THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Semester 1 2013

MATH1141
HIGHER MATHEMATICS 1A

(1) TIME ALLOWED – 2 hours

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(7) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

(8) A SHORT TABLE OF INTEGRALS IS APPENDED TO THE PAPER

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

Use a separate book clearly marked Question 1

1. i) Evaluate the following limits:

a)

$$\lim_{x \to \infty} \frac{10x^2 + 3x + \sin x}{5x^2 + 3x - 2},$$

b)

$$\lim_{x \to 0} \frac{e^{3x} - 1}{\sin(7x)}.$$

ii) A function $f : [0, 5] \to \mathbb{R}$ has the following properties:

- $\lim\limits_{x \to 2^+} f(x) = 3,$
- $\lim\limits_{x \to 2^-} f(x) = 1,$
- $f(2) = 4.$

Draw a possible sketch of the graph of f. (You do not need to give a formula for your function.)

iii) a) State the definitions of $\cosh x$ and $\sinh x$ in terms of the exponential function.

b) Prove that $\cosh^2 x - \sinh^2 x = 1$.

iv) Let $z = 5 + 5i$ and $w = 2 + i$.

a) Find $2z + 3\overline{w}$.

b) Find $z(w - 1)$.

c) Find z/w.

v) Suppose that $(x + iy)(3 + 4i) = 13 + 9i$, where $x, y \in \mathbb{R}$.
Find the value of x and the value of y.

vi) Let the set S in the complex plane be defined by

$$S = \Big\{ z \in \mathbb{C} \, : \, |z| \le 3 \text{ and } 0 \le \operatorname{Im}(z) \le 3 \Big\}.$$

a) Sketch the set S on a labelled Argand diagram.

b) By considering your sketch, or otherwise, find the area of the region defined by S.

vii) Consider the following MAPLE session.

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> with(LinearAlgebra):
> A:=<<0,1,-1>|<1,0,1>|<-1,1,0>>;
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$$A := \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

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> B:=A^2;
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$$B := \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

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> C:=A^3;
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$$C := \begin{bmatrix} -2 & 3 & -3 \\ 3 & -2 & 3 \\ -3 & 3 & -2 \end{bmatrix}$$

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> F:=3*A-C;
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$$F := \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Without carrying out any row reduction, use the above Maple session to find the inverse of the matrix A.

Use a separate book clearly marked Question 2

2. i) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^3 & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0. \end{cases}$$

a) Explain why f is differentiable everywhere and determine $f'(x)$.

b) Explain why the function g defined by $g(x) = f'(x)$ is continuous at $x = 0$.

c) Use the definition of the derivative to determine whether g is differentiable at $x = 0$.

ii) Consider the function $f(x) = \dfrac{1}{1+x}$ defined on $[0, 1]$ and let P be the partition $\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}\}$.

a) Show that the lower Riemann sum $L_P(f)$ is given by

$$L_P(f) = \sum_{k=1}^{n} \frac{1}{n+k}.$$

b) Assuming that the limits of the upper and lower Riemann sums are equal, evaluate

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k}.$$

iii) Let A, B and C be three points in the plane with corresponding position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

a) Let M be the midpoint of the line joining A and B. What is the position vector $\mathbf{m}$ of M?

b) Write a parametric vector equation for the line through C and M.

c) Suppose that

$$(\mathbf{b}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{a}) = \frac{1}{2}|\mathbf{b}-\mathbf{a}|\,|\mathbf{c}-\mathbf{a}| \quad \text{and} \quad (\mathbf{c}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = \frac{1}{2}|\mathbf{c}-\mathbf{b}|\,|\mathbf{a}-\mathbf{b}|.$$

Explain why the triangle ABC is equilateral.

iv) Consider the system of equations

$$\begin{aligned} x + y - z &= 2 & (1) \\ x - y + 3z &= 6 & (2) \\ x^2 + y^2 + z^2 &= 10 & (3) \end{aligned}$$

[Note that equation (3) is NOT linear.]

a) Give, in parametric vector form, the set of points which satisfy the first two equations (that is, (1) and (2)).

b) Describe this solution set geometrically.

c) Using the answer to (a), or otherwise, find all the points which satisfy all three equations.

Use a separate book clearly marked Question 3

3. i) Find the shortest distance from the plane

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \lambda_1 \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}, \qquad \lambda_1, \lambda_2 \in \mathbb{R}$$

to the point $\mathbf{p} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$.

ii) Let $p(z) = z^4 - z^3 - z^2 - z + 2$. Denote the roots of p by $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, where α_1 is an **integer**.

a) Find the value of α_1.

b) Given that at least one of the roots of p is not real, deduce how many of the roots are real.

c) By considering the sum of the roots, or otherwise, prove that at least one of the roots has negative real part.

d) Prove that $|\alpha_j| > \frac{1}{2}$ for $j = 1, 2, 3, 4$.

iii) Which of the following statements are true **for all** non-zero 2×2 matrices $A, B, C \in M_{2,2}(\mathbb{R})$? For those statements which are not always true, give a counterexample.

a) $AB = BA$.

b) $\det(AB) = \det(BA)$.

c) If $\det(AB) = \det(AC)$ then $\det(B) = \det(C)$.

d) If $AB = AC$ then $B = C$.

iv) a) Define what it means for a set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ to be an **orthonormal set** in $\mathbb{R}^n$.

b) Let M be the matrix whose columns consist of the n orthonormal vectors, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in $\mathbb{R}^n$. By considering $M^T M$ or otherwise, find, with reasons, all possible values for $\det(M)$.

Use a separate book clearly marked Question 4

4. i) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \int_0^{x^3} e^{-t^2}\,dt.$$

a) Determine, with reasons,

$$\lim_{t\to\infty} t^2 e^{-t^2}.$$

b) Does the improper integral

$$I = \int_0^{\infty} e^{-t^2}\,dt$$

converge? Give reasons for your answer.

c) Find all critical points and asymptotes of f.

d) Carefully sketch the graph of f, clearly indicating the above information and any other relevant features.

ii) Let f be a differentiable function on (a, b), and take $c \in (a, b)$. Define

$$q(x) = \frac{f(x) - f(c) - f'(c)(x - c)}{(x - c)^2},$$

where $a < x < b$ and $x \neq c$.

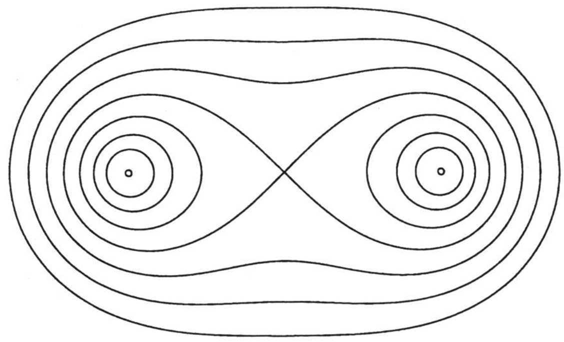
Show that if $f''(c)$ exists, then

$$\lim_{x\to c} q(x) = \frac{f''(c)}{2}.$$

iii) An oval of Cassini is a curve on the (x, y)-plane defined implicitly by

$$\left(x^2 + y^2\right)^2 - 2\left(x^2 - y^2\right) + 1 = b.$$

The shape of the curve depends on the value of the positive constant b. The plot below shows ovals of Cassini for several different values of b.



a) Show that the points on an oval of Cassini where the tangent is horizontal either lie on the unit circle $x^2 + y^2 = 1$ or lie on the y-axis.

b) Determine all such points (x, y) for which the corresponding tangent is horizontal and state carefully for which values of $b > 0$ these exist.

iv) Suppose $f : [0, 2] \to [0, 8]$ is continuous and differentiable on its domain.

a) By considering the function $g(x) = f(x) - x^3$, prove that there is a real number $\xi \in [0, 2]$ such that $f(\xi) = \xi^3$, stating any theorems you use.

b) Now suppose that $f(0) = 0$ and $f(2) = 8$. Explain why $f'(\eta) = 4$ for some real $\eta \in (0, 2)$, stating any theorems you use.

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BASIC INTEGRALS

$$\int \frac{1}{x}\,dx = \ln|x| + C = \ln|kx|, \qquad C = \ln k$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

$$\int a^x\,dx = \frac{1}{\ln a}a^x + C, \qquad a \neq 1$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$$

$$\int \operatorname{cosec}^2 ax\,dx = -\frac{1}{a}\cot ax + C$$

$$\int \tan ax\,dx = \frac{1}{a}\ln|\sec ax| + C$$

$$\int \cot ax\,dx = \frac{1}{a}\ln|\sin ax| + C$$

$$\int \sec ax\,dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C$$

$$\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C$$

$$\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C$$

$$\int \operatorname{sech}^2 ax\,dx = \frac{1}{a}\tanh ax + C$$

$$\int \operatorname{cosech}^2 ax\,dx = -\frac{1}{a}\coth ax + C$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$\begin{aligned}\int \frac{dx}{a^2-x^2} &= \frac{1}{a}\tanh^{-1}\frac{x}{a} + C, && |x| < a\\ &= \tfrac{1}{a}\coth^{-1}\tfrac{x}{a} + C, && |x| > a > 0\\ &= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C, && x^2 \neq a^2\end{aligned}$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} + C, \qquad x \geqslant a > 0$$