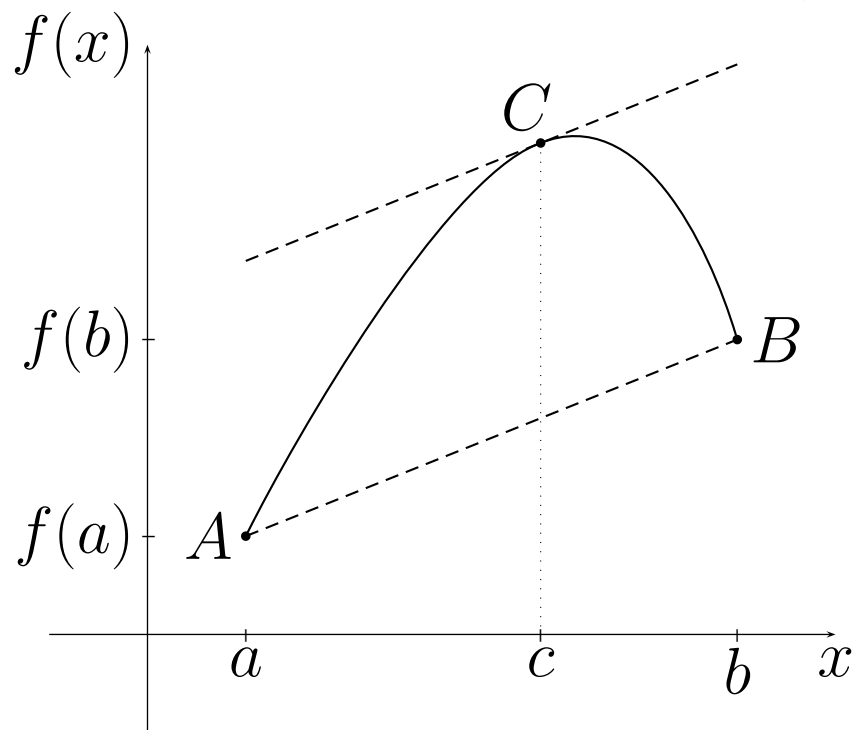


The mean value theorem

This section is about an odd looking result that stands at the foundations of much of calculus, the **Mean Value Theorem (MVT)**.



The mean value theorem. Suppose that a function f is **continuous** on $[a, b]$ and **differentiable** on (a, b) . Then, there exists at least one real number c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Briefly speaking the theorem says:

If you travel from point A to point B , then at some stage during the journey you must have been doing exactly your average speed.

Example. A car enters a tunnel at a speed of 30km/h and after 1 minute leaves the tunnel at a speed of 40 km/h. The length of the tunnel is 1km. Did the driver break the speed limit of 50km/h?

Remark. In the above theorem, it is required that f is **continuous** on the **closed** interval but **differentiable** only on the **open** interval!

Example. Find counterexamples which demonstrate that the continuity and differentiability requirements must be met.

Example. Apply the mean value theorem to the function f defined by $f(x) = x^5$ and $a = -1$, $b = 4$. Find the value(s) of c which satisfy the conclusion of the theorem.

Applications of the mean value theorem (MVT).

Before we prove the theorem, it is worth asking why this is interesting.

1. MVT is behind important facts that you will have just looked at intuitively, for example if $f'(x) \geq 0$ on (a, b) then f is ‘increasing’ on (a, b) .

2. MVT is used in the proof of the Fundamental Theorem of Calculus:

$$\text{if } F' = f \text{ then } \int_a^b f(x) dx = F(b) - F(a).$$

3. MVT allows you to prove inequalities like

$$x - \frac{x^3}{3} \leq \sin x \leq x \text{ for } 0 \leq x \leq 1.$$

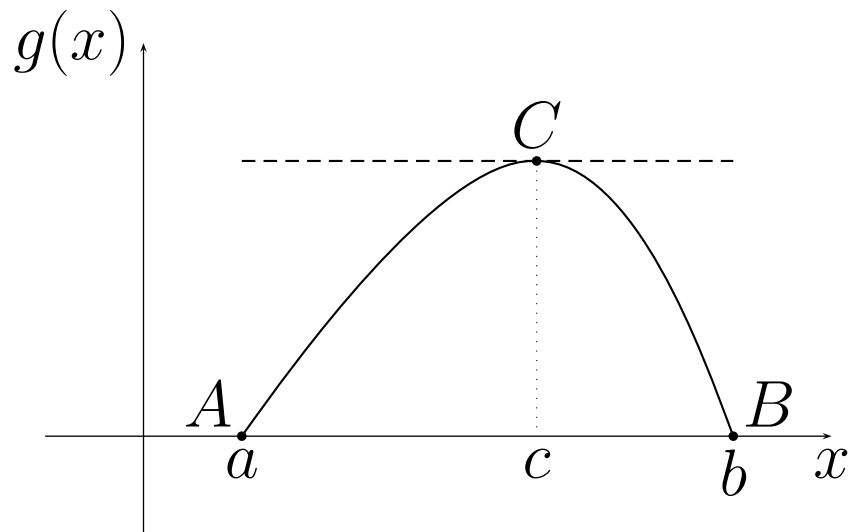
4. MVT is used to prove L'Hôpital's Rule which is used for

$$\lim_{x \rightarrow 0} \frac{\tan x - \sin x}{x^3}.$$

5. Counting zeros, generalized max-min tests, ...

Proof of the mean value theorem

If $f(a) = f(b) = 0$ then the mean value theorem reduces to **Rolle's theorem**.



Rolle's theorem. Suppose that a function g is continuous on $[a, b]$ and differentiable on (a, b) . If, in addition,

$$g(a) = g(b) = 0$$

then there exists a c in (a, b) such that $g'(c) = 0$.

Proving inequalities using the mean value theorem

An important application of the mean value theorem is in proving inequalities.

General ‘philosophy.’ Apply the mean value theorem to an appropriate function f and find a lower or upper bound for $f'(c)$ on (a, b) .

Example. Show that

$$e^x > 1 + x \quad \text{for all } x > 0.$$

Hint: Fix $x > 0$ and make the identification $[a, b] = [0, x]$ for this fixed $x > 0$.

Example. It is known that any polynomial ‘grows faster than’ the logarithm. For, instance, show that

$$\ln x < x - 1 \quad \text{for all } x > 1.$$



Error bounds

A second application of the mean value theorem is in calculating error bounds.

Question: How much bigger than $\frac{1}{2}$ can $\sin 31^\circ$ be?

Let $f(x) = \sin x$, and choose the interval $[a, b]$ with $a = \frac{\pi}{6}$ and $b = \frac{31\pi}{180}$ as above. (b is just 31° in radians and a is such that $\sin a = \frac{1}{2}$.)

Then

$$\frac{f(b) - f(a)}{b - a} = \frac{\sin b - \frac{1}{2}}{\pi/180}.$$

Applying MVT to f , there is some number $c \in (a, b)$ such that

$$f'(c) = \cos c = \frac{\sin b - \frac{1}{2}}{\pi/180}. \quad \text{In other words, } \sin b = \frac{1}{2} + \frac{\pi \cos c}{180}.$$

In fact $\frac{1}{\sqrt{2}} < \cos c < \frac{\sqrt{3}}{2}$ as $\frac{\pi}{6} < c < \frac{\pi}{4}$.

$$\text{Thus, } 0.5123 < \frac{1}{2} + \frac{\pi}{180\sqrt{2}} < \sin b < \frac{1}{2} + \frac{\pi\sqrt{3}}{360} < 0.5152$$

The sign of a derivative

Definition. Let a function f be defined on an interval I . We say that

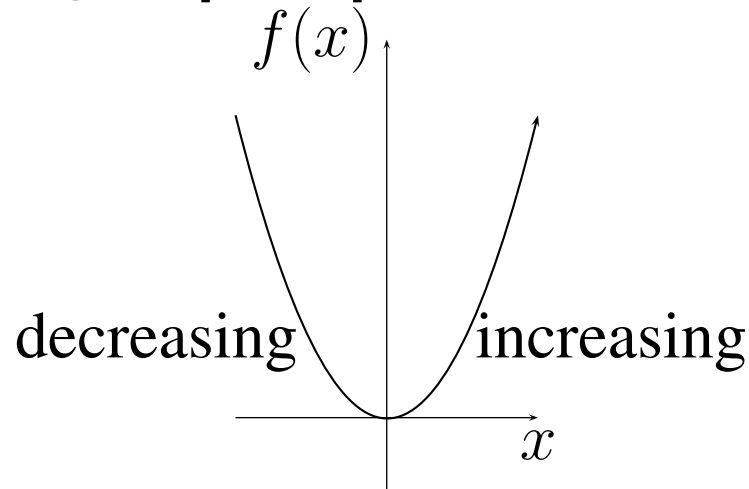
- f is increasing on I if for every two points x_1 and x_2 in I ,

$$x_1 < x_2 \quad \text{implies that} \quad f(x_1) < f(x_2).$$

- f is decreasing on I if for every two points x_1 and x_2 in I ,

$$x_1 < x_2 \quad \text{implies that} \quad f(x_1) > f(x_2).$$

Example. The function $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^2$ is increasing on $[0, 10)$ and decreasing on $[-5, 0]$.



Theorem. Let f be continuous on $[a, b]$ and differentiable on (a, b) .

- If $f'(x) > 0$ for all x in (a, b) then f is increasing on $[a, b]$.
- If $f'(x) < 0$ for all x in (a, b) then f is decreasing on $[a, b]$.
- If $f'(x) = 0$ for all x in (a, b) then f is constant on $[a, b]$.

The above theorem may be directly used to classify stationary points.







Example. Find and classify all stationary points of the function $f : \mathbb{R} \rightarrow \mathbb{R}$ whose derivative is given by

$$f'(x) = (x - 4)(x - 1)(x + 5)^2.$$

Stationary points: Set $f'(x) = 0$.

Result: $x = 4$, $x = 1$ and $x = -5$.

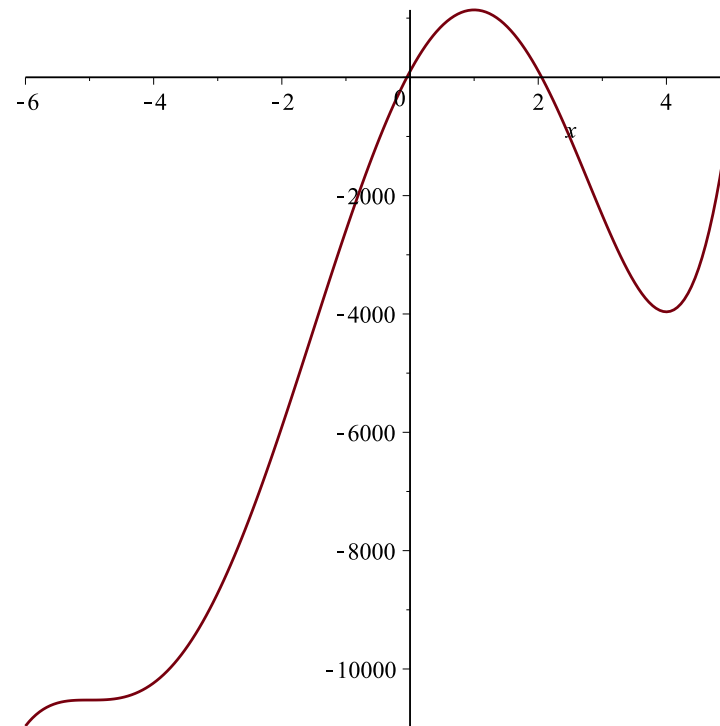
Classification: Investigate $f'(x)$ in a ‘small’ neighbourhood of any stationary point.

	-5^-	-5	-5^+	1^-	1	1^+	4^-	4	4^+
$x - 4$	—	—	—	—	—	—	—	0	+
$x - 1$	—	—	—	—	0	+	+	+	+
$(x + 5)^2$	+	0	+	+	+	+	+	+	+
$f'(x)$	+	0	+	+	0	—	—	0	+
Gradient									

Result:

- $x = 4$: local minimum point
- $x = 1$: local maximum point
- $x = -5$: horizontal point of inflexion

Look at the graph: `>plot(x5/5 + 5x4/4 - 7x3 - 85x2/2 + 100x, x = -6..5)`



The second derivative and applications

Another (potential) method for classifying the stationary points of a function f involves the **second derivative of f** , which is denoted by

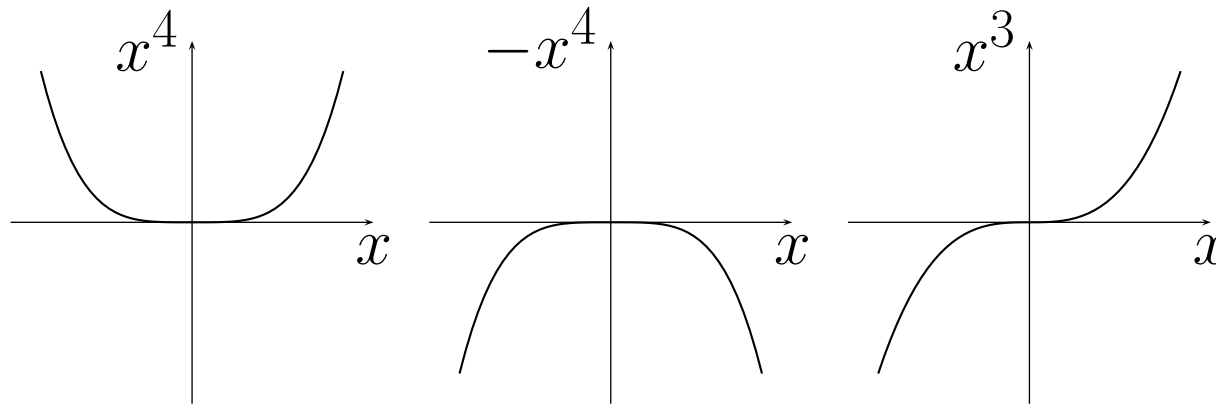
$$f'' \quad \text{or} \quad \frac{d^2y}{dx^2}, \quad \text{or} \quad y''$$

if we set $y = f(x)$.

Theorem (The second derivative test). Suppose that a function f is **twice differentiable** on (a, b) and that $c \in (a, b)$.

- If $f'(c) = 0$ and $f''(c) > 0$ then c is a local minimum point of f .
- If $f'(c) = 0$ and $f''(c) < 0$ then c is a local maximum point of f .

Remark. If $f'(c) = f''(c) = 0$, no conclusion may be drawn!



- If $f(x) = x^4$ then $f'(0) = f''(0) = 0$ and there is a local **minimum** at 0.
- If $f(x) = -x^4$ then $f'(0) = f''(0) = 0$ and there is a local **maximum** at 0.
- If $f(x) = x^3$ then $f'(0) = f''(0) = 0$ and there is a horizontal **point of inflexion** at 0.

Hence if $f'(c) = f''(c) = 0$ then it is best to classify the stationary point c by examining the sign of the derivative on either side of c !

Exercise. Find and classify the stationary points of the function $f : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f(x) = x^3 - 6x^2 + 9x - 5.$$

Critical points, maxima and minima

Question. How does one find global maxima or minima?

Definition. Suppose that f is defined on $[a, b]$. We say that a point c in $[a, b]$ is a **critical point** for f on $[a, b]$ if c satisfies one of the following properties:

- c is an endpoint a or b of the interval $[a, b]$,
- f is not differentiable at c ,
- f is differentiable at c and $f'(c) = 0$.

Theorem. Suppose that f is continuous on $[a, b]$. Then, f has a global maximum and global minimum on $[a, b]$. Moreover, the global maximum point and the global minimum point are both critical points for f on $[a, b]$.

Example. Suppose that the function $f : \mathbb{R} \rightarrow \mathbb{R}$ is given by the rule

$$f(x) = |x^2 - 3x - 4|.$$

Find the global maximum and global minimum values of f on the interval $[0, 5]$.

Counting zeros

Strategy. Using the information where the function is increasing and decreasing, about all maxima and minima, we may roughly sketch the graph and see how many times the graph intersects Ox .

Example. Determine the number of (real) zeros of

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f(x) = x^4 - x^3 - 3x^2 - 8x - 5$$

and give an approximate location for each zero.

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Antiderivatives

Remark. The velocity of a particle is the time-**derivative** of its position. Accordingly, the position of a particle may be regarded as an **antiderivative** of its velocity.

Definition. Suppose that f is continuous on an open interval I . A function F is said to be an **antiderivative** (or a **primitive**) of f on I if

$$F'(x) = f(x) \quad \text{for all } x \in I.$$

The process of finding an antiderivative of a function is called **antidifferentiation**.

Remark. It is evident that if F is an antiderivative of f then G defined by

$$G(x) = F(x) + C,$$

where C is an arbitrary constant, is also an antiderivative of f .

Theorem. Suppose that f is a continuous function on an open interval I and that F and G are two antiderivatives of f on I . Then, there exists a real constant C such that

$$G(x) = F(x) + C$$

for all x in I .

Some well-known antiderivatives are given below.

Function	Antiderivative
x^r , where r is rational and $r \neq -1$	$\frac{1}{r+1}x^{r+1} + C$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$
e^{ax}	$\frac{1}{a}e^{ax} + C$
$\frac{f'(x)}{f(x)}$	$\ln f(x) + C$

L'Hôpital's rule

Question. What is the limit of the 'indeterminate expression'

$$\lim_{x \rightarrow 0} \frac{x \sin x}{1 - \cos x} ?$$

Theorem (l'Hôpital's rule). Suppose that f and g are both differentiable functions in a neighbourhood of some $a \in \mathbb{R}$ and that either one of the two following conditions hold:

- $f(x) \rightarrow 0$ and $g(x) \rightarrow 0$ as $x \rightarrow a$
- $f(x) \rightarrow \infty$ and $g(x) \rightarrow \infty$ as $x \rightarrow a$.

If

$$\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

exists then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}.$$

Remark. The theorem also holds for

- limits as $x \rightarrow \infty$ or $x \rightarrow -\infty$
- one-sided limits (as $x \rightarrow a^+$ or $x \rightarrow a^-$).

L'Hôpital's rule is proved using the mean value theorem!

Example. Determine the limit

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x}.$$

Remark. l'Hôpital's rule may be applied iteratively.

Example.

$$\lim_{x \rightarrow 0} \frac{x - \sin x}{x^2} = \frac{1}{2} \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = \dots \text{ see above.}$$

Remark. It is important that the limit exists after a finite number of applications of l'Hôpital's rule!

Exercise. What is the limit

$$\lim_{x \rightarrow \infty} \frac{2x - \sin x}{3x + \sin x}$$

and why can l'Hôpital's rule **not** be applied?

Example. Find $\lim_{x \rightarrow 0^+} x \ln x$.

Example. Find $\lim_{x \rightarrow 0} \frac{\tan x - \sin x}{x^3}$.