Problem Set 3

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Problem 1

$$\dot{\underline{x}} = A\underline{x} + Bu$$

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

a) Controllability matrix:

$$\mathcal{C} = \begin{pmatrix} B & AB & A^2B \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 8 \\ 0 & 4 & 32 \\ 1 & 5 & 25 \end{pmatrix}$$

Since $\text{rank}(\mathcal{C}) = 3$ the system is controllable.

b) $u(t) = -K\underline{x}(t) = -k_1x_1 - k_2x_2 - k_3x_3$

$$\bar{A} = A - BK$$

$$= \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ -k_1 & -k_2 & 5-k_3 \end{pmatrix}$$

The characteristic polynomial of $\bar{A}$ is

$$\Delta(\lambda) = \det \bar{A}$$

$$= \cdots \text{ maple } <3 \cdots$$

$$= \lambda^3 + (K_3 - 9)\lambda^2 + (23 + 4K_2 - 4K_3)\lambda + 8K_1 - 4K_2 + 3K_3 - 15$$

c) We want to have $\bar{\lambda}_1 = -1$, $\bar{\lambda}_2 = -2$, $\bar{\lambda}_3 = -3$.

That means that

$$\Delta(\lambda) = (\lambda - \bar{\lambda}_1)(\lambda - \bar{\lambda}_2)(\lambda - \bar{\lambda}_3)$$

$$= \lambda^3 + (-\bar{\lambda}_3 - \bar{\lambda}_2 - \bar{\lambda}_1)\lambda^2 + (\bar{\lambda}_2\bar{\lambda}_3 + \bar{\lambda}_1\bar{\lambda}_2 + \bar{\lambda}_1\bar{\lambda}_3)\lambda + (-\bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3)$$

Equating this with the polynomial from (b) we get

$$K_3 - 9 = -\bar{\lambda}_1 - \bar{\lambda}_2 - \bar{\lambda}_3 = \alpha_1$$

$$23 + 4K_2 - 4K_3 = \bar{\lambda}_1\bar{\lambda}_2 + \bar{\lambda}_2\bar{\lambda}_3 + \bar{\lambda}_1\bar{\lambda}_3 = \alpha_2$$

$$8K_1 - 4K_2 + 3K_3 - 15 = -\bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3 = \alpha_3$$

to simplify calculations

We then have

$$K_3 = \alpha_1 + 9$$

$$\Rightarrow \quad K_2 = \frac{1}{4}\left(\alpha_2 + 4K_3 - 23\right)$$

$$= \frac{1}{4}\left(\alpha_2 + 4\alpha_1 + 13\right)$$

$$\Rightarrow \quad K_1 = \frac{1}{8}\left(\alpha_3 + 4K_2 - 3K_3 + 15\right)$$

$$= \frac{1}{8}\left(\alpha_3 + \alpha_2 + 4\alpha_1 + 13 - 3\alpha_1 - 27 + 15\right)$$

$$= \frac{1}{8}\left(\alpha_1 + \alpha_2 + \alpha_3 + 1\right)$$

For our case with $\bar{\lambda}_1 = -1, \bar{\lambda}_2 = -2, \bar{\lambda}_3 = -3$ we get $\alpha_1 = 6$, $\alpha_2 = 11$, $\alpha_3 = 6$.

This gives

$$K_1 = \frac{1}{8}\left(6 + 11 + 6 + 1\right) = 3$$

$$K_2 = \frac{1}{4}\left(4 \cdot 6 + 11 + 13\right) = 12$$

$$K_3 = 6 + 9 = 15$$

So

$$\underline{\underline{K = [3, 12, 15]}}$$

Problem 2

$$\ddot{y} + \dot{y} + 2y = 2u$$

a) Write $x_1 = y$, $x_2 = \dot{y}$.

Then

$$\dot{x}_1 = \dot{y} = x_2$$

and

$$\dot{x}_2 = \ddot{y} = -\dot{y} - 2y + 2u$$
$$= -x_2 - 2x_1 + 2u$$
$$= -2x_1 - x_2 + 2u$$

It is then clear that

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

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b) We want $u(t) = u_{eq} + \tilde{u}(t)$ where u_{eq} satisfies

$$0 = A\underline{x}_{eq} + Bu_{eq} \qquad (1)$$

$$r = C\underline{x}_{eq} \qquad (2)$$

(1) gives

$$\begin{aligned}
\underline{x}_{eq} &= -A^{-1}Bu_{eq} \\
&= -\begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 2 \end{pmatrix} u_{eq} \\
&= -\frac{1}{2}\begin{pmatrix} -1 & -1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} u_{eq} \\
&= -\frac{1}{2}\begin{pmatrix} -2 \\ 0 \end{pmatrix} u_{eq} \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_{eq} \\
&= \underline{\underline{F u_{eq}, \quad \underline{F} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}}
\end{aligned}$$

Inserting the result from (1) in (2) we get

$$\begin{aligned}
r = C\underline{x}_{eq} &= CFu_{eq} \\
&= \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} u_{eq} = 1 \cdot u_{eq} \\
&= \underline{G u_{eq}, \quad G = 1}
\end{aligned}$$

c) $\underline{x} = \underline{x}_{eq} + \underline{\tilde{x}}$, $y = r + \tilde{y}$

Using this in the state-space-equation gives

$$\underline{\dot{x}} = \underbrace{\cancel{\underline{\dot{x}}_{eq}}}_{\text{constant}} + \underline{\dot{\tilde{x}}} = \underline{\dot{\tilde{x}}}$$

$$\Rightarrow \underline{\dot{\tilde{x}}} = A(\underline{x}_{eq} + \underline{\tilde{x}}) + B(u_{eq} + \tilde{u})$$

$$= \underbrace{\cancel{(A\underline{x}_{eq} + Bu_{eq})}}_{=0,\ \text{equilibrium}} + A\underline{\tilde{x}} + B\tilde{u}$$

$$= A\underline{\tilde{x}} + B\tilde{u}$$

$$r + \tilde{y} = C(\underline{x}_{eq} + \underline{\tilde{x}})$$

$$= C\underline{x}_{eq} + C\underline{\tilde{x}}$$

$$= r + C\underline{\tilde{x}}$$

$$\Rightarrow \tilde{y} = C\underline{\tilde{x}}$$

d) $\tilde{u} = -\underline{K}\,\underline{\tilde{x}}$, $\underline{K} = R^{-1}B^TP$

where P solves

$$A^TP + PA - PBR^{-1}B^TP + Q = 0 \qquad (3)$$

We know that we can write $P = \begin{pmatrix} P_1 & P_2 \\ P_2 & P_3 \end{pmatrix}$

Writing out (3):

$$\begin{pmatrix} 0 & -2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} P_1 & P_2 \\ P_2 & P_3 \end{pmatrix} + \begin{pmatrix} P_1 & P_2 \\ P_2 & P_3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix} - \begin{pmatrix} P_1 & P_2 \\ P_2 & P_3 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} 4^{-1} \begin{pmatrix} 0 & 2 \end{pmatrix}\begin{pmatrix} P_1 & P_2 \\ P_2 & P_3 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} = 0$$

$$\begin{pmatrix} -2P_2 & -2P_3 \\ P_1 - P_2 & P_2 - P_3 \end{pmatrix} + \begin{pmatrix} -2P_2 & P_1 - P_2 \\ -2P_3 & P_2 - P_3 \end{pmatrix} - \begin{pmatrix} P_1 & P_2 \\ P_2 & P_3 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} P_1 & P_2 \\ P_2 & P_3 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} = 0$$

$$\text{———} \quad || \quad \text{———} - \begin{pmatrix} 0 & P_2 \\ 0 & P_3 \end{pmatrix}\begin{pmatrix} P_1 & P_2 \\ P_2 & P_3 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} = 0$$

$$\text{———} \quad || \quad \text{———} - \begin{pmatrix} P_2^2 & P_2P_3 \\ P_2P_3 & P_3^2 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} = 0$$

From this we get 4 equations for P_1, P_2, P_3.

(4) $-2P_2 - 2P_2 - P_2^2 + 5 = 0$

$\Leftrightarrow P_2^2 + 4P_2 - 5 = 0$

$\Leftrightarrow (P_2 - 1)(P_2 + 5) = 0$

(5) $-2P_3 + P_1 - P_2 - P_2 P_3 = 0$

$\Leftrightarrow P_1 - P_2 - 2P_3 - P_2 P_3 = 0$

$\Leftrightarrow P_1 - P_2 - (2 + P_2) P_3 = 0$

$\Leftrightarrow P_1 = P_2 + (2 + P_2) P_3$

(6) $P_2 - P_3 + P_2 - P_3 - P_3^2 + 1 = 0$

$\Leftrightarrow P_3^2 + 2P_3 - (1 + 2P_2) = 0$

$P_2 = 1$:

$$P_3^2 + 2P_3 - 3 = 0$$

$$P_3 = \frac{-2 \pm \sqrt{4 - 4 \cdot (-3)}}{2} = -1 \pm 2 = -3, 1$$

$P_2 = -5$:

$$P_3^2 + 2P_3 + 9 = 0 \quad \times$$

no real solutions!

Eq (5) then gives

$$p_1 = p_2 + (2+p_2)\,p_3$$
$$= 1 + (2+1)\,p_3$$
$$= 1 + 3p_3$$

p_3	p_1
-3	-8
1	4

So we have two solutions

$$P = \begin{pmatrix} -8 & 1 \\ 1 & -3 \end{pmatrix}, \quad P = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$$

We require P to be positive definite and since P is Hermitian this is equivalent to requiring positive eigenvalues.

Since clearly $\begin{pmatrix} -8 & 1 \\ 1 & -3 \end{pmatrix}$ has negative eigenvalues we choose

$$\underline{\underline{P = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}}}$$

(which has positive eigenvalues).

Using $K = R^{-1}B^TP$ we get

$$K = \frac{1}{4}\begin{pmatrix} 0 & 2 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 2 & 2 \end{pmatrix} = \underline{\underline{\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}}}$$

e) With $u(t) = -Kx(t) + pr(t)$ we get

$$\dot{x} = (A - BK)x + Bpr(t)$$

Assuming stability and taking $t \to \infty$ we get

$$0 = (A - BK)x_{eq} + Bpr \quad , r = \lim_{t\to\infty} r(t)$$

$$= (A - BK)Fr + Bpr$$

$$\Leftrightarrow (BK - A)F = Bp$$

$$(BK - A)F = \left(\begin{pmatrix} 0 \\ 2 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix}\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \left(\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix}\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

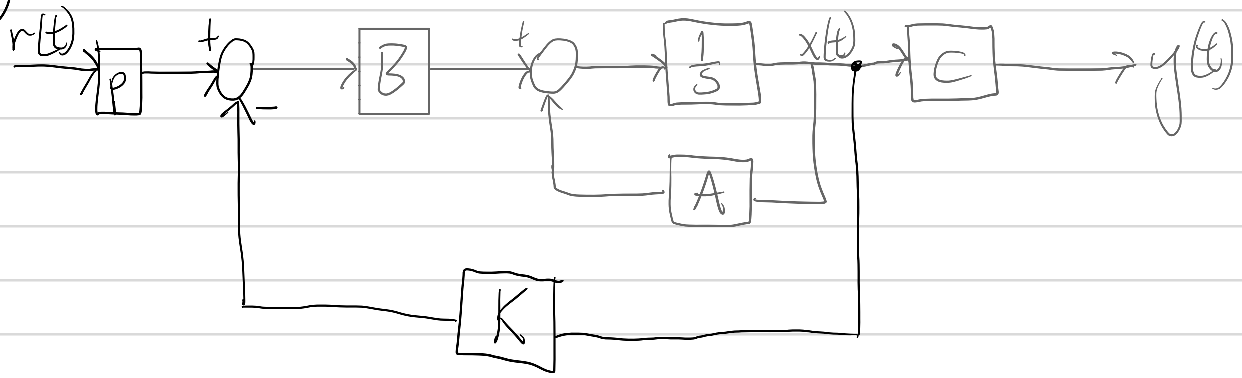
$$= \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 \\ 3 \end{pmatrix} = Bp = \begin{pmatrix} 0 \\ 2 \end{pmatrix} p$$

$$\Rightarrow \underline{\underline{p = \frac{3}{2}}}$$

$$\underline{\underline{u(t) = -\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix} x(t) + \frac{3}{2} r(t)}}$$

f)



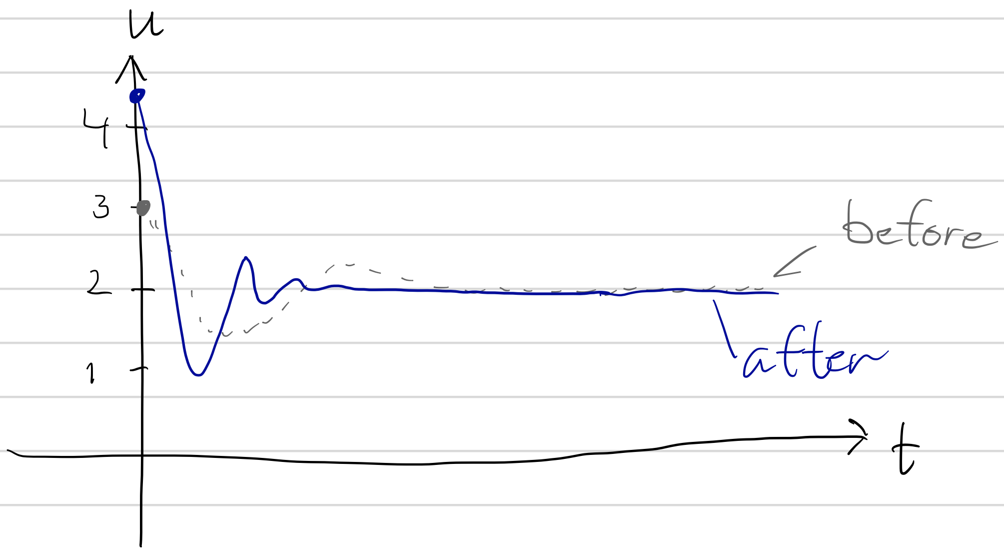
g) Q and R can be altered in two complementary ways.

1) Q can be increased such that the regulator will more heavily prioritize getting x to the referance value.

Specifically $Q = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, with $a > 5$ will give better results.

2) R can be decreased such that more "thrust" is used.

Making these changes will increase u(t) such that is has a higher initial value and also a greater dip beneath the referance.



Problem 3

a) The controllability matrix $\mathcal{C}$ is

$$\mathcal{C} = [B \quad AK]$$
$$= \begin{bmatrix} 2 & \begin{pmatrix} -4 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix} \\ 0 & \end{bmatrix}$$
$$= \begin{bmatrix} 2 & -8 \\ 0 & 6 \end{bmatrix}.$$

Rank $\mathcal{C} = 2$ so the system is observable.

b) With $e(t) = x(t) - \hat{x}(t)$ we have

$$\dot{e} = \dot{x} - \dot{\hat{x}} = [Ax + \cancel{Bu}] - [A\hat{x} + \cancel{Bu} + L(y - C\hat{x} - Du)]$$
$$= A(x - \hat{x}) - (Ly - C(x-e) - Du)$$
$$= Ae - LCe - \cancel{L(y - Cx - Du)}^{=0}$$
$$= (A - LC)e$$

c) Note that since $C = \begin{pmatrix} 0 & 1 \end{pmatrix}$ is 1×2 we require that L is 2×1 such that LC is 2×2 (same as A).

$$\Rightarrow \quad L = \begin{pmatrix} L_1 & L_2 \end{pmatrix}^T.$$

This gives

$$A - LC = \begin{pmatrix} -4 & 0 \\ 3 & -1 \end{pmatrix} - \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & -L_1 \\ 3 & -1-L_2 \end{pmatrix}$$

The characteristic polynomial is then

$$\Delta(\lambda) = (-4-\lambda)(-1-L_2-\lambda) + 3L_1$$

$$= (\lambda+4)(\lambda+1+L_2) + 3L_1$$

$$= \lambda^2 + (5+L_2)\lambda + 4 + 4L_2 + 3L_1$$

We want the eigenvalues to be -8 and -10 so

$$\Delta(\lambda) = (\lambda+8)(\lambda+10) = \lambda^2 + 18\lambda + 80$$

$$\Rightarrow \quad 5 + L_2 = 18 \Rightarrow L_2 = 13$$

$$4 + 4L_2 + 3L_1 = 80 \Rightarrow L_1 = \frac{1}{3}(80 - 4 - 4 \cdot 13)$$

$$= 8$$

So we get $\underline{\underline{L = \begin{pmatrix} 8 & 13 \end{pmatrix}^T}}$

c)