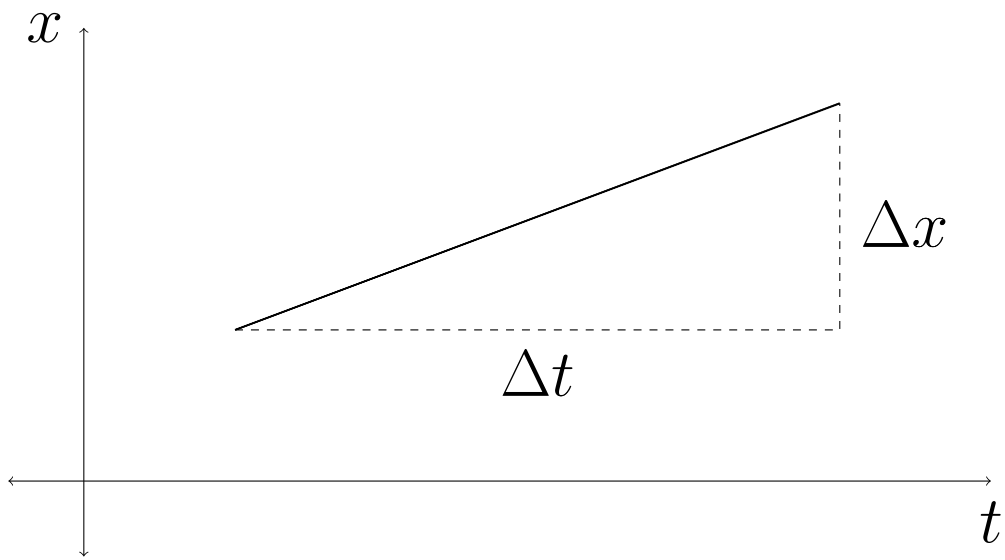
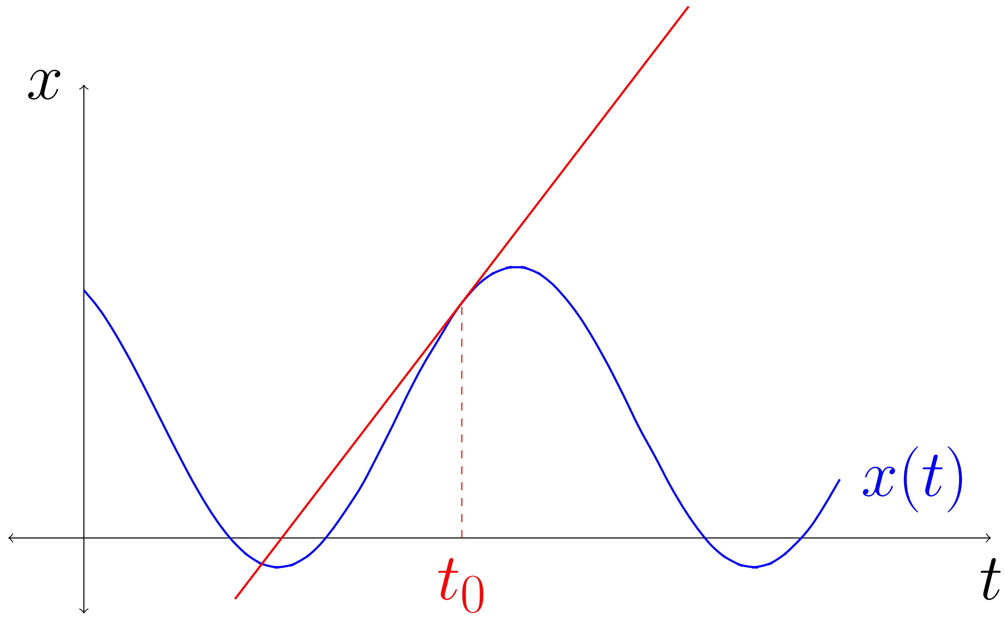
Differentiable functions

If you are travelling at a constant speed, then it is easy to find this speed by measuring a distance Δx and how long it takes for you to travel that distance Δt: speed $= \dfrac{\Delta x}{\Delta t}$.

If you graph the distance x covered against time t, you just get a straight line, and the speed is the slope of this line.

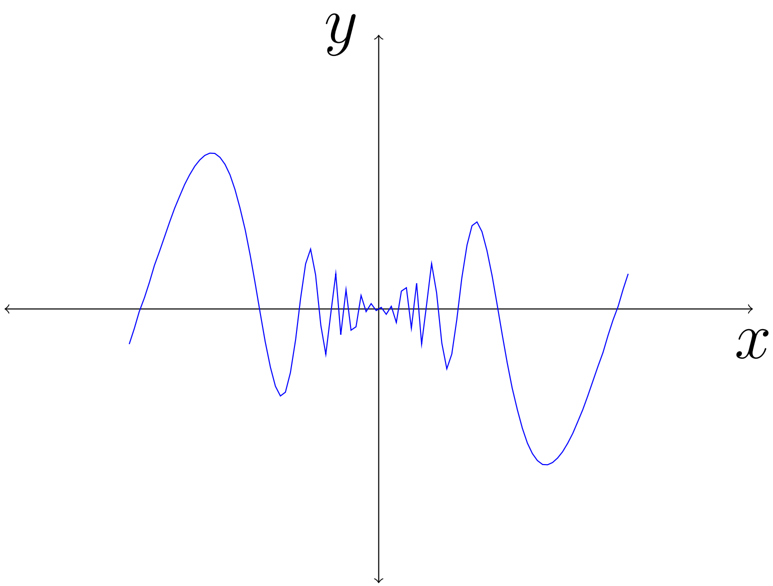


If your speed is varying, then the graph of $x(t)$ is more complicated. The speed at time t_0 is given by the slope of the graph at t_0. Finding this slope is a now much more complicated problem geometrically.



For a general graph of $y = f(x)$, deciding whether it even makes sense to talk about the slope at a point can be difficult.

Examples: $f(x) = \begin{cases} |x|^{1.1} \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0. \end{cases}$



Definition. Suppose that f is defined on some open interval containing the point x. If

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists then it is called the derivative (or, slope) of f at x and we say that f is differentiable at x.

The derivative of f at x is denoted by

$$f'(x) \quad \text{or} \quad \frac{df}{dx}(x) \quad \text{or} \quad \frac{d}{dx}f(x).$$

Remark. The ratio

$$\frac{f(x+h) - f(x)}{h}$$

is called the difference quotient for f at the point x.

Easy example. Let $f(x) = x^3$. Find, *using the definition of derivative*, $f'(2)$.

Suppose that $h \neq 0$. Then

$$f(2+h) = (2+h)^3 = 8 + 12h + 6h^2 + h^3$$

and so

$$\begin{aligned}\frac{f(2+h) - f(2)}{h} &= \frac{8 + 12h + 6h^2 + h^3 - 8}{h} \\ &= 12 + 6h + h^2 \\ &\to 12 \quad \text{as } h \to 0.\end{aligned}$$

Thus $f'(2) = 12$.

Harder example. Let $f(x) = \sin(x^3)$.
Find (using the definition of derivative) $f'(2)$.
Suppose that $h \neq 0$. Then

$$f(2+h) = \sin\big((2+h)^3\big) = \sin(8 + 12h + 6h^2 + h^3)$$

and so

$$\frac{f(2+h) - f(2)}{h} = \frac{\sin(8 + 12h + 6h^2 + h^3) - \sin(8)}{h}$$
$$\to ???$$

That limit looks hard!

What calculus provides us with a set of simple rules for

- Recognising which functions are differentiable.
- Finding the derivatives 'symbolically'.

As before you

- Use the definition to prove that a few very simple functions (e.g. constants, $f(x) = x$ or $f(x) = \sin x$) are differentiable.
- Prove differentiation rules to deal with most more complicated functions.
- Only resort to the definition when the differentiation rules don't apply.

Exercise 1. If $f(x) = C$ (constant), then, for all $x \in \mathbb{R}$, $f'(x) = 0$.

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} 0 = 0.$$

Exercise 2. If $f(x) = x$, then, for all $x \in \mathbb{R}$, $f'(x) = 1$.

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)-x}{h} = \lim_{h\to 0} \frac{h}{h} = 1.$$

Exercise 3. Use the binomial theorem to show that if $f(x) = x^n$ with $n \in \mathbb{Z}^+$, then $f'(x) = nx^{n-1}$.

Hard example. Let $f(x) = \sin x$. Show that $f'(x) = \cos x$. To do this you need the identity

$$\sin(A + B) = \sin A \cos B + \cos A \sin B.$$

Using this we see that if $h \neq 0$ then

$$\begin{aligned}\frac{\sin(x+h) - \sin(x)}{h} &= \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\ &= \sin(x)\,\frac{\cos(h) - 1}{h} + \cos(x)\,\frac{\sin(h)}{h}\end{aligned}$$

Now you need: $\displaystyle\lim_{h\to 0} \frac{\cos(h) - 1}{h} = 0$, $\qquad \displaystyle\lim_{h\to 0} \frac{\sin(h)}{h} = 1$. Using these we see that

$$\frac{\sin(x+h) - \sin(x)}{h} \to \cos x$$

as $h \to 0$.

Theorem. If f is differentiable at a then f is continuous at a.

Proof. It is seen that

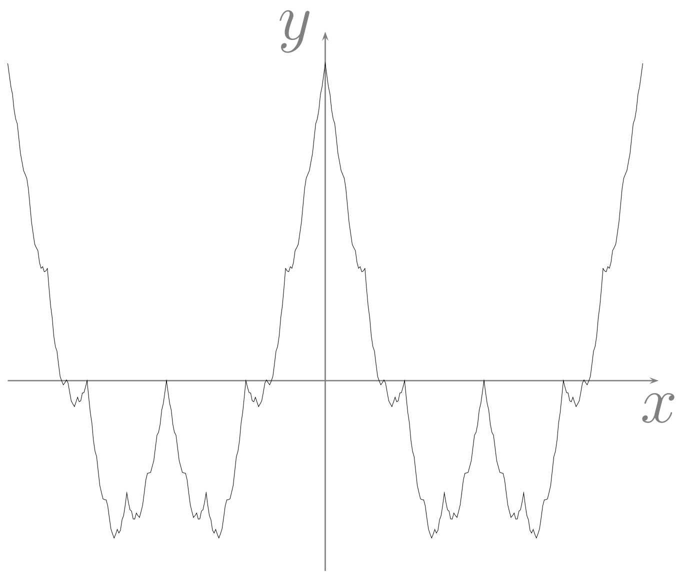
$$\begin{aligned}
\lim_{h\to 0} f(a+h) - f(a) &= \lim_{h\to 0} \left(\frac{f(a+h) - f(a)}{h} \times h \right) \\
&= \left(\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \right) \times \lim_{h\to 0} h \\
&= f'(a) \times 0 = 0
\end{aligned}$$

since f is differentiable at a. Accordingly,

$$\lim_{h\to 0} f(a+h) = f(a).$$

Corollary. If f is not continuous at a then it is not differentiable at a.

Remark Differentiability is a much stronger property than continuity. While every differentiable function is continuous, there exist functions that are continuous everywhere but differentiable nowhere. In fact, functions that are differentiable everywhere are a very rare breed, even among the continuous functions. An example of a function that is continuous everywhere but differentiable nowhere is the *Weierstrass function* whose graph is shown below.



Rules for differentiation

Many differentiable functions may be constructed via addition, subtraction, multiplication and division ...

Theorem. Suppose that f and g are differentiable functions at x. Then,

- $(f+g)'(x) = f'(x) + g'(x)$
- $(cf)'(x) = cf'(x), \quad \text{where } c \text{ is a constant}$
- $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$ (product rule)
- $(f/g)'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$ (quotient rule)

  provided that $g(x) \neq 0$.

... and function composition:

Theorem. Suppose that g is differentiable at the point x and f is differentiable at the point $g(x)$. Then,

$$(f \circ g)'(x) = f'(g(x))g'(x) \quad (\text{chain rule}).$$

Exercise. Use the product rule and induction on n to prove that

$$\frac{d}{dx}x^n = nx^{n-1}$$

for $n \in \mathbb{N}$.

Example. Find the derivative of

$$f(x) = \left[\sin\left(\frac{x}{x^2+1}\right)\right]^2.$$

Solution. We have

$$\begin{aligned}
f'(x) &= \left(\left[\sin\left(\frac{x}{x^2+1}\right)\right]^2\right)' \\
&= 2\left[\sin\left(\frac{x}{x^2+1}\right)\right]\left(\sin\left(\frac{x}{x^2+1}\right)\right)' && \text{chain rule} \\
&= 2\sin\left(\frac{x}{x^2+1}\right)\cos\left(\frac{x}{x^2+1}\right)\left(\frac{x}{x^2+1}\right)' && \text{chain rule} \\
&= \sin\left(\frac{2x}{x^2+1}\right)\frac{1\cdot(x^2+1)-x\cdot 2x}{(x^2+1)^2} && \text{quotient rule} \\
&= \sin\left(\frac{2x}{x^2+1}\right)\frac{1-x^2}{(x^2+1)^2}
\end{aligned}$$

Proofs of differentation rules

Proof of product rule. Suppose that f and g are differentiable at the point x. The difference quotient of fg at x gives

$$\begin{aligned}
&\frac{(fg)(x+h)-(fg)(x)}{h} \\
&\qquad = \frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\
&\qquad = g(x+h)\frac{f(x+h)-f(x)}{h} + f(x)\frac{g(x+h)-g(x)}{h}.
\end{aligned}$$

Since the function g is differentiable at the point x (we have just proved that), it is continuous at that point! Hence,

$$g(x+h) \to g(x) \quad \text{as} \quad h \to 0.$$

Accordingly,

$$\frac{(fg)(x+h)-(fg)(x)}{h} \quad \to \quad g(x)f'(x)+f(x)g'(x)$$

as $h \to 0$.

Proofs of the other differentiation rules are found in most undergraduate calculus textbooks. You should prove all of them yourself. It gives you a great pleasure!

Implicit differentiation

Idea. On use of the chain rule, determine the derivative of a function which is implicitly defined.

Example. Determine the tangent line to the curve defined by

$$x^4 - x^2y^2 + y^4 = 13$$

at the point $(2, 1)$.

Write

$$x^4 - x^2y(x)^2 + y(x)^4 = 13$$

and differentiate both sides:

$$4x^3 - 2xy(x)^2 - 2x^2y(x)y'(x) + 4y(x)^3y'(x) = 0.$$

Evaluate at $(x, y) = (2, 1)$ and solve for $y'(2)$:

$$32 - 4 - 8y'(2) + 4y'(2) = 0 \quad \Rightarrow \quad y'(2) = 7.$$

Equation of tangent line at $(2, 1)$:

$$y - 1 = 7(x - 2).$$

Theorem. Suppose that q is a rational number. Then

$$\frac{d}{dx}x^q = qx^{q-1}.$$

Proof. Since q is a rational number, there exist integers m and n such that

$$q = \frac{m}{n}.$$

Write

$$y = x^q = x^{m/n}$$

and take the nth power of both sides, leading to

$$y^n = x^m.$$

Differentiation of both sides with respect to x yields

$$ny^{n-1}\frac{dy}{dx} = mx^{m-1}.$$

Hence,

$$\begin{aligned}\frac{dy}{dx} &= \frac{mx^{m-1}}{ny^{n-1}}\\ &= \frac{m}{n}\frac{x^{m-1}}{x^{q(n-1)}}\\ &= qx^{(m-1)-qn+q}\\ &= qx^{q-1}\end{aligned}$$

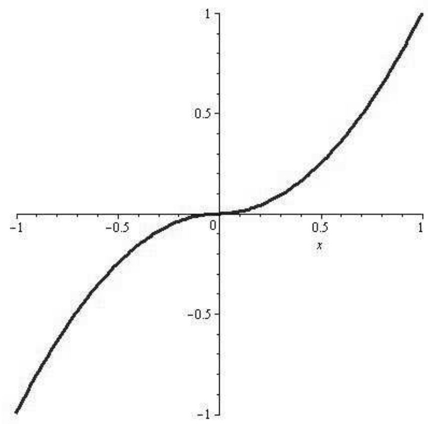
as required.

Split functions

Example. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} -x^2, & x < 0, \\ x, & x = 0, \\ x^2, & x > 0. \end{cases}$$

At $x = 0$, $f(x) = x$ and hence, at a first glance, you could guess that $f'(0) = 1$.



NO! The formulas and rules we have so far are only OK if we have the same expression for $f(x)$ on a little open interval.

Remember, we are calculating $\lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ so we need to be using the same formula for $f(x+h)$ and $f(x)$.

For 'split functions' like this, you need to be rather careful about what happens **at the join**. This usually involves taking right and left limits separately at the join point.

Here

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{h^2-0}{h} = \lim_{h\to 0^+} h = 0$$

and similarly

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{-h^2-0}{h} = 0.$$

Thus $f'(0) = \lim\limits_{h\to 0} \dfrac{f(0+h)-f(0)}{h}$ exists and equals 0.

At a point like 1, we can just use the usual method of finding $f'(x)$ as **at and near** this point $f(x)$ is given by a single polynomial formula.

$$f'(x) = \begin{cases} -2x, & x < 0, \\ 0, & x = 0, \\ 2x, & x > 0. \end{cases}$$

Theorem. Suppose that a is a fixed real number and that a function f is defined by the rule

$$f(x) = \begin{cases} p(x) & \text{if} \quad x \geq a \\ q(x) & \text{if} \quad x < a, \end{cases}$$

where p and q are defined on some open interval containing a. If f is continuous at a and $p'(a) = q'(a)$ then f is differentiable at $x = a$.

Remark. Note that the requirement of f being continuous at a is equivalent to demanding that $p(a) = q(a)$ since p and q are continuous at a.

Example. Suppose that $f : (0, \infty) \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 4\sqrt{x} & \text{if} \quad 0 < x \leq 1 \\ bx^2 + c & \text{if} \quad x > 1, \end{cases}$$

where b and c are real numbers. Find all possible values of b and c such that f is (i) continuous at $x = 1$ and (ii) differentiable at $x = 1$.

Solution. In our case, both functions $p(x) = 4\sqrt{x}$ and $q(x) = bx^2 + c$ are continuous.

(i). f is continuous at $x = 1$ if $p(1) = q(1)$. We have $p(1) = 4$ and $q(1) = b + c$, that is if

$$b + c = 4, \tag{1}$$

the function f is continuous at $x = 1$.

(ii). We have $p'(x) = \frac{2}{\sqrt{x}}$ and $q'(x) = 2bx$. Hence $p'(1) = 2$ and $q'(1) = 2b$. That is, the function f is differentiable at $x = 1$ if

$$2 = 2b, \text{ or, equivalently, } b = 1.$$

From equality (1), we now obtain that

$$c = 3.$$

Thus, the function

$$f(x) = \begin{cases} 4\sqrt{x} & \text{if} \quad 0 < x \leq 1 \\ x^2 + 3 & \text{if} \quad x > 1, \end{cases}$$

is continuous and differentiable at $x = 1$.

Derivatives and function approximation

By definition, if a function f is differentiable at a, we have

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

so that

$$f'(a) \approx \frac{f(x) - f(a)}{x - a}$$

if x is 'sufficiently close' to a. Thus,

$$f(x) \approx f(a) + f'(a)(x - a)$$

and the right-hand side may be regarded as an 'approximation' of $f(x)$ in a neighbourhood of $x = a$.

Problem. Solve $f(x) = x^4 + \cos x + 2\sin x = 1.1$.

In this example, take $x_0 = 0$. Then $f(x_0) = 1$ and

$$f'(x) = 4x^3 - \sin x + 2\cos x,$$

so $f'(x_0) = 2$.

Thus

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) = 1 + 2x.$$

Now solving $1 + 2x = 1.1$ is easy: $x = 0.05$. This (hopefully) gives a reasonable approximation to $f(x) = 1.1$.

This works fine as long as $|x - x_0|$ is small.

Derivatives and rates of change

Many physical processes involve quantities (such as temperature, volume, concentration, velocity) that change with time but may not be independent of each other. Their rates of change may then be obtained by careful application of the chain rule.

Example. A spherical balloon is being inflated and its radius is increasing at a constant rate of 6 mm/sec. At what rate is its volume increasing when the radius of the balloon is 20 mm?

Let $V(t)$ be the volume of the balloon and $r(t)$ be its radius at time t. Alternatively, let $\tilde{V}(r)$ be the volume of the balloon as a function of its radius r given by

$$\tilde{V}(r) = \frac{4}{3}\pi r^3$$

so that

$$\frac{d\tilde{V}}{dr} = 4\pi r^2.$$

Then, the chain rule implies that

$$\frac{dV}{dt} = \frac{d\tilde{V}}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

We are told that $\frac{dr}{dt} = 6$ so that

$$\frac{dV}{dt} = 4\pi(20)^2 \times 6 = 9600\pi$$

at $r = 20$.

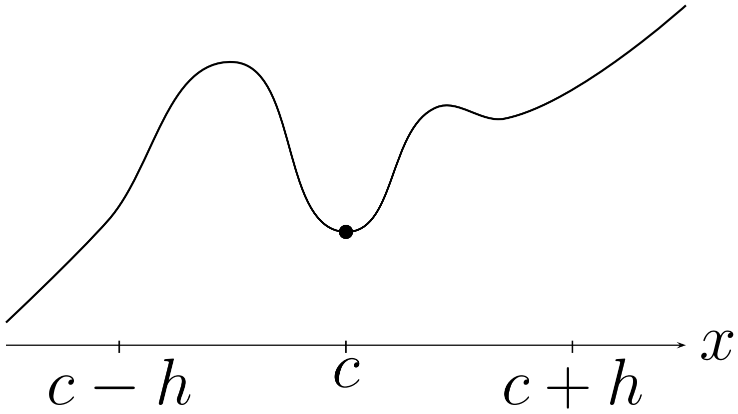
Hence, the volume is increasing at a rate of 9600π mm^3/sec when the radius is 20 mm.

The above example illustrates an approach to solving such problems.

1. Define variables for the quantities involved.
2. Write down what is known in terms of these variables and their derivatives.
3. Write down what you need to find in terms of these variables and their derivatives.
4. Write down anything else you know that relates the variables (for example, a volume or area formula).
5. Use the chain rule (or implicit differentiation) to find the relevant derivative.

Local maximum, local minimum and stationary points

In this section, we begin to develop a systematic approach to locating maxima and minima. A complete approach will be presented in the next chapter.



Local minimum point c

Definition. Let f be defined on some interval I.

- We say that a point c in I is a local minimum point if there exists an $h > 0$ such that

$$f(c) \leq f(x) \quad \text{for all} \quad x \in (c-h, c+h) \cap I.$$

- We say that a point d in I is a local maximum point if there exists an $h > 0$ such that

$$f(x) \leq f(d) \quad \text{for all} \quad x \in (d-h, d+h) \cap I.$$

Theorem. Suppose that f is defined on (a, b) and has a local maximum or minimum point at c for some c in (a, b). If f is differentiable at c then $f'(c) = 0$.

Definition. If a function f is differentiable at a point c and $f'(c) = 0$ then c is called a stationary point of f.

The converse of Theorem is false! That is, you can easily have $f'(c) = 0$ with f having neither a local max nor a local min at c. For instance, take $f(x) = x^3$.

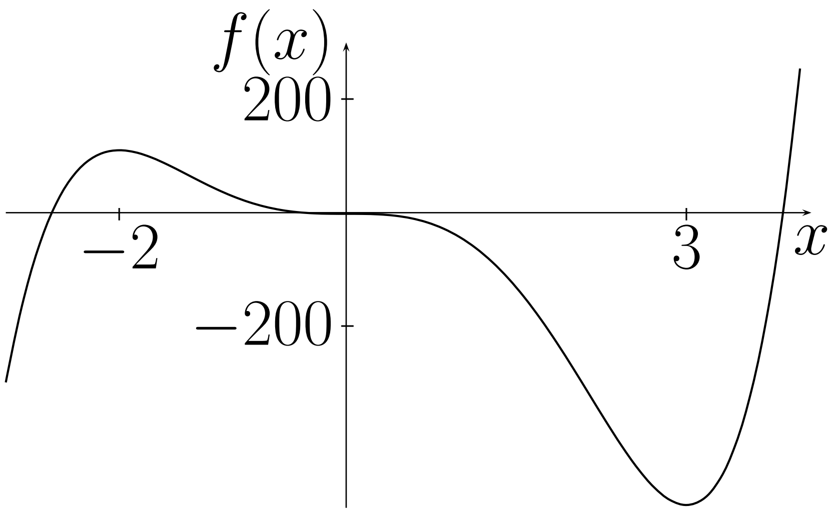
Of course it is easy to give examples of functions where the local extreme points occur at points of nondifferentiability ($f(x) = |x|$).

Nonetheless, our method for hunting down local extreme points will involve reducing the possible points to consider from the infinite set (a, b) to a finite set.

Example. Find all the stationary, maximum and minimum points of the function $f : [-3, 4] \to \mathbb{R}$ defined by

$$f(x) = 4x^5 - 5x^4 - 40x^3 - 2$$

.



Differentiation yields

$$\begin{aligned} f'(x) &= 20x^4 - 20x^3 - 120x^2 \\ &= 20x^2(x^2 - x - 6) \\ &= 20x^2(x+2)(x-3). \end{aligned}$$

Result:

- $x = -3$: local minimum point
- $x = -2$: stationary point: local maximum point
- $x = 0$: stationary point: point of inflection
- $x = 3$: stationary point: global minimum point
- $x = 4$: global maximum point