Integration

I'll assume that you can find $\int \sin(x)\,dx$, $\int x\sin(x^2)\,dx$, or $\int_0^3 (x^3 + e^x)\,dx$.

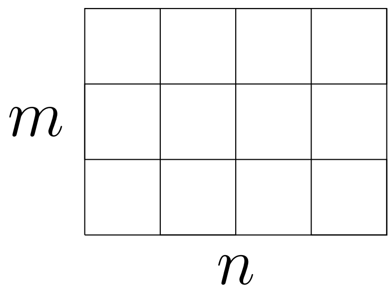
In this chapter we'll look at:

- What **area** actually means.
- The formal definition of $\displaystyle\int_a^b f(x)\,dx$.
- The area function $A(x) = \displaystyle\int_0^x f(t)\,dt$.
- The Fundamental Theorems of Calculus.
- Integration by substitution and by parts.
- Improper integrals $\displaystyle\int_0^\infty f(x)\,dx$.

Calculus links two hard problems from geometry; finding **slopes** (or **tangent lines** and finding **areas**. Of these, the problem of finding areas (of regions with curved boundaries) has the longer history. The main ideas that we shall look at in terms of defining area go back to the great ancient Greek mathematician **Archimedes**.

Area

Consider a rectangular region which we want to tile with square tiles. If the region is n tiles wide and m tiles high, then you need nm tiles to cover the region.



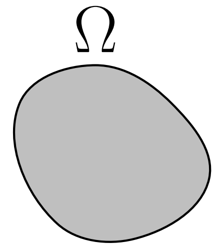
In the ancient civilizations of Babylon, Egypt etc, they wanted to keep track of how much land people owned (for taxation and other purposes). (A side remark: they **still** want to know how much land and other stuff people own for exactly the same purpose!)

The question is **what is the area?**

Formally, we demand that any definition of an area satisfy the following axioms:

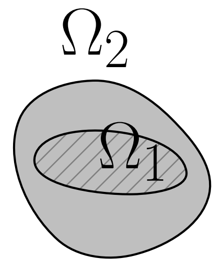
(A1) If Ω is a region of the plane then

$$\text{area}(\Omega) \geq 0.$$



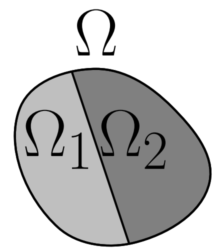
(A2) If one region Ω_1 is contained in another region Ω_2, then

$$\text{area}(\Omega_1) \leq \text{area}(\Omega_2).$$



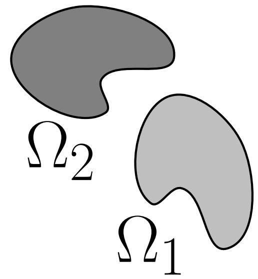
(A3) If the area of a region Ω is partitioned into two smaller disjoint regions Ω_1 and Ω_2, then

$$\text{area}(\Omega) = \text{area}(\Omega_1) + \text{area}(\Omega_2).$$



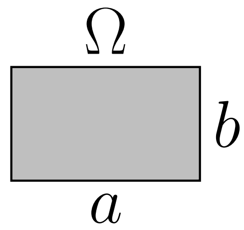
(A4) If Ω_1 and Ω_2 are congruent regions then

$$\text{area}(\Omega_1) = \text{area}(\Omega_2).$$

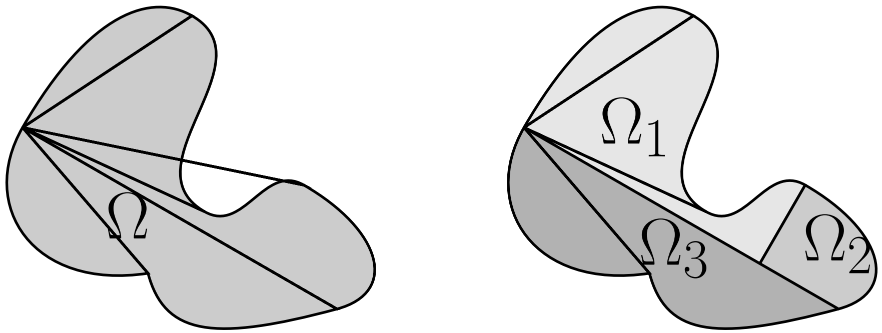


(A5) If Ω is a rectangle of length a and height b then

$$\text{area}(\Omega) = ab.$$



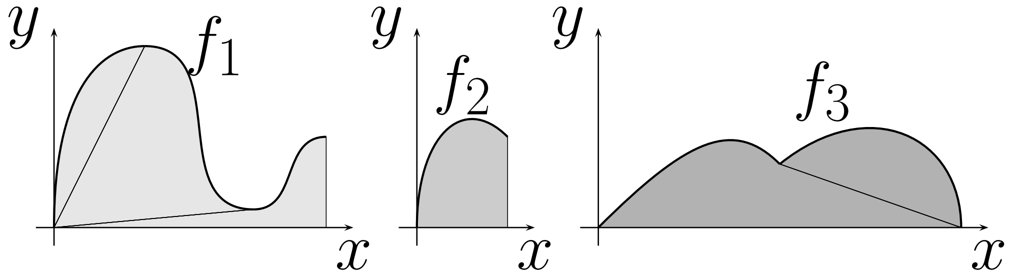
Question. How could one calculate the area of the region below?



Answer. Apply axiom (A3) and conclude that

$$\operatorname{area}(\Omega) = \operatorname{area}(\Omega_1) + \operatorname{area}(\Omega_2) + \operatorname{area}(\Omega_3).$$

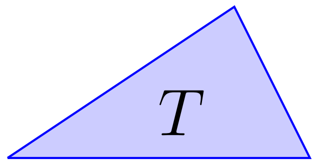
Then, rotate and translate each subregion, so that application of axiom (A4) implies that each of $\operatorname{area}(\Omega_1)$, $\operatorname{area}(\Omega_2)$ and $\operatorname{area}(\Omega_3)$ is equal to the area under the graph of a function.

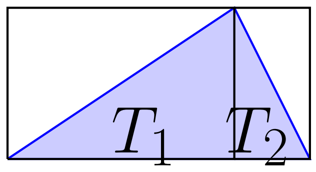


This procedure can be done for any region in the plane with a 'reasonable' boundary.

Consider first an easy region.

Suppose that T is a triangle.





$$\mathrm{Area}(T) = \mathrm{Area}(T_1) + \mathrm{Area}(T_2) = \frac{1}{2}\mathrm{Area}(\mathrm{rectangle}).$$

Thus the area of a triangle should be

$$\frac{\text{base} \times \text{height}}{2}.$$

From this you can make sense of the area of any polygon.

To build up from polygons to more general regions, you need to approximate them somehow by more and more complicated polygons, and then apply something like:

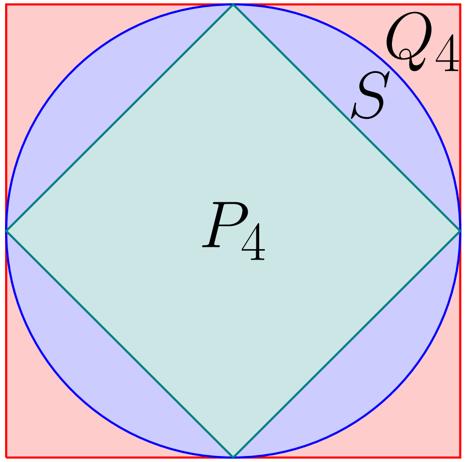
(A6) If $\{\Omega_i\}$ are disjoint sets then $\text{Area}(\cup_i \Omega_i) = \sum_i \text{Area}(\Omega_i)$.

Archimedes (287—212BC) used these properties to find or estimate the areas inside circles and parabolas, with perhaps the first explicit use of limits in mathematics.

One of Archimedes best ideas was to find polygons P_n, Q_n so that if S is a circle of radius 1, then

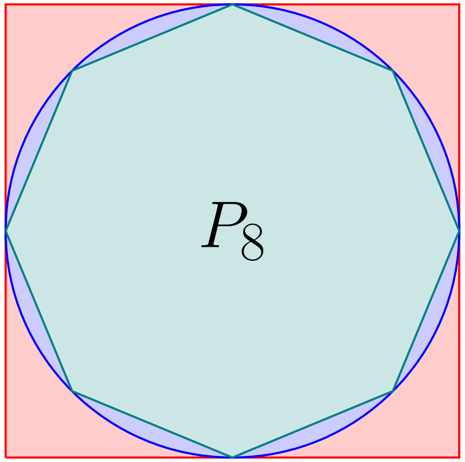
$$P_n \subseteq S \subseteq Q_n$$

and $\text{Area}(Q_n) - \text{Area}(P_n) \to 0$ as $n \to \infty$.



$$\text{Area}(P_4) \leq \text{Area}(S) \leq \text{Area}(Q_4)$$

$$2 \leq \pi \leq 4$$



$$\text{Area}(P_8) \leq \text{Area}(S) \leq \text{Area}(Q_4)$$

$$\ldots$$

$$\frac{223}{71} \leq \pi \leq \frac{22}{7}$$

Archimedes used polygons with $6, 12, 24, 48$ and 96 sides to conclude that

$$3.1408 \leq \pi \leq 3.1429.$$

Newton and **Leibniz** understood these ideas, but did not have rigorous definitions to give proper proofs. They did however see the deep link between tangents and areas (more on this soon!).

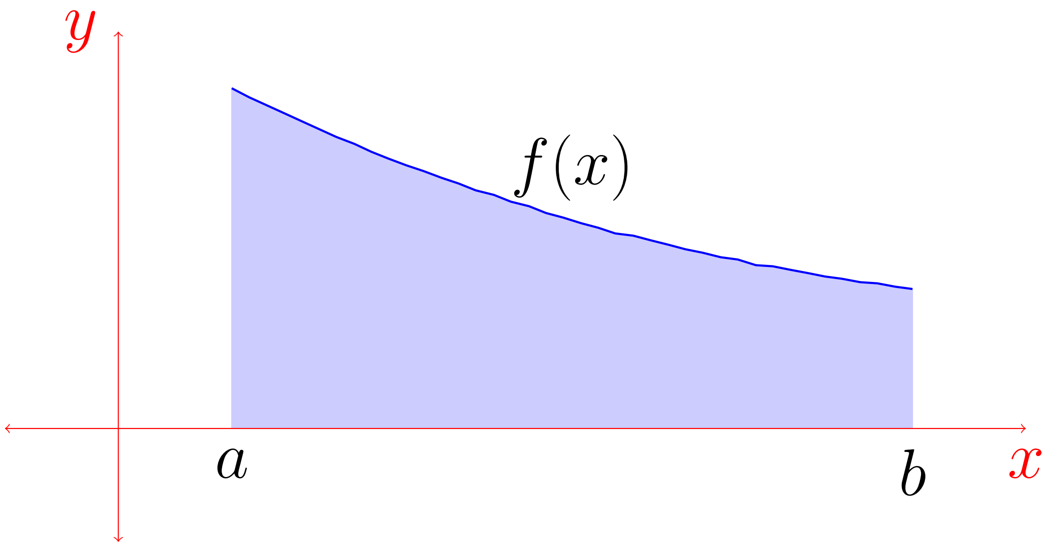
A rigorous definition of integral was introduced by **Bernhard Riemann** (1826-1866), and that is the one we shall use in this course.

Riemann's integral is based solely on approximating areas with **rectangles** (rather than general polygons), but otherwise follows Archimedes in seeking to bound the area from above and below.

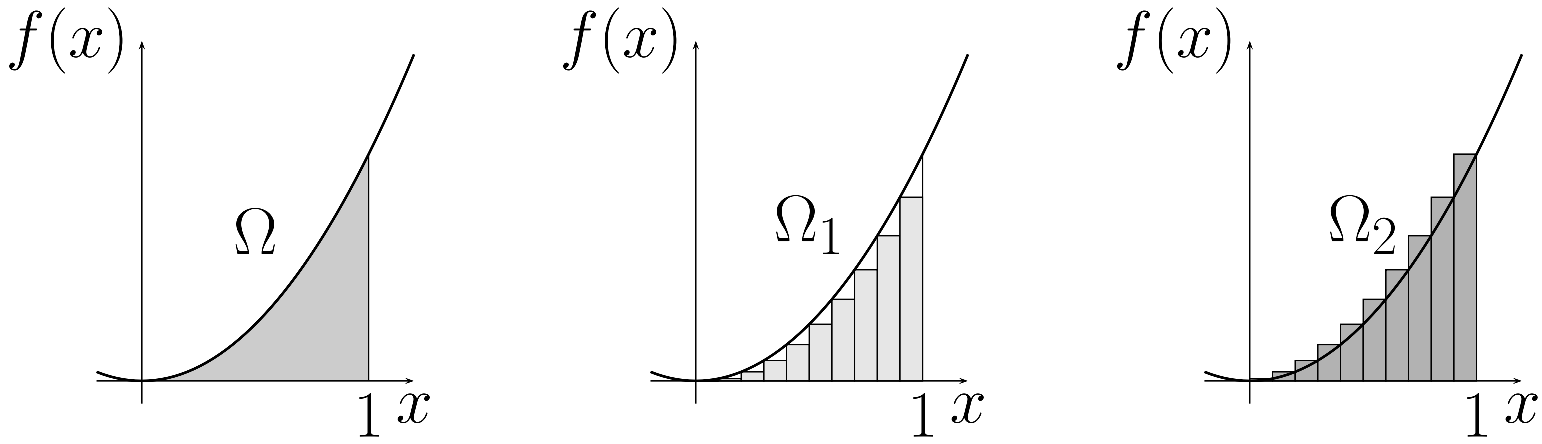
The areas that Riemann considered are those of the form

$$S = \{(x, y) \in \mathbb{R}^2 \,:\, a \le x \le b,\ 0 \le y \le f(x)\}$$

for some bounded positive function f.



Example. Suppose that $f : [0,1] \to \mathbb{R}$, $f(x) = x^2$. Let Ω denote the region bounded by the graph of f, the x-axis and the lines $x = 0$ and $x = 1$.



Idea. Find lower and upper bounds for area(Ω) by choosing appropriate 'approximations' Ω_1 and Ω_2 of the region Ω in terms of n rectangles.

It is evident that

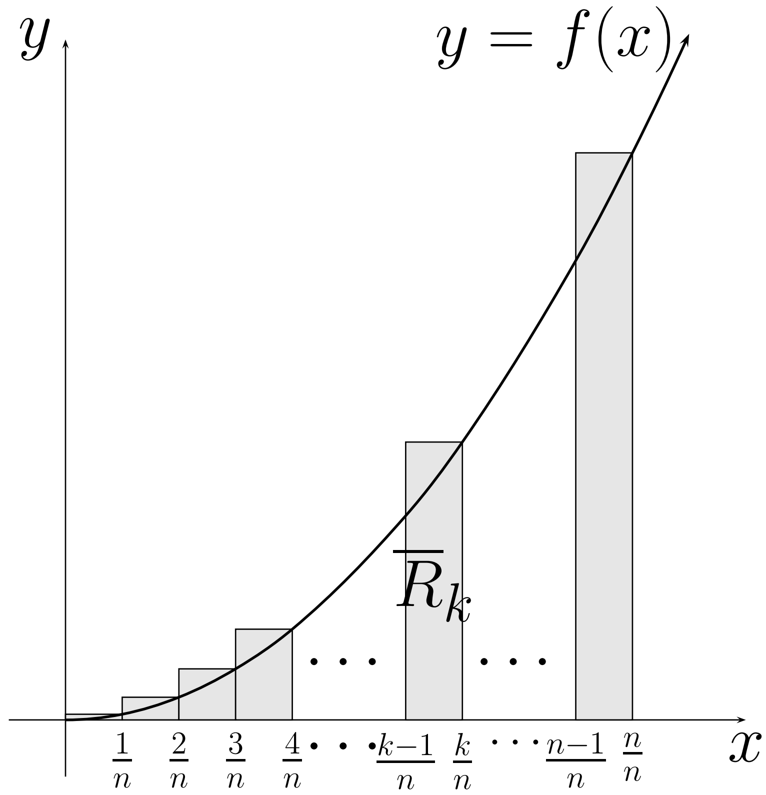
$$\mathrm{area}(\Omega_1) \leq \mathrm{area}(\Omega) \leq \mathrm{area}(\Omega_2).$$

If $\lim_{n\to\infty} \mathrm{area}(\Omega_1) = \lim_{n\to\infty} \mathrm{area}(\Omega_2)$ then

$$A = \mathrm{area}(\Omega) = \lim_{n\to\infty} \mathrm{area}(\Omega_1) = \lim_{n\to\infty} \mathrm{area}(\Omega_2).$$

Explicit evaluation of the bounds.

We begin by subdividing the interval $[0,1]$ into n subintervals $[0, \frac{1}{n}], [\frac{1}{n}, \frac{2}{n}], [\frac{2}{n}, \frac{3}{n}], \ldots, [\frac{n-1}{n}, 1]$.



The set $\mathcal{P}_n$ given by

$$\mathcal{P}_n = \{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, \frac{n-1}{n}, 1\}$$

which divides the interval $[0,1]$ into these subintervals is called a partition of $[0,1]$.

Let $\overline{R}_k$ denote the area of the kth rectangle. Then

$$\overline{R}_k = \text{width} \times \text{height} = \frac{1}{n} \times f(\tfrac{k}{n}) = \frac{k^2}{n^3}.$$

If $\overline{S}_{\mathcal{P}_n}(f)$ denotes the total area of the shaded region then

$$\overline{S}_{\mathcal{P}_n}(f) = \sum_{k=1}^{n} \overline{R}_k = \frac{1}{n^3} \sum_{k=1}^{n} k^2.$$

One may show by induction (exercise!) that

$$\sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1)$$

and hence

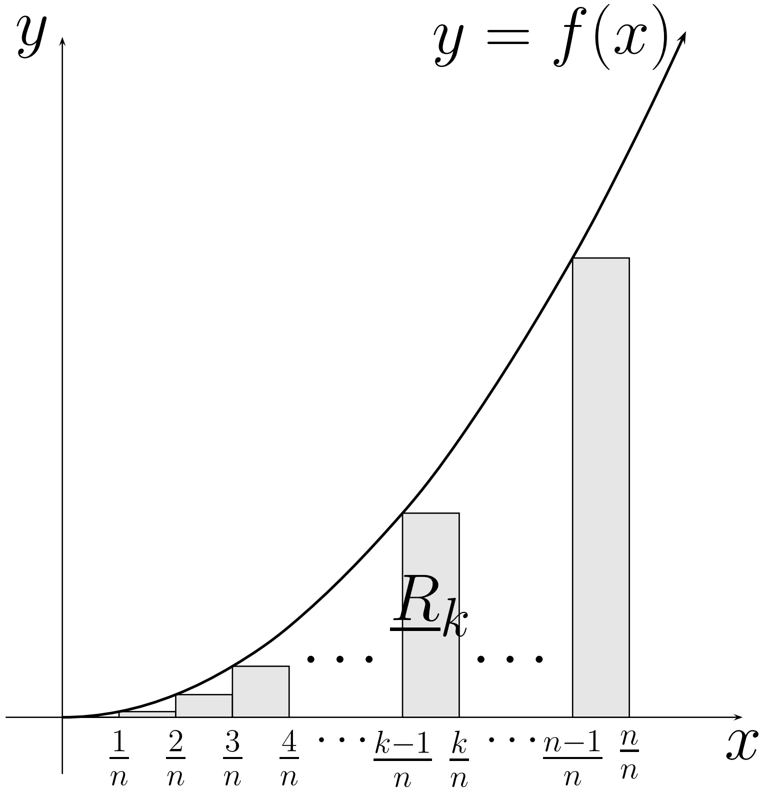
$$\overline{S}_{\mathcal{P}_n}(f) = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}.$$

The quantity $\overline{S}_{\mathcal{P}_n}(f)$ is called the upper Riemann sum of f with respect to the partition $\mathcal{P}_n$.

Axiom (A2) now implies that

$$A \leq \overline{S}_{\mathcal{P}_n}(f) = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}.$$

In a similar manner, a lower bound is obtained:



The area $\underline{R}_k$ of the kth rectangle is given by

$$\underline{R}_k = \frac{1}{n} \times f(\tfrac{k-1}{n}) = \frac{(k-1)^2}{n^3}.$$

The sum of all the areas of the rectangles is called the lower Riemann sum for the function f over the partition $\mathcal{P}_n$ and is denoted by $\underline{S}_{\mathcal{P}_n}(f)$. We obtain

$$\underline{S}_{\mathcal{P}_n}(f) = \sum_{k=1}^{n} \underline{R}_k = \frac{1}{n^3} \sum_{k=1}^{n} (k-1)^2$$

so that

$$\underline{S}_{\mathcal{P}_n}(f) = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}.$$

Axiom (2) therefore implies that

$$A \geq \underline{S}_{\mathcal{P}_n}(f) = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}.$$

Hence, for every positive integer n, the inequality

$$\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2} \leq A \leq \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$$

gives an upper and a lower bound for A.

Conclusion. In the limit $n \to \infty$, we obtain

$$A = \frac{1}{3},$$

regardless of the actual definition of A as long as it is compatible with the axioms (A1)-(A5)!

Remark. The process of calculating upper and lower Riemann sums and taking a limit of the above type (provided it exists) is called integration.

Riemann sums

Now, we generalise the approach above to an arbitrary positive bounded function f (we shall frequently assume, that f is **continuous**).

Suppose that f is a bounded function on $[a, b]$ and that $f(x) \geq 0$ for all x in $[a, b]$.

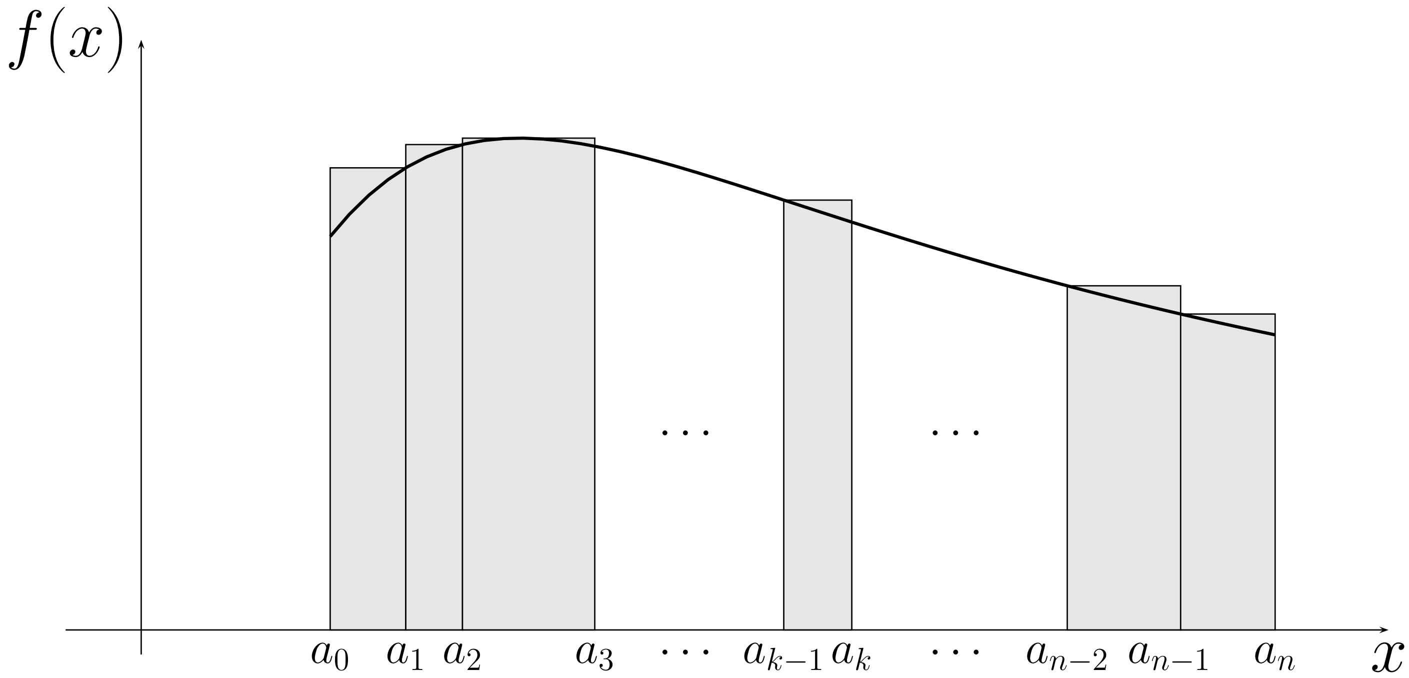
Definition. A finite set $\mathcal{P}$ of points in $\mathbb{R}$ is said to be a partition of $[a, b]$ if

$$\mathcal{P} = \{a_0, a_1, a_2, \ldots, a_n\}$$

and

$$a = a_0 < a_1 < a_2 < \ldots < a_n = b.$$

Suppose that $\mathcal{P}$ is a partition of $[a, b]$:

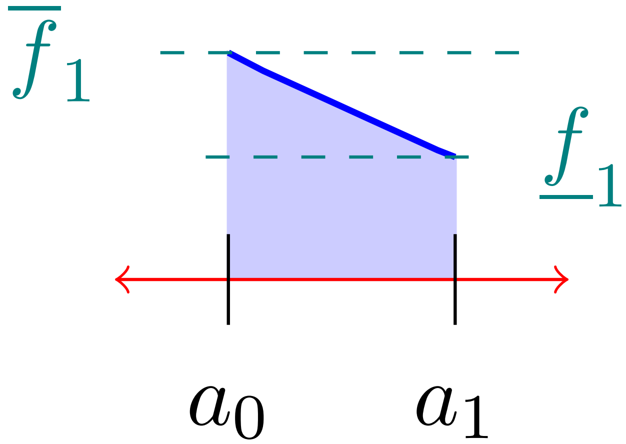


Note. The points of $\mathcal{P}$ need not be evenly spaced.

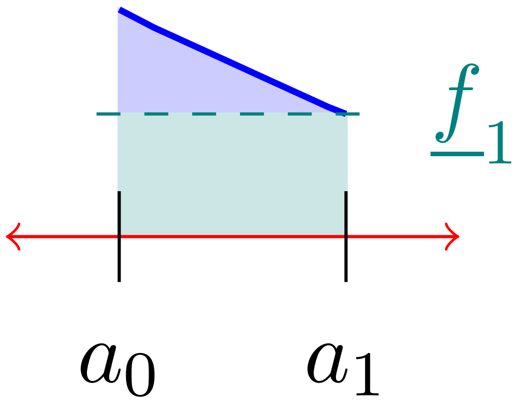
As f is bounded on $[a_0, a_1]$ we can define (see MATH1241)

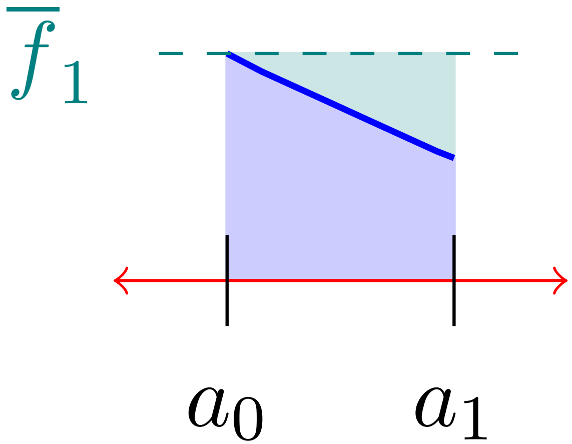
- $\overline{f}_1$ = least upper bound of f on $[a_0, a_1]$.
- $\underline{f}_1$ = greatest lower bound of f on $[a_0, a_1]$.

If the function f is continuous on $[a, b]$, then $\overline{f}_1$ (respectively, $\underline{f}_1$) is the maximum (respectively, minimum) value of the function f on $[a_0, a_1]$.



The rectangle with base $[a_0, a_1]$ and height $\underline{f}_1$ is the largest rectangle that sits entirely under the graph of f on this interval.





The rectangle with base $[a_0, a_1]$ and height $\overline{f}_1$ is the smallest rectangle that sits entirely above the graph of f on this interval.

The area of the kth rectangle in the above figure is

$$\text{width} \times \text{height} = (a_k - a_{k-1}) \times \overline{f}_k.$$

The upper Riemann sum $\overline{S}_{\mathcal{P}}(f)$ for f with respect to the partition $\mathcal{P}$ is defined by

$$\overline{S}_{\mathcal{P}}(f) = \sum_{k=1}^{n} (a_k - a_{k-1}) \overline{f}_k$$

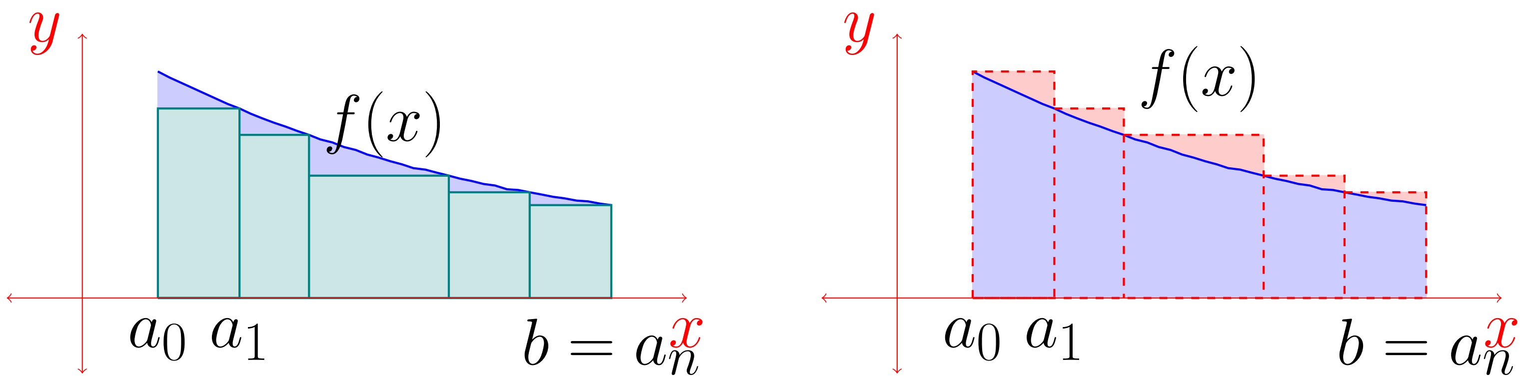
which is the total area of the rectangles in the above figure.

Likewise, the lower Riemann sum $\underline{S}_{\mathcal{P}}(f)$ for f with respect to the partition $\mathcal{P}$ is defined by

$$\underline{S}_{\mathcal{P}}(f) = \sum_{k=1}^{n} (a_k - a_{k-1}) \underline{f}_k. \tag{1}$$

You can do the same on each interval of the partition. Sticking these all together gives that

$$\underline{S}_{\mathcal{P}}(f) \leq \mathrm{Area}(\Omega) \leq \overline{S}_{\mathcal{P}}(f).$$



As you add more points to $\mathcal{P}$, $\underline{S}_{\mathcal{P}}$ gets bigger and bigger.

Similarly, as you add more points, $\overline{S}_{\mathcal{P}}$ decreases.

Definition. Suppose that $f : [a, b] \to \mathbb{R}$ is bounded. If there is a **unique** number A such that for every partition $\mathcal{P}$ of $[a, b]$, $\underline{S}_{\mathcal{P}} \leq A \leq \overline{S}_{\mathcal{P}}$ then we say that

- f is **Riemann integrable** over $[a, b]$,
- the **area** under the graph of f is A.

The number A is called the **definite integral** of f from a to b, written

$$A = \int_a^b f(x)\, dx.$$

Remark. The function f is called the integrand of the definite integral, while the points a and b are called the limits of the definite integral.

The notation

$$\int_a^b f(x)\,dx$$

is due to Leibniz. It evolved from a slightly different way of writing down lower and upper Riemann sums. For example, $\overline{S}_{\mathcal{P}}(f)$ may be written as

$$\overline{S}_{\mathcal{P}}(f) = \sum_{k=1}^{n} f(\overline{x}_k)\Delta x_k,$$

where $\Delta x_k = a_k - a_{k-1}$ and f attains its maximum value on $[a_{k-1}, a_k]$ at the point $\overline{x}_k$.

When taking a limit as before, Δx_k was replaced with dx and the symbol $\sum$ was replaced with an elongated 'S' ('S' stands for 'sum').

The use of the variable x is only tradition. You can of course use any other sensible variable, such as

$$I = \int_a^b f(u)\,du$$

and many authors avoid this completely by just writing

$$I = \int_a^b f.$$

If you were observant you may have noticed that I dropped the requirement that $f(x) \geq 0$ in the definition of Riemann integrability. The definitions of $\underline{S}_{\mathcal{P}}$ and $\overline{S}_{\mathcal{P}}$ didn't depend on this. This does however mean that we can have $\int_a^b f$ being negative. Thus, you might think of $\int_a^b f$ as being the **signed area** under the graph of f, where regions below the x-axis have negative area.

If f is a function which is Riemann integrable but not necessarily non-negative then it is natural to refer to

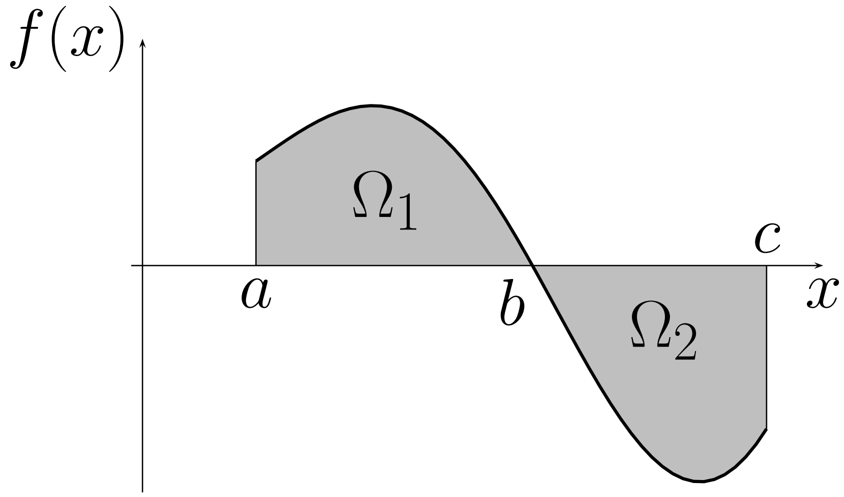
$$\int_a^b f(x)\,dx$$

as the signed area under the graph of the function f from a to b and to

$$\text{area}(\Omega) = \int_a^b |f(x)|\,dx$$

as the unsigned area under the graph of the function f from a to b provided that the latter integral exists.

Consider the following example:



In this case, the "unsigned" area is

$$\operatorname{area}(\Omega) = \int_a^b f(x)\,dx - \int_b^c f(x)\,dx.$$

Basic properties of the Riemann integral

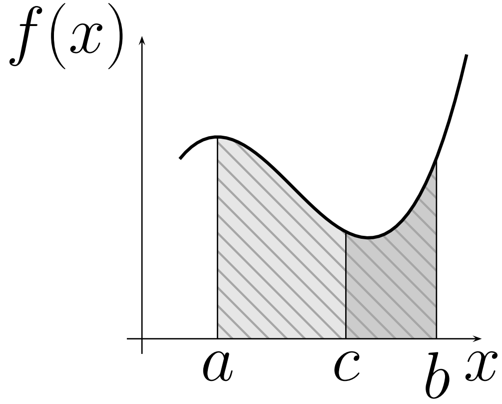
Theorem. Suppose that $f, g : [a, b] \to \mathbb{R}$ are integrable. Then,

(i) (Linearity) $\alpha f + \beta g$ is integrable for any $\alpha, \beta \in \mathbb{R}$ with

$$\int_a^b (\alpha f + \beta g)(x)\, dx = \alpha \int_a^b f(x)\, dx + \beta \int_a^b g(x)\, dx.$$

(ii) If $a < c < b$ then

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

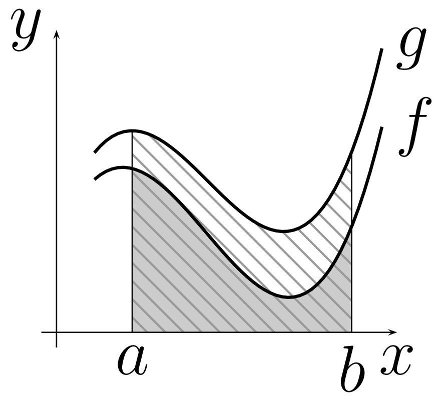


(iii) If $f(x) \geq 0$ for all x in $[a, b]$ then

$$\int_a^b f(x)\, dx \geq 0.$$

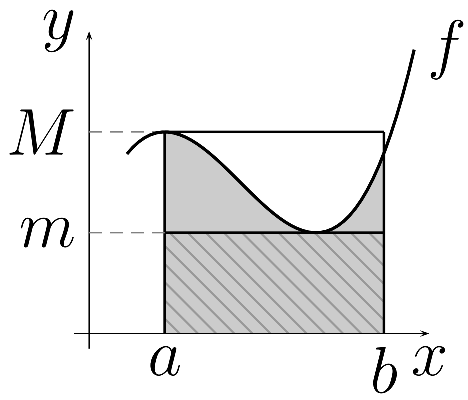
(iv) If $f(x) \leq g(x)$ for all x in $[a, b]$ then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$



(v) If $m \leq f(x) \leq M$ for all x in $[a, b]$ then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$



(vi) If $|f|$ is integrable on $[a, b]$ then

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$$

Sketch of proof.

(iii) If $f(x) \geq 0$ then

$$0 \leq \underline{S}_{\mathcal{P}}(f) \leq \int_a^b f(x)\,dx.$$

(iv) Apply (iii) to $h = g - f$.

(v) From (iv), it follows that

$$\int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx.$$

(vi) From

$$-|f(x)| \leq f(x) \leq |f(x)| \qquad \forall x \in [a,b]$$

and (iv), we deduce that

$$-\int_a^b |f(x)|\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx.$$

Hence,

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$$

It is convenient to introduce the following definition ...

Definition. Suppose that $b < a$ and that f is integrable on $[b, a]$. Then,

$$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$$

and

$$\int_a^a f(x)\, dx = 0.$$

... since, for instance, the following version of (ii) is still valid.

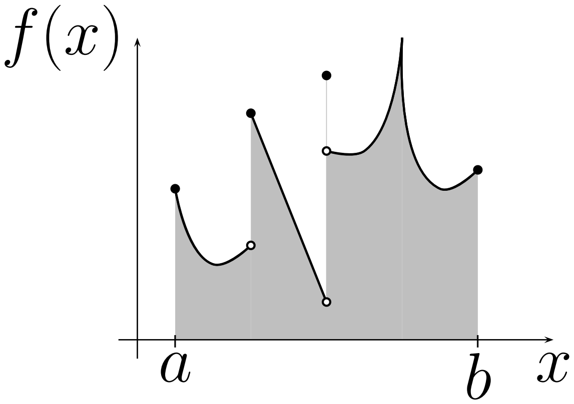
Remark. Suppose that a, b and c are real numbers and that f is integrable over some interval containing a, b and c. Then

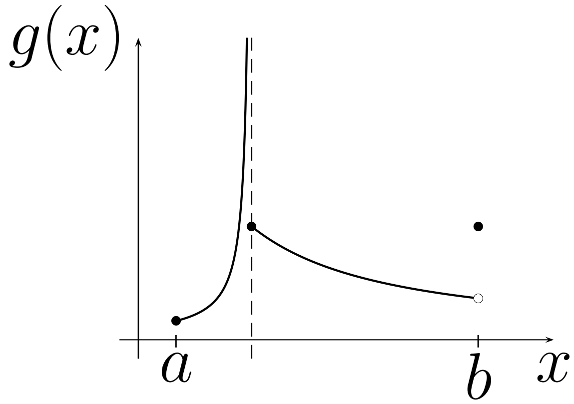
$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

A sufficient condition for a function to be Riemann integrable

Definition. A function $f : [a, b] \to \mathbb{R}$ is said to be piecewise continuous if it is continuous on $[a, b]$ at all except perhaps a finite number of points.

Examples.





Both functions f and g are piecewise continuous but f is bounded while g is not!

Theorem. If f is bounded and piecewise continuous on $[a, b]$ then f is Riemann integrable on $[a, b]$.

Proof. Proving this is not so easy — it takes $3\frac{1}{2}$ pages in SH&E! You really need to use the definitions carefully.

We skip the proof.

Example. Suppose that $f : [0,1] \to \mathbb{R}$,

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

Let $\mathcal{P} = \{a_0, a_1, \ldots, a_n\}$ be any partition of $[0,1]$. On any interval $[a_{k-1}, a_k]$ the best bounds that you could get are that

$$\underline{f}_k = 0 \leq f(x) \leq 1 = \overline{f}_k.$$

Thus

$$\underline{S}_{\mathcal{P}} = \sum_{k=1}^{n} (a_k - a_{k-1}) 0 = 0$$

while

$$\overline{S}_{\mathcal{P}} = \sum_{k=1}^{n} (a_k - a_{k-1}) 1 = a_n - a_0 = 1.$$

Hence, there is no unique number I satisfying

$$\underline{S}_{\mathcal{P}}(f) \leq I \leq \overline{S}_{\mathcal{P}}(f).$$

Thus, Riemann's definition fails to determine an area for this region, and we say that this function is not Riemann integrable. In particular, f is not piecewise continuous.

Remark. There exist more sophisticated ways of 'measuring' areas, volumes etc. such as Lebesque integration.

- If f is Riemann integrable then it is Lebesque integrable.
- If f is not Riemann integrable, it may still be Lebesque integrable. The Lebesque integral of the above example is 0!
- However, there exist regions in $\mathbb{R}^2$ to which it is impossible to assign an area in any meaningful way!

This leads to the **Banach-Tarski 'paradox'**: it is possible to take the unit ball in $\mathbb{R}^3$, break it into finitely many pieces, then translate and rotate to form two balls of exactly the same size as the original. The pieces however need to be so complicated that they don't themselves have a well-defined volume.

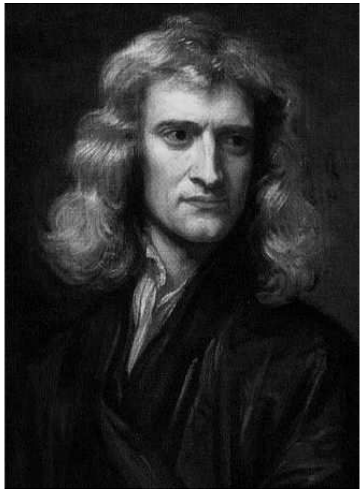
The definition of the Riemann integral is fine, but

- Calculating $\underline{S}_{\mathcal{P}}$ and $\overline{S}_{\mathcal{P}}$ seems very hard.
- It is hard work to show that a function is Riemann integrable.
- Using the definition finding $\int_a^b f$ looks awful!

All of these complaints are valid!

Before Newton and Leibniz, each area calculation required the sort of hard work we did earlier to show that $\displaystyle\int_0^1 x^2\,dx = \frac{1}{3}$, and was a major feat of computational skill.

Terminology: From now on, we refer to Riemann integrable functions as merely 'integrable'.

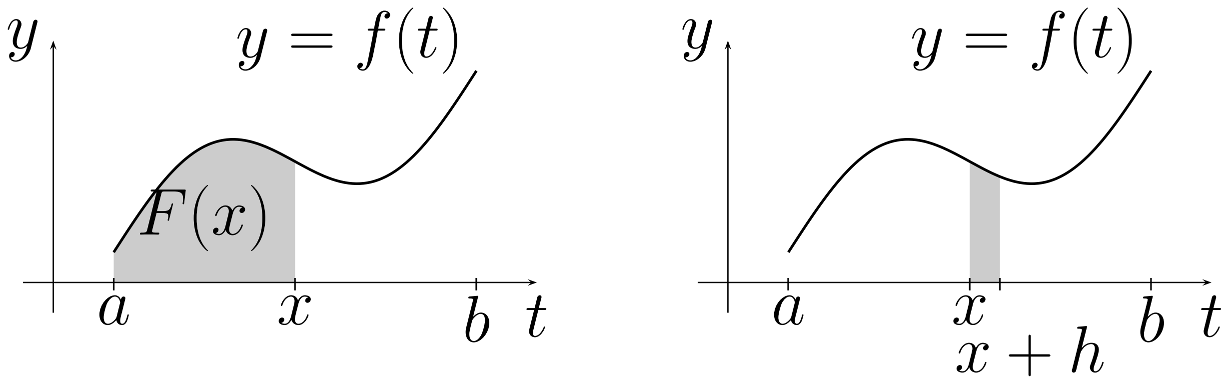
 

The remarkable insights of Newton and Leibniz were

1. Finding areas is a sort of inverse to finding tangents, and
2. As one can find tangents by using simple ‘symbolic’ differentiation rules instead of actually taking limits, you can find areas by applying the differentiation rules backwards.

The first fundamental theorem of calculus

Idea (Leibniz, Newton). How does the integral (area) change as a boundary changes?



Suppose that a function f is continuous and therefore integrable on an interval $[a, b]$.

We define the area function F by

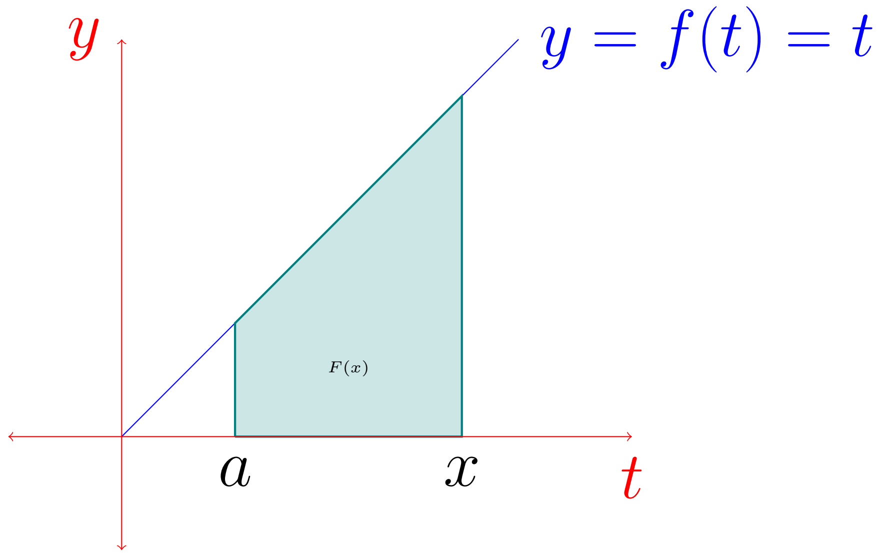
$$F(x) = \int_a^x f(t)\,dt, \quad x \in [a, b].$$

For some simple f it is possible to calculate $F(x)$ explicitly.

Example. $f(t) = 1$.

The region up to x is a rectangle with $a \leq t \leq x$ and $0 \leq y \leq 1$. This has area $(x-a) \cdot 1 = x - a$, so $F(x) = \int_a^x 1dt = x - a$.

Example. $f(t) = t$.



The region under the graph is now a trapezoid, $a \leq t \leq x$ and $0 \leq y \leq x$. This has area $F(x) = (x-a)\frac{(x+a)}{2} = \frac{1}{2}(x^2 - a^2)$.

Clearly, if $f(t) \geq 0$ then $F(x)$ increases as x increases. In fact:

Theorem. If f is bounded (and integrable) on $[a, b]$ then the area function F is continuous on $[a, b]$.

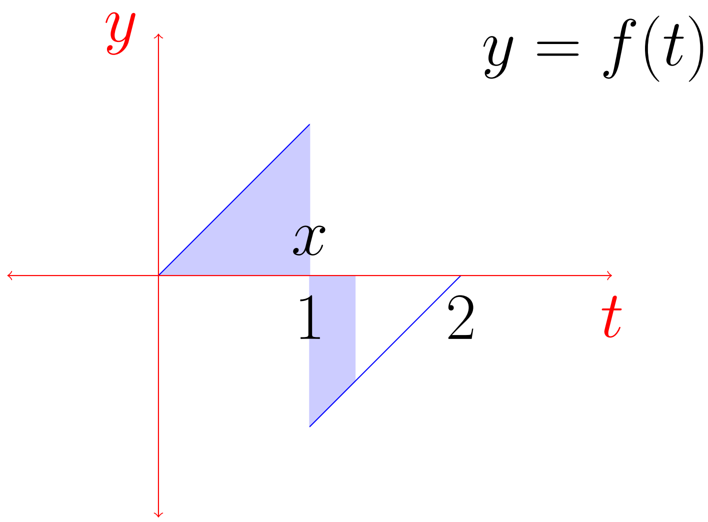
Proof. Suppose that $|f(t)| \leq M$ on $[a, b]$. Then, for any $x \in [a, b)$ and $h > 0$ we have

$$\begin{aligned} |F(x+h) - F(x)| &= \Big|\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\Big| \\ &= \Big|\int_x^{x+h} f(t)\,dt\Big| \\ &\leq \int_x^{x+h} |f(t)|\,dt \leq Mh \to 0 \end{aligned}$$

as $h \to 0^+$. It follows easily that $F(u) \to F(x)$ as $u \to x^+$. Dealing with $h < 0$ is similar. $\square$.

Example. Suppose that

$$f(t) = \begin{cases} t, & 0 \le t \le 1, \\ t-2, & 1 < t \le 2. \end{cases}$$



Sketch

$$F(x) = \int_0^x f(t)\,dt$$

for $0 \le x \le 2$.

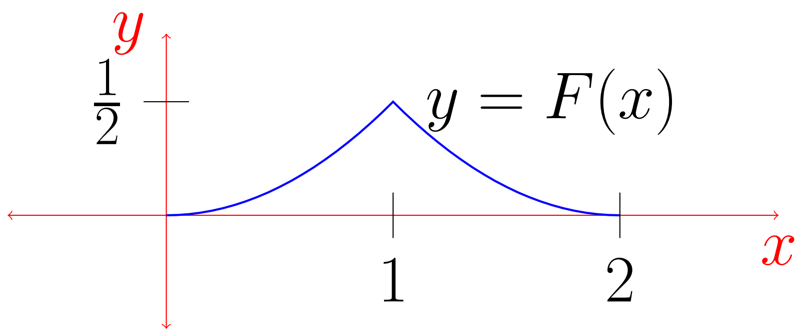
If $0 \leq x \leq 1$, then the region is a triangle with area $F(x) = \frac{x^2}{2}$.

If $1 \leq x \leq 2$ then $F(x) = \frac{1}{2}$ minus the area of a trapezoid. Signed area of the trapezoid is

$$(x-1)\frac{f(1)+f(x)}{2} = \frac{(x-1)(x-3)}{2}.$$

Thus,

$$F(x) = \begin{cases} \frac{x^2}{2}, & 0 \leq x \leq 1, \\ \frac{(x-2)^2}{2}, & 1 < x \leq 2. \end{cases}$$



Theorem. (The First Fundamental Theorem of Calculus). Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function. Then, the function $F : [a, b] \to \mathbb{R}$ defined by

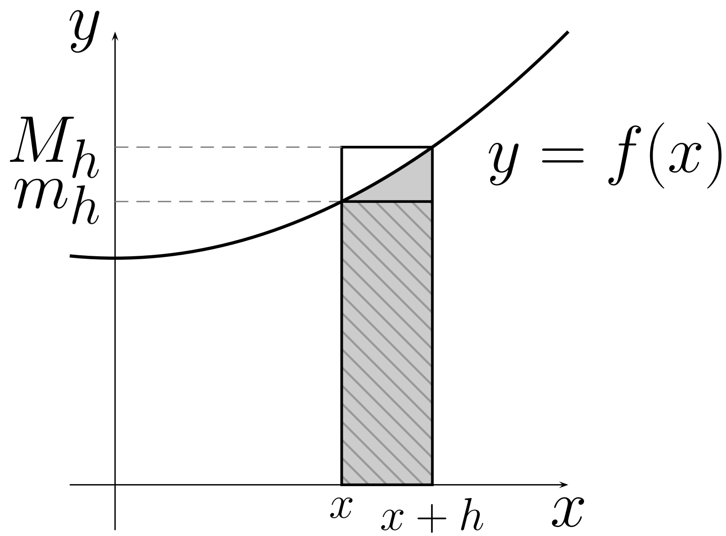
$$F(x) = \int_a^x f(t)\,dt \tag{2}$$

is continuous on $[a, b]$ and differentiable on (a, b) with

$$F'(x) = f(x)$$

for all x in (a, b).

Proof.



Since the function f is continuous on $[a, b]$, it is bounded and integrable on $[a, b]$. Therefore, by the previous theorem, we have that F is continuous on $[a, b]$.

<u>Differentiability.</u>

Suppose that $x \in (a, b)$ and choose an $h > 0$ such that $x + h \in (a, b)$. Then,

$$F(x + h) - F(x) = \int_x^{x+h} f(t)\, dt.$$

Since f is continuous on $[a, b]$, it attains a minimum value m_h and a maximum value M_h on $[x, x+h]$, that is,

$$m_h \leq f(t) \leq M_h$$

for all $t \in [x, x+h]$.

We therefore conclude that

$$m_h h \leq \int_x^{x+h} f(t)\,dt \leq M_h h$$

and hence

$$m_h h \leq F(x+h) - F(x) \leq M_h h$$

so that

$$m_h \leq \frac{F(x+h) - F(x)}{h} \leq M_h$$

since $h > 0$.

Since m_h is the minimum value of the function f on the interval $[x, x+h]$ and since f is continuous, there exists $x_0 \in [x, x+h]$ such that $f(x_0) = m_h$. Using again continuity of the function f, we obtain that

$$\lim_{h \to 0^+} m_h = f(x).$$

Arguing similarly, we have

$$\lim_{h \to 0^+} M_h = f(x).$$

Hence, by the pinching theorem,

$$\lim_{h \to 0^+} \frac{F(x+h) - F(x)}{h} = f(x).$$

In a similar manner, one can show that

$$\lim_{h \to 0^-} \frac{F(x+h) - F(x)}{h} = f(x).$$

Hence, F is differentiable on (a, b) and

$$F'(x) = f(x)$$

for all x in (a, b). $\square$

For the moment, this result is pretty, but not very helpful if you want to find areas!

It will be very important soon as many functions are **defined as area functions**, eg

$$\ln(x) := \int_1^x \frac{1}{t}\, dt$$

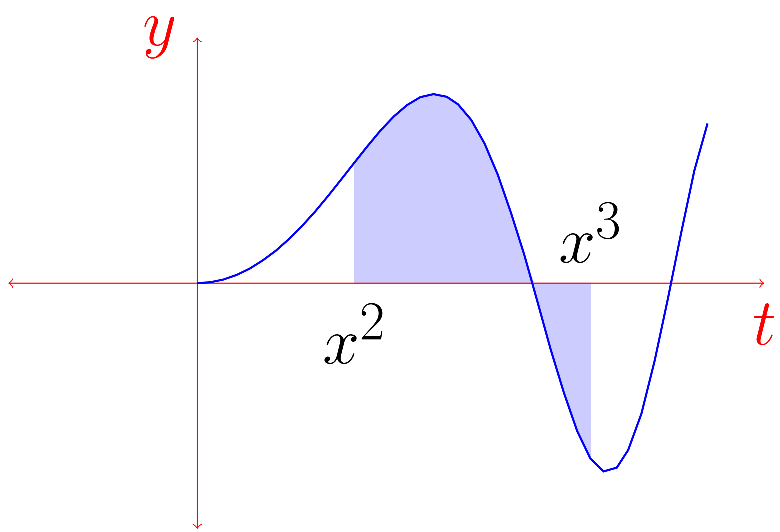
$$\operatorname{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

$$\operatorname{Si}(x) := \int_0^x \frac{\sin t}{t}\, dt.$$

The FTC1 tells us that these functions are continuous and differentiable.

Typical test/exam question: What is

$$\frac{d}{dx}\int_{x^2}^{x^3} \sin(t^2)\,dt?$$



How fast is the area changing?

Solution. Let
$$A(x) = \int_0^x \sin(t^2)\, dt.$$
Then A is differentiable by the FTC1, with $A'(x) = \sin(x^2)$.

We need to find $\frac{d}{dx}\left(A(x^3) - A(x^2)\right)$.

This just follows from the chain rule:
$$\begin{aligned}
\frac{d}{dx}\left(A(x^3) - A(x^2)\right) &= A'(x^3)\frac{d}{dx}\left(x^3\right) - A'(x^2)\frac{d}{dx}\left(x^2\right) \\
&= \sin((x^3)^2) \cdot 3x^2 - \sin((x^2)^2) \cdot 2x \\
&= 3x^2 \sin(x^6) - 2x \sin(x^4).
\end{aligned}$$

Second Fundamental Theorem of Calculus

Suppose that $f : [a, b] \to \mathbb{R}$. Recall that F is an antiderivative of f if $F'(x) = f(x)$ for all x.

A consequence of the FTC1 is that **every continuous function has an antiderivative:** $A(x) = \displaystyle\int_a^x f(t)\,dt$ will always do!

There exists a fast way of calculating an integral if an explicit antiderivative is known.

Theorem. (The second fundamental theorem of calculus). Suppose that f is a continuous function on $[a, b]$. If F is an antiderivative of f on $[a, b]$, that is,

$$F'(x) = f(x)$$

for all $x \in [a, b]$, then

$$\int_a^b f(t)\, dt = F(b) - F(a).$$

Proof. By the Mean Value Theorem, we know that two antiderivatives can only differ by a constant and hence

$$F(x) = \int_a^x f(t)\,dt + C$$

for some constant C. Accordingly,

$$F(b) - F(a) = \int_a^b f(t)\,dt + C - 0 - C = \int_a^b f(t)\,dt.$$

$\square$

Notation. We frequently use the notation

$$F(x)\Big|_a^b \qquad \text{or} \qquad \Big[F(x)\Big]_a^b$$

for the expression $F(b) - F(a)$.

The reason why the Second Fundamental Theorem of Calculus is useful is that just as we can find derivatives by applying some simple symbolic rules, we can often (but not always!) find an antiderivative by 'applying the differentiation rules backwards'.

Simple example. Find the area under the curve $y = \sin x$ between $x = 0$ and $x = \pi$.

We know from differentiation that $F(x) = -\cos x$ is **an** antiderivative for $\sin x$. So

$$\int_0^\pi \sin t \, dt = F(\pi) - F(0) = 1 - (-1) = 2.$$

The major problem is that applying the rules backwards is much more complicated than applying them forwards. eg, finding the derivative of $f(x) = x^2 \cos x^2$ is much easier than finding an antiderivative for it!

The FTC2 is only useful for calculating areas if you can somehow be clever enough to come up with a formula for an antiderivative in terms of 'non-area functions'. Sometimes this can't be done. This doesn't mean that the area can't be found, just that you need to find it some other way.

Example. Find $\displaystyle\int_2^3 \frac{1}{t}\,dt$.

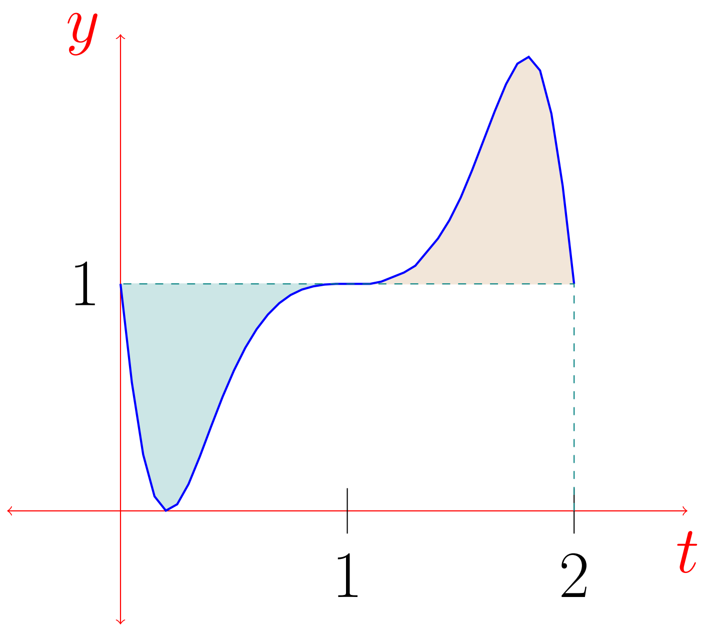
We know that an antiderivative for $\frac{1}{t}$ is $\ln t \ldots$.

Problem: $\ln x$ will be **defined** as $\displaystyle\int_1^x \frac{1}{t}\,dt$ so we'd just be saying that $\displaystyle\int_2^3 \frac{1}{t}\,dt = \int_1^3 \frac{1}{t}\,dt - \int_1^2 \frac{1}{t}\,dt$, which isn't much help!

Example. Find $\displaystyle\int_0^2 1 + \sin(\pi(t-1)^3)\,dt$.

Solution. Here you can't find an expression for this antiderivative apart from the corresponding area function.

But look at the graph:



Total area = rectangle base 2, height 1 **plus** brown region **less** green region. But by symmetry the coloured regions are equal. Hence $\displaystyle\int_0^2 1+\sin(\pi(t-1)^3)\,dt = 2.$

Notation. Since every antiderivative of f just differs is just an area function for f, plus some constant, we usually write $\displaystyle\int f$ or $\displaystyle\int f(x)\,dx$ (or ...) to denote a general antiderivative for f. For example

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \qquad (\text{for } n \neq -1)$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \frac{2\,x+1}{\sqrt{3\,x+2}}\,dx = \frac{2}{27}\sqrt{3\,x+2}\,(1+6\,x) + C$$

where C is any constant. This general form of the antiderivative is called the **indefinite integral** of f.

Checking these formulas is easy — you just differentiate the RHS. The challenge is to find the RHS.

Integration by substitution

The Chain Rule tells us that $\frac{d}{dx}f(g(x)) = f'(g(x))\,g'(x)$ so if you can recognise a function as being in the form $f'(g(x))\,g'(x)$, then you know how to find an antiderivative.

Sometimes this is obvious.

Example. Find $\int 2x \sin x^2 \, dx$.

$2x \sin x^2$ is just the derivative of the composition $-\cos x^2$, so

$$\int 2x \sin x^2 \, dx = -\cos x^2 + C.$$

Sometimes it is not so easy to see.

Example. Find $\int \frac{1}{x \ln x}\, dx$.
If $f(x) = g(x) = \ln x$ then

$$f'(g(x))\, g'(x) = \frac{1}{\ln x} \cdot \frac{1}{x}$$

so

$$\int \frac{1}{x \ln x}\, dx = f(g(x)) = \ln \ln x + C.$$

In complicated cases it helps to change variables via a **'substitution'**.

Theorem. Suppose that f and g' are continuous. Then

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

('Let $u = g(x)$')

Proof. Let F be an antiderivative of f. Then by the FTC2

$$\int_{g(a)}^{g(b)} f(u)\, du = F(g(b)) - F(g(a)).$$

Note that

$$(F \circ g)'(x) = F'(g(x))\, g'(x) = f(g(x))g'(x),$$

so $F \circ g$ is an antiderivative for the left-hand integrand. Thus

$$\begin{aligned}\int_a^b f(g(x))\, g'(x)\, dx &= (F \circ g)(b) - (F \circ g)(a) \\ &= F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(u)\, du.\end{aligned}$$

Often you can't quite recognise the $f(g(x))\,g'(x)$ part. You can still change variables. Formally writing things out via the Chain Rule as above is a bit tedious, and we usually use a more mechanical shorthand:

$$\begin{aligned}\text{Let } u &= g(x)\\ du &= g'(x)\,dx\end{aligned}$$

or sometimes

$$\begin{aligned}\text{Let } x &= h(u)\\ dx &= h'(u)\,du\end{aligned}$$

If we are doing a definite integral we need to remember to change the limits of integration!

Example. Find $I = \displaystyle\int \frac{2\,x+1}{\sqrt{3\,x+2}}\,dx$.

Solution: Let $u = \sqrt{3\,x+2}$, so $du = \dfrac{3}{2\sqrt{3\,x+2}}\,dx$.

Thus, replacing x with $\frac{u^2-2}{3}$ and $\frac{dx}{\sqrt{3\,x+2}}$ with $\frac{2}{3}\,du$, we have

$$\begin{aligned} I &= \int \Big(\frac{2(u^2-2)}{3}+1\Big)\frac{2}{3}\,du \\ &= \int \frac{4}{9}u^2 - \frac{2}{9}\,du \\ &= \frac{4}{27}u^3 - \frac{2}{9}u + C \\ &= \frac{4}{27}(3x+2)^{3/2} - \frac{2}{9}\sqrt{3\,x+2} + C. \end{aligned}$$

Note. The last step always consists of rewriting everything in terms of the original variable!

Example. Find $I = \displaystyle\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx$.

Solution. Let $x = 2\sin u$, so $dx = 2\cos u\,du$.

Now

$$4 - x^2 = 4 - 4\sin^2 u = (2\cos u)^2.$$

When $x = 0$, $u = 0$ and when $x = 1$, $\sin u = \frac{1}{2}$ and so $u = \frac{\pi}{6}$.

Thus

$$I = \int_0^{\pi/6} \frac{1}{2\cos u}\,2\cos u\,du = \int_0^{\pi/6} 1\,du = \frac{\pi}{6}.$$

Note that here it is much harder to see this as integrating $f(g(x))\,g'(x)$.

(Check: here $f(t) = 1$ and $g(x) = \sin^{-1}\frac{x}{2}$)

Seeing a good substitution is partly art and partly science. The more you do, the easier it will become.

Application. Suppose that f is a continuous function and a is a real number.

(i) If f is even then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

(ii) If f is odd then

$$\int_{-a}^{a} f(x)\,dx = 0.$$

(iii) If f is periodic with period T then

$$\int_{a}^{a+T} f(x)\,dx = \int_{0}^{T} f(x)\,dx.$$

Exercise. Find

$$\int_{\frac{\pi}{17}}^{\frac{37\pi}{51}} \sin^{99} 3x\,dx.$$

Solution. Let $f(x) = \sin^{99} 3x$. It is known, that f is a periodic function with period $\frac{2\pi}{3}$. Since $\frac{37\pi}{51} = \frac{\pi}{17} + \frac{2\pi}{3}$, by (iii) we have

$$\int_{\frac{\pi}{17}}^{\frac{37\pi}{51}} \sin^{99} 3x \, dx = \int_0^{\frac{2\pi}{3}} \sin^{99} 3x \, dx.$$

Applying again (iii) with $a = -\frac{\pi}{3}$ we obtain

$$\int_0^{\frac{2\pi}{3}} \sin^{99} 3x \, dx = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sin^{99} 3x \, dx.$$

Now, since the function f is odd (check this!), by (ii) we finally have that

$$\int_{\frac{\pi}{17}}^{\frac{37\pi}{51}} \sin^{99} 3x \, dx = 0.$$

Integration by parts

The Product Rule leads to another important technique for finding antiderivatives: $(fg)' = f'g + fg'$ and so $fg = \int f'g + \int fg'$. Rearranging this gives the **integration by parts formulae**:

$$\int fg' = fg - \int f'g$$

$$\int_a^b f(x)g'(x)\,dx = \big[f(x)g(x)\big]_a^b - \int_a^b f'(x)g(x)\,dx.$$

Roughly speaking, this means that if you are trying to integrate a product of two functions, you can convert it to an integral involving the product of the antiderivative of one and the derivative of the other.

Again, sometimes it is obvious that one can apply integration by parts, and sometimes not.

Example. Find $I = \displaystyle\int x \sin x \, dx$.

*[**Think**: Differentiating x makes it simpler; integrating $\sin x$ doesn't make it worse.]*

Typical setting out is

$$\begin{aligned} \text{Let } u &= x, \quad v' = \sin x, \\ u' &= 1, \quad v = -\cos x. \end{aligned}$$

Then

$$\begin{aligned} I &= uv - \int u'v \\ &= -x\cos x + \int \cos x \, dx \\ &= -x\cos x + \sin x + C \end{aligned}$$

(which can easily be checked by differentiating the answer!)

Example. Find $I = \int_1^e \ln x \, dx$.
Solution. It is less obvious here to choose:

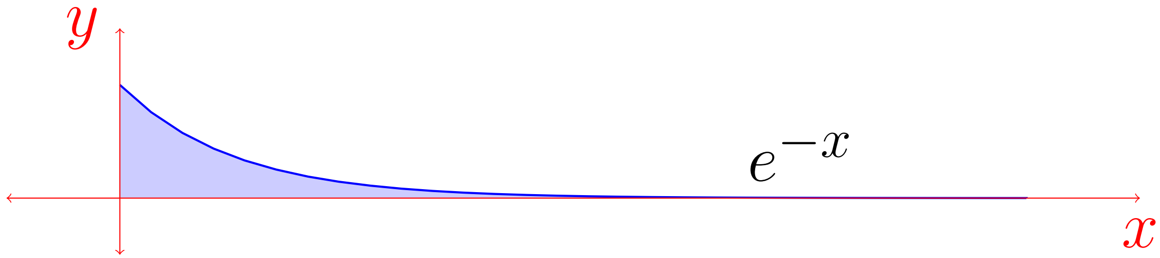
$$u = \ln x, \quad v' = 1,$$
$$u' = \frac{1}{x}, \quad v = x.$$

Then

$$\begin{aligned} I &= \left[uv\right]_1^e - \int_1^e u'v \\ &= \left[x \ln x\right]_1^e - \int_1^e 1 \, dx \\ &= e - 0 - (e - 1) = 1. \end{aligned}$$

Unbounded regions

Does it make any sense to talk about the 'area' in the first quadrant underneath the graph of $y = e^{-x}$?



We can't do what we did before because you can't possibly fit this unbounded region **inside** a finite union of polygons.

On the other hand, there is a limit to the total area of any polygons that you could ever put inside the region. This is roughly the idea behind the following definition.

Definition.

- Suppose that $f : [0, \infty) \to \mathbb{R}$ has area function $F(x) = \int_0^x f(t)\,dt$. If

$$\lim_{x \to \infty} F(x)$$

exists and equals L then we shall say that the **improper integral**

$$\int_0^\infty f(t)\,dt$$

converges and write

$$\int_0^\infty f(t)\,dt = L.$$

We say that f is **integrable** over $[0, \infty)$.

- Suppose that

$$\int_a^R f(x)\,dx$$

does not have a limit as $R \to \infty$. Then, we say that f is **not integrable** over $[a, \infty)$ and the improper integral

$$\int_a^\infty f(x)\,dx$$

is said to be **divergent**.

Example. Let $f(t) = e^{-t}$. Then

$$F(x) = \int_0^x e^{-t}\,dt = -e^{-x} + 1.$$

Now

$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty} -e^{-x} + 1 = 1$$

and so

$$\int_0^\infty e^{-t}\,dt = 1$$

gives the area of the unbounded region discussed earlier.

For historical reasons, people almost always take $\lim_{R\to\infty} F(R)$ rather than $\lim_{x\to\infty} F(x)$ but of course this is just a notational convention. Thus, you'll usually write:

$$\int_0^\infty e^{-x}\,dx = \lim_{R\to\infty}\int_0^R e^{-x}\,dx = \lim_{R\to\infty}\left(-e^{-R}+1\right) = 1$$

and so the improper integral converges.

You can define $\int_a^\infty f(x)\,dx$ or $\displaystyle\int_{-\infty}^a f(x)\,dx$ in a similar way by taking limits of integrals over finite intervals, eg

$$\int_{-\infty}^a f(x)\,dx = \lim_{R\to-\infty}\int_R^a f(x)\,dx.$$

Defining $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ takes more care.

Example. Calculate $\displaystyle\int_{-\infty}^{\infty} x\,dx$.

As before:

$$\lim_{R\to\infty}\int_{-R}^{R} x\,dx = \lim_{R\to\infty}\left[x^2/2\right]_{-R}^{R} = \lim_{R\to\infty} 0 = 0$$

which doesn't seem exactly right!?!

The problem here is **not** that f is unbounded:

$$\int_0^R \frac{2x}{1+x^2}\,dx = \left[\ln(1+x^2)\right]_0^R = \ln(1+R^2) \to \infty$$

whereas

$$\int_{-R}^{R} \frac{2x}{1+x^2}\,dx = \left[\ln(1+x^2)\right]_{-R}^{R} = 0 \to 0$$

as $R \to \infty$. This is a bit like writing $\infty - \infty = 0$???

Definition. We say that f is integrable over $(-\infty, \infty)$ if f is integrable over both $(-\infty, 0]$ and $[0, \infty)$. In this case, we write

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx.$$

If f is not integrable on either of the intervals $(-\infty, 0]$ or $[0, \infty)$ then we say that the improper integral

$$\int_{-\infty}^{\infty} f(x)\,dx$$

diverges.

Thus, the integral $\displaystyle\int_{-\infty}^{\infty} x\,dx$ diverges!

Example. Find $I = \displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$.

We have

$$\int_0^R \frac{1}{1+x^2}\,dx = \left[\tan^{-1} x\right]_0^R = \tan^{-1} R \to \frac{\pi}{2},$$

as $R \to \infty$, while for $S < 0$

$$\int_S^0 \frac{1}{1+x^2}\,dx = \left[\tan^{-1} x\right]_S^0 = -\tan^{-1} S \to \frac{\pi}{2},$$

as $S \to -\infty$.

Thus

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$$

converges and equals

$$\frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

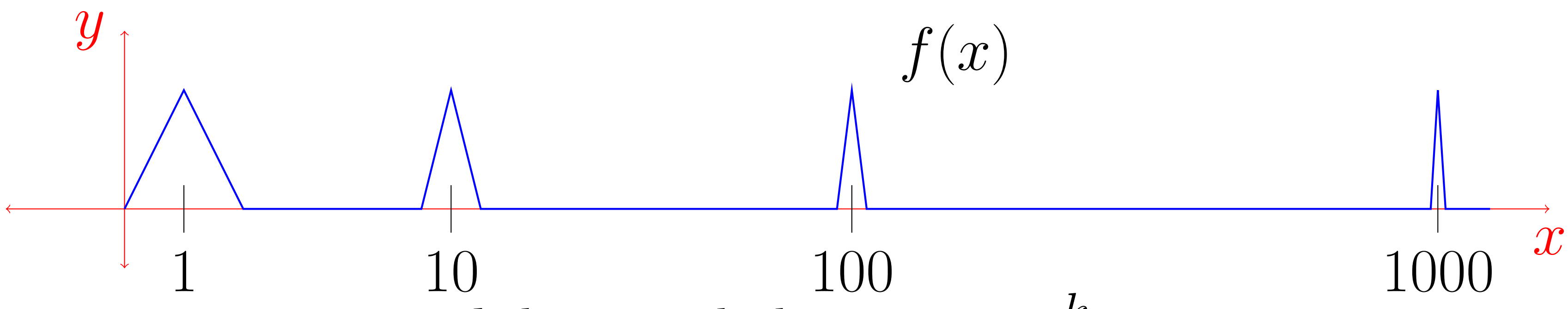
When presented with an improper integral there are two different questions:

- Does the integral converge?
- If it does converge, what is its value.

For many problems the first question is more important than the second. In any case, there are many problems where answering the second question is impossible.

Roughly, for $\displaystyle\int_0^\infty f(x)\,dx$ to converge, you want that $f(x) \to 0$ as $x \to \infty$. Actually, this condition is **neither necessary nor sufficient!**

Example. $\displaystyle\int_0^\infty f(x)\,dx$ converges but $f(x) \not\to 0$.



Example. $f(x) \to 0$ but $\displaystyle\int_0^\infty f(x)\,dx$ **diverges**.

Let $f(x) = (x+1)^{-1/2}$. For $R > 0$,

$$\int_0^R f(x)\,dx = \left[2\sqrt{x+1}\right]_0^R = 2\sqrt{R+1} - 2 \to \infty$$

as $R \to \infty$ so the integral diverges.

The p-test

Question. When does $\int_1^\infty \frac{1}{x^p}\, dx$ converge?

Clearly you want $p > 0$ or else $\frac{1}{x^p} \not\to 0$ as $x \to \infty$, which would be very bad for convergence.

Theorem. (p-test) The improper integral

$$\int_1^\infty \frac{1}{x^p}\, dx$$

is convergent if $p > 1$ and divergent if $p \leq 1$.

Proof. If $p \neq 1$ then

$$\begin{aligned}\int_1^R x^{-p}\,dx &= \left[\frac{x^{1-p}}{1-p}\right]_1^R \\ &= \frac{R^{1-p}-1}{1-p} \\ &\to \begin{cases} \frac{1}{p-1} & \text{when } 1-p<0 \\ \infty & \text{when } 1-p>0 \end{cases}\end{aligned}$$

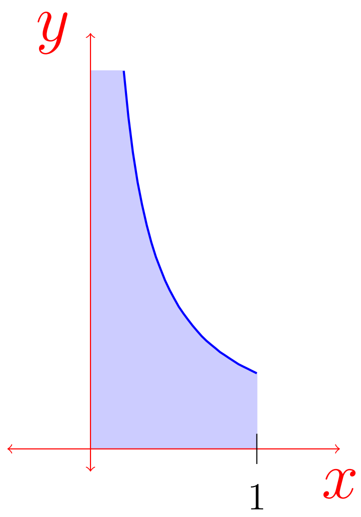
as $R \to \infty$.

If $p = 1$ then

$$\int_1^R \frac{1}{x}\,dx = \Big[\ln x\Big]_1^R = \ln R - \ln 1 \to \infty \quad \text{as } R \to \infty.$$

$\square$

Remark. Note that the p-test concerns integrals starting from $x = 1$. Starting at $x = 0$ is more complicated as $\frac{1}{x^p}$ isn't bounded as $x \to 0^+$.



You can also take limits to define things like $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$. These converge if and only if $p < 1$. Many students got caught in the 2013 exam trying to apply the p-test to $\displaystyle\int_0^\infty \frac{1}{x^2}\,dx$.

The Comparison Test

Most functions that we consider are positive and **decreasing**. For such functions what matters is how fast $f(x) \to 0$ as $x \to \infty$. The p-test gives us a guide.

For a general positive decreasing function f:

- if $f(x) \to 0$ faster than $\dfrac{1}{x^p}$ for some $p > 1$ then $\displaystyle\int_1^\infty f(x)\,dx$ converges;
- if $f(x) \to 0$ slower than $\dfrac{1}{x^p}$ for some $p < 1$ then $\displaystyle\int_1^\infty f(x)\,dx$ diverges;

This follows from the Comparison Test:

Theorem (Comparison test). Suppose that f and g are integrable functions and that

$$0 \leq f(x) \leq g(x)$$

for $x \geq a$.

(i) If $\displaystyle\int_a^\infty g(x)\,dx$ converges then $\displaystyle\int_a^\infty f(x)\,dx$ converges.

(ii) If $\displaystyle\int_a^\infty f(x)\,dx$ diverges then $\displaystyle\int_a^\infty g(x)\,dx$ diverges.

Sketch of proof. This follows from

$$0 \leq \int_a^R f(x)\,dx \leq \int_a^R g(x)\,dx$$

and the Least Upper Bound Axiom (MATH1241).

Example. Discuss the convergence of

$$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx.$$

The integrand is even and hence

$$I = 2\int_0^{\infty} e^{-x^2}\, dx = 2\int_0^{1} e^{-x^2}\, dx + 2\int_1^{\infty} e^{-x^2}\, dx.$$

On the other hand, we know that

$$e^{x^2} > x^2 \qquad \Rightarrow \qquad e^{-x^2} < \frac{1}{x^2}$$

for $x \neq 0$ and that the p-integral

$$\int_1^{\infty} \frac{1}{x^2}\, dx$$

converges. Since $\int_0^1 e^{-x^2}\, dx$ is a finite number, we conclude that I converges.

Typical questions

Example. Discuss the convergence of $\displaystyle\int_1^\infty \frac{1}{x^4+2}\,dx$.

Solution. Let $f(x) = \dfrac{1}{x^4+2}$ and let $g(x) = \dfrac{1}{x^4}$.

Now $f(x) \le g(x)$ for all $x \ge 1$ and, **by the p-test,** $\int_1^\infty g(x)\,dx$ converges. Hence, **by the Comparison Test**, $\int_1^\infty f(x)\,dx$ converges too. (Always name the theorems that you are using!)

Example. Discuss the convergence of $\displaystyle\int_e^\infty \frac{\ln x}{\sqrt{x}}\,dx$.

Solution. Let $f(x) = \dfrac{\ln x}{\sqrt{x}}$ and let $g(x) = \dfrac{1}{\sqrt{x}}$. Then, for all $x \ge e$, $f(x) \ge g(x)$ and, by the p-test, $\int_e^\infty g(x)\,dx$ diverges, so by the Comparison Test, $\int_e^\infty f(x)\,dx$ diverges too.

Example. Discuss the convergence of $\displaystyle\int_e^\infty \frac{1}{x\ln x}\,dx$.

This is tricky, since $\frac{1}{x\ln x} \le \frac{1}{x}$ (which doesn't help) and $\frac{1}{x\ln x} \ge \frac{1}{x^{1+\epsilon}}$ for all $\epsilon > 0$ (which doesn't help either).

You have to do this one directly:

$$\int_e^R \frac{1}{x\ln x}\,dx = \big[\ln(\ln x)\big]_e^R = \ln\ln R - 0 \to \infty$$

as $R \to \infty$, so the integral diverges.

What about $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^2}\,dx$?

$$\int_e^R \frac{1}{x(\ln x)^2}\,dx = \left[\frac{-1}{\ln x}\right]_e^R = \frac{-1}{\ln R} + 1 \to 1$$

as $R \to \infty$, so the integral converges.

Comparison Test - Pro version

Example. Discuss the convergence of $\displaystyle\int_2^\infty \frac{1}{x^4-2}\,dx$.

Here the inequalities are a bit fiddlier. You could use

$$\frac{1}{x^4-2} = \frac{1}{x^2+\sqrt{2}} \times \frac{1}{x^2-\sqrt{2}} \leq \frac{1}{x^2+\sqrt{2}} < \frac{1}{x^2}$$

but all that really matters is that $\frac{1}{x^4-2}$ behaves like $\frac{1}{x^4}$ for large x.

Instead of using inequalities to estimate integrands, one often uses a 'dominant term analysis' such as

$$f(x) = \frac{\sqrt{\sin x + x^2}}{2x^4-1}$$

'behaves like' $g(x) = \frac{1}{2x^3}$ for large x and hence one expects the convergence of the two associated improper integrals to be the same.

The precise formulation of this idea is as follows:

Theorem (Limit form of the comparison test.) Suppose that f and g are nonnegative and bounded on $[a, \infty)$. If

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$$

and $0 < L < \infty$ then either

$$\text{both} \quad \int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx \quad \text{converge}$$

or

$$\text{both} \quad \int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx \quad \text{diverge.}$$

This means that you can usually determine whether an improper integral converges just by picking out the **dominant terms**.

Example. Discuss the convergence of $\displaystyle\int_2^\infty \frac{\sqrt{3x^2 + \sin x}}{x^4 - 2}\,dx$.

Here

$$f(x) = \frac{\sqrt{3x^2 + \sin x}}{x^4 - 2}$$

behaves like $g(x) = \frac{1}{x^3}$ for large x, in the sense that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \sqrt{3}.$$

By the p-test $\int_2^\infty g(x)\,dx$ converges and so, by the limit form of the comparison test,

$$\int_2^\infty \frac{\sqrt{3x^2 + \sin x}}{x^4 - 2}\,dx$$

converges.

Trickier examples

Example. Discuss the convergence of $\displaystyle\int_1^\infty \frac{\sqrt{x+1}-\sqrt{x}}{x}\,dx$.

Solution. Let
$$f(x) = \frac{\sqrt{x+1}-\sqrt{x}}{x}.$$
Note that
$$f(x) = \frac{\sqrt{x+1}-\sqrt{x}}{x} \cdot \frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}} = \frac{1}{x\sqrt{x+1}+x\sqrt{x}}.$$
Let $g(x) = \frac{1}{2x^{3/2}}$. For all $x > 1$, $f(x) \leq g(x)$, and by the p-test $\int_1^\infty g(x)\,dx$ converges. Therefore by the Comparison Test,
$$\int_1^\infty f(x)\,dx$$
converges too.

The Gamma Function

Consider the improper integral $\int_0^\infty e^{-t}\, t^2\, dt$.

You can check that

$$\int_0^R e^{-t}\, t^2\, dt = 2 - e^{-R}\left(R^2 - 2R - 2\right) \to 2$$

as $R \to \infty$, so this integral converges.

By mathematical induction, you can prove that for any integer $n \geq 0$,

$$\int_0^\infty e^{-t}\, t^n\, dt = n!.$$

This gives us a way to define factorials for numbers other than integers:

$$x! := \int_0^\infty e^{-t}\, t^x\, dt.$$

Actually, for historical reasons (blame Legendre), we don't quite do this! The extension of the factorial function to non integers is called the Γ function, and is defined for $x > 0$ by

$$\Gamma(x) := \int_0^\infty e^{-t}\, t^{x-1}\, dt.$$

Thus $\Gamma(n) = (n-1)!$.

This function turns out to pop up in all sorts of areas and is one of the most important 'special functions' in mathematics. Euler first wrote this in 1730 as

$$n! = \int_0^1 (-\ln(s))^n\, ds.$$

This converts to the usual version setting $t = -\ln s$. Indeed, in this case, $s = e^{-t}$, $ds = -e^{-t}dt$ and t varies from ∞ to 0. Thus,

$$\int_0^1 (-\ln(s))^n \, ds = \int_\infty^0 t^n \cdot (-e^{-t})dt = \int_0^\infty e^{-t}t^n dt.$$

What next?

- In MATH1241 you'll concentrate on the question of which functions have an antiderivative which can be written in terms of elementary functions. You'll look at lots of integrals of rational functions and trig functions.
- Recurrence formulae for integrals:

$$\int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-1} x \, dx.$$

- A double integral $\displaystyle\iint_\Omega f(x,y)\,dx\,dy$ gives the volume under the graph of a function of two variables, over a region $\Omega \subseteq \mathbb{R}^2$. You essentially do these by reducing them to two one variable integrals. The complication is that Ω might be a messy subset of $\mathbb{R}^2$.

- If $\delta(x, y, z)$ is the mass density at point (x, y, z) inside a planet E then the triple integral $\iiint_E \delta(x, y, z)\, dx\, dy\, dz$ would give the total mass of the planet.

Summary

- Unlikely to be asked to find an area by Riemann sums, but they could be snuck in as an estimation or limit question.
- There may be an easy (for you) question involving integration by substitution and/or by parts. You won't be given any hints!
- They will certainly test improper integrals!
- For the comparison test, you need to get a good feeling for which ones converge and which diverge, so that you know what you are trying to prove. Get used to finding the dominant terms in the integrand.
- You need to explain the reasoning that you use. Name all the tests that you use. (p-test, comparison test (both versions)).