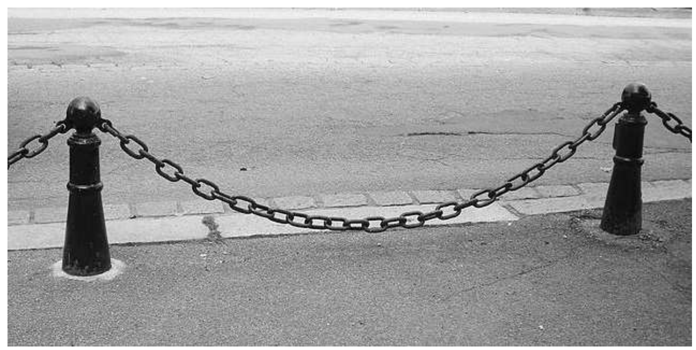
The hyperbolic functions

Question: What shape does a hanging chain make?



Question: What is $\displaystyle\int \frac{dx}{\sqrt{x^2+25}}$?

The answers to both these questions involve a family of functions known as the **hyperbolic functions**.

Hyperbolic sine and cosine functions

The **hyperbolic cosine function** $\cosh : \mathbb{R} \to \mathbb{R}$ is defined by

$$\cosh x := \frac{e^x + e^{-x}}{2}.$$

The **hyperbolic sine function** $\sinh : \mathbb{R} \to \mathbb{R}$ (pronounced 'shine') is defined by

$$\sinh x := \frac{e^x - e^{-x}}{2}.$$

Questions:

1. These are just simple combinations of the exponential function, so why bother with giving them names?
2. What have they got to do with cos and sin?
3. Why 'hyperbolic'?

Although the graphs of these functions are nothing like those of cos and sin, they have a fantastic range of identities that mimic those of the standard trig functions. We'll be able to use these to find antiderivatives for a whole range of new functions.

Properties

1. cosh and sinh are differentiable with

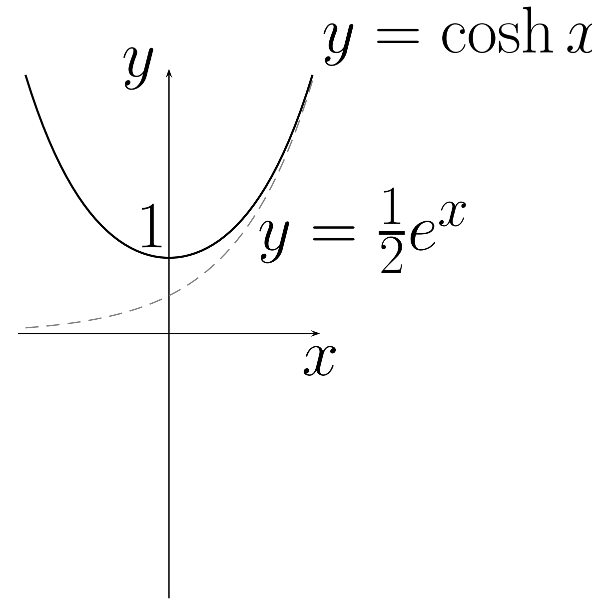
$$\frac{d}{dx}(\sinh x) = \cosh x, \qquad \frac{d}{dx}(\cosh x) = \sinh x$$

so that $\cosh x$ and $\sinh x$ obey the differential equation

$$\frac{d^2y}{dx^2} = y.$$

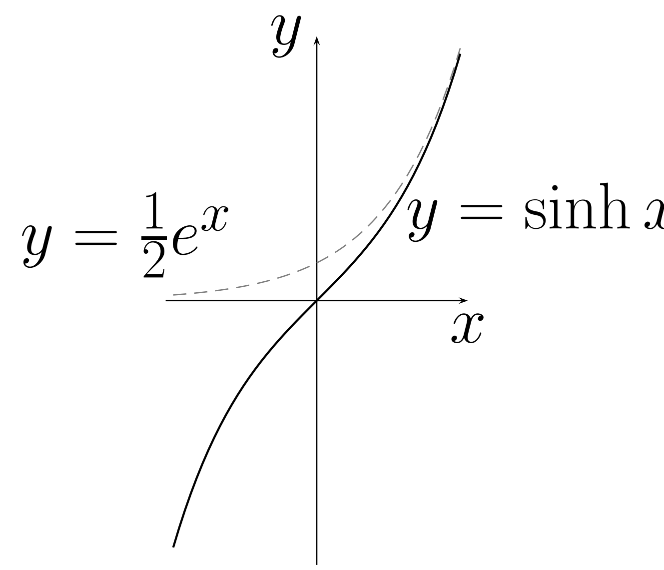
Properties of the $\cosh$ **function.**

- $\cosh$ is an even function.
- $\cosh 0 = 1$.
- $\cosh$ is decreasing on $(-\infty, 0)$, stationary at 0 and increasing on $(0, \infty)$.
- $\cosh x \geq 1$ for all x in $\mathbb{R}$.
- $\cosh x$ gets arbitrarily close to $\frac{1}{2}e^{\pm x}$ as $x \to \pm\infty$.



Properties of the $\sinh$ **function.**

- $\sinh$ is an odd function.
- $\sinh 0 = 0$.
- $\sinh$ is increasing on $(-\infty, \infty)$ and has a point of inflexion at 0.
- $\sinh x < 0$ for $x < 0$ and $\sinh x > 0$ for $x > 0$.
- $\sinh x$ gets arbitrarily close to $\pm\frac{1}{2}e^{\pm x}$ as $x \to \pm\infty$.



Theorem. The hyperbolic functions are related by

$$\cosh^2 x - \sinh^2 x = 1.$$

Remark. The similarity to relations such as

$$\cos^2 x + \sin^2 x = 1, \quad \frac{d}{dx}\cos x = -\sin x, \quad \frac{d}{dx}\sin x = \cos x$$

explains the words cosine and sine in the hyperbolic functions.

The term hyperbolic is motivated in the following manner:

Example. Sketch the curve $\gamma(t)$ defined by

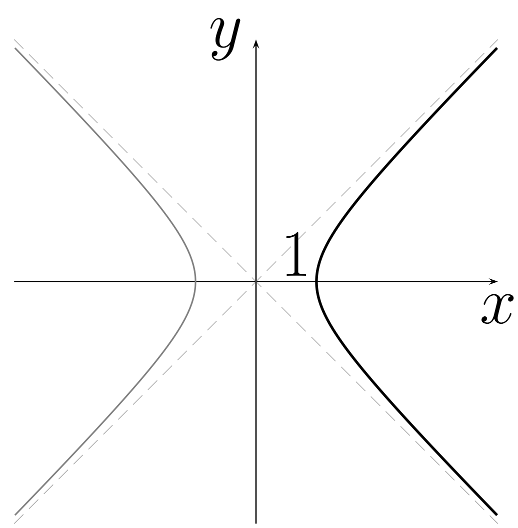
$$\gamma(t) = \big(x(t), y(t)\big) = (\cosh t, \sinh t), \quad t \in \mathbb{R}.$$

Elimination of the parameter t leads to

$$[x(t)]^2 - [y(t)]^2 = \cosh^2 t - \sinh^2 t = 1$$

so that γ parametrises the branch of the hyperbola

$$x^2 - y^2 = 1, \qquad x > 0.$$



The other branch of the hyperbola is parametrised by

$$\big(x(t), y(t)\big) = (-\cosh t, \sinh t).$$

Other hyperbolic functions

Other hyperbolic functions are defined in analogy with the trigonometric functions according to

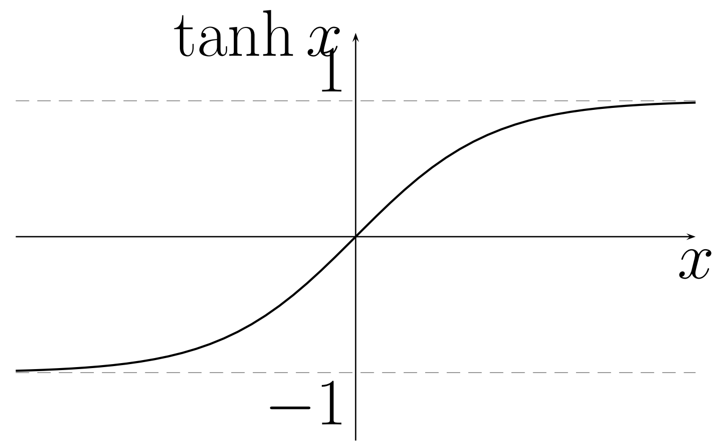
$$\tanh x = \frac{\sinh x}{\cosh x}, \qquad \coth x = \frac{\cosh x}{\sinh x},$$
$$\operatorname{sech} x = \frac{1}{\cosh x}, \qquad \operatorname{cosech} x = \frac{1}{\sinh x}.$$

Recall that

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Properties of the tanh **function.**

- tanh is an odd function.
- $\tanh 0 = 0$.
- tanh is increasing on $(-\infty, \infty)$ and has a point of inflexion at 0.
- $\tanh x < 0$ for $x < 0$ and $\tanh x > 0$ for $x > 0$.
- $\lim_{x \to \pm\infty} \tanh x = \pm 1$.
- $\frac{d}{dx} \tanh x = \operatorname{sech}^2 x > 0$.



Hyperbolic identities

‘Difference of squares’ identities.

$$\cosh^2 x - \sinh^2 x = 1$$

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\coth^2 x - 1 = \operatorname{cosech}^2 x$$

‘Sum and difference’ formulae.

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$

‘Double-angle’ formulae.

$$\sinh(2x) = 2\sinh x \cosh x$$
$$\cosh(2x) = \cosh^2 x + \sinh^2 x$$
$$\tanh(2x) = \frac{2\tanh x}{1 + \tanh^2 x}.$$

Exercise. Prove the first two ‘sum and difference’ formulae and, hence, derive the third.

Hyperbolic derivatives and integrals

The following derivatives may be readily verified:

$$\frac{d}{dx}\sinh x = \cosh x, \qquad \frac{d}{dx}\cosh x = \sinh x$$

$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x, \qquad \frac{d}{dx}\coth x = -\operatorname{cosech}^2 x$$

$$\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x\ \tanh x$$

$$\frac{d}{dx}\operatorname{cosech} x = -\operatorname{cosech} x\ \coth x.$$

Corresponding indefinite integrals are, for instance,

$$\int \sinh x\, dx = \cosh x + C, \qquad \int \operatorname{sech}^2 x\, dx = \tanh x + C.$$

Example. Determine the definite integral

$$I = \int_0^{(\ln 2)^2} \frac{\operatorname{sech}^2 \sqrt{x}}{\sqrt{x}}\, dx.$$

The inverse hyperbolic functions

Recall the graphs of sinh and tanh are increasing functions and hence are $1-1$.

cosh however is not $1-1$ and so we need to restrict the domain to $[0,\infty)$.

For inverses then we are dealing with

$$\begin{aligned} \cosh &: [0,\infty) \to [1,\infty), & \cosh^{-1} &: [1,\infty) \to [0,\infty) \\ \sinh &: \mathbb{R} \to \mathbb{R}, & \sinh^{-1} &: \mathbb{R} \to \mathbb{R} \\ \tanh &: \mathbb{R} \to (-1,1), & \tanh^{-1} &: (-1,1) \to \mathbb{R}. \end{aligned}$$

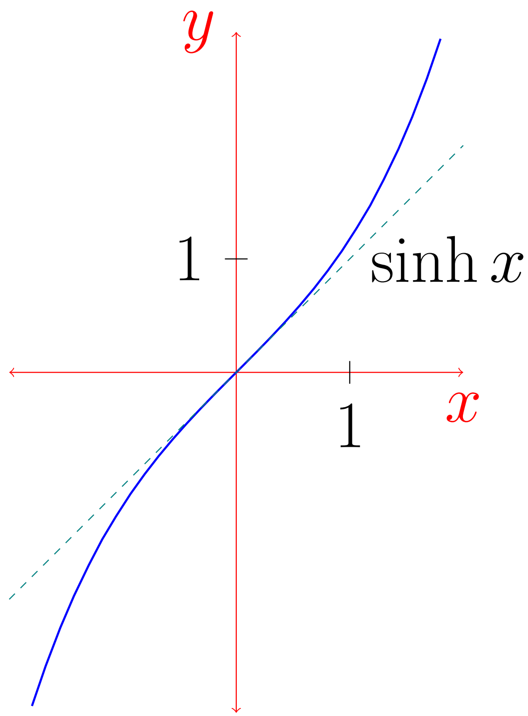
Of course we can get the graphs of these functions by just reflecting the graphs of cosh, sinh and tanh.

Inverse hyperbolic sine

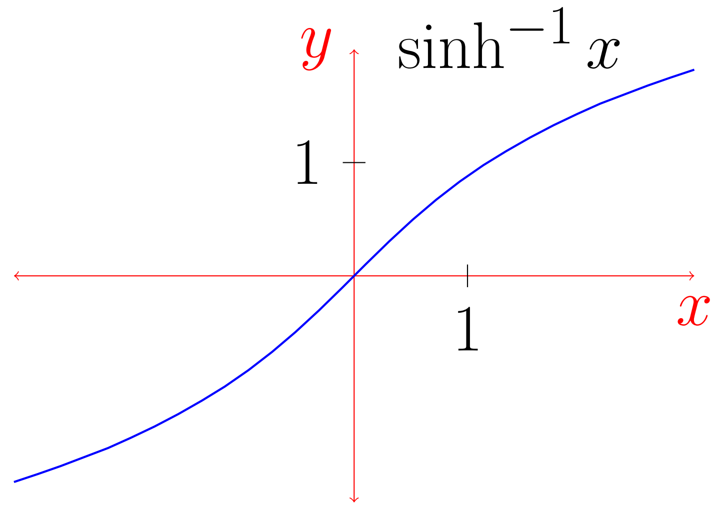
$$
\begin{aligned}
y = \sinh x \iff y &= \frac{e^x - e^{-x}}{2} \\
&\iff e^x - 2y - e^{-x} = 0 \\
&\iff \left(e^x\right)^2 - 2ye^x - 1 = 0 \\
&\iff e^x = \frac{2y \pm \sqrt{4y^2 + 4}}{2} = y \pm \sqrt{y^2 + 1} \\
&\iff e^x = y + \sqrt{y^2 + 1}
\end{aligned}
$$

$$y - \sqrt{y^2 + 1} < 0$$

$$\iff x = \sinh^{-1} y = \ln\left(y + \sqrt{y^2 + 1}\right).$$



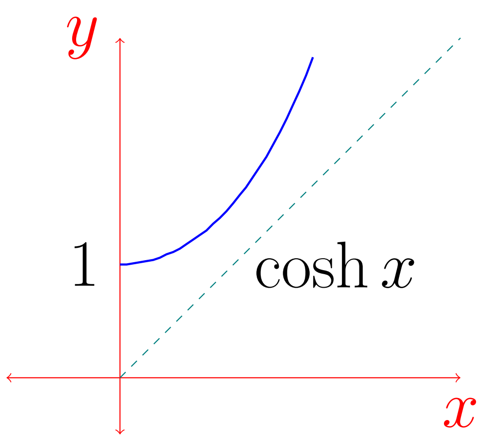
y
1
$\sinh x$
1
x

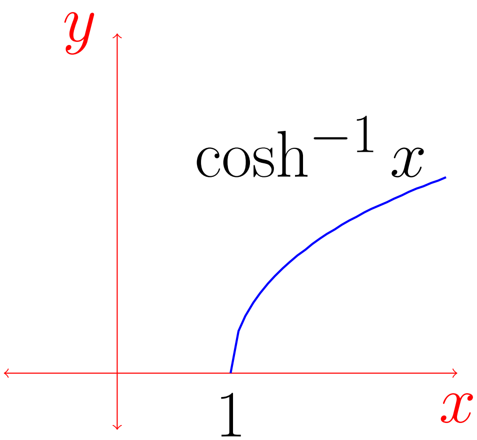


y
$\sinh^{-1} x$
1
1
x

Inverse hyperbolic cosine

As on the last slide, you can show that $\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right)$

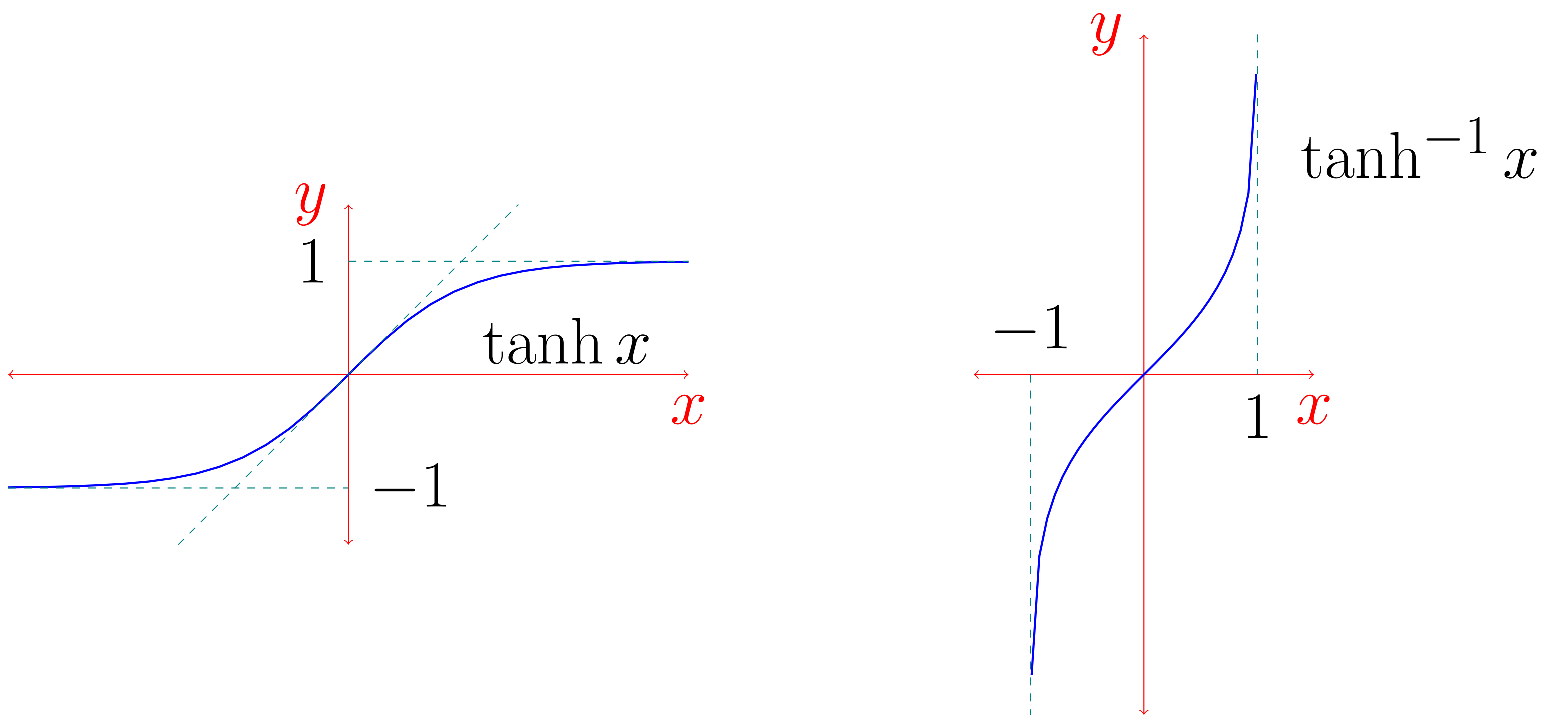




Inverse hyperbolic tangent

Here we have:

$$\tanh^{-1} x = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right) .$$



Example. Evaluate

$$\sinh\left(\cosh^{-1}\frac{4}{3}\right).$$

The main interest in these inverse hyperbolic functions however is in their derivatives:

$$\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{x^2+1}}, \quad x \in \mathbb{R},$$
$$\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}, \quad x > 1,$$
$$\frac{d}{dx}\tanh^{-1} x = \frac{1}{1-x^2}, \quad -1 < x < 1.$$

Thus, the inverse hyperbolic functions provide antiderivatives for some relatively simple functions which we otherwise can't integrate.

There are two ways to prove these:

1. use the formulae in terms of ln on the previous slide, and a bit of algebra.
2. use the Inverse Function Theorem:

Example. Use the inverse function theorem to confirm that

$$\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{x^2+1}}.$$

Example. Find $I = \displaystyle\int \frac{dx}{\sqrt{x^2+4}}$.

Example. Find $I = \displaystyle\int \frac{dx}{4x-3-x^2}$.

Example. Determine the indefinite integral

$$\int \frac{dx}{\sqrt{x^2-2x+10}}.$$