Norwegian university of science and technology
Department of engineering cybernetics

Contact for questions during exam:
Name: PhD student Tor Aksel Heirung
Tel: 91120005

English version

Exam in TTK4135

Optimization and Control

Optimalisering og regulering

Friday June 10, 2011

Duration: 0900 - 1300

Combination of allowed help remedies:
D - No printed or hand-written notes.
Certified calculator with empty memory.

In the Appendix potentially useful information is included.
The grades will be available by July 1.

1 LP and Simplex (42%)

a (4%) Are LP-problems convex optimization problems? If yes; are they also strictly convex problems? Please substantiate the answers.

b (2%) Does the Simplex method use the gradient of the Lagrange function or the objective function to compute the next iteration point?

c (6%) Compute the KKT-conditions for the following LP-problem on standard form.

$$\begin{array}{rc} & \min_{x \in \mathbb{R}^3} \; 3x_1 + 2x_2 + x_3 \\ & 2x_1 + x_2 + x_3 = 8 \\ s.t. & x_1 - x_2 - x_3 = 1 \\ & x \geq 0 \end{array} \tag{1}$$

d (12%) (1) has two basic feasible points. Please find them. Check the KKT-conditions for these two points and confirm that the one which satisfies the KKT-conditions coincides with the solution.

e (6%) Show how the following LP-problem

$$\begin{array}{rc} & \min_{x \in \mathbb{R}^3} \; 3x_1 + x_2 + x_3 \\ & 2x_1 + x_2 + x_3 \leq 2 \\ s.t. & x_1 - x_2 - x_3 \leq -1 \\ & x \geq 0 \end{array}$$

can be transformed into a LP-problem on standard form as shown in the Appendix (7).

f (8%) The dual problem of the LP-problem on standard form, see Appendix (7), is given by

$$\begin{array}{rc} & \max_{\lambda \in \mathbb{R}^m} \; b^T \lambda \\ s.t. & A^T \lambda \leq c \end{array} \tag{2}$$

Show that the KKT-conditions for the dual problem are equal to the KKT-conditions for the (primal) LP-problem on standard form.

g (4%) Discuss the correspondence between solutions of the (primal) standard problem (7) and the dual problem (2). Hint: Existence of solutions, unboundedness and infeasibility might be relevant.

2 MPC and optimal control (30%)

a (6%) The solution to the infinite horizon LQ-problem form.

$$\min_{x_1,x_2,..,u_0,u_1,...} f_\infty = \frac{1}{2}\sum_{i=0}^{\infty}\{{x_i}^T Q x_i + {u_i}^T P u_i\}, \quad Q \succeq 0, \quad P \succ 0$$

$$s.t. \quad x_{i+1} = Ax_i + Bu_i, \ 0 \le i \le \infty$$

is given by the controller $u_i = Kx_i$ where K is computed via the Riccati equation. If we include constraints (upper and lower bounds) on the control inputs and some outputs $y_i = Dx_i$ (called CV by S.O.Hauger), a dual-mode MPC control law at time $i = 0$ can be specified by

$$u_i = \begin{cases} Kx_i + c_i, & i \in \{0, 1, ..., L-1\} \\ Kx_i, & i \ge L \end{cases} \tag{3}$$

The prediction horizon is divided into two parts; $i \in \{0, 1, ..., L-1\}$ and $i \ge L$. Why may we assume $c_i = 0$ for $i \ge L$, and how does this influence the choice of L.

b (6%) Discuss the optimization problem to compute c_i in (3). A detailed formulation is not required.

c (4%) Assume that the linear model in (9) is replaced by a nonlinear model $x_{i+1} = g(x_i, u_i)$ in the optimization problem (8)-(12). What kind of optimization problem is this? Suggest an algorithm for solving this problem.

d (4%) The prediction horizon is a key tuning parameter in MPC. Explain how you would choose this parameter. You may use a figure to explain this.

e (4%) What does control input blocking (called MV blocking by S.O.Hauger) mean and why is it important? Explain using a figure to show how control input blocking works.

f (6%) Consider a time-invariant two-dimensional weighting matrix $Q = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix}$ in a quadratic objective function as in (8). Assume that the acceptable variation of y_1 is ± 10 about the setpoint and the acceptable variation for y_2 is ± 0.1 about the setpoint. Suggest a reasonable ratio $\frac{q_1}{q_2}$ for the choice of q_1 and q_2.

3 Various topics (28%)

a (6%) Assume an unconstrained 2-dimensional minimization problem ($x \in \mathbb{R}^2$) where we apply the Nelder-Mead method. Assume the following ordered points.

$$\begin{aligned} x^1 &= (0.7, 0.8)^T, \quad f(x^1) = 10 \\ x^2 &= (0.5, 0.8)^T, \quad f(x^2) = 20 \\ x^3 &= (0.7, 0.5)^T, \quad f(x^3) = 30 \end{aligned}$$

The reflection point x^{refl} for $x \in \mathbb{R}^n$ is given by

$$\begin{aligned} & x^{refl} \stackrel{def}{=} g(-1) \\ & \text{where} \\ g(t) &= \overline{x} + t(x^{n+1} - \overline{x}) \\ \overline{x} &= \frac{1}{n}\sum_{i=1}^{n} x^i \end{aligned}$$

The points x^1, x^2, x^3 have been ordered in a specific way. Explain this ordering.

Assume that $f(x^{refl}) = 35$. In this case we try to perform inside contraction. What is meant by inside contraction?

b (6%) Assume an unconstrained minimization problem with $f(x) = \frac{1}{2}x^TQx - b^Tx$, $Q = Q^T \succeq 0$ with a search direction p_k at iteration k. Assume $\nabla f(x_k)^T p_k < 0$. Find the optimal step length α, i.e. the value of α which minimizes $f(x_k + \alpha p_k)$.

c (16%) Consider the problem

$$\begin{aligned} & \min_{x \in \mathbb{R}2} \; -2x_1 + x_2 \\ s.t. \quad & (1 - x_1)^3 - x_2 \geq 0 \\ & 0.25x_1^2 + x_2 - 1 \geq 0 \end{aligned}$$

The optimal solution is $x^* = (0, 1)^T$ where both constraints are active.

- Do the LICQ conditions hold at this point?
- Are the KKT conditions satisfied at this point?
- Are the 2nd order necessary conditions satisfied?
- Are the 2nd order sufficient conditions satisfied?

Appendix

Part 1 Optimization problems and optimality conditions

$\mathcal{E}$ and $\mathcal{I}$ given below are two finite sets of indices.

General optimization problem. f and c_i are differentiable functions:

$$\begin{aligned} &\min_{x\in\mathbb{R}^n} f(x) \\ &c_i(x) = 0, \qquad i \in \mathcal{E} \\ &c_i(x) \geq 0, \qquad i \in \mathcal{I} \end{aligned} \tag{4}$$

The Lagrangian function is given by

$$\mathcal{L}(x,\lambda) = f(x) - \sum_{i\in\mathcal{E}\cup\mathcal{I}} \lambda_i c_i(x)$$

The KKT-conditions for (4) are given by:

$$\begin{aligned} \nabla_x \mathcal{L}(x^*,\lambda^*) &= 0 \\ c_i(x^*) &= 0, \qquad i \in \mathcal{E} \\ c_i(x^*) &\geq 0, \qquad i \in \mathcal{I} \\ \lambda_i^* &\geq 0, \qquad i \in \mathcal{I} \\ \lambda_i^* c_i(x^*) &= 0, \qquad i \in \mathcal{E}\cup\mathcal{I} \end{aligned} \tag{5}$$

2nd order (sufficient) conditions for (4) are given by:

$$w \in \mathcal{C}(\lambda^*) \Leftrightarrow \begin{cases} \nabla c_i(x^*)^T w = 0 & \text{for all } i \in \mathcal{E} \\ \nabla c_i(x^*)^T w = 0 & \text{for all } i \in \mathcal{A}(x^*)\cap\mathcal{I} \text{ with } \lambda_i^* > 0 \\ \nabla c_i(x^*)^T w \geq 0 & \text{for all } i \in \mathcal{A}(x^*)\cap\mathcal{I} \text{ with } \lambda_i^* = 0 \end{cases}$$

Theorem (Second-Order Sufficient Conditions)

Suppose that for some feasible point $x^* \in R^n$ there is a Lagrange multiplier vector λ^* such that the KKT conditions (5) are satisfied. Suppose also that

$$w^T \nabla_{xx}\mathcal{L}(x^*,\lambda^*) w > 0, \qquad \text{for all } w \in \mathcal{C}(\lambda^*),\ w \neq 0. \tag{6}$$

Then x^* is a strict local solution for (4).

LP-problem on standard form:

$$\begin{aligned} & \min_{x \in \mathbb{R}^n} \ f(x) = c^T x \\ s.t. \quad & \quad Ax = b \\ & \quad x \geq 0 \end{aligned} \tag{7}$$

where $A \in R^{m \times n}$ and $rank(A) = m$.

QP-problem on standard form:

$$\begin{aligned} & \min_{x \in \mathbb{R}^n} \ f(x) = \tfrac{1}{2} x^T G x + x^T d \\ s.t. \quad & \quad a_i^T x = b_i, \qquad i \in \mathcal{E} \\ & \quad a_i^T x \geq b_i, \qquad i \in \mathcal{I} \end{aligned}$$

where $G = G^T$. Alternatively, the equalities can be written $Ax = b$, $A \in \mathbb{R}^{m \times n}$.

Iterative method:

$$\begin{gathered} x_{k+1} = x_k + \alpha_k p_k \\ x_0 \ given \\ x_k, p_k \in \mathbb{R}^n, \ \alpha_k \in \mathbb{R} \end{gathered}$$

p_k is the search direction and α_k is the line search parameter.

Part 2 Linear quadratic control of discrete dynamic systems

A typical optimal control problem on the time horizon 0 to n might take the form

$$\begin{aligned}\min \quad f_0 = \frac{1}{2}\sum_{i=0}^{n-1}&\{(y_i - y_{ref,i})^T Q_i (y_i - y_{ref,i}) \\ &+ (u_i - u_{i-1})^T P_i (u_i - u_{i-1})\} \\ &+ \frac{1}{2}(y_n - y_{ref,n})^T S (y_n - y_{ref,n})\end{aligned} \tag{8}$$

subject to equality and inequality constraints

$$x_{i+1} = A_i x_i + B_i u_i,\ 0 \le i \le n-1 \tag{9}$$
$$y_i = H x_i$$
$$x_0 = \text{given (fixed)} \tag{10}$$
$$U_L \le u_i \le U_U,\ 0 \le i \le n-1 \tag{11}$$
$$Y_L \le y_i \le Y_U,\ 1 \le i \le n \tag{12}$$

where system dimensions are given by

$$\begin{aligned} u_i &\in \mathbb{R}^m \\ x_i &\in \mathbb{R}^l \\ y_i &\in \mathbb{R}^j \end{aligned}$$

The subscript i refers to the sampling instants. That is, subscript $i+1$ refers to the sample instant one sample interval after sample i. Note that the sampling time between each successive sampling instant is constant. Further, we assume that the control input u_i is constant between each sample.

Theorem: Assume that $x_{ref,i} = 0$, $u_{ref,i} = 0$, $0 \le i \le n$ and that $H = I$, i.e. $y_i = x_i$. The solution of (8), (9) and (10) is given by $u_i = K_i x_i$, $0 \le i \le n-1$ where the feedback gain matrix is derived by

$$\begin{aligned} K_i &= -P_i^{-1} B_i^T R_{i+1}(I + B_i P_i^{-1} B_i^T R_{i+1})^{-1} A_i,\ 0 \le i \le n-1 \\ R_i &= Q_i + A_i^T R_{i+1}(I + B_i P_i^{-1} B_i^T R_{i+1})^{-1} A_i,\ 0 \le i \le n-1 \\ R_n &= S \end{aligned}$$