Sets, inequalities and functions

Sets of numbers

A set is a collection of distinct objects. The objects in a set are called the elements or members of the set.

- The set $\mathbb{N}$ of natural numbers is given by
$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}.$$
- The set $\mathbb{Z}$ of integers is given by
$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$
- The set $\mathbb{Q}$ of rational numbers consists of numbers of the form $\frac{p}{q}$ where p, q are integers and $q \neq 0$.
- There are numbers, such that $\sqrt{2}$, which are not rational numbers.

Assume that $x^2 = 2$ and that $x = p/q$ where q is non-zero and p, q are integers with no factors in common than $1, -1$. Then

$$(p/q)^2 = 2,$$

so that

$$p^2 = 2q^2. \tag{1}$$

The right-hand side has a factor of 2 and hence so does the left hand side. The implies that p has a factor of 2 so that

$$p = 2p'$$

for some integer p'. Substituting back into (1) yields

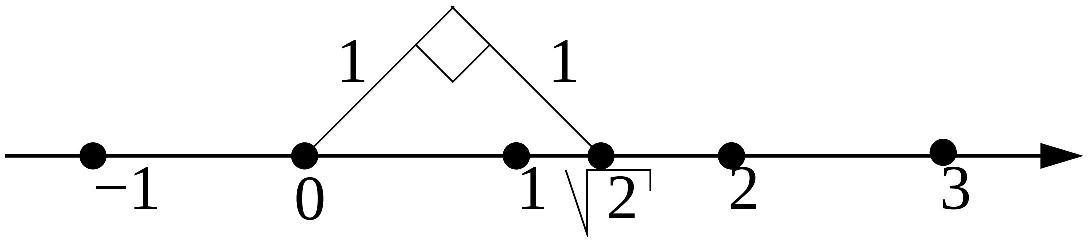
$$(2p')^2 = 2q^2$$

and arithmetic shows that

$$2(p')^2 = q^2.$$

Similarly, q has a factor of 2. But then p and q must have a factor of 2 in common. This contradicts the fact that p, q were assumed to have no factors in common other than $1, -1$. Our conclusion is that no rational number has square equal to 2.

- $\sqrt{2}$ and numbers such that $\sqrt{3}$, π, e are examples of irrational numbers.
- The totality of all rational and irrational numbers is called the set of real numbers, $\mathbb{R}$, and is represents by the real line.
- The following figure gives us understanding where we should put the number $\sqrt{2}$ on a number line.



Notation.
If x is a member of a set A, then we write $x \in A$.
If x is not a member of A then we write $x \notin A$.

Example.

$$2 \in \mathbb{N}, \quad -12 \notin \mathbb{N}, \quad \frac{22}{7} \notin \mathbb{Z}, \quad \sqrt{2} \notin \mathbb{Q}, \quad \sqrt{2} \in \mathbb{R}.$$

Exercise.

$$-\frac{1}{2} \square \mathbb{Q}, \quad -12 \square \mathbb{Q}, \quad 0 \square \mathbb{R}, \quad \sqrt{5} \square \mathbb{Q}, \quad 1 \square \mathbb{N}.$$

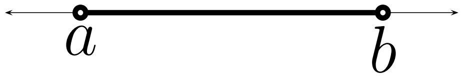
Remark. The set

$$\{x \in \mathbb{R} : x \notin \mathbb{Q}\}$$

is the set of all real numbers x such that (":") x is not an element of $\mathbb{Q}$.

Notation for intervals. Suppose that a and b are real numbers and that $a < b$. Then

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}$$



$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$$

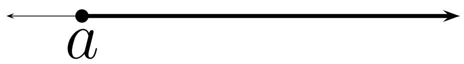
$$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$$

$$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$$

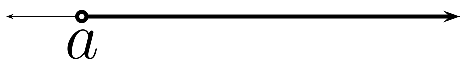
An interval $[a, b]$ that includes its endpoints a and b is called a closed interval, while an interval (a, b) that excludes its endpoints is called an open interval. The intervals $[a, b)$ and $(a, b]$ are neither open nor closed.

Rays of the real line using the symbol ∞.

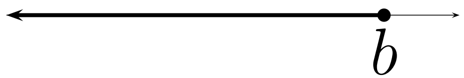
$$[a, \infty) = \{x \in \mathbb{R} : a \leq x\}$$



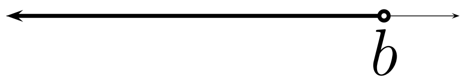
$$(a, \infty) = \{x \in \mathbb{R} : a < x\}$$



$$(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$$



$$(-\infty, b) = \{x \in \mathbb{R} : x < b\}$$



$$(-\infty, \infty) = \mathbb{R}$$

Definition. We say that a set A is a subset of a set B if every element of A is an element of B. If A is a subset of B then we also say that B contains the set A.

Examples.

- $\mathbb{N}$ is a subset of $\mathbb{Z}$, and $\mathbb{Z}$ is a subset of $\mathbb{Q}$, and $\mathbb{Q}$ is a subset of $\mathbb{R}$.
- $\{0, 2, 3\}$ is a subset of $\{0, 1, 2, 3, 5\}$.
- $(-1, 2]$ is not a subset of $[0, \infty)$.
- $\{1\}$ is a subset of $[0, \infty)$.
- Any set is a subset of itself.
- $(1, 3)$ is a subset of $[1, 3)$.

Solving inequalities

Remember:

- You can always add or subtract the same "thing" from both sides.
- You can always multiply or divide both sides by a positive quantity.
- You can't multiply or divide by zero!
- If you multiply by a negative quantity you need to swap the direction of the inequality.

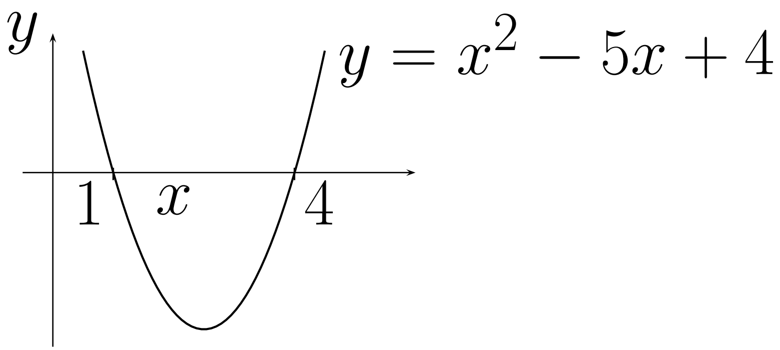
Often solving an inequality turns into solving an equality.

Two types of inequalities deserve special attention: quadratic inequalities and rational inequalities.

Examples. (a) Solve the quadratic inequality

$$x^2 + 4 \geq 5x.$$

First we solve the quadratic equation $x^2 + 4 = 5x$, which is equivalent to $x^2 - 5x + 4 = 0$.
The roots are $x_1 = 1$, $x_2 = 4$
Put them on the plane

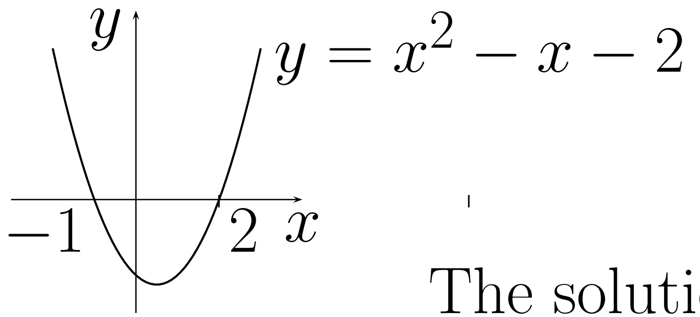


So the solution of the inequality is $x \leq 1$ or $x \geq 4$.

(b) Solve the rational inequality

$$\frac{1}{x+1} < \frac{1}{(x-2)}.$$

Let $x \neq -1$ and $x \neq 2$.
Multiply both sides of the inequality by the positive number $(x-2)^2(x+1)^2$.
We have $(x-2)^2(x+1) < (x-2)(x+1)^2$.
Open the brackets $x^3 - 3x^2 + 4 < x^3 - 3x - 2$.
Finally, we have $-3x^2 + 3x + 6 < 0$ or $x^2 - x - 2 > 0$.
Root of the equation $x^2 - x - 2 = 0$ are $x_1 = 2$, $x_2 = -1$.



The solution is $x < -1$ or $x > 2$.

The absolute value, of a real number x is denoted by $|x|$ and defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

The number $|x|$ may be interpreted as the "size" or "magnitude" of the number x. It can be also viewed as a distance from x to the origin.

Properties. Suppose that x and y are real numbers. Then

- $|-x| = |x|$,
- $|xy| = |x||y|$,
- $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$ provided that $y \neq 0$,
- $|x+y| \leq |x| + |y|$ (the triangle inequality).

The following facts are useful for solving inequalities.

- For every real number x,

$$|x| = \sqrt{x^2}, \quad |x|^2 = x^2.$$

- Geometrically, the number $|x-y|$ is interpreted as the *distance* from x to y (or from y to x).
- **Example.** How small does $|x-2|$ need to be to ensure that $|x^2-4| < 1$?

  Think of $|x-2|$ as the distance from x to 2. So this is really asking, 'How close does x have to be to 2 so that x^2 is within 1 of 4?'
- For any positive real number a,

$$|x| < a \quad \Leftrightarrow \quad x^2 < a^2 \quad \Leftrightarrow \quad -a < x < a.$$

- For any positive real number a,

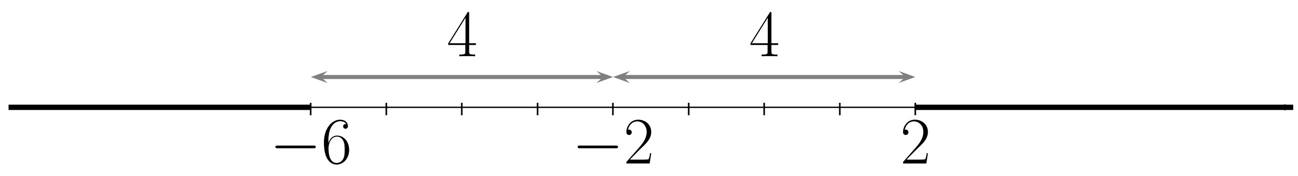
$$|x| > a \quad \Leftrightarrow \quad x^2 > a^2 \quad \Leftrightarrow \quad x < -a \quad \text{or} \quad x > a.$$

Examples. Solve the following inequalities.

(a) $$|x+2| \geq 4$$

Geom. solution:

The distance from -2 to x is greater than or equal to 4



So the solution is $x \leq -6$ or $x \geq 2$.

Alg. solution:

The inequality $|x+2| \geq 4$ is
equivalent to $x+2 \geq 4$ or $x+2 \leq -4$,
which is equivalent to $x \geq 2$ or $x \leq -6$.

(b)
$$\frac{|x+5|}{|x-1|} < 1$$

Suppose that $x \neq 1$.
Multiply both sides of the inequality by the positive number $|x-1|$ and obtain $|x+5| < |x-1|$.
Squaring both (positive) sides, we have $(x+5)^2 < (x-1)^2$.
Expanding and solving, we have $x < -2$.

Proving inequalities. (a) Prove that for all $x, y \geq 0$,

$$\frac{x+y}{2} \geq \sqrt{xy}.$$

$(\sqrt{x} - \sqrt{y})^2 \geq 0 \quad \Rightarrow \quad x + y - 2\sqrt{xy} \geq 0 \quad \Rightarrow$
$x + y \geq 2\sqrt{xy} \quad \Rightarrow \quad \frac{x+y}{2} \geq \sqrt{xy}.$

(b) Prove that for $x > 0$,

$$x + \frac{1}{x} \geq 2.$$

Using (a) with $y = \frac{1}{x}$,
we have $\frac{x+\frac{1}{x}}{2} \geq \sqrt{x\frac{1}{x}} = 1.$
Multiplying by 2, we obtain $x + \frac{1}{x} \geq 2.$

Disproving inequalities

To prove that an inequality (or equality) does not hold, it is enough to give one example, for which the inequality (or equality) does not work.

Example.

Is it true or false (and why) that

$$\text{if } \ a > b, \quad \text{then } \ |a| > |b|.$$

The claim is evidently true for positive a and b.
Therefore, we will look for the example among negative numbers.
Take, for example, $a = -1$ and $b = -2$.
Then $a > b$ and $|a| = 1$, $|b| = 2$.
So $|a| > |b|$ is false for this example.
Thus the claim can not be true in general. So it is false.

Functions

A function $f : A \to B$ is a rule which assigns to every element x belonging to a set A exactly one element $f(x)$ belonging to a set B, that is $x \mapsto f(x)$.

Terminology.

- A is called the domain of the function f, that is

$$A = \mathrm{Dom}(f) = \{\text{all allowable inputs}\}.$$

- B is called the codomain of f, that is

$$B = \mathrm{Codom}(f) = \{\text{all allowable outputs}\}.$$

- The range of f is

$$\begin{aligned}\mathrm{Range}(f) &= \{f(x) : x \in A\} \\ &= \{\text{all outputs that actually occur}\}.\end{aligned}$$

Example.

$$\begin{aligned} f : [1, \infty) &\to \mathbb{R} \\ x &\mapsto \sqrt{x-1}. \end{aligned}$$

$$\mathrm{Dom}(f) = [1, \infty), \quad \mathrm{Codom}(f) = \mathbb{R}, \quad \mathrm{Range}(f) = [0, \infty).$$

Remarks.

- f denotes a function, while $f(x) \in B$ is a number, namely the value of f at the point $x \in A$.
- The codomain of f may be changed but it must contain all the outputs of f.
- The statement

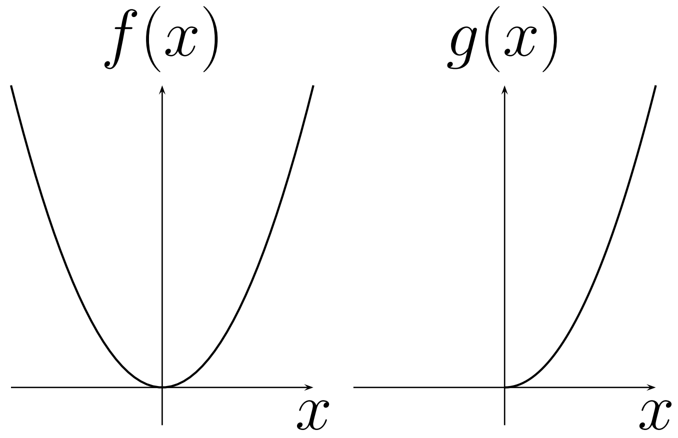
$$\text{‘}f(x) = \sqrt{x} \text{ for all } x \text{ in } [0, \infty)\text{’}$$

may be abbreviated as

$$f(x) = \sqrt{x} \qquad \forall x \in [0, \infty).$$

- Functions which are defined by the same rule but have different domains are not the same. For example, consider

$$\begin{aligned} f : \mathbb{R} \to \mathbb{R}, &\quad f(x) = x^2 \\ g : [0, \infty) \to \mathbb{R}, &\quad g(x) = x^2 \end{aligned}$$



Natural domain. If, for whatever reason, the domain of a function is not defined then we may choose the natural domain or maximal domain, that is the largest possible domain for which the rule makes sense (for real numbers).

Examples.

(a) Find the maximal domain for

$$f(x) = \frac{1}{x^2 + x - 2}.$$

The domain of f consists of all x such that $x^2 + x - 2 \neq 0$.
Solving this quadratic equality, we obtain that $x \neq 1$ and $x \neq -2$.
So $\mathrm{Dom}(f) = \mathbb{R} \setminus \{-2, 1\}$.

(b) Find the maximal domain and the range for

$$f(x) = \sqrt{\cos x}.$$

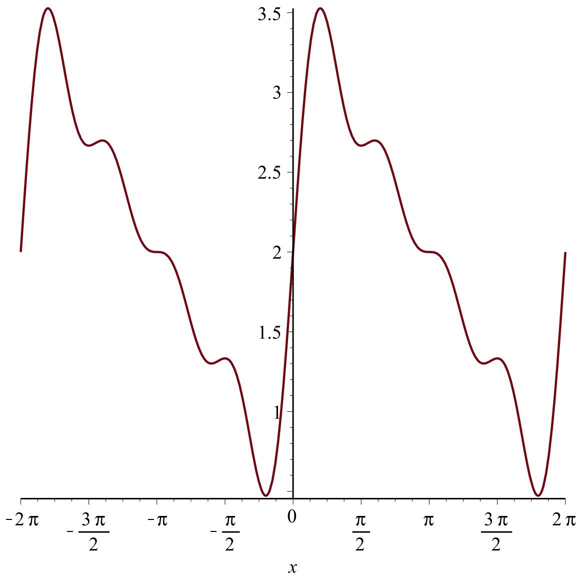
The domain of f consists of all x such that $\cos x \geq 0$.
This is equivalent to $x \in [-\pi/2 + 2\pi n, \pi/2 + 2\pi n]$, where $n \in \mathbb{Z}$.
To determine the range, note that $0 \leq \cos x \leq 1$ for all $x \in \mathrm{Dom}(f)$.
Thus, $0 \leq \sqrt{\cos x} \leq 1$ for all $x \in \mathrm{Dom}(f)$.
So $\mathrm{Range}(f) = [0, 1]$.

Remark. We distinguish between the range and the codomain of a function since it is often difficult to find the range of a function. For example, what is the range of

$$2 + \sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 4x}{4}\ ?$$

Using MAPLE:

>plot(2+sin(x)+(1/2)*sin(2*x)+(1/3)*sin(3*x)+(1/4)*sin(4*x), x = -2*Pi..2*Pi)



Combining functions. If f and g are two functions with the same domain, then one can combine f and g to form new functions.

Definition. Suppose that $f : A \to B$ and $g : A \to B$ are real-valued functions. Then, the functions $f+g$, $f-g$, $f \cdot g$ and f/g are defined by the rules

$$\begin{aligned}
(f+g)(x) &= f(x) + g(x) && \forall x \in A \\
(f-g)(x) &= f(x) - g(x) && \forall x \in A \\
(f \cdot g)(x) &= f(x)g(x) && \forall x \in A \\
(f/g)(x) &= \frac{f(x)}{g(x)} && \forall x \in A \text{ such that } g(x) \neq 0,
\end{aligned}$$

Example. (a) Find $(f+g)(0)$ and the maximal domains of f/g, where

$$f(x) = 1 + x^2, \quad g(x) = \cos(x).$$

Since $\mathrm{Dom}(f) = \mathrm{Dom}(g) = \mathbb{R}$,
the function $f+g$ is well-defined and $\mathrm{Dom}(f+g) = \mathbb{R}$.
$(f+g)(0) = f(0) + g(0) = 1 + 1 = 2$.
The function f/g is defined for all x such that $g(x) \neq 0$.
$\cos x \neq 0$ is equivalent $x \neq \pi/2 + \pi n$, where $n \in \mathbb{Z}$.
Thus, $\mathrm{Dom}(f/g) = \mathbb{R} \setminus \{\pi/2 + \pi n : n \in \mathbb{Z}\}$.

(b) Let
$$f(x) = \sqrt{x}, \quad g(x) = \frac{1}{x-1}.$$
Think how the function $f+g$ can be defined.

Note that $\mathrm{Dom}(f) = \{x : x \geq 0\}$, and $\mathrm{Dom}(g) = \{x : x \neq 1\}$.
The function $f+g$ can be defined only for points x which belong to both domains $\mathrm{Dom}(f)$ and $\mathrm{Dom}(g)$.
That is $\mathrm{Dom}(f+g) = \{x : x \geq 0 \text{ and } x \neq 1\}$.
So $(f+g)(x) = \sqrt{x} + \frac{1}{x-1}$, for all $x \in \mathrm{Dom}(f+g)$.

Definition. Suppose that

$$f : C \to D \qquad \text{and} \qquad g : A \to B$$

are functions such that $\operatorname{Range}(g)$ is a subset of $\operatorname{Dom}(f)$. Then the composition

$$f \circ g : A \to D$$

is defined by the rule

$$(f \circ g)(x) = f(g(x)) \qquad \forall x \in A.$$

Example. Let the functions f and g be given by the rules

$$f(x) = \sqrt{x}, \quad g(x) = \cos(x) - 2$$

Find if exist, $(f \circ g)$ and $(g \circ f)$.

Note that $\mathrm{Dom}(f) = \{x : x \geq 0\}$, and $\mathrm{Dom}(g) = \mathbb{R}$.
$\mathrm{Range}(f) = \{x : x \geq 0\}$, and $\mathrm{Range}(g) = [-3, -1]$.
To define $f \circ g$ we need $\mathrm{Range}(g)$ to be a subset of $\mathrm{Dom}(f)$, which is false.
Hence, for all $x \in \mathrm{Dom}(g)$, the value $f(g(x))$ is indefinite, and so $f \circ g$ can not be defined.
To define $g \circ f$ we need $\mathrm{Range}(f)$ to be a subset of $\mathrm{Dom}(g)$, which is true.
Thus, $g \circ f$ is defined for all points from $\mathrm{Dom}(f)$.
$(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \cos\sqrt{x} - 2.$

Polynomials. A function $f : \mathbb{R} \to \mathbb{R}$ is called a polynomial if

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where $n \in \mathbb{N}$ is the degree and the coefficients $a_0, a_1, \ldots, a_n$ are real numbers with the leading coefficient $a_n \neq 0$.

Examples.

The function p defined by $p(x) = 2x^3 - 5x$, for $x \in \mathbb{R}$ is a polynomial with degree 3 and the leading coefficient 2.

The function p defined by $p(x) = 3$, for $x \in \mathbb{R}$ is also a polynomial with degree 0 and the leading coefficient 3.

Rational functions. Suppose that p and q are polynomials. The function f defined by the rule

$$f(x) = \frac{p(x)}{q(x)}, \quad \mathrm{Dom}(f) = \{x \in \mathbb{R} : q(x) \neq 0\}$$

is called a rational function.

Examples.

The function f defined by

$$f(x) = \frac{1}{x^2 + x - 2}, \qquad \mathrm{Dom}(f) = \mathbb{R} \setminus \{-2, 1\}$$

is rational.

The function f defined by

$$f(x) = x - 1 + \frac{3}{x^2 + 3} = \frac{(x^2 + 3)(x - 1) + 3}{x^2 + 3}$$

is also rational and $\mathrm{Dom}(f) = \mathbb{R}$.

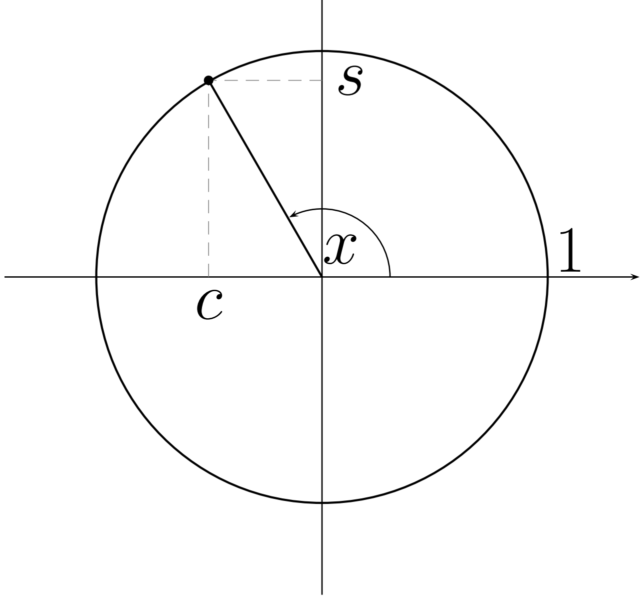
Definition. The trigonometric functions

$$\sin : \mathbb{R} \to \mathbb{R} \qquad \text{and} \qquad \cos : \mathbb{R} \to \mathbb{R}$$

are defined by

$$\sin x = s \qquad \text{and} \qquad \cos x = c,$$

where s and c are defined in a geometric manner:



The following properties are immediate from the definition.

- $\mathrm{Dom}(\sin) = \mathrm{Dom}(\cos) = \mathbb{R}$.
- $\mathrm{Range}(\sin) = \mathrm{Range}(\cos) = [-1, 1]$.
- sin and cos are periodic of period 2π, that is

$$\sin(x + 2\pi) = \sin x, \quad \cos(x + 2\pi) = \cos x.$$

- cos is an even function, that is

$$\cos(-x) = \cos x.$$

- sin is an odd function, that is

$$\sin(-x) = -\sin x.$$

- $\sin^2 x + \cos^2 x = 1$.

Other trigonometric functions with suitable domains are defined by

$$\tan x = \frac{\sin x}{\cos x}, \qquad \cot x = \frac{\cos x}{\sin x},$$
$$\sec x = \frac{1}{\cos x}, \quad \operatorname{cosec} x = \frac{1}{\sin x}.$$

The six trigonometric functions are related by various identities and formulae (which you are supposed to know):

- complementary identities

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x,$$
$$\cos\left(\frac{\pi}{2} - x\right) = \sin x$$

- Pythagorean identities

$$\begin{aligned} \cos^2 x + \sin^2 x &= 1, \\ 1 + \tan^2 x &= \sec^2 x, \\ \cot^2 x + 1 &= \operatorname{cosec}^2 x / \end{aligned}$$

- the sum and difference formulae

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y,$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}.$$

- double-angle formulae

$$\sin(2x) = 2 \sin x \cos x,$$

$$\cos(2x) = \cos^2 x - \sin^2 x,$$

$$\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x}.$$

The elementary functions

The elementary functions are all those functions that can be constructed by combining (a finite number of) polynomials, exponentials, logarithms, roots and trigonometric functions (including the inverse trigonometric functions) via function composition, addition, subtraction, multiplication and division. For example,

$$\begin{aligned} f(x) &= e^{\sin x} + x^2, \\ g(x) &= \frac{\ln x - \tan x}{\sqrt{x}}, \\ h(x) &= \sqrt[3]{x^4 - 2x^2 + 5}, \\ k(x) &= |x| = \sqrt{x^2}. \end{aligned}$$

Every rational function is an elementary function.

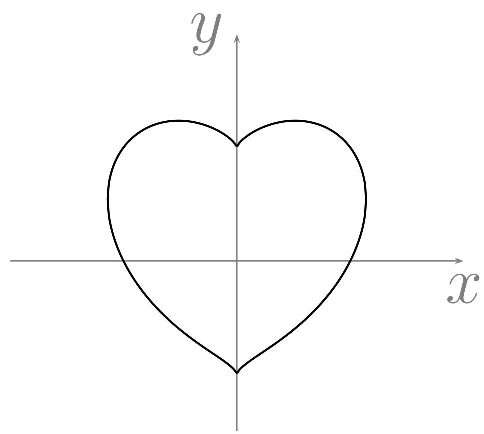
However, there exist important functions which are not elementary!

Implicitly defined functions

Many curves on the plane can be described as all those points (x, y) on the plane that satisfy some equation involving x and y. For example, consider the equation

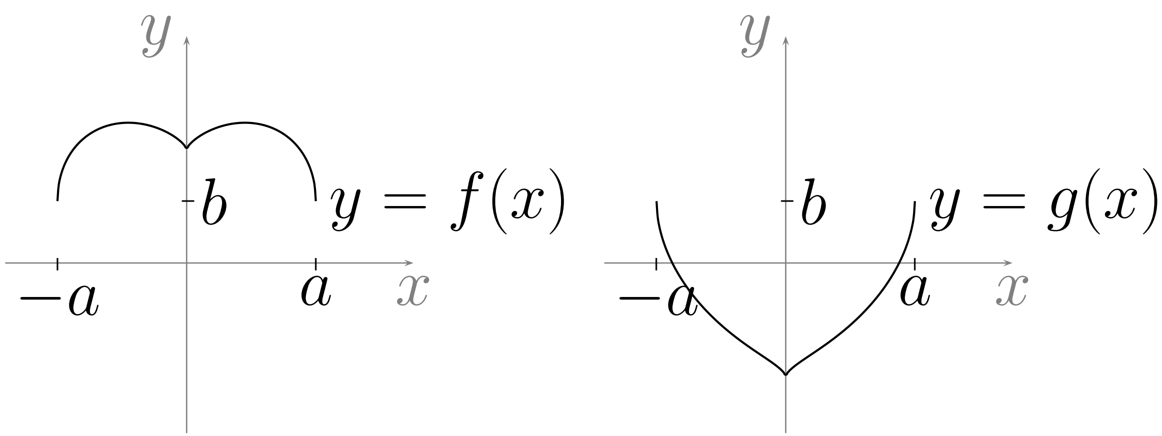
$$(x^2 + y^2 - 1)^3 - x^2y^3 = 0. \qquad (\heartsuit)$$

The set of points (x, y) satisfying this equation are shown on the graph below.



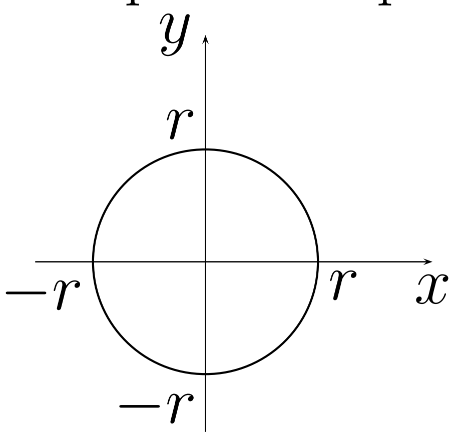
Properties.

- There exist several y-values for some x-values. Hence, the curve cannot be the graph of one function of x.
- The curve may be decomposed into two curves which may be regarded as the graphs of two functions, f and g, say.

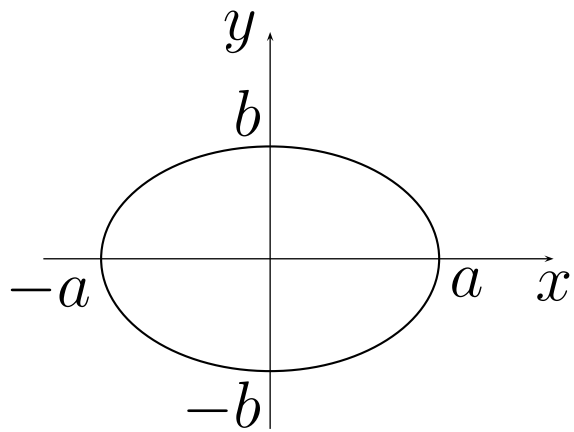


- We say that the functions f and g are implicitly defined by the relation ($\heartsuit$).

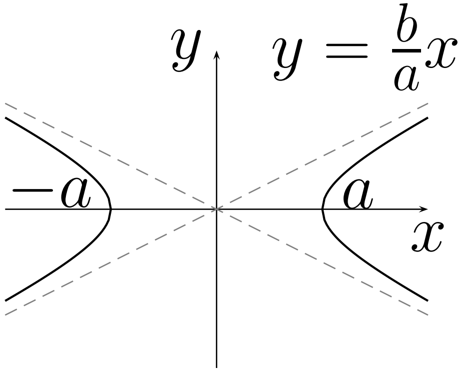
Other examples of implicitly defined functions are conic sections:



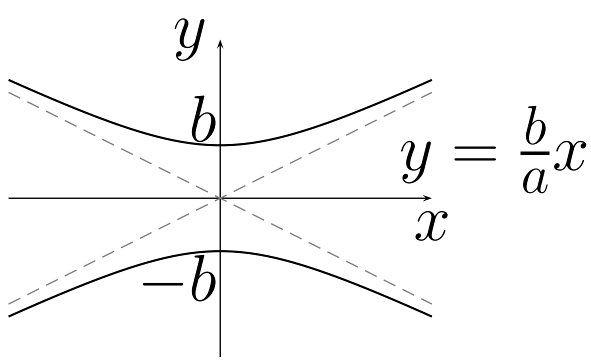
Circle: $x^2 + y^2 = r^2$



Ellipse: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$



Hyperbola: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$



Hyperbola: $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$

Sometimes it is better leave things in the implicit form. For example, for

$$x^4 + \sin(y^4) - x^2 + 200xy - y^2 = 95,$$

you have no choice!