THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Semester 1 2012

MATH1131
MATHEMATICS 1A

(1) TIME ALLOWED – 2 hours

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

(8) A SHORT TABLE OF INTEGRALS WILL BE SUPPLIED

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

Use a separate book clearly marked Question 1

1. i) Let $u = 3 + 2i$ and $w = 1 - 5i$.

a) Find $u - 2w$ in **Cartesian form.**

b) Find u/w in **Cartesian form.**

ii) Let $z = \sqrt{3} - i$.

a) Calculate $|z|$ and $\text{Arg}(z)$.

b) Express z in polar form.

c) Hence, or otherwise, express $z^{10} + (\overline{z})^{10}$ in **Cartesian form.**

iii) Evaluate the determinant

$$\begin{vmatrix} 1 & -1 & 4 \\ 0 & 2 & 7 \\ 0 & 3 & 1 \end{vmatrix}.$$

iv) a) Evaluate $\displaystyle\lim_{x\to\infty} \frac{3x^2 + \sin(2x^2)}{x^2}$.

b) Evaluate $\displaystyle\lim_{x\to 0} \frac{3x^2 + \sin(2x^2)}{x^2}$.

v) Consider the curve in the plane defined by

$$x^2 - 5x\sin y + y^2 = 4.$$

Find the equation of the tangent line to this curve at the point $(2, 0)$.

vi) Let $p(x) = x^5 + 5x + 7$.

a) Explain why p has at least one real root.

b) Prove that p has exactly one real root.

Use a separate book clearly marked Question 2

2. i) Use De Moivre's Theorem to show that

$$\cos(3\theta) = 4\cos^3\theta - 3\cos\theta.$$

ii) Find the intersection of the line $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\lambda \in \mathbb{R}$, with the plane $5x - 2y + z = 17$.

iii) Consider the following MAPLE session:

```
> with(LinearAlgebra):
> A:=<<0,0,1>|<0,1,0>|<-1,0,0>>;
```

$$A := \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

```
> A^2;
```

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

```
> A^3;
```

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

```
> A^4;
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Using the Maple session above, find the **inverse** of A^{2001}.

iv) The points C and D have position vectors

$$\mathbf{c} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{d} = \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix}.$$

a) Find the cross product $\mathbf{c} \times \mathbf{d}$.

b) Hence, or otherwise, find the area of the parallelogram with adjacent sides OC and OD, where O is the origin.

v) Evaluate the indefinite integral

$$\int x^4 \ln x \, dx.$$

vi) Consider the three functions:

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = \frac{x^2}{1+x^2},$$

$$g : (0,3) \to \mathbb{R}, \qquad g(x) = (x-1)^2,$$

$$h : [1,5] \to \mathbb{R}, \qquad h(x) = \sqrt{1 + \ln x + \sin x \cos x}.$$

Only **one** of these functions has a maximum value (on its given domain). Which one is it? Give reasons for your answer.

vii) Sketch the polar curve $r = 2 - 2\cos\theta$. You should show any lines of symmetry, and clearly identify where the curve intersects the x and y axes.

viii) Suppose that $y = x^{\sin x}$. Find $\dfrac{dy}{dx}$.

Use a separate book clearly marked Question 3

3. i) The points A and B in $\mathbb{R}^3$ have position vectors

$$\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}.$$

a) Find a parametric vector equation of the line l passing through A and B.

b) By evaluating an appropriate dot product, show that the line l from part (a) is perpendicular to the line $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix} + \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} t;\ t \in \mathbb{R}.$

ii) Let $P = \begin{pmatrix} 1 & 2 & 1 \\ 3 & -1 & 4 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 5 & 0 \end{pmatrix}$.

a) Evaluate PQ^T.

b) What is the size of P^TQ?

c) Does the matrix product PQ exist? Explain your answer.

iii) A system of three equations in three unknowns x, y and z has been reduced to the following echelon form

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & \alpha^2 - 9 & \alpha - 3 \end{array}\right).$$

a) For which value of α will the system have no solution?

b) For which value of α will the system have infinitely many solutions?

c) For the value of α determined in part (b), find the general solution.

iv) The number of \$10, \$20, and \$50 notes in the cash register at Bill's Burger Barn is x, y and z respectively. The total value of all the notes in the register is \$1020. There are 44 notes in total. Also, the number of \$10 notes is equal to the sum of the number of \$20 notes and the number of \$50 notes.

a) Explain why $x + 2y + 5z = 102$.

b) By setting up two further equations and solving the system of three equations in three unknowns x, y and z, determine how many of each type of note is in the cash register.

v) The (non-zero) point Q has position vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$. The vector $\overrightarrow{OQ}$ makes angles α, β and γ respectively with the X, Y and Z axes.

a) By considering the vector $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ show that

$$a = \sqrt{a^2 + b^2 + c^2}\cos\alpha.$$

b) Deduce that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

c) If the angles α and β are complementary, that is, their sum is $90°$, what can be said about the vector $\overrightarrow{OQ}$?

Use a separate book clearly marked Question 4

4. i) Find a quadratic function $q(x) = x^2 + bx + c$ such that the function

$$h(x) = \begin{cases} e^{3x}, & \text{if } x \leq 0 \\ q(x), & \text{if } x > 0 \end{cases}$$

is differentiable at $x = 0$.

ii) Each of the following calculations is expressed in MAPLE. Write each in normal mathematical notation and **evaluate**.

a) `arcsin(sin(7*Pi/3));`

b) `diff(int(exp(t^2),t=0..x^2),x);`

iii) Prove that $\displaystyle\lim_{x\to\infty} \frac{x^2-2}{x^2+3} = 1$ as follows:
Given any real number $\epsilon > 0$, find a real number M (expressed in terms of ϵ), such that if $x > M$ then $\left|\dfrac{x^2-2}{x^2+3} - 1\right| < \epsilon$.

iv) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = x^3 + \sinh x + 1.$$

a) Explain why f has a differentiable inverse g.

b) What is the domain of g?

c) Evaluate $g'(1)$.

v) A chemical process produces Factor X, which flows into a 50 litre tank which is initially empty.
At time $t \geq 0$, Factor X flows into the tank at the rate of $\dfrac{100}{10+t^2}$ litres per hour.
Will the tank eventually overflow? Explain your answer.