Chapter 2: Vector Geometry

Definition

The **dot product** (or **scalar product**) of two vectors

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

in $\mathbb{R}^n$ is

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + ... a_n b_n.$$

Properties of the dot product:

- $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$
- $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
- $(\lambda \mathbf{a}) \cdot \mathbf{b} = \lambda(\mathbf{a} \cdot \mathbf{b})$
- $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
- $|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$ (Cauchy-Schwartz inequality)

Proving the Cauchy-Schwartz inequality

Assuming $\mathbf{b} \neq \mathbf{0}$, we have

$$0 \leq (\mathbf{a} - \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b}) \cdot (\mathbf{a} - \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b})$$

$$= \mathbf{a}\cdot\mathbf{a} + (\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}})^2(\mathbf{b}\cdot\mathbf{b}) - 2\mathbf{a}\cdot(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b})$$

$$= \mathbf{a}\cdot\mathbf{a} - \frac{(\mathbf{a}\cdot\mathbf{b})^2}{\mathbf{b}\cdot\mathbf{b}},$$

so

$$(\mathbf{a}\cdot\mathbf{b})^2 \leq (\mathbf{a}\cdot\mathbf{a})\cdot(\mathbf{b}\cdot\mathbf{b})$$

and

$$|\mathbf{a}\cdot\mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|.$$

We define the **angle** between two nonzero vectors by the formula

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta.$$

This makes sense by the Cauchy-Schwartz inequality, since

$$-1 \leq \cos \theta \leq 1.$$

In $\mathbb{R}^2$ and $\mathbb{R}^3$, this is **the same as** the usual geometric notion of angle.

In particular,

$$\mathbf{a} \cdot \mathbf{b} = 0$$

if and only if **a** and **b** are perpendicular, or **orthogonal**.

Ex: Find the dot product of and the angle between the vectors **a** and **b** if $\mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

Answer: The dot product is

$$\mathbf{a} \cdot \mathbf{b} = 1 \cdot 2 + (-1) \cdot 1 + 2 \cdot 1 = 3.$$

For the angle θ, we have

$$|\mathbf{a}| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6} \text{ and } |\mathbf{b}| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}.$$

So

$$3 = \sqrt{6} \cdot \sqrt{6} \cos\theta,$$

and

$$\theta = \cos^{-1}(\frac{1}{2}) = \frac{\pi}{3}.$$

Ex: Show that $\begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$ is orthogonal to $\begin{pmatrix} 3 \\ 1 \\ 11 \end{pmatrix}$.

Answer: Take the dot product:

$$\begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 11 \end{pmatrix} = 2 \cdot 3 + 5 \cdot 1 + (-1) \cdot 11 = 0.$$

Ex: Write down a unit vector which is perpendicular to both

$$\begin{pmatrix} 2 \\ -6 \\ -3 \end{pmatrix} \text{ and } \begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix}.$$

Answer: Let $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be such a vector. Then we have the equations

$$\mathbf{x} \cdot \begin{pmatrix} 2 \\ -6 \\ -3 \end{pmatrix} = \mathbf{x} \cdot \begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix} = 0, \text{ and } \mathbf{x} \cdot \mathbf{x} = 1,$$

or

$$2x - 6y - 3z = 0, \quad 4x + 3y - z = 0, \quad x^2 + y^2 + z^2 = 1.$$

Adding twice the second equation to the first, we get

$$10x - 5z = 0, \text{ or } z = 2x.$$

Substituting $z = 2x$ in the second equation, we get

$$2x + 3y = 0, \text{ or } y = -\frac{2}{3}x.$$

Finally, writing the third equation in terms of x, we get

$$x^2 + (-\frac{2}{3}x)^2 + (2x)^2 = \frac{49}{9}x^2 = 1,$$

so

$$x = \pm\frac{3}{7}, \quad y = \mp\frac{2}{7}, \quad z = \pm\frac{6}{7}.$$

Ex: Prove that the altitudes of a triangle are concurrent.

Let O, A, B be vertices of a triangle, which we think of as points in $\mathbb{R}^n$ (with O being the origin). Let G be the point where the altitudes of $\overrightarrow{OA}$ and $\overrightarrow{OB}$ intersect.

Claim: $\overrightarrow{OG}$ is perpendicular to $\overrightarrow{AB}$.

From the claim, it follows that the thrid altitude also passes through G. (Why?)

Proof of claim: Since the altitude of $\overrightarrow{OA}$ passes through G, we have $\overrightarrow{GB} \perp \overrightarrow{OA}$, and

$$(B - G) \cdot A = 0.$$

Similarly, Since the altitude of $\overrightarrow{OB}$ passes through G, we have

$$(A - G) \cdot B = 0.$$

Then

$$G \cdot (B - A) = G \cdot B - G \cdot A = A \cdot B - B \cdot A = 0.$$

So $\overrightarrow{OG}$ is indeed perpendicular to $\overrightarrow{AB}$.

Definition

A set of vectors is **orthogonal** if every pair of them is orthogonal. An othrogonal set is **orthonormal** if all the vectors have length 1.

Ex: In $\mathbb{R}^3$ the standard basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is an orthonormal set. Give an example of an orthonormal set which doesn't contain any of the standard basis vectors.

Answer: An easy example is

$$\{-\mathbf{i}, -\mathbf{j}, -\mathbf{k}\}.$$

Another example is

$$\{\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}\}.$$

Ex: Suppose that $\{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}\}$ is an orthonormal set in $\mathbb{R}^n$, (with $n \geq 3$) and

$$\mathbf{b} = c_1\mathbf{a_1} + c_2\mathbf{a_2} + c_3\mathbf{a_3}.$$

Find the scalars c_1, c_2, c_3.

Answer: Lets take the dot product of each of the $\mathbf{a_i}$ with $\mathbf{b}$.

$$\mathbf{b} \cdot \mathbf{a_1} = (c_1\mathbf{a_1} + c_2\mathbf{a_2} + c_3\mathbf{a_3}) \cdot \mathbf{a_1} = c_1\mathbf{a_1} \cdot \mathbf{a_1} + c_2\mathbf{a_2} \cdot \mathbf{a_1} + c_3\mathbf{a_3} \cdot \mathbf{a_1}$$

$$= c_1 \cdot 1 + c_2 \cdot 0 + c_3 \cdot 0 = c_1.$$

Similarly,

$$\mathbf{b} \cdot \mathbf{a_2} = c_2 \text{ and } \mathbf{b} \cdot \mathbf{a_3} = c_3.$$

The **projection** of a vector **a** onto a nonzero vector **b** is obtained by dropping a perpendicular from **a** to the line spanned by **b**; the projection is then the vector given by the point of intersection.

Definition

The projection of **a** onto $\mathbf{b} \neq \mathbf{0}$ is

$$\mathrm{proj}_{\mathbf{b}}\mathbf{a} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b}.$$

Ex: Find the projection of $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ onto $\begin{pmatrix} -2 \\ 3 \\ 6 \end{pmatrix}$.

Answer: Let $\mathbf{a} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -2 \\ 3 \\ 6 \end{pmatrix}$.

Then

$$\mathbf{a} \cdot \mathbf{b} = (2 \cdot (-2) + (-3) \cdot 3 + 1 \cdot 6) = -7$$

and

$$|\mathbf{b}|^2 = (-2)^2 + 3^2 + 6^2 = 49.$$

So

$$\mathrm{proj}_{\mathbf{b}}\mathbf{a} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\right) \mathbf{b} = \frac{-7}{49} \begin{pmatrix} -2 \\ 3 \\ 6 \end{pmatrix} = -\frac{1}{7} \begin{pmatrix} -2 \\ 3 \\ 6 \end{pmatrix}.$$

Ex: Find the length of the projection of $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ onto $\begin{pmatrix} 5 \\ 6 \end{pmatrix}$.

Answer: The length of the projection of $\mathbf{a} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ onto $\mathbf{b} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$ is given by

$$|\mathrm{proj}_{\mathbf{b}}\mathbf{a}| = |\left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b}| = |\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}|\cdot|\mathbf{b}| = \frac{|\mathbf{a}\cdot\mathbf{b}|}{|\mathbf{b}|}.$$

We have $|\mathbf{a}\cdot\mathbf{b}| = |-8| = 8$ and $|\mathbf{b}| = \sqrt{61}$.

So the length of the projection is

$$\frac{8}{\sqrt{61}}.$$

Definition

The **cross product** of two vectors

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

in $\mathbb{R}^3$ is

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}.$$

The cross product of **a** and **b** is a vector which is orthogonal to both **a** and **b**.

A way to remember this formula is through the use of **determinants**, which we will learn more about later in the course.

We write

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

The determinant of a 2×2 matrix is given by

$$\begin{vmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{vmatrix} = ad - bc.$$

Ex: Use the determinant formula to find the cross product of $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ -2 \\ -3 \end{pmatrix}$.

Answer:

$$\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 6 \\ -2 \\ -3 \end{pmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 2 \\ 6 & -2 & -3 \end{vmatrix}$$

$$= \mathbf{i}\begin{vmatrix} -1 & 2 \\ -2 & 3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & 2 \\ 6 & -3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & -1 \\ 6 & -2 \end{vmatrix}$$

$$= \mathbf{i}[(-1)\cdot(-3) - 2\cdot(-2)] - \mathbf{j}[2\cdot(-3) - 2\cdot 6] + \mathbf{k}[2\cdot(-2) - (-1)\cdot 6].$$

$$= \begin{pmatrix} 7 \\ 18 \\ 2 \end{pmatrix}.$$

Properties of Cross Products:

- $\mathbf{a} \times \mathbf{a} = \mathbf{0}$
- $\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b})$
- $\mathbf{a} \times (\lambda \mathbf{b}) = \lambda(\mathbf{a} \times \mathbf{b}) = (\lambda \mathbf{a}) \times \mathbf{b}$
- $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$

The cross product is neither commutative nor associative!

For the three standard basis vectors in $\mathbb{R}^3$ we have

$$\mathbf{e_1} \times \mathbf{e_2} = \mathbf{e_3}, \quad \mathbf{e_2} \times \mathbf{e_3} = \mathbf{e_1}, \quad \mathbf{e_3} \times \mathbf{e_1} = \mathbf{e_2}.$$

The cross product also has a geometric interpretation.

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta,$$

where θ is the angle between **a** and **b**.

Ex: If $\mathbf{a}, \mathbf{b} \neq \mathbf{0}$ and $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ explain why the vectors must be parallel.

Answer: Since

$$|\mathbf{a}||\mathbf{b}| \sin(\theta) = 0,$$

we must have $\sin\theta = 0$, and therefore $\theta = 0$ or $\theta = \pi$.

Ex: Show that

$$|\mathbf{a} \times \mathbf{b}|^2 + (\mathbf{a} \cdot \mathbf{b})^2 = |\mathbf{a}|^2 |\mathbf{b}|^2.$$

Answer: We have

$$|\mathbf{a} \times \mathbf{b}|^2 + (\mathbf{a} \cdot \mathbf{b})^2 = (|\mathbf{a}||\mathbf{b}| \sin\theta)^2 + (|\mathbf{a}||\mathbf{b}| \cos\theta)^2$$

$$= |\mathbf{a}|^2 |\mathbf{b}|^2 (\sin^2\theta + \cos^2\theta) = |\mathbf{a}|^2 |\mathbf{b}|^2.$$

Given a parallelogram S in $\mathbb{R}^{\mathbf{3}}$, defined by

$$S = \{\lambda \mathbf{a} + \mu \mathbf{b} : 0 \leq \lambda \leq 1, \quad 0 \leq \mu \leq 1\},$$

the area of S is given by $|\mathbf{a}||\mathbf{b}| \sin \theta = |\mathbf{a} \times \mathbf{b}|$.

Ex: Find the area of the parallelogram spanned by

$$\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \text{ and } \begin{pmatrix} -1 \\ 3 \\ 7 \end{pmatrix}.$$

The cross product of the two vectors is

$$\begin{pmatrix} 24 \\ -13 \\ 9 \end{pmatrix},$$

so the area of the parallelogram is

$$\sqrt{24^2 + (-13)^2 + 9^2} = \sqrt{826}.$$

Definition

The scalar triple product of three vectors $\mathbf{a}, \mathbf{b},$ and $\mathbf{c}$ in $\mathbb{R}^3$ is

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

Ex: Find the scalar triple product of

$$\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix}, \text{ and } \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}.$$

Answer: The cross product of the second two vectors is

$$\begin{pmatrix} 9 \\ 23 \\ 1 \end{pmatrix}.$$

Taking the dot product of this with the first vector, we get -2.

Properties of the scalar triple product:

- $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
- If any pair of $\mathbf{a}, \mathbf{b},$ and $\mathbf{c}$ are parallel then the scalar triple product is 0.

The scalar triple product can be written as a determinant:

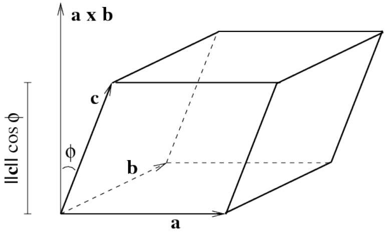
$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

The absolute value of scalar triple product is

$$|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| = |\mathbf{a} \times \mathbf{b}||\mathbf{c}||\cos\phi| = |\mathbf{a}||\mathbf{b}||\mathbf{c}||\cos\phi||\sin\theta|,$$

where ϕ is the angle between $\mathbf{c}$ and the perpendicular to the plane spanned by $\mathbf{a}$ and $\mathbf{b}$, and θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

This is the volume of the **parallelepiped** spanned by $\mathbf{a}, \mathbf{b},$ and $\mathbf{c}$.



Ex: Find the volume of the parallelepiped spanned by

$$\begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -4 \\ 6 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \\ -5 \end{pmatrix}.$$

Answer: The cross product of the first two vectors is

$$\begin{pmatrix} -20 \\ -9 \\ 4 \end{pmatrix}.$$

Taking the dot product with the third vector, we get the scalar triple product of 22 , which is the volume of the parallelopiped.

Recall that the vector equation of a plane is

$$\mathbf{x} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}.$$

The vector $\mathbf{b} \times \mathbf{c}$ is perpendicular to the plane, and every point on the plane satisfies

$$(\mathbf{b} \times \mathbf{c}) \cdot (\mathbf{x} - \mathbf{a}) = 0.$$

More generally, if we are given a point on a plane and a vector $\mathbf{n}$ perpendicular to the plane, the equation of the plane is

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{a}) = 0.$$

This is called the **point normal form** of the plane.

Given a plane in point normal form

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{a}) = 0,$$

we can write

$$n_1(x_1 - a_1) + n_2(x_2 - a_2) + n_3(x_3 - a_3) = 0,$$

or

$$n_1x_1 + n_2x_2 + n_3x_3 + (-a_1n_1 - a_2n_2 - a_3n_3) = 0.$$

This is the Cartesian equation of the plane.

Ex: Find the point normal form and hence the Cartesian equation of the plane passing through $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ with normal $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

Answer: The point normal form is

$$(\mathbf{x} - \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}) \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 0.$$

The Cartesian equation is

$$(x-2)\cdot 1 + (y-(-1))\cdot(-2) + (z-3)\cdot 1 = 0,$$

or

$$x - 2y + z = 7.$$

Ex: Find the Cartesian form of the plane whose vector equation is

$$\mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -2 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix}.$$

Answer: A normal vector to the plane is given by

$$\begin{pmatrix} -1 \\ -2 \\ 4 \end{pmatrix} \times \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ 14 \\ 10 \end{pmatrix}.$$

The point-normal form of the plane is then

$$(\mathbf{x} - \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}) \cdot \begin{pmatrix} 12 \\ 14 \\ 10 \end{pmatrix} = 0,$$

and the Cartesian equation is

$$12x + 14y + 10z = 30.$$

Ex: Convert to point-normal form the plane with equation

$$2x - 3y + 4z = 12.$$

Answer: A normal vector is given by

$$\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix},$$

and a point on the plane is

$$\begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix},$$

so a point normal form is

$$(\mathbf{x} - \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}) \cdot \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} = 0.$$

We can use vector projections to find distances between points and lines or planes.

The distance from a point P to a line containing the points A and B is the length of $\overrightarrow{QP}$, where $\overrightarrow{AQ}$ is the projection of $\overrightarrow{AP}$ onto $\overrightarrow{AB}$.

Note that

$$\overrightarrow{QP} = \overrightarrow{AP} - \operatorname{proj}_{\overrightarrow{AB}} \overrightarrow{AP} = \overrightarrow{AP} - \frac{\overrightarrow{AP} \cdot \overrightarrow{AB}}{\overrightarrow{AB} \cdot \overrightarrow{AB}} \overrightarrow{AB} .$$

Ex: Find the shortest distance from the point $P = \begin{pmatrix} 0 \\ 3 \\ 8 \end{pmatrix}$ to the line

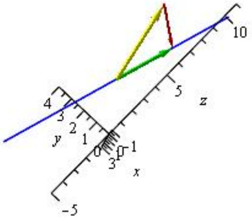
$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}.$$

Answer: We use the previous formula, where two points on the line are

$$A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},\ B = \begin{pmatrix} 2 \\ 1 \\ 7 \end{pmatrix},\ \text{and}\ \overrightarrow{AB} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix},\quad \overrightarrow{AP} = \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix}.$$

Then the distance from the point to the line is

$$\left| \overrightarrow{AP} - \frac{\overrightarrow{AP} \cdot \overrightarrow{AB}}{\overrightarrow{AB} \cdot \overrightarrow{AB}} \overrightarrow{AB} \right| = \left| \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \right| = 3.$$



This is a picture of the previos example: the green arrow is $\overrightarrow{AB}$, the yellow arrow is $\overrightarrow{AP}$, and the distance is the length of the red arrow (since in this case $\mathrm{proj}_{\overrightarrow{AB}}\,\overrightarrow{AP}=\overrightarrow{AB}$).

A similar method works to find the distance between a point and a plane.

Ex: Find the shortest distance from the point $P = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ to the plane

$$\mathbf{x} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}.$$

Answer: Let $A = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$. Then the distance is the length of the projection of $\overrightarrow{AP} = \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix}$ onto a normal vector of the plane.

A normal vector to the plane is

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix},$$

and the length of the projection is

$$\frac{|\overrightarrow{AP} \cdot \mathbf{n}|}{|\mathbf{n}|} = \frac{5}{3}.$$