THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Semester 1 2011

MATH1141
HIGHER MATHEMATICS 1A

(1) TIME ALLOWED – 2 hours

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

(8) A SHORT TABLE OF INTEGRALS WILL BE SUPPLIED

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

Use a separate book clearly marked Question 1

1. i) Let $z = -1 - i$.

a) Find $|z|$.

b) Find $\text{Arg}(z)$.

c) Use the polar form of z to evaluate z^{102} and then express your answer in **Cartesian form**.

ii) a) Simplify $(2 + 4i)^2$.

b) Hence, or otherwise, solve the quadratic equation $z^2 - 4z + (7 - 4i) = 0$.

iii) Sketch the following region on the Argand diagram

$$S = \{z \in \mathbb{C} : 0 \leq \text{Arg}(z - i) \leq \frac{\pi}{4}\}.$$

iv) Evaluate the limit

$$\lim_{x \to \infty} \frac{1}{x - \sqrt{x^2 - 6x - 4}} .$$

v) Evaluate the improper integral

$$\int_1^{\infty} x^{-5/4}\, dx .$$

vi) A curve in the plane is defined implicitly by the equation

$$x^2 - 3xy^2 + 11 = 0 .$$

a) Show that the curve has slope at the point (x, y) given by

$$\frac{dy}{dx} = \frac{2x - 3y^2}{6xy} .$$

b) Find the equation of the tangent to the curve at the point $(1, 2)$.

c) Write a Maple command to plot the curve in the region $1 \leq x \leq 4$ and $-5 \leq y \leq 5$.

Use a separate book clearly marked Question 2

2. i) You may use the following Maple session to assist you in answering the question below.

```
> with(LinearAlgebra):
> A:=<<1,2,1>|<1,3,a>|<-1,a,3>>;
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$$A := \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & a \\ 1 & a & 3 \end{bmatrix}$$

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> t:=<1,3,2>;
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$$t := \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

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> B:=<A|t>;
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$$B := \begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 3 & a & 3 \\ 1 & a & 3 & 2 \end{bmatrix}$$

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> G:=GaussianElimination(B);
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$$G := \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & a+2 & 1 \\ 0 & 0 & 6-a^2-a & 2-a \end{bmatrix}$$

For which values of a will the system $A\mathbf{x} = \mathbf{t}$ have:

a) no solutions,

b) unique solution,

c) infinitely many solutions?

ii) Evaluate the determinant

$$\begin{vmatrix} 2 & 0 & -1 \\ 1 & 3 & 0 \\ 5 & 7 & 3 \end{vmatrix}.$$

iii) Find the point of intersection, if any, of the line $\boldsymbol{x} = \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\lambda \in \mathbb{R}$, with the plane

$$\boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \nu \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \mu, \nu \in \mathbb{R}.$$

iv) Consider the function f defined by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

a) Given that $\lim_{x \to \infty} xe^{-x} = 0$, evaluate the limit

$$\lim_{h \to 0} \frac{e^{-1/h^2}}{h}.$$

b) Using the definition of a derivative, determine whether f is differentiable at $x = 0$.

v) Consider the function f defined by

$$f(x) = \begin{cases} \dfrac{2}{x} & \text{for } x < -1, \\ x^2 - 1 & \text{for } -1 \leq x \leq \dfrac{6}{\pi}, \\ \left(\dfrac{72}{\pi^2} - 2\right) \sin \dfrac{1}{x} & \text{for } x > \dfrac{6}{\pi}. \end{cases}$$

a) Find all critical points of f and determine their nature. In particular, distinguish between local and global extrema.

b) Find all horizontal asymptotes for f.

c) Given that f does not have a point of inflexion, sketch the graph of f, clearly indicating all of the above information.

Use a separate book clearly marked Question 3

3. i) Let
$$S = e^{i\theta} + \frac{e^{3i\theta}}{3} + \frac{e^{5i\theta}}{3^2} + \frac{e^{7i\theta}}{3^3} + \cdots .$$

a) Prove that
$$S = \frac{3(3e^{i\theta} - e^{-i\theta})}{10 - 6\cos(2\theta)}.$$

b) Hence, or otherwise, find the sum
$$T = \sin(\theta) + \frac{\sin(3\theta)}{3} + \frac{\sin(5\theta)}{3^2} + \frac{\sin(7\theta)}{3^3} + \cdots .$$

ii) Suppose that $\mathbf{u}$ and $\mathbf{v}$ are non-zero, non-parallel vectors in $\mathbb{R}^3$ of the same magnitude. Prove that $\mathbf{u} - \mathbf{v}$ is perpendicular to $\mathbf{u} + \mathbf{v}$.

iii) Let $A = (a_{ij})$ be a real $n \times n$ matrix and let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ be the standard basis vectors for $\mathbb{R}^n$.

a) Prove that $\boldsymbol{e}_i^T A \boldsymbol{e}_j = a_{ij}$ for all $1 \le i, j \le n$.

b) Prove that if A is symmetric then $\boldsymbol{x}^T A \boldsymbol{y} = (A\boldsymbol{x})^T \boldsymbol{y}$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.

c) Conversely, prove that if $\boldsymbol{x}^T A \boldsymbol{y} = (A\boldsymbol{x})^T \boldsymbol{y}$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, then A is symmetric.

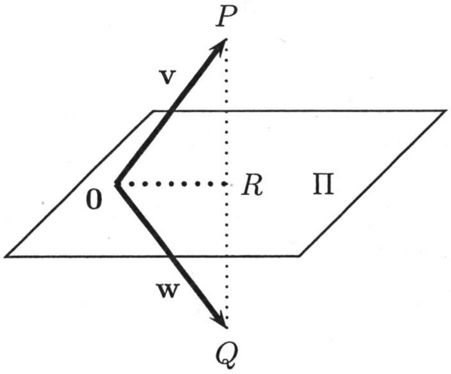
iv) The matrix

$$M = \frac{1}{9}\begin{pmatrix} 1 & 4 & -8 \\ 4 & 7 & 4 \\ -8 & 4 & 1 \end{pmatrix}$$

has the following property.

If $\boldsymbol{v}$ is the position vector of any point P in $\mathbb{R}^3$, then $\boldsymbol{w} = M\boldsymbol{v}$ is the vector obtained by reflecting $\boldsymbol{v}$ in a fixed plane Π, which passes through the origin.

Hence, in the diagram, $|\boldsymbol{v}| = |\boldsymbol{w}|$ and $|RP| = |RQ|$, where R is the foot of the perpendicular from P (or Q) to the plane.



a) Show that M reflects the vector $\boldsymbol{a} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ to the vector $-\boldsymbol{a}$.

b) Hence write down the Cartesian equation of Π.

c) Find a **non-zero** vector $\boldsymbol{u}$ such that $M\boldsymbol{u} = \boldsymbol{u}$.

d) Find the shortest distance from the point B with position vector $\boldsymbol{b} = \begin{pmatrix} 6 \\ -6 \\ 0 \end{pmatrix}$ to the plane.

Use a separate book clearly marked Question 4

4. i) a) Using L'Hôpital's rule or otherwise, indicate why

$$\lim_{x\to\infty} e^{-x}x^n = 0$$

for any $n \in \mathbb{N}$.

b) Show that for all $x \geq 1$,

$$e^{-x}x^n < Cx^{-2},$$

where C is a positive constant.

c) Hence, or otherwise, show that the improper integral

$$\int_1^\infty e^{-x}x^n\,dx$$

converges for any $n \in \mathbb{N}$.

ii) Suppose that $f : [0,2] \to [0,12]$ is continuous on its domain and twice differentiable on $(0,2)$. Further suppose that $f(0) = 0, f(2) = 12$.

a) Explain why $f'(c) = 6$ for some real number $c \in (0,2)$.

b) Suppose further that $f'(0) = 0$, prove that $f''(d) > 3$ for some real number $d \in (0,c)$.

iii) Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = x - a\tanh x$$

for some constant $a \in \mathbb{R}$.

a) Explain why $\operatorname{sech} x < 1$ for $x > 0$.

b) By considering the derivative, or otherwise, find the values of a for which $f(x) > 0$ for all $x > 0$? Give reasons for your answer

c) For which values of a does the equation

$$x = a\tanh x$$

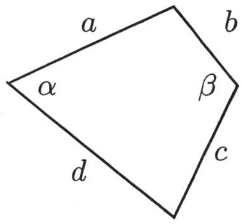
have a positive solution?

iv) The area $A(t)$ of an arbitrary convex quadrilateral with given side lengths a, b, c, d depends on the sum $t = \alpha + \beta$ of either pair of opposite angles, and is given by

$$A(t) = \sqrt{(s-a)(s-b)(s-c)(s-d) - \frac{1}{2}abcd(1+\cos t)},$$

where

$$s = \frac{1}{2}(a+b+c+d).$$



a) Explain why the area A of a convex quadrilateral, with fixed side lengths a, b, c, d, is maximal if the sum of either pair of opposite angles is π.

b) Show that the area function $A : [0, \pi] \to \mathbb{R}$ as defined above is invertible, and that the inverse function B is differentiable on $(A(0), A(\pi))$.

c) Show that

$$B'(A_0) = \frac{4A_0}{abcd},$$

where

$$A_0 = \sqrt{(s-a)(s-b)(s-c)(s-d) - \frac{1}{2}abcd}.$$