THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Semester 1 2013

MATH1131
MATHEMATICS 1A

(1) TIME ALLOWED – 2 hours

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

(8) A SHORT TABLE OF INTEGRALS IS APPENDED TO THE PAPER

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

Use a separate book clearly marked Question 1

1. i) Evaluate the following limits:

a)

$$\lim_{x\to\infty} \frac{10x^2 + 3x + \sin x}{5x^2 + 3x - 2},$$

b)

$$\lim_{x\to 0} \frac{e^{3x} - 1}{\sin(7x)}.$$

ii) A function $f : [0, 5] \to \mathbb{R}$ has the following properties:

- $\lim_{x\to 2^+} f(x) = 3$,
- $\lim_{x\to 2^-} f(x) = 1$,
- $f(2) = 4$.

Draw a possible sketch of the graph of f. (You do not need to give a formula for your function.)

iii) a) State the definitions of $\cosh x$ and $\sinh x$ in terms of the exponential function.

b) Prove that $\cosh^2 x - \sinh^2 x = 1$.

iv) Let $z = 5 + 5i$ and $w = 2 + i$.

a) Find $2z + 3\overline{w}$.

b) Find $z(w - 1)$.

c) Find z/w.

v) Suppose that $(x + iy)(3 + 4i) = 13 + 9i$, where $x, y \in \mathbb{R}$.
Find the value of x and the value of y.

vi) Let the set S in the complex plane be defined by

$$S = \left\{ z \in \mathbb{C} \,:\, |z| \le 3 \text{ and } 0 \le \mathrm{Im}(z) \le 3 \right\}.$$

a) Sketch the set S on a labelled Argand diagram.

b) By considering your sketch, or otherwise, find the area of the region defined by S.

vii) Consider the following MAPLE session.

```
> with(LinearAlgebra):
> A:=<<0,1,-1>|<1,0,1>|<-1,1,0>>;
```

$$A := \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

```
> B:=A^2;
```

$$B := \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

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> C:=A^3;
```

$$C := \begin{bmatrix} -2 & 3 & -3 \\ 3 & -2 & 3 \\ -3 & 3 & -2 \end{bmatrix}$$

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> F:=3*A-C;
```

$$F := \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Without carrying out any row reduction, use the above Maple session to find the inverse of the matrix A.

Use a separate book clearly marked Question 2

2. i) A function h is defined by

$$h(x) = \begin{cases} ax^2 + 3x, & \text{if } x \geq 1 \\ 2x + d, & \text{if } x < 1. \end{cases}$$

Given that h is differentiable at $x = 1$, find the values of a and d.

ii) Evaluate

$$\int_0^{\ln 2} 9xe^{3x}\, dx.$$

iii) Find the equation of the tangent at the origin to the curve implicitly defined by

$$e^x + \sin y = xy + 1.$$

iv) Sketch the polar curve whose equation in polar coordinates is given by

$$r = 1 + \cos 2\theta.$$

v) Let $z = \sqrt{2} - \sqrt{2}i$.

a) Find $|z|$.

b) Find $\text{Arg}(z)$.

c) Use the polar form of z to evaluate z^6. Express your answer in **Cartesian form**.

vi) Let $A = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}$.

a) Evaluate AB or explain why this product does not exist.

b) Evaluate AB^T or explain why this product does not exist.

vii) a) Find, in polar form, all solutions to the equation $z^5 = -1$, where $z \in \mathbb{C}$.

b) Hence, or otherwise, express $z^5 + 1$ as a product of real linear and real quadratic factors.

Use a separate book clearly marked Question 3

3. i) Determine the point of intersection of the line

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad \text{for} \quad t \in \mathbb{R}.$$

and the plane $4x - 5y + 3z = 0$.

ii) Let $M = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & 1 \\ 0 & 2 & \alpha \end{pmatrix}$.

a) Evaluate the determinant of M.

b) Determine the value(s) of α for which M does **not** have an inverse.

c) Find the inverse of M when $\alpha = 1$.

iii) Let $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$.

a) Find the cross product $\mathbf{u} \times \mathbf{v}$.

b) Hence find the **Cartesian** equation of the plane parallel to $\mathbf{u}$ and $\mathbf{v}$ and passing through the point $\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$.

iv) Let $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ 4 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 0 \\ 3 \\ -3 \\ \beta \end{pmatrix}$ be two vectors in $\mathbb{R}^4$.

a) Find the value of β so that the vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

b) For the value $\beta = 1$, find the projection, $\text{proj}_{\mathbf{u}}(\mathbf{v})$, of $\mathbf{v}$ onto $\mathbf{u}$.

v) Let A, B and D be three points on some circle with centre C in $\mathbb{R}^2$ with position vectors

$$\mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad \text{and} \quad \mathbf{d} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}.$$

a) Let M be the midpoint of the line joining A and B. Find the position vector $\mathbf{m}$ of the point M.

b) Find a non-zero vector $\mathbf{u}$ that is perpendicular to $\overrightarrow{AB}$.

c) Hence or otherwise, find the parametric vector equation for the line whose points are equidistant from A and B.

d) Given that the parametric vector equation for the line whose points are equidistant from B and D is $\mathbf{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\mu \in \mathbb{R}$, find the centre C of the circle.

Please see over ...

Use a separate book clearly marked Question 4

4. i) Find $\displaystyle\int \frac{\cos(\ln(x))}{x}\,dx.$

ii) Use the Pinching theorem to evaluate

$$\lim_{x\to\infty} e^{-x}\sin(x).$$

iii) Show that the improper integral $\displaystyle\int_0^\infty \frac{dx}{x^2+e^x}$ converges.

iv) The following calculation is expressed in MAPLE

```
> F:=diff(int(sin(sqrt(t)),t=0..x^2),x);
```

a) Write the calculation using standard mathematical notation.

b) Evaluate F.

v) Let $p(x) = x^3 + 4x - 7$.

a) Use the Intermediate Value theorem to show that p has at least one real root in the interval $[1, 2]$.

b) Show that p has **exactly** one real root in the interval $[1, 2]$.

c) Let g be the inverse of p and α be the unique root of p, whose existence is guaranteed in part b). Express $g'(0)$ in terms of α.

vi) Use the Mean Value Theorem to prove that, for $x > 0$,

$$\ln(1+x) > \frac{x}{1+x}.$$

BASIC INTEGRALS

$$\int \frac{1}{x}\,dx = \ln|x| + C = \ln|kx|, \qquad C = \ln k$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

$$\int a^x\,dx = \frac{1}{\ln a}a^x + C, \qquad a \neq 1$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$$

$$\int \operatorname{cosec}^2 ax\,dx = -\frac{1}{a}\cot ax + C$$

$$\int \tan ax\,dx = \frac{1}{a}\ln|\sec ax| + C$$

$$\int \cot ax\,dx = \frac{1}{a}\ln|\sin ax| + C$$

$$\int \sec ax\,dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C$$

$$\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C$$

$$\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C$$

$$\int \operatorname{sech}^2 ax\,dx = \frac{1}{a}\tanh ax + C$$

$$\int \operatorname{cosech}^2 ax\,dx = -\frac{1}{a}\coth ax + C$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$\begin{aligned}\int \frac{dx}{a^2-x^2} &= \frac{1}{a}\tanh^{-1}\frac{x}{a} + C, && |x| < a\\ &= \tfrac{1}{a}\coth^{-1}\tfrac{x}{a} + C, && |x| > a > 0\\ &= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C, && x^2 \neq a^2\end{aligned}$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} + C, \qquad x \geqslant a > 0$$