Gravity

PHYS 1121 and 1131, Session 2, 2010, UNSW

Also see http://www.animations.physics.unsw.edu.au/mechanics/chapter11_gravity.html

- **context in physics** (& history)
- **Newton's law of gravity**
- **Cavendish measures G** (and thus m_{Earth})
- **the gravitational field**
- **Gravitational potential energy in a non-uniform field**
- **escape velocity**
- **Planetary motion**
  Kepler's laws, and Newton's laws
- **Orbits and energy**
- **Limitations to Newton's laws**

Gravity: where does it fit in? *this page not examinable*

Gravity [general relativity]	Electric force*	Weak nuclear force	Strong nuclear force	Colour force
gravitons	photons	mesons		gluons

electro-weak

Grand Unified Theories

Some tries for classical gravity

Theories Of Everything

* Electromagnetism "unified" by Maxwell, but especially by Einstein: Magnetism may be simplistically considered as the relativistic correction to electric interactions which applies when charges move.
http://www.phys.unsw.edu.au/einsteinlight/jw/module2_FEB.htm

- Only gravity and electric force have macroscopic ("infinite") range.

$m_{graviton}? = m_{photon} = 0$

- Gravity weakest, but dominates on large scale. *Why?*

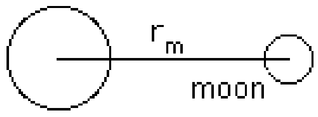
Greeks to Galileo:

i) things fall to the ground ('natural' places)

ii) planets etc move (variety of reasons)

but no connection (in fact, natural *vs* supernatural)

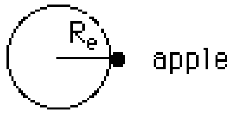
Newton's calculation: accel^n of moon (centripetal)



$$= r_m \omega_m^2$$

$$= (3.8\ 10^8\ \mathrm{m})\left(\frac{2\pi}{27.3\ 24\ 3600}\right)^2$$

$$= 2.7\ 10^{-3}\ \mathrm{m.s^{-2}}$$



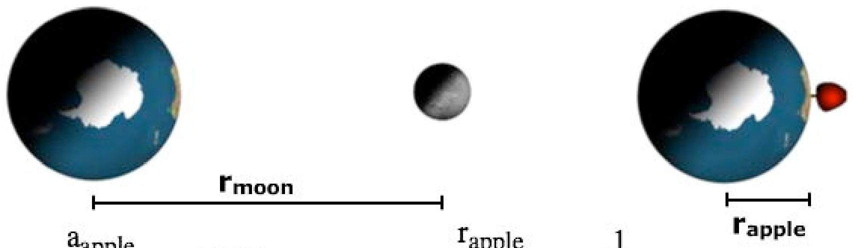
accel^n of "apple" = 9.8 ms^{-2}

$$\frac{a_{apple}}{a_{moon}} = 3600; \qquad \frac{r_m}{R_e} = \frac{385000\ \mathrm{km}}{6370\mathrm{km}} = 60;$$

$$\left(\frac{r_m}{R_e}\right)^2 = 3600$$

Newton's brilliant idea: What if the apple and the moon accelerate according to the same law? –> What if every body in the universe attracts every other, inverse square law?

Mechanics > Gravity > 11.1 F_g proportional to $1/r^2$



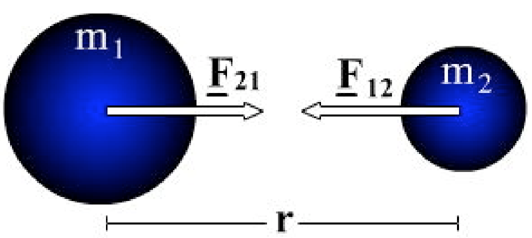
$$\frac{a_{apple}}{a_{moon}} = 3600 \qquad \frac{r_{apple}}{r_{moon}} = \frac{1}{60}$$

$$\frac{F_{apple}}{F_{moon}} = \frac{m_{apple}a_{apple}}{m_{moon}a_{moon}} = \frac{m_{apple}}{m_{moon}}\left(\frac{r_{moon}}{r_{apple}}\right)^2$$

$$\frac{F_{apple}}{F_{moon}} = \frac{\dfrac{m_{apple}}{r_{apple}{}^2}}{\dfrac{m_{moon}}{r_{moon}{}^2}} \qquad F \propto \frac{m}{r^2}$$

Mechanics > Gravity > 11.1 F_g proportional to $1/r^2$

Newton's law of universal gravitation



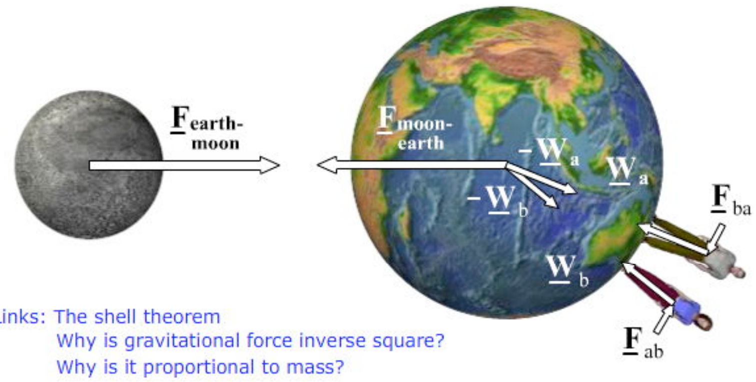
$$F_{grav} = -G\frac{m_1 m_2}{r^2}$$

Negative sign means
$\underline{\mathbf{F}}$ // – $\underline{\mathbf{r}}$

Why is it inverse square? Wait for Gauss' law in electricity.

Mechanics > Gravity > 11.1 F_g proportional to $1/r^2$

Newton's third law $\quad \mathbf{F}_{12} = -\mathbf{F}_{21}$



Links: The shell theorem
Why is gravitational force inverse square?
Why is it proportional to mass?

All are Newton pairs

Why $\propto \frac{1}{r^2}$? Newton knew Kepler's empirical law:

For planets, $r^3 \propto T^2$ (r = orbit radius, T = period)

Now if $\quad a_{centrip} \propto F \propto \frac{1}{r^2}$

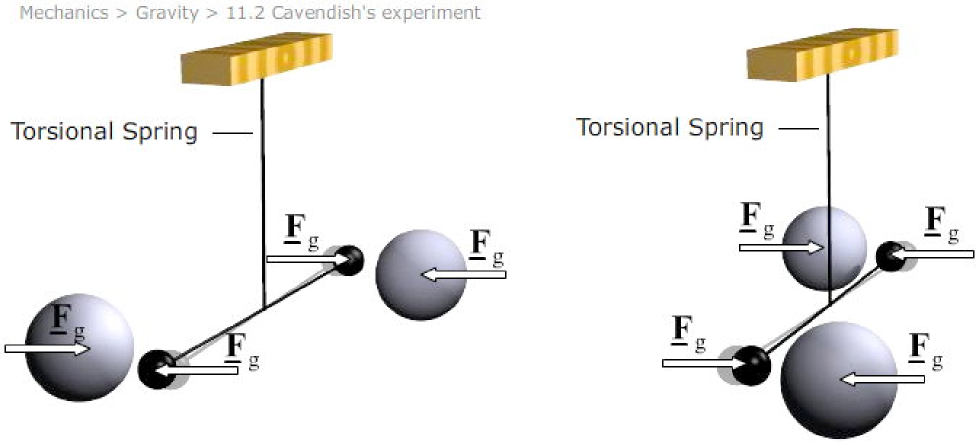
then $\quad$ constant $= a_{centrip} r^2 = r\omega^2 r^2 = r^3\omega^2$

Planet	r from sun	T	ω	$r\omega^2$
	million km	Ms	$rad.s^{-1}$	ms^{-2}
Mercury	58	7.62	$8.25\ 10^{-7}$	$3.95\ 10^{-5}$
Venus	108	19.4	$3.23\ 10^{-7}$	$1.13\ 10^{-5}$
Earth	150	31.6	$1.99\ 10^{-7}$	$5.94\ 10^{-6}$
etc				

so calculate:	$r^3\omega^2$
mercury	$1.31\ 10^{20}\ m^3s^{-2}$
venus	$1.32\ 10^{20}\ m^3s^{-2}$
earth	$1.33\ 10^{20}\ m^3s^{-2}$
etc	

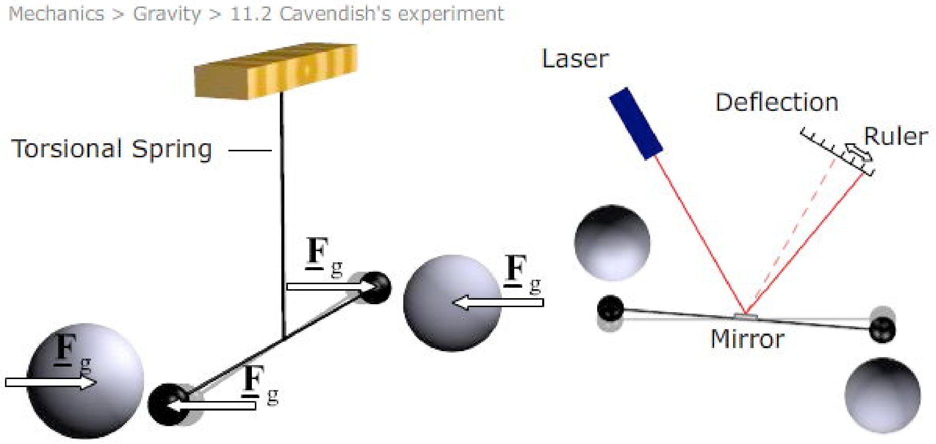
more later

How big is G? **Cavendish's experiment** (1798)



$$F = -G\frac{m_1 m_2}{r^2}$$

From deflection and spring constant, calculate F, know m_1 and m_2, $\therefore$ can calculate G.



$$F_g = -G\frac{m_1 m_2}{r^2} \qquad G = 6.67 \times 10^{-11}\ \mathrm{Nm^2kg^{-2}}$$

$$G = 6.67\ 10^{-11}\ \mathrm{Nm^2kg^{-2}} \qquad \textit{or}\ \ m^3kg^{-1}s^{-2}$$

Now also weight of m: $|W| = mg \cong G\frac{m.M_e}{r_e^2}$

$\therefore$ Cavendish first calculated mass of the earth:

$$M_e = \frac{g r_e^2}{G} = \frac{9.8\ \mathrm{m.s^{-2}} \times (6.37\ 10^6\ \mathrm{m})^2}{6.67\ 10^{-11}\ \mathrm{Nm^2kg^{-2}}}$$

$$= 6.0\ 10^{24}\ \mathrm{kg}$$

(Get other solar system masses from their moons etc)

Some numbers

What is force between two students in adjacent chairs? Between two oil tankers at 100 m?

$$F = -G\frac{m_1 m_2}{r^2}$$

Students: $m \sim 70$ kg, $r \sim 0.4$ m –> 2 μN

Tankers: $m \sim 10^8$ kg, $r \sim 100$ m –> 70 N

Conclusion: usually can neglect gravity unless at least one of the bodies is of astronomical size.

What happens when more there are ≥ 3 bodies?

Superposition principle.

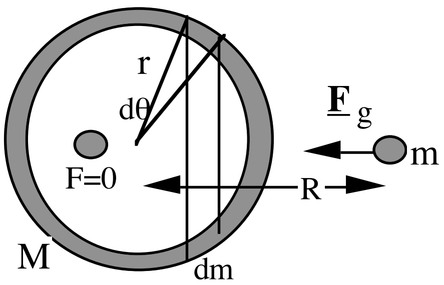
$\underline{\mathbf{F}}_{\text{all objects together}} = \Sigma\, \underline{\mathbf{F}}_{\text{individual}}$

or $\underline{\mathbf{F}}_1 = \sum_i \underline{\mathbf{F}}_{1i}$ force on m_1 due to masses m_i

continuous body $\underline{\mathbf{F}}_1 = \int_{\text{body}} d\,\underline{\mathbf{F}}$

Shell theorem

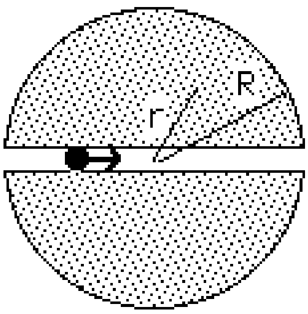
A uniform shell of mass M causes the same gravitational force on a body outside is as does a point mass M located at the centre of the shell, and zero force on a body inside it.



Proof by integrating x components of F due to dm. Not required

$$\underline{\mathbf{F}}_g = \frac{GMm}{R^2}$$

Example. If ρ_{earth} were uniform (it isn't), how long would it take for a mass to fall through a hole through the earth to the other side?



$$M_r = \rho.\frac{4}{3}\pi r^3$$

$$\therefore\ F_r = -G\frac{m\rho.\frac{4}{3}\pi r^3}{r^2}$$

$$F = -Kr$$

where $K = Gm\rho.\frac{4}{3}\pi$ is a constant

$\therefore$ motion is simple harmonic motion with $\omega = \sqrt{\frac{K}{m}}$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{G\rho.\frac{4}{3}\pi}} = \frac{2\pi}{\sqrt{GM/R^3}}$$

(Units of G are $m^3kg^{-1}s^{-2}$, units of ρ are $kg.m^{-3}$ correct.)

= .... = 84 minutes

$\therefore$ falls through (one half cycle) in 42 minutes

(actually faster for real density profile)

Gravity near Earth's surface

$$W = |F_g| = G\frac{Mm}{{r_e}^2}$$

$$W = mg_o = G\frac{M_e m}{r^2}$$

define g_o as acceleration in an inertial (non-rotating) frame

$$g_o = G\frac{M_e}{r^2}$$

Usually, $r \cong R_e$, but

$$g_o = G\frac{M_e}{(R_e+h)^2} = g_s\left(\frac{R_e}{R_e+h}\right)^2$$

$$= g_s\left(\frac{1}{1+h/R_e}\right)^2$$

where g_s is g_o at surface

Other complications:

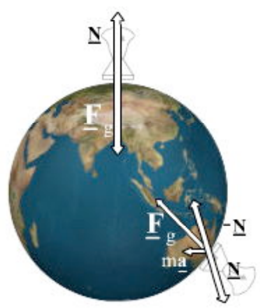
i) Earth is not uniform (especially the crust)

useful for prospecting

ii) Earth is not spherical

iii) Earth rotates (*see Foucault pendulum in foyer of OMB. Do your own experiment*)

Mechanics > Gravity > 11.3 Acceleration of falling objects



At poles, $\mathbf{\underline{F}}_g + \mathbf{\underline{N}} = 0$

At latitude θ, $\mathbf{\underline{F}}_g + \mathbf{\underline{N}} = m\mathbf{\underline{a}}$

weight measured $= -\mathbf{\underline{N}} = \mathbf{\underline{F}}_g - m\mathbf{\underline{a}}$

where $a = r\omega^2 = (R_e \cos\theta)\omega^2$

$= 0.034\ \mathrm{ms^{-2}}$ at equator

$= 0$ at poles

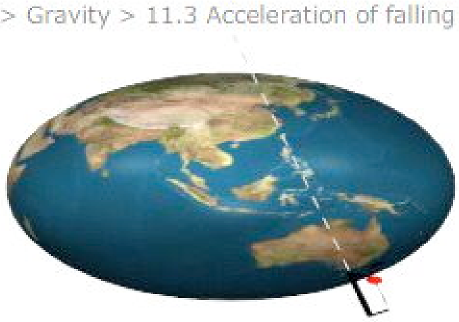
Link: Revise Circular Motion

We usually define, for a 'stationary' object near the earth's surface (ie stationary in the rotating frame)

$$-\mathbf{\underline{g}} = -\frac{\mathbf{\underline{N}}}{m} = -\frac{\mathbf{\underline{F}} - m\mathbf{\underline{a}}}{m}$$

So $\mathbf{\underline{g}}$ is greatest at the poles, least at the equator, and does not (quite) point towards centre.

Mechanics > Gravity > 11.3 Acceleration of falling objects

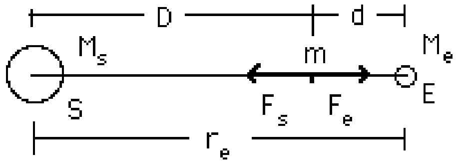


horizontal is at right angles to $\mathbf{g}$

Earth is flattened at poles

Puzzle: Save the moon

How far from the earth is the point at which the gravitational attractions towards the earth and that towards the sun are equal and opposite? Compare with distance earth-moon (380,000 km)



$$|F_e| = |F_s|$$

$$\frac{GM_e m}{d^2} = \frac{GM_s m}{(r_e - d)^2}$$

$$M_e(r_e - d)^2 = M_s d^2$$

$$r_e^2 - 2r_e d + d^2 = \frac{M_s}{M_e} d^2$$

$$\left(\frac{M_s}{M_e} - 1\right) d^2 + 2r_e d - r_e^2 = 0$$

$$d \cong \frac{r_e}{\sqrt{M_s/M_e}} = \frac{1.5\ 10^{11}\ \text{m}}{\sqrt{2.0\ 10^{30}\text{kg}/6.0\ 10^{24}\text{kg}}} = 260{,}000\ \text{km}$$

But the distance earth to moon = 380,000 km *?*

Gravitational field. A field is ratio of force on a particle to some property of the particle. For gravity, (gravitational) mass is the property:

$$\frac{\mathbf{F}_{grav}}{m} = \mathbf{g} = \mathbf{g}(\mathbf{r}) \quad \text{is a vector field (it has a vector value at all points in space)}$$

cf electric field $\frac{\mathbf{F}_{elec}}{q} = \mathbf{E}(\mathbf{r})$

(later in syllabus)

Gravitational potential energy. Revision:

Potential energy

For a **conservative** force $\underline{\mathbf{F}}$ (i.e. one where work done against it, $W = W(\underline{\mathbf{r}})$) we can define potential energy U by $\Delta U = W_{against}$. i.e.

$$\Delta U = -\int_i^f \underline{\mathbf{F}} \cdot \underline{\mathbf{dr}}$$

near Earth's surface, $\underline{\mathbf{F}}_g = m\underline{\mathbf{g}} \cong$ constant

$$= -\int_i^f (-mg\underline{\mathbf{k}}) \cdot (dx\underline{\mathbf{i}} + dy\underline{\mathbf{j}} + dz\underline{\mathbf{k}})$$

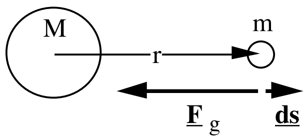
$$= mg\,\underline{\mathbf{k}} \cdot \underline{\mathbf{k}} \int_i^f dz$$

$$= mg\,(z_f - z_i)$$

choose reference at $z_i = 0$, so

$$U = mgz$$

Gravitational potential energy of m *and* M.



$$\Delta U_g \equiv -\int_i^f \underline{\mathbf{F}}_g \cdot \underline{\mathbf{ds}} = \int_i^f F_g dr$$

$$= \int_i^f G\frac{Mm}{r^2}\,dr$$

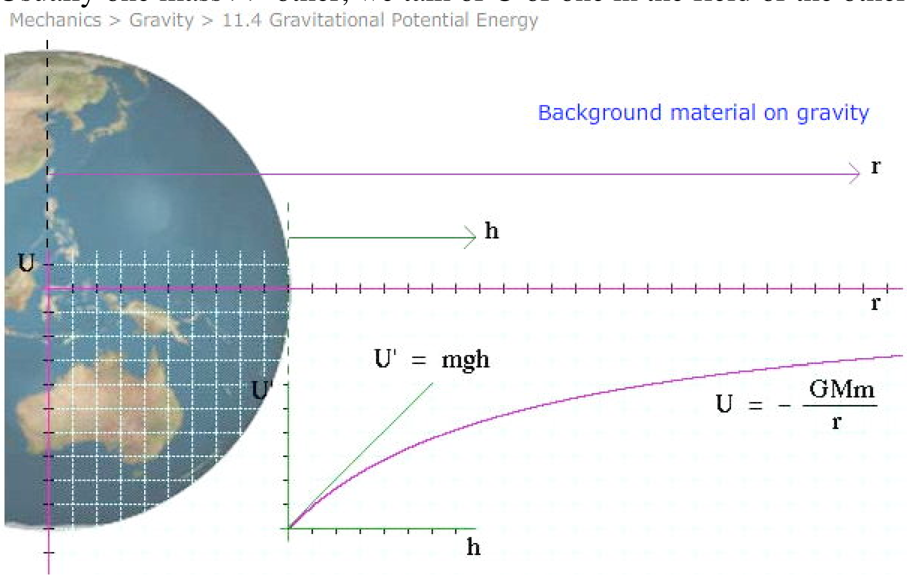
$$= -GMm\left[\frac{1}{r_f} - \frac{1}{r_i}\right]$$

Convention: take $r_i = \infty$ as reference

$$U(r) = -\frac{GMm}{r}$$

U = work to move one mass from ∞ to r in the field of the other. Always negative.

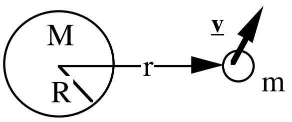
Usually one mass >> other, we talk of U of one in the field of the other, but it is U of both.



Escape "velocity". *"What goes up **sometimes** comes down"*

Escape "velocity" is **minimum** speed v_e required to escape, i.e. to get to a large distance ($r \to \infty$).

Newton's calculation:



Projectile in space: no non-conservative forces so conservation of mechancial energy

$$K_i + U_i = K_f + U_f$$

$$\frac{1}{2}\,mv_e^2 - \frac{GMm}{R} = 0 + 0$$

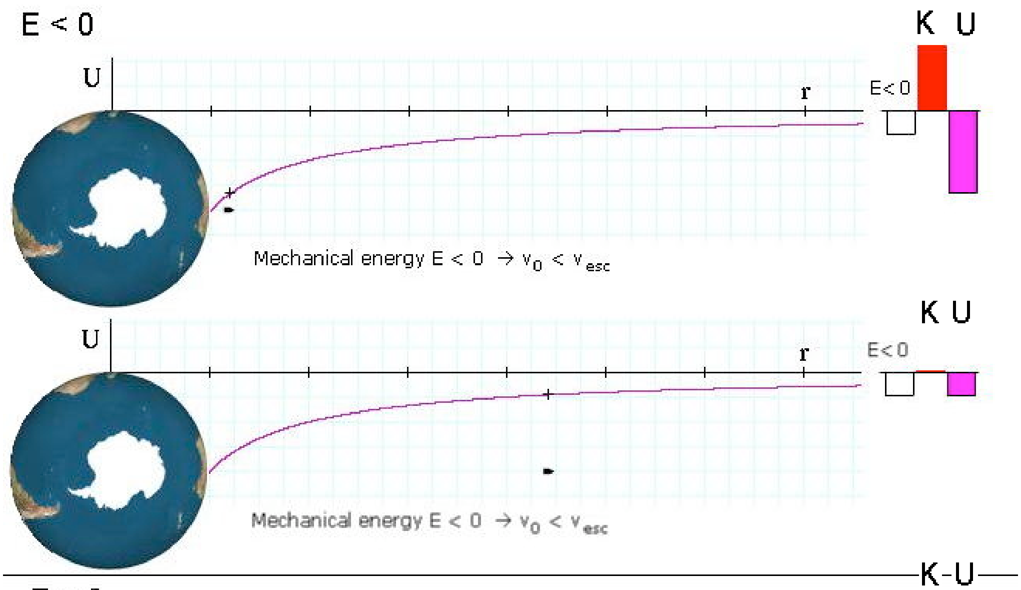
$$v_{esc} = \sqrt{\frac{2GM}{R}}$$

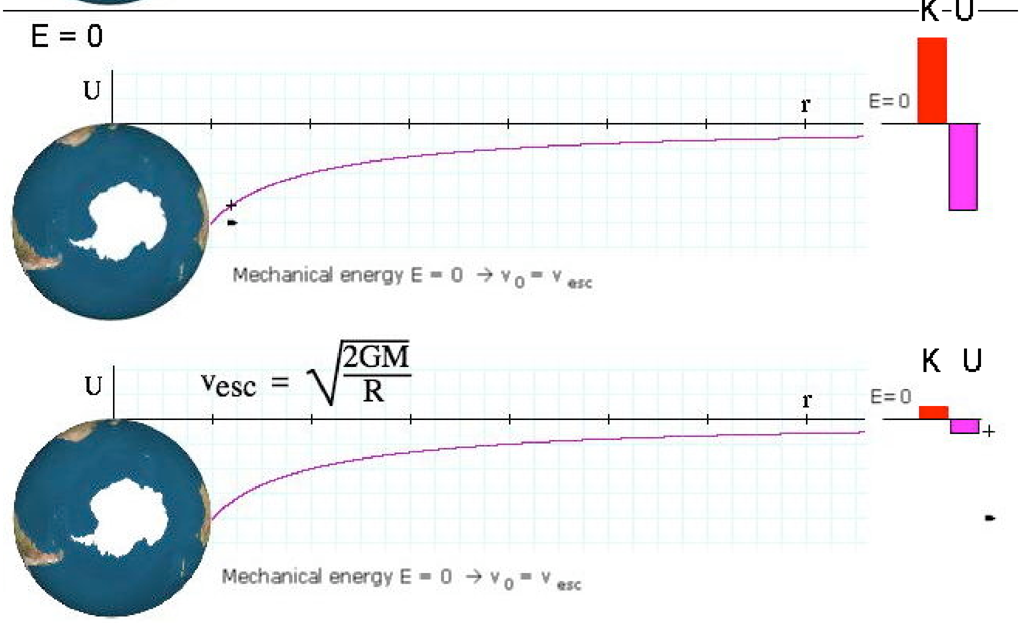
For Earth:

$$v_{esc} = \sqrt{\frac{2\ 6.67\ 10^{-11}\ m^3kg^{-1}s^{-2}\ 5.98\ 10^{24}\ kg}{6.37\ 10^6\ m}}$$

$$= 11.2\ km.s^{-1} = 40{,}000\ k.p.h.$$

Put launch sites near equator: $v_{eq} = R_e\omega_e = 0.47\ km.s^{-1}$





Question: What is the relation between M and R such that $v_{escape} = c$?

$$c = \sqrt{\frac{2GM}{R}}$$

$$R = \frac{2GM}{c^2}$$ *radius of Newtonian black hole* (Mitchell, 1783)

For the Earth,

$$R_{BH} = \frac{2 * 6.67\ 10^{-11}\ m^3kg^{-1}s^{-2} * 5.98\ 10^{24}\ kg}{(3\ 10^8\ m/s)^2}$$

$= 9$ mm

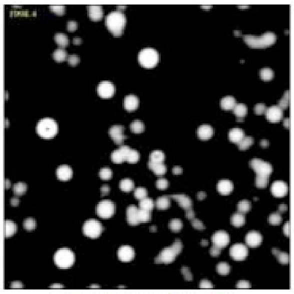
For the sun

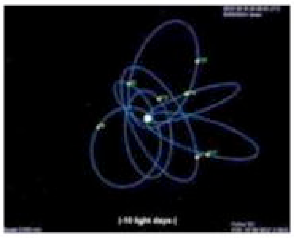
$$R_{BH} = \frac{2 * 6.67\ 10^{-11}\ m^3kg^{-1}s^{-2} * 1.99\ 10^{30}\ kg}{(3\ 10^8\ m/s)^2}$$

$= 3$ km

Mechanics > Gravity > 11.5 Escape velocity

Stars near the centre of our galaxy





Animation

Telescope images

courtesy of Max-Planck-Institut für extraterrestrische Physik

Question In Jules Verne's "From the Earth to the Moon", the heros' spaceship is fired from a cannon*. If the barrel were 100 m long, what would be the average acceleration in the barrel?

$$v_f^2 - v_i^2 = 2as$$

$$a = \frac{v_e^2 - 0}{2s} = (11.2\ km.s^{-1})2 \times 100\ m$$

$$= 630{,}000\ ms^{-2} = 64{,}000\ g$$

* why? If you burn all the fuel on the ground, you don't have to accelerate and to lift it. *Much* more efficient.

Planetary motion

Some history *– not in the syllabus but interesting:*

"The music of the spheres" - Plato

Leucippus & Democritus C5 B.C.

heliocentric universe

Hipparchus (C2 BC) & Ptolemy (C2 AD) geocentric universe

Tycho Brahe (1546-1601) - very many, very careful, naked eye observations.

Johannes Kepler joined him. He fitted the data to these *empirical* laws:

Kepler's laws: (*this is in syllabus*)

1 All planets move in elliptical orbits, with the sun at one focus.

Except for Pluto, these ellipses are ≅ circles

$M_{sun} >> m_{planet}$, *so sun is ≅ c.m.*

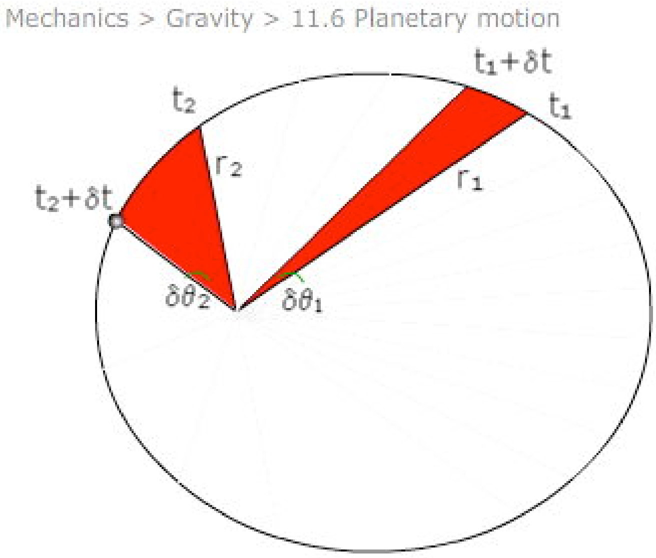
2 A line joining the planet to the sun sweeps out equal areas in equal time.

Slow at apogee (distant), fast at perigee (close)

3 The square of the period ∝ the cube of the semi-major axis

Slow for distant, fast for close

Newton's explanations:



Area $= \frac{1}{2}$ r.rδθ

i.e. for same δt, $\frac{1}{2} r^2 \delta\theta = \text{constant}$

Conservation of angular momentum **L** . Sun at c.m.

$\therefore \quad |\underline{\mathbf{L}}| = |\underline{\mathbf{r}} \times \underline{\mathbf{p}}| = |\underline{\mathbf{r}} \times m\underline{\mathbf{v}}|$ *momentum = **p**. see later* *wait til we do rotation*

$= mrv_{tangential}$

$= mr.r\omega = mr^2\frac{\delta\theta}{\delta t}$

$= \frac{m}{\delta t} r^2\delta\theta = \text{constant.}$

Conservation of **L** $\Rightarrow$ Kepler 2.

Law of periods: *(we consider only circular orbits)*

Kepler 3: $T^2 \propto r^3$

Newton 2: $F = ma$

$$F = m r\omega^2$$

$$G\frac{Mm}{r^2} = mr\left(\frac{2\pi}{T}\right)^2$$

$$T^2 = \left(\frac{4\pi^2}{GM}\right) r^3$$

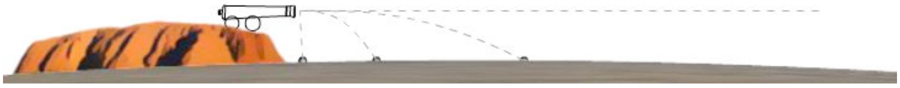
(works for ellipses with semi-major axis a instead of r)

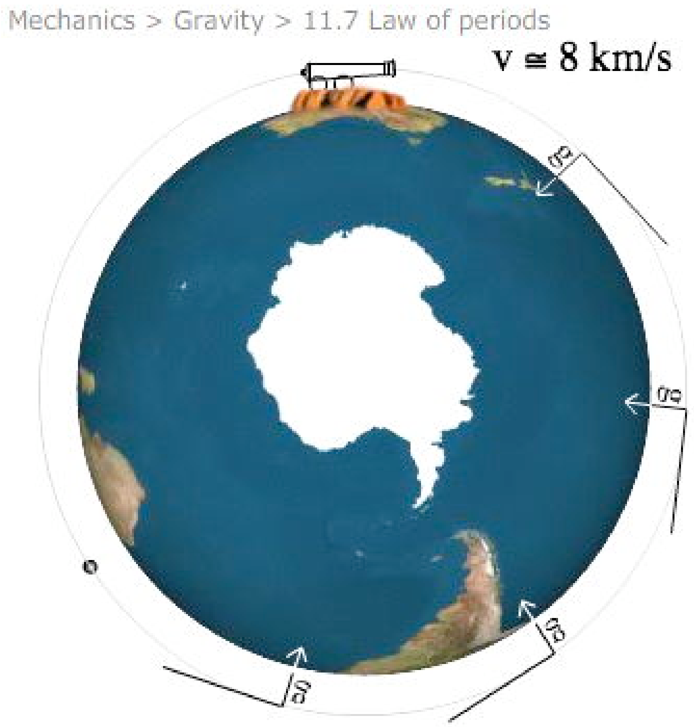
Newton 2 & Newton's gravity $\Rightarrow$ Kepler 3

Newton 2 & Newton's gravity also $\Rightarrow$ Kepler 1

but the algebra is long. See e.g. Newton "Principia Mathematica" or Bradbury "Theoretical mechanics" Wiley 1968

Newton's cannon





Example What is the period of the smallest earth orbit? ($r \cong R_e$) What is period of the moon? (r_{moon} = $3.82\ 10^8$ m)

$$T_1 = \sqrt{\left(\frac{4\pi^2}{GM}\right) r^3} = ...$$

$$= \sqrt{\frac{4\ \pi^2}{6.67\ 10^{-11}\ 5.98\ 10^{24}}\ (6.37\ 10^6)^3}\ \ \text{s}$$

$$= 84 \text{ min}$$

For moon, either directly, or else use Kepler 3: $T^2 \propto r^3$

$$\frac{T_2}{T_1} = \left(\frac{r_2}{r_1}\right)^{3/2} = \left(\frac{3.82\ 10^8}{6.37\ 10^6}\right)^{3/2} = 464$$

$$T_2 = 464\ T_1 = 27 \text{ days}$$

For other planets: most have moons, so the mass of the planet can be calculated from

$$T^2 = \left(\frac{4\pi^2}{GM}\right) r^3$$

Orbits and energy

No non-conservative forces do work, so mechanical energy is constant:

$$E = K + U$$

$$= \frac{1}{2} mv^2 - \frac{GMm}{r}$$

Let's remove v to get E(r). Consider circular orbit:

$$\frac{v^2}{r} = a_c = \frac{F}{m} = \frac{GMm}{r^2 m}$$ *multiply by mr/2 to get expression for K:*

$$\therefore \quad \frac{1}{2} mv^2 = \frac{1}{2}\frac{GMm}{r}$$ *so high orbits (large r) are slow (low K)*

$$E = K + U$$

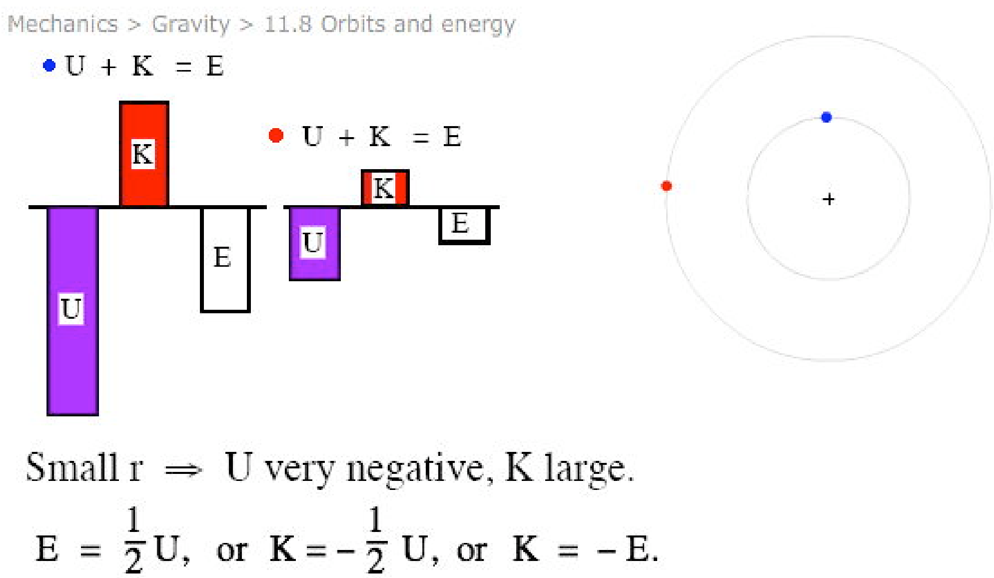
$$= \frac{1}{2}\frac{GMm}{r} - \frac{GMm}{r}$$

$$= -\frac{GMm}{2r}$$

i.e. $E = \frac{1}{2} U$, or $K = -\frac{1}{2} U$, or $K = -E$.

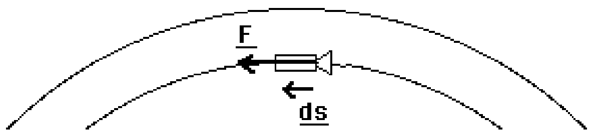
Small r $\Rightarrow$ U very negative, K large.

(inner planets fast, outer slow)



Remember: Large, slow orbits have high energy, small, fast orbits have low energy. Logical? Good. Now try this one:

Example A spacecraft in orbit fires rockets while pointing forward. Is its new orbit faster or slower?



F // **ds** $\therefore$ Work done on craft

$W = \int \mathbf{F} \cdot \mathbf{ds} > 0.$

$\therefore \quad E = -\dfrac{GMm}{2r}$ increases, i.e. it becomes less negative. (R is larger). K = - E, $\therefore$ K smaller, so it travels *more slowly*.

called "Speeding down"

Quantitatively:

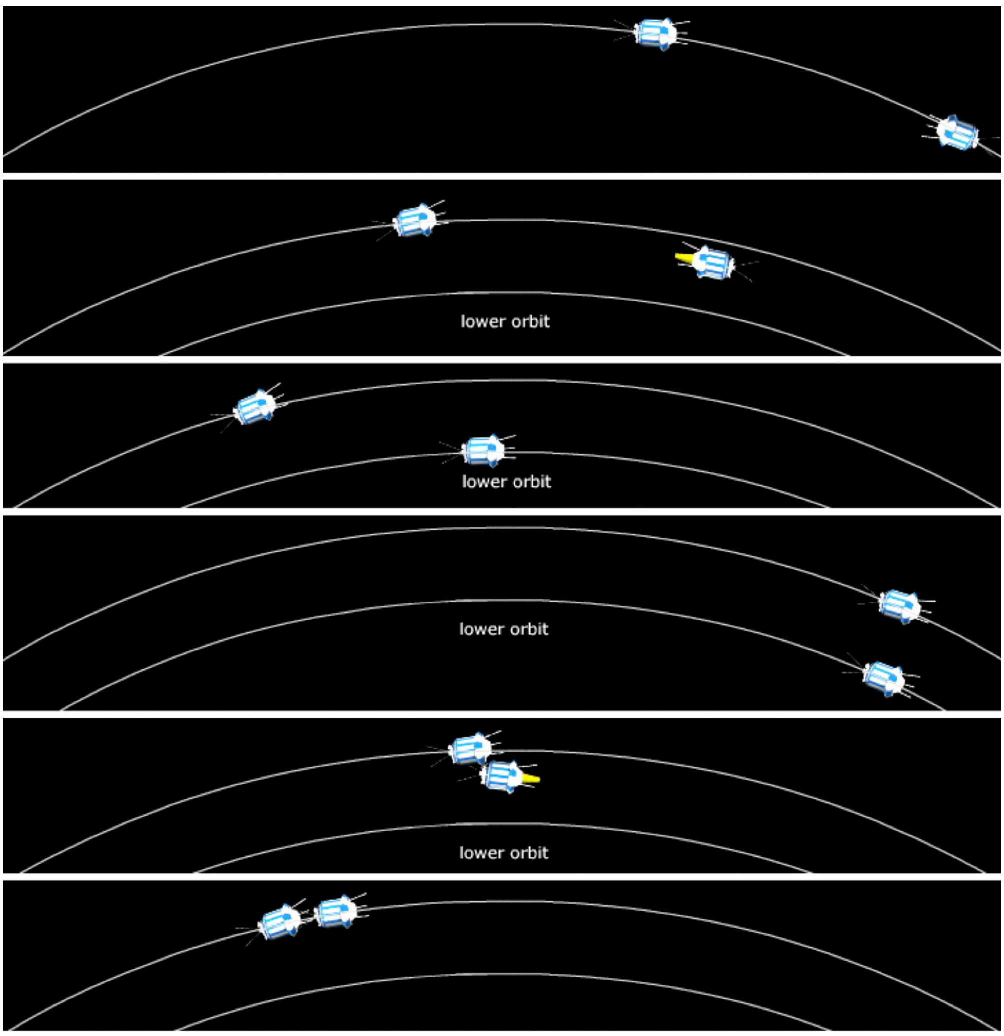
$K_i = -E_i \quad K_f = -E_f = -(E_i + \Delta E)$

$K_f = K_i - \Delta E$

$\frac{1}{2} mv_f^2 = \frac{1}{2} mv_f^2 - W$

Looks odd, but need lots of work to get to a high, slow orbit.

Manœuvring in orbit.



To catch up, trailing craft fires engines *backwards*, and loses energy. It thus falls to a lower orbit where it travels faster, until it catches up. It then fires its engines *forwards* in order to slow down: i.e. it climbs back to the original, slower orbit.

Example: In what orbit does a satellite remain above the same point on the equator?

Called the Clarke Geosynchronous Orbit

i) Period of orbit = period of earth's rotation

ii) Must be circular so that ω constant

$$T = 23.9 \text{ hours}$$

$$T^2 = \left(\frac{4\pi^2}{GM}\right) r^3$$

$$r = \sqrt[3]{\frac{GMT^2}{4\pi^2}} = ...$$

$$= 42{,}000 \text{ km}$$ popular orbit!

Limits to Newtonian mechanics

Mechanics > Gravity > 11.8 Orbits and energy

Newtonian gravity accurate if $\frac{\lvert U_{grav}\rvert}{mc^2} \ll 1$ otherwise use General Relativity.	At earth's surface $\frac{\lvert U_{grav}\rvert}{mc^2} = 7\text{x}10^{-10}$
Newtonian mechanics accurate if $\frac{v}{c} \ll 1$ otherwise use Special Relativity.	For a jet airliner $\frac{v}{c} < 10^{-6}$
Newtonian mechanics accurate if $\frac{\text{momentum} * \text{size}}{\text{Planck's constant}} \gg 1$ otherwise, use Quantum Mechanics.	For a small molecule at room temperature $\frac{\text{momentum} * \text{size}}{\text{Planck's constant}} \gtrsim 10$

Links: Background material on gravity Revise Circular Motion

For *very* high precision (e.g. GPS) you can't neglect special or general relativity

For *very* small things (e.g. electrons, cold small atoms) you can't neglect quantum mechanics.