

NORWEGIAN UNIVERSITY
OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF TELECOMMUNICATIONS

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EXAMINATION IN COURSE TTT4120 DIGITAL SIGNAL PROCESSING

Date: Friday Desember 3rd, 2010
Time: 09.00 - 13.00

Permitted aids: D–No printed or handwritten material allowed.
Specified, simple calculator allowed..

INFORMATION

- The examination consists of 4 problems.
 - Problem 1 concerns analysis of digital filters.
 - Problem 2 concerns rational process models.
 - Problem 3 concerns fixed point implementation.
 - Problem 4 concerns Wiener filters.
 - All tasks are weighted equally.
 - A list of formulas can be found in the appendix.
- All answers should be justified!
- The teacher will visit you twice, the first time around 10.00 and the second time around 11.45.

Problem 1

1a) Given a causal, linear and time invariant (LTI) system with unit pulse response $h(n)$.

Give the region of convergence (ROC) in the z -plane for the system.

Where must the zeros and poles be placed in the z -plane?

Answer : ROC is $|z| > |a|$ and $|a| < 1$

No restriction on zeros, however poles must be inside unit circle, i.e. $\max_k |p_k| < |a|$

1b) A causal LTI filter is described by the following difference equation :

$$y(n) + \frac{1}{6}y(n-1) - \frac{1}{6}y(n-2) = 2x(n) + \frac{1}{6}x(n-1), \quad n = -\infty, \infty \quad (1)$$

Show that the transfer function $H(z)$ is given by :

$$H(z) = \frac{2 + \frac{1}{6}z^{-1}}{(1 + \frac{1}{2}z^{-1})(1 - \frac{1}{3}z^{-1})} \quad (2)$$

Answer :

$$Y(z) + \frac{1}{6}Y(z)z^{-1} - \frac{1}{6}Y(z)z^{-2} = 2X(z) + \frac{1}{6}X(z)z^{-1} \Rightarrow$$

$$Y(z)(1 + \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}) = X(z)(2 + \frac{1}{6}z^{-1}) \Rightarrow$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2 + \frac{1}{6}z^{-1}}{1 + \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}}$$

It is easy to see that the roots of the denominator is respectively $z = -\frac{1}{2}$ and $z = \frac{1}{3}$

1c) Give the ROC area in the z-plane for the filter in task 1b.

Is the filter stable?

Does the filter have minimum phase?

Answer : $\max[|p_1|, |p_2|] = \frac{1}{2} \Rightarrow \text{ROC} : |z| > \frac{1}{2}$

The causality and the ROC together guarantees stability

Minimum phase requires that both poles and zeros are inside the unit circle. We have a zeros at $n_1 = -\frac{1}{12}$. Thus the filter has minimum phase.

1d) Show that the unit impulse response of the filter is given by :

$$h(n) = \begin{cases} (-\frac{1}{2})^n + (\frac{1}{3})^n & n \geq 0 \\ 0 & n < 0 \end{cases} \quad (3)$$

Answer :

$$\frac{2 + \frac{1}{6}z^{-1}}{1 + \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}} = \frac{C}{1 + \frac{1}{2}z^{-1}} + \frac{D}{1 - \frac{1}{3}z^{-1}} \Rightarrow$$

$$C + D = 2$$

$$(-\frac{C}{3} + \frac{D}{2})z^{-1} = \frac{1}{6}z^{-1} \Rightarrow -2C + 3D = 1$$

which give $C = D = 1$!

Thus :

$$H(z) = \frac{1}{1 + \frac{1}{2}z^{-1}} + \frac{1}{1 - \frac{1}{3}z^{-1}} \Rightarrow$$

$$h(n) = \begin{cases} (-\frac{1}{2})^n + (\frac{1}{3})^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

Problem 2

2a) Sketch the DF2 and the parallel form structures for the filter in task 1.

2b) The autocorrelation sequence of a general filter is defined as :

$$r_{hh}(m) = \sum_{n=-\infty}^{\infty} h(n)h(n+m) \quad m = -\infty, \infty \quad (4)$$

Show that the filter in task 1 has the following autocorrelation sequence :

$$r_{hh}(m) = \begin{cases} \frac{46}{21}(-\frac{1}{2})^m + \frac{111}{56}(\frac{1}{3})^m & m \geq 0 \\ r_{hh}(-m) & m < 0 \end{cases} \quad (5)$$

Answer : we assume $m \geq 0$

$$r_{hh}(m) = \sum_n [(-\frac{1}{2})^n + (\frac{1}{3})^n][(-\frac{1}{2})^{n+m} + (\frac{1}{3})^{n+m}] = A(-\frac{1}{2})^m + B(\frac{1}{3})^m$$

where

$$\begin{aligned} A &= \sum_n [(-\frac{1}{2})^n + (\frac{1}{3})^n](-\frac{1}{2})^n = \sum_n (\frac{1}{4})^n + (-\frac{1}{6})^n \\ B &= \sum_n [(-\frac{1}{2})^n + (\frac{1}{3})^n](\frac{1}{3})^n = \sum_n (-\frac{1}{6})^n + (\frac{1}{9})^n \end{aligned}$$

Utilizing that the sums go from $0 \rightarrow \infty$ we get :

$$\begin{aligned} A &= \frac{1}{1 - \frac{1}{4}} + \frac{1}{1 + \frac{1}{6}} = \frac{4}{3} + \frac{6}{7} = \frac{4 * 7 + 3 * 6}{3 * 7} = \frac{46}{21} \approx 2.2 \\ B &= \frac{1}{1 + \frac{1}{6}} + \frac{1}{1 - \frac{1}{9}} = \frac{6}{7} + \frac{9}{8} = \frac{6 * 8 + 7 * 9}{7 * 8} = \frac{111}{56} \approx 2.0 \end{aligned}$$

2c) White noise $w(n)$ with power $\sigma_w^2 = 1$ is input to the filter in task 2a.

Which kind of parametric processes are respectively the internal signals of the DF2 structure and the output signal?

Answer : The difference equation can be rewritten to include the internal sequence $g(n)$ in the DF2-structure :

$$g(n) = -\frac{1}{6}g(n-1) + \frac{1}{6}g(n-2) + w(n) \quad (6)$$

$$y(n) = 2g(n) + \frac{1}{6}g(n-1) \quad (7)$$

The internal sequence $g(n)$ is thus an AR[2]-process, while the output sequence $y(n)$ is an ARMA[1,2]-process.

2d) Find, by using linear prediction (i.e. the Yule-Walker equations), the process parameters for the best AR[1]-model of the filter output signal $y(n)$.

Answer : The Yule-Walker equations for the best AR[1] -model are given by

$$\gamma_{yy}(1) = -a\gamma_{yy}(0) \Rightarrow a = -\frac{\gamma_{yy}(1)}{\gamma_{yy}(0)}$$

$$\gamma_{ff}(0) = \gamma_{yy}(0) + a\gamma_{yy}(1)$$

where $f(n)$ is the prediction error.

Using equation 5 and $\sigma_w^2 = 1$ we have

$$\begin{aligned} \gamma_{yy}(0) &= \sigma_w^2 r_{hh}(0) = A + B \approx 4.2 \\ \gamma_{yy}(1) &= \sigma_w^2 r_{hh}(1) = -\frac{A}{2} + \frac{B}{3} \approx -1.1 + 0.67 = -0.43 \end{aligned}$$

Thus we find

$$a = -\frac{-0.43}{4.2} \approx 0.10$$

$$\sigma_{ff}^2 = 4.2 - 0.43 * 0.10 \approx 4.16$$

Problem 3

The discrete filter with structures as in task 2a is to be implemented in fixed point representation using $B + 1$ bits and dynamic range $[-1, 1)$.

Rounding (quantization) is performed after each multiplication and the rounding error can be regarded as white noise with power $\sigma_e^2 = \frac{2^{-2B}}{12}$. All the rounding noise sources lead to a resulting noise signal $z(n)$ with power σ_z^2 at the output.

- 3a)** Find the resulting noise power σ_z^2 at the filter output as a function of σ_e^2 for the DF2 structure.

Answer : We have four multiplication sources. Two of them are in the feedback part, (i.e. equation 6) and two in the forward part (i.e. equation 7). The corresponding rounding noise sources in the feedback part must go through the total filter $h(n)$ to reach the output. The noise sources in the forward part are in fact already at the output.

$$\text{Thus } \sigma_z^2 = 2\sigma_e^2 r_{hh}(0) + 2\sigma_e^2 = (2 * 4.2 + 2)\sigma_e^2 = 10.4\sigma_e^2$$

- 3b)** Also find the resulting noise power σ_z^2 at the filter output as a function of σ_e^2 for the parallel structure.

Answer : The parallel structure consists of two first order filters with unit pulse responses (see equation 3) given by $h_1(n) = (-\frac{1}{2})^n u(n)$ and $h_2(n) = (\frac{1}{3})^n u(n)$. We need the energy (zero'th autocorrelation value) for them; i.e.

$$\begin{aligned} \sum_{n=0}^{\infty} h_1^2(n) &= \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^{2n} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3} \\ \sum_{n=0}^{\infty} h_2^2(n) &= \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{2n} = \sum_{n=0}^{\infty} \left(\frac{1}{9}\right)^n = \frac{1}{1 - \frac{1}{9}} = \frac{9}{8} \end{aligned}$$

$$\text{Thus the resulting noise power at the output is } \sigma_z^2 = \frac{4}{3}\sigma_e^2 + \frac{9}{8}\sigma_e^2 = \frac{59}{24}\sigma_e^2 \approx 2.5\sigma_e^2$$

Note that the parallel structure is much better than the DF2-structure with respect to noise power due to rounding!

3c) The filter input $x(n)$ has full dynamic range, i.e. $x_{max} = \max_n |x(n)| = 1$.

Show that one has to scale the input by $2/7$ (downscaling by $7/2$) in order to avoid overflow in the parallel structure.

Answer : We have one internal summary node in each of the first order branches. In addition we of course have the summary at output $y(n)$. From the input to the internal nodes we have unit pulse responses corresponding to the two terms in $h(n)$ in equation 3.

Thus we have :

$$\begin{aligned}\sum_{n=0}^{\infty} |h_1(n)| &= \sum_{n=0}^{\infty} |(-\frac{1}{2})^n| = \sum_{n=0}^{\infty} (\frac{1}{2})^n = \frac{1}{1 - \frac{1}{2}} = 2 \\ \sum_{n=0}^{\infty} |h_2(n)| &= \sum_{n=0}^{\infty} |(\frac{1}{3})^n| = \sum_{n=0}^{\infty} (\frac{1}{3})^n = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2}\end{aligned}$$

The following is wrong : Further, it is obvious from equation 3 that the sum of magnitudes of the total unit pulse response $h(n)$ is the sum of the above two values, i.e. $2 + \frac{3}{2} = \frac{7}{2}$. Thus it is this last value $\frac{7}{2}$ which we have to use as a scale factor.

The correct is : From eq. 3 in task 1d we easily see that $h(n) > 0$, $n = 2m$ (equal time indexes) and that $h(n) < 0$, $n = 2m + 1$ (unequal/odd time indexes). Thus $h(n) = h_e(n) + h_o(n)$ where

$$h_e(m) = h(2m) = [(-\frac{1}{2})^{2m} + (\frac{1}{3})^{2m}]u(m) = [(\frac{1}{4})^m + (\frac{1}{9})^m]u(m)$$

$$h_o(m) = h(2m + 1) = [(-\frac{1}{2})^{2m+1} + (\frac{1}{3})^{2m+1}]u(m) = -[\frac{1}{2}(\frac{1}{4})^m - \frac{1}{3}(\frac{1}{9})^m]u(m)$$

where the terms in the brackets are always positive, i.e. $h_e(m) > 0$ and $h_o(m) < 0$. Thus we have $|h(n)| = h(2n) - h(2n + 1)$

$$\begin{aligned}\sum_{m=0}^{\infty} h_e(m) &= \frac{4}{3} + \frac{9}{8} = \frac{8 * 4 + 3 * 9}{3 * 8} = \frac{59}{24} \\ \sum_{m=0}^{\infty} -h_o(m) &= \frac{1}{2} \frac{4}{3} - \frac{1}{3} \frac{9}{8} = \frac{2}{3} - \frac{3}{8} = \frac{8 * 2 - 3 * 3}{3 * 8} = \frac{7}{24}\end{aligned}$$

Thus the sum is given by $\sum_{m=0}^{\infty} |h(n)| = \frac{59+7}{24} = \frac{11}{4}$ which is larger than the two scaling factor alternatives above.

3d) Find the reduction in signal to noise ratio $SNR = \sigma_y^2 / \sigma_z^2$ due to scaling at the output of the parallel structure.

Answer : Scaling down the input from $x(n)$ to $x(n)/S$ leads to that the output also is scaled by the same amount, i.e $y(n)/S$. Thus the output power σ_y^2 is reduced by a factor S^2 . The noise power σ_z^2 is independent of the scaling factor as scaling is done before internal multiplications (and thus roundings). Thus the SNR is reduced by a factor $S^2 = (\frac{11}{4})^2 = \frac{121}{16} \approx 7.6$ (or with old solution $(\frac{7}{2})^2 = \frac{49}{4} = 12.25$)

Problem 4

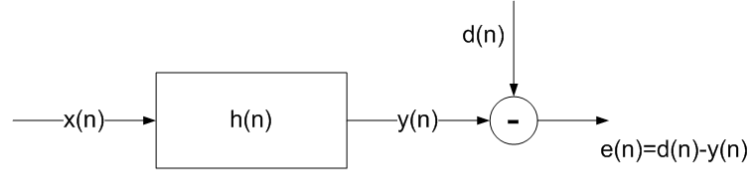


Figure 1: The general Wiener filter

A stationary signal $s(n)$ is contaminated by white additive noise $w(n)$, i.e. the observed signal is $x(n) = s(n) + w(n)$.

A Wiener filter $h(n)$ will minimize the mean square error $\sigma_e^2 = E[e^2(n)] = E[(d(n) - y(n))^2]$. The statistical properties of $s(n)$, $d(n)$ and the noise power σ_w^2 are supposed to be known.

4a) Show that the mean square error can be written as

$$\sigma_e^2 = \gamma_{dd}(0) - 2 \sum_k h(k) \gamma_{dx}(k) + \sum_k \sum_l h(k) h(l) \gamma_{xx}(l - k) \quad (8)$$

Answer : When we replace $y(n)$ by $h(n) * x(n) = \sum_k h(k) x(n - k)$ we get :

$$\begin{aligned} \sigma_e^2 &= E[d^2(n)] - 2E[d(n) \sum_k h(k) x(n - k)] + E[(\sum_k h(k) x(n - k))(\sum_l h(l) x(n - l))] \Rightarrow \\ \sigma_e^2 &= \gamma_{dd}(0) - 2 \sum_k h(k) E[d(n) x(n - k)] + \sum_k \sum_l h(k) h(l) E[x(n - k) x(n - l)] \Rightarrow \\ \sigma_e^2 &= \gamma_{dd}(0) - 2 \sum_k h(k) \gamma_{dx}(k) + \sum_k \sum_l h(k) h(l) \gamma_{xx}(l - k) \end{aligned}$$

4b) Derive the formula for the FIR Wiener filter of length M

Answer : The sums in the MSE-expression is now finite ($0 \rightarrow M - 1$). Thus we can define two M -dimensional column vectors \vec{h} and $\vec{\gamma}_{dx}$ and a $M \times M$ matrix $\underline{\gamma}_{xx}(k, l) = \gamma_{xx}(k - l)$.

The MSE can now be written $MSE = \gamma_{dd}(0) - 2\vec{h}^T \vec{\gamma}_{dx} + \vec{h}^T \underline{\gamma}_{xx} \vec{h}$

We now find the optimal filter which gives MMSE by setting the gradient equal to zero :

$$\nabla_{\vec{h}} MSE = -2\vec{\gamma}_{dx} + 2\underline{\gamma}_{xx} \vec{h} = \vec{0}$$

Thus we have :

$$\underline{\gamma}_{xx} \vec{h} = \vec{\gamma}_{dx} \quad \Rightarrow \quad \vec{h} = \underline{\gamma}_{xx}^{-1} \vec{\gamma}_{dx} \quad (9)$$

since the matrix is Toeplitz and thus always invertable.

4c) Assume a filter length of $M = 3$. Describe the differences in the formula when the FIR filter is to be used for respectively :

- straight noise reduction, i.e. $d(n) = s(n)$
- smoothing, i.e. $d(n) = s(n - 1)$
- prediction, i.e. $d(n) = s(n + 1)$

Answer : since $w(n)$ is white noise it is uncorrelated both with $s(n)$ and with $d(n)$. Thus $\gamma_{dx}(l) = \gamma_{ds}(l)$ and $\gamma_{xx}(l) = \gamma_{ss}(l) + \sigma_w^2 \delta(l)$

- straight noise reduction, i.e. $d(n) = s(n) \rightarrow \gamma_{ds}(l) = \gamma_{ss}(l)$. The left formula in equation 9 can be written elementwise as :

$$\gamma_{ss}(l) = \sum_{k=0}^{M-1} h(k) \gamma_{xx}(l - k) \quad l = 0, 1, 2$$

- smoothing, i.e. $d(n) = s(n - 1) \rightarrow \gamma_{ds}(l) = \gamma_{ss}(l - 1)$. The left formula in equation 9 can be written elementwise as :

$$\gamma_{ss}(l - 1) = \sum_{k=0}^{M-1} h(k) \gamma_{xx}(l - k) \quad l = 0, 1, 2$$

- prediction, i.e. $d(n) = s(n + 1) \rightarrow \gamma_{ds}(l) = \gamma_{ss}(l + 1)$. The left formula in equation 9 can be written elementwise as :

$$\gamma_{ss}(l + 1) = \sum_{k=0}^{M-1} h(k) \gamma_{xx}(l - k) \quad l = 0, 1, 2$$

Thus the difference lies in the indexes of $\gamma_{ss}(l)$ on the left hand side :

- for $d(n) = s(n)$ we use $\vec{\gamma}_{dx} = [\gamma_{ss}(0), \gamma_{ss}(1), \gamma_{ss}(2)]^T$
- for $d(n) = s(n - 1)$ we use $\vec{\gamma}_{dx} = [\gamma_{ss}(1), \gamma_{ss}(0), \gamma_{ss}(1)]^T$
- for $d(n) = s(n + 1)$ we use $\vec{\gamma}_{dx} = [\gamma_{ss}(1), \gamma_{ss}(2), \gamma_{ss}(3)]^T$

Some basic equations and formulaes.

A. Sequences :

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha} \iff |\alpha| < 1$$

$$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}$$

B. Linear convolution :

$$y(n) = h(n) * x(n) = \sum_k h(k)x(n-k) = \sum_k x(k)h(n-k)$$

$$Y(z) = H(z)X(z) \Rightarrow Y(f) = H(f)X(f) \Rightarrow Y(k) = H(k)X(k) \quad k = 0, \dots, N-1$$

C. Transforms :

$$H(z) = \sum_n h(n)z^{-n} \Rightarrow H(f) = \sum_n h(n) e^{-j2\pi n f}$$

$$H(k) = \sum_n h(n) e^{-j2\pi n k/N} \quad k = 0, \dots, N-1$$

$$h(n) = \sum_k H(k) e^{j2\pi n k/N} \quad n = 0, \dots, N-1$$

D. The sampling theorem :

$$x(n) = x(nT_s) = x_a(t)|_{t=nT_s} \quad n = -\infty, \dots, \infty$$

$$X(f) = X(F/F_s) = F_s \sum_k X_a[(f-k)F_s]$$

E. Autocorrelation, energy spectrum and Parsevals theorem :

Given a sequence $x(n)$ with finite energy E_x :

$$\text{Autocorrelation : } r_{xx}(m) = \sum_n x(n)x(n+m) \quad m = -\infty, \dots, \infty$$

$$\text{Energy spectrum : } S_{xx}(z) = X(z)X(z^{-1}) \Rightarrow S_{xx}(f) = |X(f)|^2$$

$$\text{Parsevals theorem: } E_x = \sum_n x^2(n) = \int_0^{2\pi} |X(f)|^2 df = \int_0^{2\pi} S_{xx}(f) df$$

F. Multirate formulaes :

Decimation where $T_{sy} = DT_{sx}$:

$$v(mT_{sy}) = \sum_k h[(mD - k)T_{sx}] x(kT_{sx}) \quad m = -\infty, \dots, \infty$$

Upsampling where $T_{sx} = UT_{sy}$:

$$y(lT_{sy}) = \sum_n h[(l - nU)T_{sy}] x(nT_{sx}) \quad l = -\infty, \dots, \infty$$

Interpolation where $T_{sy} = DT_{sv} = \frac{D}{U}T_{sx}$:

$$y(lT_{sy}) = \sum_m h[(lD - mU)T_{sv}] x(mT_{sx}) \quad l = -\infty, \dots, \infty$$

G. Autocorrelation, power spectrum and Wiener-Khintchin theorem :

Given a stationary, ergodic sequence $x(n)$ with infinite energy :

$$\text{Autocorrelation : } \gamma_{xx}(m) = E[x(n)x(n+m)] \quad m = -\infty, \dots, \infty$$

$$\text{Power spectrum: } \Gamma_{xx}(z) = Z[\gamma_{xx}(m)] \Rightarrow$$

$$\text{Wiener-Khintchin : } \Gamma_{xx}(f) = DTFT[\gamma_{xx}(m)] = \sum_m \gamma_{xx}(m) e^{-j2\pi mf}$$

H. The Yule-Walker and Normal equations where $a_0 = 1$:

Yule-Walker equations :
$$\sum_{k=0}^P a_k \gamma_{xx}(m-k) = \sigma_f^2 \delta(m) \quad m = 0, \dots, P$$

Normal equations:
$$\sum_{k=1}^P a_k \gamma_{xx}(m-k) = -\gamma_{xx}(m) \quad m = 1, \dots, P$$