Linsys 6

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Problem 1

$$\dot{x} = Ax + Bw$$
$$y = Cx$$

$$A = \begin{pmatrix} 0 & 1 \\ -8 & -6 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad R_w(\tau) = 4\delta(\tau)$$

a) Since $R_w(\infty) = 0$, we must have

$$\mu_w = 0.$$

b) Note that $R_w(0) = E(w^2) = E\left((w-\mu_w)^2\right) = \sigma_w^2$

This means that $\sigma_w^2 = \text{"}4\delta(0)\text{"} = \infty$

c) $S_w(j\omega) = \mathcal{F}\{R_w(\tau)\}$

$$= \int_{-\infty}^{\infty} 4\delta(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$= \underline{\underline{4}}$$

d) $\hat{g}(s) = C(s\mathbb{I}-A)^{-1}B$

$$(s\mathbb{I}-A)^{-1} = \begin{pmatrix} s & -1 \\ 8 & s+6 \end{pmatrix}^{-1}$$

$$= \frac{1}{s(s+6)+8} \begin{pmatrix} s+6 & 1 \\ -8 & s \end{pmatrix}$$

$$= \frac{1}{(s+2)(s+4)} \begin{pmatrix} s+6 & 1 \\ -8 & s \end{pmatrix}$$

$$\hat{g}(s)(s+2)(s+4) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} s+6 & 1 \\ -8 & s \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} s+6 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= s+6+2$$

$$\Rightarrow \quad \hat{g}(s) = \frac{s+8}{(s+2)(s+4)} = \underline{\underline{\frac{s+8}{s^2+6s+8}}}$$

e) $g(t) = \mathcal{L}\{\hat{g}(s)\}$

$$\hat{g}(s) = \frac{s+8}{(s+2)(s+4)} = \frac{\alpha_1}{s+2} + \frac{\alpha_2}{s+4}$$

$$\Leftrightarrow \quad (s+4)\alpha_1 + (s+2)\alpha_2 = s+8$$

$s=-2$: $2\alpha_1 = 6 \Rightarrow \alpha_1 = 3$

$s=-4$: $-2\alpha_2 = 4 \Rightarrow \alpha_2 = -2$

This gives $\hat{g}(s) = \frac{3}{s+2} - \frac{2}{s+4}$

$$\Rightarrow \underline{\underline{g(t) = 3e^{-2t} - 2e^{-4t}}}$$

f) $y(t) = \int_0^t g(\tau)\, w(t-\tau)\, d\tau$

$$\bar{\mu}_y = \lim_{t\to\infty} E[y(t)]$$

$$= \lim_{t\to\infty} \int_0^t g(\tau)\, E[w(t-\tau)]\, d\tau$$

$$\underline{\underline{= 0}}$$

h) We compute the power spectral density

$$S_y(j\omega) = \hat{g}(j\omega)\hat{g}(-j\omega)\underbrace{S_w(j\omega)}_{=4}$$

$$= 4 \frac{j\omega+8}{(j\omega+2)(j\omega+4)} \cdot \frac{-j\omega+8}{(-j\omega+2)(-j\omega+4)}$$

$$= 4 \frac{(64+\omega^2)}{(4+\omega^2)(16+\omega^2)}$$

$$= \frac{\alpha_1}{4+\omega^2} + \frac{\alpha_2}{16+\omega^2}$$

$$\Rightarrow 4(64+\omega^2) = \alpha_1(16+\omega^2) + \alpha_2(4+\omega^2)$$

$$\omega = 4i: \quad 4 \cdot (64-16) = \alpha_2(4-16)$$

$$\Leftrightarrow \quad \alpha_2 = -16$$

$$\omega = 2i: \quad 4(64-4) = \alpha_1(16-4)$$

$$\Leftrightarrow \quad \alpha_1 = 20$$

$$\Rightarrow S_y(j\omega) = \frac{20}{4+\omega^2} - \frac{16}{16+\omega^2}$$

f) $y(t) = g(t) * w(t)$

$$\Rightarrow \bar{\mu}_y = E(y(t)) = g(t) * E(w(t))$$
$$= g(t) * 0$$
$$= \underline{\underline{0}}$$

g) $$\sigma_y^2 = E\left[y^2\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_y(j\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{20}{4+\omega^2} - \frac{16}{16+\omega^2}\right)d\omega$$

$$= \frac{5}{2\pi}\int_{-\infty}^{\infty}\frac{1}{1+\left(\frac{\omega}{2}\right)^2}\, d\omega - \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{1+\left(\frac{\omega}{4}\right)^2}\, d\omega$$

$$= \frac{5}{\cancel{2}\pi}\cdot \cancel{2}\arctan\left(\frac{\omega}{2}\right)\Big|_{-\infty}^{\infty} - \frac{4}{2\pi}\arctan\left(\frac{\omega}{4}\right)\Big|_{-\infty}^{\infty}$$

$$= \frac{5}{\cancel{\pi}}\left(\frac{\cancel{\pi}}{2} + \frac{\cancel{\pi}}{2}\right) - \frac{2}{\cancel{\pi}}\left(\frac{\cancel{\pi}}{2} + \frac{\cancel{\pi}}{2}\right)$$

$$= 5 - 2$$

$$= \underline{\underline{3}}$$

Problem 2

$$x_{k+1} = Ax_k + Bu_k + Gw_k$$

$$y_k = Cx_k + Du_k + Hv_k$$

where $E[w_k] = 0$, $E[v_k] = 0$

$$E[w_k w_i] = \begin{cases} Q & i=k \\ 0 & i \neq k \end{cases}$$

$$E[v_k v_i] = \begin{cases} R & i=k \\ 0 & i \neq k \end{cases}$$

$$E[w_k v_i^T] = 0$$

a) $\hat{y}_k^- = E[y_k]$

$= E[Cx_k + Du_k]$

$= C\,E[x_k] + Du_k$ since u_k is known

$= C\hat{x}_k^- + Du_k$

$$\hat{x}_k = \hat{x}_k^- + K_k\left(y_k - \hat{y}_k^-\right)$$

$$P_k^- = E\left[\left(x_k - \hat{x}_k^-\right)\left(x_k - \hat{x}_k^-\right)^T\right]$$

$$P_k = E\left[\left(x_k - \hat{x}_k\right)\left(x_k - \hat{x}_k\right)^T\right]$$

b) From (1) and (2) we get

$$y_k - \hat{y}_k^- = Cx_k + Du_k + Hv_k - C\hat{x}_k^- - Du_k$$

$$= C\left(x_k - \hat{x}_k^-\right) + Hv_k$$

$$x_k - \hat{x}_k = x_k - \hat{x}_k^- - K_k\left(y_k - \hat{y}_k^-\right)$$

$$= \left(x_k - \hat{x}_k^-\right) - K_k C\left(x_k - \hat{x}_k^-\right) - K_k H v_k$$

$$= \left(\mathbb{I} - K_k C\right)\left(x_k - \hat{x}_k^-\right) - K_k H v_k$$

$$\Rightarrow \left(x_k - \hat{x}_k\right)^T = \left(x_k - \hat{x}_k^-\right)^T\left(\mathbb{I} - K_k C\right)^T - v_k^T H^T K_k^T$$

$$\Rightarrow \left(x_k - \hat{x}_k\right)\left(x_k - \hat{x}_k\right)^T = \left(\mathbb{I} - K_k C\right)\left(x_k - \hat{x}_k^-\right)\left(x_k - \hat{x}_k^-\right)^T\left(\mathbb{I} - K_k C\right)^T$$

$$- K_k H v_k \left(x_k - \hat{x}_k^-\right)^T\left(\mathbb{I} - K_k C\right)^T$$

$$- \left(\mathbb{I} - K_k C\right)\left(x_k - \hat{x}_k^-\right) K_k^T H^T v_k^T$$

$$+ K_k H v_k v_k^T H K_k^T$$

Taking the expected value, we get

$$P_k = E\left[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T\right]$$

$$= (\mathbb{I} - K_k C)\, E\left[(x_k - \hat{x}_k^-)(x_k - \hat{x}_k^-)^T\right] (\mathbb{I} - K_k C)^T$$

$$- K_k H \; E\left[v_k (x_k - \hat{x}_k^-)^T\right] (\mathbb{I} - K_k C)^T$$

$$- (\mathbb{I} - K_k C)\, E\left[(x_k - \hat{x}_k^-) v_k^T\right] H^T K_k^T$$

$$+ K_k H \; E\left[v_k v_k^T\right] H^T K_k^T$$

It is reasonable to assume that $x_k - \hat{x}_k^-$ and v_k are uncorrelated and since $E[v_k] = 0$ we get

$$\underline{\underline{P_k = (\mathbb{I} - K_k C)\, P_k^- (\mathbb{I} - K_k C)^T + K_k H R H^T K^T}}$$

c) We expand the parenthesis and get

$$P_k = (\mathbb{I} - K_k C) P_k^- (\mathbb{I} - C^T K_k^T) + K_k H R H^T K_k^T$$

$$= P_k^- - K_k C P_k^- - P_k^- C^T K_k^T + K_k C P_k^- C^T K^T + K_k H R H^T K_k^T$$

$$= P_k^- - K_k C P_k^- - P_k^- C^T K_k^T + K_k \left(C P_k^- C^T + H R H^T \right) K_k^T$$

d) $$\frac{d\, tr(P_k^-)}{d K_k} = 0$$

$$\frac{d\, tr(K_k C P_k^-)}{d K_k} = C P_k^-$$

Note that $P_k^- = (P_k^-)^T$ so we also have

$$\frac{d\, tr(P_k^- C^T K_k^T)}{d K_k} = \frac{d}{d K_k} tr\left((C P_k^-)^T K_k^T \right) = C P_k^-$$

If $S = CP_k^- C^T + HRH^T$, then

$S^T = S$

since both P_k^- and R are symmetric.

Using the given formula, we then get:

$$\frac{d}{dK_k} tr(K_k S K_k^T) = 2SK_k^T$$
$$= 2\left(CP_k^- C^T + HRH^T\right)K_k^T$$

Combing results we get

$$\frac{d}{dK_k} tr(P_k) = 0 - CP_k^- - CP_k^- + 2\left(CP_k^- C^T + HRH^T\right)K_k^T$$
$$= -2CP_k^- + 2\left(CP_k C^T + HRH^T\right)K_k^T$$

e) Setting $\frac{d}{dK_k} tr(P_k) = 0$ we get

$$0 = -2CP_k^- + 2(CP_k^-C^T + HRH^T)K_k^T$$

$$\Rightarrow \underbrace{CP_k^-}_{\text{symmetric}} = \underbrace{(CP_k^-C^T + HRH^T)}_{\text{symmetric}}K_k^T$$

$$\Leftrightarrow P_k^-C^T = K_k(CP_kC^T + HRH^T)$$

$$\Leftrightarrow \underline{\underline{K_k = P_k^-C^T(CP_kC^T + HRH^T)^{-1}}}$$

f) $$\hat{x}_{k+1}^- = E[x_{k+1}] = E[Ax_k + Bu_k + Gw_k]$$

$$= AE[x_k] + Bu_k + G\underbrace{\cancel{E[w_k]}}_{=0}$$

$$= A\hat{x}_k + Bu_k$$

g) $P_{k+1}^{-} = E\left[\left(x_{k+1} - \hat{x}_{k+1}^{-}\right)\left(x_{k+1} - \hat{x}_{k+1}^{-}\right)^T\right]$

$$x_{k+1} - \hat{x}_{k+1}^{-} = A x_k + \cancel{B u_k} + G w_k - A \hat{x}_k - \cancel{B u_k}$$

$$= A\left(x_k - \hat{x}_k\right) + G w_k$$

$$\left(x_{k+1} - \hat{x}_{k+1}^{-}\right)^T = \left(x_k - \hat{x}_k\right)^T A^T + w_k^T G^T$$

$$\Rightarrow P_{k+1}^{-} = E\left[A\left(x_k - \hat{x}_k\right)\left(x_k - \hat{x}_k\right)^T A^T\right.$$

$$+ G w_k \left(x - \hat{x}_k\right)^T A^T + A\left(x_k - \hat{x}_k\right) w_k^T G^T$$

$$\left. + G w_k w_k^T G^T\right]$$

$$= A P_k A^T + 0 + 0 + G Q G^T$$

$$\underline{\underline{= A P_k A^T + G Q G^T}}$$

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad D = 4$$

$$G = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad H = \begin{pmatrix} -1 & 1 \end{pmatrix}, \quad Q = 2, \quad R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

k	0	1
$\hat{x}_k^-$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$
$\hat{x}_k$	$\begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix}$	$\begin{pmatrix} 4/21 \\ 15/14 \end{pmatrix}$
P_k^-	$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 13/4 & -\frac{3}{4} \\ -\frac{3}{4} & \frac{9}{4} \end{pmatrix}$
u_k	1	-1
y_k	3	-4
$\hat{y}_k^-$	4	$-\frac{7}{2}$
K_k	$\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$	$\begin{pmatrix} 13/21 \\ -1/7 \end{pmatrix}$
P_k	$\begin{pmatrix} 1 & 0 \\ 0 & 1/4 \end{pmatrix}$	

h) We compute K_0

$$K_0 = P_0^- C^T \left(C P_0^- C^T + HRH^T \right)^{-1}$$

$$HRH^T = HH^T = \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 2$$

$$CP_0^- C^T = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2$$

$$K_0 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} (2+2)^{-1}$$

$$= \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$$

$$\hat{y}_0^- = C\hat{x}_0^- + Du_0$$

$$= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 4 \cdot 1$$

$$= 4$$

$$\hat{x}_0 = \hat{x}_0^- + K_0 (y_0 - \hat{y}_0^-)$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} (3 - 4)$$

$$= \underline{\underline{\begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix}}}$$

$$\hat{x}_1^- = A\hat{x}_0 + Bu_0$$

$$= \begin{pmatrix} -1 & 1 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} 1 = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}}}$$

$$P_0 = (\mathbb{I} - C K_0) P_0^- (\mathbb{I} - C K_0)^T + K_0 H R H^T K_0^T$$

$$C K_0 = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} = \frac{1}{2}$$

$$\Rightarrow P_0 = \left(1 - \frac{1}{2}\right) \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \left(1 - \frac{1}{2}\right) + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} 2 \begin{pmatrix} \frac{1}{2} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1/4 \end{pmatrix}$$

$$P_1^- = A P_0 A^T + G Q G^T$$

$$= \begin{pmatrix} -1 & 1 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1/4 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 1 & -3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot 2 \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 1/4 & -3/4 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 5/4 & -3/4 \\ -3/4 & 9/4 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 13/4 & -3/4 \\ -3/4 & 9/4 \end{pmatrix}$$

$$K_1 = P_1^- C^T \left(C P_1^- C^T + HRH^T \right)^{-1}$$

$$= \begin{pmatrix} 13/4 & -3/4 \\ -3/4 & 9/4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 13/4 & -3/4 \\ -3/4 & 9/4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \right)^{-1}$$

$$= \begin{pmatrix} 13/4 \\ -3/4 \end{pmatrix} \frac{4}{21}$$

$$= \begin{pmatrix} 13/21 \\ -1/7 \end{pmatrix}$$

$$\hat{y}_1^- = C \hat{x}_1^- + D u_1$$

$$= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} + 4 \cdot (-1)$$

$$= \frac{1}{2} - 4$$

$$= -\frac{7}{2}$$

$$\hat{x}_1 = \hat{x}_1^- + K_1 \left(y_1 - \hat{y}_1^- \right)$$

$$= \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} + \begin{pmatrix} 13/21 \\ -1/7 \end{pmatrix} \left(-4 - \left(-\frac{7}{2} \right) \right)$$

$$= \begin{pmatrix} 4/21 \\ 15/14 \end{pmatrix}$$