Solution Suggestion
Exam - TTK4115 Linear System Theory
December 10, 2008

AAE (2008-12-05)

Problem 1

a) We start by identifying the state and input matrices as:

$$A = \begin{bmatrix} 1 & 1 \\ -1 & -4 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

The eigenvalues of A can be found solving the characteristic polynomial:

$$\Delta(\lambda) = \det(\lambda I - A) = \left| \begin{bmatrix} \lambda - 1 & -1 \\ 1 & \lambda + 4 \end{bmatrix} \right| = \lambda^2 + 3\lambda - 3 = 0$$

This yields:

$$\lambda_{1,2} = -\frac{3}{2} \pm \frac{\sqrt{21}}{2}$$
$$\lambda_1 = 0.7913$$
$$\lambda_2 = -3.791$$

We also need to find two eigenvectors for the matrix, to perform the similarity transform. This is found using the expression:

$$(A - \lambda_i I)q_i = 0$$

The first eigenvalue yields:

$$(A - \lambda_1 I)q_1 = \begin{bmatrix} 1 + \frac{3}{2} - \frac{\sqrt{21}}{2} & 1 \\ -1 & -4 + \frac{3}{2} - \frac{\sqrt{21}}{2} \end{bmatrix} \begin{bmatrix} q_{11} \\ q_{21} \end{bmatrix} = 0$$

We can solve the above equation set by i.e. back substitution. Choosing

$$q_{21} = 1$$

yields

$$q_{11} = -\frac{1}{1 + \frac{3}{2} - \frac{\sqrt{21}}{2}} \cdot q_{21} = -\frac{2}{5 - \sqrt{21}}$$

The second eigenvalue yields:

$$(A - \lambda_2 I)q_2 = \begin{bmatrix} 1 + \frac{3}{2} + \frac{\sqrt{21}}{2} & 1 \\ -1 & -4 + \frac{3}{2} + \frac{\sqrt{21}}{2} \end{bmatrix} \begin{bmatrix} q_{12} \\ q_{22} \end{bmatrix} = 0$$

Choosing

$$q_{22} = 1$$

yields

$$q_{12} = -\frac{1}{1 + \frac{3}{2} + \frac{\sqrt{21}}{2}} \cdot q_{21} = -\frac{2}{5 + \sqrt{21}}$$

The eigenvector matrix can now be found as:

$$Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} = \begin{bmatrix} -\frac{2}{5-\sqrt{21}} & -\frac{2}{5+\sqrt{21}} \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -4.791 & -0.2087 \\ 1.000 & 1.000 \end{bmatrix}$$

To perform the similarity transform, we need the inverse of Q:

$$Q^{-1} = \frac{1}{q_{11}q_{22} - q_{12}q_{21}} \begin{bmatrix} q_{22} & -q_{12} \\ -q_{21} & q_{11} \end{bmatrix} = -\frac{1}{\sqrt{21}} \begin{bmatrix} 1 & \frac{2}{5+\sqrt{21}} \\ -1 & -\frac{2}{5-\sqrt{21}} \end{bmatrix} = \begin{bmatrix} -0.2182 & -0.04554 \\ 0.2182 & 1.046 \end{bmatrix}$$

The equivalent state equations can now be found as (see Def. 4.1, p. 95 in Chen):

$$\bar{A} = Q^{-1}AQ = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} + \frac{\sqrt{21}}{2} & 0 \\ 0 & -\frac{3}{2} - \frac{\sqrt{21}}{2} \end{bmatrix} = \begin{bmatrix} 0.7913 & 0 \\ 0 & -3.791 \end{bmatrix}$$

$$\bar{B} = Q^{-1}B = -\frac{1}{\sqrt{21}} \begin{bmatrix} 1 & \frac{2}{5+\sqrt{21}} \\ -1 & -\frac{2}{5-\sqrt{21}} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{4}{5\sqrt{21}+21} \\ \frac{4}{5\sqrt{21}-21} \end{bmatrix} = \begin{bmatrix} -0.09109 \\ 2.091 \end{bmatrix}$$

b) To find the transition matrix for the (original) system, we can start by finding the transition matrix for the transformed system as:

$$\bar{\Phi}(t) = \mathrm{e}^{\bar{A}t} = \begin{bmatrix} \mathrm{e}^{\lambda_1 t} & 0 \\ 0 & \mathrm{e}^{\lambda_2 t} \end{bmatrix}$$

Now the transition matrix for the original system should be given by:

$$\Phi(t) = \mathrm{e}^{At} = Q\mathrm{e}^{\bar{A}t}Q^{-1} = \begin{bmatrix} -\frac{2}{5-\sqrt{21}} & -\frac{2}{5+\sqrt{21}} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{e}^{\lambda_1 t} & 0 \\ 0 & \mathrm{e}^{\lambda_2 t} \end{bmatrix} \left(-\frac{1}{\sqrt{21}}\right) \begin{bmatrix} 1 & \frac{2}{5+\sqrt{21}} \\ -1 & -\frac{2}{5-\sqrt{21}} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{2}{5\sqrt{21}-21}\mathrm{e}^{\lambda_1 t} - \frac{2}{5\sqrt{21}+21}\mathrm{e}^{\lambda_2 t} & \frac{1}{\sqrt{21}}\mathrm{e}^{\lambda_1 t} - \frac{1}{\sqrt{21}}\mathrm{e}^{\lambda_2 t} \\ -\frac{1}{\sqrt{21}}\mathrm{e}^{\lambda_1 t} + \frac{1}{\sqrt{21}}\mathrm{e}^{\lambda_2 t} & -\frac{2}{5\sqrt{21}+21}\mathrm{e}^{\lambda_1 t} + \frac{2}{5\sqrt{21}-21}\mathrm{e}^{\lambda_2 t} \end{bmatrix} =$$

$$= \begin{bmatrix} 1.046\mathrm{e}^{\lambda_1 t} - 0.04554\mathrm{e}^{\lambda_2 t} & 0.2182\mathrm{e}^{\lambda_1 t} - 0.2182\mathrm{e}^{\lambda_2 t} \\ -0.2182\mathrm{e}^{\lambda_1 t} + 0.2182\mathrm{e}^{\lambda_2 t} & -0.04554\mathrm{e}^{\lambda_1 t} + 1.046\mathrm{e}^{\lambda_2 t} \end{bmatrix}$$

Problem 2

a) We are given the transfer matrix:

$$G(s) = (g_{11}(s), g_{12}(s)) = \left(\frac{-5}{s+1}, \frac{s^2 - 5s + 6}{s^2 + s}\right)$$

The numerator of $g_{11}(s)$ has a degree of zero, and the denominator of $g_{11}(s)$ has a degree of one, the transfer function $g_{11}(s)$ is therefore strictly proper. The numerator of $g_{12}(s)$ has a degree of two, and the denominator of $g_{11}(s)$ has a degree of two, the transfer function $g_{12}(s)$ is therefore proper. The transfer matrix $G(s)$ is therefore a proper rational matrix, and according to Theorem 4.2 (p. 101 in Chen), the transfer matrix is realizable.

b) To find a realization, we should decompose the transfer matrix as:

$$G(s) = G_{sp}(s) + G(\infty)$$

Using l'Hospital's rule, we find:

$$G(\infty) = (0, 1) = D$$

The strictly proper transfer matrix in the above decomposition should be:

$$G_{sp}(s) = G(s) - G(\infty) = \left(\frac{-5}{s+1}, \frac{-6s+6}{s^2+s}\right)$$

The least common denominator for all the elements in $G_{sp}(s)$ is

$$d(s) = s^2 + s$$

which yields:

$$G_{sp}(s) = \frac{1}{d(s)}(N(s)) = \frac{1}{s^2+s}\left(\begin{bmatrix} -5 & -6 \end{bmatrix} s + \begin{bmatrix} 0 & 6 \end{bmatrix}\right)$$

We now have everything needed to set up a realization in controllable canonical form:

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} -5 & -6 & 0 & 6 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \end{bmatrix} u$$

c) We can compute the controllability matrix as

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$$

where

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}, \; AB = \begin{bmatrix} -1 & 0 \\ 0 & -2 \\ 0 & 2 \end{bmatrix}, \; A^2B = \begin{bmatrix} -1 & 0 \\ 0 & 2 \\ 0 & -2 \end{bmatrix}$$

so

$$\mathcal{C} = \begin{bmatrix} 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 2 & 0 & -2 & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 & -2 \end{bmatrix}$$

where each row is linearly independent, so the rank is three, and therefore the system is controllable. Next, we can compute the observability matrix as

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$$

where

$$C = \begin{bmatrix} -5 & -3 & 3 \end{bmatrix}, \; CA = \begin{bmatrix} 5 & 6 & 0 \end{bmatrix}, \; A^2B = \begin{bmatrix} -5 & -6 & 0 \end{bmatrix}$$

so

$$\mathcal{O} = \begin{bmatrix} -5 & -3 & 3 \\ 5 & 6 & 0 \\ -5 & -6 & 0 \end{bmatrix}$$

reduced to row echelon form is

$$\mathcal{O}' = \begin{bmatrix} -5 & -3 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

where we only have two non-zero rows, so the rank must be two, thus the system is not observable.

d) The system is not a minimal realization, since it is not both controllable and observable (see Theorem 7.M2, p. 207 in Chen).

Problem 3

a) BIBO stability: A bounded-input bounded-output (BIBO) system is a system where all bounded inputs results in a bounded output. It is defined for the zero-state response, that is, it is applicable only if the system is initially relaxed (see p. 122 in Chen).

Lyapunov stability: A system is stable in the sense of Lyapunov if every finite initial state results in a bounded response. This is therefore the stability of the zero-input response, that is, when no forcing input is applied to the system (see p. 130 in Chen).

Asymptotic stability: A system is asymptotically stable if it is stable in the sense of Lyapunov, and in addition, the response approaches zero as time goes to infinity (see p. 130 in Chen).

b) The answer is Theorem 5.4 p. 120 in Chen.

Lyapunov stability: Eigenvalues of A must have zero or negative real parts, and eigenvalues with zero real parts must be simple roots (multiplicity one) of the minimal polynomial of A.

Asymptotic stability: Eigenvalues of A must have negative real parts.

c) The eigenvalues of A are:

$$\Delta(\lambda) = \det(\lambda I - A) = \left| \begin{bmatrix} \lambda+2 & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & 0 & \lambda \end{bmatrix} \right| = \lambda^2(\lambda+2) \;\Rightarrow\; \lambda_{1,2} = 0,\; \lambda_3 = -2$$

The A matrix given in the problem is already on Jordan form, and the Jordan block for the repeated eigenvalue is of order two. This means that the minimal polynomial of A has an eigenvalue with zero real part of multiplicity two, and thus, the system is not stable in the sense of Lyapunov (or asymptotically stable for that matter).

d) The usage of the Lyapunov equation, $A^{\mathrm{T}}M + MA = -I$, is straight forward (remember that M is a symmetric matrix, that is $m_{12} = m_{21}$):

$$\begin{bmatrix} 0 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} + \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} \begin{bmatrix} 0 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

This yields an equation set such as:

$$\begin{aligned} -2m_{12} &= -1 \\ -2m_{11} + m_{12} - m_{22} &= 0 \\ -2m_{11} + m_{12} - m_{22} &= 0 \\ -4m_{12} + 2m_{22} &= -1 \end{aligned}$$

Solving the equation set yields

$$m_{11} = 0,\; m_{12} = m_{21} = \frac{1}{2},\; m_{22} = \frac{1}{2}$$

or

$$M = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

which has principal minors 0 and $-\frac{1}{4}$, which are both not positive, and therefore the M matrix is not positive definite (Sylvester criterion), and the system is not stable.

Problem 4

a) The principle of the linear quadratic regulator (LQR) is to find the controller input $u(t)$ which, for the system $\dot{x} = Ax + Bx$, minimizes the cost function

$$J = \int_0^\infty [x^{\mathrm{T}}Qx + u^{\mathrm{T}}Ru]\mathrm{d}t$$

where x is the state vector, and u is the input vector. Q is the weight matrix for the state variables, and R is the weight matrix for the inputs. The weight matrixes must be real, symmetric, and positive definite. The relative size of the elements of Q and R enforces tradeoffs between the magnitude of the control action and the speed of response. The system must be controllable, and all states must be available as measurements or estimates. The minimizing input will be given by

$$u = -Kx$$

where $K = R^{-1}B^{\mathrm{T}}P_c^*$ and P_c^* is found by solving the continuous time algebraic Riccati equation:

$$A^{\mathrm{T}}P_c^* + P_c^*A - P_c^*BR^{-1}B^{\mathrm{T}}P_c^* + Q = 0$$

Shifting the equilibrium to x^* can be done using the controller $u = Px^* - Kx$, where P is chosen such that $x(t \to \infty) = x^*$.

Problem 5

a) A white noise process, $F(t)$, is a stationary random process with spectral density function

$$S_F(j\omega) = \sigma^2$$

and autocorrelation function

$$R_F(\tau) = \sigma^2\delta(\tau).$$

The mean value is $\mathrm{E}[F(t)] = 0$, and thus the variance is $R_F(0) = \infty$. If the process is Gaussian, the realizations of the random variable $F(t)$ are drawn from a Gaussian distribution with a zero mean and variance σ^2.

Gaussian white noise can also be defined as the derivative of the Wiener process.

b) The Wiener process, $X(t)$, is the integral of a Gaussian white noise process, $X(t) = \int_0^t F(u)\mathrm{d}t$. The mean value is

$$\mathrm{E}[X(t)] = 0$$

the mean-square value, or the variance, since the mean value is zero, is

$$\mathrm{E}[X^2(t)] = t$$

which means that it is not a stationary process (the variance is not independent of time). Since the process is not stationary, the autocorrelation function is given in terms of t_1 and t_2, not the time difference $\tau = t_2 - t_1$, and is given as:

$$R_X(t_1, t_2) = \begin{cases} t_2, & t_1 \geq t_2 \\ t_1, & t_2 < t_1 \end{cases}$$

c) See Fig. 1 for the block diagram. A realization of $G(s)$ can be found as:

$$(2+s)X_1(s) = 10U(s) \;\Rightarrow\; \dot{x}_1 = -2x_1 + 10u$$

From the description of the system we know that an unknown, slowly time-varying disturbance is affecting the input, and it is modeled as a Wiener process, thus the expression

$$\dot{x}_2 = \beta w_1$$

is the Wiener process, since w_1 is unity (Gaussian) white noise. The state x_2 is thus the disturbance, and is added to the input. The output is the state of the given system, x_1, and it is corrupted by a white noise component, thus the output equation is:

$$y = x_1 + \gamma w_2$$

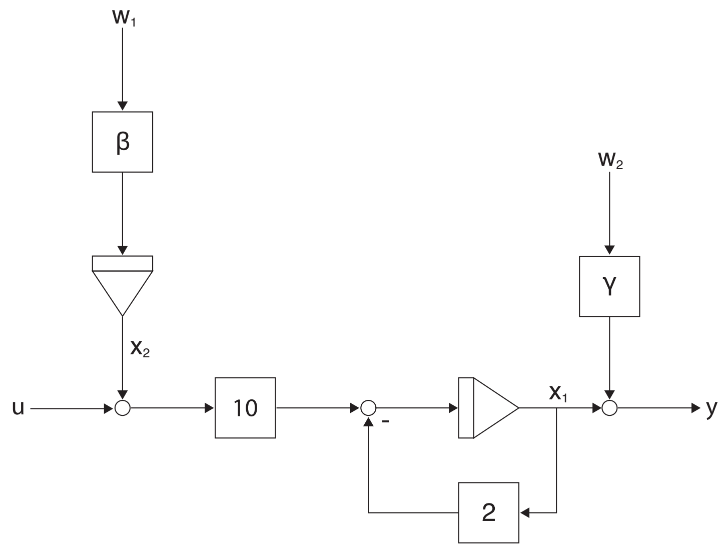

Figure 1: Block diagram for Problem 5 c).

d) The parameters β and γ are scaling parameters for the standard deviations of the noise inputs. This can be seen from

$$E[y] = E[x_1 + \gamma w_2] = x_1 + \gamma E[w_2] = x_1$$

and thus

$$E[(y - E[y])^2] = E[(x_1(t) + \gamma w_2(t) - x_1)^2] = \gamma^2 E[{w_2}^2]$$

and, likewise:

$$E[(\dot{x}_2 - E[\dot{x}_2])^2] = E[(\beta w_1 - \beta E[w_1])^2] = \beta^2 E[{w_1}^2]$$

The parameter β will also determine the variation rate for the disturbance x_2, as $E[(x_2(t))^2] = \beta^2 t$. The higher the value of β is, the faster the variance of x_2 will grow.

e) First, we should check that the system is observable. The state matrix is

$$A = \begin{bmatrix} -2 & 10 \\ 0 & 0 \end{bmatrix}$$

and the measurement matrix is

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

which means the observability matrix is

$$\mathcal{O} = \begin{bmatrix} 1 & 0 \\ -2 & 10 \end{bmatrix}$$

which has full rank, and therefore the system is observable. Next, since we have a continuous system, we must discretize the system using e.g. Euler discretization:

$$x((k+1)\Delta t) = (I + \Delta t A)x(k\Delta t) + \Delta t B u(k\Delta t) + \Delta t G w_1(k\Delta t)$$
$$y(k\Delta t) = Cx(k\Delta t) + Hw_2(k\Delta t)$$

where

$$B = \begin{bmatrix} 10 \\ 0 \end{bmatrix}, \ G = \begin{bmatrix} 0 \\ \beta \end{bmatrix}, \ H = \begin{bmatrix} \gamma \end{bmatrix}$$

Δt is the sample time, and k is the time step.

Further we need to specify the process and noise covariance, which should in this case be

$$Q_k = E\left[\begin{bmatrix} 0 \\ \Delta t \beta w_1 \end{bmatrix} \begin{bmatrix} 0 & \Delta t \beta w_1 \end{bmatrix}\right] = \begin{bmatrix} 0 & 0 \\ 0 & \Delta t^2 \beta^2 \end{bmatrix} = GQG^{\mathrm{T}}\Delta t$$

from Equation (7.1.9), and

$$R_k = [\gamma^2] = R\frac{1}{\Delta t}$$

from Equation (7.1.13).

Discuss... If we assume that we start from a relaxed state, we set the initial error covariance to

$$P_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and the initial state estimate to:

$$\hat{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now we can estimate the system's states using the following scheme. Entering with the initial estimates and error covariance matrix, we compute:

I. The prediction step/time update

(1) Project the state ahead:

$$\hat{x}_k^- = A\hat{x}_{k-1} + Bu_k$$

(2) Project the error covariance ahead:

$$\hat{P}_k^- = A\hat{P}_{k-1}A^{\mathrm{T}} + Q_k$$

II. The correction step/measurement update

(3) Compute the Kalman gain:

$$K_k = P_k^- C^{\mathrm{T}}(CP_k^- C^{\mathrm{T}} + R)^{-1}$$

(4) Update estimate with measurement y_k:

$$\hat{x}_k = \hat{x}_k^- + K_k(y_k - C\hat{x}_k^-)$$

(5) Update the error covariance:

$$\hat{P}_k = (I - K_k C)\hat{P}_k^-$$

The steps in part I and II are repeated until termination of the filter. We can also start with part II instead of part I if we supply x_0^- and P_0^- instead of x_0 and P_0. If the system is linear time invariant, and the noise has stationary statistics, as in this case, the stationary Kalman filter can be used, which simplifies the computation.

f) Perhaps the most salient problem with roundoff errors in the context of Kalman filters, is the possibility for the error covariance matrix P to become non-positive definite. If this happens, the state estimates will diverge. One technique for improving numerical stability is to avoid zero diagonal elements in the Q matrix. I.e. for the filter in part e) above, we should add a small positive quantity to the upper left element of Q_k. Other techniques that should reduce roundoff problems are:

1. Using high-precision arithmetic.
2. Symmetrize the P and P^- matrices, e.g. by only using the upper triangular parts in computations.
3. Use the U-D factorization form of the Kalman filter.

$P \leftarrow \frac{1}{2}(P + P^{\mathrm{T}})$

Use symmetric formula in (5).