THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Semester 1 2012

MATH1141
HIGHER MATHEMATICS 1A

(1) TIME ALLOWED – 2 hours

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

(8) A SHORT TABLE OF INTEGRALS WILL BE SUPPLIED

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

Use a separate book clearly marked Question 1

1. i) Let $u = 3 + 2i$ and $w = 1 - 5i$.

a) Find $u - 2w$ in **Cartesian form**.

b) Find u/w in **Cartesian form**.

ii) Let $z = \sqrt{3} - i$.

a) Calculate $|z|$ and $\mathrm{Arg}(z)$.

b) Express z in polar form.

c) Hence, or otherwise, express $z^{10} + (\overline{z})^{10}$ in **Cartesian form**.

iii) Evaluate the determinant

$$\begin{vmatrix} 1 & -1 & 4 \\ 0 & 2 & 7 \\ 0 & 3 & 1 \end{vmatrix}.$$

iv) a) Evaluate $\displaystyle\lim_{x\to\infty} \frac{3x^2 + \sin(2x^2)}{x^2}$.

b) Evaluate $\displaystyle\lim_{x\to 0} \frac{3x^2 + \sin(2x^2)}{x^2}$.

v) Consider the curve in the plane defined by

$$x^2 - 5x\sin y + y^2 = 4.$$

Find the equation of the tangent line to this curve at the point $(2, 0)$.

vi) Let $p(x) = x^5 + 5x + 7$.

a) Explain why p has at least one real root.

b) Prove that p has exactly one real root.

Use a separate book clearly marked Question 2

2. i) Consider the function $f : (0, 2\sqrt{\pi}\,] \to \mathbb{R}$ defined by

$$f(x) = x^2 + \cos(x^2).$$

a) Find all critical points of f and determine their nature.

b) Explain why f is invertible, state the domain of f^{-1} and find $f^{-1}(5\pi/2)$.

c) Where is f^{-1} differentiable?

ii) Let

$$f(x) = \int_0^{x^2-9x} e^{-t^2}\, dt.$$

a) Use the Mean Value Theorem to show that f has a stationary point x_0 in the interval $[0, 9]$.

b) Find the value of x_0 and determine the nature of the stationary point.

iii) Suppose that z lies on the unit circle in the complex plane.

a) Show that $z + \dfrac{1}{z}$ is real.

b) Find the maximum value of $z + \dfrac{1}{z}$.

iv) Use De Moivre's theorem to prove that

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1.$$

v) Consider the plane P with parametric vector form

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}, \qquad \lambda_1, \lambda_2 \in \mathbb{R}.$$

a) Does the point $\mathbf{a} = \begin{pmatrix} 4 \\ 4 \\ -7 \end{pmatrix}$ lie on P?

b) Is the vector $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ parallel to P?

c) Is the vector $\mathbf{c} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ orthogonal to P?

Use a separate book clearly marked Question 3

3. i) Suppose that A is a 4×4 matrix with $\det(A) = 5$. The matrix B is the result of performing the following three elementary row operations on A:

1. first multiply the 3rd row by 7;
2. then replace the second row with twice the first row plus the second row;
3. then swap the first and last rows.

What is the value of $\det(B)$?

ii) Consider the line in $\mathbb{R}^3$,

$$x - 4 = -y = z - 5.$$

a) Write this line in parametric vector form.

b) Find the point on the line closest to the origin $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

iii) Consider the following Maple session, which defines a matrix A and a vector $\mathbf{b} \in \mathbb{R}^3$:

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> with(LinearAlgebra):
> A:=<<1,3,2>|<1,2,a>|<-2,2*a,4>>;
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$$A := \begin{bmatrix} 1 & 1 & -2 \\ 3 & 2 & 2a \\ 2 & a & 4 \end{bmatrix}$$

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> b:=<1,2,-2>;
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$$\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

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> GaussianElimination(<A|b>);
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$$\begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & -1 & 2a+6 & -1 \\ 0 & 0 & -4+2a^2+2a & -2-a \end{bmatrix}$$

For which values of a will the system $A\mathbf{x} = \mathbf{b}$ have

a) a unique solution,

b) no solutions,

c) infinitely many solutions?

iv) A matrix $Q \in M_{nn}(\mathbb{R})$ is said to be *nilpotent* (of degree 2) if $Q^2 = \mathbf{0}$, the zero matrix.

a) Give an example of a non-zero 2×2 nilpotent matrix.

b) Explain why a nilpotent matrix cannot be invertible.

Suppose now that $S, Q \in M_{nn}(\mathbb{R})$ commute, that S is invertible and that Q is nilpotent (of degree 2).

c) Prove that $S^{-1}Q = QS^{-1}$.

d) Show that $S + Q$ is invertible by finding an integer k such that

$$(S + Q)(S^{-1} - S^{-k}Q) = I.$$

v) Consider the non-zero vector $\mathbf{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ in $\mathbb{R}^3$ which makes angles α, β, γ with the three coordinate axes respectively.

By considering dot products with the standard basis vectors $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$, (or otherwise), prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

vi) Suppose that A is an $n \times n$ matrix with the property that every vector $\mathbf{b} \in \mathbb{R}^n$ can be written uniquely as a linear combination of the **columns** of A. Prove that every vector $\mathbf{b} \in \mathbb{R}^n$ can also be written uniquely as a linear combination of the **rows** of A.

Use a separate book clearly marked Question 4

4. i) Consider the polar curve $r = 1 + \cos 2\theta$.

a) Prove that the curve is symmetric about the x axis and also about the y axis.

b) Sketch the curve.
(You are NOT required to find the derivative.)

ii) Let $f(x) = \tanh x$ (the hyperbolic tangent).

a) Express $\tanh x$ in terms of exponentials.

b) Sketch the graph $y = f(x)$.

c) Show that

$$\lim_{x\to\infty} \frac{1 - \tanh x}{e^{-2x}} = \frac{1}{2}.$$

d) Explain why the improper integral

$$\int_0^\infty (1 - \tanh x)\, dx$$

converges.

e) Compute

$$\int_0^\infty (1 - \tanh x)\, dx,$$

justifying your calculations.

iii) Suppose that f is a function whose derivative is continuous and hence bounded on $[a, b]$, with $|f'(x)| \leq L$ for all $x \in [a, b]$.

a) Show that for any $n > 0$,

$$\int_a^b f(x) \sin nx\, dx = \frac{K(n)}{n} + \frac{1}{n}\int_a^b f'(x) \cos nx\, dx,$$

where $K(n) = f(a)\cos(na) - f(b)\cos(nb)$.

b) Explain why

$$\left|\int_a^b f'(x) \cos nx\, dx\right| \leq (b - a)L.$$

c) Find, with reasons,

$$\lim_{n\to\infty} \int_a^b f(x) \sin nx\, dx.$$