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Problem 1

a) Let $z = -1 + i\sqrt{3}$

$$\arg z = \arctan\left(\frac{\sqrt{3}}{-1}\right) + \pi$$

$$= \frac{2\pi}{3}$$

$$|z| = \sqrt{(-1)^2 + (\sqrt{3})^2}$$

$$= 2$$

So we write $z = 2e^{\frac{2\pi}{3}i}$

which makes it easy to compute z^3, $|z|^6$

$$z^3 = 2^3 e^{3\cdot\left(\frac{2\pi}{3}\right)i} = 8e^{2\pi i} = \underline{8}$$

$$|z|^6 = 2^6 = 64$$

b) Want to solve $z^3 = 8i$ which should have three solutions.

$$|z^3| = |z|^3 = |8i| = 8$$

$$\Rightarrow |z| = \sqrt[3]{8} = 2$$

$$\arg(z^3) = 3\arg z = \arg 8i = \frac{\pi}{2} + 2\pi k$$

for $k = 0, \pm 1, \pm 2, \ldots$

$$\Rightarrow \arg z = \frac{\pi}{6} + \frac{2\pi}{3}k$$

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For $k = 0, 1, 2$ we get

$$\arg z_1 = \frac{\pi}{6}$$

$$\arg z_2 = \frac{5\pi}{6}$$

$$\arg z_3 = \frac{3\pi}{2}$$

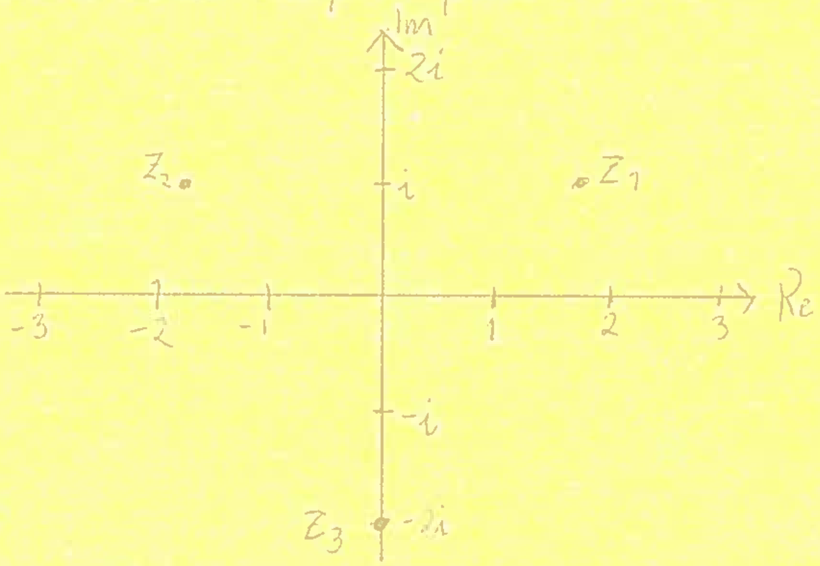
$k = 3$ corresponds to $k = 0$ so we're done.
The three numbers ~~that~~ satisfy $z^3 = 8i$ are

$$z_1 = 2e^{\frac{\pi}{6}i} = \sqrt{3} + i$$

$$z_2 = 2e^{\frac{5\pi}{6}i} = -\sqrt{3} + i$$

$$z_3 = 2e^{\frac{3\pi}{2}i} = 0 - 2i = -2i$$

Drawn in the complex plane:



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Problem 2

a) The associated homogeneous equation is

$$y'' + 6y' + 9y = 0 \qquad (E_h)$$

which has char. eq.

$$r^2 + 6r + 9 = 0$$
$$\iff (r+3)^2 = 0$$

which has a repeated root $r=-3$. This means that e^{-3t} and te^{-3t} form a fundamental set of solutions so the general solution is

$$y_h = Ae^{-3t} + Bte^{-3t}$$

A, B constants.

b) Using the method of undetermined coefficients we try $y_p = A\cos t + B\sin t$

$$\Rightarrow y_p' = -A\sin t + B\cos t$$

$$y_p'' = -A\cos t - B\sin t$$

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A, B should then satisfy:

$$-A\cos t - B\sin t + 6(-A\sin t + B\cos t) + 9(A\cos t + B\sin t)$$
$$= \cos t$$

$$\Leftrightarrow (-A + 6B + 9A)\cos t + (-B - 6A + 9B)\sin t = \cos t$$

$$\Rightarrow \quad \text{(i) } 6B + 8A = 1$$
$$\text{(ii) } 8B - 6A = 0$$

$$\text{(ii)} \Rightarrow A = \frac{8}{6}B = \frac{4}{3}B$$

$$\text{(i): } 6B + 8A = 6B + 8\left(\frac{4}{3}B\right) = 1$$

$$= \frac{50}{3}B$$

$$\Rightarrow B = \frac{3}{50}$$

$$\Rightarrow A = \frac{4}{3}\cdot\frac{3}{50} = \frac{4}{50} = \frac{2}{25}$$

So $y_p = \frac{2}{25}\cos t + \frac{3}{50}\sin t$

is a particular solution.

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c) The general solution of (1) is

$$y = y_h + y_p$$

$$= Ae^{-3t} + Bte^{-3t} + \frac{2}{25}\cos t + \frac{3}{50}\sin t$$

$$\Rightarrow y' = -3Ae^{-3t} + Be^{-3t} + Bt(-3)e^{-3t}$$

$$- \frac{2}{25}\cos t + \frac{3}{50}\sin t$$

$$\Rightarrow y(0) = A + B\cdot 0 + \frac{2}{25} = A + \frac{2}{25}$$

$$y'(0) = -3A + B + \frac{3}{50}$$

Since we want $y(0) = y'(0) = 0$ we get

$$A + \frac{2}{25} = 0 \Rightarrow A = -\frac{2}{25}$$

$$-3A + B + \frac{3}{50} = 0 \Rightarrow B = -\frac{3}{50} + 3A$$

$$= -\frac{3}{10}$$

The unique solution is then

$$y(t) = -\frac{2}{25}e^{-3t} - \frac{3}{10}te^{-3t} + \frac{2}{10}\cos t + \frac{3}{50}\sin t$$

Problem 3

$a \in \mathbb{R}$ $A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$

a) $\vec{x}' = A\vec{x}$

If $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ then we have the

We'll compute eigenvalues λ_1, λ_2 and corresponding eigenvectors $\vec{v}_1, \vec{v}_2$.

Eigenvalues: $\det(A - \lambda I) = 0$

$\iff \begin{vmatrix} -\lambda & a \\ -a & -\lambda \end{vmatrix} = 0$

$\iff 0 = (-\lambda)^2 + a^2$

$\Rightarrow \lambda_{1,2} = \pm ai$

Eigenvector for $\lambda = +ai$

$(A - ai)\vec{x} = \vec{0}$

$\Rightarrow \begin{pmatrix} -ai & a \\ -a & -ai \end{pmatrix} \sim \begin{pmatrix} a & ai \\ -a & -ai \end{pmatrix} \sim \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}$

So an eigenvector for $\lambda = ai$ is

$\vec{v}_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$.

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We have a pair of complex conjugated e. values and an eigenvector for $\lambda = ai$.

~~That~~ A fundamental system of (real) solutions is then given by

$$\mathrm{Re}(\vec{v}e^{\lambda t}) \quad \text{and} \quad \mathrm{Im}(\vec{v}e^{\lambda t})$$

$$\vec{v}\cdot e^{\lambda t} = \begin{pmatrix}-i\\1\end{pmatrix}e^{ai} = \begin{pmatrix}-i\\1\end{pmatrix}(\cos at + i\sin at)$$

$$= \begin{pmatrix}-i\cos at + \sin at\\ \cos at + i\sin at\end{pmatrix} = \begin{pmatrix}\sin at - i\cos at\\ \cos at + i\sin at\end{pmatrix}$$

Let $\vec{u}_1 = \mathrm{Re}(\vec{v}e^{\lambda t}) = \begin{pmatrix}\sin at\\ \cos at\end{pmatrix}$

$\vec{u}_2 = \mathrm{Im}\,\vec{v}e^{\lambda t} = \begin{pmatrix}-\cos at\\ \sin at\end{pmatrix}$

Then $\vec{u}_1, \vec{u}_2$ form a fundamental set of real solutions.

b) The general real solution is then

$$\vec{x}(t) = A\vec{u}_1 + B\vec{u}_2$$

$$\Rightarrow \vec{x}(0) = A\begin{pmatrix}0\\1\end{pmatrix} + B\begin{pmatrix}-1\\0\end{pmatrix}$$

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So for our case where $\vec{x}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, we get

$$A \cdot 0 - B = 2 \implies B = -2$$
$$A \cdot 1 + B \cdot 0 = 1 \implies A = 1$$

$$\vec{x}(t) = \begin{pmatrix} \sin at \\ \cos at \end{pmatrix} - 2 \begin{pmatrix} -\cos at \\ \sin at \end{pmatrix}$$

solves the initial value problem.

Problem 4

Let $\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} 3 \\ 6 \\ -1 \end{pmatrix}$

a) We want to solve

$$c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = \vec{p} = \begin{pmatrix} 2 \\ 4 \\ -10 \end{pmatrix}$$

$$\iff (\vec{u} \mid \vec{v} \mid \vec{w}) \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \vec{p}.$$

which we solve with an augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 2 & 4 & 6 & 4 \\ 1 & 6 & -1 & -10 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & -4 & -12 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 5 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -3 \end{array}\right)$$

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So we have

$$c_1 = 8 + 5c_3$$
$$c_2 = -3 + c_3$$
$$c_3 \text{ free}$$

So for instance $c_1 = 8, c_2 = -3, c_3 = 0$ give

$$8\vec{u} - 3\vec{v} + 0\vec{w} = \vec{p}$$

b) We try by row reducing following the same method as (a.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 2 & 4 & 6 & 5 \\ 1 & 6 & -1 & 6 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 1 \\ 1 & 6 & -1 & 6 \end{array}\right)$$

The second row tells us we have an inconsistent system so we cannot write $\vec{q}$ as a linear combination of $\vec{u}, \vec{v}, \vec{w}$.

c) No because ~~every~~ ~~vect~~ vectors in $\mathbb{R}^3$ can be reached (are in span$\{\vec{u}, \vec{v}, \vec{w}\}$).
Three vectors in $\mathbb{R}^3$ are linearly independent iff. span{"all three vectors"} $= \mathbb{R}^3$.

d) They are linearly dependent so
$\underline{\det A = 0}$

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Problem 5

a)
$$\left(\begin{array}{ccc|ccc} 2 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 4 & 2 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 2 & 2 & 0 & 1 & 0 & 0 \\ 4 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1/2 & 0 & 0 \\ 0 & -2 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & 0 & 1/2 \\ 0 & 1 & 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right)$$

So $A^{-1} = \begin{pmatrix} -1/2 & 0 & 1/2 \\ 1 & 0 & -1/2 \\ 0 & 1 & 0 \end{pmatrix}$

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b) $T: \mathbb{R}^3 \to \mathbb{R}^3$

First we note that

$$T\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Since A is invertible it is one-to-one and onto, so T is thus one-to-one (and onto).

Problem 6

Let $A = \begin{pmatrix} 1 & 2 & 0 & 3 & 1 \\ 2 & 4 & -1 & 5 & 4 \\ 3 & 6 & -1 & 8 & 5 \\ 5 & 4 & 8 & -1 & 1 \end{pmatrix}$

a) $$A \sim \begin{pmatrix} 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & -1 & -1 & 2 \\ 0 & 0 & -1 & -1 & 2 \\ 0 & -6 & 8 & -16 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 3 & 1 \\ 0 & -6 & 8 & -16 & -4 \\ 0 & 0 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

b) A basis for col$A$ is given by the pivot columns of A. So we get that

$$\mathcal{B}_{col} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 6 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -1 \\ 8 \end{pmatrix} \right\}$$

is a basis of col$A$.

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The rank of A is equal to the number of pivot columns, so

$$\text{Rank } A = 3$$

c) Since A is 4×5 and Rank $A = 3$. we get $\dim(\text{Nul}A) = 5-3=2$ by using the rank theorem

d) $\dim \text{Row}A = \dim \text{col}A = \text{Rank}A = 3$

So $\dim \text{Row}A = 3$.

For $\dim \text{Nul}A^T$ we note that A^T is 5×4 and $\text{Rank } A = \text{Rank } A^T$.

An argument for this is that:

$$\text{col}A = \text{Row}A^T$$

$$\Rightarrow \underbrace{\dim \text{Col}A}_{\text{Rank}A} = \underbrace{\dim \text{Row}A^T}_{\text{Rank}A^T}$$

By the rank theorem we get

$$\dim \text{Nul}A^T = 4 - \underset{\text{Rank}A^T}{3} = \underline{\underline{1}}$$

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Problem 7

We'll organize the information in a stochastic matrix and find a steady state probability vector for the system.

To: ↓	From (the day before) Below	Equal	Above
Below	.70	.10	.10
Equal	.20	.80	.20
Above	.10	.10	.70

Green indicates implied information.

So we can see that the stochastic matrix for the system is

$$P = \begin{pmatrix} .7 & .1 & .1 \\ .2 & .8 & .2 \\ .1 & .1 & .7 \end{pmatrix}$$

We note that since P is stochastic, it has a unique probabillity vector $\vec{q}$ such that

$$P\vec{q} = \vec{q},$$

and $\vec{q}$ reflects the long term behaviour of the system.

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So we want to solve

$$P\vec{q} = \vec{q} \iff (P-I)\vec{q} = \vec{0}$$

We solve this with an augmented matrix:

$$\begin{pmatrix} .7-1 & .1 & .1 \\ .2 & .8-1 & .2 \\ .1 & .1 & .7-1 \end{pmatrix} \sim \begin{pmatrix} -.3 & .1 & .1 \\ .2 & -.2 & .2 \\ .1 & .1 & -.3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1/3 & -1/3 \\ 1 & -1 & 1 \\ 1 & 1 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1/3 & -1/3 \\ 0 & -4/3 & 4/3 \\ 0 & 4/3 & -5/3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1/3 & -1/3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$q_1 = q_3$, $q_2 = 2q_3$, q_3 free

So $\vec{q}' = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

We divide by the sum of the entries and get

$$\vec{q} = \frac{1}{1+2+1} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{matrix} 1/5 \\ 2/5 \\ 1/5 \end{matrix} \begin{pmatrix} 1/4 \\ 2/4 \\ 1/4 \end{pmatrix} = \begin{pmatrix} .25 \\ .5 \\ .25 \end{pmatrix}$$

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The probabillities are thus as follows:

Below 0°C: 25% chance

Equal to 0°C: 50% chance

Above 0°C: 25% chance

The skier should prep her skies for 0°C weather ~~snow~~.

Problem 8

If we "pretend" $y = mx + c$ fits the data we get the equations:

$$
\begin{aligned}
m \cdot 0 + c &= 4 \\
m \cdot 1 + c &= -1 \\
m \cdot 2 + c &= 1 \\
m \cdot 3 + c &= -3 \\
m \cdot 4 + c &= -1
\end{aligned}
$$

which we write as a matrix equation

$$
\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} m \\ c \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 4 \\ -1 \\ 1 \\ -3 \\ -1 \end{pmatrix}}_{\vec{b}}
$$

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Problem 9

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

a) Have to verify that $\det(A-2I)=0$ and $A\vec{u} = \lambda\vec{u}$. (for some λ).

$$\det(A-2I) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$$ because of linear dependence.

$$A\vec{u} = = \begin{pmatrix} 3+1+1 \\ 1+3+1 \\ 1+1+3 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix} = 5\vec{u}$$

This shows that $\lambda = 5$ is an eigenvalue for $\vec{u}$.

b) Lets compute ~~the~~ a basis for the eigenspace of $\lambda = 2$. We do this by solving

$$(A-2I)\vec{x} = \vec{0}$$

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$$(A-2I) \sim \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow x_1 = -x_2 - x_3$

x_2, x_3 free

A basis for Nul$(A-2I)$ (this is ~~dim~~) is:

$$\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

We have that $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ is a basis for the eigenspace of $\lambda = 5$.

Since A is 3×3 and the total dimension of all eigenspaces (of A) is $2+1=3$, there can't be any more eigenvalues.

c) Since A is symmetric A must be orthogonally diagonalizable. So we can factor A into $A = PDP^{-1}$ where P is invertible, D diagonal.

We know that $P = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ and

$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ satisfy $A = PDP^{-1}$

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c) Since A is symmetric it is orthogonally diagonalizable.

The eigenspaces of two distinct eigenvalues are mutually orthogonal, ~~but~~ so we only have to make sure the basis of the eigenspace of $\lambda=2$ is an orthogonal basis.

$$\begin{pmatrix}-1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}-1\\0\\1\end{pmatrix} = 1 \neq 0$$

So we construct a new basis using Gram-Schmidt:

$$\vec{v}_1 = \begin{pmatrix}-1\\1\\0\end{pmatrix}$$

$$\vec{v}_2 = \begin{pmatrix}-1\\0\\1\end{pmatrix} - \frac{\vec{v}_1\cdot\begin{pmatrix}-1\\0\\1\end{pmatrix}}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 = \begin{pmatrix}-1\\0\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}-1\\1\\0\end{pmatrix}$$

$$= \begin{pmatrix}-1/2\\-1/2\\1\end{pmatrix}$$

$$= -\frac{1}{2}\begin{pmatrix}1\\1\\-2\end{pmatrix}$$

The basis $\left\{\begin{pmatrix}-1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\-2\end{pmatrix}\right\}$ is an orthogonal basis for the eigenspace of $\lambda=2$.

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If we let $P = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & -2 & 1 \end{pmatrix}$ and

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

then we have orthoganally diagonalized A into

$A = PDP^{-1}$.

Problem 10

Let $W \subseteq \mathbb{R}^n$ be a subspace and $W^\perp$ be its orthogonal complement.

a) We can define $W^\perp$ by

$$W^\perp = \{\vec{x}\, c : \vec{x} \cdot \vec{w} = 0,\ \vec{w} \in W\}$$

Since $\vec{w} \in \mathbb{R}^n$, $\vec{x} \cdot \vec{w}$ only makes sense for $\vec{x} \in \mathbb{R}^n$. This shows that $W^\perp \subseteq \mathbb{R}^n$.

To show it is a subspace we have to check that for $\vec{u}, \vec{v} \in W^\perp$;

1) $(\vec{u} + \vec{v}) \in W^\perp$

2) $c \cdot \vec{u} \in W^\perp$ $\quad$ c scalar.

3) $\vec{0} \in W^\perp$

1) $\vec{u} + \vec{v} = \vec{x}$

$$\begin{aligned} \vec{x} \cdot \vec{w} &= (\vec{u} + \vec{v}) \cdot \vec{w} \quad \text{for some } \vec{w} \in W \\ &= \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w} \\ &= 0 + 0 \quad \text{since } \vec{u}, \vec{v} \in W^\perp \\ &= 0 \end{aligned}$$

So $\vec{x} \cdot \vec{w} = 0$ so $\vec{x} = \vec{u} + \vec{v} \in W^\perp$

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2) $(c\cdot\vec{u})\circ\vec{w} = c\cdot(\vec{u}\cdot\vec{w})$, $\vec{w}\in W$

$= c\cdot 0$

$= 0$

So $c\vec{u}\in W^\perp$

3) $\vec{0}\circ\vec{w} = 0$ for any $\vec{w}$ (that makes sense)

So $\vec{0}\in W^\perp$

Since all three conditions are satisfied and $W^\perp$ is a subset of $\mathbb{R}^n$, $W^\perp$ is a subspace of $\mathbb{R}^n$.

b) Assume $\vec{w}\in(W\cap W^\perp)$

Then $\vec{w}$ satisfies

$$\vec{w}\circ\vec{v} = \vec{0} \text{ for all } \vec{v}\in W$$

Specifically it satisfies

$$\vec{w}\circ\vec{w} = \vec{0}$$

$$\Leftrightarrow \|\vec{w}\|^2 = 0$$

$$\Leftrightarrow \|\vec{w}\| = 0$$

So $\vec{w} = \vec{0}$ since it has length 0.

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c) Let $\{\vec{w}_1, \ldots, \vec{w}_r\}$ be a basis of W.
Let $\{\vec{v}_1, \ldots, \vec{v}_s\}$ be a basis of $W^\perp$.

We note that ~~all~~ basis is linearly independent. ~~Also, we~~
We also note that ~~every angel pair $\vec{w}_i$~~ $\vec{w}_i, \vec{v}_j$

$$\vec{w}_i \cdot \vec{v}_j = 0$$

So all vectors of the basis of W are orthogonal to all vectors of the basis of $W^\perp$ (by definition of orthogonal complement).

This tells us that the set $\{\vec{w}_1, \ldots, \vec{w}_r, \vec{v}_1, \ldots, \vec{v}_s\}$ has $r+s$ linearly independent vectors.

We want to show that $\text{span}\{\vec{w}_1, \ldots, \vec{w}_r, \vec{v}_1, \ldots, \vec{v}_s\} = \mathbb{R}^n$ or $r+s=n$.

Consider the matrix $A = (\vec{w}_1 | \ldots | \vec{w}_r | \vec{v}_1 | \cdots | \vec{v}_s)$.

its rank is $r+s$ because all collumns are linearly independent. Now consider Nul A or more spesifically dim Nul A.
The size of A is $n \times (r+s)$

The rank theorem tells us that

$$\text{Rank}A + \dim \text{Nul}A = (r+s)$$
$$\Rightarrow \dim\text{Nul}A = (r+s)-(r+s)$$
$$= 0$$

So the dimension of the nullspace of A is zero. This equivalent to A being invertible which means A must square. This means that

$$r+s = n.$$

The basis has n linearly independent vectors and all are inn $\mathbb{R}^n$, so it must be a basis of $\mathbb{R}^n$.

Note: span of basis = col A = $\mathbb{R}^n$

Therefore the basis spans $\mathbb{R}^n$.