Curve sketching.

You all know how to sketch the graph of $y = x^2 - 4x$ or $y = \frac{1}{1-x^2}$.
In this section we will look at

- Additional information you can put into a sketch.
- Sketching curves that don't come in Cartesian form
- implicitly defined curves, such as $x^2 + \frac{y^2}{4} = 1$.
- given by a parameter, such as

$$x(t) = \sin t \cos t \ln |t|, \quad y(t) = \sqrt{|t|}\,\cos t$$
$$t \in [-1, 1], \quad t \neq 0.$$

- in polar coordinates, such as

$$x = r\cos\theta, \quad y = r\sin\theta, \quad \text{where} \quad r = \cos 4\theta.$$

Curves defined by a Cartesian equation $y = f(x)$

Many high school students always start curve sktching by differentiating f.

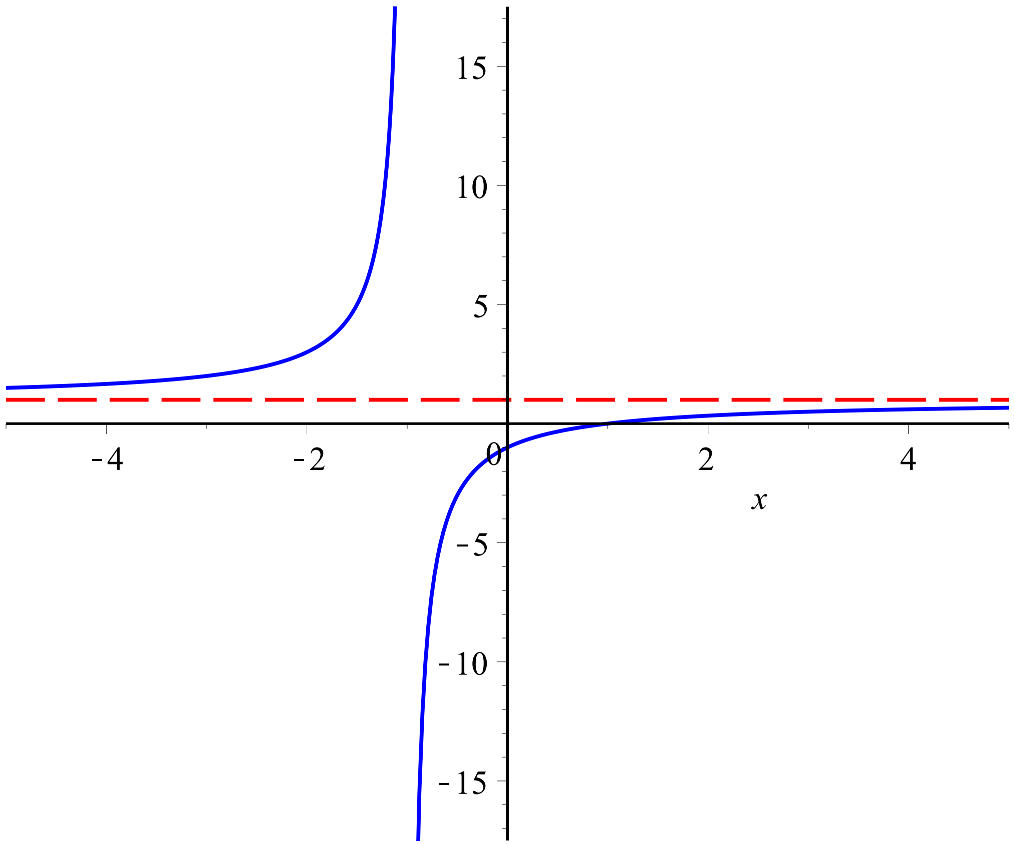
Resist this urge and instead use the following checklist:

- The domain of f.
- How does $f(x)$ behave as $x \to \infty$ or $x \to -\infty$?
- Are there any asymptotes?
- Does f have any symmetries?
- What are the x and y-intercepts?
- Where should you find stationary points?

Some of these may be irrelevant, or very hard, for some functions, but you should think about them all before you proceed too far.

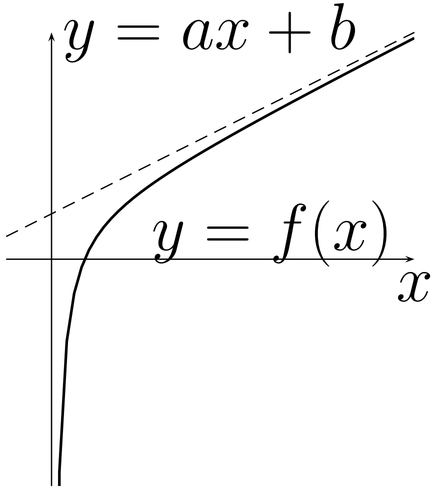
Asymptotic behaviour

It is usually pretty easy to decide whether $f(x)$ approaches a limit as $x \to \pm\infty$.



A graph of the rational function $f(x) = \dfrac{x-1}{x+1}$ and (where applicable), its asymptotes.

A particular type of asymptotic behaviour is where $f(x)$ gets closer and closer to a **straight line**.



Definition. Suppose that $a \neq 0$ and b are real numbers. We say that a straight line given by the equation

$$y = ax + b$$

is an oblique asymptote for a function f if

$$\lim_{x \to \infty} \big(f(x) - (ax + b)\big) = 0$$

or

$$\lim_{x \to -\infty} \big(f(x) - (ax + b)\big) = 0.$$

Remark. If f is a rational function with

$$f(x) = \frac{p(x)}{q(x)}, \quad \deg(p) = \deg(q) + 1$$

then the oblique asymptotes of f may be determined by polynomial division.

Example. Find the oblique asymptotes to the function f defined by

$$f(x) = \frac{x(x-1)}{x-2}, \text{ for all } \quad x \neq 2.$$

Dividing polynomials, we have
$f(x) = \frac{x(x-1)}{x-2} = x + 1 + \frac{2}{x-2}$.
Thus, the line $y = x + 1$ is an oblique asymptote as $x \to \infty$ and as $x \to -\infty$.

Example. Find the oblique asymptotes for

$$f(x) = \frac{(x-2)|x| + \sin x}{x}.$$

For x large and positive

$f(x) = \frac{(x-2)x+\sin x}{x} = (x-2) + \frac{\sin x}{x}$

so $\lim_{x\to\infty}\big(f(x) - (x-2)\big) = \lim_{x\to\infty} \frac{\sin x}{x} = 0.$

Thus $y = x - 2$ is an oblique asymptote as $x \to \infty$.

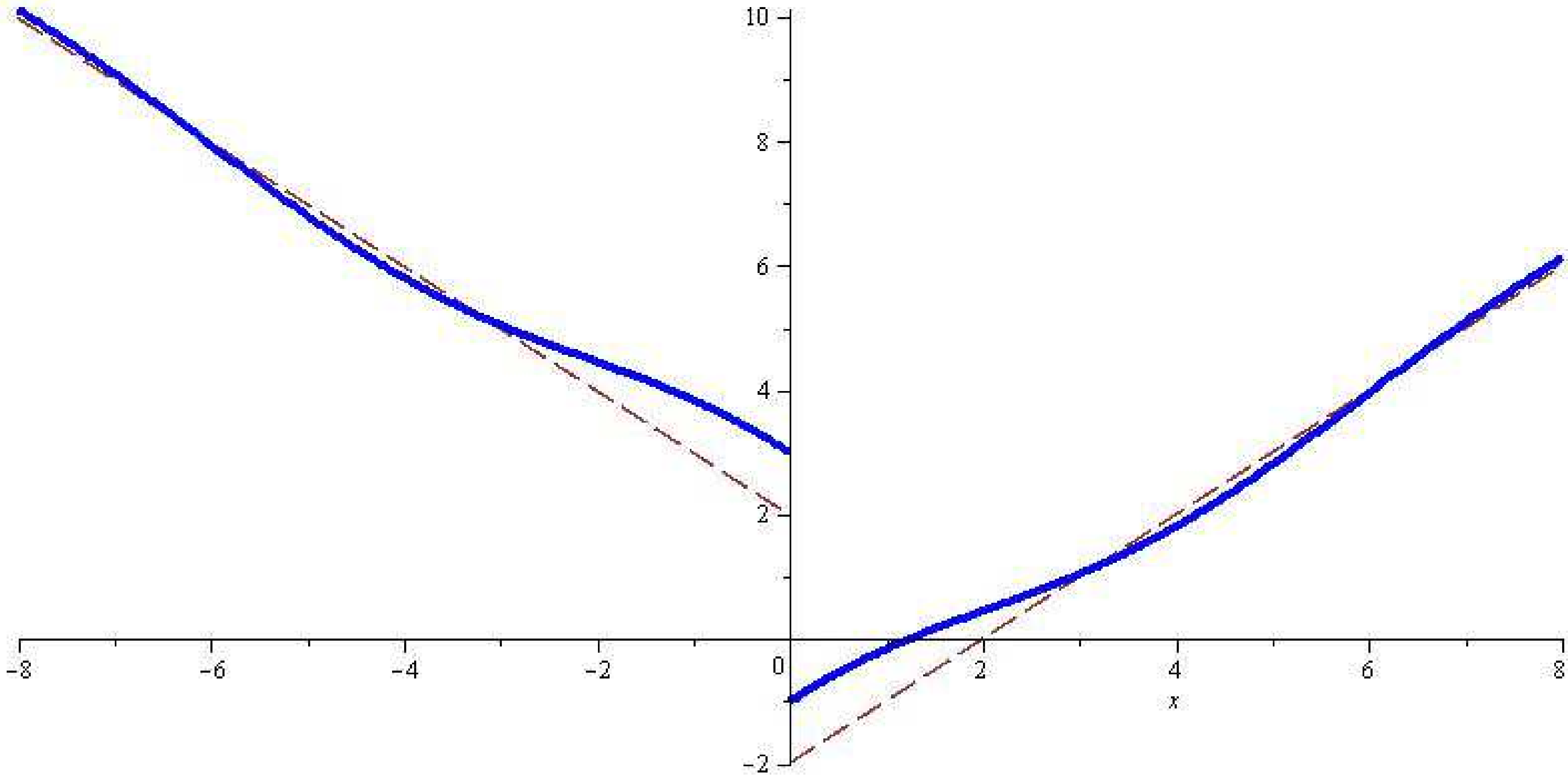
For x large and negative,

$f(x) = \frac{-(x-2)x+\sin x}{x} = (2-x) + \frac{\sin x}{x}$

so the oblique asymptote as $x \to -\infty$ is $y = 2 - x$.

In an exam they will tell you if you are meant to find the oblique asymptotes.

See next slide for the graph of $f(x) = \dfrac{(x-2)|x| + \sin x}{x}$



10
8
6
4
2
-2
-8
-6
-4
-2
0
2
4
6
8
x

Symmetries

Identify any symmetries:

- f is even if $f(-x) = f(x)$ for all $x \in \mathrm{Dom}(f)$.
- f is odd if $f(-x) = -f(x)$ for all $x \in \mathrm{Dom}(f)$.
- f is periodic of period T if $f(x+T) = f(x)$ for all $x \in \mathrm{Dom}(f)$.

Example. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \frac{|\sin x|}{2 + \cos(2x)}.$$

- f is even since $f(-x) = f(x)$ for all $x \in \mathbb{R}$.
- f is of period π since $f(x+\pi) = f(x)$ for all $x \in \mathbb{R}$.

Curve Sketching Example.

Sketch $f(x) = \dfrac{(x-1)|x| + x}{x-1}$.

Go through the checklist!

Domain. There is a problem at $x = 1$. So the domain is $\mathbb{R} \setminus \{1\}$.

Asymptotics. The vertical asymptote is the line $x = 1$. Note that $f(x) \to \infty$ as $x \to 1^+$ and $f(x) \to -\infty$ as $x \to 1^-$.

To find oblique asymptotes, it is useful to rewrite f as

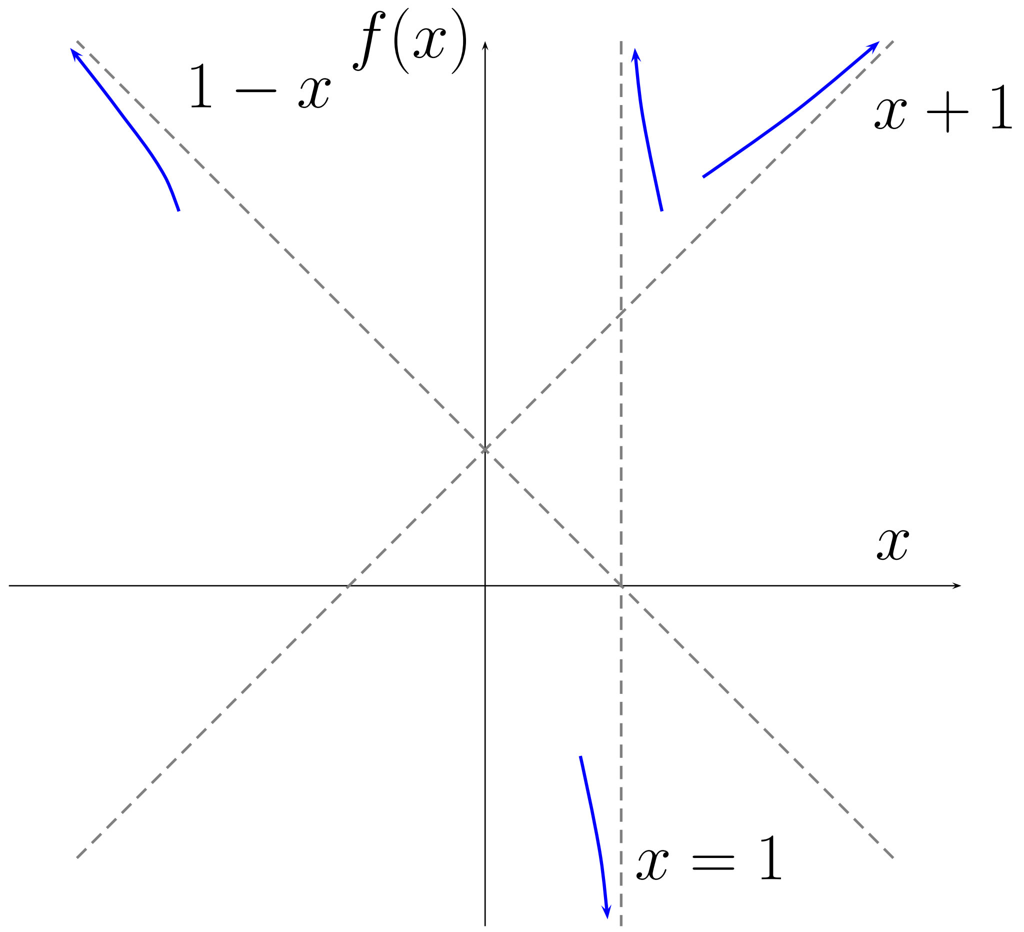
$$f(x) = \begin{cases} x + \frac{x}{x-1}, & x \geq 0, \\ -x + \frac{x}{x-1}. & x < 0 \end{cases} = \begin{cases} x + 1 + \frac{1}{x-1}, & x \geq 0, \\ -x + 1 + \frac{1}{x-1}. & x < 0. \end{cases}$$

It follows that

$f(x) \sim x + 1$ for large positive x and remains above the line $x + 1$.

$f(x) \sim 1 - x$ for large negative x and remains below the line $1 - x$.

Let us illustrate the information we have:



Symmetries.

$$\begin{array}{ll} f(-x) = f(x)?, & \text{No,} \\ f(-x) = -f(x)? & \text{No,} \\ \text{Periodic?} & \text{No.} \end{array}$$

Intercepts. When $x = 0$ and $f(x) = 0$?

Clearly, $f(0) = 0$. Thus, the y-intercept is 0.

If $f(x) = 0$, then $(x-1)|x| + x = 0$.
Two cases:
$x \geq 0$: $\quad (x-1)x + x = 0 \iff x = 0.$

$x < 0$: $\quad (x-1)(-x) + x = 0 \iff -x^2 + 2x = 0$ no solution.

Thus, the x-intercept is 0.

Stationary points.

Using the expanded form for f we see that

$$f'(x) = \begin{cases} 1 + \frac{-1}{(x-1)^2}, & x > 0, \\ -1 + \frac{-1}{(x-1)^2}. & x < 0. \end{cases}$$

We also have that f is not differentiable at 0 (prove it!).

Clearly $f'(x) < 0$ for all $x < 0$, f is decreasing there and there are no stationary points there.

For $x > 0$, $f'(x) = 0 \iff (x-1)^2 = 1 \iff x - 1 = \pm 1$.

Thus there is a stationary point at $x = 2$.

	$(-\infty, 0)$	$(0, 1)$	$(1, 2)$	$(2, \infty)$
$f'(x)$	$-$	$-$	$-$	$+$
$f(x)$	$\searrow$	$\searrow$	$\searrow$	$\nearrow$

Thus, the stationary point at 2 must be a local min.

What happens at 0?

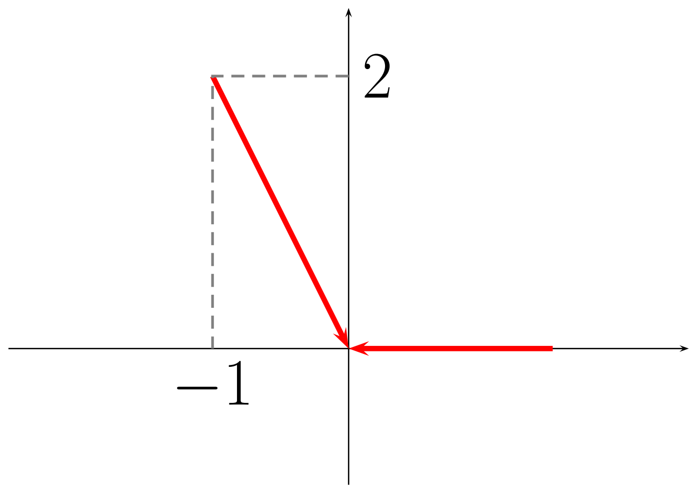
Recall that $f'(x_0)$ is a gradient (slope) at the point x_0.

- If $x \to 0^+$, then $f'(x) \to 0$.

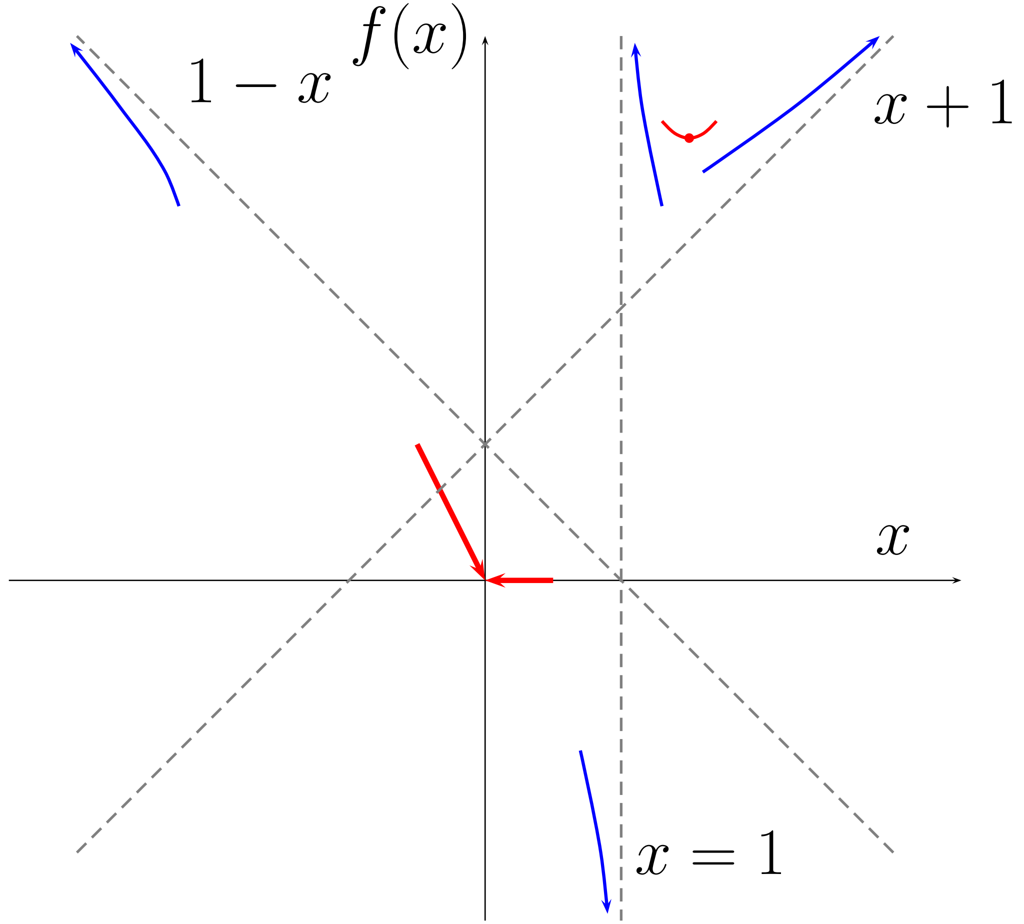
  This means that as we go to 0 from the right, the curve comes to 0 with the gradient 0.

- If $x \to 0^-$, then $f'(x) \to -2$.

  This means that as we go to 0 from the left, the curve comes to 0 with the gradient -2.

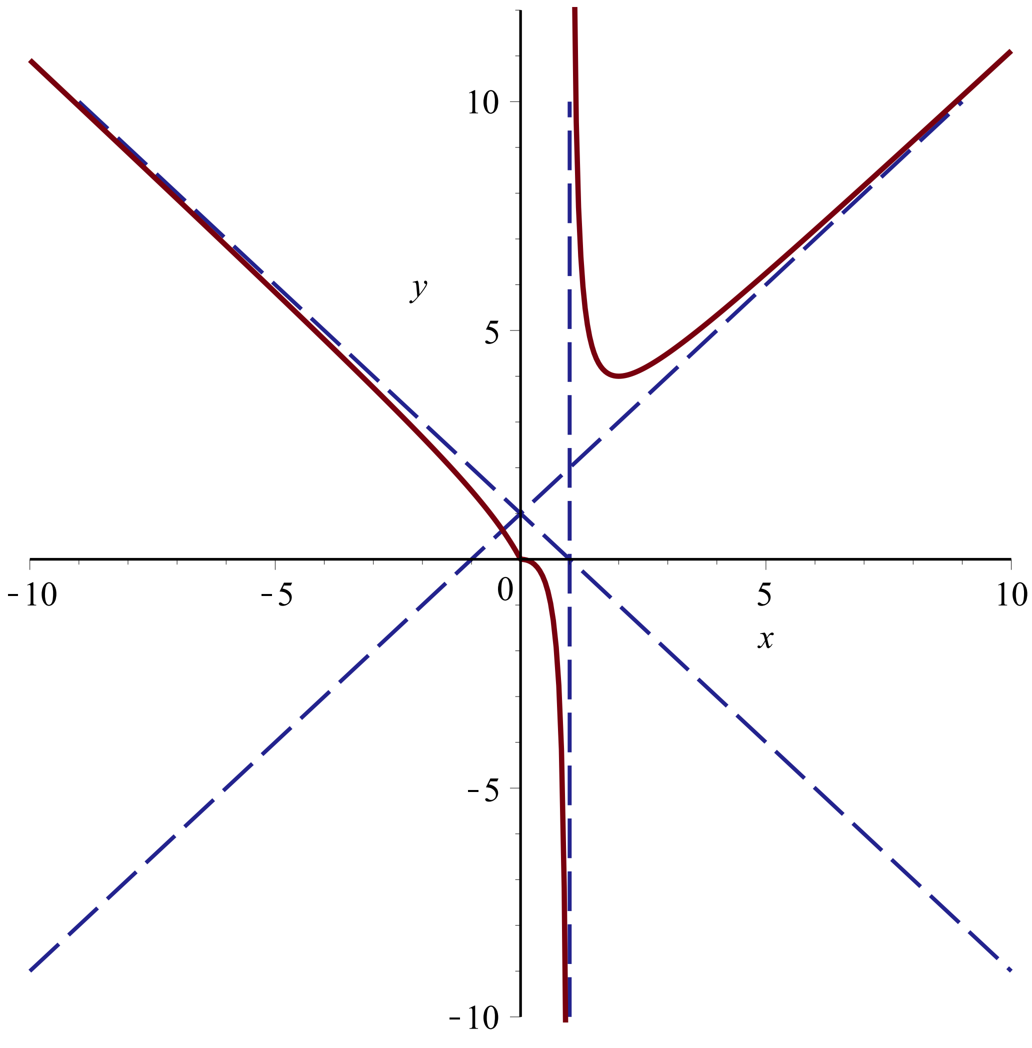


Let's add this information to the graph:



Now we are ready to draw the graph (I am using Maple):

```
> restart; with(Student[Calculus1]); with(plots);
> f := ((x-1)*abs(x)+x)/(x-1); P := implicitplot(Asymptotes(f, x), x = -10 .. 10, y = -10 .. 10, color = navy,
thickness = 1, linestyle = dash); Q := plot(f, x = -10 .. 10, discont = true, thickness = 2); display(P, Q)
```



Parametrically defined curves

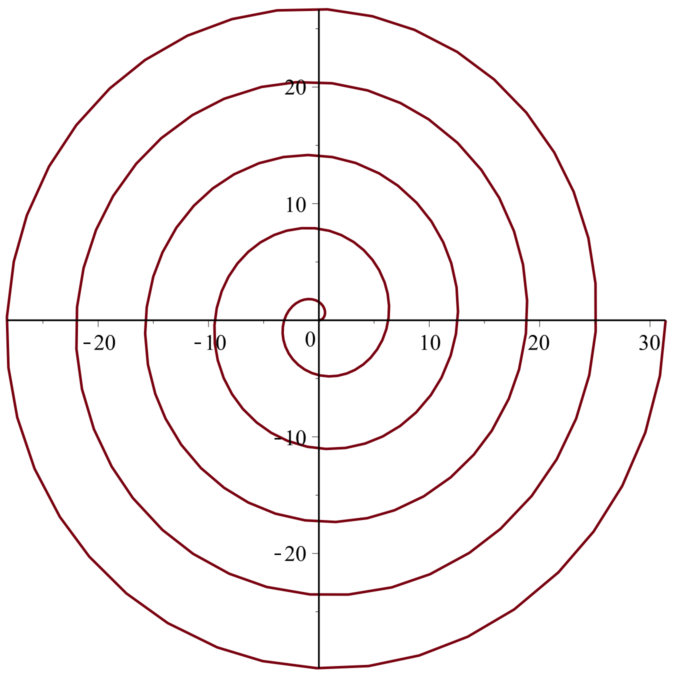
Parametrically defined curves in a plane are given by

$$\big(x(t), y(t)\big), \quad t \in A,$$

where t is the parameter and A is a given domain.

Parametrically defined curve may be interpreted as a path of the motion of a particle on a plane. At every moment t you are given a position $(x(t), y(t))$ of the particle.

For example, $\gamma(t) = (t\cos(t), t\sin(t))$, $t \in [0, 10\pi]$.



A curve in Cartesian form $y = f(x)$ can be always written parametrically $(x(t), y(t)) = (t, f(t))$.

Given a curve $(x(t), y(t))$ sometimes you can see a relationship between x and y.

- Sometimes you can write y as a function of x or vice versa. For example, if
$$(x(t), y(t)) = (t + 1, t^2 - 1)$$
then $y = (x - 1)^2 - 1$ which is obviously a parabola.
- Or, if
$$(x(t), y(t)) = (3 \cos t, 2 \sin t)$$
then
$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$
which is an ellipse.

But often you can't do this!

Sketching parametrically defined curves.

Example. Sketch the curve

$$\gamma(t) = \big(x(t), y(t)\big) = (t^2 - 1, t^3 - 1), \;\; t \in \mathbb{R}.$$

Possible values of $x(t)$ and $y(t)$:
As $t \in \mathbb{R}$, we see that

$$x(t) \in [-1, \infty), \quad y(t) \in (-\infty, \infty.)$$

Intercepts:

$x(t) = 0$ if and only if $t = \pm 1$ so that we obtain the points

$$\gamma(-1) = (0, -2), \quad \gamma(1) = (0, 0).$$

In addition, $y(t) = 0$ if and only if $t = 1$.

Vector derivatives:

Consider the 'tangent vector'

$$\gamma'(t) = \big(x'(t), y'(t)\big).$$

If one interprets $\gamma(t)$ as the position of a particle at the time t then $\gamma'(t)$ is the velocity of the particle at that time and $\frac{\gamma'(t)}{|\gamma'(t)|}$ is a unit vector (normalized tangent vector), which shows the direction of the motion.

We will justify later that the slope of a parametrised curve $\gamma(t) = \big(x(t), y(t)\big)$ is given by

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$$

at all points with $x'(t) \neq 0$.

Here,

$$\gamma'(t) = (2t, 3t^2), \quad \frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{3}{2}t$$

so that

$$\gamma'(t) \text{ points } \nwarrow \quad \text{ for } \quad t < 0$$

and

$$\gamma'(t) \text{ points } \nearrow \quad \text{ for } \quad t > 0.$$

Note that $\gamma'(0) = 0$ so that the 'particle stops' at $t = 0$!

In fact, there exists a cusp at $\gamma(0) = (-1, -1)$ since the normalised tangent vector has the property

$$\lim_{t \to 0^\pm} \frac{\gamma'(t)}{|\gamma'(t)|} = \lim_{t \to 0^\pm} \frac{(2t, 3t^2)}{\sqrt{4t^2 + 9t^4}} = (\pm 1, 0).$$

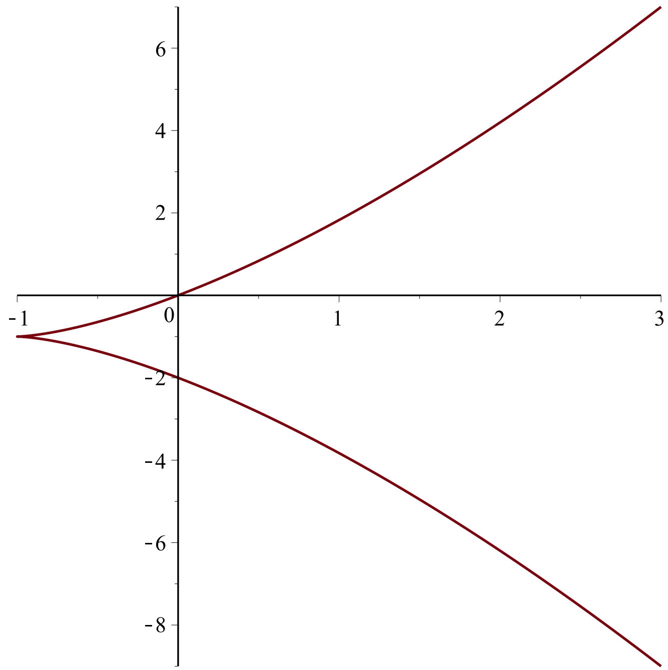
Thus, the curve does not have a 'proper' tangent vector at the point $\gamma(0) = (-1, -1)$!

Limiting behaviour:

As $t \to \pm\infty$,

$$\lim_{t\to\pm\infty} \frac{\gamma'(t)}{|\gamma'(t)|} = \lim_{t\to\pm\infty} \frac{(2t, 3t^2)}{\sqrt{4t^2 + 9t^4}} = (0, 1).$$

Accordingly, the normalised tangent vector becomes 'vertical at infinity'.



Justification of slope formula: By definition,

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0} \frac{y(t+h)-y(t)}{x(t+h)-x(t)} \\
&= \lim_{h\to 0} \left[\left(\frac{y(t+h)-y(t)}{h} \right) \Big/ \left(\frac{x(t+h)-x(t)}{h} \right) \right] \\
&= \left(\lim_{h\to 0} \frac{y(t+h)-y(t)}{h} \right) \Big/ \left(\lim_{h\to 0} \frac{x(t+h)-x(t)}{h} \right) \\
&= \frac{y'(t)}{x'(t)}
\end{aligned}$$

provided that $x(t)$ and $y(t)$ are differentiable and $x'(t) \neq 0$.

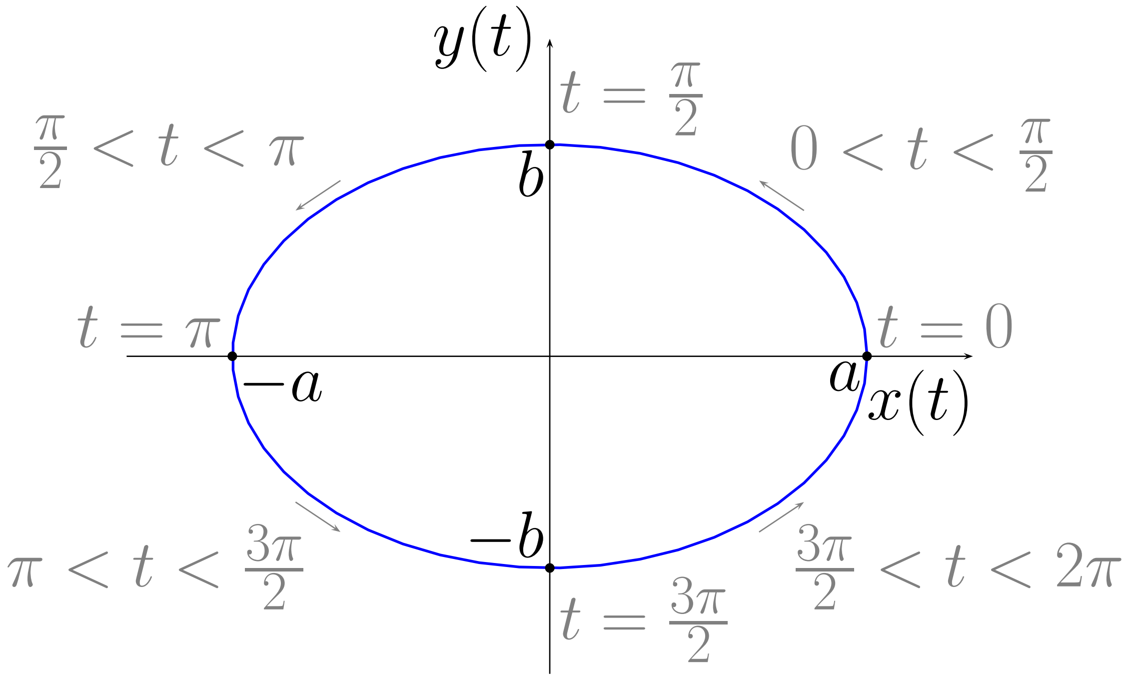
Parametrisation of conic sections

The ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

with semi-axes a and b admits the parametrisation

$$x(t) = a\cos t, \quad y(t) = b\sin t, \qquad 0 \leq t < 2\pi.$$



Each point (x, y) of the ellipse corresponds to a unique $t \in [0, 2\pi)$.

The table below lists some commonly used parametrisations of conic sections.

Conic section	Cartesian equation	Parametric equation
Parabola	$4ay = x^2$	$x(t) = 2at$ $y(t) = at^2$
Circle	$x^2 + y^2 = a^2$	$x(t) = a\cos t$ $y(t) = a\sin t$
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$x(t) = a\cos t$ $y(t) = b\sin t$
Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$x(t) = a\sec t$ $y(t) = b\tan t$

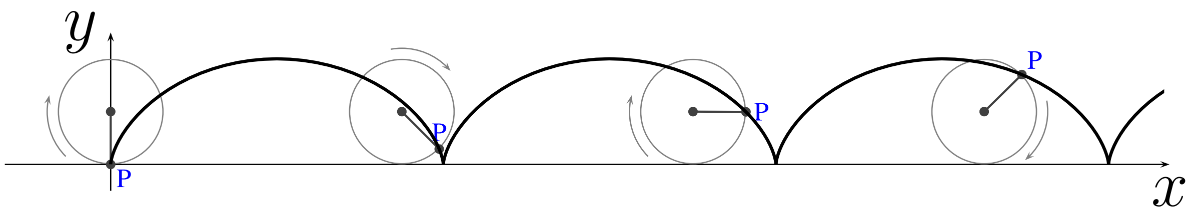
The cycloid and curve of fastest descent

Question. Find the shape of a curve that a particle should follow if it is to ‘slide’ without friction in the minimum time from a higher point A to a lower point B (not directly beneath it) under the influence of gravity.

Curve of fastest descent.

Such a curve is known as a curve of fastest descent or a brachistochrone (which, in Greek, means ‘shortest time’).

Answer. The curve of fastest descent from A to B is the unique arc of an (inverted) cycloid whose tangent at A is vertical.



The cycloid.

Description. A circle of radius r rolls along the x-axis, starting from the origin as shown above. Show that the locus $(x(t), y(t))$ of the point P on the edge of the circle which satisfies $(x(0), y(0)) = (0, 0)$ is given by

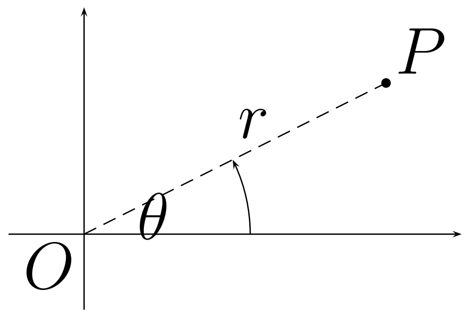
$$x(t) = r(t - \sin t),$$
$$y(t) = r(1 - \cos t),$$

where $t \geq 0$.

Curves defined by polar coordinates

Many problems in mathematics are easier to solve if one chooses a suitable coordinate system. Usually we use Cartesian coordinates. Here, we focus on polar coordinates.

Every point P in a plane can be specified by (r, θ), where
$r \geq 0$ is the distance of P from the origin and
$0 \leq \theta < 2\pi$ is the angle (taken in the anticlockwise direction) between OP and the positive horizontal axis.



The pair (r, θ) is called polar coordinates of P.

Note. If P is the origin then $r = 0$ and θ is not defined.

Polar coordinates (r, θ) and Cartesian coordinates (x, y) of a point P are related by

$$x = r\cos\theta, \quad y = r\sin\theta$$

and

$$r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}$$

provided that $x \neq 0$.

Note. Finding Cartesian coordinates of a point P given in terms of polar coordinates is easy but care must be taken in the opposite case.

Example. Find the polar coordinates of the point P with Cartesian coordinates

$$P = (x, y) = (-3, \sqrt{3}).$$

$r = \sqrt{(-3)^2 + (\sqrt{3})^2} = \sqrt{12} = 2\sqrt{3}$

$\tan\theta = \frac{\sqrt{3}}{-3} = -\frac{1}{\sqrt{3}}$. Thus, $\theta = \pi - \pi/6 = 5\pi/6$ or $\theta = 2\pi - \pi/6$.

Since, $x < 0$ and $y > 0$, we choose $\theta = 5\pi/6$.

Basic sketches of polar curves.
Many curves can be described by equations of the form

$$r = f(\theta) \quad \text{or} \quad \theta = g(r)$$

so that we obtain the parametrically defined curves

$$\gamma(\theta) = (r\cos\theta, r\sin\theta) = \big(f(\theta)\cos\theta, f(\theta)\sin\theta\big)$$

or

$$\gamma(r) = (r\cos\theta, r\sin\theta) = \big(r\cos g(r), r\sin g(r)\big),$$

where θ or r plays the role of the parameter.

Remark. Polar forms of equations may be simpler or more involved compared to their Cartesian counterparts.

Examples. Find the polar forms of the

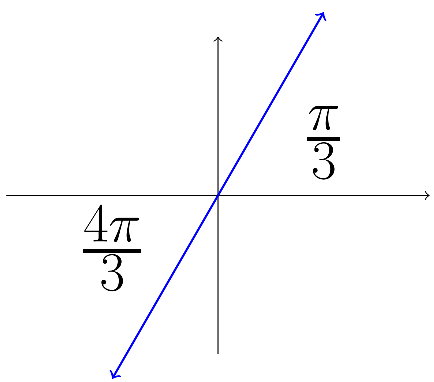
(a) straight line

$$y = \sqrt{3}\,x.$$

(b) circle

$$x^2 + y^2 = 4.$$

(a) becomes $r \sin\theta = \sqrt{3} r \cos\theta$, which is $\sin\theta = \sqrt{3}\cos\theta$ or $\tan\theta = \sqrt{3}$.
It follows that $\theta = \pi/3$ or $\theta = 4\pi/3$.



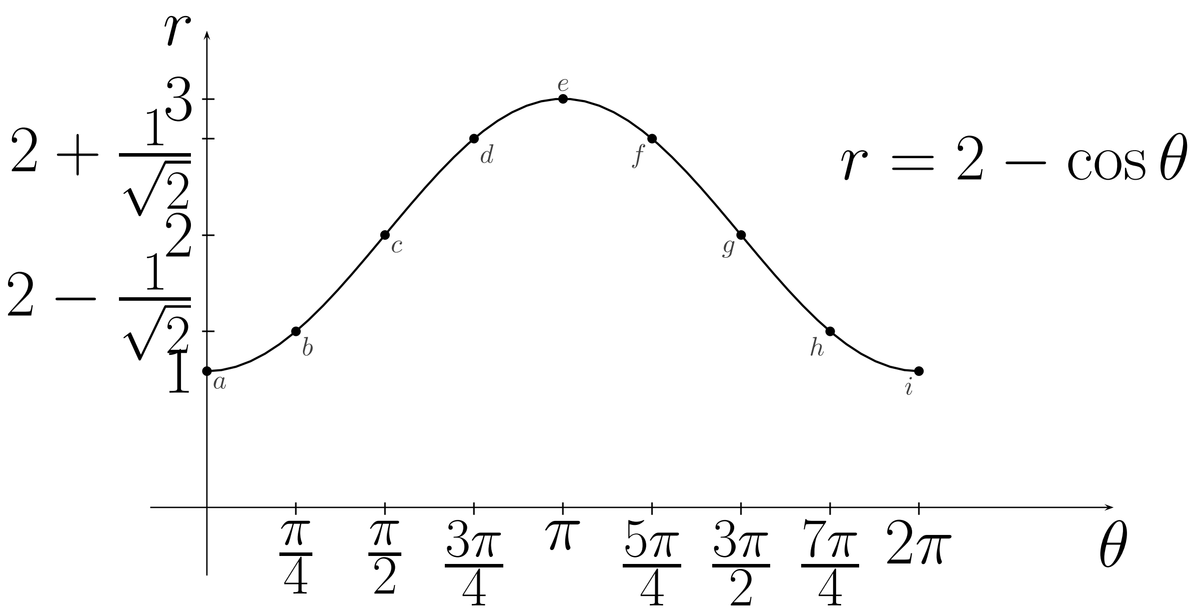
(b) is $r^2 \cos^2\theta + r^2 \sin^2\theta = 4$
$r^2 = 4$ or $r = 2$.

Remark. In order to sketch a curve represented by an equation in polar form, it may be helpful to begin with an r vs θ sketch.

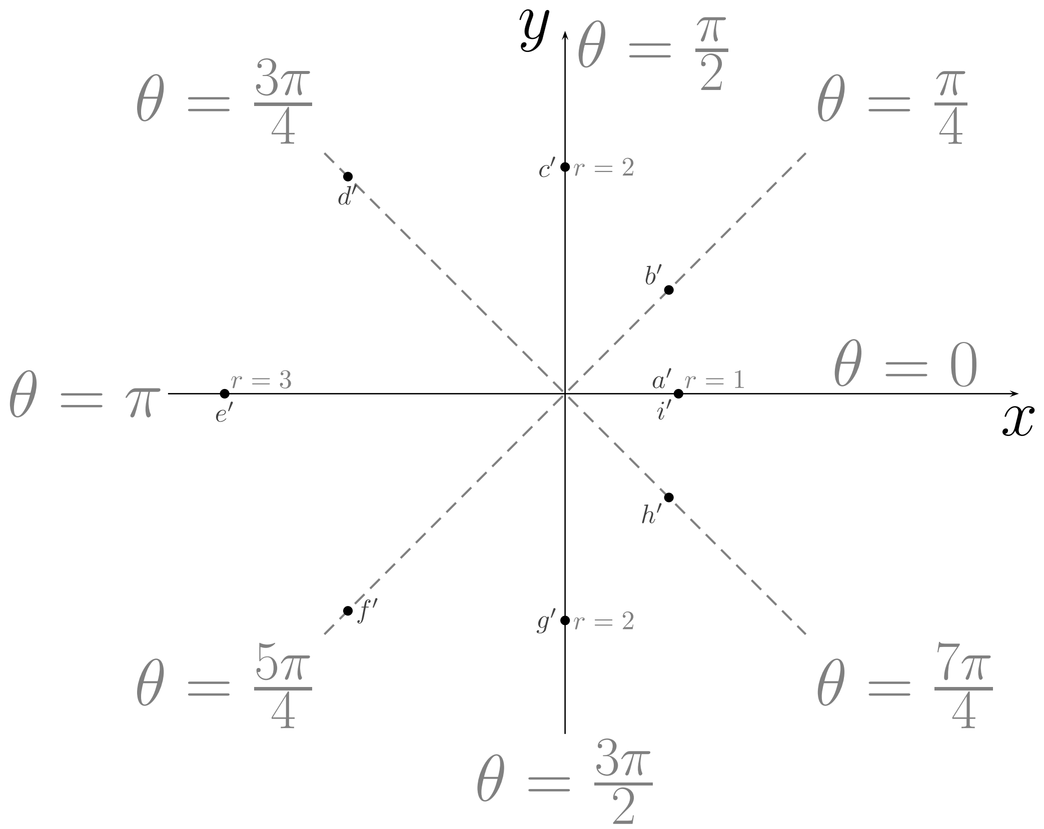
Example. Sketch the polar curve defined by

$$r = 2 - \cos\theta, \qquad 0 \leq \theta \leq 2\pi.$$

We first graph r against θ ...



... and then mark the corresponding points on the (x, y)-plane:

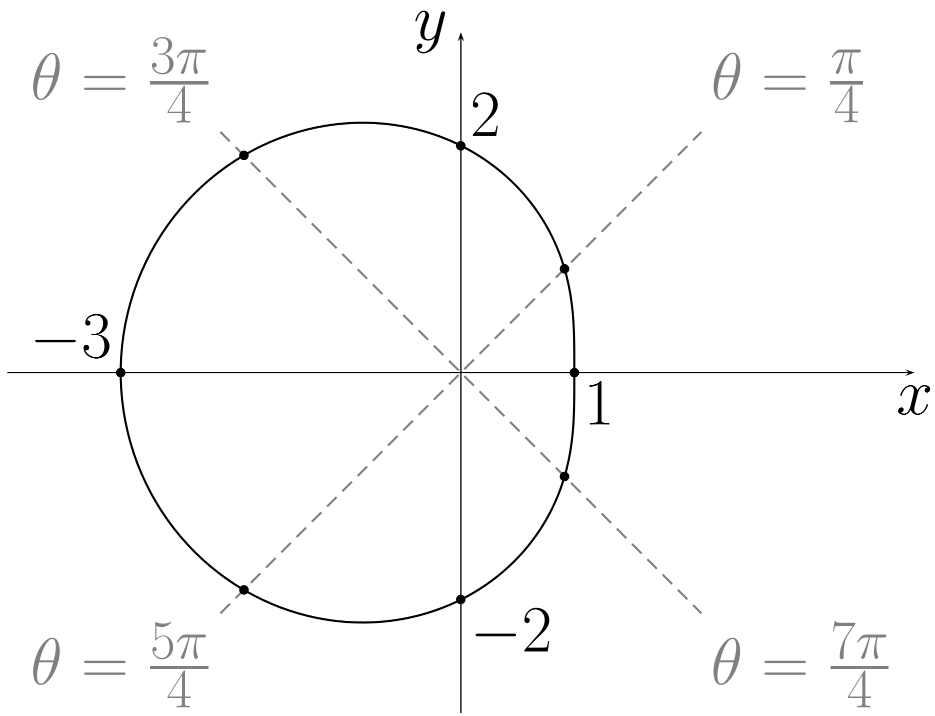


Now:

- As θ increases from 0 to π, r increases from 1 to 3. Hence, the distance r from the origin to points on the curve increases from 1 to 3.
- As θ increases from π to 2π, r decreases from 3 to 1. Hence, the distance r from the origin to points on the curve decreases from 3 to 1.

These considerations lead to the final sketch.

$r = 2 - \cos\theta$



Symmetries.

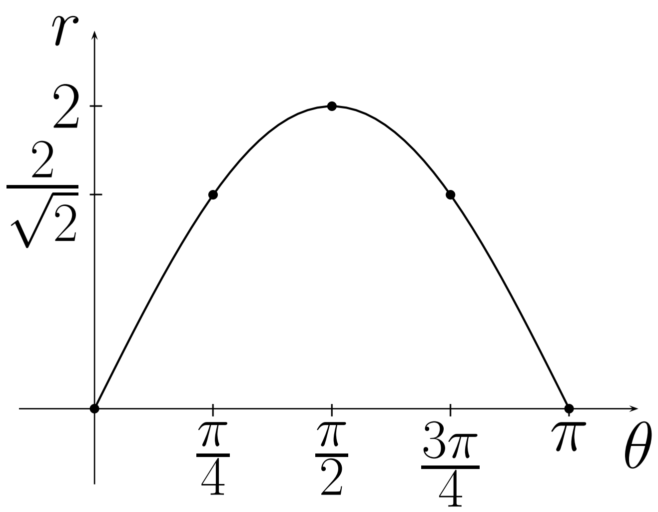
- If $f(-\theta) = f(\theta)$ then the polar curve is symmetric about the x-axis.
- If $f(\pi - \theta) = f(\theta)$ then the polar curve is symmetric about the y-axis.
- If f is 2π–periodic then it suffices to consider θ in the range $0 \leq \theta < 2\pi$.

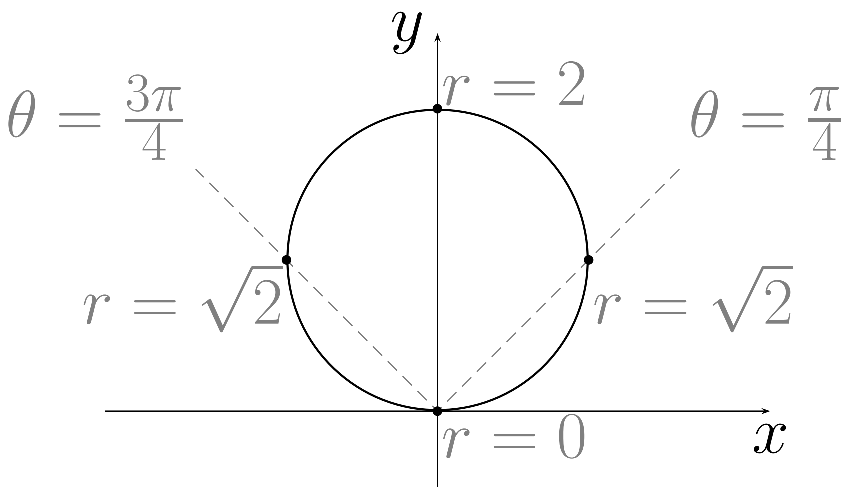
Example. Sketch the curve described by the polar equation

$$r = 2\sin\theta, \qquad 0 \leq \theta \leq \pi$$

and show that it constitutes a circle.

Graphing r against θ, followed by y against x leads to the following





The curve looks as though it is a circle, an observation that can be confirmed by rewriting $r = 2\sin\theta$ in terms of x and y:

$$r = 2\sin\theta$$
$$r^2 = 2r\sin\theta$$
$$x^2 + y^2 = 2y$$
$$x^2 + y^2 - 2y + 1 = 1$$
$$x^2 + (y-1)^2 = 1.$$

Therefore the polar curve is a circle with Cartesian centre $(0, 1)$ and radius 1.

Sketching polar curves using calculus.

Suppose that a curve can be expressed in polar form as

$$r = f(\theta).$$

Since the curve's parametric form is given by

$$\gamma(\theta) = \big(x(\theta), y(\theta)\big) = (r\cos\theta, r\sin\theta),$$

the tangent vector reads

$$\gamma'(\theta) = \big(x'(\theta), y'(\theta)\big).$$

Thus, horizontal tangents are obtained by solving

$$y'(\theta) = 0 \quad \textbf{but} \quad x'(\theta) \neq 0,$$

while vertical tangents correspond to

$$x'(\theta) = 0 \quad \textbf{but} \quad y'(\theta) \neq 0.$$

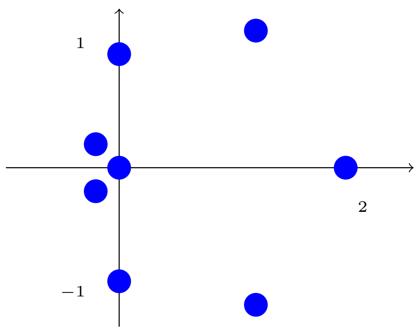
Example. Sketch the curve described by the polar equation

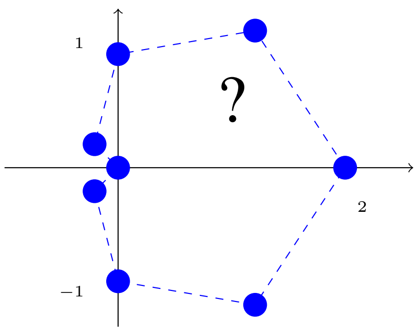
$$r = 1 + \cos\theta, \quad \theta \in [0, 2\pi].$$

You can begin by just plotting some points:

θ	r
0	2
$\pi/4$	$1 + 1/\sqrt{2}$
$\pi/2$	1
$3\pi/4$	$1 - 1/\sqrt{2}$
π	0

Note that $r(-\theta) = r(\theta)$ so the curve is symmetric in the x-axis.





Think about how r is changing as θ changes:
As θ increases from 0 to π, r decreases from 2 to 0.

A tangent vector $\gamma'(\theta) = (x'(\theta), y'(\theta))$ for the curve is given by

$$
\begin{aligned}
x'(\theta) &= -\sin\theta\cos\theta - (1+\cos\theta)\sin\theta && = \tfrac{dr}{d\theta}\cos\theta - r\sin\theta \\
&= -\sin\theta(1+2\cos\theta) \\
y'(\theta) &= -\sin\theta\sin\theta + (1+\cos\theta)\cos\theta && = \tfrac{dr}{d\theta}\sin\theta + r\cos\theta \\
&= \cos^2\theta - 1 + \cos\theta + \cos^2\theta \\
&= (2\cos\theta - 1)(\cos\theta + 1)
\end{aligned}
$$

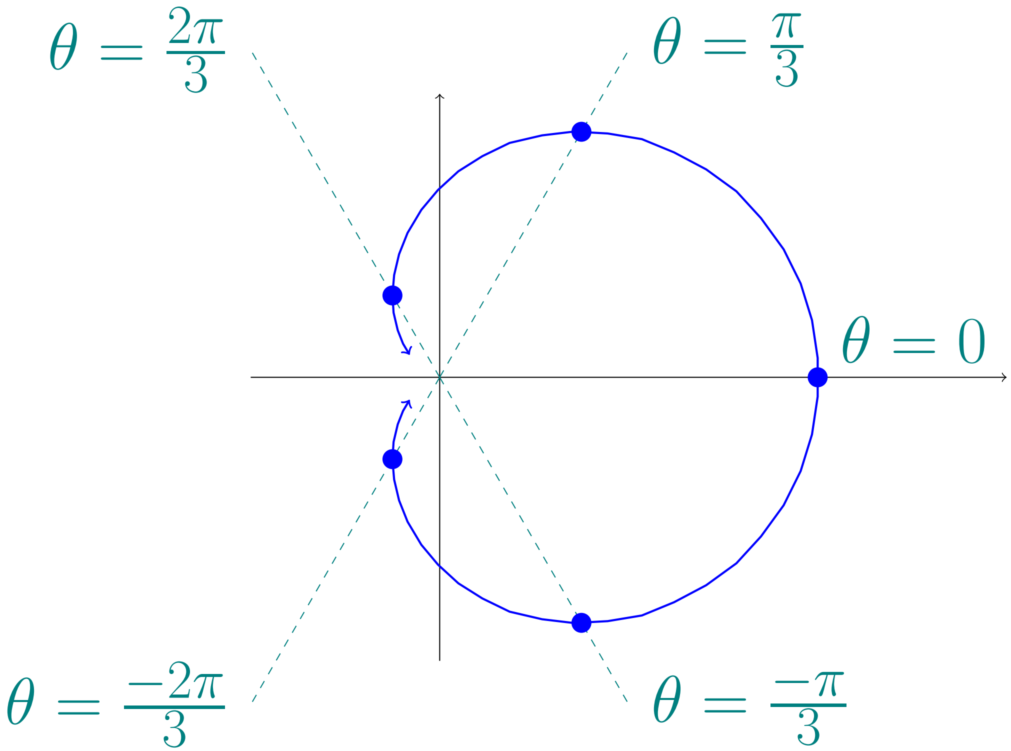
For a horizontal tangent you want $y'(\theta) = 0$ (but $x'(\theta) \neq 0$). Thus, either

- $2\cos\theta - 1 = 0$ so $\cos\theta = \frac{1}{2}$ or $\theta = \pm\pi/3$,
- $\cos\theta = -1$ so $\theta = \pi$.

For a vertical tangent you want $x'(\theta) = 0$ (but $y'(\theta) \neq 0$). Thus, either

- $\sin\theta = 0$, so $\theta = 0$ or π,
- $1 + 2\cos\theta = 0$ so $\cos\theta = -\frac{1}{2}$ so $\theta = \pm 2\pi/3$.

What we have so far tells us that the picture looks like



Horizontal tangents at $\theta = \pm\frac{\pi}{3}$.
Vertical tangents at $\theta = \pm\frac{2\pi}{3}$ and at $\theta = 0$.

Deciding what happens when $\theta = \pi$ is a bit tricky as $x'(\pi) = y'(\pi) = 0$.

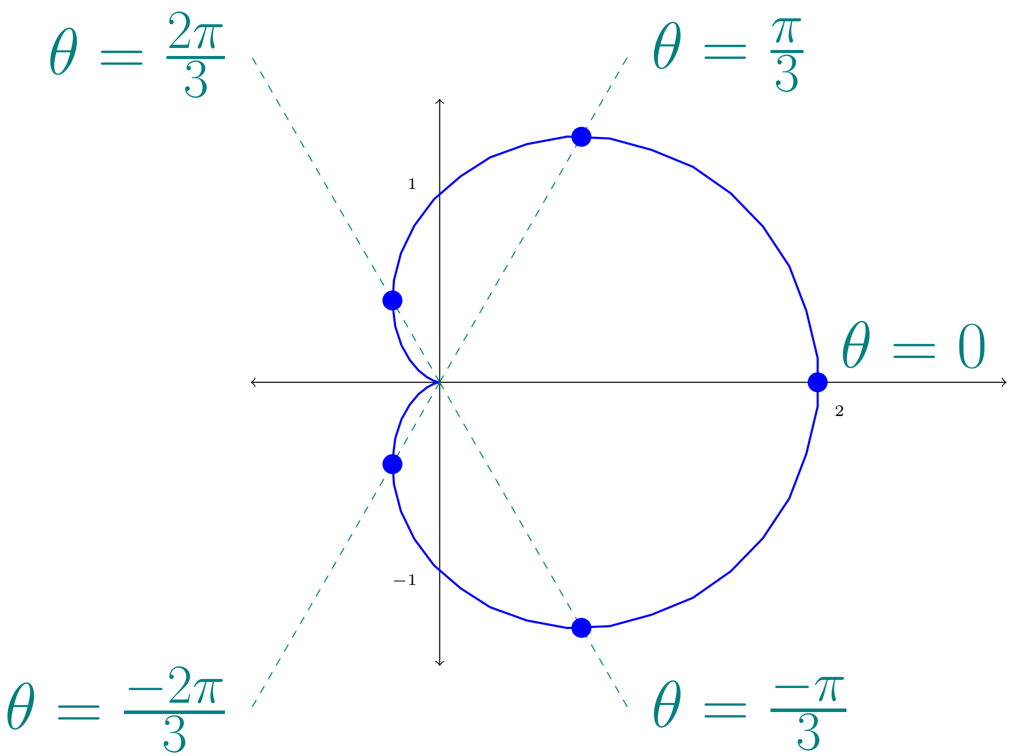
What is happening to the slope near $\theta = \pi$?

Calculate the slope as $\theta \to \pi$,

$$\begin{aligned}\lim_{\theta\to\pi} \frac{y'(\theta)}{x'(\theta)} &= \lim_{\theta\to\pi} \frac{(2\cos\theta - 1)(\cos\theta + 1)}{-\sin\theta(1 + 2\cos\theta)} \\ &= -\left(\frac{2\cos\pi - 1}{1 + 2\cos\pi}\right) \cdot \lim_{\theta\to\pi} \frac{\cos\theta + 1}{\sin\theta} \\ &\overset{L'H}{=} -\left(\frac{-3}{-1}\right) \cdot \lim_{\theta\to\pi} \frac{-\sin\theta}{\cos\theta} = 0.\end{aligned}$$

Thus the slope is going to zero near $\theta = \pi$, and so the curve has a cusp at that point.

The full picture is thus:



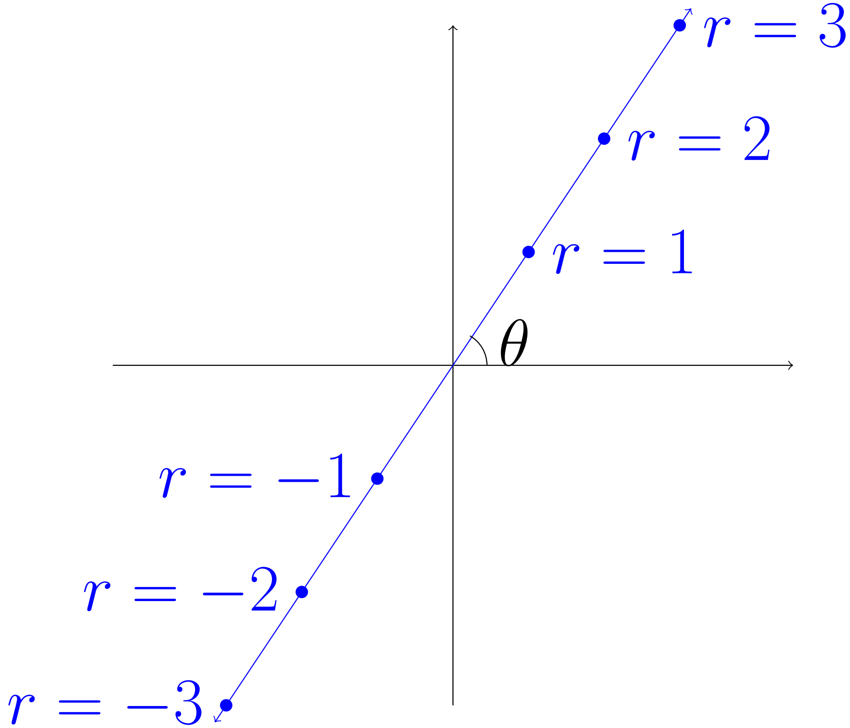
This shape is called a cardiod.

Allow r to be negative

It turns out to be useful to allow r to be negative!

Maple, for example allows this.

You can think of the r coordinate as measuring the position of a point along a ray at angle θ from the origin. If we extend this ray to a line, we get to specify points in the opposite direction using negative values of r.



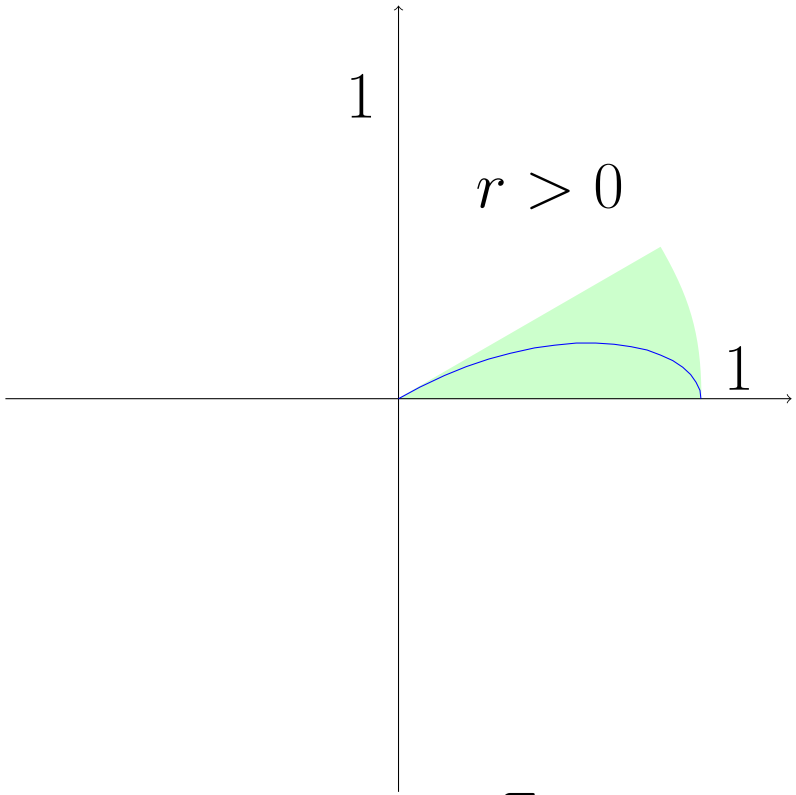
eg, $(x, y) = (\sqrt{2}, \sqrt{2})$ has polar coordinates

$(r, \theta) = (2, \pi/4)$ or

$(r, \theta) = (-2, 5\pi/4)$

Example. Sketch the polar curve $r = \cos 3\theta$.

As θ increases from 0 to $\pi/6$, r decreases from 1 to 0.

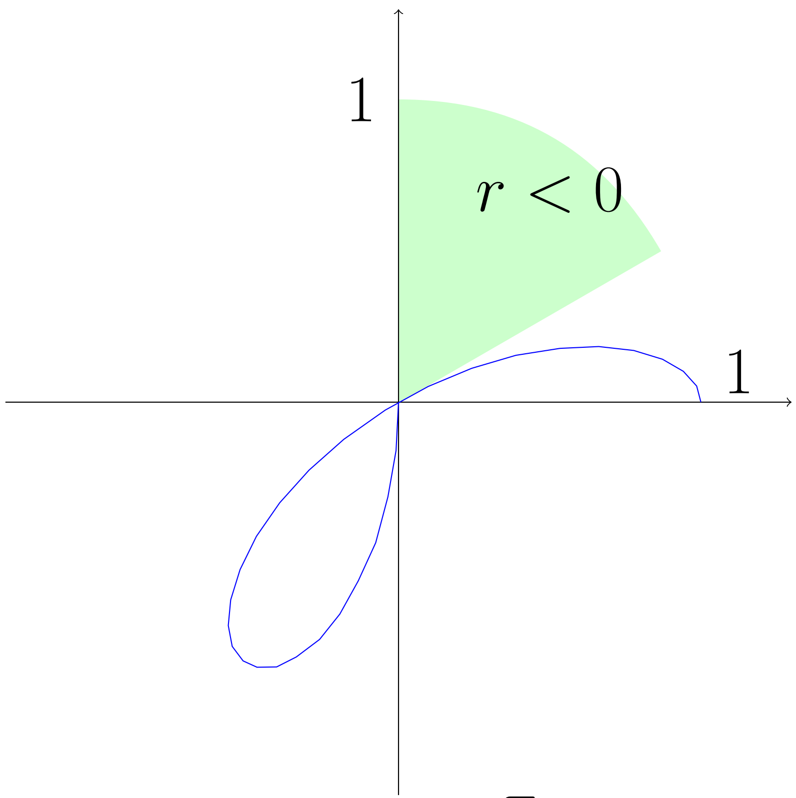


$$0 \leq \theta \leq \frac{\pi}{6}$$

Allow r to be negative.

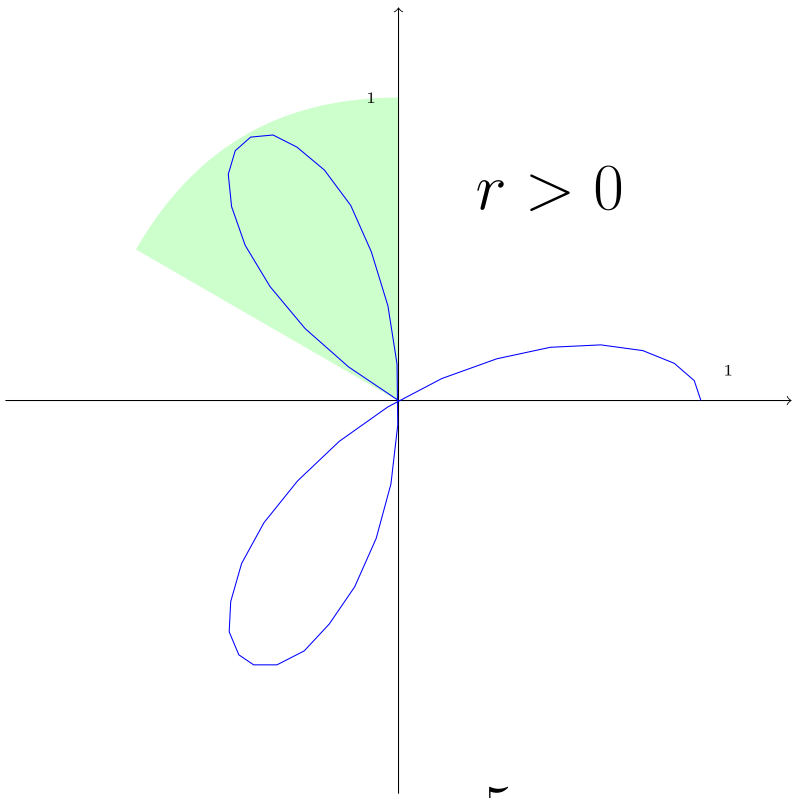
As θ increases from $\pi/6$ to $\pi/3$, r decreases from 0 to -1.

As θ increases from $\pi/3$ to $\pi/2$, r increases from -1 to 0.



$$0 \leq \theta \leq \frac{\pi}{2}$$

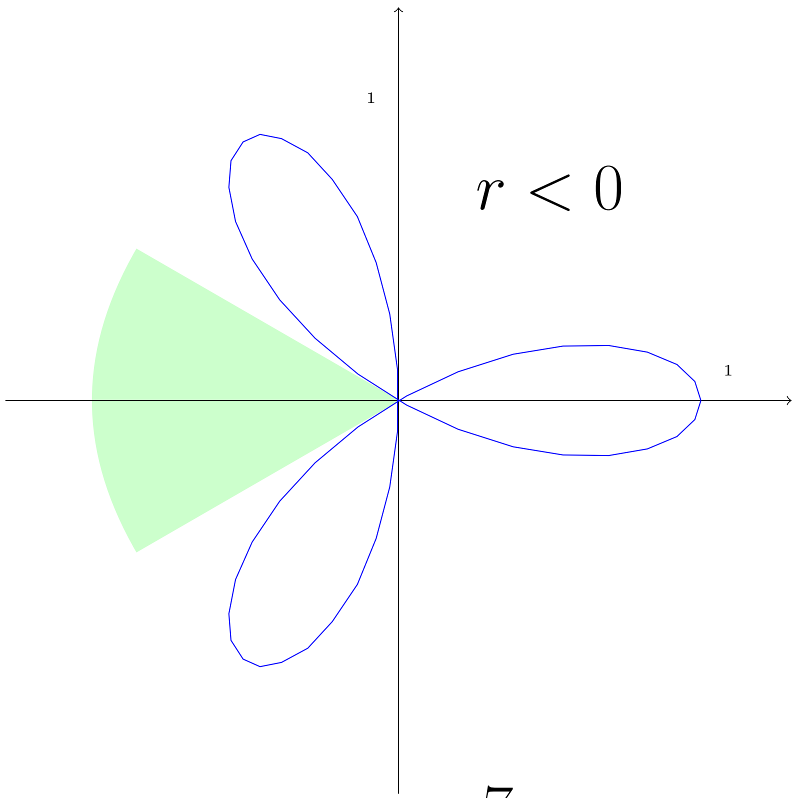
As θ increases from $\pi/2$ to $2\pi/3$, r increases from 0 to 1.
As θ increases from $2\pi/3$ to $5\pi/6$, r decreases from 1 to 0.



$r > 0$

$$0 \leq \theta \leq \frac{5\pi}{6}$$

As θ increases from $5\pi/6$ to π , r decreases from 0 to -1.
As θ increases from π to $7\pi/6$, r increases from -1 to 0.



$$0 \leq \theta \leq \frac{7\pi}{6}$$

The curve retraces every π.