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Solution - TTK 4115 Linear systems theory

16. desember 2015, 09:00 – 13:00

Oppgave 1 (8 %)

Forklar følgende begreper:
Explain the following concepts:

a) (4%)
Eksakt diskretisering.
Exact discretization.

b) (4%)
Realisering.
Realization.

Solution

a) Observability is a structural property of systems that refers to the ability of reconstructing the initial state of a system $x(t_0)$ from the knowledge of inputs $u(t)$ and outputs $y(t)$ for $t \geq t_0$. It corresponds to a geometric relationship between the system matrices (A, C), and it is a fundamental requirement for the design of state estimators.

b) Discretization refers to the process of transforming a continuous-time system, described by differential equations, into a discrete-time system, described by difference equations. This is is typically done in the presence of sampled measurements and digital controllers. Among several discretization methods, exact discretization is one of the most accurate and it consists in computing the general solution of the continuous-time system and then evaluating the exponential matrix and the convolution integral at the fixed sampling time $T > 0$ (under the assumption that inputs are constant on each sampling interval). The adjective "exact" is motivated by the fact the solution of the discretized system coincides with the one of the original continuous-time system at any time step $t_k = kT,\ k \geq 0$.

c) Given a rational and proper matrix function $\hat{G}(s)$, a realization is any state-space model (A, B, C, D) such that $\hat{G}(s)$ is the corresponding transfer matrix, i.e.

$$\hat{G}(s) = C(sI - A)^{-1}B + D$$

A realization is said to be minimal if its state has the least achievable dimension; in particular, this corresponds to the requirement that state-space model (A, B, C, D) is both controllable and observable.

Oppgave 2 (21 %)

Gitt systemet:
Given the system

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} 1 & 2 \\ -3 & -6 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ b \end{bmatrix} u, \qquad b \in \mathbb{R} \\ y &= \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{x} \end{aligned}$$

a) (9%)
Transformer systemet til Jordanform:
Transform the system to Jordan form:

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= \mathbf{J}\hat{\mathbf{x}} + \hat{\mathbf{B}}u \\ y &= \hat{\mathbf{C}}\hat{\mathbf{x}} \end{aligned}$$

b) (4%)
Hvilke av disse egenskapene innehar systemet? Forklar dine valg.
Which of these properties does the system possess? Explain your choice.

A Marginalt stabilt, *Marginally stable*

B Ustabilt, *Unstable*

C Asymptotisk stabilt, *Asymptotically stable*

c) (2%)
Er systemet observerbart?
Is the system observable?

d) (6%)
Bestemme for hvilke verdier av $b \in \mathbb{R}$ systemet er: (Forklar dine valg)
Determine for which values of $b \in \mathbb{R}$ the system is: (Explain your choice)

1) styrbart, *controllable*

2) BIBO stabilt, *BIBO stable*

Solution

a) The Jordan form of the system can be derived by means of a transformation matrix $\mathbf{T}$ such that

$$\mathbf{J} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}, \quad \hat{\mathbf{B}} = \mathbf{T}^{-1}\mathbf{B}, \quad \hat{\mathbf{C}} = \mathbf{C}\mathbf{T}$$

Compute the characteristic polynomial of $\mathbf{A}$:

$$p_{\mathbf{A}}(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = (1-\lambda)(-6-\lambda) + 6 = \lambda^2 + 5\lambda = \lambda(\lambda+5).$$

We can easy compute the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -5$.
To obtain the desired transformation matrix, we first need to compute the eigenvectors $\mathbf{v}_1 = [v_{1,1}\ v_{2,1}]^T$ and $\mathbf{v}_2 = [v_{1,2}\ v_{2,2}]^T$. The first eigenvector must satisfy the identity

$$(\mathbf{A} - 0\mathbf{I})\mathbf{v}_1 = 0 \quad \Leftrightarrow \quad \left\{ \begin{array}{rl} v_{1,1} + 2v_{2,1} & = 0 \\ -3v_{1,1} - 6v_{2,1} & = 0 \end{array} \right.$$

which gives

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

The second eigenvector must satisfy the identity

$$(\mathbf{A} - (-5)\mathbf{I})\mathbf{v}_2 = 0 \quad \Leftrightarrow \quad \left\{ \begin{array}{rl} 6v_{1,2} + 2v_{2,2} & = 0 \\ -3v_{1,2} - 1v_{2,2} & = 0 \end{array} \right.$$

which gives

$$\mathbf{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

Now one has

$$\mathbf{T} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} = [\mathbf{v}_1 \ \mathbf{v}_2] = \begin{bmatrix} v_{1,1} & v_{1,2} \\ v_{2,1} & v_{2,2} \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 1 & 3 \end{bmatrix},$$

with

$$\mathbf{T}^{-1} = \frac{1}{\det \mathbf{T}} \begin{bmatrix} t_{22} & -t_{12} \\ -t_{21} & t_{11} \end{bmatrix} = \begin{bmatrix} -\frac{3}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

The system matrices in the Jordan form are then readily obtained:

$$\mathbf{J} = \begin{bmatrix} 0 & 0 \\ 0 & -5 \end{bmatrix} \quad \hat{\mathbf{B}} = \begin{bmatrix} -\frac{3}{5} - \frac{b}{5} \\ \frac{1}{5} + \frac{2b}{5} \end{bmatrix} \quad \hat{\mathbf{C}} = [-1 \quad 2].$$

b) As found in the previous point, the eigenvalues of the system are $\lambda_1 = 0,\ \lambda_2 = -5$: since only one of them has strictly negative real part, the system turns out to be not asymptotically stable. However, as the Jordan block associated to the zero eigenvalue λ_1 is of dimension 1×1, the system is marginally stable. Hence, the correct answer is **A**.

c) The observability matrix of the system is

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -4 \end{bmatrix},$$

and hence $\det \mathcal{O} = -2 \neq 0$ this proving system observability.

d1) Compute the controllability matrix

$$\mathcal{C} = [\mathbf{B} \quad \mathbf{AB}] = \begin{bmatrix} 1 & 1 + 2b \\ b & -3 - 6b \end{bmatrix}$$

In order to check controllability, we look at the values of b such that $\det \mathcal{C} = 0$:

$$\det \mathcal{C} = -2b^2 - 7b - 3 = 0$$

$$\Downarrow$$

$$b = \frac{7 \pm \sqrt{49 - 24}}{-2} \quad \Rightarrow \ b_1 = -3,\ b_2 = -1/2.$$

In conclusion, the system is controllable for any $b \in \mathbb{R}$ with $b \neq -3,\ b \neq -1/2$.

d2) To investigate BIBO stability we need to check the poles of the transfer function of the system:

$$\begin{aligned}
\hat{g}(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \\
&= [1 \quad 1] \begin{bmatrix} \dfrac{6+s}{5s+s^2} & \dfrac{2}{5s+s^2} \\ \dfrac{-3}{5s+s^2} & \dfrac{-1+s}{5s+s^2} \end{bmatrix} \begin{bmatrix} 1 \\ b \end{bmatrix} \\
&= \frac{3+b+s(1+b)}{5s+s^2} = \frac{n(s)}{d(s)}
\end{aligned}$$

The poles are $p_1 = 0, p_2 = -5$: in order to achieve BIBO stability, we need to cancel out the zero pole p_1, and hence we must require s to be a factor of the numerator $n(s)$:

$$n(s) = s \cdot c_0 \Rightarrow \begin{cases} b = -3 \\ c_0 = 1 + b = -2 \end{cases}$$

With this choice, the transfer function becomes

$$\hat{g}(s) = \frac{-2}{5+s}.$$

In conclusion, the system is BIBO stable only for $b = -3$.

Oppgave 3 (14 %)

Lyapunovs ligning er:
The Lyapunov equation is:

$$\mathbf{A}^T\mathbf{M} + \mathbf{M}\mathbf{A} = -\mathbf{N}$$

a) (14%)
Bruk Lyapunovs ligning til å vise at systemet nedenfor er asymptotisk stabilt.
Use the Lyapunov equation to show that the system below is asymptotically stable.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \quad \mathbf{A} = \begin{bmatrix} -1 & -2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Hint: Use a diagonal $\mathbf{N}$ (one possible choice is $\mathbf{N} = \mathbf{I}$)

Solution

Two different methods can be used.

♠ **First method**. Set $\mathbf{N} = \mathbf{I}$. We first note that the matrix $\mathbf{A}$ is block diagonal, and hence we can restrict to consider a block diagonal $\mathbf{M}$ matrix

$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} & 0 \\ m_{12} & m_{22} & 0 \\ 0 & 0 & p_{33} \end{bmatrix}$$

The corresponding Lyapunov reads as

$$\begin{bmatrix} -2m_{11} + 4m_{12} & -2m_{11} - 5m_{12} + 2m_{22} & 0 \\ -2m_{11} - 5m_{12} + 2m_{22} & -4m_{12} - 8m_{22} & 0 \\ 0 & 0 & -2p_{33} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

and we see immediately that $m_{33} = 1/2$. We have to solve the equations

$$\begin{aligned} &-2m_{11} + 4m_{12} = -1 \\ &-2m_{11} - 5m_{12} + 2m_{22} = 0 \\ &-4m_{12} - 8m_{22} = -1 \end{aligned}$$

and verify that the solutions fulfill the positive definiteness constraints

$$m_{11} > 0, \quad m_{11}m_{22} - m_{12}^2 > 0.$$

Using Gauss elimination method, the system can be rewritten in the equivalent triangular form

$$\begin{aligned} -2m_{11} - 5m_{12} + 2m_{22} &= 0 \\ -4m_{12} - 8m_{22} &= -1 \\ 20m_{22} &= \tfrac{13}{4} \end{aligned}$$

and hence one gets

$$m_{11} = \frac{7}{20}, \quad m_{12} = -\frac{3}{40}, \quad m_{22} = \frac{13}{80}.$$

Now $m_{11} = \frac{7}{20} > 0$ and

$$m_{11}m_{22} - m_{12}^2 = \frac{41}{800} > 0,$$

that also implies

$$\det \mathbf{M} = (m_{11}m_{22} - m_{12}^2)m_{33} > 0.$$

In conclusion, we have shown that the matrix

$$\mathbf{M} = \begin{bmatrix} \frac{7}{20} & -\frac{3}{40} & 0 \\ -\frac{3}{40} & \frac{13}{80} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

is positive definite and since by construction it satisfies the Lyapunov equation with a positive definite $\mathbf{N}$, the asymptotic stability of the system has been proved.

♣ **Second method**. Set

$$\mathbf{N} = \begin{bmatrix} n_1 & 0 & 0 \\ 0 & n_2 & 0 \\ 0 & 0 & n_3 \end{bmatrix}$$

with $n_\ell,\ \ell = 1, 2, 3$ to be specified. Accordingly, select a diagonal $\mathbf{M}$

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}$$

and compute the corresponding Lyapunov equation

$$\begin{bmatrix} -2m_1 & -2m_1 + 2m_2 & 0 \\ -2m_1 + 2m_2 & -8m_2 & 0 \\ 0 & 0 & -2m_3 \end{bmatrix} = \begin{bmatrix} -n_1 & 0 & 0 \\ 0 & -n_2 & 0 \\ 0 & 0 & -n_3 \end{bmatrix}.$$

From the equation

$$-2m_1 + 2m_2 = 0$$

we get $m_1 = m_2$, and as long as we select $n_2 = 4n_1$, any possible choice of coefficients $m_1 = m_2 > 0$ with $m_3 = n_3/2$ will work well. The simplest choice is for instance $n_1 = 2nq_2 = 8, n_3 = 2$, which leads to the positive definite diagonal matrix

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Oppgave 4 (18 %)

Gitt følgende system
Consider the system

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x} \end{aligned}$$

a) (2%)
Sjekk om systemet er kontrollerbart og styrbart
Check controllability and observability

b) (16%)
Gitt observeren
Given the observer

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}(y - \mathbf{C}\hat{\mathbf{x}})$$

og tilbakekoblingen
and the dynamic feedback control

$$u = -\mathbf{K}\hat{\mathbf{x}},$$

bestem forsterkningsmatrisene $\mathbf{K}, \mathbf{L}$ slik at
determine the gains $\mathbf{K}, \mathbf{L}$ *such that*

A) egenverdiene til lukket-sløyfe matrisen $\mathbf{A} - \mathbf{BK}$ er $\lambda_1 = -1, \lambda_2 = -2$,
the eigenvalues of the closed-loop matrix $\mathbf{A} - \mathbf{BK}$ *are* $\lambda_1 = -1, \lambda_2 = -2$,

B) egenverdien til feiltilstandssystemmatrisen $\mathbf{A} - \mathbf{LC}$ er $\mu_1 = -3, \mu_2 = -5$.
the eigenvalues of the error system matrix $\mathbf{A} - \mathbf{LC}$ *are* $\mu_1 = -3, \mu_2 = -5$.

Solution

a) We compute controllability and observability matrices and check that they have full-rank

$$\mathcal{C} = [\mathbf{B}\ \mathbf{AB}] = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \Rightarrow \det \mathcal{C} = 1 \neq 0$$

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow \det \mathcal{O} = 1 \neq 0$$

b) The system is controllable and observable, and thanks to the separation principle the eigenvalues of the closed-loop system and the eigenvalues of the estimation error system can be assigned independently. The desired characteristic polynomials for the two subsystems are

$$p^{\dagger}_{\mathbf{A}-\mathbf{BK}}(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 + 3\lambda + 2$$

$$p^{\dagger}_{\mathbf{A}-\mathbf{LC}}(\lambda) = (\lambda - \mu_1)(\lambda - \mu_2) = \lambda^2 + 8\lambda + 15$$

Set $\mathbf{K} = [k_1 \ k_2]$ and compute the characteristic polynomial

$$\begin{aligned} p_{\mathbf{A}-\mathbf{BK}}(\lambda) &= \det(\mathbf{A} - \mathbf{BK} - \lambda \mathbf{I}) = \det \begin{bmatrix} 1 - k_1 - \lambda & 1 - k_2 \\ -k_1 & 3 - k_2 - \lambda \end{bmatrix} \\ &= \lambda^2 + (k_1 + k_2 - 4)\lambda - 2k_1 - k_2 + 3 \end{aligned}$$

Imposing the equality

$$p_{\mathbf{A}-\mathbf{BK}}(\lambda) = p^{\dagger}_{\mathbf{A}-\mathbf{BK}}(\lambda)$$

yields the equations

$$\begin{aligned} k_1 + k_2 - 4 &= 3 \\ -2k_1 - k_2 + 3 &= 2 \end{aligned}$$

We obtain the coefficients $k_1 = -6, \quad k_2 = 13$.
Set $\mathbf{L} = [l_1 \ l_2]^T$ and compute the characteristic polynomial

$$\begin{aligned} p_{\mathbf{A}-\mathbf{LC}}(\lambda) &= \det(\mathbf{A} - \mathbf{LC} - \lambda \mathbf{I}) = \det \begin{bmatrix} 1 - l_1 - \lambda & 1 \\ -l_2 & 3 - \lambda \end{bmatrix} \\ &= \lambda^2 + (l_1 - 4)\lambda - 3l_1 + l_2 + 3 \end{aligned}$$

Imposing the equality

$$p_{\mathbf{A}-\mathbf{LC}}(\lambda) = p^{\dagger}_{\mathbf{A}-\mathbf{LC}}(\lambda)$$

yields the equations

$$\begin{aligned} l_1 - 4 &= 8 \\ -3l_1 + l_2 + 3 &= 15 \end{aligned}$$

We obtain the coefficients $l_1 = 12, \quad l_2 = 48$.

Oppgave 5 (17%)

a-b) (4%)

Spektraltetthetsfunksjonen til en stasjonær stokastisk prosess $x(t)$ er gitt ved:
The spectral density function of a stationary random process $x(t)$ is given by:

$$S_x(j\omega) = \frac{4}{\omega^4 + 5\omega^2 + 4}$$

Dette gir $S_x(s)$ polene ± 1 og ± 2, vis dette.
This gives $S_x(s)$ the poles ± 1 and ± 2, show this.

$\bigtriangledown$

Solution

$$\begin{aligned} S_x(s) &= \frac{4}{(s+1)(s-1)(s+2)(s-2)} = \frac{4}{(s^2-1)(s^2-4)} \\ &= \frac{4}{s^4-5s^2+4} = \frac{4}{(j\omega)^4 - 5(j\omega)^2 + 4} = \frac{4}{\omega^4+5\omega^2+4} \end{aligned}$$

because

$$j^2 = -1; \quad j^4 = 1$$

$\bigtriangleup$

$x(t)$ kan representeres som filtrert hvitstøy hvor spektraltettheten til hvitstøyen på inngangen er $S_v(j\omega) = 1$.Finn transferfunksjonen $G(s)$ til dette filteret.

$x(t)$ can be represented as filtered white noise where the spectral density of the white input noise is $S_v(j\omega) = 1$. Find the transfer function $G(s)$ of this filter.

$\bigtriangledown$

Solution

From L9-S24 Shaping filter: If we can use spectral factorization on a given PSD, $S_x(j\omega)$, the part with poles and zeroes in the left half-plane, $G(j\omega)$, is the shaping filter for unit white noise, because

$$S_x(j\omega) = G(j\omega)\, G(-j\omega) \cdot 1$$

Here:

$$G(s) = \frac{2}{(s+1)(s+2)} = \frac{2}{s^2+3s+2}$$

$\bigtriangleup$

c) (13%)

Gitt transferfunksjonen:
Given the transfer function:

$$\frac{x(s)}{v(s)} = G(s) = \frac{2}{(s+1)(s+2)}$$

La $v(t)$ være en hvitstøyprosess med spektraltetthet $S_v(j\omega) = 1 = \tilde{Q}$. Finn differensiallikninga som beskriver utviklinga av variansen til $x(t)$ og finn stasjonærverdien til variansen for $x(t)$.

Let $v(t)$ be a white noise process with spectral density $S_v(j\omega) = 1 = \tilde{Q}$. Find the differential equation which describe the variance of $x(t)$ and find the stationary value of the variance of $x(t)$.

▽

Solution

$$\begin{aligned}\frac{x(s)}{v(s)} &= \frac{2}{s^2+3s+2}\\ s^2x(s)+3sx(s)+2x(s) &= 2v(s)\end{aligned}$$

This is equal to

$$\ddot{x}(t)+3\dot{x}(t)+2x(t)=2v(t)$$

in the time domain. Choosing $x_1(t)=x(t)$ and $x_2(t)=\dot{x}(t)$ we get the state space model

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -2x_1-3x_2+2v\end{aligned}$$

On vector form:

$$\underline{\dot{x}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{F}\underline{x} + \underbrace{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}_{G} v$$

The covariance equation for this system is

$$\dot{P} = FP + PF^T + G\tilde{Q}G^T; \quad P(t_0)$$

where the variance of $x(t)$ is element $(1,1)$ in the matrix $P(t)$. The initial covariance matrix $P(t_0)$ must be given in order to find the transient solution, but is not requisite for the stationary solution.

The stationary solution to this equation when $\tilde{Q}=1$ is given by

$$\begin{aligned}\dot{P}(\infty) &= 0\\ \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}P(\infty)+P(\infty)\begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}+\begin{bmatrix} 0 \\ 2 \end{bmatrix}\begin{bmatrix} 0 & 2 \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\end{aligned}$$

Defining (P is symmetric)

$$P(\infty) = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

we get the component equations

$$\begin{aligned}\begin{bmatrix} p_{12} & p_{22} \\ -2p_{11}-3p_{12} & -2p_{12}-3p_{22} \end{bmatrix}+\begin{bmatrix} p_{12} & -2p_{11}-3p_{12} \\ p_{22} & -2p_{12}-3p_{22} \end{bmatrix}+\begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\\ \begin{bmatrix} 2p_{12} & -2p_{11}-3p_{12}+p22 \\ -2p_{11}-3p_{12}+p_{22} & -4p_{12}-6p_{22} \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & -4 \end{bmatrix}\end{aligned}$$

and the stationary solutions are

$$
\begin{aligned}
p_{12} &= 0 \\
p_{22} &= \frac{4}{6} = \frac{2}{3} \\
p_{11} &= \frac{1}{2}p_{22} = \frac{1}{3}
\end{aligned}
$$

The stationary variance of $x(t)$ is therefor $\frac{1}{3}$.

△

Oppgave 6 (22%)

a) (2%)

Skriv opp likningene for systemet det diskrete kalmanfilteret er en optimalt estimator for.

Write the system equations for which the discrete Kalman filter is an optimal estimator.

▽

Solution

$$
\begin{aligned}
&\underline{x}_{k+1} = \Phi_k \underline{x}_k + \Lambda_k \underline{u}_k + \Gamma_k \underline{v}_k \\
&\underline{z}_k = H_k \underline{x}_k + \underline{w}_k
\end{aligned}
$$

$$
\begin{array}{ll}
E\{\underline{x}_0\} = \bar{\underline{x}}_0 & E\left\{(\underline{x}_0 - \bar{\underline{x}}_0)(\underline{x}_0 - \bar{\underline{x}}_0)^T\right\} = \bar{P}_0 \\
E\{\underline{v}_k\} = \underline{0} & E\left\{\underline{v}_k \underline{v}_l^T\right\} = \delta_{kl} Q_k \\
E\{\underline{w}_k\} = \underline{0} & E\left\{\underline{w}_k \underline{w}_l^T\right\} = \delta_{kl} R_k \\
 & E\left\{\underline{x}_0 \underline{v}_k^T\right\} = \underline{0} \\
 & E\left\{\underline{x}_0 \underline{w}_k^T\right\} = \underline{0} \\
 & E\left\{\underline{v}_k \underline{w}_l^T\right\} = \underline{0}
\end{array}
$$

△

b) (2%)

Skriv opp likningene for systemet det kontinuerlige kalmanfilteret er en optimalt estimator for.

Write the system equations for which the continuous Kalman filter is an optimal estimator.

▽

Solution

$$
\begin{aligned}
&\underline{\dot{x}}(t) = F(t)\underline{x}(t) + L(t)\underline{u}(t) + G(t)\underline{v}(t)\\
&\underline{z}(t) = H(t)\underline{x}(t) + \underline{w}(t)
\end{aligned}
$$

$$
\begin{array}{ll}
E\left\{\underline{x}(t_0)\right\} = \underline{\bar{x}}_0 & E\left\{\left(\underline{x}(t_0) - \underline{\bar{x}}_0\right)\left(\underline{x}(t_0) - \underline{\bar{x}}_0\right)^T\right\} = \bar{P}_0\\
E\left\{\underline{v}(t)\right\} = \underline{0} & E\left\{\underline{v}(t)\underline{v}^T(\tau)\right\} = \delta(t-\tau)\tilde{Q}(t)\\
E\left\{\underline{w}(t)\right\} = \underline{0} & E\left\{\underline{w}(t)\underline{w}^T(\tau)\right\} = \delta(t-\tau)\tilde{R}(t)\\
 & E\left\{\underline{x}(t_0)\underline{v}^T(t)\right\} = \underline{0}\\
 & E\left\{\underline{x}(t_0)\underline{w}^T(t)\right\} = \underline{0}\\
 & E\left\{\underline{v}(t)\underline{w}^T(\tau)\right\} = \underline{0}
\end{array}
$$

△

c) (6%)
Gitt målingene
Given the measurements

$$z_k = x + w_k; \quad w_k \sim \mathcal{N}(0, R) \tag{1}$$

av den ukjent konstanten x så kan vi beregne et estimatet ved
of the unknown constant x. The estimate is then given by

$$\hat{x}_k = \frac{1}{k}\sum_{i=1}^{k} z_i \tag{2}$$

Utled en tilsvarende rekursiv form av denne likninga som den vi har for det diskret kalmanfilteret (med likningene splittet i tidsoppdatering og måleoppdatering), dvs finn likningne for beregning av $\hat{x}_k$ og K_k.

Derive a comparable recursive equation to the discrete Kalman filter equations (time and measurement update equations), i.e derive the equations for calculating $\hat{x}_k$ and K_k.

▽

Solution

$$
\begin{aligned}
\hat{x}_k &= \frac{1}{k}\sum_{i=1}^{k} z_i = \frac{1}{k}z_k + \frac{1}{k}\sum_{i=1}^{k-1} z_i = \frac{1}{k}z_k + \frac{k-1}{k}\underbrace{\frac{1}{k-1}\sum_{i=1}^{k-1} z_i}_{\hat{x}_{k-1}}\\
&= \frac{1}{k}z_k + \frac{k-1}{k}\hat{x}_{k-1} = \hat{x}_{k-1} + \frac{1}{k}\left(z_k - \hat{x}_{k-1}\right)
\end{aligned}
$$

Since the state is a constant we may set

$$x_{k+1} = x_k = x$$

giving

$$\bar{x}_{k+1} = \hat{x}_k$$

Defining the gain

$$K_k = \frac{1}{k}$$

we may write the following recursive equation for estimating x_k based on k measurements:

$$\begin{aligned} \bar{x}_{k+1} &= \hat{x}_k; \quad \hat{x}_0 = 0 \\ \hat{x}_k &= \bar{x}_k + K_k \left(z_k - \bar{x}_k \right) \\ K_k &= \frac{1}{k} \end{aligned}$$

△

d) (6%)
Gitt systemet
Given the system

$$x_{k+1} = x_k; \quad x_0 \sim \mathcal{N}\left(\hat{x}_0, \hat{P}_0\right) \tag{3}$$

$$z_k = x_k + w_k; \quad w_k \sim \mathcal{N}\left(0, R\right) \tag{4}$$

Finn formlene for $\hat{P}_k$ og K_k fra likningene for det diskrete kalmanfilteret. Sammenlikn kalmanfilterforsterkninga funnet i punktene (6.c) og (6.d) og kommenter.

Derive the formulas for $\hat{P}_k$ and K_k from the discrete Kalman filter equations. Compare the Kalman filter gains found in (6.c) and (6.d) and give a comment.

▽

Solution

The discrete Kalman filter is

$$\begin{aligned} \underline{\bar{x}}_{k+1} &= \Phi \underline{\hat{x}}_k; \quad \underline{\hat{x}}_0 \text{ given}; \quad \bar{P}_{k+1} = \Phi \hat{P}_k \Phi^T + \Gamma Q \Gamma^T; \quad \hat{P}_0 \text{ given} \\ \underline{\hat{x}}_k &= \underline{\bar{x}}_k + K_k \left(\underline{z}_k - H \underline{\bar{x}}_k \right); \quad \hat{P}_k = \left(I - K_k H \right) \bar{P}_k \\ K_k &= \bar{P}_k H^T \left(H \bar{P}_k H^T + R \right)^{-1} \end{aligned}$$

For our scalar system

$$\Phi = 1; \ \Gamma = 0; \ Q = 0; \ H = 1$$

giving the equations

$$
\begin{aligned}
\bar{P}_{k+1} &= \hat{P}_k \\
K_k &= \hat{P}_{k-1}\left(\hat{P}_{k-1}+R\right)^{-1} = \frac{\hat{P}_{k-1}}{\hat{P}_{k-1}+R} \\
\hat{P}_k &= \left(1-\frac{\hat{P}_{k-1}}{\hat{P}_{k-1}+R}\right)\hat{P}_{k-1} = \frac{\hat{P}_{k-1}}{\hat{P}_{k-1}+R}R = \frac{\hat{P}_{k-1}}{1+\hat{P}_{k-1}/R} \\
K_k &= \hat{P}_k/R
\end{aligned}
$$

Lets solve the rekursion for $\hat{P}_k$

$$
\begin{aligned}
\hat{P}_1 &= \frac{\hat{P}_0}{1+\hat{P}_0/R} \\
\hat{P}_2 &= \frac{\frac{\hat{P}_0}{1+\hat{P}_0/R}}{1+\frac{\hat{P}_0/R}{1+\hat{P}_0/R}} = \frac{\hat{P}_o}{1+\hat{P}_0/R+\hat{P}_0/R} = \frac{\hat{P}_0}{1+2\hat{P}_0/R} \\
\hat{P}_3 &= \frac{\frac{\hat{P}_0}{1+2\hat{P}_0/R}}{1+\frac{\hat{P}_0/R}{1+2\hat{P}_0/R}} = \frac{\hat{P}_0}{1+2\hat{P}_0/R+\hat{P}_0/R} = \frac{\hat{P}_0}{1+3\hat{P}_0/R}
\end{aligned}
$$

From this sequence we find by induction that

$$
\begin{aligned}
\hat{P}_k &= \frac{\hat{P}_0}{1+k\hat{P}_0/R} \\
K_k &= \frac{\hat{P}_0/R}{1+k\hat{P}_0/R}
\end{aligned}
$$

In B.c??? where we had no a priori information we found the gain

$$K_k = \frac{1}{k}$$

this is the same as we get here when $k \to \infty$. The effect of the a priori information diminish due to all the new informations in $z_1, z_2, \cdots, z_k$

$$K_k = \frac{\hat{P}_0/R}{1+k\hat{P}_0/R} = \frac{1}{k+\frac{R}{\hat{P}_0}} \to \frac{1}{k} \text{ when } k \text{ is large}$$

$\triangle$

e) (6%)
Gitt systemet
Given the system

$$\dot{x} = 0; \quad x(0) \sim \mathcal{N}\left(\hat{x}(0), \hat{P}(0)\right) \tag{5}$$

$$z(t) = x(t) + w(t); \quad w(t) \sim \mathcal{N}\left(0, \tilde{R}\right) \tag{6}$$

Skriv opp differentiallikninga for $\hat{P}(t)$ og løs den. Finn $K(t)$
Find the differential equation for $\hat{P}(t)$ *and solve it and calculate* $K(t)$.

▽

Solution

The continuous Kalman filter is

$$\begin{aligned}
\dot{\hat{\underline{x}}}(t) &= F\hat{\underline{x}}(t) + K(t)\left(\underline{z}(t) - H\hat{\underline{x}}(t)\right); \quad \hat{\underline{x}}(t_0) \text{ given} \\
\dot{\hat{P}}(t) &= F\hat{P}(t) + \hat{P}(t)F^T + G\tilde{Q}G^T - \hat{P}(t)H^T\tilde{R}^{-1}H\hat{P}(t); \quad \hat{P}(t_0) \text{ given} \\
K(t) &= \hat{P}(t)H^T\tilde{R}^{-1}
\end{aligned}$$

For our scalar system

$$F = 0;\ G = 0;\ \tilde{Q} = 0;\ H = 1$$

giving the differential equation

$$\dot{\hat{P}} = -\frac{\hat{P}^2}{\tilde{R}}$$

This equation can be solved by the method of separation of variables

$$\begin{aligned}
\dot{\hat{P}} &= \frac{d\hat{P}}{dt} = -\frac{\hat{P}^2}{\tilde{R}} \\
\frac{d\hat{P}}{\hat{P}^2} &= -\frac{dt}{\tilde{R}} \\
\int_{\hat{P}_0}^{\hat{P}(t)} \frac{d\hat{P}}{\hat{P}^2} &= -\int_0^t \frac{dt}{\tilde{R}} \\
-\Big|_{\hat{P}_0}^{\hat{P}(t)} \frac{1}{\hat{P}} &= -\frac{t}{\tilde{R}} \\
-\frac{1}{\hat{P}(t)} + \frac{1}{\hat{P}_0} &= -\frac{t}{\tilde{R}} \\
\frac{1}{\hat{P}(t)} &= \frac{1}{\hat{P}_0} + \frac{t}{\tilde{R}} \\
\hat{P}(t) &= \frac{1}{\frac{1}{\hat{P}_0} + \frac{t}{\tilde{R}}} = \frac{\hat{P}_o}{1 + t\hat{P}_0/\tilde{R}}
\end{aligned}$$

That is

$$
\begin{aligned}
\hat{P}(t) &= \frac{\hat{P}_o}{1 + t\hat{P}_0/\tilde{R}} \\
K(t) &= \frac{\hat{P}_o/\tilde{R}}{1 + t\hat{P}_0/\tilde{R}}
\end{aligned}
$$

$\triangle$