Continuous functions

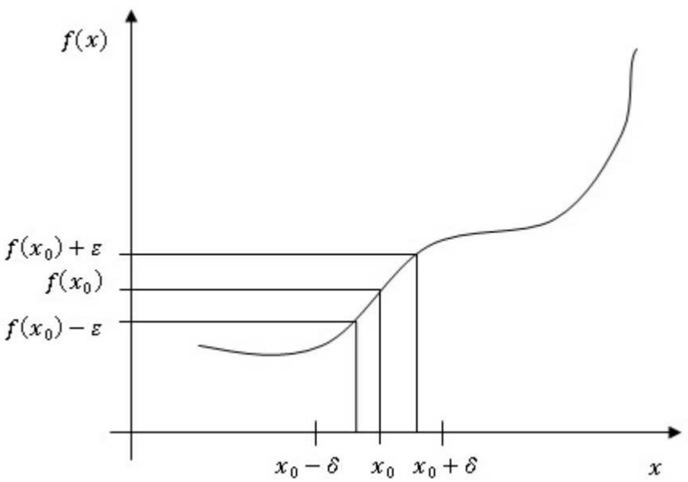
Recall that: f is continuous at $x_0 \in \mathbb{R}$ if

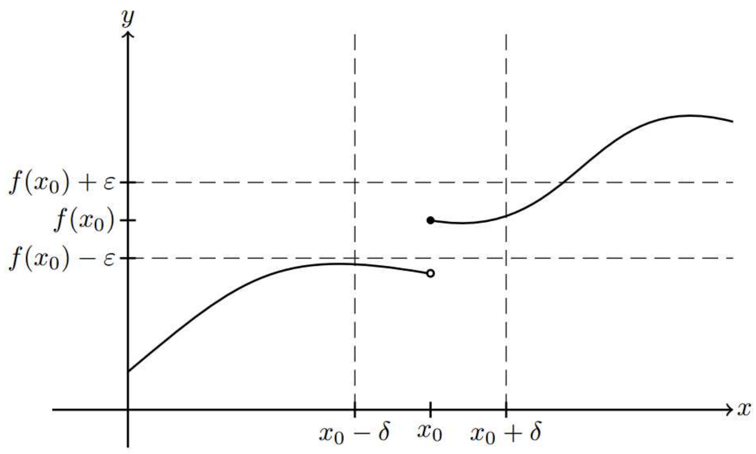
$$\lim_{x \to x_0} f(x) = f(x_0).$$

The value $f(x_0)$ needs to be defined and the limit needs to exist!

Formal definition is: f is continuous at $x_0 \in \mathbb{R}$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(x_0)| < \epsilon \quad \text{whenever} \quad |x - x_0| < \delta.$$





Combining functions.

Theorem. Suppose that the functions f and g are continuous at a point x_0. Then

$$f+g, \quad f-g, \quad fg$$

are continuous at x_0. If $g(x_0) \neq 0$ then

$$f/g$$

is also continuous at x_0.

Proof. Suppose that f and g are continuous at x_0. Then,

$$\lim_{x \to x_0} f(x) = f(x_0), \qquad \lim_{x \to x_0} g(x) = g(x_0)$$

by the definition of continuity at a point. Therefore,

$$\begin{aligned}
\lim_{x \to x_0} (f+g)(x) &= \lim_{x \to x_0} (f(x) + g(x)) && \text{(def. of } f+g\text{)} \\
&= \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x) && \text{(limit rule)} \\
&= f(x_0) + g(x_0) && (f, g \text{ cont.}) \\
&= (f+g)(x_0) && \text{(def. of } f+g\text{)}.
\end{aligned}$$

Hence $f+g$ is continuous at x_0.

The proofs that the functions $f-g$, fg and f/g are continuous at x_0 are similar.

If a function f is continuous at every point $x_0 \in \mathbb{R}$, then f is called continuous everywhere.

Claim. The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = x$$

is continuous everywhere.

Proof. Take $x_0 \in \mathbb{R}$ and $\epsilon > 0$. Define $\delta = \epsilon$ and assume that

$$|x - x_0| < \delta.$$

Then,

$$|f(x) - f(x_0)| = |x - x_0| < \delta = \epsilon.$$

Thus, we conclude that

$$\text{for every } \epsilon > 0, \text{ there exists a number } \delta > 0 \text{ such that}$$

$$|f(x) - f(x_0)| < \epsilon \quad \text{whenever} \quad |x - x_0| < \delta.$$

Simple exercise. Prove that the constant function is continuous everywhere.

Example. Show that polynomials and rational functions are continuous at every point of their respective domains.

Any polynomial can be obtained from f and constant functions via addition and multiplication, e.g.
$x^3 - 4x^2 + 5 = [(x \times x \times x)] + [(-4) \times x \times x] + 5,$
and hence is continuous everywhere.

Any rational function is of the form $\dfrac{p(x)}{q(x)},$
where p and q are two (continuous) polynomials, and is therefore continuous at every point x_0 for which $q(x_0) \neq 0$.

Example. Show that the functions $\sin$ and $\cos$ are continuous everywhere.

For $x \in \mathbb{R}$ we put the point on the unit circle such that the distance from $(1,0)$ to this point is x around the circumference. Let $P(x) = (\cos x, \sin x)$. Suppose $x, x_0 \in \mathbb{R}$. The direct distance from $P(x_0)$ to $P(x)$ must be less than the distance along the curve.

So $|P(x_0) - P(x)| < |x - x_0|$.

Squaring distances: $|P(x_0) - P(x)|^2 < (x - x_0)^2$,

$(\sin x - \sin x_0)^2 + (\cos x - \cos x_0)^2 < (x - x_0)^2$,

$(\sin x - \sin x_0)^2 \leq (x - x_0)^2$,

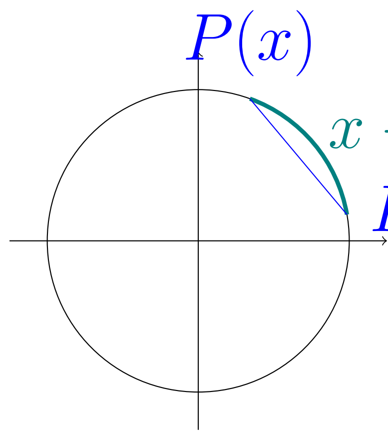
$0 \leq |\sin x - \sin x_0| \leq |x - x_0|$.

Since $\lim_{x \to x_0} |x - x_0| = 0$,

by the Pinching Theorem we conclude $\lim_{x \to x_0} \sin x = \sin x_0$.

Thus $\sin$ is continuous everywhere.

The proof for $\cos$ is the same.



Even larger classes of continuous functions may be obtained in the following manner:

Theorem. Suppose that f is continuous at x_0 and that g is continuous at $f(x_0)$. Then $g \circ f$ is continuous at x_0.

Proof.

$$\begin{aligned}
\lim_{x \to x_0} (g \circ f)(x) &= \lim_{x \to x_0} (g(f(x)) && \text{(def. of } g \circ f\text{)} \\
&= g(\lim_{x \to x_0} f(x)) && \text{(cont. of } g\text{)} \\
&= g(f(x_0)) && \text{(cont. of } f\text{)} \\
&= (g \circ f)(x_0). && \text{(def. of } g \circ f\text{)}
\end{aligned}$$

Hence $g \circ f$ is continuous at x_0.

Example. Why is $f(x) = \sqrt{\cos^2(x) + 3}$ continuous everywhere?

Short answer: It is a combination of continuous functions and hence is continuous.

Longer answer: Let $g_1(x) = \cos x$, let $g_2(x) = x^2 + 3$ and let $g_3(x) = \sqrt{x}$. Then

$$f(x) = g_3(g_2(g_1(x)))$$

Now g_1, g_2 and g_3 are continuous everywhere they are defined.
The range of g_2 lies inside the domain of g_3 so the domain of f is all of $\mathbb{R}$.
Hence the composition f is also continuous everywhere.

Continuity on the interval

Definition. Suppose that f is a real-valued function defined on an open interval (a, b). We say that f is continuous on (a, b) if f is continuous at every point of the interval (a, b).

Definition. Suppose that f is a real-valued function defined on a closed interval $[a, b]$. We say that

- f is continuous at the endpoint a if

$$\lim_{x \to a^+} f(x) = f(a),$$

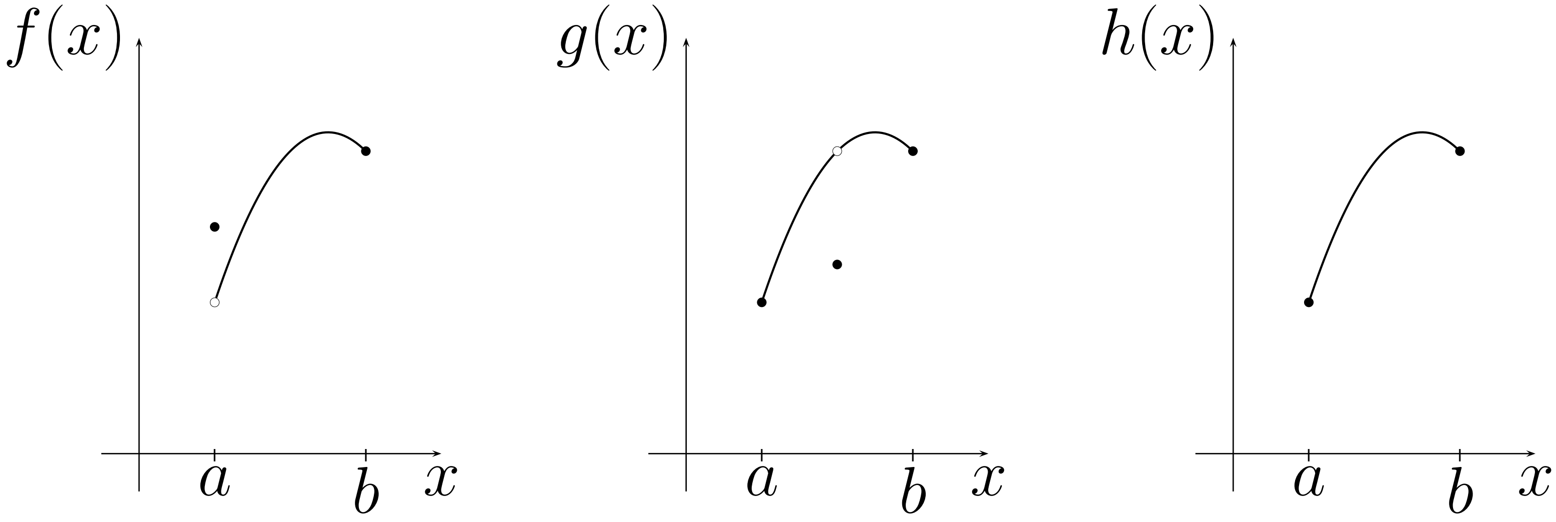
- f is continuous at the endpoint b if

$$\lim_{x \to b^-} f(x) = f(b),$$

- f is continuous on $[a, b]$ if f is continuous on (a, b) and at each of the endpoints a and b.

Example. The function $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ is continuous on $[0, \infty)$.

Example. Consider the functions f, g and h, whose graphs are shown below.



All three functions are defined on the interval $[a, b]$.

- f is continuous on the open interval (a, b) and at the endpoint b.
- g is continuous at the endpoints a and b but not continuous on the open interval (a, b).
- h is continuous on the closed interval $[a, b]$.

The intermediate value theorem

Look at the following two claims:

1) A plane takes off and after 12 minutes it is at 20,000 feet. At some point, it must have passed through an altitude of 10,000 feet.

2) Yesterday Shonky Services shares were \$2.34 a share. Today they are trading at \$1.47 a share. At some point they must have been trading at \$2.00 a share.

The first of these is true, the second not. The difference lies in the properties of the two functions involved:

$A(t) =$ altitude at time t

$S(t) =$ share price at time t.

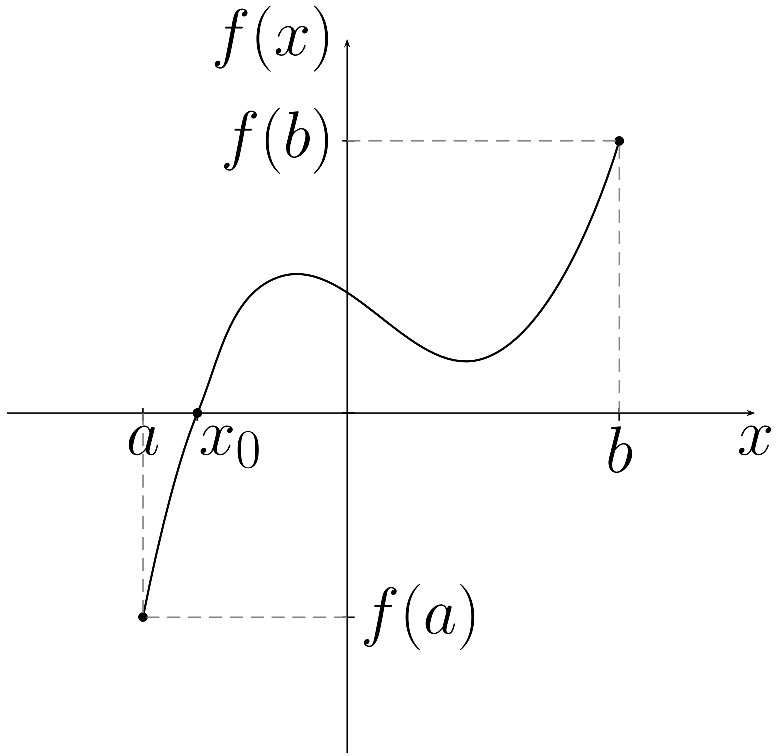
The first is a continuous function on a nice domain $[0, 12]$. The second is much more complicated!

The intermediate value theorem (IVT) (version 1)

Theorem. Suppose that the function $f : [a, b] \mapsto \mathbb{R}$ is continuous. Assume also that

$$f(a) < 0 \quad \text{and} \quad f(b) > 0.$$

Then there is a point x_0 in the open interval (a, b) at which $f(x_0) = 0$.

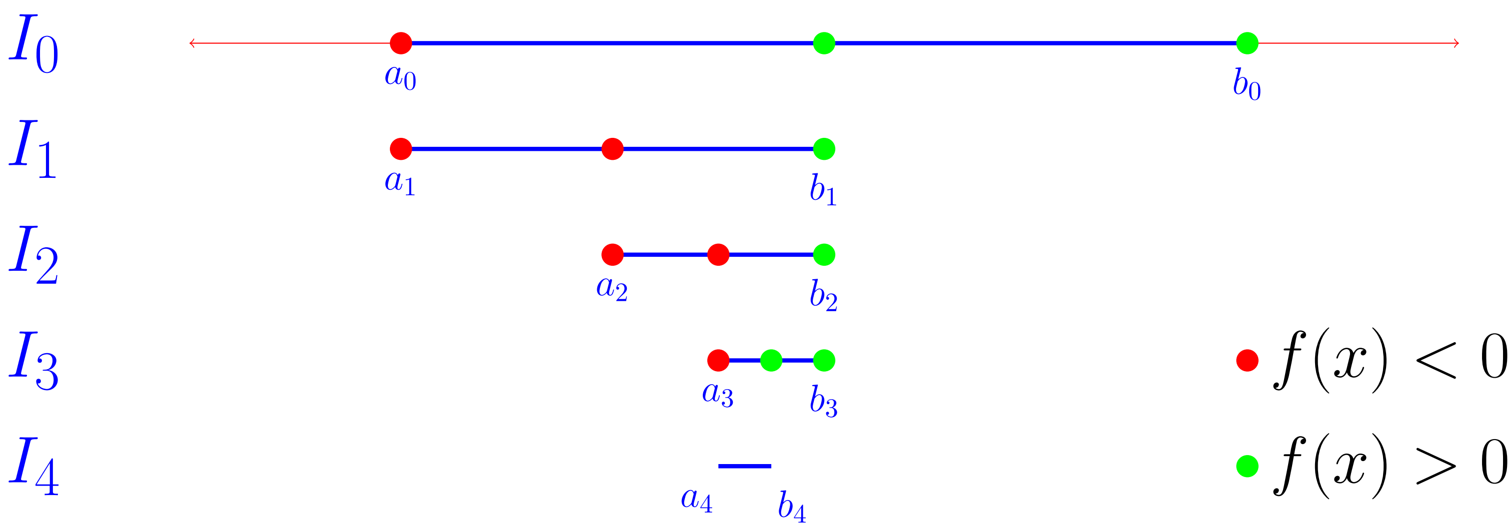


Proof. We will recursively define a sequence of nested, closed subintervals of $[a, b]$ whose endpoints converge to a point in $[a, b]$ at which $f(x) = 0$.

Let $a_0 = a$ and $b_0 = b$ and let m_0 be the midpoint of the interval $I_0 = [a_0, b_0]$.

Calculate $f(m_0)$. If this is 0 we have found c! If $f(m_0) < 0$ then let $I_1 = [a_1, b_1] := [m_0, b_0]$. If $f(m_0) > 0$ then let $I_1 = [a_1, b_1] := [a_0, m_0]$. In either of these cases the function changes sign on I_1.

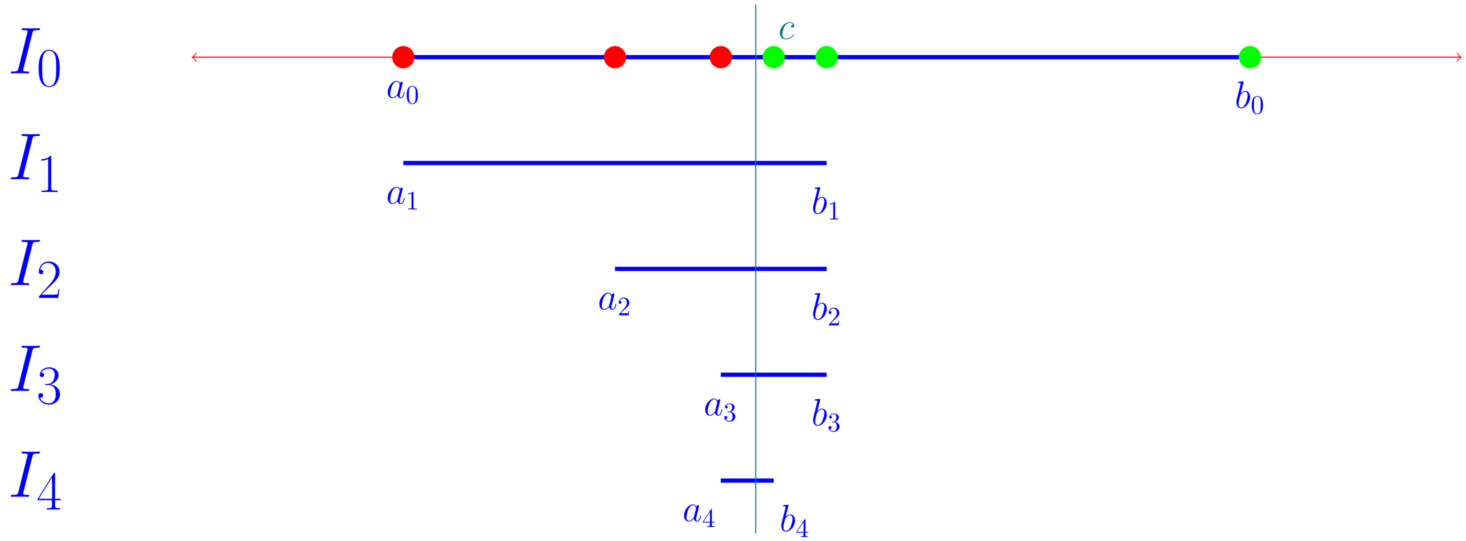
Now repeat this, looking at what happens at the midpoint of I_1. Indeed continuing, we get a nested sequence of closed intervals $I_n = [a_n, b_n]$ with $f(a_n) < 0$ and $f(b_n) > 0$.



By the construction $|a_n - b_n| = 2^{-n}(b-a)$ and

$$a_0 \le a_1 \le a_2 \le \cdots \le b_2 \le b_1 \le b_0.$$

The sequences $\{a_n\}$ and $\{b_n\}$ must converge to a common value, say c.



Suppose $f(c) > 0$. Let $\epsilon = f(c)/2$. By continuity, there exists $\delta > 0$ such that if $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$. Whatever δ is, there is some a_n with $|a_n - c| < \delta$ and hence $|f(a_n) - f(c)| < \epsilon$.

This says that $$f(c) - \epsilon < f(a_n) < f(c) + \epsilon$$

But $f(c)/2 > 0$ and $f(a_n) \le 0$ which is impossible!

Similarly, it impossible to have $f(c) < 0$. Hence $f(c) = 0$. $\blacksquare$

The proof relies crucially on the fact that $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist.

This is the point where we need to know that, unlike the rationals, the reals don't have any holes!

This is usually stated as the Least Upper Bound (LUB) Property of the real numbers.

'LUB Property': If $a_1 \le a_2 \le a_3 \le \ldots$ is a nondecreasing sequence of real numbers and there is some upper bound M for the sequence, ie

$$a_k \le M, \qquad \text{for all } k,$$

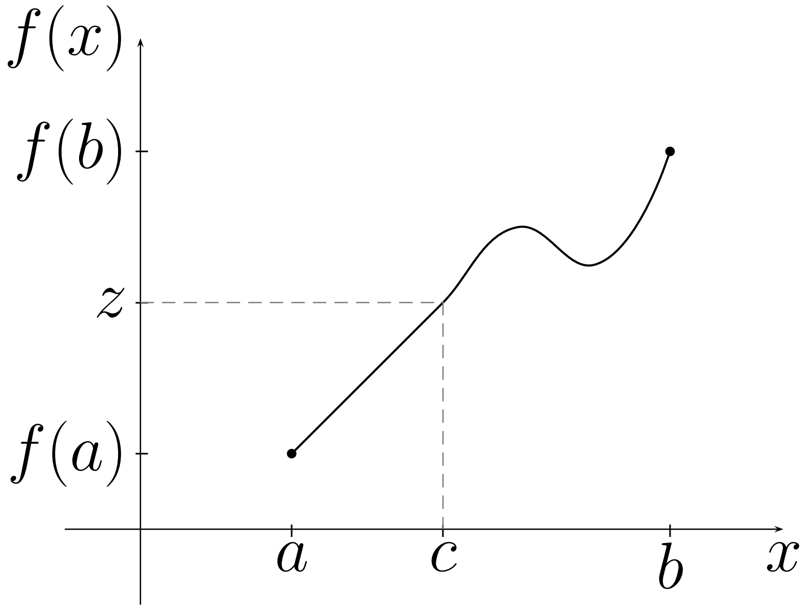
then a_k converges to some real number L.

This is false if real is replaced by rational!

(which is why the theorem is false in $\mathbb{Q}$)

The intermediate value theorem (version 2)

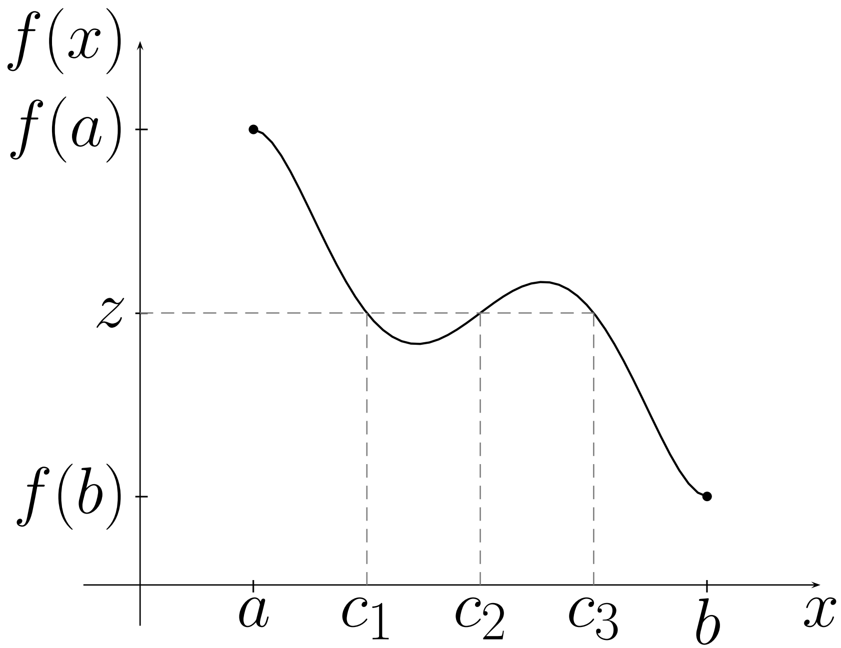
The intermediate value theorem. Suppose that f is continuous on the closed interval $[a, b]$. If z lies between $f(a)$ and $f(b)$ then there exists at least one real number c in $[a, b]$ such that $f(c) = z$.



Proof is left as an exercise.
Hint: Apply the intermediate value theorem (version 1) to the function $g(x) = f(x) - z$.

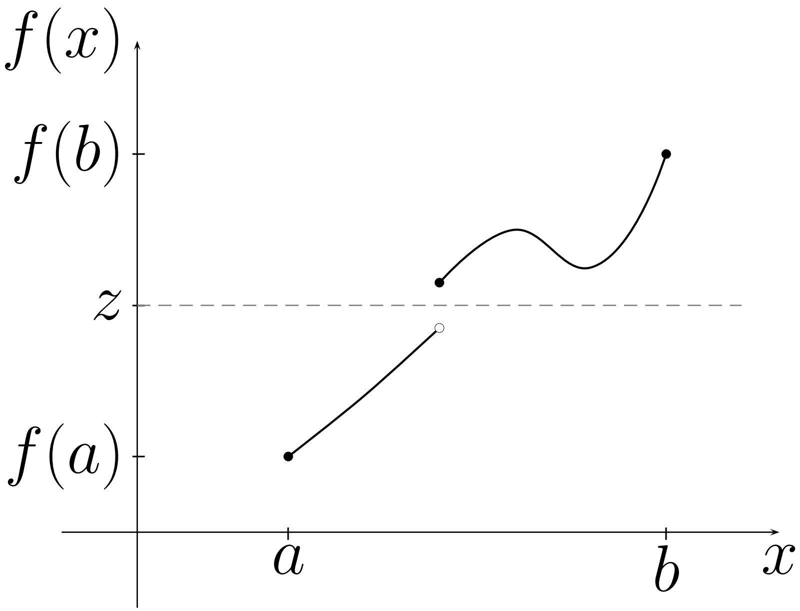
Remarks.

• The number c in $[a, b]$ may not be unique.



There exists three numbers c_i with $f(c_i) = z$.

- Continuity of f is crucial.



For all $c \in [a, b]$, we see that $f(c) \neq z$.

Example. Find $x > 0$ such that $\cos x = x$.

Let $f(x) = x - \cos x$. (So f is continuous everywhere!)
Now $f(0) = -1$ and $f(\pi/2) = \pi/2$ so there exists (at least one point) $c \in [0, \pi/2]$ such that $f(c) = 0$, that is $\cos c = c$.

One may calculate the intervals $[a_n, b_n]$ in the proof of the IVT:

a_n	b_n	m_n	$f(m_n)$
0.00000	1.5708	0.78540	0.07829
0.00000	0.78540	0.39270	-0.53118
0.39270	0.78540	0.58905	-0.24242
0.58905	0.78540	0.68722	-0.08579
0.68722	0.78540	0.73631	-0.00464
0.73631	0.78540	0.76086	0.03662

So $c \in (0.73631, 0.78540)$. This computational method works fine — but only if we know that the solution actually exists!
Exercise. Why is there only one solution in $[0, \pi/2]$?

Example. Show that there exists a solution $c \in [1,2]$ of the equation

$$\sqrt{c} = c^2 - 1$$

and approximate its value.

Consider the function $f(x) = \sqrt{x} - x^2 + 1$.
Since $f(1) = 1 > 0$ and $f(2) = \sqrt{2} - 3 < 0$, by IVT we have that there exists $c \in [1,2]$ such that $f(c) = 0$.
That is $\sqrt{c} - c^2 + 1 = 0$ or $\sqrt{c} = c^2 - 1$.
Let's find an approximate value of c.
$f(1.5) \sim -0.026 < 0 \Rightarrow c \in [1, 1.5]$
$f(1.25) \sim 0.55 > 0 \Rightarrow c \in [1.25, 1.5]$
$f(1.375) \sim 0.28 > 0 \Rightarrow c \in [1.375, 1.5]$
$f(1.4375) \sim 0.13 > 0 \Rightarrow c \in [1.4375, 1.5]$
$f(1.46875) \sim 0.05 > 0 \Rightarrow c \in [1.46875, 1.5]$
$f(1.484375) \sim 0.01 > 0 \Rightarrow c \in [1.484375, 1.5]$
$f(1.4921875) \sim -0.005 < 0 \Rightarrow c \in [1.484375, 1.4921875]$

Example. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and

$$\lim_{x\to-\infty} f(x) = -1, \qquad \lim_{x\to\infty} f(x) = 1.$$

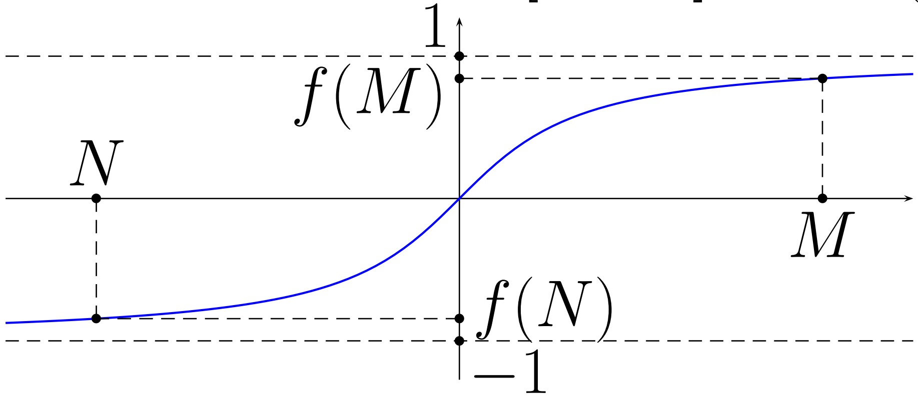
Show that there exists a point $c \in \mathbb{R}$ such that $f(c) = 0$.

(We can't apply IVT directly as we are not working on a closed and bounded interval.)

Let $\epsilon = \frac{1}{2}$. As $\lim_{x\to\infty} f(x) = 1$ there exists M such that for all $x \geq M$, $|f(x) - 1| < \frac{1}{2}$. In particular, $f(M) > \frac{1}{2} > 0$.

Similarly, as $\lim_{x\to-\infty} f(x) = -1$ there exists N such that for all $x \leq N$, $|f(x) - (-1)| < \frac{1}{2}$. In particular, $f(N) < -\frac{1}{2} < 0$.

Now apply the IVT to f on the closed and bounded interval $[N, M]$ to deduce that there is a number $c \in [N, M]$ with $f(c) = 0$.



The maximum-minimum theorem

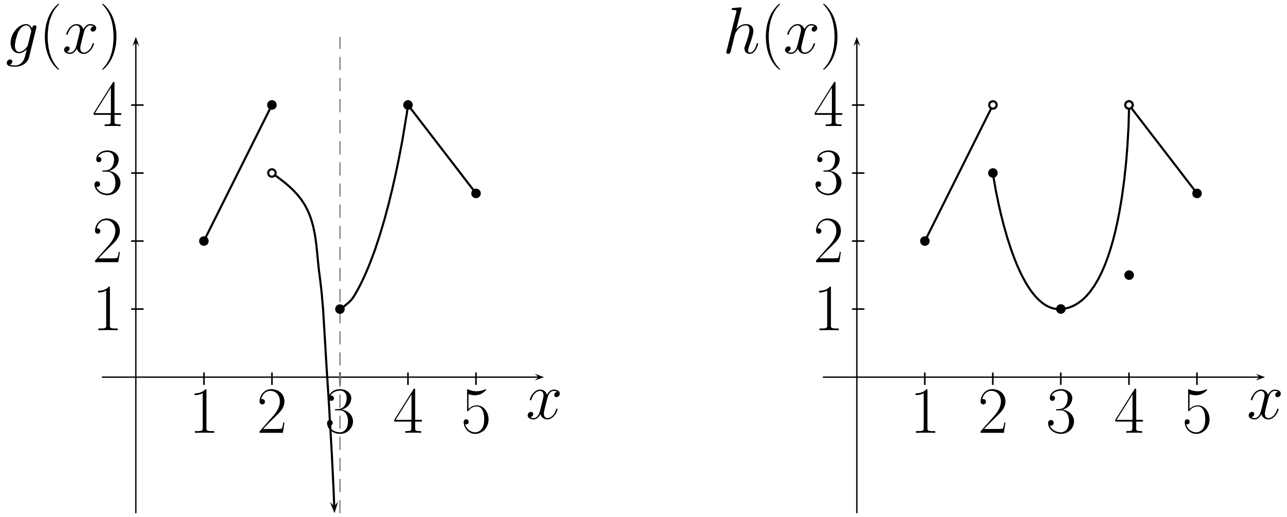
Definition. Suppose that f is defined on a closed interval $[a, b]$.

- We say that a point c in $[a, b]$ is an absolute minimum point for f on $[a, b]$ if
$$f(c) \leq f(x) \quad \text{for all} \quad x \in [a, b].$$
The corresponding value $f(c)$ is called the absolute minimum value of f on $[a, b]$. If f has an absolute minimum point on $[a, b]$ then we say that f attains a minimum on $[a, b]$.

- We say that a point d in $[a, b]$ is an absolute maximum point for f on $[a, b]$ if
$$f(x) \leq f(d) \quad \text{for all} \quad x \in [a, b].$$
The corresponding value $f(d)$ is called the absolute maximum value of f on $[a, b]$. If f has an absolute maximum point on $[a, b]$ then we say that f attains a maximum on $[a, b]$.

An absolute maximum point and an absolute minimum point are sometimes referred to as a global maximum point and a global minimum point.

Example. Consider the functions g and h, which are illustrated below.



The absolute minimum and maximum points of g and h on $[1, 5]$ are recorded in the following table.

	g	h
Absolute minimum points	none	3
Absolute minimum value	n.a.	1
Absolute maximum points	2, 4	none
Absolute maximum value	4	n.a.

This example shows that a function $f : [a, b] \to \mathbb{R}$ need not have an absolute maximum point (or an absolute minimum point) on $[a, b]$.

The maximum-minimum theorem. If f is continuous on a closed interval $[a, b]$ then f attains a minimum and maximum on $[a, b]$. That is, there exist points c and d in $[a, b]$ such that

$$f(c) \leq f(x) \leq f(d)$$

for all x in $[a, b]$.

Proof is a bit complicated and, therefore, is omitted. It must use the facts that

1. the domain $[a, b]$ is closed and bounded.
2. f is continuous on $[a, b]$
3. the LUB-Property

If you drop any of these conditions the theorem is false!

Bounded Functions.

Definition. Suppose that $f : A \to \mathbb{R}$.
We say that f is bounded on A if there exists some positive number M such that $|f(x)| \leq M,$ for all $x \in A$.

The domain A is a clearly vital part of this definition. The function $f(x) = x^2$ is bounded on the domain $[0, 100]$, but not on the domain $\mathbb{R}$.

The Max-Min Theorem implies that
Theorem. If f is continuous on a closed and bounded interval $[a, b]$, then it is a bounded function on $[a, b]$.

Don't get bounded intervals and bounded functions confused!

Note that a function can be bounded without having a maximum or minimum value:

Examples. $f : [0, \infty] \to \mathbb{R}$, $f(x) = \dfrac{x^2}{1 + x^2}$.

It is clear that for all $x \in [0, \infty)$, $0 \leq f(x) \leq 1$.

The lower bound of 0 is achieved and is a minimum value.

The upper bound of 1 is the least upper bound for the function, but it is not achieved and f has no maximum value.

This is an example that we can not drop the assumption that the interval $[a, b]$ is closed in the Max-Min theorem.

Indeed, using MAPLE we may see:

>plots[interactive]$(x^2/(x^2+1))$