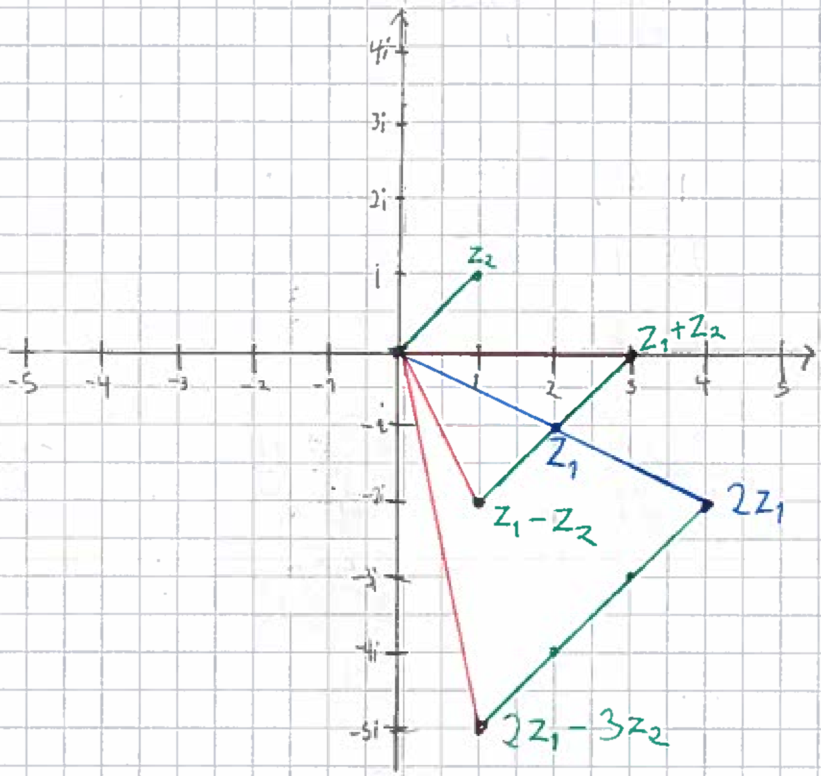
Øving 2

1.3.1 $\quad z_1 = 2-i \quad , \quad z_2 = 1+i$



$z_1+z_2 = 3$

$z_1-z_2 = 1-2i$

$2z_1-3z_2 = 1-5i$

1.3.5

a) $\left|\frac{1+2i}{-2-i}\right| = \frac{|1+2i|}{|-2-i|} = \frac{\sqrt{1^2+2^2}}{\sqrt{(-2)^2+(-1)^2}} = \underline{1}$

b) $|(1+i)(2-3i)(-3+4i)|$

$= |1-i||2-3i||-3+4i|$

$= \sqrt{1^2+(-1)^2} \cdot \sqrt{2^2+(-3)^2} \cdot \sqrt{(-3)^2+4^2}$

$= \sqrt{2}\sqrt{13} \cdot \sqrt{25}$

$= \underline{5\sqrt{26}}$

c) $\left|\frac{i(2+i)^3}{(1-i)^2}\right| = \frac{|i| \cdot |2+i|^3}{|1-i|^2}$

$= \frac{1 \cdot \left(\sqrt{2^2+1^2}\right)^3}{\left(\sqrt{1^2+(-1)^2}\right)^2}$

$= \underline{\frac{5\sqrt{5}}{2}}$

d) $\left|\frac{(\pi+i)^{100}}{(\pi-i)^{100}}\right| = \frac{|\pi+i|^{100}}{|\pi-i|^{100}}$

$(\pi+i) = \overline{(\pi-i)}$

so: $|\pi+i| = |\pi-i|$

$\Rightarrow \left|\frac{(\pi+i)^{100}}{(\pi-i)^{100}}\right| = \underline{1}$

oppg 1,3,7

a) $\arg\left(-\frac{1}{2}\right) = \pi$, $\left|-\frac{1}{2}\right| = \frac{1}{2}$

$-\frac{1}{2} = \underline{\frac{1}{2}e^{i\pi}}$

b) $\arg(-3+3i) = \frac{3\pi}{4}$, $|-3+3i| = \sqrt{(-3)^2+3^2}$

$= \sqrt{2\cdot 3^2}$

$= 3\sqrt{2}$

$-3+3i = \underline{3\sqrt{2}\cdot e^{i\frac{3\pi}{4}}}$

c) $\arg(-\pi i) = -\frac{\pi}{2}$ $|-\pi i| = \pi$

$-\pi i = \underline{\pi \cdot e^{-i\frac{\pi}{2}}}$

d) $z = -2\sqrt{3} - 2i$

$\arg(z) = \arctan\left(\frac{-2}{-2\sqrt{3}}\right) = \frac{\pi}{6} - \pi = -\frac{5\pi}{6}$

$|z| = \sqrt{(-2\sqrt{3})^2 + (-2)^2} = \sqrt{12+4} = 4$

$z = \underline{4 \cdot e^{-i\frac{5\pi}{6}}}$

e) $\arg(1-i) = -\frac{\pi}{4}$, $\arg(-\sqrt{3}+i) = \arctan\left(\frac{1}{-\sqrt{3}}\right)$

$= -\frac{\pi}{6} + \pi = \frac{5\pi}{6}$

$\Rightarrow \arg((1-i)(-\sqrt{3}+i)) = -\frac{\pi}{4} + \frac{5\pi}{6} = \frac{7\pi}{12}$

$|(1-i)(-\sqrt{3}+i)| = |1-i|\cdot|-\sqrt{3}+i|$

$= \sqrt{2}\cdot\sqrt{4}$

$= 2\sqrt{2}$

$(1-i)\cdot(-\sqrt{3}+i) = \underline{2\sqrt{2}\cdot e^{i\frac{7\pi}{12}}}$

f) $(\sqrt{3}-i)^2$

$|\sqrt{3}-i| = \sqrt{\sqrt{3}^2+(-1)^2} = 2$

$\arg(\sqrt{3}-i) = \arctan\left(\frac{-1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

$\Rightarrow \arg((\sqrt{3}-1)^2) = 2\cdot\left(-\frac{\pi}{6}\right) = -\frac{\pi}{3}$

$(\sqrt{3}-i)^2 = 2^2 \cdot e^{-i\frac{\pi}{3}}$

$= 4e^{-i\frac{\pi}{3}}$

g) $z_1 = -1+\sqrt{3}i$, $z_2 = 2+2i$

$\arg(z_1) = \arctan\left(\frac{\sqrt{3}}{-1}\right) = -\frac{\pi}{3}+\pi = \frac{2\pi}{3}$

$|z_1| = \sqrt{(-1)^2+(\sqrt{3})^2} = 2$

$\arg(z_2) = \pi/4$

$|z_2| = \sqrt{2^2+2^2} = 2\sqrt{2}$

$\arg(z_1 z_2) = \frac{2\pi}{3}+\frac{\pi}{4} = \frac{11\pi}{12}$

$\frac{-1+\sqrt{3}i}{2+2i} = \frac{2}{2\sqrt{2}}) \cdot e^{i\frac{11\pi}{12}}$

$= \frac{\sqrt{2}}{2} \cdot e^{i\frac{11\pi}{12}}$

h) $\arg(-\sqrt{7}) = -\pi$

$\arg(1+i) = \pi/4$

$\arg(\sqrt{3}+i) = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

$$\arg\left(\frac{-\sqrt{7}(1+i)}{\sqrt{3}+i}\right) = -\pi + \frac{\pi}{4} - \frac{\pi}{6} = -\frac{11\pi}{12}$$

$$\left|\frac{-\sqrt{7}\cdot(1+i)}{\sqrt{3}+i}\right| = \frac{|-\sqrt{7}|\cdot|1+i|}{|\sqrt{3}+i|}$$

$$= \frac{\sqrt{7}\cdot\sqrt{2}}{2}$$

$$= \frac{\sqrt{7}}{\sqrt{2}}$$

$$\frac{-\sqrt{7}\cdot(1+i)}{\sqrt{3}+i} = \frac{\sqrt{14}}{2} e^{-i\frac{11\pi}{12}}$$

1.3.13 (a), (b), (c) and (d) are all true

1.4.1 a) $e^{-i\pi/4} = \cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)$

$$= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$$

b) $$\frac{e^{1+3i\pi}}{e^{-1+i\pi/2}} = e^{1+3i\pi+1-i\pi/2}$$

$$= e^2 \cdot e^{i\frac{5}{2}\pi}$$

$$= e^2\cdot\left(\cos\left(\frac{5}{2}\pi\right) + i\sin\left(\frac{5}{2}\pi\right)\right)$$

$$= e^2(0+i)$$

$$= ie^2$$

c) $e^{i} = \cos 1 + i \sin 1$

$e^{e^{i}} = e^{\cos 1 + i\sin 1}$

$= e^{\cos 1} \cdot e^{i \sin 1}$

$= e^{\cos 1} \cdot (\cos(\sin(1)) + i \sin(\sin(1)))$

$= e^{\cos 1} \cos(\sin(1)) + i \cdot e^{\cos 1} \sin(\sin(1))$

1,4,3 a) $\arg\left(\frac{1-i}{3}\right) = -\frac{\pi}{4}$

$\left|\frac{1-i}{3}\right| = \frac{\sqrt{2}}{3}$

$\frac{\sqrt{2}}{3} \cdot e^{-i\pi/4}$

b) $\arg(-8\pi(1+\sqrt{3}i)) = \pi + \arctan\left(\frac{\sqrt{3}}{1}\right)$

$= \frac{4}{3}\pi$

$= -\frac{2\pi}{3}$

$|-8\pi \cdot (1+\sqrt{3}i)| = |-8\pi| \cdot |1+\sqrt{3}i|$

$= 8\pi \cdot 2$

$= 16\pi$

$-8\pi(1+\sqrt{3}i) = 16\pi \cdot e^{-i2\pi/3}$

c) $\arg((1+i)^6) = 6 \cdot \arg(1+i)$

$= 6 \cdot \frac{\pi}{4}$

$= \frac{3\pi}{2}$

$|(1+i)^6| = |1+i|^6$

$= (\sqrt{2})^6$

$= 8$

$(1+i)^6 = 8 \cdot e^{i3\pi/2}$

1,4,7 let $z = a + bi$

then $z + 2\pi i = a + bi + 2\pi i$

$= a + (b + 2\pi) i$

$e^z = e^{a+bi} = e^a \cdot e^{bi}$

$= e^a \cdot (\cos b + i \sin b)$

$e^{z+2\pi i} = e^a \cdot e^{i(b+2\pi)}$

$= e^a (\cos(b+2\pi) + i \sin(b+2\pi))$

For både cosinus og sinus gjelder

$\cos\theta = \cos(\theta + 2\pi)$

$\sin\theta = \sin(\theta + 2\pi)$

Så da kan vi skrive

$e^a(\cos(b+2\pi) + i\sin(b+2\pi))$

$= e^a(\cos b + i \sin b) \quad (= e^z)$

$\blacksquare$

1,5,5

a) $z^4 = -16$

$$|z^4| = |-16| \Rightarrow |z| = 2$$

$$\arg(z) = \frac{\arg(-16) + 2\pi k}{4}$$

$$= -\frac{\pi}{4} + \frac{\pi}{2}k$$

$k=-1$: $\arg(z_1) = -\frac{3\pi}{4}$

$k=0$: $\arg(z_2) = -\frac{\pi}{4}$

$k=1$: $\arg(z_3) = \frac{\pi}{4}$

$k=2$: $\arg(z_4) = \frac{3\pi}{4}$

Løsningene er:

$$z_1 = 2e^{-i\frac{3\pi}{4}}$$

$$z_2 = 2e^{-i\pi/4}$$

$$z_3 = 2e^{i\pi/4}$$

$$z_4 = 2e^{i\frac{3\pi}{4}}$$

b) $z^5 = 1$

$|z^5| = |1| = 1$

$|z| = \sqrt[5]{1} = 1$

$\arg(z^5) = \dfrac{\arg(1) + 2\pi k}{5}$

$= \dfrac{0}{5} + \dfrac{2\pi k}{5}$

$= \dfrac{2\pi k}{5}$

$k=0: \quad \arg(z_1) = 0$

$k=1: \quad \arg(z_2) = \dfrac{2\pi}{5}$

$k=2: \quad \arg(z_3) = \dfrac{4\pi}{5}$

$k=3: \quad \arg(z_4) = \dfrac{6\pi}{5} = -\dfrac{4\pi}{5}$

$k=4: \quad \arg(z_5) = \dfrac{8\pi}{5} = -\dfrac{2\pi}{5}$

Løsningene er:

$z_1 = 1 \cdot e^{i \cdot 0} = 1$

$z_2 = 1 \cdot e^{i\frac{2\pi}{5}} = e^{i\frac{2\pi}{5}}$

$z_3 = 1 \cdot e^{i\frac{4\pi}{5}} = e^{i\frac{4\pi}{5}}$

$z_4 = 1 \cdot e^{-i\frac{4\pi}{5}} = e^{-i\frac{4\pi}{5}}$

$z_5 = 1 \cdot e^{-i\frac{2\pi}{5}} = e^{-i\frac{2\pi}{5}}$

c) $z^4 = i$

$|z^4| = |i| = 1$

$|z| = 1$

$$\arg(z) = \frac{\arg(i) + 2\pi K}{4}$$

$$= \frac{\pi/2}{4} + \frac{\pi}{2}K$$

$$= \frac{\pi}{8} + \frac{\pi}{2}K$$

$K=0$: $\arg(z_1) = \pi/8$

$K=1$: $\arg(z_2) = 5\pi/8$

$K=2$: $\arg(z_3) = 9\pi/8 = -7\pi/8$

$K=3$: $\arg(z_4) = 13\pi/8 = -3\pi/8$

Så løsningene er:

$$z_1 = e^{i\pi/8}$$

$$z_2 = e^{i5\pi/8}$$

$$z_3 = e^{-i7\pi/8}$$

$$z_4 = e^{-i3\pi/8}$$

d)
$$z^3 = 1-\sqrt{3}i$$

$$|z^3| = |1-\sqrt{3}| = \sqrt{1^2+(-\sqrt{3})^2}$$
$$= \sqrt{4} = 2$$
$$|z| = \sqrt[3]{2}$$

$$\arg(z) = \frac{\arg(-\sqrt{}\, i)+2\pi k}{3}$$
$$= \frac{-\pi/3}{3} + \frac{2\pi \cdot k}{3}$$
$$= -\frac{\pi}{9} + \frac{2\pi}{3}k$$

$k=-1$: $\arg(z_1) = -\frac{7\pi}{9}$

$k=0$: $\arg(z_2) = -\frac{\pi}{9}$

$k=1$: $\arg(z_3) = \frac{5\pi}{9}$

Så løsningene er:

$$z_1 = \sqrt[3]{2}\cdot e^{-i7\pi/9}$$
$$z_2 = \sqrt[3]{2}\cdot e^{-i\pi/9}$$
$$z_3 = \sqrt[3]{2}\cdot e^{i5\pi/9}$$

e) $z^2 = -1 + i$

$|z^2| = |-1+i| = \sqrt{2}$

$|z| = \sqrt[4]{2}$

$$\arg(z^2) = \frac{\arg(-1+i) + 2\pi k}{2}$$

$$= \frac{3\pi/4}{2} + \frac{2\pi k}{2}$$

$$= \frac{3\pi}{8} + \pi k$$

$K=-1$: $\arg(z_1) = \frac{3\pi}{8} - \pi = -\frac{5\pi}{8}$

$K=0$: $\arg(z_2) = \frac{3\pi}{8} + 0 = \frac{3\pi}{8}$

Så løsningene er:

$$z_1 = \sqrt[4]{2} \cdot e^{-i5\pi/8}$$

$$z_2 = \sqrt[4]{2} \cdot e^{i3\pi/8}$$

f)

$$\frac{2i}{1+i} = \frac{2i(1-i)}{(1+i)(1-i)}$$

$$= \frac{2i-2i^2}{1^2+1}$$

$$= \frac{2+2i}{2}$$

$$= 1+i$$

$$z^6 = 1+i$$

$$|z^6| = |1+i| = \sqrt{2}$$

$$|z| = \sqrt[12]{2}$$

$$\arg(z^6) = \frac{\arg(1+i)+2\pi K}{6}$$

$$= \frac{\pi/4}{6} + \frac{\pi K}{3}$$

$$= \frac{\pi}{24} + \frac{\pi K}{3}$$

$K=-2$: $\arg(z_1) = -\frac{5\pi}{8}$

$K=-1$: $\arg(z_2) = -\frac{7\pi}{24}$

$K=0$: $\arg(z_3) = \frac{\pi}{24}$

$K=1$: $\arg(z_4) = \frac{3\pi}{8}$

$K=2$: $\arg(z_5) = \frac{17\pi}{24}$

$K=3$: $\arg(z_6) = \frac{25\pi}{24} = \frac{-23\pi}{25}$

Så løsningene er:

$$z_0 = \sqrt[12]{2}\, e^{-i\frac{23\pi}{24}}$$

$$z_1 = \sqrt[12]{2}\, e^{-i5\pi/8}$$

$$z_2 = \sqrt[12]{2}\, e^{-i7\pi/24}$$

$$z_3 = \sqrt[12]{2}\, e^{i\pi/24}$$

$$z_4 = \sqrt[12]{2}\, e^{i3\pi/8}$$

$$z_5 = \sqrt[12]{2}\, e^{i17\pi/24}$$

1.5.7 $z^2 - 2z + i = 0$

$$z_{1,2} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot i}}{2 \cdot 1}$$

$$= \frac{2 \pm \sqrt{4 - 4i}}{2}$$

$$= 1 \pm \sqrt{1-i}$$

$w^2 = 1 - i \Rightarrow z_{1,2} = 1 + w$

Regner ut de to løsningene til w.

$$|w^2| = |1-i| = \sqrt{2}$$

$$|w| = \sqrt[4]{2}$$

$$\arg(w^2) = \frac{\arg(1-i) + 2\pi k}{2}$$

$$= \frac{-\pi/4}{2} + \pi k$$

$$= -\pi/8 + \pi k$$

$\arg(w_1) = -\pi/8 \quad (k=0)$

$\arg(w_2) = 7\pi/8 \quad (k=1)$

Så løsningene til z er:

$$z_1 = 1 + \sqrt[4]{2}\, e^{-i\pi/8}$$

og $z_2 = 1 + \sqrt[4]{2}\, e^{i7\pi/8}$

eller alternativt (og enklere)

$$z_1 = 1 + \sqrt{1-i}$$

$$z_2 = 1 - \sqrt{1-i}$$

1.5.9 $z^3 - 3z^2 + 6z - 4 = 0 \qquad (*)$

Ser (etter litt prøving og feiling) at $z=1$ er en løsning:

$$z=1: \quad 1^3 - 3\cdot 1^2 + 6\cdot 1 - 4 = 0 \ \checkmark$$

Det betyr at $(z-1)$ er en faktor i $(*)$.
Kan da redusere $(*)$ til et 2 grads polynom.

$$(z^3 - 3z^2 + 6z - 4) : (z-1) = \underline{\underline{z^2 - 2z + 4}}$$

$$\begin{array}{l} \underline{z^3 - z^2} \\ \quad -2z^2 + 6z - 4 \\ \quad \underline{-2z^2 + 2z} \\ \qquad\quad 4z - 4 \\ \qquad\quad \underline{4z - 4} \\ \qquad\qquad 0 \end{array}$$

Løser $z^2 - 2z + 4 = 0$ for å finne de to resterende løsningene til $(*)$.

$$z_{1,2} = \frac{2 \pm \sqrt{(-2)^2 - 4\cdot 1\cdot 4}}{2\cdot 1}$$
$$= 1 \pm \sqrt{1-4}$$
$$= 1 \pm \sqrt{-3}$$
$$= 1 \pm \sqrt{3}\, i$$

Så løsningene til $(*)$ er

$$\underline{\underline{z_1 = 1,\ z_2 = 1 + \sqrt{3}i \text{ og } z_3 = 1 - \sqrt{3}i}}$$

(Det er et tredjegradspolynom så vet at det ikke har flere enn tre løsninger)