Matte 4K, øving 9

gruppe 2, Rendell Cale

Ønsker tilbakemelding :)

Godkjent

17.1:

5) . For $x=a$ we have

$$w = (a+iy)^2$$
$$= a^2+2ayi-y^2$$
$$= a^2-y^2+2ayi$$

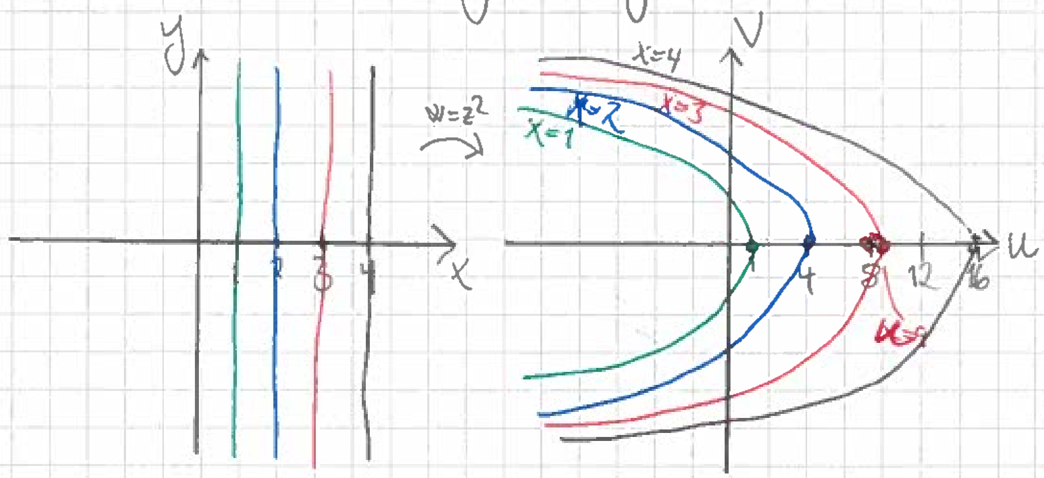
For $x=1,2,3,4$ we then get the curves:

$x=1$: $1-y^2+2yi$

$x=2$: $4-y^2+4yi$

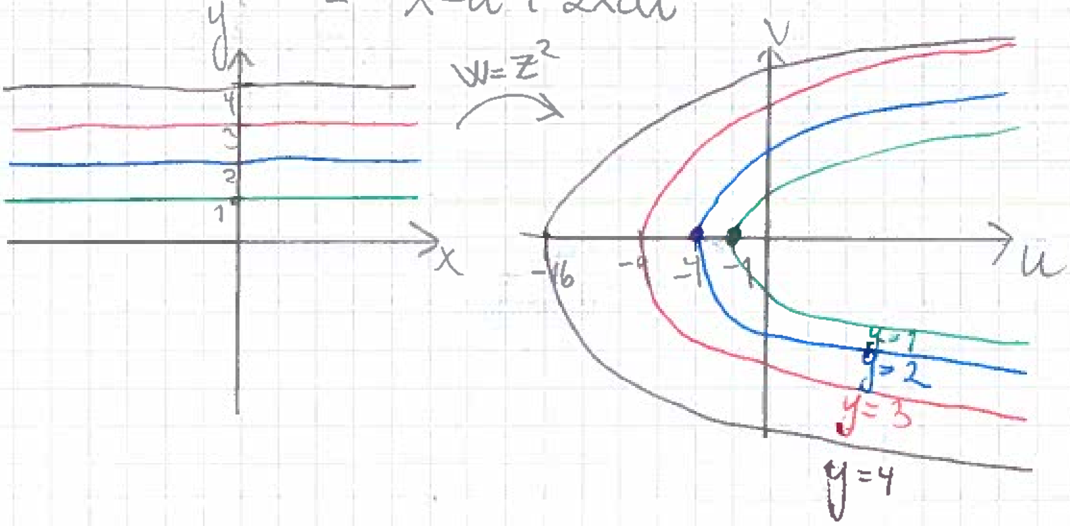
$x=3$: $9-y^2+6yi$

$x=4$: $16-y^2+8yi$

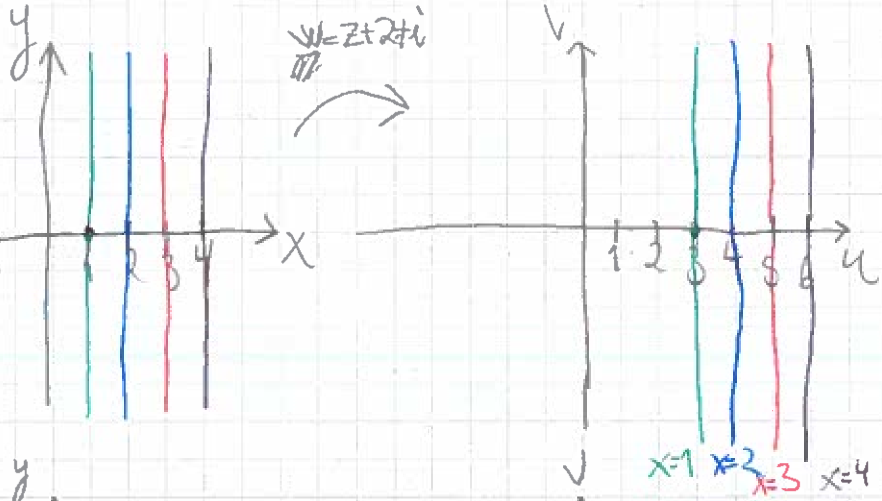


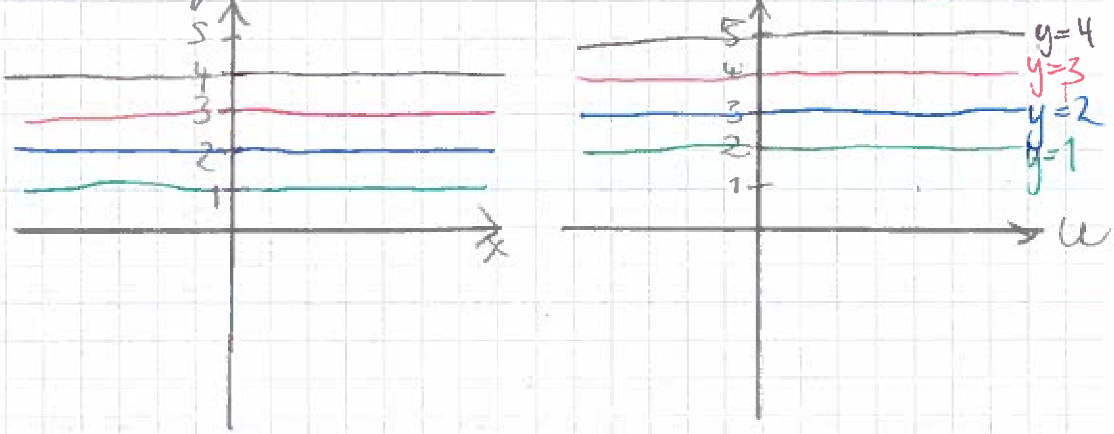
$y = a$ gives

$$\begin{aligned} w &= (x+ia)^2 \\ &= x^2 + 2xai - a^2 \\ &= x^2 - a^2 + 2xai \end{aligned}$$



8)





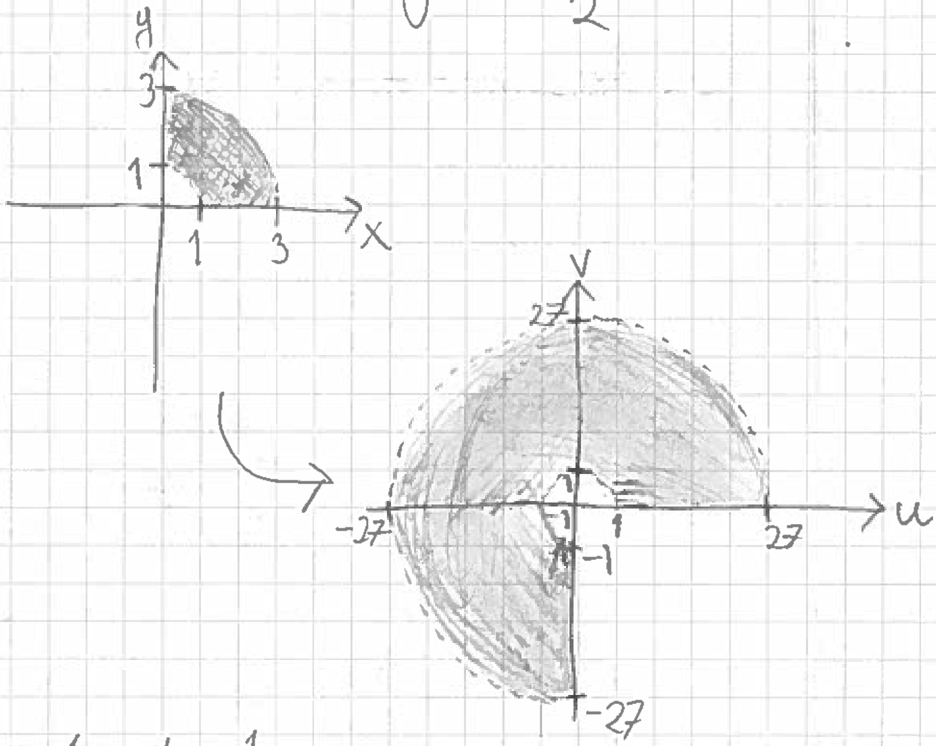
11) $1 < |z| < 3$, $0 < \operatorname{Arg} z < \pi/2$

$w = z^3$

Note that $|w| = |z|^3$ and $\arg w = 3 \arg z$

This gives $1 < |w| < 27$

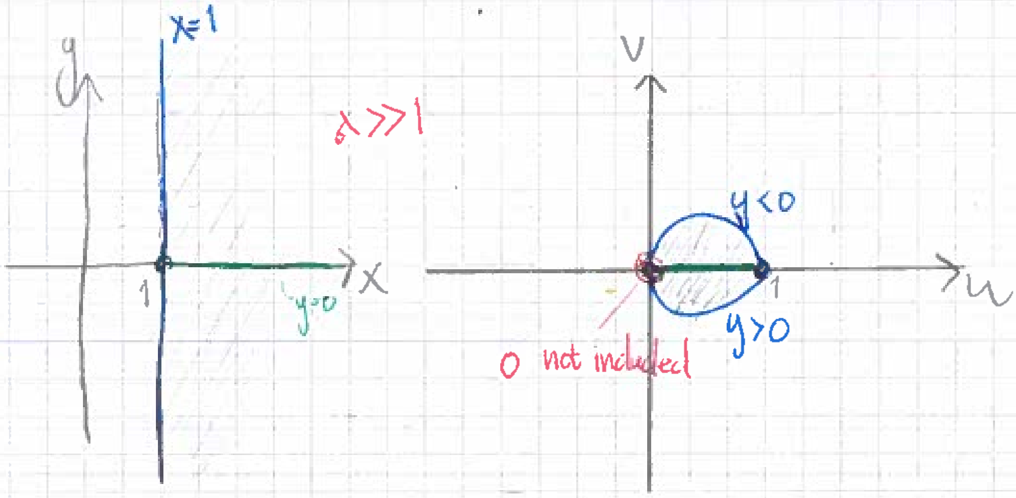
and $0 < \arg w < \frac{3\pi}{2}$



13) $x \geq 1$, $w = \frac{1}{z}$

$$w = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$$

$x = 1$ gir $w = \frac{1-iy}{1+y^2}$



15) $f(z) = az^3 + bz^2 + cz + d$

This is not conformal when the derivative is 0.

$$f'(z) = 3az^2 + 2bz + c$$

Solving $f'(z_0) = 0$ gives

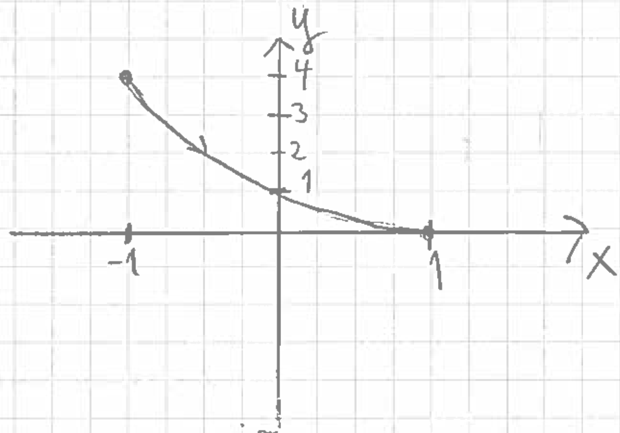
$$z_0 = \frac{-2b \pm \sqrt{4b^2 - 4 \cdot 3a \cdot c}}{2 \cdot 3 \cdot a}$$

$$z_0 = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}$$

For $z = z_0$ as above, $f(z)$ is non-conformal.

14.1:

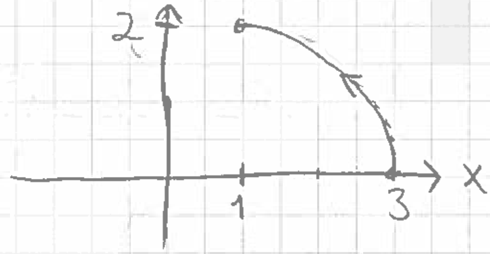
4) $z(t) = t + (1-t)^2 i$, $(-1 \leq t \leq 1)$



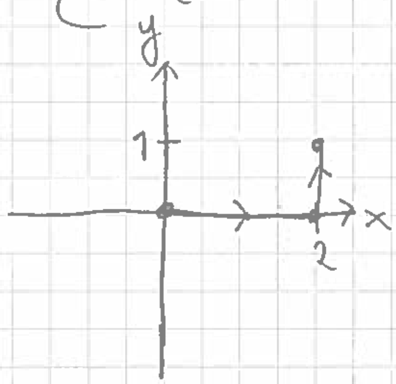
7) $z(t) = 1 + 2e^{i\frac{\pi}{4}t}$, $0 \leq t \leq 2$

$\Leftrightarrow z(\tau) = 1 + 2e^{i\tau}$, $0 \leq \tau \leq \frac{\pi}{2}$, $\tau = \frac{\pi}{4}t$

$2e^{i\tau}$, $0 \leq \tau \leq \frac{\pi}{2}$ is a quarter circle with radius 2.



12) $z(t) = \begin{cases} 2t, & 0 \leq t \leq 1 \\ 2 + i(t-1), & 1 < t \leq 2 \end{cases}$



17) $|z+a-ib|=r$, clockwise

$|z-z_0|=r$ defines a circle with radius r around z_0

$$|z+a-ib| = |z-\underbrace{(-a+ib)}_{z_0=-a+ib}|$$

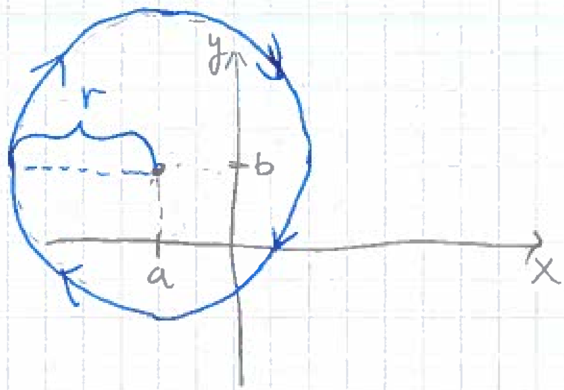
$z(t) = z_0 + r(\cos t + i\sin(-t))$, $0 \leq t \leq 2\pi$

is a working parametrisation, but lets simplify it:

$$\cos(t) + i\sin(-t) = \cos t - i\sin t = e^{-it}$$

So we have

$$\underline{z(t) = -a+ib+re^{-it}}$$



21) $f(z) = \operatorname{Re} z = x$

$F(z) = \frac{1}{2}x^2 + C$, C = constant (complex)

is such that $F'(z) = f(z)$

By the first method:

$$I_1 = \int_C \operatorname{Re} z \, dz = \frac{1}{2}x^2 \Big|_{1+i}^{5+5i}$$

$$= \frac{1}{2}(5^2 - 1^2)$$

$$= \underline{12}$$

but f is not analytic so this doesn't make any sense.

Using the second method:

$$I_2 = \int_C \operatorname{Re} z \, dz = \int_1^5 \operatorname{Re}(z(t)) \, \dot{z}(t) \, dt$$

where $z(t) = (1+i)t$, $1 < t < 5$

This gives $\operatorname{Re}(z) = t$

$\dot{z}(t) = 1+i$

So $I_2 = \int_1^5 t(1+i) \, dt$

$\Leftrightarrow \quad I_2 = (1+i)\left[\frac{1}{2}t^2\right]_1^5$

$= \frac{(1+i)}{2}(25-1)$

$= \underline{\underline{12+12i}}$

23) $\int_C e^z dz$, C: shortest path from $\frac{\pi}{2}i$ to πi

Note that e^z is analytic so

$$\int_C e^z dz = e^{\pi i} - e^{\frac{\pi}{2}i}$$

$$= \underline{\underline{-1-i}}$$

35) $L = \sqrt{\underbrace{(5-1)^2}_{\text{x-axis}} + \underbrace{(5-1)^2}_{\text{y-axis}}} = 4\sqrt{2}$

M is such that $|Re(z)| \leq M$ for all z on C
so $M = |Re(5+5i)| = 5$

This gives us an upper bound of

$$M \cdot L = 5 \cdot 4\sqrt{2}$$

$$= 20\sqrt{2}$$

14.2:

14) $f(z) = \frac{1}{z}$

C: unit circle, counter clockwise

$f(z)$ is not defined (and thus not analytic) at the origin, which is a point in the unit circle, so Cauchy's Integral Theorem does not apply.

$$f(z) = \frac{1}{z} = \frac{z}{|z|^2}$$

$$C: z(t) = e^{it}, \quad 0 \le t \le 2\pi$$

$$\int_C f(z)\,dz = \int_0^{2\pi} f(e^{it})\, i e^{it}\, dt$$

$$= i \int_0^{2\pi} e^{2it}\, dt$$

$$= i \frac{1}{2i}\left(e^{4\pi i} - 1\right)$$

$$\underline{= 0}$$

22) $\displaystyle\int_C \mathrm{Re}(z)\,dz = I$

$f(z) = \mathrm{Re}(z) = x$ is not analytic so Cachy's Integral Theorem does not apply.

$C = C_1 \cup C_2$

where C_1: upperhalf of unit circle,

$z_1(t) = e^{it}, 0 \leq t \leq \pi$

C_2: line from -1 to 1

$z_2(t) = -1 + t, 0 \leq t \leq 2$

$I = I_1 + I_2$

where $I_1 = \displaystyle\int_{C_1} \mathrm{Re}(z)\,dz$

and $I_2 = \displaystyle\int_{C_2} \mathrm{Re}(z)\,dz$

$$I_1 = i\int_0^{\pi} \cos t \, e^{it}\,dt$$

$$\cos t = \frac{e^{it} + e^{-it}}{2} \text{ so}$$

$$I_1 = \frac{i}{2}\int_0^{\pi} (e^{it} + e^{-it}) e^{it}\,dt$$

$$= \frac{i}{2}\int_0^{\pi} e^{2it} + 1 \; dt$$

$$= \frac{i}{2}\left[\frac{e^{2it}}{2i} + t\right]_0^{\pi}$$

$$= \frac{i}{2}\left[\cancel{\frac{1}{2i}} + \pi - \cancel{\frac{1}{2i}} - 0\right]$$

$$= \underline{\frac{\pi}{2} i}$$

$$I_2 = \int_0^2 (-1+t)\cdot 1\,dt$$

$$= \left[-t + \frac{1}{2}t^2\right]_0^2$$

$$= (-2+2+0-0)$$

$$= 0$$

So $$\int_C \mathrm{Re}(z)\,dz = \frac{\pi}{2}i + 0 = \frac{\pi}{2}i$$

27) $f(z) = \frac{\cos z}{z}$

C is the intersection of $|z|=1$ and $|z|=3$, but this doesn't exist so

$$\underline{\int_C f(z)\,dz = 0}$$