THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Semester 1 2014

MATH1141
HIGHER MATHEMATICS 1A

(1) TIME ALLOWED – 2 hours

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

(8) A SHORT TABLE OF INTEGRALS IS APPENDED TO THE PAPER

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

Use a separate book clearly marked Question 1

1. i) Let $z = 7 + i$ and $w = 4 + 3i$.

a) Find $2z - \overline{w}$ in $a + ib$ form.

b) Find $5(w - i)/z$ in $a + ib$ form.

c) Find $|zw|$.

d) Find $\operatorname{Arg}(zw)$.

e) Hence, or otherwise, show that $\operatorname{Arg}(z) + \operatorname{Arg}(w) = \dfrac{\pi}{4}$.

f) Use the polar form of zw to evaluate $(zw)^{40}$.

ii) Consider the following system of equations:

$$\begin{array}{rcrcrcl} x & - & y & - & z & = & 1 \\ x & - & 3y & + & z & = & 1 \\ 2x & - & 3y & - & z & = & 2. \end{array}$$

a) Write the system in augmented matrix form and reduce it to row echelon form.

b) Solve the system.

iii) Evaluate the limits

a)

$$\lim_{x \to \infty} \frac{6x^2 + \sin x}{4x^2 + \cos x};$$

b)

$$\lim_{x \to 0} \frac{e^{2x} - 2x - 1}{4x^2}.$$

iv) A function g is defined by

$$g(x) = \begin{cases} \dfrac{|x^2 - 16|}{x - 4} & \text{if } x \neq 4 \\ \alpha & \text{if } x = 4. \end{cases}$$

By considering the left and right hand limits at $x = 4$, show that no value of α can make g continuous at the point $x = 4$.

v) Let $f(x) = x^5 + x^3 + x - 2$.

a) Prove that f has at least one real root in the interval $[0, 2]$, naming any theorems you use.

b) State, with reasons, the number of real roots of f.

Use a separate book clearly marked Question 2

2. i) Use a substitution to find the integral

$$\int \frac{dx}{x(1 + (\log x)^2)}.$$

ii) a) Give the definitions of $\sinh x$ and $\cosh x$ in terms of the exponential function.

b) Use your definitions to prove that $\sinh(2x) = 2\sinh x \cosh x$.

iii) Evaluate the integral

$$\int_0^{\pi/3} x\sin(2x)\, dx.$$

iv) Simplify the matrix expression $(A^T A)^{-1}(A^T A)^T$, where A is an invertible matrix.

v) Consider the three points A, B, C in $\mathbb{R}^3$ with position vectors $\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$, and $\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$ respectively.

a) Find a parametric vector form for the plane Π that passes through points A, B, and C.

b) Calculate the cross product $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC}$, showing your working.

c) Hence, or otherwise, find a **Cartesian** equation for the plane Π.

d) Find the area of the triangle ABC.

e) Find the minimal distance from the point $P \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix}$ to the plane Π.

vi) Consider the following MAPLE session.

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> with(LinearAlgebra):
> A := <<0,1,1>|<1,-1,-1>|<-2,1,0>>;
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$$A := \begin{bmatrix} 0 & 1 & -2 \\ 1 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

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> B := A^2;
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$$B := \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & -3 \\ -1 & 2 & -3 \end{bmatrix}$$

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> C:=A^3;
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$$C := \begin{bmatrix} 2 & -3 & 3 \\ -2 & 2 & 1 \\ -1 & 0 & 4 \end{bmatrix}$$

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> F := 2A + B + C;
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$$F := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Using the above MAPLE session, or otherwise, find the 3×3 matrix which is the inverse of the matrix A.

Use a separate book clearly marked Question 3

3. i) Let $g(x) = 3x - \cos 2x - 1$, $x \in \mathbb{R}$. Explain why g has a differentiable inverse function $h = g^{-1}$ and calculate $h'(-2)$.

ii) a) State carefully the Mean Value Theorem.

b) Use the Mean Value Theorem to prove that if $a < b$ then

$$0 < \tan^{-1} b - \tan^{-1} a \leq b - a.$$

c) Using (b) or otherwise, prove that the improper integral

$$I = \int_1^\infty \tan^{-1}\left(t + \frac{1}{t^2}\right) - \tan^{-1} t \, dt$$

converges.

iii) Use the ϵ-M definition of the limit to prove that

$$\lim_{x \to \infty} \frac{e^x}{\cosh x} = 2.$$

iv) Consider the polar curve $r = 1 + \cos 4\theta$.

a) Determine the values of $\theta \in [0, 2\pi]$ for which r has the smallest and largest values.

b) Hence, or otherwise, sketch this polar curve. (You are not required to find the slope.)

v) For $x > 0$, let $f(x) = x^{x \ln x}$.

a) Evaluate $f'(x)$.

b) Determine the values of x for which $f'(x) > 0$ and the values of x for which $f'(x) < 0$.

c) Given that $\lim\limits_{x \to 0+} f(x) = 1$, sketch the graph $y = f(x)$ for $0 \leq x \leq 2$.

Use a separate book clearly marked Question 4

4. i) Find the conditions on b_1, b_2, b_3 which ensure that the following system has a solution.

$$\begin{array}{rcrcrcl} 2x & & & - & 4z & = & b_1 \\ 3x & + & y & - & 2z & = & b_2 \\ -2x & - & y & & & = & b_3 \end{array}$$

ii) Let I, J and K be the points in $\mathbb{R}^3$ whose position vectors are the three standard basis vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ respectively.

By considering vectors of the form $\begin{pmatrix} x \\ x \\ x \end{pmatrix}$, find the position vector of a point A, not the origin, such that the distances from A to I, J and K are all 1.

iii) Consider the complex matrix $A = \begin{pmatrix} 2 & i \\ 1+i & \alpha \end{pmatrix}$.

a) Find A^{-1} in the case when $\alpha \in \mathbb{R}$.

b) Find all values of $\alpha \in \mathbb{C}$ for which $\det(A^2) = -1$.

iv) You may assume that $(z^9 - 1) = (z^3 - 1)(z^6 + z^3 + 1)$.

a) Explain why the roots of $z^6 + z^3 + 1 = 0$ are $e^{\pm\frac{2\pi i}{9}}, e^{\pm\frac{4\pi i}{9}}, e^{\pm\frac{8\pi i}{9}}$.

b) Divide $z^6 + z^3 + 1$ by z^3 and let $x = z + \dfrac{1}{z}$.
Find a cubic equation satisfied by x.

c) Deduce that $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{8\pi}{9} = 0$.

v) The norm $\|M\|$ of an $n \times n$ matrix M is the maximum value that $|M\mathbf{u}|$ takes for all unit vectors $\mathbf{u} \in \mathbb{R}^n$.

a) Show that for any vector $\mathbf{x} \in \mathbb{R}^n$,

$$|M\mathbf{x}| \leq \|M\||\mathbf{x}|.$$

b) Suppose that M and N are any two $n \times n$ matrices. By considering $MN\mathbf{u}$, or otherwise, show that

$$\|MN\| \leq \|M\|\|N\|.$$

c) What is the norm of the matrix $\begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$?

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BASIC INTEGRALS

$$\int \frac{1}{x}\,dx = \ln|x| + C = \ln|kx|, \qquad C = \ln k$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

$$\int a^x\,dx = \frac{1}{\ln a}a^x + C, \qquad a \neq 1$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$$

$$\int \operatorname{cosec}^2 ax\,dx = -\frac{1}{a}\cot ax + C$$

$$\int \tan ax\,dx = \frac{1}{a}\ln|\sec ax| + C$$

$$\int \cot ax\,dx = \frac{1}{a}\ln|\sin ax| + C$$

$$\int \sec ax\,dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C$$

$$\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C$$

$$\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C$$

$$\int \operatorname{sech}^2 ax\,dx = \frac{1}{a}\tanh ax + C$$

$$\int \operatorname{cosech}^2 ax\,dx = -\frac{1}{a}\coth ax + C$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$\begin{aligned}\int \frac{dx}{a^2 - x^2} &= \frac{1}{a}\tanh^{-1}\frac{x}{a} + C, && |x| < a\\ &= \tfrac{1}{a}\coth^{-1}\tfrac{x}{a} + C, && |x| > a > 0\\ &= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C, && x^2 \neq a^2\end{aligned}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\frac{x}{a} + C, \qquad x \geqslant a > 0$$