Department of Engineering Cybernetics
Faculty of Information Technology, Mathematics and Electrical Engineering
Norwegian University of Science and Technology (NTNU)

Contact for questions during exam:
PhD student Tor Aksel Heirung
Tel: 91120005
Professor Rolf Henriksen
Tel: 92608199

English version

Exam in TTK4135

Optimization and Control

Optimalisering og regulering

Saturday June 2, 2012

Time: 09:00 – 13:00

Combination of allowed help remedies:
D – No printed or hand-written notes.
Certifed calculator with empty memory.

In the Appendix potentially useful information is included.
The grades will be available by June 23.

1 Various Topics (36 %)

General Optimization Problems

Consider the general constrained optimization problem (A.1).

a (4 %) Under what conditions on

- the objective function $f(x)$,
- the equality constraint functions $c_i(x),\ i \in \mathcal{E}$, and
- the inequality constraint functions $c_i(x),\ i \in \mathcal{I}$

is the problem (A.1) convex?

b (2 %) Reformulate (A.1) as a maximization problem.

c (4 %) Assume that there are no inequality constraints in (A.1), i.e., $\mathcal{I} \in \emptyset$. Define a suitable merit function for use in an SQP algorithm.

d (8 %) Let (A.1) be unconstrained, i.e., $\mathcal{I} \in \emptyset$ and $\mathcal{E} \in \emptyset$. Assume that f is convex. Show that any local minimizer x^* is a global minimizer. A correct proof gives full score; a verbal explanation may give some credit.

Linear Programming

Consider the LP problem in the standard form (A.6).

e (4 %) Define a *basic feasible point.*

f (6 %) Show how the LP problem

$$
\begin{aligned}
\min \quad & 3x_1 + 2x_2 + x_3 && \text{(1a)}\\
\text{s.t.} \quad & 2x_1 + 2x_2 + x_3 \le 3 && \text{(1b)}\\
& x_1 - \ \ x_2 - x_3 \le -1 && \text{(1c)}\\
& x \ge 0 && \text{(1d)}
\end{aligned}
$$

can be transformed into an LP problem in standard form (A.6).

g (8 %) The dual problem of the LP problem in standard form is given by

$$
\begin{aligned}
\max \quad & b^\top \lambda && \text{(2a)}\\
\text{s.t.} \quad & A^\top \lambda \le c && \text{(2b)}
\end{aligned}
$$

Show that the KKT conditions for the dual problem are equal to the KKT conditions for the original LP problem.

2 Quadratic Programming (QP) (36 %)

Consider the QP problem in the standard form (A.7).

a (4 %) Define the active set $\mathcal{A}(x)$ for (A.7). The mathematical definition must be included for full score.

b (8 %) Assume that there are no inequality constraints in (A.7), i.e., $\mathcal{I} \in \emptyset$. Derive the KKT conditions for the equality constrained QP problem and formulate the conditions using the KKT matrix (that is, formulate the KKT conditions as a matrix equation).

c (8 %) Assume that there are no inequality constraints in (A.7), i.e., $\mathcal{I} \in \emptyset$. State the necessary conditions for the vector x^* satisfying the KKT conditions to be the unique global solution of (A.7) (with $\mathcal{I} \in \emptyset$).

Hint: One of the conditions involves the basis for the nullspace of A.

d (6 %) Let (A.7) be unconstrained, i.e., $\mathcal{I} \in \emptyset$ and $\mathcal{E} \in \emptyset$, and assume that G is positive definite ($G > 0$).

- Derive the Newton direction p_k^{N}.
- Show that p_k^{N} is a descent direction.
- Show that the iteration algorithm $x_{k+1} = x_k + p_k^{\mathrm{N}}$ always converges to the optimum in one step.

e (10 %) A farmer wants to grow two different kinds of crop, C (e.g., carrot) and R (e.g., rutabaga) in a field F of size 100 000 m^2 which is available to him. Growing 1 tonne C in F requires an area of 4 000 m^2, whereas growing 1 tonne of R requires an area of 3 000 m^2. In addition, the two crops require different amounts of fertilizer. C requires 60 kg fertilizer per tonne grown, whereas R requires 80 kg fertilizer per tonne grown. The price of 1 kg fertilizer is 1. The market available for the farmer is very local and depends strongly upon the supply (measured in tonnes grown) of the crops. The selling price thus depends upon the supply.

- The price of crop C is $7000 - 200x_1$ per tonne, where x_1 is the number of tonnes grown of C.
- The price of crop R is $4000 - 140x_2$ per tonne, where x_2 is the number of tonnes grown of R.

Federal regulations allow the farmer to use at most a total of 2000 kg of fertilizer. The farmer wants to maximize the total profit. Formulate the above problem as a quadratic program. (Do not solve the model!)

3 Optimal Control and MPC (28 %)

Consider the optimal control problem for a linear dynamic system described by (A.9) on a finite horizon of length n. Q_i, $1 \leq i \leq n$, and S are symmetric positive semidefinite matrices ($Q_i \geq 0$, $S \geq 0$), whereas P_i, $1 \leq i \leq n$, are symmetric positive definite matrices ($P_i > 0$).

a (10 %) Assume that there are no inequality constraints in this problem, i.e., we remove (A.9e) and (A.9f).

- Specify the Lagrangean function for (A.9a)–(A.9d).
- State the KKT conditions for (A.9a)–(A.9d).
- We want to implement an optimal controller (LQ controller) for this problem. The solution is given by Theorem 2 (at the end of the appendix). What type of controller is this (i.e., is it linear/nonlinear, time-varying/time-invariant, state feedback/output feedback)?

b (6 %) Consider now the infinite horizon (the limit as n tends to infinity) optimal control problem

$$\min \quad f^{\infty} = \frac{1}{2} \sum_{i=0}^{\infty} \left\{ x_i^\top Q x_i + u_i^\top P u_i \right\} \tag{3a}$$

$$\text{s.t.} \quad x_{i+1} = A x_i + B u_i, \quad 0 \leq i \leq \infty \tag{3b}$$

- The controller is in this case given by $u_i = K x_i$. How is K computed? Specify the equation(s).
- What conditions on A, B, and D ($Q = D^\top D$) must be satisfied for asymptotic stability of optimal closed-loop system?

The remainder of this problem concerns MPC where the constraints (A.9e) and (A.9f) must be taken into account.

c (4 %) Give a brief explanation of the MPC principle. Please use a sketch in your explanation.

d (4 %)

- What are generally considered the main reasons for the success of MPC?
- All practical systems exhibit nonlinear behavior. Why, however, is the majority of MPC applications still based on linear models?

e (4 %) Infeasibility handling is important in MPC design, since disturbances or model inaccuracies can push the state outside of the feasibility region, i.e., the QP problem in the MPC has no feasible solution. A practical implementation must be able to handle this. Suggest a method for that.

Institutt for teknisk kybernetikk
Fakultet for informasjonsteknologi, matematikk og elektroteknikk
Norges teknisk-naturvitenskapelige universitet (NTNU)

Kontaktperson under eksamen:
PhD student Tor Aksel Heirung
Tel: 91120005
Professor Rolf Henriksen
Tel: 92608199

Utgave/Utgåve: bokmål/nynorsk

Eksamen i TTK4135

Optimalisering og regulering

Optimization and Control

Lørdag 2. Juni 2012

Tid: 09:00 – 13:00

Tillatte hjelpemidler / Tilletne hjelpemiddel:
D – Ingen trykte eller skrevne hjelpemidler. / Inga trykte eller skrevne hjelpemiddel.
Godkjent kalkulator med tomt minne. / Godkjend kalkulator med tomt minne.

Nyttig informasjon finnes i vedlegg. / Nyttig informasjon finns i vedlegg.
(Denne informasjonen er gitt på engelsk for å samsvare med pensumlitteraturen som den er hentet ifra.)
Sensur faller 23. juni. / Sensur fell 23. juni.

1 Diverse emner (36 %)

Generelle optimeringsproblemer

Betrakt det generelle optimeringsproblemet med bibetingelser (A.1).

a (4 %) Under hvilke betingelser på

- objektfunksjonen $f(x)$,
- likhetsbetingelsesfunksjonene $c_i(x),\ i \in \mathcal{E}$, og
- ulikhetsbetingelsesfunksjonene $c_i(x),\ i \in \mathcal{I}$

er problemet (A.1) konvekst?

b (2 %) Reformuler (A.1) til et maksimeringsproblem.

c (4 %) Anta at det ikke er noen ulikhetsbetingelser i (A.1), altså $\mathcal{I} \in \emptyset$. Definer en egnet merit-funksjon for bruk i en SQP-algoritme.

d (8 %) La (A.1) være ubeskranket ("unconstrained"), altså $\mathcal{I} \in \emptyset$ og $\mathcal{E} \in \emptyset$. Anta at f er konveks. Vis at ethvert lokalt minimum ("local minimizer") x^* er et globalt minimum ("global minimizer"). Et korrekt bevis gir full uttelling; en tekstlig forklaring kan gi delvis uttelling.

Lineær programmering

Betrakt LP-problemet på standardformen (A.6).

e (4 %) Definer et "*basic feasible point*".

f (6 %) Vis hvordan LP-problemet

$$
\begin{aligned}
\min \quad & 3x_1 + 2x_2 + x_3 && \text{(1a)}\\
\text{s.t.} \quad & 2x_1 + 2x_2 + x_3 \leq 3 && \text{(1b)}\\
& x_1 - \ \ x_2 - x_3 \leq -1 && \text{(1c)}\\
& x \geq 0 && \text{(1d)}
\end{aligned}
$$

kan transformeres til et LP-problem på standardformen (A.6).

g (8 %) Det duale problemet til LP-problemet på standardform er gitt av

$$
\begin{aligned}
\max \quad & b^\top \lambda && \text{(2a)}\\
\text{s.t.} \quad & A^\top \lambda \leq c && \text{(2b)}
\end{aligned}
$$

Vis at KKT-betingelsene for det duale problemet er like KKT-betingelsene for det originale LP-problemet.

2 Kvadratisk programmering (QP) (36 %)

Betrakt QP-problemet på standardformen (A.7).

a (4 %) Definer det aktive settet ("the active set") $\mathcal{A}(x)$ for (A.7). Den matematiske definisjonen må være med for full uttelling.

b (8 %) Anta at det ikke er noen ulikhetsbetingelser i (A.7), altså $\mathcal{I} \in \emptyset$. Utled KKT-betinglesene for det likhetsbeskrankede ("equality constrained") QP-problemet og formuler betingelsene med KKT-matrisen (det vil si: formuler KKT-betinglesene som en matriseligning).

c (8 %) Anta at det ikke er noen ulikhetsbetingelser i (A.7), altså $\mathcal{I} \in \emptyset$. Oppgi de nødvendige betinglesene ("the necessary conditions") for at en vektor x^* som tilfredsstiller KKT-betingelsene skal være den unike globale løsningen ("the unique global solution") av (A.7) (med $\mathcal{I} \in \emptyset$).

Hint: En av betingelsene har å gjøre med basisen for nullrommet til A.

d (6 %) La (A.7) være ubeskranket ("unconstrained"), altså $\mathcal{I} \in \emptyset$ og $\mathcal{E} \in \emptyset$, og anta at G er positiv definitt ($G > 0$).

- Utled Newton-retningen p_k^{N}.
- Vis at p_k^{N} er en minkende retning ("a descent direction").
- Vis at iterasjonsalgoritmen $x_{k+1} = x_k + p_k^{\mathrm{N}}$ alltid konvergerer til optimum i ett steg.

e (10 %) En bonde ønsker å dyrke to forskjellige grønnsaker, C (f.eks. gulrot) og R (f.eks. kålrot) i en åker F som er 100 000 m^2 stor. Å dyrke 1 tonn C i F krever et område med areal 4 000 m^2, mens å dyrke 1 tonn av R krever et område med areal 3 000 m^2. I tillegg trenger de to grønnsakene forskjellige mengder gjødsel. C trenger 60 kg gjødsel per dyrket tonn, mens R trenger 80 kg gjødsel per dyrket tonn. 1 kg gjødsel har en pris på 1. Bonden skal selge grønnsakene i et veldig lokalt marked der etterspørsel er sterkt avhengig av tilbud (målt i antall tonn han har dyrket). Altså vil salgsprisen avhenge av hvor mye grønnsaker han tilbyr.

- Grønnsak C selges for $7000 - 200x_1$ per tonn, hvor x_1 er antall tonn dyrket C.
- Grønnsak R selges for $4000 - 140x_2$ per tonn, hvor x_2 er antall tonn dyrket R.

Lovverket tillater bonden å bruke inntil 2000 kg gjødsel totalt. Bonden å ønsker å maksimere total profitt. Formuler problem over som et kvadratisk program. (Ikke løs modellen!)

3 Optimalregulering og MPC (28 %)

Betrakt optimalreguleringsproblemet for et lineært dynamisk system beskrevet av (A.9) over en endelig horisont av lengde n. Q_i, $1 \le i \le n$, og S er symmetriske positiv semidefinitte matriser ($Q_i \ge 0$, $S \ge 0$), mens P_i, $1 \le i \le n$, er symmetriske positiv definitte matriser ($P_i > 0$).

a (10 %) Anta at det ikke er noen ulikhetsbetingelser i dette problemet, vi fjerner altså (A.9e) og (A.9f).

- Spesifiser Lagrange-funksjonen for (A.9a)–(A.9d).
- Oppgi KKT-betingelsene for (A.9a)–(A.9d).
- Vi ønsker å implementere en optimal regulator (LQ-regulator) for dette problemet. Løsningen er gitt av Theorem 2 (se slutten av Appendix). Hva slags type regulator er dette (det vil si, er den lineær/ulineær, tidsvarierende/tidsinvariant, tilstandstilbakekobling/utgangstilbakekobling)?

b (6 %) Betrakt nå optimalreguleringsproblemet med uendelig horisont (grensen når n går mot uendelig)

$$\min \quad f^{\infty} = \frac{1}{2}\sum_{i=0}^{\infty}\left\{x_i^\top Q x_i + u_i^\top P u_i\right\} \tag{3a}$$

$$\text{s.t.} \quad x_{i+1} = Ax_i + Bu_i, \quad 0 \le i \le \infty \tag{3b}$$

- Regulatoren er i dette tilfellet gitt av $u_i = Kx_i$. Hvordan beregnes K? Spesifiser ligningen(e).
- Hvilke betingelse må A, B og D ($Q = D^\top D$) tilfredsstille for asymptotisk stabilitet i det optimale lukket-sløyfe-systemet?

Resten av denne oppgaven handler om MPC der beskrankningnene (A.9e) og (A.9f) må tas hensyn til.

c (4 %) Gi en kort forklaring av MPC-prinsippet. Vennligst bruk en skisse i forklaringen.

d (4 %)

- Hva regnes vanligvis som hovedgrunnene til MPCs suksess?
- Alle praktiske systemer har ulineær oppførsel. Hvorfor er da flertallet av MPC-anvendelser basert på lineære modeller?

e (4 %) Håndering av ugyldighet ("infeasibility handling") er viktig i MPC-design siden forstyrrelser eller unøyaktigheter i modellen kan dytte tilstanden utenfor det gyldige området ("the feasibility region"), altså at QP-problemet i MPC-en ikke har noen gyldig løsning ("no feasible solution"). En praktisk implementasjon må kunne håndtere dette. Foreslå en metode for det.

Appendix

Part 1 Optimization Problems and Optimality Conditions

A general formulation for constrained optimization problems is

$$\min_{x\in\mathbb{R}^n} \quad f(x) \tag{A.1a}$$
$$\text{s.t.} \quad c_i(x) = 0, \qquad i \in \mathcal{E} \tag{A.1b}$$
$$c_i(x) \geq 0, \qquad i \in \mathcal{I} \tag{A.1c}$$

where f and and the functions c_i are all smooth, differentiable, real-valued functions on a subset of $\mathbb{R}^n$, and $\mathcal{E}$ and $\mathcal{I}$ are two finite sets of indices.

The Lagrangean function for the general problem (A.1) is

$$\mathcal{L}(x,\lambda) = f(x) - \sum_{i\in\mathcal{E}\cup\mathcal{I}} \lambda_i c_i(x) \tag{A.2}$$

The KKT-conditions for (A.1) are given by:

$$\nabla_x \mathcal{L}(x^*,\lambda^*) = 0 \tag{A.3a}$$
$$c_i(x^*) = 0, \qquad i \in \mathcal{E} \tag{A.3b}$$
$$c_i(x^*) \geq 0, \qquad i \in \mathcal{I} \tag{A.3c}$$
$$\lambda_i^* \geq 0, \qquad i \in \mathcal{I} \tag{A.3d}$$
$$\lambda_i^* c_i(x^*) = 0, \qquad i \in \mathcal{E}\cup\mathcal{I} \tag{A.3e}$$

2nd order (sufficient) conditions for (A.1) are given by:

$$w \in \mathcal{C}(x^*,\lambda^*) \Leftrightarrow \begin{cases} \nabla c_i(x^*)^\top w = 0 & \text{for all } i \in \mathcal{E} \\ \nabla c_i(x^*)^\top w = 0 & \text{for all } i \in \mathcal{A}(x^*)\cap\mathcal{I} \text{ with } \lambda_i^* > 0 \\ \nabla c_i(x^*)^\top w \geq 0 & \text{for all } i \in \mathcal{A}(x^*)\cap\mathcal{I} \text{ with } \lambda_i^* = 0 \end{cases} \tag{A.4}$$

Theorem 1: (Second-Order Sufficient Conditions) *Suppose that for some feasible point $x^* \in \mathbb{R}^n$ there is a Lagrange multiplier vector λ^* such that the KKT conditions* (A.3) *are satisfied. Suppose also that*

$$w^\top \nabla^2_{xx}\mathcal{L}(x^*,\lambda^*)w > 0, \qquad \text{for all } w \in \mathcal{C}(x^*,\lambda^*),\ w \neq 0. \tag{A.5}$$

Then x^* is a strict local solution for (A.1).

LP problem in standard form:

$$\min_{x} \quad f(x) = c^\top x \tag{A.6a}$$
$$\text{s.t.} \quad Ax = b \tag{A.6b}$$
$$x \geq 0 \tag{A.6c}$$

where $A \in \mathbb{R}^{m\times n}$ and rank $A = m$.

QP problem in standard form:

$$\min_{x} \quad f(x) = \frac{1}{2}x^\top Gx + x^\top c \tag{A.7a}$$
$$\text{s.t.} \quad a_i^\top x = b_i, \qquad i \in \mathcal{E} \tag{A.7b}$$
$$a_i^\top x \geq b_i, \qquad i \in \mathcal{I} \tag{A.7c}$$

where G is a symmetric $n \times n$ matrix, $\mathcal{E}$ and $\mathcal{I}$ are finite sets of indices and c, x and $\{a_i\}, i \in \mathcal{E} \cup \mathcal{I}$, are vectors in $\mathbb{R}^n$. Alternatively, the equalities can be written $Ax = b$, $A \in \mathbb{R}^{m\times n}$.

Iterative method:

$$x_{k+1} = x_k + \alpha_k p_k \tag{A.8a}$$
$$x_0 \text{ given} \tag{A.8b}$$
$$x_k, p_k \in \mathbb{R}^n, \;\; \alpha_k \in \mathbb{R} \tag{A.8c}$$

p_k is the search direction and α_k is the line search parameter.

Part 2 Linear quadratic control of discrete dynamic systems

A typical optimal control problem on the time horizon 0 to n might take the form

$$\begin{aligned}\min\ f_0 = \frac{1}{2}\sum_{i=0}^{n-1}&\big\{(y_i - y_{\mathrm{ref},i})^\top Q_i (y_i - y_{\mathrm{ref},i}) \\ &+ (u_i - u_{i-1})^\top P_i (u_i - u_{i-1})\big\} \\ &+ \frac{1}{2}(y_n - y_{\mathrm{ref},n})^\top S (y_n - y_{\mathrm{ref},n})\end{aligned} \tag{A.9a}$$

subject to equality and inequality constraints

$$x_{i+1} = A_i x_i + B_i u_i,\ 0 \le i \le n-1 \tag{A.9b}$$
$$y_i = H x_i \tag{A.9c}$$
$$x_0 = \text{given (fixed)} \tag{A.9d}$$
$$U_L \le u_i \le U_U,\ 0 \le i \le n-1 \tag{A.9e}$$
$$Y_L \le y_i \le Y_U,\ 1 \le i \le n \tag{A.9f}$$

where system dimensions are given by

$$u_i \in \mathbb{R}^m \tag{A.9g}$$
$$x_i \in \mathbb{R}^l \tag{A.9h}$$
$$y_i \in \mathbb{R}^j \tag{A.9i}$$

The subscript i refers to the sampling instants. That is, subscript $i+1$ refers to the sample instant one sample interval after sample i. Note that the sampling time between each successive sampling instant is constant. Further, we assume that the control input u_i is constant between each sample.

Theorem 2: *Assume that* $x_{\mathrm{ref},i} = 0$, $u_{\mathrm{ref},i} = 0$, $0 \le i \le n$ *and that* $H = I$, *i.e.,* $y_i = x_i$. *The solution of* (A.9a), (A.9b), *and* (A.9d) *is given by* $u_i = K_i x_i$, $0 \le i \le n-1$, *where the feedback gain matrix is derived by*

$$K_i = -P_i^{-1} B_i^T R_{i+1} (I + B_i P_i^{-1} B_i^T R_{i+1})^{-1} A_i,\ 0 \le i \le n-1 \tag{A.10a}$$
$$R_i = Q_i + A_i^T R_{i+1} (I + B_i P_i^{-1} B_i^T R_{i+1})^{-1} A_i,\ 0 \le i \le n-1 \tag{A.10b}$$
$$R_n = S \tag{A.10c}$$