Chapter 1: Introduction to Vectors

The concept of a vector arose in physics.

Definition

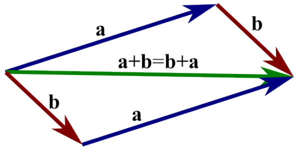
A **vector** is a quantity that has both a size, or magnitude, and a direction.

- In physics, a vector is contrasted with a **scalar** quantity, which posseses a magnitude but no direction.
- A vector is represented geometrically by an arrow connecting two points. The length of this arrow is the magnitude of the vector, and the direction is pointed to by the arrowhead.
- Two vectors are **parallel** if they have the same, or the opposite, direction.
- Two vectors are equal if they have the same magnitude and same direction.
- The **zero vector** is a special case: it does not have a direction.

We will denote vectors by bold letters. The length of a vector $\mathbf{v}$ will be denoted by $|\mathbf{v}|$.

We can add two vectors $\mathbf{a}$ and $\mathbf{b}$ by drawing $\mathbf{a}$ as an arrow between points P and Q and $\mathbf{b}$ as an arrow between points Q and R; then $\mathbf{a} + \mathbf{b}$ is the vector represented by the arrow from P to R.

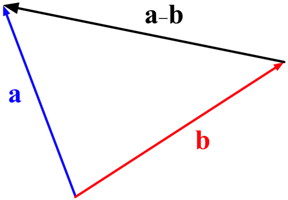
This is the **triangle method**.



An equivalent method is the **parallelogram method**:

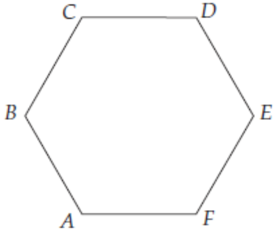
We draw $\mathbf{a}$ from P to Q and $\mathbf{b}$ from P to R. Let S be the point such that $PQSR$ is a parallelogram. Then $\mathbf{a} + \mathbf{b}$ is the vector represented by the arrow from P to S.

To subtract a vector **b** from a vector **a**, we add to **a** the vector with the same magnitude as **b** but with the opposite direction, called **-b**.



The **Triangle Inequality:** Since the arrows representing the vectors **a**, **b**, and $\mathbf{a} + \mathbf{b}$ can be drawn in a triangle, we have $|\mathbf{a}| + |\mathbf{b}| \geq |\mathbf{a} + \mathbf{b}|$.

Example: Suppose $ABCDEF$ is a regular hexagon with the vector $\mathbf{p}$ on the side AB and vector $\mathbf{q}$ on the side BC. Express the vectors on the sides: CD, DE, EF, FA and the diagonals AC, AD, AE in terms of $\mathbf{p}$ and $\mathbf{q}$.



Answer: We use the geometric fact (which can be deduced by dividing the hexagon into six equilateral triangles) that

$$\overrightarrow{AD} = 2\,\overrightarrow{BC} = 2\mathbf{q}.$$

Then since

$$\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD},$$

we have

$$\overrightarrow{CD} = 2\mathbf{q} - (\mathbf{p} + \mathbf{q}) = \mathbf{q} - \mathbf{p}.$$

By symmetry,

$$\overrightarrow{FA} = -\,\overrightarrow{CD} = \mathbf{p} - \mathbf{q}.$$

Similarly, the other sides and diagonals can be deduced by addition, subtraction, and symmetry.

We can also **scale** vectors by multiplying them by a number, or **scalar**:

$$\mathbf{a} \to c\mathbf{a}, c \in \mathbb{R}.$$

- Multiplying by a number between 0 and 1 shrinks the vector
- Multiplying by 1 does nothing
- Multiplying by a number greater than 1 expands the vector
- Multiplying by -1 reverses the direction
- Multiplying by 0 gives $\mathbf{0}$, the zero vector

Vector operations satisfy a number of important properties:

- $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ (Commutativity)
- $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$ (Associativity)
- $\mathbf{a} + \mathbf{0} = \mathbf{a}$
- $\mathbf{a} + \textbf{-a} = \mathbf{0}$
- $c(d\mathbf{a}) = cd\mathbf{a}$
- $1\mathbf{a} = \mathbf{a}$
- $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
- $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$

These properties define a very important concept in mathematics: a **vector space**.

Why are commutativity and associativity of vector addition true?

Linear independence:

Lets pretend the earth is flat! If I walk x kilometers east and then y kilometers north, is it possible that I arrive back where I started?

Yes, but only if x and y are both 0!

This notion is formalized as follows.

A collection of vectors $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n$ is **linearly independent** if the only way to write

$$a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + ... + a_n\mathbf{u}_n = \mathbf{0}$$

is if all the scalars $a_1, a_2, ..., a_n$ are 0.

Otherwise the set of vectors is **linearly dependent**.

Example: Suppose $\mathbf{a}, \mathbf{b}$ are non-zero vectors. Show that

$$\mathbf{p} = \mathbf{a} + \mathbf{b}, \mathbf{q} = \mathbf{b} - \mathbf{c}, \mathbf{r} = \mathbf{a} - \mathbf{b} + \mathbf{c}, \text{ and } \mathbf{s} = \mathbf{b} + \frac{1}{2}\mathbf{c}$$

are linearly dependent.

Answer: We would like to write

$$x\mathbf{p} + y\mathbf{q} + z\mathbf{r} + w\mathbf{s} = \mathbf{0},$$

with not all of x, y, z, w equal to 0.

Expanding the left side, we get

$$\begin{aligned} x\mathbf{p}+y\mathbf{q}+z\mathbf{r}+w\mathbf{s} &= x(\mathbf{a}+\mathbf{b})+y(\mathbf{b}-\mathbf{c})+z(\mathbf{a}-\mathbf{b}+\mathbf{c})+w(\mathbf{b}+\frac{1}{2}\mathbf{c}) \\ &= (x+z)\mathbf{a} + (x+y-z+w)\mathbf{b} + (-y+z+\frac{w}{2})\mathbf{c} \end{aligned}$$

Therefore $x\mathbf{p} + y\mathbf{q} + z\mathbf{r} + w\mathbf{s}$ will equal $\mathbf{0}$ if

$$x + z = x + y - z + w = -y + z + \frac{w}{2} = 0.$$

This has the general solution

$$y = -\frac{4}{3}x, \quad z = -x, \quad w = -\frac{2}{3}x.$$

For example, we can take

$$x = 1, \quad y = -\frac{4}{3}, \quad z = -1, \quad w = -\frac{2}{3}$$

and we will have

$$x\mathbf{p} + y\mathbf{q} + z\mathbf{r} + w\mathbf{s} = 0.$$

Therefore $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}$ are linearly dependent.

Ex: Prove (using vectors) that the line joining the midpoint of two sides of a triangle is parallel to the third side and half its length.

Answer: Let ABC be a triangle, let P be the midpoint of AB, and let Q be the midpoint of BC.

Then we have

$$\overrightarrow{AP} = \frac{1}{2}\overrightarrow{AB} \text{ and } \overrightarrow{QC} = \frac{1}{2}\overrightarrow{BC}.$$

We can write

$$\begin{aligned}\overrightarrow{AC} = \overrightarrow{AP} + \overrightarrow{PQ} + \overrightarrow{QC} &= \frac{1}{2}\overrightarrow{AB} + \overrightarrow{PQ} + \frac{1}{2}\overrightarrow{BC} \\ &= \overrightarrow{PQ} + \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{BC}) = \overrightarrow{PQ} + \frac{1}{2}\overrightarrow{AC}\end{aligned}$$

So we get

$$\overrightarrow{PQ} = \frac{1}{2}\overrightarrow{AC}.$$

Ex: Prove that the medians of a triangle meet at a point G which divides each median in the ratio 2:1.

There are really two assertions here:

(1) The three medians all intersect at a single point.

(2) That point divides each median in the ratio 2:1.

We will prove:

(2') Whenever two medians of a triangle intersect, the intesection point is $2/3$ of the way along each median (from the vertex end).

This implies both (1) and (2). (Why?)

We can identify vectors with points in space as follows.

Fix a point O, which we call the origin. Then for every point in space P, the **position vector** of P is the vector represented by the arrow from O to P.

Conversely, given any vector $\mathbf{v}$, we can draw the arrow representing $\mathbf{v}$ starting from O; the endpoint of the arrow is a point P.

This establishes a 1-**to**-1 **correspondence** between vectors and points in space.

A more algebraic way to think of points in space is through the use of **coordinates**, which are numbers giving a position in space with respect to certain axes in some frame of reference.

For example, if I tell you that I am five blocks east and two blocks south of City Hall, then I have given you my coordinates with respect to the East-West and North-South axes of the city.

In this case we can call City Hall the **origin**, and then if our units are city blocks, my coordinates are $(5, -2)$

(where we have chosen East-West as our first axis, with East being positive, and North-South as the second axis, with North being positive.)

Another example of coordinates is latitude and longitude, which give positions of places on earth in terms of angles with certain great circles. However curved spaces are more complicated and we don't deal with these here.

The **Euclidean space** $\mathbb{R}^n$ is the set of n-tuples of numbers

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ . \\ . \\ a_n \end{pmatrix}.$$

We think of each such n-tuple as a point in space, with each coordinate a_i giving the distance along some axis from a fixed origin, which is the point $\begin{pmatrix} 0 \\ 0 \\ . \\ . \\ 0 \end{pmatrix}$.

In 2 dimensions, we labels the axes by X and Y, and in 3-dimensions by X, Y, and Z. In higher dimensions we usually use subscripts, e.g. $X_1, X_2, ..., X_n$.

We can add points in $\mathbb{R}^n$ using the rule

$$\begin{pmatrix} a_1 \\ a_2 \\ . \\ . \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ . \\ . \\ a_n + b \end{pmatrix},$$

and we can multiply by scalars using

$$c \begin{pmatrix} a_1 \\ a_2 \\ . \\ . \\ a_n \end{pmatrix} = \begin{pmatrix} c \cdot a_1 \\ c \cdot a_2 \\ . \\ . \\ c \cdot a_n \end{pmatrix}.$$

If we identify each point P in $\mathbb{R}^n$ with the vector OP, where O is the origin in $\mathbb{R}^n$, then this addition and scalar multiplication are **the same ones** defined earlier for geometric vectors.

We think of points in $\mathbb{R}^n$ as vectors.

The vector represented by the arrow from a point P to another point Q in $\mathbb{R}^n$ can also be drawn from O to $Q - P$; therefore it corresponds to the point $Q - P$.

Ex: Find the vectors $\overrightarrow{PQ}$ and $\overrightarrow{QP}$ if

$$P = \begin{pmatrix} 7 \\ -1 \\ 3 \end{pmatrix} \text{ and } Q = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}.$$

Answer:

$$\overrightarrow{PQ} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} - \begin{pmatrix} 7 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -5 \\ 2 \\ -6 \end{pmatrix}, \quad \overrightarrow{QP} = -\overrightarrow{PQ} = \begin{pmatrix} 5 \\ -2 \\ 6 \end{pmatrix}$$

Recall that two (nonzero) vectors are parallel if one is a scalar multiple of the other.

Ex: Suppose that

$$A = \begin{pmatrix} 0 \\ 0 \end{pmatrix} B = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, C = \begin{pmatrix} 3 \\ 5 \end{pmatrix}, D = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

are the position vectors for four points A, B, C, D. Prove that the quadrilateral $ABCD$ is a parallelogram.

Answer: We have

$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \overrightarrow{DC} \text{ and } \overrightarrow{BC} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \overrightarrow{AD};$$

since the opposite sides of ABCD are the same, it is a parallelogram.

The **standard basis vectors** in $\mathbb{R}^2$ are the vectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

which are often denoted by $\mathbf{i}$ and $\mathbf{j}$.

Observe that every vector in $\mathbb{R}^2$ can be written in terms of these basis vectors. For example $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ can be written as $\mathbf{i} - 2\mathbf{j}$.

In 3-dimensions, the basis vectors, $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Every vector in $\mathbb{R}^3$ can be expressed in terms of these basis vectors

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}.$$

In higher dimensions, we label the basis vectors as $\mathbf{e_1}, \mathbf{e_2}$,... etc.

For example, in $\mathbb{R}^4$ we have

$$\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{e_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{e_4} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

For a point $\begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbb{R}^2$, the distance from the origin is given by the Pythagorean Theorem as

$$\sqrt{x^2 + y^2}.$$

Similarly, in three dimensions, the distance of $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ from the origin is given by

$$\sqrt{x^2 + y^2 + z^2}.$$

More generally we define the length of a vector $\mathbf{v} = \begin{pmatrix} a_1 \\ a_2 \\ . \\ . \\ a_n \end{pmatrix}$ in $\mathbb{R}^n$ to be $|v| = \sqrt{a_1^2 + a_2^2 + ... + a_n^2}$.

The length of a vector **v** connecting two points

$$P = \begin{pmatrix} a_1 \\ a_2 \\ . \\ . \\ a_n \end{pmatrix} \text{ and } Q = \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{pmatrix}$$

is the length of $P - Q$, or the distance from P to Q:

$$|\mathbf{v}| = |P - Q| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + ... + (a_n - b_n)^2}.$$

Ex: Find the distance between $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ and between $\begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 6 \\ -1 \end{pmatrix}$.

Answer: The distances are

$$\sqrt{(1-3)^2 + (-2-(-4))^2} = \sqrt{8}$$

and

$$\sqrt{((-1)-3)^2 + (2-6)^2 + (5-(-1))^2} = \sqrt{68}$$

The "length function" $|\cdot|$ has the following important properties:

1. $|\mathbf{a}| \geq 0$.
2. $|\mathbf{a}| = 0$ if and only if $\mathbf{a} = 0$.
3. $|c\mathbf{a}| = |c||\mathbf{a}|$, for $c \in \mathbb{R}$.
4. $|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$.

These properties define the concept of a **norm**.

A vector which has unit length is called a **unit vector**. Any vector can be made into a unit vector by dividing by its length.

Ex: Find a unit vector parallel to the vector $\mathbf{a} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$.

Answer: The length of **a** is

$$|\mathbf{a}| = \sqrt{2^2 + (-3)^2 + 1^2} = \sqrt{14}$$

and

$$\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{1}{\sqrt{14}}\mathbf{a} = \frac{1}{\sqrt{14}} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

is a unit vector parallel to **a**.

Ex: Suppose A and B are points with position vectors $\mathbf{a}$ and $\mathbf{b}$. Find a vector (in terms of $\mathbf{a}$ and $\mathbf{b}$) which bisects the angle AOB, where O is the origin.

Answer: The vector

$$\frac{\mathbf{a}}{|\mathbf{a}|} + \frac{\mathbf{b}}{|\mathbf{b}|}$$

will bisect the angle AOB.

A line passing through the origin O and another point P is just the set of all scalar multiples of the vector $\mathbf{u} = OP$.

We write the **parametric vector equation** of such a line as

$$\mathbf{x} = \lambda \mathbf{u}.$$

The scalar variable λ is called a **parameter**.

We say that such a line (through the origin!) is the **span** of $\mathbf{u}$, and write

$$\text{span}(\mathbf{u}) = \{\lambda \mathbf{u} : \lambda \in \mathbb{R}\}.$$

Ex: In $\mathbb{R}^2$ what is the span of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$? What is span$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$?

Answer: span$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the x-axis; span$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the line $y = x$.

For a line that does not pass through the origin, we can still represent the line parametrically using a direction vector, but we need to first add a starting point:

$$\mathbf{x} = \mathbf{u}_0 + \lambda \mathbf{u}_1$$

The line is $\operatorname{span}(\mathbf{u}_1)$ shifted over by the vector $\mathbf{u}_0$.

If we are given two points P and Q on the line, we can take

$$\mathbf{u}_0 = P \text{ and } \mathbf{u}_1 = Q - P.$$

Ex: Find the vector equation of the line passing through the point P with position vector $\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$ and parallel to the vector $\begin{pmatrix} 1 \\ 6 \\ -4 \end{pmatrix}$.

Answer: $\mathbf{x} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 6 \\ -4 \end{pmatrix}$.

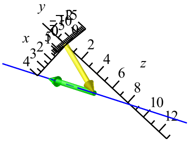


Figure : $\mathbf{x} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 6 \\ -4 \end{pmatrix}$.

Ex: Find the vector equation of the line passing through the two points P, Q with position vectors $P = \begin{pmatrix} -1 \\ 2 \\ 6 \end{pmatrix}$ and $Q = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix}$.

Answer: A direction vector is given by

$$\overrightarrow{PQ} = Q - P = \begin{pmatrix} 5 \\ -4 \\ -3 \end{pmatrix},$$

and the line is

$$\mathbf{x} = \begin{pmatrix} -1 \\ 2 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -4 \\ -3 \end{pmatrix}.$$

Ex: Find the vector equation of the line in 2-dimensions with cartesian equation $y = 2x + 1$.

Answer: We can choose the two points $(0, 1)$ and $(1, 3)$ to obtain the direction vector $(1, 2)$, and then the line is

$$\mathbf{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Ex: Does the point $\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$ lie on the line

$$\mathbf{x} = \begin{pmatrix} 3 \\ 4 \\ 11 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}?$$

Answer: We need to solve

$$\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 11 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix},$$

which gives the equations

$$-1 = 3 + 2\lambda, \quad 2 = 4 + \lambda, \quad 3 = 11 + 4\lambda,$$

which has the solution $\lambda = -2$.

We can write down an equation for a **line segment** by restricting the parameter to an interval.

Ex: Find a vector equation of the line segment from

$$P = \begin{pmatrix} -3 \\ -1 \\ 4 \end{pmatrix} \text{ to } Q = \begin{pmatrix} 2 \\ -2 \\ 6 \end{pmatrix}.$$

Answer:

$$\mathbf{x} = \begin{pmatrix} -3 \\ -1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -1 \\ 7 \end{pmatrix}, \ \lambda \in [0, 1].$$

Given a vector equation for a line

$$\mathbf{x} = \mathbf{u}_0 + \lambda \mathbf{u}_1$$

or

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix},$$

we can eliminate the parameter by solving the component equations

$$x = x_0 + \lambda x_1$$

$$y = y_0 + \lambda y_1$$

$$z = z_0 + \lambda z_1$$

The result is the **Cartesian equation** of the line.

Ex: Find the Cartesian equations of the lines

$$\mathbf{x} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

and

$$\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}.$$

Answer:

$$(1) \quad x = -1 + 3\lambda, \quad y = 2 - 2\lambda, \quad z = 3 + \lambda,$$

$$\lambda = \quad \mathbf{\frac{x+1}{3} = \frac{2-y}{2} = z - 3}$$

$$(2) \quad x = 2 + 2\mu, \quad y = 1, \quad z = -5 - \mu,$$

$$\mu = \quad \mathbf{\frac{x-2}{2} = 5 - z} \text{ and } \mathbf{y{=}1}.$$

Two lines are parallel if and only if their direction vectors are parallel.

Ex: Find the equation of the line passing through $\begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$ and parallel to

$$\frac{x+1}{3} = \frac{y-1}{-2} = \frac{z+6}{-4}.$$

Answer: In vector form, the second line is

$$\mathbf{x} = \begin{pmatrix} -1 \\ 1 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix}.$$

Then the line we want is

$$\mathbf{x} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix}.$$

Ex: Find the intersection (if possible) of the lines

$$\mathbf{x} = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \text{ and } \mathbf{x} = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}.$$

Answer: We have three equations for the intersection:

$$-1 + \lambda = 3 + 2\mu, \quad 2\lambda = 5 + \mu, \quad 3 - \lambda = -2 - 3\mu,$$

which have the solution

$$\mu = -1, \quad \lambda = 2,$$

corresponding to the point $\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$.

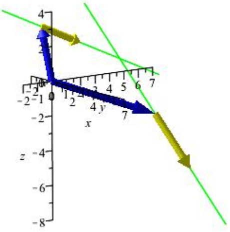


Figure : The intersection of $\mathbf{x} = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$. They intersect at the point $\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$, corresponding to $\mu = -1$ and $\lambda = 2$.

Just as a line through the origin is the span of a single (nonzero) vector, a **plane** through the origin is the span of two (non-parallel) vectors:

$$\text{span}\{\mathbf{a}, \mathbf{b}\} = \{\lambda\mathbf{a} + \mu\mathbf{b} : \lambda, \mu \in \mathbb{R}\}.$$

This is the set of points that we can reach by stretching each of the two vectors **a** and **b** by some proportions and then adding them together.

The vector equation of the plane throught the origin is:

$$\mathbf{x} = \lambda\mathbf{a} + \mu\mathbf{b}.$$

Ex: Describe the span of

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} \right\}.$$

Repeat for

$$\text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \right\}.$$

Answer: The first is a plane through the origin; the second is a line through the origin.

Ex: Find the vector equation of the plane passing through the origin parallel to the vectors $\begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 4 \\ 7 \end{pmatrix}$.

Answer:

$$\mathbf{x} = \lambda \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 4 \\ 7 \end{pmatrix}.$$

Just as for a line, we can write the vector equation of a plane not passing through the origin by shifting the span of two vectors by a starting point :

$$\mathbf{x} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}.$$

So we need a point in the plane and two direction vectors. Given three points P, Q, and R in the plane, we can take

$$\mathbf{a} = P, \quad \mathbf{b} = P - Q, \text{ and } \mathbf{c} = P - R.$$

Ex: Find the vector equation of the plane passing through the point P with position vector

$$\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$$

and parallel to the vectors

$$\begin{pmatrix} 1 \\ 6 \\ -4 \end{pmatrix} \text{ and } \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}.$$

Answer:

$$\mathbf{x} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 6 \\ -4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}.$$

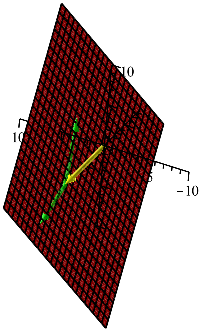


Figure : $\mathbf{x} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 6 \\ -4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}.$

Ex: Find the vector equation of the plane passing through the three points P, Q, R with position vectors

$$P = \begin{pmatrix} -1 \\ 2 \\ 6 \\ 2 \end{pmatrix}, \quad Q = \begin{pmatrix} 4 \\ -2 \\ 3 \\ -1 \end{pmatrix}, \text{ and } R = \begin{pmatrix} 1 \\ 7 \\ -2 \\ 5 \end{pmatrix}.$$

Answer:

$$\mathbf{x} = P + \lambda(Q - P) + \mu(R - P)$$

$$= \begin{pmatrix} -1 \\ 2 \\ 6 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -4 \\ -3 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 5 \\ -8 \\ 3 \end{pmatrix}.$$

To describe regions in the plane, we can restrict the values of the parameters.

Ex: Find the vector equation of the parallelogram in $\mathbb{R}^3$ whose vertices are the origin and the points

$$P = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \quad Q = \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix}, \text{ and } R = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}.$$

Answer:

$$\mathbf{x} = \lambda \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix}, \ \lambda, \mu \in [0, 1].$$

What about the triangle whose vertices are $O, P,$ and Q?

As with lines, we can eliminate the two parameters in the vector equation to find the Cartesian equation of a plane (though this can be annoying to do in practice).

Ex: Find the cartesian equation of the plane

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}.$$

Answer: We have the equations

$$x = 1 + 2\lambda - \mu, \quad y = 2 + 4\lambda, \quad z = 3 + 3\mu,$$

which we can combine into

$$x = 1 + 2 \cdot \frac{y-2}{4} - \frac{z-3}{3},$$

or

$$x - \frac{y}{2} + \frac{z}{3} = 1.$$

We can reverse this and go from a Cartesian equation to a vector equation.

Ex: Find the vector equation of the plane

$$3x - 6y + 2z = 12.$$

Answer: First lets find 3 points:

$$(4, 0, 0), (0, -2, 0), \text{ and } (0, 0, 6).$$

Then the vector equation is

$$\mathbf{x} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ -2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -4 \\ 0 \\ 6 \end{pmatrix}.$$

Ex: Find the intersection of the planes

$$2x + y - z = 10 \text{ and } 3x + 4y + 2z = 29.$$

Answer: Twice the first equation plus the second gives

$$7x + 6y = 49.$$

We can find two points in the line of intersection:

$$(7, 0, 4) \text{ and } (1, 7, -1).$$

Then the line is:

$$\mathbf{x} = \begin{pmatrix} 7 \\ 0 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -6 \\ 7 \\ -5 \end{pmatrix}.$$

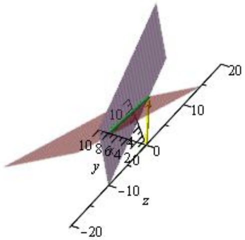


Figure : Intersection of the planes $2x + y - z = 10$ and $3x + 4y + 2z = 29$.

Ex: Show that the line

$$\mathbf{x} = \lambda \begin{pmatrix} 11 \\ -5 \\ -3 \end{pmatrix}$$

is parallel to the plane

$$\mathbf{x} = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix} + \nu \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix}.$$

Answer: We need to solve

$$\begin{pmatrix} 11 \\ -5 \\ -3 \end{pmatrix} = \mu \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix} + \nu \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix},$$

or

$$11 = 4\mu - \nu, \quad -5 = 2\mu + 3\nu, \quad -3 = 6\mu + 5\nu,$$

which has the solutions

$$\mu = 2 \text{ and } \nu = -3.$$

Ex: Find the intersection of

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ -2 \end{pmatrix} \text{ and } 2x + 3y - z = 29.$$

Answer: Write

$$x = 1 - \lambda, \quad y = 2 + 4\lambda, \quad z = 3 - 2\lambda.$$

Substituting in the equation for the plane, we get

$$2(1 - \lambda) + 3(2 + 4\lambda) - (3 - 2\lambda) = 29,$$

so at the intersection we must have $\lambda = 2$ and the point of intersection is $\begin{pmatrix} -1 \\ 10 \\ -1 \end{pmatrix}$.