NTNU – Trondheim
Norwegian University of
Science and Technology

Out: March 07, 2018
Recommended completion: April 17, 2018

Exercise 10

TTK4130 Modeling and Simulation

Problem 1 (Robotic manipulator)

We wish to model a robotic manipulator with the configuration shown in Figure 1.

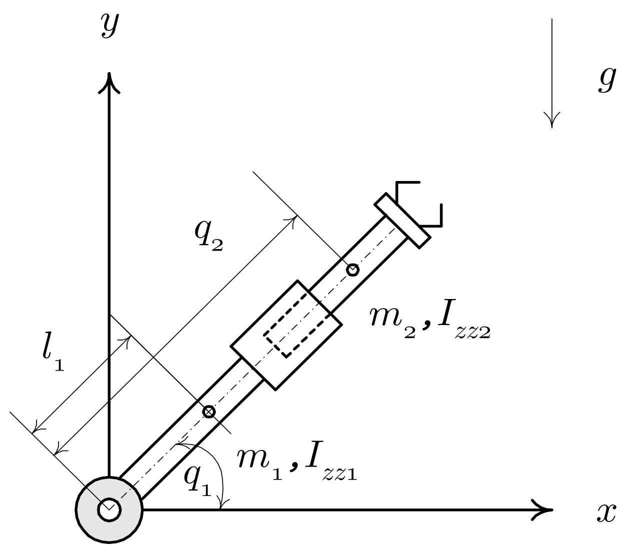

Figure 1: Manipulator

The manipulator has two degrees of freedom (that is, two generalized coordinates). We will use Lagrange's equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = \tau_i, \qquad i = 1, 2$$

to set up the equations of motion for the manipulator, where

$$\mathcal{L} = T - U = \text{kinetic energy} - \text{potential energy} \tag{1}$$

and q_1 and q_2 are the generalized coordinates (see Figure 1). The axis x and y can be assumed fixed, that is, axes in an inertial system.

We will disregard mass and inertia of the motors in this problem. The moment of inertia of the first arm is denoted I_{zz1}, while the moment of inertia of the second arm is I_{zz2} (each referenced to the center of mass of the respective arm). The dots on the figure marks the centers of mass for each arm. The arrow marked g illustrates the direction of gravity.

(a) Find the total kinetic energy, T, for the manipulator, and show that it can be written on the form $T = \frac{1}{2}\dot{\mathbf{q}}^\mathsf{T}\mathbf{M}(\mathbf{q})\dot{\mathbf{q}}$ where $\mathbf{q} = \begin{pmatrix} q_1 & q_2 \end{pmatrix}^\mathsf{T}$ and

$$\mathbf{M}(\mathbf{q}) = \begin{pmatrix} m_1 l_1^2 + I_{zz1} + I_{zz2} + m_2 q_2^2 & 0 \\ 0 & m_2 \end{pmatrix}.$$

(b) Find the potential energy, U, for the manipulator.

(c) Derive the equations of motion for the manipulator by use of Lagrange's equation.

(d) In this problem you should show that the equations of motion in (c) can be written

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \boldsymbol{\tau}. \tag{2}$$

Explain how several choices are possible for $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$. Show that when you use the Christoffel symbols (cf. eq. (8.57)–(8.58) in the book), then

$$\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{pmatrix} m_2 q_2 \dot{q}_2 & m_2 q_2 \dot{q}_1 \\ -m_2 q_2 \dot{q}_1 & 0 \end{pmatrix}.$$

What is the vector $\mathbf{g}(\mathbf{q})$?

(e) What matrix properties do the matrices $\mathbf{M}(\mathbf{q})$ and $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ possess?

(f) Show (using the matrices developed in this problem) that the matrix $\dot{\mathbf{M}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ is skew-symmetric when $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ has been defined by use of the Christoffel symbols.

(g) Show that the derivative of the energy function $E(\mathbf{q}, \dot{\mathbf{q}}) = T(\mathbf{q}, \dot{\mathbf{q}}) + U(\mathbf{q})$ is

$$\dot{E}(\mathbf{q}, \dot{\mathbf{q}}) = \dot{\mathbf{q}}^{\mathsf{T}} \boldsymbol{\tau}.$$

Hint: Use $T = \frac{1}{2}\dot{\mathbf{q}}^{\mathsf{T}}\mathbf{M}(\mathbf{q})\dot{\mathbf{q}}$ and (2), do not insert the detailed model. Use that $\frac{\partial U}{\partial \mathbf{q}} = \mathbf{g}^{\mathsf{T}}(\mathbf{q})$.
What can we say about passivity of the manipulator?

Problem 2 (Tank with liquid)

A tank with area A is filled with an incompressible liquid with (constant) density ρ and level h. The liquid volume is then $V = Ah$ and the mass of the liquid in the tank is $m = V\rho$. Liquid enters the tank through a pipe with mass flow $w_i = \rho A_i v_i$, where A_i is the pipe cross section, and v_i is the velocity (constant over the cross section). Liquid leaves the tank through a second pipe with mass flow $w_u = \rho A_u v_u$ where A_u is the cross section of the pipe and v_u is the velocity.

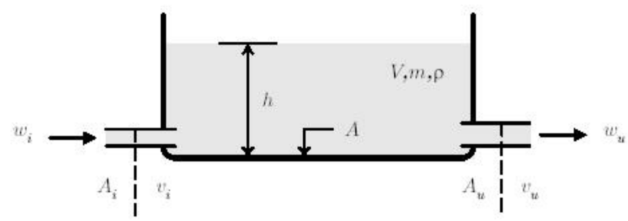

Figure 2: Tank with liquid

Use a mass balance for the tank to set up a differential equation for the level h.

Problem 3 (Stirred tank (Exam 2015))

In this problem, we consider a stirred tank that cools an inlet stream, see Figure 3. The tank is cooled by a "jacket" that contains a fluid of (presumably) lower temperature than the tank. The inlet stream to the tank has density ρ, temperature T_1, and massflowrate w_1. The outflow from the tank is

$$w_2 = Cu\sqrt{h},$$

where C is a constant and u is the valve opening. The liquid level is h. You can assume that the outflow is controlled such that the level does not exceed the height of the jacket.

The inlet and outlet massflowrates for the jacket is matched such that the jacket is always filled with fluid ($w_3 = w_4$). The cooling fluid has density ρ_c, and the inlet stream to the jacket has temperature T_3. Since the tank is stirred, we assume homogenous conditions, that is, the temperature T is the same everywhere in the tank. Similarly, we assume that the temperature T_c is the same everywhere in the jacket.

The cross-sectional area of the tank is A. The volume of the jacket is V_c.

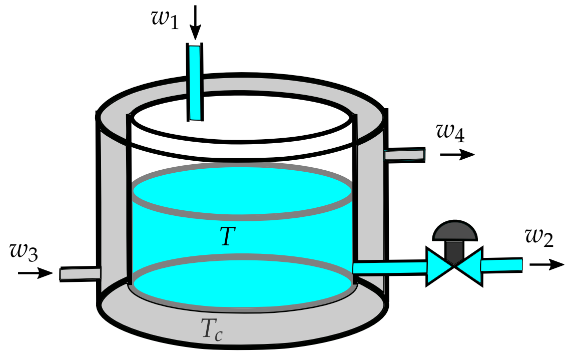

Figure 3: Tank with cooling jacket.

The heat transfer from the tank to the jacket is

$$Q = Gh(T - T_c),$$

where h is the height of the liquid in the tank, and G a (constant) heat transfer coefficient. We assume that the jacket (and tank) is well insulated from the surroundings, meaning there are no other heat losses.

We assume both fluids incompressible, meaning that specific internal energy and enthalpy both can be assumed equal and proportional to temperature, with constant of proportionality being c_p and c_{pc} for the two fluids, respectively.

(a) Set up differential equations for the temperatures T in the tank and T_c in the jacket, and the level h in the tank.

Problem 4 (Compressor, momentum balance, Bernoulli's equation)

A compressor takes in air with pressure p_0 and velocity $v_0 = 0$ from the surroundings. The air flows through a duct into the compressor. For control, it would be beneficial to have a measurement of the mass flow into the compressor. However, this measurement is not available.

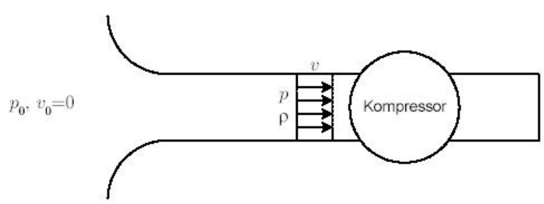

Figure 4: Compressor

Instead, there is a pressure measurement in the duct, giving a measurement p. How can the mass flow w and velocity v be found from this measurement? Assume that the density ρ in the duct is constant and known, there is no friction, and that the velocity is uniform over the cross-section where the pressure transmitter is located.

Hint: Use (the stationary) Bernoulli's equation.

Problem 5 (Mixing, reactions (Exam 2010))

An incompressible liquid of substance C enters a perfectly mixed tank (a continuous stirred tank reactor, CSTR) with mass flow w_C and temperature T_C. In the tank, the substance reacts (e.g. due to the presence of a catalyst) to form the substance D with a rate JV, where J is the reaction rate per unit volume, and $V = Ah$ is the volume of the tank. The tank then consists of a mixture of C and D, which leaves the tank with mass flow w and temperature T. The mass of substance C in the tank is denoted m_C, and the mass of substance D is denoted m_D.

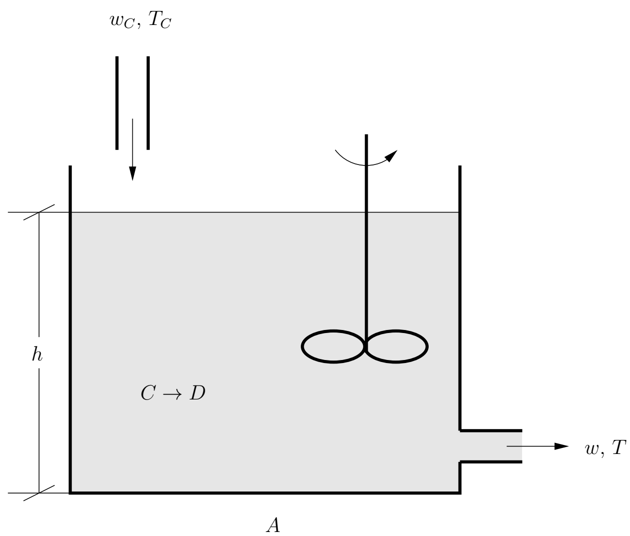

Figure 5: Tank reactor

(a) Set up a differential equation for the level of the tank. (Hint: Use the ordinary overall mass balance. Assume that the average density ρ is constant.)

(b) In a material volume V_m, the following holds:

$$\frac{\mathrm{D}}{\mathrm{D}t} \iiint\limits_{V_m} \rho_C \mathrm{d}V = - \iiint\limits_{V_m} J \mathrm{d}V.$$

Use this together with the appropriate form of the transport theorem to explain that the mass balance for substance C on integral form in a fixed control volume V_f is

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint\limits_{V_f} \rho_C \mathrm{d}V = - \iiint\limits_{V_f} J \mathrm{d}V - \iint\limits_{\partial V_f} \rho_C \vec{v} \cdot \vec{n} \mathrm{d}A.$$

(In this particular case, the natural control volume, the volume of liquid in the tank, is not fixed, but this can be ignored since $\rho_C \vec{v}_c \cdot \vec{n} = 0$ – expansion of the volume does not accumulate more of substance C.)

(c) Use this to write up the mass balance for the mass of substance C in the tank ($\frac{\mathrm{d}}{\mathrm{d}t} m_C = \ldots$). Assume here, and for the rest of the problem, that J is proportional to the density of substance C, $J = k \frac{m_C}{V}$, and that the outflow of substance C is proportional to the mass ratio of substance C to the total mass in the tank, and the total outflow, $w_{C,out} = \frac{m_C}{m_C + m_D} w$.

(d) What is the mass balance equation on integral form for substance D (in a fixed volume)? Use this to write up the mass balance of substance D.

(e) Check that the solution in (c) and (d) agrees with the answer in (a).

The final question is optional:

(f) Set up a differential equation for the temperature in the tank. Assume that the heat generated by the reaction is proportional to J, with proportionality constant c. Disregard kinetic energy, potential energy and pressure work. Assume no 'heat flux' (the tank is well insulated). Assume the internal energy is $u = c_p T$.