DigSig 10

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Problem 1

Since $s[n] = 0.9\, s[n-1] + v[n]$, we have

$$\gamma_{ss}[\ell] = \frac{0.9}{1 - 0.9^2} (0.9)^{|\ell|} = \frac{90}{19} (0.9)^{|\ell|}.$$

We want to reduce the noise in the signal so we use

$$\gamma_{SS}[\ell] = \gamma_{DS}[\ell]$$

and since white noise is uncorrelated with other signals, we have

$$\gamma_{DS}[\ell] = \gamma_{DX}[\ell]$$

Since $v[n]$ is a white noise signal, we have

$$\gamma_{XX}[\ell] = \gamma_{SS}[\ell] + \sigma_v^2 \delta[\ell]$$

This gives the $M \times M$ autocorrelation matrix

$$\Gamma_{xx} = \begin{bmatrix} \gamma_{xx}[0] & \gamma_{xx}[1] & \gamma_{xx}[2] \\ \vdots & \ddots & \\ \vdots & & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} 4.827 & 4.263 & 3.837 \\ 4.263 & 4.827 & 4.263 \\ 3.837 & 4.263 & 4.827 \end{bmatrix}$$

The cross-correlation vector is given by

$$\underline{\gamma_{Dx}} = \begin{bmatrix} \gamma_{Dx}[0] & \gamma_{Dx}[1] & \gamma_{Dx}[2] \end{bmatrix}^T$$

$$= \begin{bmatrix} 4.737 & 4.263 & 3.837 \end{bmatrix}^T$$

Solving $\Gamma_{xx} \underline{h} = \underline{\gamma_{Dx}}$ for $\underline{h}$, we get

$$\underline{h} = \begin{pmatrix} h[0] \\ h[1] \\ h[2] \end{pmatrix} = \begin{pmatrix} 0.9149 \\ 0.0701 \\ 0.0058 \end{pmatrix}$$

So the coeffs. are

$$\underline{\underline{h[0] = 0.9149, \; h[1] = 0.0701, \; h[2] = 0.0058}}$$

Problem 2

$$H_1(z) = \frac{z^{-1} - \frac{1}{2}}{1 - \frac{1}{2}z^{-1}} \quad , \quad H_2(z) = \frac{1}{1 + \frac{1}{2}z^{-1}}$$

a) $$H(z) = \frac{z^{-1} - \frac{1}{2}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{2}z^{-1}\right)}$$

$$= \frac{A}{1 - \frac{1}{2}z^{-1}} + \frac{B}{1 + \frac{1}{2}z^{-1}}$$

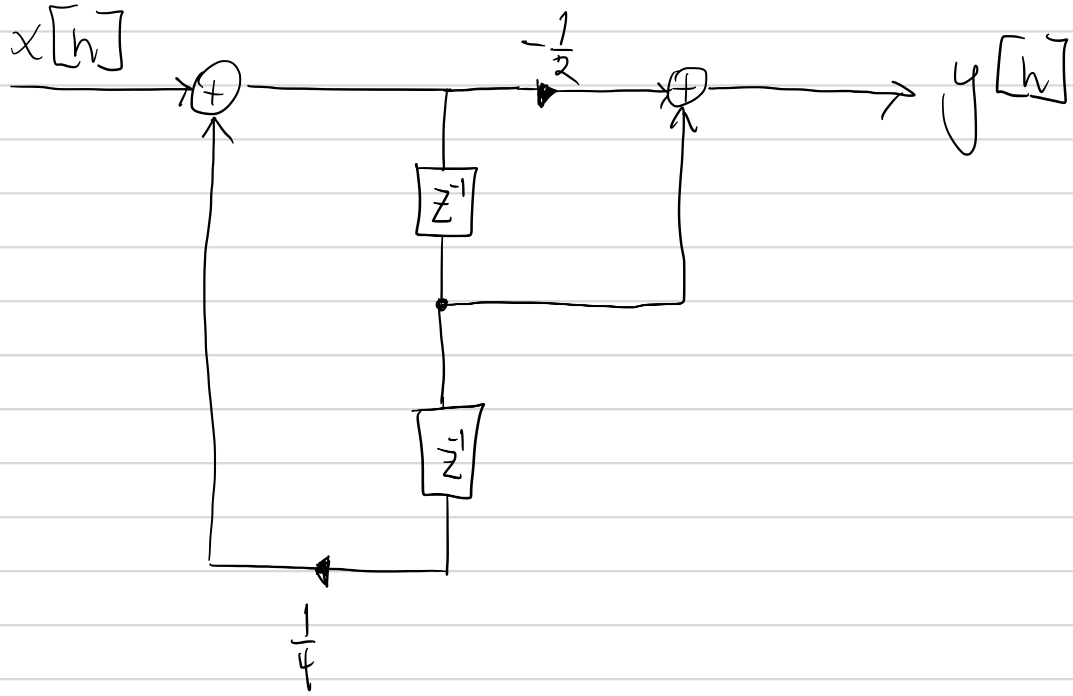
$$\Rightarrow \quad z^{-1} - \frac{1}{2} = A\left(1 + \frac{1}{2}z^{-1}\right) + B\left(1 - \frac{1}{2}z^{-1}\right)$$

$$\Rightarrow \quad A + B = -\frac{1}{2} \quad , \quad \frac{A}{2} - \frac{B}{2} = 1$$

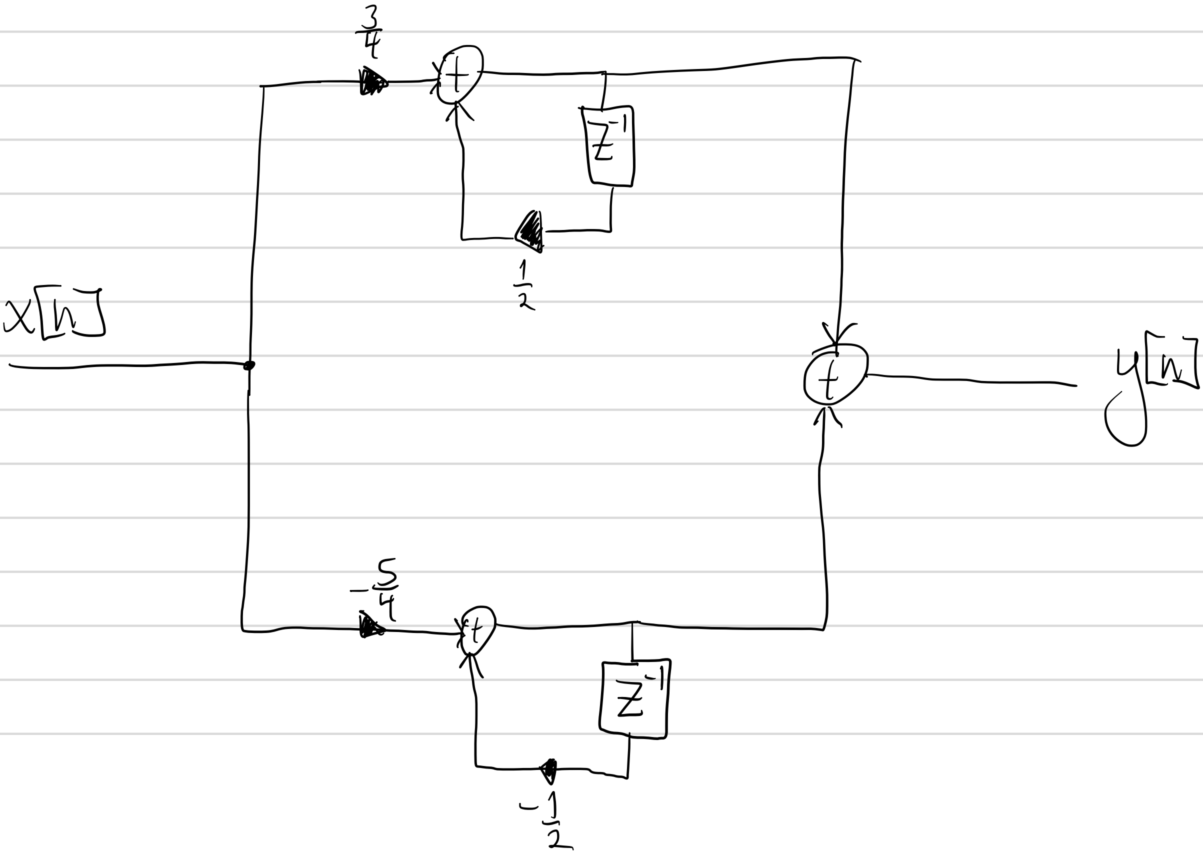
$$\Leftrightarrow \quad B = -\frac{5}{4} \quad , \quad A = \frac{3}{4}$$

$$\Rightarrow \quad H(z) = \frac{\frac{3}{4}}{1 - \frac{1}{2}z^{-1}} + \frac{-\frac{5}{4}}{1 + \frac{1}{2}z^{-1}}$$

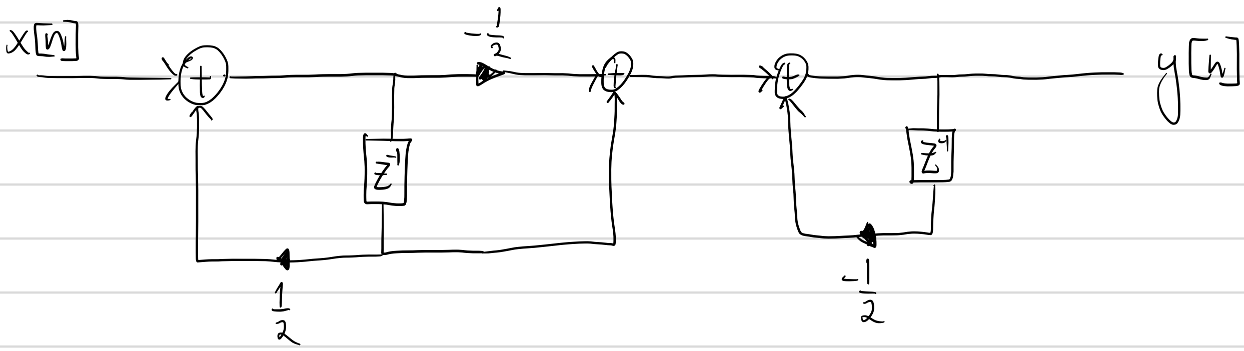
b) DF2: $H(z) = \dfrac{-\frac{1}{2} + z^{-1}}{1 - \frac{1}{4}z^{-2}}$



Parallel:

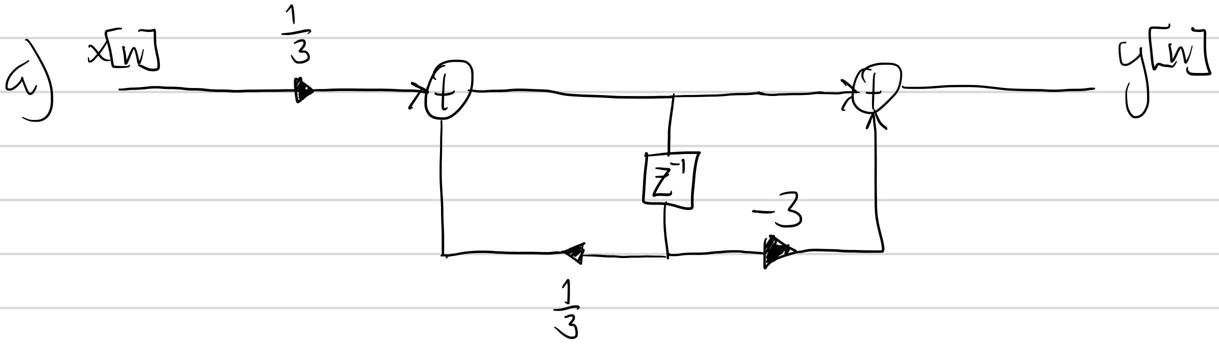


Cascade:



Problem 3

a)



b) Since the filter is causal we must have $h(n)=0, n<0$.

$$H(z) = \frac{1}{3}\frac{1}{1-\frac{1}{3}z^{-1}} - \frac{z^{-1}}{1-\frac{1}{3}z^{-1}}$$

$$= \frac{1}{3}\left(\frac{1}{1-\frac{1}{3}z^{-1}}\right) - z^{-1}\left(\frac{1}{1-\frac{1}{3}z^{-1}}\right)$$

We know that $\mathcal{Z}^{-1}\left\{\frac{1}{1-\frac{1}{3}z^{-1}}\right\} = \left(\frac{1}{3}\right)^n$ so

$$h[n] = \frac{1}{3}\left(\frac{1}{3}\right)^n - \left(\frac{1}{3}\right)^{n-1} = -\frac{8}{9}\left(\frac{1}{3}\right)^{n-1}, \ n>0.$$

Since $H(z) = h[0] + h[1]z^{-1} + \dots$ we must have

$$h[0] = \lim_{z\to\infty} H(z) = \lim_{z\to\infty} \frac{1}{3}\cdot\frac{1-\frac{3}{z}}{1-\frac{1}{3z}}$$

$$= \underline{\underline{\frac{1}{3}}}$$

So $h[n] = \begin{cases} -\frac{8}{9}\left(\frac{1}{3}\right)^{n-1} & , n>0 \\ \frac{1}{3} & , n=0 \\ 0 & , n<0 \end{cases}$

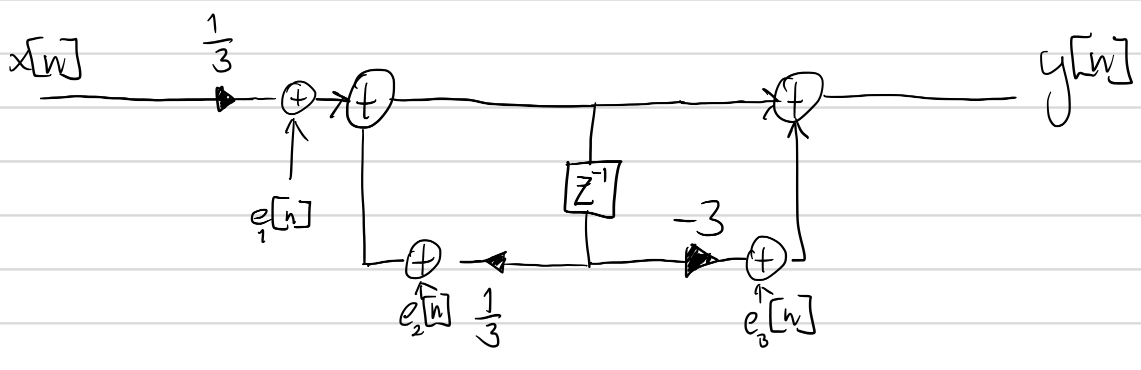
c) The error associated with rounding quantisation will be uniformly distributed in

$$-\frac{2^{-B}}{2} < \varepsilon < \frac{2^{-B}}{2}$$

so it has mean $\mu_\varepsilon = 0$ and noise power

$$\sigma_\varepsilon^2 = \frac{2^{-2B}}{12}.$$

d) The quantization happens after each multiplication so we have three systems to consider:



Power from e_1 and e_2 are the same, so combined they give filter output:

$$\sigma_{12}^2 = 2 \cdot \sigma_\varepsilon^2 \sum_{k=-\infty}^{\infty} h[k]^2$$

$$\sigma_{12}^2 = 2 \cdot \sigma_\varepsilon^2 \left(\left(\frac{1}{3}\right)^2 + \left(\frac{-8}{9}\right)^2 \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^{2n} \right)$$

$$= 2 \cdot \sigma_\varepsilon^2 \left(\frac{1}{9} + \frac{64}{81} \cdot \frac{1}{1-\left(\frac{1}{3}\right)^2} \right)$$

$$= 2 \sigma_\varepsilon^2$$

The third quantization e_3 is feed directly to the output, so

$$\sigma_3^2 = \sigma_\varepsilon^2 .$$

In total when then have that the noise power of the filter is

$$\sigma_q^2 = \sigma_{12}^2 + \sigma_3^2$$

$$= 2\sigma_\varepsilon^2 + \sigma_\varepsilon^2$$

$$= 3\sigma_\varepsilon^2$$

e) The largest scaling factor which guarantees no overflow is

$$G = \frac{1}{\sum_k |h[k]|}, \quad \text{since } |x[n]| \leqslant 1$$

$$\sum_k |h[k]| = \frac{1}{3} + \frac{8}{9} \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k$$

$$= \frac{1}{3} + \frac{8}{9} \cdot \frac{1}{1-\frac{1}{3}}$$

$$= \frac{5}{3}$$

$$\Rightarrow \underline{\underline{G = \frac{3}{5}}}$$

- Since the noise isn't scaled down, the S/N will go down.
- With B = 7 bits, we have

$$\sigma_q^2 = 3\,\sigma_\varepsilon^2$$

$$= 3\,\frac{2^{-2B}}{12}$$

$$= 2^{-16}$$

The outsignal will have power

$$\sigma_y^2 = \sigma_{xx}^2 \sum h[k]^2 + \sigma_q^2$$

$$= \sigma_x^2 + \sigma_q^2 .$$

This gives

$$SNR = \frac{\sigma_y^2}{\sigma_q^2} = 1 + \frac{\sigma_x^2}{\sigma_q^2}$$

$$= \underline{\underline{1 + 2^{16} \cdot \sigma_x^2}}$$