Norwegian University of Science and Technology
Department of Electronics and Telecommunications

TTT4120 Digital Signal Processing
Suggested Solutions for Final Exam Fall 2013

Problem 1

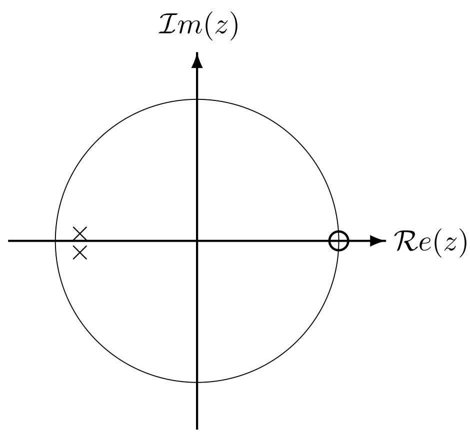
(a) We take the Z transform of the difference equation

$$Y(z) = -1.8z^{-1}Y(z) - 0.81z^{-2}Y(z) + X(z) - z^{-1}X(z)$$

which simplifies to

$$\frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 + 1.8z^{-1} + 0.81z^{-2}}$$

(b) The filter has one zero at $z = 1$ and a second order pole at $z = -0.9$ because $1 + 1.8z^{-1} + 0.81z^{-2} = (1 + 0.9z^{-1})^2 = 0 \Rightarrow z = -0.9, -0.9$ are the poles. Here is how the poles and zeros are scattered on the z-axis.



The filter has a zero at $z = 1$ which corresponds to $\omega = 0$. This means that the filter has small values around low frequencies. The filter has its poles at $z = -0.9$ which corresponds to $\omega = \pi$. This means high values for higher frequencies. Thus the filter shows a transition from lower to higher amplitudes while going from lower to higher frequencies and that is in line with a highpass filter profile.

(c) DF I and DF II structures are shown in figures 1 and 2.

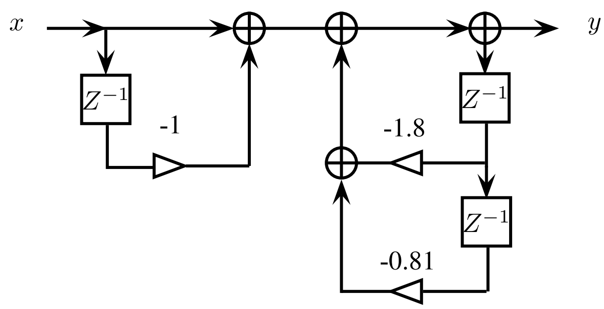


Figure 1: DF I

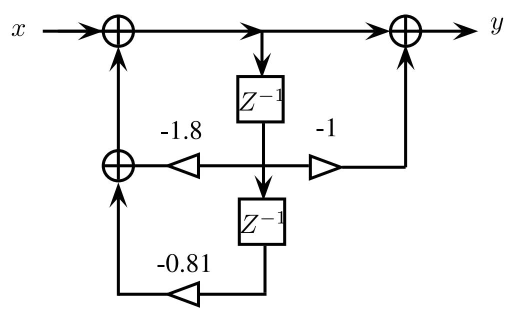


Figure 2: DF II

The direct form structures are simple and easy to implement but very sensitive to changes and rounding wrt to their gains. The DF II structure is canonical (minimal number of delay elements).

(d) The impulse response to the given filter (shown in figure 1 in the question sheet) can be given as

$$H(z) = \frac{0.5 + 0.3z^{-1}}{1 - 0.8z^{-1}}$$

We know that $\sigma_y^2 = r_{hh}(0)\sigma_x^2$. Given that $\sigma_x^2 = 1$, it remains sufficient to find $r_{hh}(0)$. We have that

$$r_{hh}(0) = \sum_n h^2(n)$$

If we now assume that

$$G(z) = \frac{1}{1 - 0.8z^{-1}}$$

we can say that

$$H(z) = 0.5G(z) + 0.3z^{-1}G(z)$$

which results in

$$h(n) = 0.5g(n) + 0.3g(n-1)$$

Clearly by taking the inverse Z transform of $G(z)$, we have that $g(n) = 0.8^n u(n)$ and therefore

$$
\begin{aligned}
h(n) &= 0.5(0.8)^n u(n) + 0.3 * (0.8)^{n-1} u(n-1) \\
&= \begin{cases} 0.875(0.8)^n, & n \geqslant 1 \\ 0.5, & n = 0 \\ 0, & n < 0 \end{cases}
\end{aligned}
$$

As a result

$$
r_{hh}(0) = 0.5^2 + 0.875^2 \sum_{n=1}^{\infty} 0.8^{2n} = 0.25 + 0.7656(\frac{0.8^2}{1-0.8^2}) = 1.611
$$

Finally, $\sigma_y^2 = 1.611 \cdot \sigma_x^2 = 1.611$

(e) If we call the total quantization error $e_T(n)$, we notice that there are three separate sources of quantization error due to three multiplications namely $e_1(n), e_2(n), e_3(n)$. Due to the fact that each quantization error can be modeled as an additive white noise source with zero mean and variance σ_e^2, we can write

$$
e_T(n) = e_1(n) * g(n) + e_2(n) * g(n) + e_3(n) * g(n)
$$

where $g(n)$ is the same as part (d). Therefore

$$
\sigma_{e_T}^2 = \sigma_{e_1}^2 r_{gg}(0) + \sigma_{e_2}^2 r_{gg}(0) + \sigma_{e_3}^2 r_{gg}(0) = 3\sigma_e^2 r_{gg}(0)
$$

while we have that

$$
r_{gg}(0) = \sum_n g^2(n) = \sum_n 0.8^{2n} = \frac{1}{1-0.8^2} = 2.778
$$

Finally we have that $\sigma_{e_T}^2 = 3 \cdot \sigma_e^2 \cdot 2.778 = 8.333 \cdot \sigma_e^2$.

(f) Following the same reasoning as (e), we see that for this different structure

$$
e_T(n) = e_1(n) * h(n) + e_2(n) + e_3(n)
$$

and therefore

$$
\sigma_{e_T}^2 = \sigma_{e_1}^2 r_{hh}(0) + \sigma_{e_2}^2 + \sigma_{e_3}^2 = \sigma_e^2(2 + r_{hh}(0)) = (2 + 1.611)\sigma_e^2 = 3.611\sigma_e^2
$$

Problem 2

(a) Taking the Z transform from both sides of the difference equation gives

$$
z^{-1}Y(z) - 5/2Y(z) + zY(z) = X(z)
$$

which results in

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{z - \frac{5}{2} + z^{-1}} = \frac{z^{-1}}{(1 - 0.5z^{-1})(1 - 2z^{-1})}$$

We can now write

$$H(z) = \frac{A}{(1 - 0.5z^{-1})} + \frac{B}{(1 - 2z^{-1})}$$

where A, B are found as

$$A = H(z)(1 - 0.5z^{-1})\big|_{z=0.5} = -\frac{2}{3}$$
$$B = H(z)(1 - 2z^{-1})\big|_{z=2} = \frac{2}{3}$$

Therefore,

$$H(z) = \frac{\frac{2}{3}}{(1 - 2z^{-1})} - \frac{\frac{2}{3}}{(1 - 0.5z^{-1})}$$

(b) We have that

- causal: $h(n) = \frac{2}{3}2^n u(n) - \frac{2}{3}(\frac{1}{2})^n u(n)$, which is not stable, also $|z| > 2$ for convergence (or, ROC does not include unit circle means the system is not stable).
- anti-causal: $h(n) = -\frac{2}{3}2^n u(-n-1) + \frac{2}{3}(\frac{1}{2})^n u(-n-1)$, which is not stable, also $|z| < 1/2$ for convergence (or, ROC does not include unit circle means the system is not stable).
- non-causal: $h(n) = -\frac{2}{3}2^n u(-n-1) - \frac{2}{3}(\frac{1}{2})^n u(n)$, which is stable, also $1/2 < |z| < 2$ for convergence (or, ROC does include unit circle means the system is stable).

(c) Given the figure, we could easily notice two zeros at $z_i = -1/2, -1/4$ and three poles at $p_i = 1/2, 1/2 + 1/2j, 1/2 - 1/2j$ where the last two are conjugate poles. Therefore the transfer function can be written as

$$\begin{aligned} H(z) &= \prod_i \frac{z - z_i}{z - p_i} \\ &= \frac{(z + 1/2)(z + 1/4)}{(z - 1/2)(z - 1/2 - 1/2j)(z - 1/2 + 1/2j)} \\ &= \frac{(z + 1/2)(z + 1/4)}{(z - 1/2)(z^2 - z + 1/2)} \end{aligned}$$

If we want the system to be stable, we should have the convergence region include the unit circle, i.e. $|z| > 1/\sqrt{2}$. This fulfilled if the system is causal. Other possible convergence regions are $1/2 < |z| < 1/\sqrt{2}$ (non-causal) and $|z| < 1/2$ (anti-causal) which do not result in a stable system.

Problem 3

(a) Using the information in the question, i.e. $z = e^{j\omega}$ and $s = j\Omega$ and definition of the bilinear transform, we have that

$$\begin{aligned}
\frac{T}{2} j\Omega &= \frac{1 - e^{-j\omega}}{1 + e^{-j\omega}} \\
&= \frac{e^{-j\omega/2}(e^{j\omega/2} - e^{-j\omega/2})}{e^{-j\omega/2}(e^{j\omega/2} + e^{-j\omega/2})} \\
&= \frac{2j[e^{j\omega/2} - e^{-j\omega/2}]/2j}{2[e^{j\omega/2} + e^{-j\omega/2}]/2} \\
&= j\frac{\sin(\omega/2)}{\cos(\omega/2)} \\
&= j\tan\frac{\omega}{2} \Rightarrow \Omega = \frac{2}{T}\tan\frac{\omega}{2}
\end{aligned}$$

NOTE! The exam problem had a typo, stating $z = e^{-j\omega}$. Using this information in the same way as above, leads to the relation

$$\Omega = -\frac{2}{T}\tan\frac{\omega}{2}$$

.

(b) We have that

$$\begin{aligned}
H_a(s) &= 2 \cdot \frac{s + \Omega_c}{s + 4\Omega_c} \\
&= 2 \cdot \frac{\frac{s}{\Omega_c} + 1}{\frac{s}{\Omega_c} + 4}
\end{aligned}$$

Then we have that

$$\Omega_c = \frac{2}{T}\tan\frac{\omega_c}{2} = \frac{1}{T} \longrightarrow T = \frac{1}{\Omega_c}$$

And

$$\frac{s}{\Omega_c} = sT = 2 \cdot \frac{1 - z^{-1}}{1 + z^{-1}}$$

Therefore,

$$\begin{aligned}
H(z) &= 2 \cdot \frac{2 \cdot \frac{1 - z^{-1}}{1 + z^{-1}} + 1}{2 \cdot \frac{1 - z^{-1}}{1 + z^{-1}} + 4} \\
&= 2 \cdot \frac{2 - 2z^{-1} + 1 + z^{-1}}{2 - 2z^{-1} + 4 + 4z^{-1}} \\
&= 2 \cdot \frac{3 - z^{-1}}{6 + 2z^{-1}} \\
&= \frac{2}{2} \cdot \frac{1 - \frac{1}{3}z^{-1}}{1 + \frac{1}{3}z^{-1}} \\
&= \frac{1 - \frac{1}{3}z^{-1}}{1 + \frac{1}{3}z^{-1}}
\end{aligned}$$

(c) We can assume

$$G(z) = \frac{1}{1 + \frac{1}{3}z^{-1}}$$

$$g(n) = \begin{cases} (-\frac{1}{3})^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Then,

$$\begin{aligned} h(n) &= \left[g(n)u(n) - \frac{1}{3}g(n-1)u(n-1) \right] \\ &= \begin{cases} 0, & n < 0 \\ g(0), & n = 0 \\ g(n) - \frac{1}{3}g(n-1), & n > 0 \end{cases} \\ &= \begin{cases} 0, & n < 0 \\ 1, & n = 0 \\ (-\frac{1}{3})^n - \frac{1}{3} \cdot (-\frac{1}{3})^{n-1} = 2(-\frac{1}{3})^n, & n > 0 \end{cases} \end{aligned}$$

Problem 4

(a) we have that

$$h(n) = 3\delta(n) + 2\delta(n-1) \Rightarrow$$
$$x(n) = e(n) * h(n) = 3e(n) + 2e(n-1) \Rightarrow$$

The process is formed as a linear combination of current and past values of the input process. This type of process is a Moving Average (MA) process. The order of the process is 1, i.e. the process is a MA(1) process.

(b) The process $e(n)$ is white noise, that means noise samples are uncorrelated, i.e. $E(e(n)e(n+l)) = 0 \quad (l \neq 0)$. Also, due to the fact that the process is zero-mean, we get $E(e^2(n)) = \sigma_e^2$. As a result

$$\gamma_{ee}(l) = E(e(n)e(n+l)) = \sigma_e^2\delta(l) = \delta(l)$$

and consequently $\Gamma_{ee}(\omega) = DTFT\{\gamma_{ee}(l)\} = 1$.

(c) The autocorrelation function to the process $x(n)$ can be obtained via

$$\begin{aligned} \gamma_{xx}(l) &= E\left(x(n)x(n+l)\right) \\ &= E\left((3e(n) + 2e(n-1))(3e(n+l) + 2e(n+l-1))\right) \\ &= E\left(9e(n)e(n+l) + 6e(n)e(n+l-1) + 6e(n-1)e(n+l) + 4e(n-1)e(n-1+l)\right) \\ &= 9\gamma_{ee}(l) + 6\gamma_{ee}(l-1) + 6\gamma_{ee}(l+1) + 4\gamma_{ee}(l) \\ &= 13\gamma_{ee}(l) + 6\gamma_{ee}(l-1) + 6\gamma_{ee}(l+1) \\ &= \begin{cases} 13, & l = 0 \\ 6, & l = \pm 1 \\ 0, & |l| > 1 \end{cases} \end{aligned}$$

As a result, we have that $\Gamma_{xx}(\omega) = 13 + 6e^{-j\omega} + 6e^{j\omega} =$
$13 + 6(e^{-j\omega} + e^{j\omega}) = 13 + 12\cos(\omega) = 1 + 24\cos^2(\omega/2)$

(d) A general expression for a first-order predictor can be formulated as $\hat{x}(n) = \rho x(n-1)$.

The prediction error is defined as
$e_p(n) = x(n) - \hat{x}(n) = x(n) - \rho x(n-1)$. The prediction error power is then equal to

$$\begin{aligned} E(e_p^2(n)) &= E((x(n) - \rho x(n-1))^2) \\ &E(x^2(n) + \rho^2 x^2(n-1) - 2\rho x(n)x(n-1)) \\ &= \gamma_{xx}(0) + \rho^2\gamma_{xx}(0) - 2\rho\gamma_{xx}(1) \\ &= (1+\rho^2)\gamma_{xx}(0) - 2\rho\gamma_{xx}(1) \end{aligned}$$

The prediction error power is a convex function of ρ. Therefore, there is ρ^* which minimizes the prediction error power. It can be obtained from

$$\begin{aligned} \frac{\partial}{\partial\rho} E(e_p^2(n)) = 0 \Rightarrow \\ 2\rho^*\gamma_{xx}(0) - 2\gamma_{xx}(1) = 0 \Rightarrow \\ \rho^* = \frac{\gamma_{xx}(1)}{\gamma_{xx}(0)} \end{aligned}$$

Extra: The minimum prediction error power, i.e. $E^*(e_p^2(n))$ is then equal to $E^*(e_p^2(n)) = \frac{\gamma_{xx}^2(0) - \gamma_{xx}^2(1)}{\gamma_{xx}(0)}$ by setting ρ^* in the definition for $E(e_p^2(n))$.

(e) For the numbers given in this part, we have that $\rho^* = \frac{\gamma_{xx}(1)}{\gamma_{xx}(0)} = \frac{6}{13}$ and therefore $E^*(e_p^2(n)) = (1 + (\frac{6}{13})^2) \cdot 13 - 2 \cdot \frac{6}{13} \cdot 6 \approx 10.231$.
The other way is also from the point extra in (d), i.e.
$E^*(e_p^2(n)) = \frac{13^2 - 6^2}{13} = \frac{133}{13} \approx 10.231$.