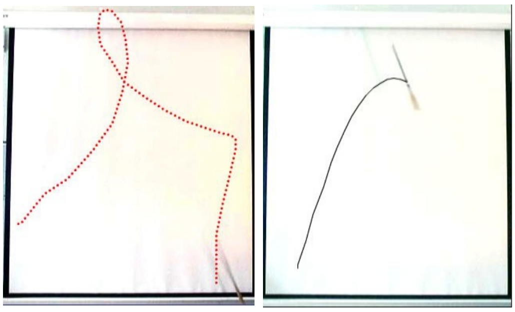
Mechanics for systems of particles and extended bodies

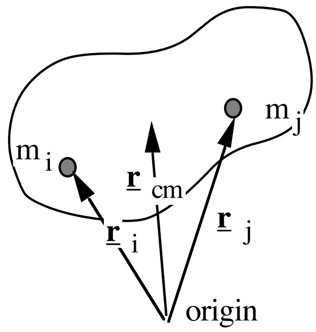
PHYS1121-1131 UNSW. Session 1



Left: trajectory of end of rod.   Right: parabola is the trajectory followed by the

Centre of mass

In a finite body, not all parts have the same acceleration. Not even if it is rigid. How to apply $\underline{\mathbf{F}} = m\,\underline{\mathbf{a}}$?

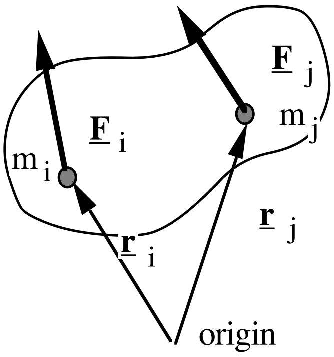


Total mass $M = \Sigma\, m_i$

Define the **centre of mass** as the point with displacement

$$\underline{\mathbf{r}}_{cm} = \frac{\Sigma\, m_i \underline{\mathbf{r}}_i}{M}$$

Why this definition? Consider n particles, m_i at positions $\underline{\mathbf{r}}_i$, $\underline{\mathbf{F}}_i$ acts on each. For each particle, N2 gives $\underline{\mathbf{F}}_i = m_i \underline{\mathbf{a}}_i$ Add these to get total force acting on all particles:



$$\Sigma\, \underline{\mathbf{F}}_i = \Sigma\, m_i \underline{\mathbf{a}}_i \qquad \textit{definition of acceleration}$$

$$= \Sigma\, m_i \frac{d^2}{dt^2} \underline{\mathbf{r}}_i$$

if masses constant, *can change order of d/dt and multiply:*

$$= \Sigma\, \frac{d^2}{dt^2}\, m_i \underline{\mathbf{r}}_i$$

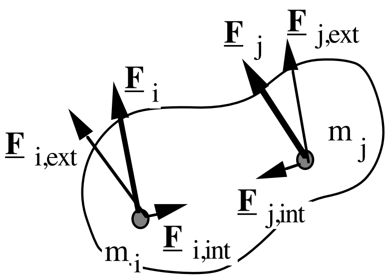
$$= \frac{d^2}{dt^2}\, \Sigma\, m_i \underline{\mathbf{r}}_i \qquad \textit{multiply top and bottom by M}$$

$$= M\frac{d^2}{dt^2}\left(\Sigma\, \frac{m_i \underline{\mathbf{r}}_i}{M}\right)$$

But we defined $\underline{\mathbf{r}}_{cm} = \dfrac{\Sigma\, m_i \underline{\mathbf{r}}_i}{M}$ Then $\Sigma\, \underline{\mathbf{F}}_i = M\dfrac{d^2}{dt^2}\underline{\mathbf{r}}_{cm} = M\,\underline{\mathbf{a}}_{cm}$

(total force) = (total mass)*(acceleration of centre of mass)

Look at forces in detail:



Each $\underline{\mathbf{F}}_i$ is the sum of internal forces (from other particles in the body/ system) and external forces (from outside the system)

$$\Sigma\, \underline{\mathbf{F}}_i = \Sigma\, \underline{\mathbf{F}}_{i,internal} + \Sigma\, \underline{\mathbf{F}}_{i,\,external}$$

Newton 3: All internal forces $\underline{\mathbf{F}}_{ij}$ between i^{th} and j^{th} particles are Newton pairs:

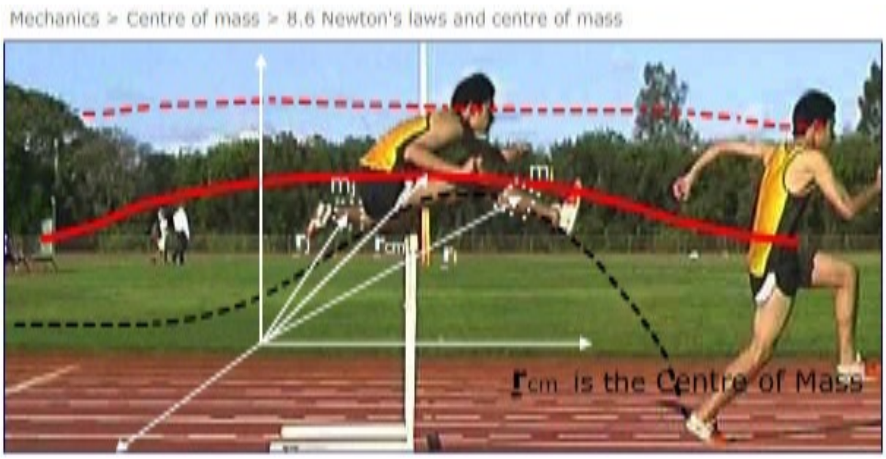
$$\underline{\mathbf{F}}_{ji} = -\,\underline{\mathbf{F}}_{ij}$$

$\therefore \quad \Sigma$ internal forces $= 0$

$$\therefore \quad \Sigma\, \underline{\mathbf{F}}_i = \Sigma\, \underline{\mathbf{F}}_{i,\,external} = \underline{\mathbf{F}}_{external}$$

$$\therefore \quad \underline{\mathbf{F}}_{external} = M\, \underline{\mathbf{a}}_{cm}$$

$$\begin{pmatrix}\text{total}\\ \textbf{external}\text{ force}\end{pmatrix} = \begin{pmatrix}\text{total}\\ \text{mass}\end{pmatrix} * \begin{pmatrix}\text{acceleration of}\\ \text{centre of mass}\end{pmatrix}$$



For n discrete particles, **centre of mass** at

$$\underline{\mathbf{r}}_{cm} = \frac{\sum m_i \underline{\mathbf{r}}_i}{\sum m_i} = \frac{\sum m_i \underline{\mathbf{r}}_i}{M} \qquad \text{(i)}$$

For a continuous body, elements of mass dm at $\underline{\mathbf{r}}$

$$\underline{\mathbf{r}}_{cm} = \frac{\int_{body} \underline{\mathbf{r}}\, dm}{\int_{body} dm} = \frac{\int_{body} \underline{\mathbf{r}}\, dm}{M} \qquad \text{(ii)}$$

This is the same equation. Really.

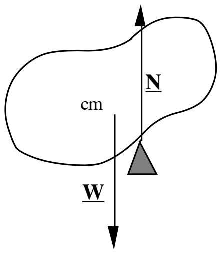
Can rearrange (i):

$$0 = \sum \frac{m_i \underline{\mathbf{r}}_i - m_i \underline{\mathbf{r}}_{cm}}{M} \quad \to$$

$$\sum m_i (\underline{\mathbf{r}}_i - \underline{\mathbf{r}}_{cm}) = 0$$

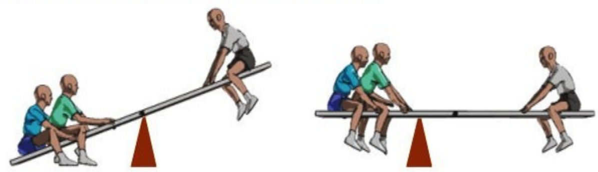
law of the see-saw

(ii) $\to$ $$\int_{body} (\underline{\mathbf{r}}_i - \underline{\mathbf{r}}_{cm})dm = 0$$



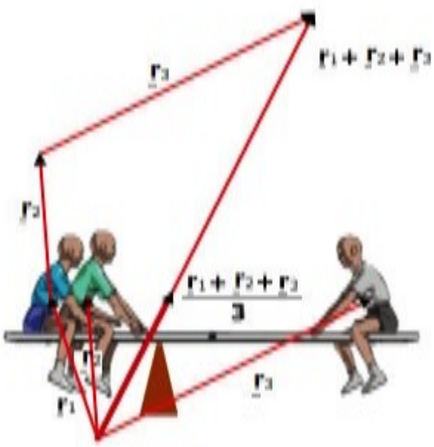
Later, when doing rotation, we'll consider which is a useful way to find c.m. experimentally. Three boys with equal mass m:

Mechanics > Centre of mass > 8.3 Finding the centre of mass

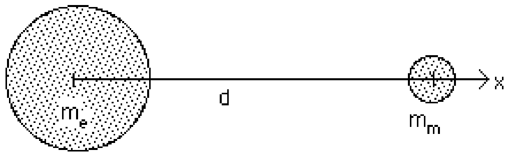


Mechanics > Centre of mass > 8.4 Examples

$$\underline{\mathbf{r}}_{cm} = \frac{m\underline{\mathbf{r}}_1 + m\underline{\mathbf{r}}_2 + m\underline{\mathbf{r}}_3}{m+m+m} = \frac{\underline{\mathbf{r}}_1 + \underline{\mathbf{r}}_2 + \underline{\mathbf{r}}_3}{3}$$



Example. Where is the c.m. of the earth moon system?



$$\underline{\mathbf{r}}_{cm} = \frac{\Sigma m_i \underline{\mathbf{r}}_i}{\Sigma m_i}$$

Take origin at centre of earth.

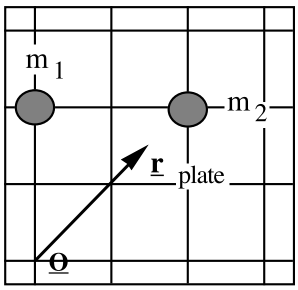
$$x_{cm} = \frac{m_e x_e + m_m x_m}{m_e + m_m}$$

$$= \frac{m_m d}{m_e + m_m}$$

$= 4{,}600$ km i.e. inside the earth.

recall: when doing Newton's 3rd, we derived the centre of rotation of this system

Example



On a square plate (mass m_p), we place m_1 and m_2 as indicated.

$m_p = 135$ g, $m_1 = 100$ g and $m_2 = 50$ g

Where is the cm of the system?

$$\underline{\mathbf{r}}_{cm} = \frac{\Sigma m_i \underline{\mathbf{r}}_i}{\Sigma m_i}$$

$$= \frac{m_p(1.5\underline{\mathbf{i}} + 1.5\mathbf{j}) + m_1(2.0\mathbf{j}) + m_2(2.0\underline{\mathbf{i}} + 2.0\mathbf{j})}{m_p + m_1 + m_2}$$

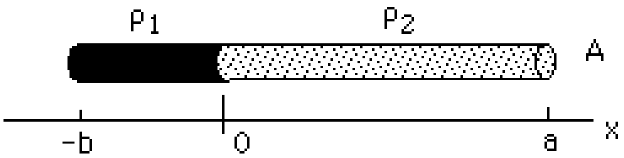
$$= \frac{(303g)\underline{\mathbf{i}} + (503g)\mathbf{j}}{285g} = 1.1\,\underline{\mathbf{i}} + 1.8\mathbf{j}$$

Check?

check that $\Sigma m_i(\underline{\mathbf{r}}_i - \underline{\mathbf{r}}_{cm}) = 0$

Example. Rod, cross-section A, made of length a of material with density ρ_2 and length b of material with density ρ_1. Where is c.m.?

If $\rho_1/ = 2\rho_2$, and a = 2b, where is cm?



$$\underline{\mathbf{r}}_{cm} = \frac{\int \mathbf{r}_i\, dm}{\int dm}$$ *How am I going to integrate dm? and over the whole body?*

$$dm = \rho dV = \rho A dx$$

$$x_{cm} = \frac{\int x dm}{\int dm}$$ *Put origin at join and ρ is constant for the integrations*

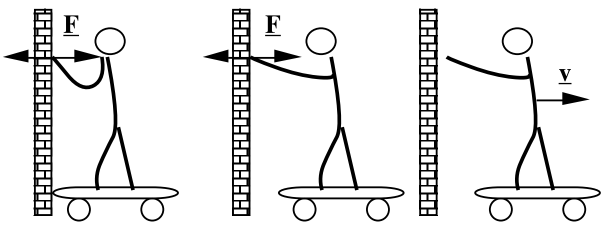
$$= \frac{\int_{-b}^{0} \rho_1 A x\, dx + \int_{0}^{a} \rho_2 A x\, dx}{\int_{-b}^{0} \rho_1 A\, dx + \int_{0}^{a} \rho_2 A\, dx}$$ *messy? But it's only four easy integrations*

$$= \frac{-\frac{1}{2}\rho_1 b^2 + \frac{1}{2}\rho_2 a^2}{\rho_1 b + \rho_2 a}$$

$$= \frac{a^2 - rb^2}{2(a + rb)}$$ where we define $r = \frac{\rho_1}{\rho_2}$

Internal *vs* external work.

Problem. Skateboarder pushes away from a wall



Point of application of $\underline{\mathbf{F}}$ does not move, ∴ normal force does no work, but K changes. Where does energy come from? *Obvious: arms!*

$$F_{ext} = Ma_{cm}$$

$$F_{ext}\, dx = Ma_{cm}\, dx_{cm} = M\frac{dv_{cm}\, dx_{cm}}{dt} = Mv_{cm}dv_{cm}$$

"Centre of mass work"

$$W_{cm} = \int_i^f F_{ext}\, dx = \left(\frac{1}{2}Mv_{cm}^2\right)_f - \left(\frac{1}{2}Mv_{cm}^2\right)_i$$

Work done = that which would have been done if F_{ext} *had acted on cm.*

Momentum

Definition: $\underline{\mathbf{p}} \equiv m\underline{\mathbf{v}}$

In relativity, we'll find that this is a low v approximation to

$$\left(\underline{\mathbf{p}} = \frac{m\underline{\mathbf{v}}}{\sqrt{1 - v^2/c^2}}\right)$$

and also that $K = (\gamma - 1)mc^2$

Generalised form of

Newton 2: $\Sigma \underline{\mathbf{F}} = \frac{d}{dt}\underline{\mathbf{p}}$

$$\Sigma \underline{\mathbf{F}} = m\frac{d}{dt}\underline{\mathbf{v}} + \underline{\mathbf{v}}\frac{d}{dt} m$$

If m constant, $\Sigma \underline{\mathbf{F}} = m\,\underline{\mathbf{a}}$ *but for the general case, use the general expression*

System of particles:

What is system? - you choose: draw a boundary around it.

$$\underline{\mathbf{P}} = \Sigma \underline{\mathbf{p}}_i \qquad \text{and} \qquad M = \Sigma m_i$$

$$\underline{\mathbf{P}} = \Sigma m_i \underline{\mathbf{v}}_i = \Sigma m_i \frac{d}{dt}\underline{\mathbf{r}}_i$$

$$= \frac{d}{dt}\Sigma m_i \underline{\mathbf{r}}_i = M\frac{d}{dt}\left(\frac{\Sigma m_i \underline{\mathbf{r}}_i}{M}\right)$$

$$\underline{\mathbf{P}} = M\underline{\mathbf{v}}_{cm}$$

If M constant: $\frac{d}{dt}\underline{\mathbf{P}} = M\underline{\mathbf{a}}_{cm}$

$$\Sigma \underline{\mathbf{F}}_i = \Sigma \frac{d}{dt}\underline{\mathbf{p}}_i = \frac{d}{dt}\underline{\mathbf{P}}$$

All internal forces are in pairs $\underline{\mathbf{F}}_{ji} = -\underline{\mathbf{F}}_{ij}$

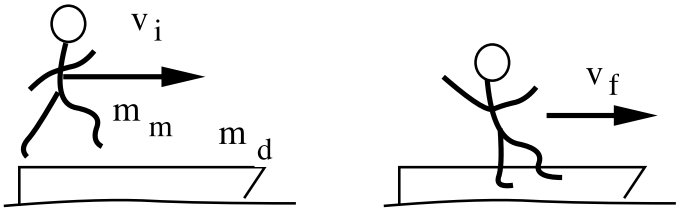
$\therefore \quad \underline{\mathbf{F}}_{ext} = \frac{d}{dt}\underline{\mathbf{P}}$ *two very important conclusions:*

i) Motion of cm is like that of particle mass M at $\underline{\mathbf{r}}_{cm}$ subjected to $\underline{\mathbf{F}}_{ext}$.

ii) ***If*** $\underline{\mathbf{F}}_{ext} = 0$, **momentum of whole system is conserved**

Note that momentum is a vector, so we have a conservation law that can apply in one or more directions.

Example 90 kg man jumps ($v_j = 5\ ms^{-1}$) into a (stationary) 30 kg dinghy. What is their final speed? (Neglect friction.)



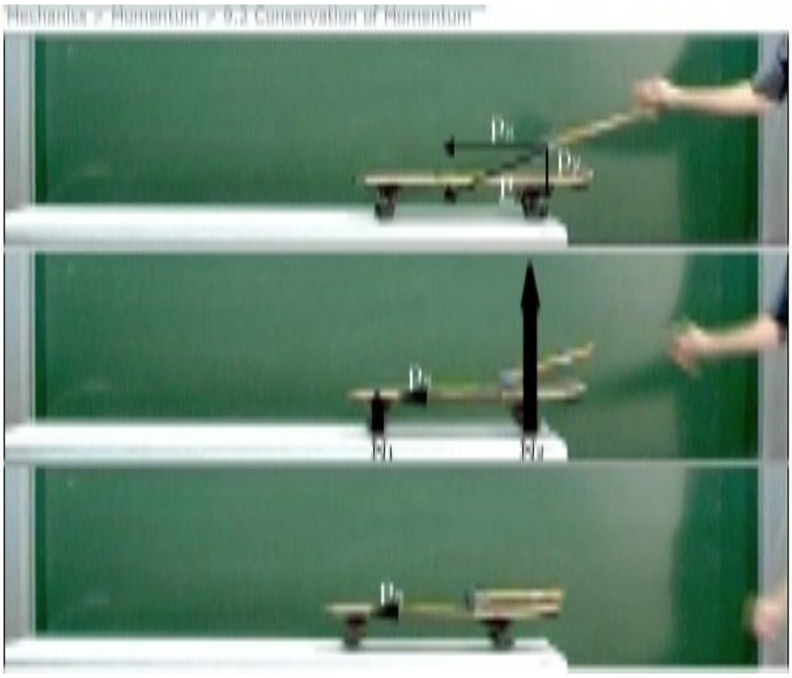
No external forces act in horizontal direction so P_x is conserved.

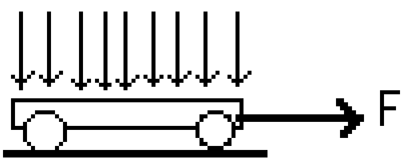
$$P_i = P_f$$

man *dinghy* *man* *dinghy*

$$m_m v_j + 0 = (m_m + m_d) v_f$$

$$v_f = \frac{m_m}{m_m + m_d} v_j$$





Example Rain falls into an open trailer (area 10 m^2) at 10 $litres.min^{-1}.m^{-2}$.

Neglecting friction, what F required to maintain constant speed of 10 ms^{-1}?

10 litres has mass 10 kg

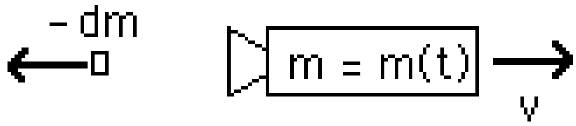
$$F_x = \frac{d}{dt}(mv_x) = m\frac{d}{dt} v_x + v_x \frac{d}{dt} m$$

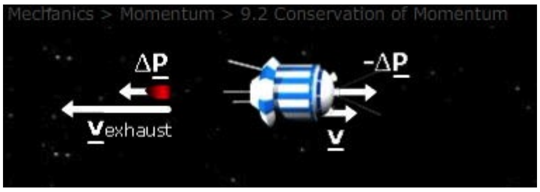
$$= 10\ ms^{-1} \times \left(\frac{10\ kg.m^{-2}}{60\ s} 10\ m^2\right)$$

$$= 17\ N.$$

Example. Rocket has mass m = m(t), which decreases as it ejects exhaust at rate $r = -\dfrac{dm}{dt}$ and at relative velocity u. What is the acceleration of the rocket?

$\left(\dfrac{dm}{dt} = \begin{matrix}\textit{rate of increase of}\\ \textit{mass of rocket}\end{matrix} < 0\right)$





No external forces act so momentum conserved. In the frame of the rocket, forwards direction:

$$dp_{rocket} + dp_{exhaust} = 0$$

$$(m+dm).dv + (-dm).(-u) = 0$$

(*neglect very small dm.dv term as small quantities go to zero*)

$$m.dv + dm.u = 0$$

$$dv = -u\frac{dm}{m}$$

$$a = \frac{dv}{dt} = -\frac{u}{m}\cdot\frac{dm}{dt}$$

$$a = \frac{ur}{m} \qquad \textit{1st rocket equation}$$

$$dv = -u\frac{dm}{m} = dv = -u\,d(\mathit{ln}\,m)$$

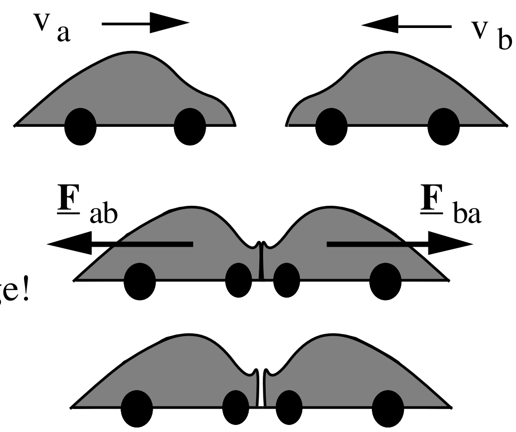
$$\int_i^f dv = v_f - v_i = u\,\mathit{ln}\frac{m_i}{m_f} \qquad \textit{2nd rocket equation}$$

need high exhaust velocity u (c?), else require $m_i >> m_f$

Collisions Definition: in a collision, "large" forces act between bodies over a "short" time.

In comparison, we shall often neglect the momentum change due to external forces.

Example 1:



$F_{ab} = F_{ba} = 0$

$F_{ab} = F_{ba} =$ large!

$F_{ab} = F_{ba} = 0$

forces that crumple cars during (brief) collision are much larger than friction force (tires - road), ∴ neglect F_{ext}.

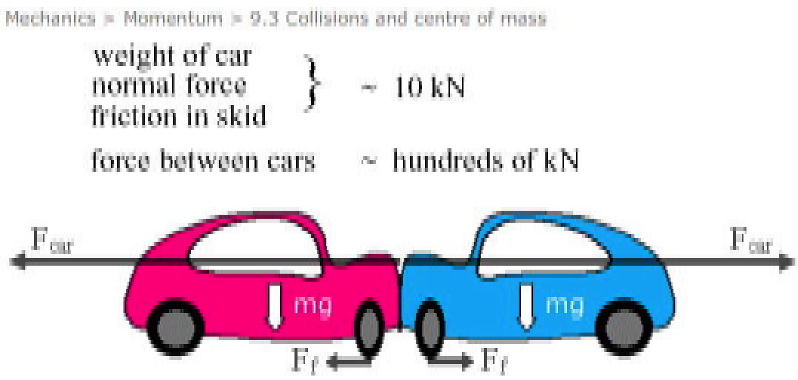
Be quantitative: suppose car decelerates from 30 kph to rest in a 20 cm 'crumple zone'.

Approximate as constant acceleration $a = (v_f^2 - v_i^2)/2\Delta x = -170\ \mathrm{ms^{-2}}$, so |force| on car during collision ~ m|a| ~ 200 kN, compared with friction at ~ 10 kN.

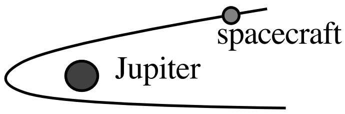
*Other safety questions while we are here: What is the force on occupants? **If** they have the same acceleration as the car, then F = ma ~ 10 kN: seat belts, air bags. What about reducing crumple zones? Forces on pedestrians? How big are crumple zones for pedestrians?*

Mechanics > Momentum > 9.3 Collisions

Video courtesy of The Australasian New Car Assessment Program (ANCAP)



Example 2



doesn't "hit" *Examples: deep space probes*

Here, start and finish of collision not well defined

At large separation before and after, $F_{ab} = F_{ba} \cong 0$

During collision (fly-by), forces are considerable.

However, $F_{grav} \propto 1/r^2$, so much smaller at large distances.

Impulse ($\underline{\mathbf{J}}$) and momentum

Newton 2 $\Rightarrow$ $$d\underline{\mathbf{p}} = \underline{\mathbf{F}}\ dt$$

$$\therefore \qquad \int_i^f d\underline{\mathbf{p}} = \int_i^f \underline{\mathbf{F}}\ dt$$ *so*

Definition: $$\underline{\mathbf{I}} \equiv \underline{\mathbf{p}}_f - \underline{\mathbf{p}}_i = \int_i^f \underline{\mathbf{F}}\ dt$$

In collisions, Impulse is integral of large internal force over short time

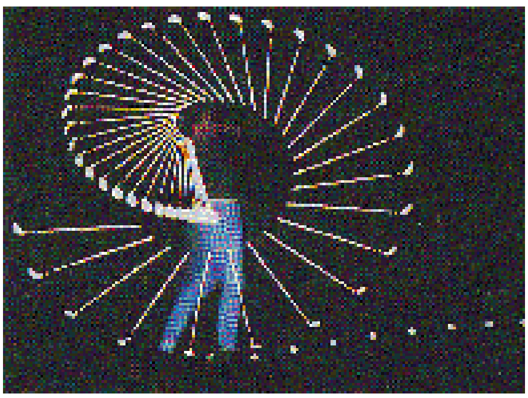
Ball *is* inflated to normal pressure.

Can get an *under*estimate of force:

$$F > \begin{pmatrix}\text{pressure}\\ \text{in ball}\end{pmatrix} * \text{deformed area}$$

$$\sim 70\ \text{kPa} * 0.02\ \text{m}^2 \sim 1\ \text{kN}$$

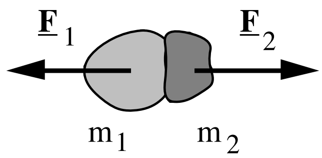
Camera flashes at equal times



When is head of club travelling fastest?

Speed of ball compared to speed of club?

Usual case: external forces small, act for small time, therefore $\int_i^f \underline{\mathbf{F}}_{ext}\,dt$ is small.



$$\Delta \underline{\mathbf{p}}_1 = \int_i^f \underline{\mathbf{F}}_1\,dt \equiv \bar{\underline{\mathbf{F}}}_1 \Delta t$$

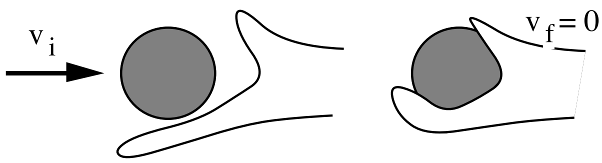
$$\Delta \underline{\mathbf{p}}_2 = \int_i^f \underline{\mathbf{F}}_2\,dt = -\int_i^f \underline{\mathbf{F}}_1\,dt$$

$$\therefore \quad \Delta \underline{\mathbf{p}}_1 = -\Delta \underline{\mathbf{p}}_2$$

$$\therefore \quad \Delta \underline{\mathbf{P}} = \Delta \underline{\mathbf{p}}_1 - \Delta \underline{\mathbf{p}}_2 = 0$$

If external forces are negligible (in any direction), then the momentum of the system is conserved (in that direction).

Example. Cricket ball, m = 156 g, travels at 45 ms^{-1}. What impulse is required to catch it? If the force applied were constant, what average force would be required to stop it in 1 ms? in 10 ms? What stopping distances in these cases?



$m = 0.156$ kg, $v_i = 45$ m.s^{-1} $\qquad v_f = 0$.

$$\underline{\mathbf{I}} \equiv \underline{\mathbf{p}}_f - \underline{\mathbf{p}}_i$$

$= m(v_f - v_i)$ to right

$= .... = 7.0$ kgms^{-1} to *left*.

$\underline{\mathbf{I}} = \int_i^f \underline{\mathbf{F}}\,dt$ $\qquad$ if $\underline{\mathbf{F}}$ constant, $\quad I = F\Delta t$.

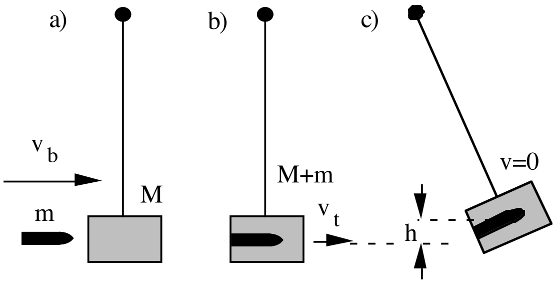
$\therefore \quad F_{av} = I/\Delta t$.

If const F $\Rightarrow$ const a. $\quad s = v_{av}\Delta t$. $\quad v_{av} = 23$ m.s^{-1}

Δt	1.0 ms	10 ms	
F	7 kN	0.7 kN	*ouch!*

s 2 cm 20 cm

Example. (*A common method to measure speed of bullet*.) Bullet (m) with v_b fired into stationary block (M) on string. (i) What is their (combined) velocity after the collision? (ii) What is the kinetic energy of the bullet? (iii) of the combination? (iv) How high does the block then swing?

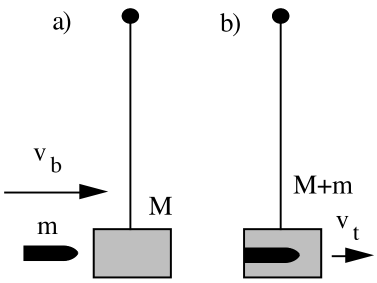


*Note the different stages and **three** diagrams:*

*a-b):collision, no horizontal external forces ∴ **momentum conserved**. Friction does work, so **mechanical energy is lost**, not conserved*

*b-c):during this phase, external forces **do** act, so **momentum is lost**, not conserved. However, there are no non-conservative forces, so **mechanical energy conserved**.*

Analyse a) to b)



No horizontal ext forces during collision ∴ momentum conserved

i) $P_{xi} = P_{xf}$

$$mv_b = (m + M)v_t$$

$$v_t = \frac{m}{m + M} v_b$$

ii) $K_b = \frac{1}{2} mv_b^2$

iii) $K_t = \frac{1}{2}(m + M)\, v_t^2 = \frac{1}{2}(m + M)\left(\frac{m}{m + M} v_b\right)^2$

$$= \frac{1}{2}\frac{m^2}{m + M} v_b^2 < K_b.$$

Conclusion: $U_i = U_f$, $K_i \neq K_f$.

Mechanical energy is ***not*** conserved - deformation of block is **not elastic**; heat is produced.

Let’s look in more detail: *little digression about elastic and inelastic collisions*

During a collision with negligible external forces,

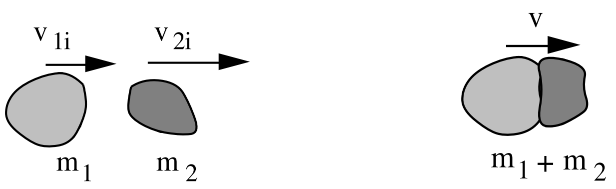
$$\underline{\mathbf{P}} = (\Sigma m)\, \underline{\mathbf{v}}_{cm} \quad \text{is conserved}$$

Σm constant $\therefore$ $\underline{\mathbf{v}}_{cm}$ is constant $\therefore$ $\frac{1}{2} M\underline{\mathbf{v}}_{cm}^2$ constant

K of c.m. is ***not*** lost. But the K of components with respect to c.m. *can* be lost.

Greatest possible loss of K: if all final velocities = $\underline{\mathbf{v}}_{cm}$, i.e. if all objects stick together after collision. Called **completely inelastic collision.**

Completely inelastic collision.



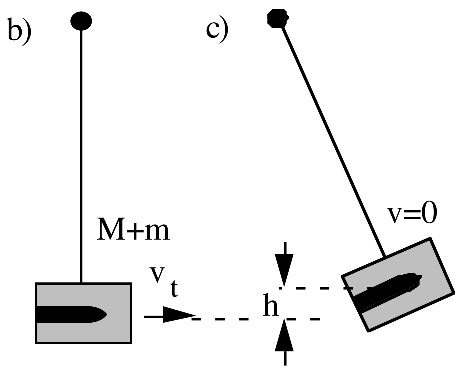
Contrast:

Completely elastic collision is one in which non-conservative forces do **no** work, so mechanical energy is conserved.

Inelastic collision is one in which non-conservative forces do some work, so mechanical energy is **not** conserved.

Completely inelastic collision is one in which all kinetic energy with respect to the centre of mass is lost: Non-conservative forces do as much work as possible, so as much mechanical energy as possible is lost

Part (iv) of previous example (b-c):



here the external forces (gravity and tension) *do* do work and change momentum. But there is no non-conservative force and so *in this part of the process* conservation of mechanical energy applies:

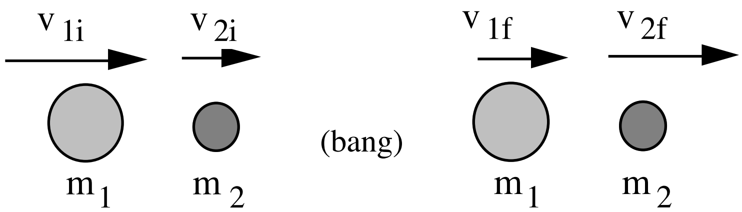
$$\Delta U + \Delta K = 0$$

$$(M + m)g\,(\Delta h - 0) + (0 - K_t) = 0$$

$$\Delta h = \ldots = \frac{1}{2}\frac{m^2}{g(m + M)^2}\, v_b^2$$

so we can rearrange and get v_b *from* Δh

Example Elastic collision in one dimension



note the before and after diagrams again

Collision: neglect external forces ⇒

$$p_i = p_f$$

$$m_1v_{1i} + m_2v_{2i} = m_1v_{1f} + m_2v_{2f} \qquad \text{(i)}$$

elastic ⇒ $K_i = K_f$

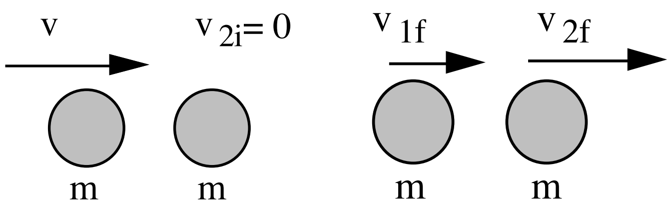
$$\frac{1}{2} m_1v_{1i}^2 + \frac{1}{2} m_2v_{2i}^2 = \frac{1}{2} m_1v_{1f}^2 + \frac{1}{2} m_2v_{2f}^2 \qquad \text{(ii)}$$

usually know m_1, m_2, v_{1i}, v_{2i}. Two unknowns (v_{1f}, v_{2f}), ∴ we can always solve.

Or: transform to frame where (e.g.) $v_1 = 0$ *can simplify the algebra*

Or: transform to centre of mass frame.

Example. Take $m_1 = m_2$, $v_{2i} = 0$, $v_{1i} = v$.



neglect external forces ⇒ $p_i = p_f$

$$mv + 0 = mv_{1f} + mv_{2f} \qquad \text{(i)}$$

$$\frac{1}{2} mv^2 + 0 = \frac{1}{2} mv_{1f}^2 + \frac{1}{2} mv_{2f}^2 \qquad \text{(ii)}$$

(i) –> $v_{2f} = v - v_{1f}$ (iii)

substitute in (ii) –>

$$\frac{1}{2} mv^2 + 0 = \frac{1}{2} mv_{1f}^2 + \frac{1}{2} m(v^2 + v_{1f}^2 - 2vv_{1f})$$

$$\therefore \quad 0 = v_{1f}^2 - vv_{1f}$$

$$0 = v_{1f}(v_{1f} - v)$$ *2 solutions*

Either: $v_{1f} = 0$ and (iii)–> $v_{2f} = v$

i.e. 1st stops dead, all p and K transferred to m_2

or: $v_{1f} = v$ and (iii)–> $v_{2f} = 0$ i.e. missed it.

Example Show that, for an elastic collision in one dimension, the relative velocity is unchanged.

i.e. show $v_{1i} - v_{2i} = v_{2f} - v_{1f}$

p and K conservation gave:

(i) $m_1(v_{1f} - v_{1i}) = -m_2(v_{2f} - v_{2i})$

(ii) $\frac{1}{2}\, m_1(v_{1f}^2 - v_{1i}^2) = -\frac{1}{2}\, m_2(v_{2f}^2 - v_{2i}^2)$

If they hit, $(v_{1f} - v_{1i}) \neq 0$, $(v_{2f} - v_{2i}) \neq 0$ *use* $a^2 - b^2 = (a - b)(a + b)$

(ii)/(i) $\Rightarrow$ $v_{1i} + v_{1f} = v_{2i} + v_{2f}$

$\therefore$ $v_{1i} - v_{2i} = v_{2f} - v_{1f}$

i.e. relative velocity the same before and after

Solve –> $v_{2f} = \dfrac{2m_1}{m_1 + m_2}\, v_{1i} + \dfrac{m_2 - m_1}{m_2 + m_1}\, v_{2i}$

Example Two similar objects, mass m, collide completely inelastically.

case 1: $v_{1i} = v$, $v_{2i} = 0$.

case 2: $v_{1i} = v$, $v_{2i} = -v$.

What energy is lost in each case?

p conserved –> $mv_{1i} + mv_{2i} = 2mv_f$

$$v_f = \frac{v_{1i} + v_{2i}}{2}$$

$$\Delta K = K_f - K_i = \frac{1}{2}(2m)\, v_f^2 - \frac{1}{2}\, mv_{1i}^2 - \frac{1}{2}\, mv_{2i}^2$$

case 1: $\Delta K = \frac{1}{2}(2m)\left(\frac{v + 0}{2}\right)^2 - \frac{1}{2}\, mv^2$

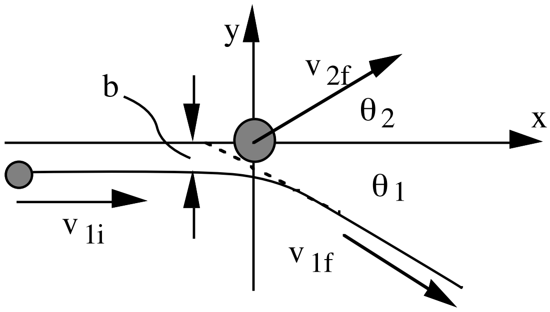
$= -\frac{1}{4}\, mv^2$

case 2: $\Delta K = \frac{1}{2}(2m)\left(\frac{0 + 0}{2}\right)^2 - \frac{1}{2}\, mv^2 - \frac{1}{2}\, mv^2$

$= -mv^2$ 4 times as much energy lost

Remember this if you have the choice in traffic, rugby etc

Elastic collisions in 2 (& 3) dimensions



Choose frame in which m_2 stationary, v_{1i} in x dir^n

b is called impact parameter (distance "off centre")

p_x conserved $m_1 v_{1i} = m v_{1f} \cos\theta_1 + m v_{2f} \cos\theta_2$

p_y conserved $0 = m v_{2f} \sin\theta_2 - m v_{1f} \sin\theta_1$

K conserved

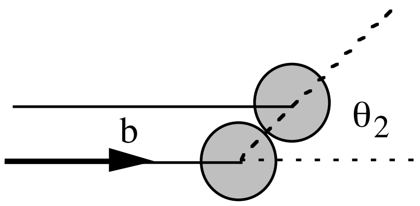
$$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m v_{1f}^2 + \frac{1}{2} m v_{2f}^2 + \Delta K \qquad \text{(iii)}$$

where $\Delta K = 0$ for elastic case

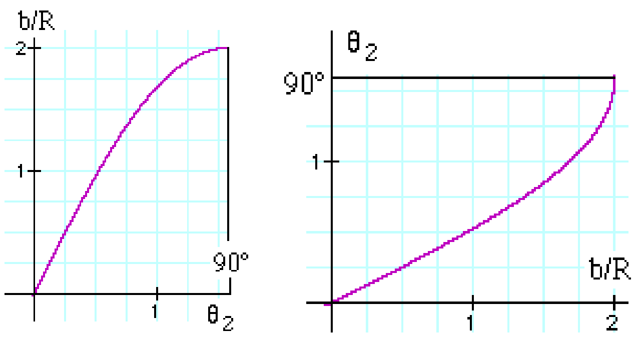
3 equations in v_{1f}, v_{2f}, θ_1 and θ_2: need more info

(often given θ_1 or θ_2)

Incidentally: for hard spheres, neglecting rotation and friction (*reasonable during collision, but not after*)



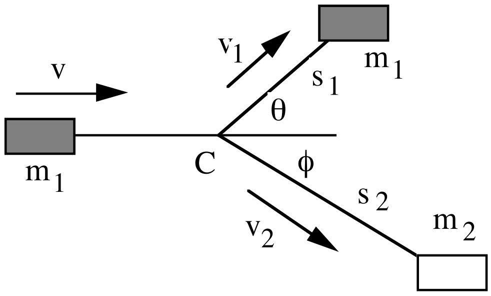
$(R + R) \sin\theta_2 = b \qquad \theta_2 = \sin^{-1} \frac{b}{2R}$



i) Note that as $\theta \to 90°$, small error in b gives large error in θ_2.

ii) Experiment on billiard table: Does $b = R$ give $\theta_2 = 30°$?

friction, rotation ignored



Example. Police report of road accident. Car 1, mass m_1 strikes stationary car m_2 at point C. They then slide to rest in positions shown. Given $\mu_k = \mu$ (assumed same for both) find the initial speed v of m_1. Can you check assumption? (*real example*)

After collision, a for both $= \dfrac{F_f}{m} = -\mu \dfrac{W}{m} = -\mu g$

$$v_f^2 - v_i^2 = 2as = -2\mu g s$$

$$0 - v_1^2 = -2\mu g s_1$$

$$v_1 = \sqrt{2\mu g s_1} \qquad v_2 = \sqrt{2\mu g s_2}$$

Neglect external forces during collision: $\Delta P = 0$

P_x: $m_1 v = m_1 v_1 \cos\theta + m_2 v_2 \cos\phi$ (i)

P_y: $0 = m_1 v_1 \sin\theta - m_2 v_2 \sin\phi$ (ii)

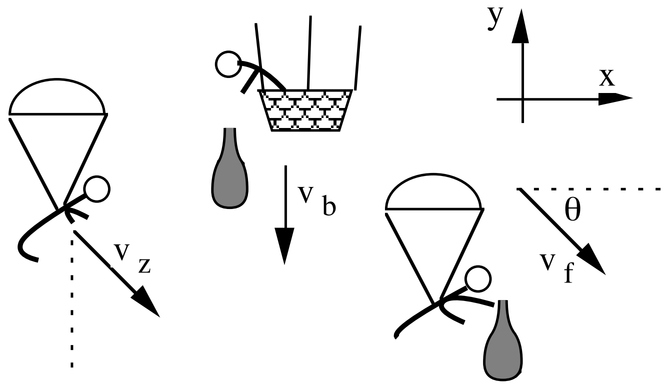
(i) $\Rightarrow$ $v = \sqrt{2\mu g s_1}\ \cos\theta + (m_2/m_1)\sqrt{2\mu_2 g s_2}\ \cos\phi$

Note the "spare" equation—we can use it to check the model or assumptions:

(ii) $\Rightarrow$ $m_1\sqrt{2\mu_1 g s_1}\ \sin\theta = m_2\sqrt{2\mu_2 g s_2}\ \sin\phi$

$$\frac{\mu_2}{\mu_1} = \frac{s_1 m_1^2 \sin^2\theta}{s_2 m_2^2 \sin^2\phi}$$

(The μ may not be the same for the two: surfaces different, orientation of wheels etc)



Balloonist Albert writes message on a bottle (1 kg) and drops it over the side. It is falling vertically at 40 $m.s^{-1}$ when caught by parachutist Zelda (m = 50 kg), travelling at 1 ms^{-1} at 45° to vertical. Collision (bottle—Zelda's hand) lasts 10 ms.

i) If only gravity acted, what is Δp for Zelda over 10 ms ?
ii) Neglecting ext forces during collision, what is the velocity of (Zelda+bottle) after collision?
iii) What impulse is applied to bottle during collision?
iv) What is the impulse applied to Zelda?
v) What is the average force during collision?
vi) Will Albert and Zelda live happily ever after?

i) due to $\underline{\mathbf{W}}$, $\Delta\underline{\mathbf{p}} = \underline{\mathbf{W}}\ \Delta t$ = .. = 5 $kgm.s^{-1}$ down

ii) Neglect ext forces ⇒ momentum conserved.

$$m_b\,\underline{\mathbf{v}}_{bi} + m_Z\,\underline{\mathbf{v}}_{Zi} = m_{(Z+b)}\,\underline{\mathbf{v}}_{(Z+b)f}$$

$$1(-40\,\underline{\mathbf{j}}) + 50(1\cos 45°\,\underline{\mathbf{i}} - 1\cos 45°\,\underline{\mathbf{j}}) = 51(v_x\,\underline{\mathbf{i}} + v_y\,\underline{\mathbf{j}})$$

$\underline{\mathbf{i}}$ dir^n: $v_x = \cos 45° \cdot \frac{50}{51} \times 1 = 0.7\ ms^{-1}$

$\underline{\mathbf{j}}$ dir^n: $v_y = \frac{-1\times 40 - 50\cos 45°}{51} = -1.5\ ms^{-1}$

$\therefore\quad |v_f| = \sqrt{v_x^2 + v_y^2} = 1.6\ ms^{-1}$

$\theta_f = \tan^{-1}\frac{v_y}{v_x} \Rightarrow$ 67° to horizontal

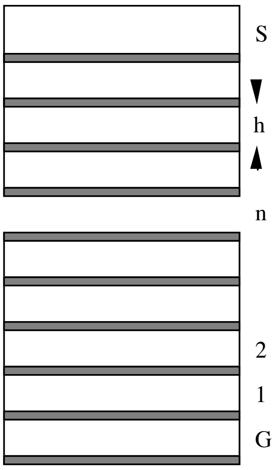
iii) $\underline{\mathbf{I}}_b \equiv \underline{\mathbf{p}}_{bf} - \underline{\mathbf{p}}_{bi} = 1\text{x}(v_x\,\underline{\mathbf{i}} + v_y\,\underline{\mathbf{j}}) - 1(-40\,\underline{\mathbf{j}})$

$= (1.6\,\underline{\mathbf{i}} + 38\,\underline{\mathbf{j}})\ kgm.s^{-1}$

iv) $\underline{\mathbf{I}}_Z = -\underline{\mathbf{I}}_b = -(1.6\,\underline{\mathbf{i}} + 38\,\underline{\mathbf{j}})\ kgm.s^{-1}$

$|\underline{\mathbf{I}}_Z| = \sqrt{1.6^2 + 38^2} = 38\ kgm.s^{-1}$

v) $\overline{\underline{\mathbf{F}}}_Z = \frac{\Delta\,\underline{\mathbf{p}}_Z}{\Delta t} = \frac{|\underline{\mathbf{I}}_Z|}{\Delta t} = .. = 380$ N



Example Controlled demolition. A building has S stories, each of height h. Explosions destroy the strength of the n^{th} floor. How long before $(n+1)^{th}$ floor hits ground, falling vertically?

Assume that the building remains intact above the explosion and inelastic collisions with the lower floors. To obtain a lower bound, assume negligible strength between lower floors.

(S−n) floors have mass (S−n)m. To get the lower estimate on falling time, assume no strength in the demolished floor, s the upper floors are in free fall (as a rigid body) for a distance h with acceleration g. So they strike the next floor with speed $v_n = \sqrt{2gh}$.

Inelastic collision with next floor gives speed v where:

$$(S-n)mv_n = (S-n+1)mv$$

Let the falling mass after any collision have initial speed v_0 and speed before the next collision be v_c.

$$v_c^2 - v_0^2 = 2gh$$

$$v_c = \sqrt{2gh + v_0^2} = v_0\left(1 + \frac{2gh}{v_0^2}\right)^{1/2}$$

For i^{th} collision

$$(S-n+i-1)mv_c(i) = (S-n+i)mv_0(i+1)$$

$$v_c(i) = \sqrt{2gh + (v_0(i))^2}$$

$$\frac{S-n+i-1}{S-n+i}\sqrt{2gh + (v_0(i))^2} = v_0(i+1)$$

$$v_0(i+1) = \left(1 - \frac{1}{S-n+i}\right)\sqrt{2gh + (v_0(i))^2}$$

$$h = v_0t + \frac{1}{2}gt^2$$

$$0 = \frac{1}{2}gt^2 + v_0t - h$$

$$t = \frac{-v_0 \pm \sqrt{v_0^2 + 2gh}}{g}$$