $-5 \leq y \leq 5$.

Use a separate book clearly marked Question 2

2. i) Use De Moivre's Theorem to prove that

$$\cos(4\theta) = 8\cos^4\theta - 8\cos^2\theta + 1.$$

ii) Consider the line ℓ with parametric vector equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} t, \quad \text{for} \quad t \in \mathbb{R}.$$

a) Give two points on the line.

b) Give a vector parallel to the line.

c) Explain why the line ℓ is perpendicular to the plane P with Cartesian equation

$$9x + 6y + 15z = 24.$$

d) Find a point on the line whose y-coordinate is 0.

iii) Consider the following Maple session:

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> with(LinearAlgebra):
> A:=<<1,1,0>|<-1,1,0>|<0,0,sqrt(2)>>;
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$$A := \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

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> A^2;
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$$\begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

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> A^4;
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$$\begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

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> A^8;
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$$\begin{bmatrix} 16 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 16 \end{bmatrix}$$

Use the above MAPLE session to find the inverse of A^7.

iv) Evaluate the limit

$$\lim_{x \to 1} \frac{(x-1)^2}{1+\cos(\pi x)} .$$

v) Evaluate the indefinite integral

$$\int x \sin(2x)\, dx.$$

vi) The function f has domain $[0, 1]$ and is defined by $f(x) = e^x + ax$, where a is a positive constant.

a) Prove that 2 is in the range of f.

b) Prove that f has an inverse function f^{-1}.

c) Find the domain of f^{-1}.

Use a separate book clearly marked Question 3

3. i) Let $A = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 \\ 0 & 5 \\ 1 & 0 \end{pmatrix}$.

a) Find AB.

b) What is the size of the matrix BA?

ii) A pet shop has x hamsters, y rabbits and z guinea pigs.
Each hamster eats $50g$ of dry food and $40g$ of fresh vegetables, and needs $1m^2$ of space.
Each rabbit eats $300g$ dry food and $320g$ of fresh vegetables, and needs $5m^2$ of space.
Each guinea pig eats $100g$ of dry food and $200g$ of fresh vegetables, and needs $3m^2$ of space.
Altogether they eat $2900g$ of dry food and $3920g$ of fresh vegetables, and need $63m^2$ of space.

a) Explain why $5x + 30y + 10z = 290$.

b) Write down a system of linear equations that determine x, y and z.

c) Reduce your system to echelon form and solve to find the number of hamsters, rabbits and guinea pigs.

iii) The points A, B and C in $\mathbb{R}^3$ have position vectors

$$\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \quad \text{and} \quad \mathbf{c} = \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}$$

respectively.
Write down a parametric vector equation of the plane passing through A, B and C.

iv) Consider the following system of linear equations.

$$\begin{aligned} x + y - z &= 2 \\ 2x + 3y + z &= 6 \end{aligned}$$

a) Using Gaussian Elimination find the general solution to the system of equations.

b) Hence, or otherwise, find a solution to the system with the property that the sum of the x, y and z coordinates of the solution is 0.

v) Suppose A, B are two points in $\mathbb{R}^3$ with position vectors $\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$ respectively. We let O denote the origin.

a) Find $|\overrightarrow{OB}|$.

b) Find the area of triangle AOB.

c) Hence, or otherwise, find the perpendicular distance from A to the line through O and B.

vi) Suppose that $\mathbf{u}$ and $\mathbf{v}$ are non-zero, non-parallel vectors in $\mathbb{R}^3$ of the same magnitude. Prove that $\mathbf{u} - \mathbf{v}$ is perpendicular to $\mathbf{u} + \mathbf{v}$.

Use a separate book clearly marked Question 4

4. i) a) Give the definition of $\cosh x$.

b) Use the definition to prove that

$$4\cosh^3 x = \cosh 3x + 3\cosh x\ .$$

ii) Find

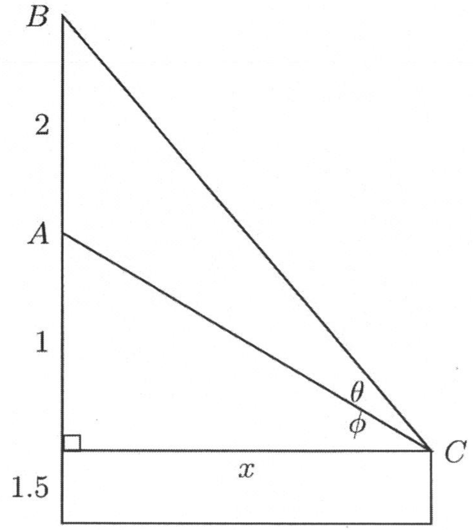
a) $\dfrac{d}{dx}\displaystyle\int_0^x \frac{\cos t}{\sqrt{1+t^2}}\,dt$;

b) $\dfrac{d}{dx}\displaystyle\int_0^{\sinh x} \frac{\cos t}{\sqrt{1+t^2}}\,dt$. Give your answer in simplest form.

iii) a) Sketch the curve whose equation in polar coordinates is $r = 6\sin 2\theta$.

b) Find the gradient, $\frac{dy}{dx}$, of this curve at the point where $\theta = \frac{1}{6}\pi$.

iv) A statue 2 metres high stands on a pillar 2.5 metres high. A person, whose eye is 1.5m above the ground, stands at a distance x metres from the base of the pillar.



The diagram shows the above information, with the person's eye being at C.

a) Prove that

$$\frac{d}{dt}\left(\cot^{-1} t\right) = \frac{-1}{1+t^2}.$$

b) Show that

$$\theta = \cot^{-1}\left(\frac{x}{3}\right) - \cot^{-1} x$$

c) Hence find the distance x that maximises the angle θ.