Chapter 4: Linear Equations and Matrices

Solving a two-by-two systems of linear equations such as

$$\begin{array}{rcrl} 2x & + & 3y & = 1 \\ 4x & - & 2y & = 2 \end{array}$$

using substitution/elimination of variables is not so hard.

But what if there are 30 equations with 30 variables?

We need a systematic approach!

Ex: Prove that the simultaneous equations
$ax + by = p, \quad cx + dy = q$ have a unique solution if and only if $ad - bc \neq 0$.

Answer: First assume that $ad - bc \neq 0$. Then at least one of the four numbers a, b, c, d is not zero. Let's say $a \neq 0$.

Then from the first equation we get

$$x = \frac{p - by}{a},$$

and substituting in the second equation we get

$$c\left(\frac{p - by}{a}\right) + dy = \left(\frac{-cb}{a} + d\right)y + \frac{cp}{a} = q.$$

Since the coefficient of y is not zero (why?), we get a unique solution for y, and hence for x as well.

The same argument works if $b \neq 0$, or if $c \neq 0$, or if $d \neq 0$. (Why?)

Now suppose $ad - bc = 0$. Assume $a \neq 0$. Then as before, we can transform the system into

$$x = \frac{p - by}{a}, \quad \frac{cp}{a} = q.$$

The second equation doesn't involve x or y, and the first equation is the equation of a line. The same argument works if $b \neq 0$, or if $c \neq 0$, or if $d \neq 0$.

If $a = b = c = d = 0$, then we get the equation $p = q = 0!$.

What does a solution to a system of linear equations mean geometrically?

In 2-dimensions: it is an intersection of lines in $\mathbb{R}^2$. For two lines:

- The lines could be parallel - no intersection
- The lines could be the same - the intersection is the full line
- The lines could intersect at a point

In 3-dimensions: it is an intersection of planes in $\mathbb{R}^3$. For two planes:

- The planes could be parallel - no intersection
- The planes could be the same - the intersection is the full plane
- The planes could intersect in a line

Example: Solve the following system of linear equations:

$$\begin{array}{ccccccc} x & + & y & + & z & = 6 \\ x & - & y & - & z & = 0 \end{array} .$$

Answer: Adding the two equations gives $2x = 6$, or $x = 3$. Substituting back into the two equations gives $y + z = 3$.

Thus the solution is $x = 3$ and $y + z = 3$.

Geometrically, these two equations represent planes which intersect in a line.

Ex: Solve the following system of linear equations:

$$\begin{array}{rcrcrl} x & + & y & + & z & = 7 \\ & - & 2y & - & z & = 0 \\ & & & & 3z & = 12 \end{array} .$$

Answer: From the third equation, we get $z = 4$, then from the second equation we get $y = -2$, and finally from the first equation we get $x = 5$.

Geometrically, these three equations represent planes which intersect at a point.

The method we use here is called *back substitution*.

It works well for systems of the form

$$\begin{array}{rcrcrl} a_1x & + & b_1y & + & c_1z & = d_1 \\ & & b_2y & + & c_2z & = d_2 \\ & & & & c_3z & = d_3 \end{array}$$

(at least when a_1, b_2, c_3 are not zero).

Ex: Solve the linear systems

$$\begin{array}{rcrcrl} x & + & 2y & + & 3z & = 6 \\ 2x & + & 4y & + & 6z & = 11 \end{array}$$

and

$$\begin{array}{rcrcrl} x & + & 2y & + & 3z & = 6 \\ 2x & + & 4y & + & 6z & = 12 \end{array} .$$

Answer: The first system has no solutions (the equations give two parallel planes).

The second system has the solution $x + 2y + 3z = 6$ (the two equations give the same plane).

We are going to learn a method called **Gaussian elimination**, which reduces any system of linear equations to a form where you can do back substitution.

Key idea: When we solve equations, the variables are really just symbols. We only care about the coefficients. When dealing with linear systems, we will actually ignore the variables altogether, and just discuss the block of coefficients.

The system

$$\begin{array}{ccccccc} x & + & y & + & z & = 6 \\ x & - & y & - & z & = 0 \end{array}$$

will be written as the block of numbers

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & -1 & -1 & 0 \end{array}\right),$$

called the **augmented matrix** of the system.

Ex: Find augmented matrix of

$$\begin{array}{rcrcrl} x & + & y & + & z & = 7 \\ & - & 2y & - & z & = 0 \\ & & & & 3z & = 12 \end{array} .$$

Answer:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & -2 & -1 & 0 \\ 0 & 0 & 3 & 12 \end{array} \right).$$

More generally, an $m \times n$ **matrix** is a rectangular block of numbers with m rows and n columns:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & . & . & a_{1n} \\ a_{21} & a_{22} & a_{23} & . & . & a_{2n} \\ a_{31} & a_{32} & a_{33} & . & . & a_{3n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{m1} & a_{m2} & a_{33} & . & . & a_{mn} \end{pmatrix}$$

A linear system of m equations in n variables has an $m \times n$ matrix of coefficients; the augmented matrix is an $m \times (n+1)$ matrix.

If $(A|\mathbf{b})$ is the augmented matrix of a linear system, we write the system in *vector form* as

$$A\mathbf{x} = \mathbf{b}.$$

Ex: Write the following system of equations in augmented matrix and in vector form:

$$\begin{array}{rcrcrcrcr} x_1 & + & 3x_2 & - & 6x_3 & + & 7x_4 & = & -2 \\ -2x_1 & & & + & 5x_3 & - & 4x_4 & = & 3 \\ 7x_1 & & & & & - & 5x_4 & = & -10 \end{array}.$$

Answer: Augmented matrix:

$$\left(\begin{array}{cccc|c} 1 & 3 & -6 & 7 & -2 \\ -2 & 0 & 5 & -4 & 3 \\ 7 & 0 & 0 & -5 & -10 \end{array} \right).$$

Vector form:

$$\begin{pmatrix} 1 & 3 & -6 & 7 \\ -2 & 0 & 5 & -4 \\ 7 & 0 & 0 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ -10 \end{pmatrix}.$$

Let's say we have a list of equations. There are a few things we can do that don't change the set of solutions.

- We can change the order of the equations.
- We can multiply any equation by a **nonzero** number.
- We can add a multiple of one equation to a different equation.

None of these operations change the set of solutions! (Why?)

The corresponding operations on (augmented) matrices are called **elementary row operations**.

- Swap two rows
- Multiply a row by a nonzero number.
- Add a multiple of one row to another row.

Key idea: If we start with a system of linear equations, convert to augmented matrix form, perform a sequence of elementary row operations, and then convert back to a system of linear equations, the solution of the new system will be **the same** as the solution of the original system.

Example: Let's try to rewrite the system

$$\begin{array}{ccccccc} 2x & - & y & + & z & = & \text{-1} \\ x & + & y & + & z & = & 4 \\ 3x & + & 2y & - & z & = & -2 \end{array}$$

in a form suitable for back substitution.

We start with the augmented matrix

$$\left(\begin{array}{ccc|c} 2 & -1 & 1 & -1 \\ 1 & 1 & 1 & 4 \\ 3 & 2 & -1 & -2 \end{array}\right).$$

We first move the 1 to the top left:

$$\left(\begin{array}{ccc|c} 2 & -1 & 1 & -1 \\ 1 & 1 & 1 & 4 \\ 3 & 2 & -1 & -2 \end{array}\right) \xrightarrow{R1 \leftrightarrow R2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & -1 & 1 & -1 \\ 3 & 2 & -1 & -2 \end{array}\right)$$

And then clear out the column beneath it:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & -1 & 1 & -1 \\ 3 & 2 & -1 & -2 \end{array}\right) \xrightarrow{R2-2R1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -3 & -1 & -9 \\ 3 & 2 & -1 & -2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -3 & -1 & -9 \\ 3 & 2 & -1 & -2 \end{array}\right) \xrightarrow{R3-3R1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -3 & -1 & -9 \\ 0 & -1 & -4 & -14 \end{array}\right)$$

Or we can combine the last two steps:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & -1 & 1 & -1 \\ 3 & 2 & -1 & -2 \end{array}\right) \xrightarrow{R2-2R1,\ R3-3R1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -3 & -1 & -9 \\ 0 & -1 & -4 & -14 \end{array}\right)$$

Next we bring the -1 up to the second row:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -3 & -1 & -9 \\ 0 & -1 & -4 & -14 \end{array}\right) \xrightarrow{R1 \leftrightarrow R2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -1 & -4 & -14 \\ 0 & -3 & -1 & -9 \end{array}\right)$$

And clear out the column beneath it:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -1 & -4 & -14 \\ 0 & -3 & -1 & -9 \end{array}\right) \xrightarrow{R3 - 3R2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -1 & -4 & -14 \\ 0 & 0 & 11 & 33 \end{array}\right)$$

This format of the final matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -1 & -4 & -14 \\ 0 & 0 & 11 & 33 \end{array}\right)$$

is called **row echelon form**.

- The first nonzero entry of each row (the **pivot**) is to the right of the pivot of the previous row.
- The entries below each pivot are all zero.
- We also require any zero rows to be at the bottom, though in this case there are no zero rows.

Any augmented matrix can be converted to row echelon form by a sequence of elementary row operations.

The virtue of row echelon form is that the corresponding system can be solved by back substitution.

For example, the matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -1 & -4 & -14 \\ 0 & 0 & 11 & 33 \end{array}\right)$$

from the previous example corresponds to the system

$$\begin{array}{rcrcrl} x & + & y & + & z & = 4 \\ & & -y & - & 4z & = -14 \\ & & & & 11z & = 33 \end{array}$$

which has the solution $z = 3,\ y = 2,\ x = -1$.

Ex: Row reduce the matrix $\left(\begin{array}{cccc|c} 0 & 2 & 3 & 3 & 5 \\ -4 & -2 & -1 & 2 & 1 \\ 1 & -1 & -2 & -2 & 0 \end{array}\right)$.

Answer:

$$\left(\begin{array}{cccc|c} 0 & 2 & 3 & 3 & 5 \\ -4 & -2 & -1 & 2 & 1 \\ 1 & -1 & -2 & -2 & 0 \end{array}\right) \xrightarrow{R1 \leftrightarrow R3} \left(\begin{array}{cccc|c} 1 & -1 & -2 & -2 & 0 \\ -4 & -2 & -1 & 2 & 1 \\ 0 & 2 & 3 & 3 & 5 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & -1 & -2 & -2 & 0 \\ -4 & -2 & -1 & 2 & 1 \\ 0 & 2 & 3 & 3 & 5 \end{array}\right) \xrightarrow{R2+4R1} \left(\begin{array}{cccc|c} 1 & -1 & -2 & -2 & 0 \\ 0 & -6 & -9 & \text{-6} & 1 \\ 0 & 2 & 3 & 3 & 5 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & -1 & -2 & -2 & 0 \\ 0 & -6 & -9 & \text{-6} & 1 \\ 0 & 2 & 3 & 3 & 5 \end{array}\right) \xrightarrow{R2 \leftrightarrow R3} \left(\begin{array}{cccc|c} 1 & -1 & -2 & -2 & 0 \\ 0 & 2 & 3 & 3 & 5 \\ 0 & -6 & -9 & \text{-6} & 1 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & -1 & -2 & -2 & 0 \\ 0 & 2 & 3 & 3 & 5 \\ 0 & -6 & -9 & \text{-6} & 1 \end{array}\right) \xrightarrow{R3+3R2} \left(\begin{array}{cccc|c} 1 & -1 & -2 & -2 & 0 \\ 0 & 2 & 3 & 3 & 5 \\ 0 & 0 & 0 & 3 & 16 \end{array}\right).$$

Ex: Row reduce the matrix $\left(\begin{array}{ccc|c} 2 & 4 & -1 & 3 \\ 3 & -7 & 10 & 1 \\ 4 & 0 & 1 & 0 \end{array}\right)$.

Answer:

$$\left(\begin{array}{ccc|c} 2 & 4 & -1 & 3 \\ 3 & -7 & 10 & 1 \\ 4 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R2 \to 2R2} \left(\begin{array}{ccc|c} 2 & 4 & -1 & 3 \\ 6 & -14 & 20 & 2 \\ 4 & 0 & 1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 2 & 4 & -1 & 3 \\ 6 & -14 & 20 & 2 \\ 4 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R2-3R1,\ R3-2R1} \left(\begin{array}{ccc|c} 2 & 4 & -1 & 3 \\ 0 & -26 & 23 & \text{-7} \\ 0 & -8 & 3 & \text{-6} \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 2 & 4 & -1 & 3 \\ 0 & -26 & 23 & \text{-7} \\ 0 & -8 & 3 & \text{-6} \end{array}\right) \xrightarrow{R3 \to 13R3} \left(\begin{array}{ccc|c} 2 & 4 & -1 & 3 \\ 0 & -26 & 23 & \text{-7} \\ 0 & -104 & 39 & \text{-78} \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 2 & 4 & -1 & 3 \\ 0 & -26 & 23 & \text{-7} \\ 0 & -104 & 39 & \text{-78} \end{array}\right) \xrightarrow{R3-4R2} \left(\begin{array}{ccc|c} 2 & 4 & -1 & 3 \\ 0 & -26 & 23 & \text{-7} \\ 0 & 0 & -53 & \text{-50} \end{array}\right).$$

Ex: Use Gaussian Elimination to solve the system of equations:

$$\begin{cases} x + y - z = 0 \\ 2x + y + 3z = -2 \\ -3x + 2y + 4z = -16 \end{cases}$$

Answer: First write the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & 1 & 3 & \text{-}2 \\ \text{-}3 & 2 & 4 & \text{-}16 \end{array} \right).$$

Then row reduce to get:

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -1 & 5 & \text{-}2 \\ 0 & 0 & 26 & \text{-}26 \end{array} \right),$$

which corresponds to the system

$$\begin{cases} x + y - z = 0 \\ \quad -y + 5z = -2 \\ \quad\quad 26z = -26 \end{cases}$$

Ex: Use Gaussian Elimination to solve the system of equations:

$$\left\{\begin{array}{ccccccccc} x_1 & + & x_2 & & & - & x_4 & = & -3 \\ 2x_1 & + & 3x_2 & - & x_3 & - & x_4 & = & -15 \\ 4x_1 & + & 2x_2 & + & 2x_3 & + & x_4 & = & -1 \end{array}\right.$$

Answer: First write the augmented matrix:

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & -3 \\ 2 & 3 & -1 & -1 & -15 \\ 4 & 2 & 2 & 1 & -1 \end{array}\right).$$

Then row reduce to get:

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & -3 \\ 0 & 1 & -1 & 1 & -9 \\ 0 & 0 & 0 & 7 & \text{-7} \end{array}\right),$$

which has the solution

$$x_4 = -1, \quad x_2 = -9 - x_4 + x_3 = -8 + x_3, \quad x_1 = -3 + x_4 - x_2 = 4 - x_3.$$

A matrix is in **reduced row echelon form** if in addition to being in row echelon form, all the pivots are 1 and there are zeros in the columns above the pivots.

For example:

$$\left(\begin{array}{cccc|c} \mathbf{1} & 0 & 0 & 4 & 1 \\ 0 & \mathbf{1} & 0 & 0 & 2 \\ 0 & 0 & \mathbf{1} & -2 & 3 \end{array}\right).$$

Ex: Find the reduced row echelon form for

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 10 \\ 2 & 1 & 3 & 1 & 4 \\ 3 & 1 & 4 & -2 & -5 \end{array}\right).$$

Answer: First we start row reducing:

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 10 \\ 0 & -1 & -1 & -5 & -16 \\ 0 & 0 & 0 & -1 & -3 \end{array}\right).$$

Now we make the second pivot one and clear out the entry above it:

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & -2 & -6 \\ 0 & 1 & 1 & 5 & 16 \\ 0 & 0 & 0 & -1 & -3 \end{array}\right).$$

Finally, we make the third pivot 1 and clear out the above column:

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right).$$

Given a linear system, we can tell immediately by looking at the row echelon form which of the following three possibilities occurr:

- No solutions
- Infinitely many solutions
- Exactly one solution

Ex:

$$\left(\begin{array}{ccc|c} 2 & 4 & -1 & 3 \\ 0 & 5 & 5 & 2 \\ 0 & 0 & 2 & 7 \end{array}\right)$$

- The third equation determines x_3
- The second equation determines x_2 (in terms of x_3)
- The first equation determines x_1 (in terms of x_2 and x_3)

Unique solution!

Ex:

$$\left(\begin{array}{ccc|c} 2 & 4 & -1 & 3 \\ 0 & 5 & 5 & 2 \\ 0 & 0 & 0 & 7 \end{array}\right)$$

The last equation is

$$0 = 7.$$

No solution!

Ex:

$$\left(\begin{array}{ccc|c} 2 & 4 & -1 & 3 \\ 0 & 5 & 5 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

- The third equation tells us nothing.
- The second equation determines x_2 in terms of x_3
- The first equation determines x_1 in terms of x_2 and x_3

Solutions corresponding to assigning an arbitrary value λ to x_3.

Infinitely many solutions!

Ex:

$$\left(\begin{array}{ccccc|c} 2 & 4 & -1 & 3 & -2 & 1 \\ 0 & 5 & 5 & 2 & 1 & 6 \\ 0 & 0 & 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Solutions are determmined by assigning arbitrary values λ and μ to x_3 and x_4.

A *leading column* in a matrix is a column containing a nonzero entry which is the first nonzero entry in its row.

Theorem

If the augmented matrix for a system of linear equations $(A|\mathbf{b})$ *is reduced to row-echelon form* $(U|\mathbf{y})$*, then*

1. *The system has* **no solution** *if the right-hand column* $\mathbf{y}$ *is a leading column.*
2. *The system has a* **unique solution** *if* $\mathbf{y}$ *is not a leading column but every column of* U *is a leading column.*
3. *The system has* **infinitely many solutions** *if* $\mathbf{y}$ *is not a leading column and at least one column of* U *is a non-leading column. The number of parameters required to describe the solutions equals the number of non-leading columns.*

Discuss the number of solutions of:

$$\left(\begin{array}{ccccc|c} 2 & 4 & -1 & 3 & -2 & 1 \\ 0 & 5 & 5 & 2 & 1 & 6 \\ 0 & 0 & 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{array}\right), \left(\begin{array}{cccc|c} 2 & 4 & -1 & 3 & -2 \\ 0 & 5 & 5 & 2 & 1 \\ 0 & 0 & 2 & 4 & 5 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right),$$

$$\left(\begin{array}{cccc|c} 2 & 4 & -1 & 3 & -2 \\ 0 & 5 & 5 & 2 & 1 \\ 0 & 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

(a) no solutions, (b) unique solution, (c) infinitely many solutions (1 parameter).

Summary of row reduction in terms of leading columns

We advance *column by column* from left to right.

- If the current column is *not* leading, do nothing and move on to next column
- If the current column *is* leading, switch rows if necessary until a leading entry (pivot) is as high as possible
- Once a pivot is as high as possible, kill all the entries beneath it by adding multiples of the row with the pivot

Once we have done all the columns, the matrix will be in row echelon form.

1. Sometimes we might do *optional* steps, such as switching rows or multiplying by scalars to get a better pivot.
2. It won't be clear which columns are leading until we get to them, as doing row operations changes which columns are leading.

Examples: find the next step in row reduction in each case

$$\left(\begin{array}{cccc|c} 2 & 4 & -1 & 3 & -2 \\ 0 & 0 & 5 & 2 & 1 \\ 0 & 5 & 2 & 4 & 5 \\ 0 & 1 & 0 & 2 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccccc|c} 2 & 4 & -1 & 3 & 1 & -2 \\ 0 & 5 & 5 & 2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 4 & 5 \\ 0 & 0 & 0 & 1 & 3 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccccccc|c} 2 & 4 & -1 & 3 & 1 & -2 & 1 & 1 \\ 0 & 5 & 5 & 2 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 4 & 5 & 3 & 4 \\ 0 & 0 & 0 & 2 & 3 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 4 & 5 & 6 & 4 \\ 0 & 0 & 0 & 3 & 3 & 0 & 0 & 0 \end{array}\right)$$

Ex: Suppose that the system with augmented matrix

$$\left(\begin{array}{ccc|c} 1 & a & a & b \\ a & 1 & a & c \\ a & a & 1 & d \end{array}\right)$$

has infinitely many solutions. What can you say about a, b, c and d?

Answer: We can row reduce to get

$$\left(\begin{array}{ccc|c} 1 & a & a & b \\ 0 & 1-a^2 & a-a^2 & c-ab \\ 0 & a-a^2 & 1-a^2 & d-ab \end{array}\right).$$

The row reduced form of the left part has a non-leading column if

$$(1-a^2) = \pm(a-a^2),$$

which has the solutions $a = 1, -\frac{1}{2}$.

For $a = 1$ the right part is not a leading column if $b = c = d$; for $a = -\frac{1}{2}$, we require $b + c + d = 0$.

A **homogenous** linear system is a system of the form

$$A\mathbf{x} = \mathbf{0},$$

which means that the right hand side of each equation is 0.

Theorem

A homogeneous system of equations always has at least one solution ($\mathbf{x} = \mathbf{0}$), and has a unique solution if and only if every column in the row-echelon form is a leading column.

Ex: The system

$$\left(\begin{array}{ccc|c} -1 & 3 & 2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

has infinitely many solutions since there is one non-leading column.

Theorem

A homogenous system of linear equations with more unknowns than equations always has infinitely many solutions.

(Why?)

Recall that we used the notation $A\mathbf{x}$ to represent the system of linear expressions

$$\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \quad \vdots \quad \cdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n. \end{array}$$

Theorem

If A is an $m \times n$ matrix and $\mathbf{x}$ and $\mathbf{y}$ are vectors with n components and λ is a scalar, then

- $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}$.
- $A(\lambda\mathbf{x}) = \lambda A\mathbf{x}$.

Proof: a) Letting $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ . \\ . \\ y_n \end{pmatrix}$, we have

$$A(\mathbf{x}+\mathbf{y}) = \left\{ \begin{array}{cccc} a_{11}(x_1+y_1) + a_{12}(x_2+y_2) + \cdots + a_{1n}(x_n+y_n) \\ a_{21}(x_1+y_1) + a_{22}(x_2+y_2) + \cdots + a_{2n}(x_n+y_n) \\ \vdots \quad \vdots \quad \cdots \quad \vdots \\ a_{m1}(x_1+y_1) + a_{m2}(x_2+y_2) + \cdots + a_{mn}(x_n+y_n) \end{array} \right\}$$

$$= \left\{ \begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \quad \vdots \quad \cdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{array} \right\} + \left\{ \begin{array}{c} a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n \\ a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n \\ \vdots \quad \vdots \quad \cdots \quad \vdots \\ a_{m1}y_1 + a_{m2}y_2 + \cdots + a_{mn}y_n \end{array} \right.$$

$$= A\mathbf{x} + A\mathbf{y}.$$

(b) is a similar calculation.

Corollary: $A(\mathbf{x_1} + \mathbf{x_2} + ... + \mathbf{x_k}) = A\mathbf{x_1} + A\mathbf{x_2} + ... + A\mathbf{x_k}$.

Theorem

If $\mathbf{u}$ *and* $\mathbf{v}$ *are solutions of the homogeneous equation* $A\mathbf{x} = \mathbf{0}$ *then so are* $\mathbf{u} + \mathbf{v}$ *and* $\lambda\mathbf{u}$*, for any* $\lambda \in \mathbb{R}$*.*

Proof: $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$ so $\mathbf{u} + \mathbf{v}$ is a solution.

Similarly for $\lambda\mathbf{u}$.

Theorem

If $\mathbf{x_1}$ and $\mathbf{x_2}$ are solutions to $A\mathbf{x} = \mathbf{b}$ then $\mathbf{x_2} = \mathbf{x_1} + \mathbf{v}$, where $\mathbf{v}$ is a solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Suppose $\mathbf{x_1}$ and $\mathbf{x_2}$ are solutions to $A\mathbf{x} = \mathbf{b}$. Let $\mathbf{v} = \mathbf{x_2} - \mathbf{x_1}$. Then

$$A\mathbf{v} = A(\mathbf{x_2} - \mathbf{x_1}) = A\mathbf{x_2} - A\mathbf{x_1} = \mathbf{b} - \mathbf{b} = \mathbf{0},$$

so $\mathbf{v}$ is a solution to the homogeneous equation.

Also, $\mathbf{x_2} = \mathbf{x_1} + \mathbf{x_2} - \mathbf{x_1} = \mathbf{x_1} + \mathbf{v}$.

Corollary: If A is a square matrix, then $A\mathbf{x} = \mathbf{b}$ has a unique solution iff $A\mathbf{x} = \mathbf{0}$ has a unique solution.

(Why is this true? What if A is not square?)

Ex: Suppose A is an $n \times n$ matrix. Prove that if a row of A is a linear combination of the other rows, then the system $A\mathbf{X} = \mathbf{b}$ does not have a unique solution.

Suppose that

$$R_n = c_1 R_1 + c_2 R_2 + ... + c_{n-1} R_{n-1}.$$

Let's do the row operations $R_n - c_1 R_1$, then $R_n - c_2 R_2$, all the way until $R_n - c_{n-1} R_{n-1}$.

Now R_n is a row of zeros! Therefore, the row reduced form has at least one row of zeros, and not all of the columns will be leading (since A is square).

Intuitively, if one row is a linear combination of the others, then the corresponding equation is redundant, so there won't be enough equations to eliminate all the variables when you do back substitution.

Ex: Let

$$A = \begin{pmatrix} -1 & 3 & 2 \\ 3 & 1 & 4 \\ 2 & 4 & 6 \end{pmatrix}$$

Find conditions on the numbers b_1, b_2, b_3 such that if

$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ then $A\mathbf{x} = \mathbf{b}$ has at least one solution.

Answer: We can row reduce the augmented matix to get

$$A = \left(\begin{array}{ccc|c} -1 & 3 & 2 & b_1 \\ 0 & 10 & 10 & b_2 + 3b_1 \\ 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array} \right).$$

Therefore the system will have at least one solution if and only if $b_3 - b_2 - b_1 = 0$.

Ex: Find conditions on λ such that the system

$$\left\{\begin{array}{ccccccc} x_1 & + & x_2 & + & \lambda x_3 & = 1 \\ x_1 & + & 2\lambda x_2 & + & x_3 & = 0 \\ 2x_1 & + & 4\lambda x_2 & + & \lambda x_3 & = -1 \end{array}\right.$$

has a unique solution, no solution, infinitely many solutions.

Answer: We can row reduce the augmented matix to get

$$\left(\begin{array}{ccc|c} 1 & 1 & \lambda & 1 \\ 0 & 2\lambda - 1 & 1 - \lambda & -1 \\ 0 & 0 & \lambda - 2 & -1 \end{array}\right).$$

- no solutions if $\lambda = 2$ or if $\lambda = \frac{1}{2}$
- unique solution otherwise

Ex: Does $\begin{pmatrix} 3 \\ 0 \\ 5 \\ 6 \end{pmatrix}$ belong to the span of $\left\{ \begin{pmatrix} 1 \\ -2 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 1 \\ 2 \end{pmatrix} \right\}$?

Answer: We need to solve the augmented system

$$\left(\begin{array}{cc|c} 1 & 0 & 3 \\ -2 & 4 & 0 \\ 3 & 1 & 5 \\ 2 & 2 & 6 \end{array} \right),$$

which reduces to:

$$\left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 22 \\ 0 & 0 & 0 \end{array} \right).$$

Since there are no solutions, it is not in the span.

Ex: Find the intersection (if any) of the line

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

and the plane

$$\mathbf{x} = \begin{pmatrix} -1 \\ -11 \\ -7 \end{pmatrix} + \alpha \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix}.$$

Answer: We want to solve the linear system

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -11 \\ -7 \end{pmatrix} + \alpha \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix}$$

Consider the augmented system

$$\left(\begin{array}{ccc|c} 1 & 3 & 3 & 2 \\ -2 & 5 & 2 & 13 \\ -5 & 1 & 1 & 6 \end{array} \right), \text{ with columns corr. to } \beta, \alpha, -\lambda$$

which reduces to:

$$\left(\begin{array}{ccc|c} 1 & 3 & 3 & 2 \\ 0 & 11 & 8 & 17 \\ 0 & 0 & 1 & \text{-2} \end{array} \right),$$

So we get the solution

$$\lambda = 2, \quad \alpha = 3, \quad \beta = -1,$$

corresponding to the point $\begin{pmatrix} 7 \\ 6 \end{pmatrix}$.

Ex: Three friends Ian, Shaun, Edna entered a coffee shop to buy coffee beans. Ian paid \$20 in total for 1 kilo of Brazilian, 2 kilos of Zinger and 3 kilos of Devil's. Shaun paid \$40 in total for 2 kilo of Brazilian, 2 kilos of Zinger and 8 kilos of Devil's. Edna paid \$82 in total for 4 kilo of Brazilian, 10 kilos of Zinger and 11 kilos of Devil's.
What was the cost of each type of coffee?

Answer: The purchases form a linear system where the prices are the unknowns. The augmented matrix is:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 20 \\ 2 & 2 & 8 & 40 \\ 4 & 10 & 11 & 82 \end{array} \right),$$

where the columns represent Brazilian, Zinger, and Devil's.

This has the unique solution: Brazilian \$10, Zinger's \$2, Devil's \$2.

Ex: A farmer intends to sow 2,000 hectares of land with oats, corn, wheat and rice. Because of the different requirements, it will take him 5 hours per hectare to plant each of oats and wheat, 7 hours per hectare to plant the corn and 9 hours per hectare to plant the rice. The cost of seed for each of the oats and corn is \$20 per hectare, for wheat \$24 per hectare and for rice it is \$28 per hectare. He has a total of 16000 hours and \$50400 available.

- Write down a matrix equation to determine the number of hectares of each grain that he can sow.
- Solve the equations, showing any restrictions on the parameters.
- Because of market prices, he wishes to sow as much rice as possible. How much should he sow of each grain in order to achieve this?

Solution: Let x_1 be the amount of oats, x_2 be the amount of corn, x_3 be the amount of wheat and x_4 be the amount of rice. The equations implied by the above information can be written in matrix form as:

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2000 \\ 5 & 5 & 7 & 9 & 16000 \\ 20 & 24 & 20 & 28 & 50400 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2000 \\ 0 & 1 & 0 & 2 & 2600 \\ 0 & 0 & 1 & 2 & 3000 \end{array}\right)$$

Hence the solutions are

$$x_1 = 3\lambda - 3600, x_2 = 2600 - 2\lambda, x_3 = 3000 - 2\lambda, x_4 = \lambda$$

so

$$1200 \leq \lambda \leq 1300.$$

To maximise x_4 we take $\lambda = 1300$ giving

$$x_1 = 300, x_2 = 0, x_3 = 400, x_4 = 1300.$$