Solution Suggestion
Exam - TTK4115 Linear System Theory
2. December, 2016

November 22, 2016

Problem 1

The plant is given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{1}$$

a) We check for controllability by checking rank

$$\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

The rank of this matrix is 2, which is full rank. The plant is controllable. Let the state feedback be given by $u = -[k_1, k_2]\mathbf{x}$ so that

$$\mathbf{A} - \mathbf{bk} = \begin{bmatrix} 0 & 1 \\ -k_1 & 1 - k_2 \end{bmatrix} \tag{2}$$

The characteristic polynomial can be computed as

$$|\lambda\mathbb{I} - (\mathbf{A} - \mathbf{bk})| = \begin{vmatrix} \lambda & -1 \\ k_1 & \lambda + k_2 - 1 \end{vmatrix} = \lambda^2 + (k_2 - 1)\lambda + k_1 \tag{3}$$

Comparison to the desired polynomial $(\lambda + 1)^2 = \lambda^2 + 2\lambda + 1$ yields the appropriate gains $k_1 = 1$ and $k_2 = 3$.

b) The system matrix whose stability is in question reads as

$$\mathbf{A}_{\text{cl}} = \mathbf{A} - \mathbf{bk}' = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \tag{4}$$

Choose $\mathbf{N} = \mathbb{I} \succ 0$ and define a symmetric, but as yet unknown, matrix

$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} \tag{5}$$

Verify that

$$\mathbf{A}_{\text{cl}}^{\mathsf{T}}\mathbf{M} + \mathbf{M}\mathbf{A}_{\text{cl}} + \mathbf{N} = \begin{bmatrix} 1 - 4m_{12} & m_{11} - 2\,(m_{12} + m_{22}) \\ m_{11} - 2\,(m_{12} + m_{22}) & 2m_{12} - 4m_{22} + 1 \end{bmatrix} = \mathbf{0} \tag{6}$$

Simple arithmetic yields the answer $m_{12} = 1/4 \Rightarrow m_{22} = 3/8 \Rightarrow m_{22} = 3/8 \Rightarrow m_{11} = 5/4$. This leaves the verification of whether $\mathbf{M}$ is positive definite or not. The method of principle minors gives an answer in the affirmative[1]

$$m_{11} = 5/4 > 0, \quad \begin{vmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{vmatrix} = \begin{vmatrix} 5/4 & 1/4 \\ 1/4 & 3/8 \end{vmatrix} = \left(\frac{1}{8}\right)^2 \begin{vmatrix} 10 & 2 \\ 2 & 3 \end{vmatrix} = \frac{26}{64} = \frac{13}{32} > 0 \tag{7}$$

The system is asymptotically stable!

c) The plant is observable, verify full rank below

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{8}$$

Observability implies that a Luenberger observer with arbitrary convergence rate can be designed

$$\dot{\hat{\mathbf{x}}} = \underbrace{\mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u}}_{\text{Simulation/Prediction}} + \underbrace{\mathbf{L}(\mathbf{y} - \hat{\mathbf{y}})}_{\text{Correction}} \tag{9}$$

The estimator combines simulation based prediction with a correction term garnered by comparing the true output with the estimated one. It is not possible to use an open loop observer for the given plant. The observer

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} \tag{10}$$

would be unstable since $\mathbf{A}$ is equipped with eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 1$.

d) A simple first order process is supplied as

$$\dot{x} = u, \quad y = x \tag{11}$$

along with an output feedback controller

$$\dot{\hat{x}} = -(k + l)\hat{x} + ly \tag{12a}$$
$$u = -k\hat{x} \tag{12b}$$

Define the estimation error as

$$e = x - \hat{x} \tag{13}$$

Expressing the equations in terms of the error yields the plant equation

$$\dot{x} = -kx + ke, \quad \dot{e} = -le \tag{14}$$

One eigenvalue is located at $-k$ whereas the other is found at $-l$. The lack of a coupling indicates the the observer gain can be chosen independently of the feedback gain k. The result extends to state feedback from estimated states in general, and is known as the separation principle.

e) The output feedback

$$u = -kyr \tag{15}$$

is to be compared to (12). The respective transfer functions are readily computed as

$$\frac{u}{y}(s) = -k, \quad \frac{u}{y}(s) = -\frac{lk}{s + (k + l)} \tag{16}$$

The transfer function describing the dynamics of (12) introduces low-pass action, and can hence be expected to yield a smoother output when y is polluted by high-frequency noise.

[1] $|c\mathbf{A}| = c^n|\mathbf{A}|$

Problem 2

This task considers PID regulators, exemplified by

$$u(s) = \left(K_p + K_d s + \frac{K_i}{s}\right) e(s) \tag{17}$$

a) The following formulation is preferable for time-domain implemenation

$$u(s) = \left(K_p + \frac{K_d s}{\tau s + 1} + \frac{K_i}{s}\right) e(s) \tag{18}$$

since it is <u>proper</u>, as opposed to (17). It is not possible to realize improper transfer functions on a standard state-space formulation.

b) A rewriting of (18) yields the transfer function

$$G(s) = \frac{u}{e}(s) = \frac{(K_i - K_d/\tau^2)s + K_i/\tau}{s^2 + s/\tau} + \left(K_p + \frac{K_d}{\tau}\right) \tag{19}$$

Using the standard realization formula supplied in the appendix, one has

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}e, \quad u = \mathbf{c}\mathbf{x} + de \tag{20}$$

where

$$\mathbf{A} = \begin{bmatrix} -\frac{1}{\tau} & 0 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} K_i - \frac{K_d}{\tau^2} & \frac{K_i}{\tau} \end{bmatrix}, \quad d = K_p + \frac{K_d}{\tau} \tag{21}$$

c) The state-space implementation

$$\dot{x}(t) = e(t), \quad u(t) = K_i x(t) + K_p e(t) \tag{22}$$

Laplace transforms as $x(s) = e(s)/s$ leading to

$$u(s) = \left(K_p + \frac{K_i}{s}\right) e(s) \tag{23}$$

This is a PI-regulator, as desired. The system (22) is to be discretized. Under the assumption that $e(t)$ is approximately constant over the time interval $t \in [(k-1)T, kT]$ integration yields

$$\int_{kT}^{(k+1)T} \dot{x}(t')\ dt' = x[k+1] - x[k] = \int_{kT}^{(k+1)T} e(t')\ dt' \simeq \int_{kT}^{(k+1)T} e[k]\ dt' = Te[k] \tag{24}$$

Hence the discretized model

$$x[k+1] = x[k] + Te[k], \quad u[k] = K_i x[k] + K_p e[k] \tag{25}$$

Comparison to

$$x[k+1] = a_d x[k] + b_d e[k], \quad u[k] = c_d x[k] + d_d e[k] \tag{26}$$

shows that $a_d = 1,\ b_d = T,\ c_d = K_i$ and $d_d = K_P$.

Problem 3

a) The continuous measurement equation for the accelerometer reads as

$$\alpha(t) = \ddot{x}(t) + \beta(t) \tag{27}$$

Integrating both sides over the sampling interval yields

$$\int_{kT}^{(k+1)T} \alpha(t')\ dt' = \int_{kT}^{(k+1)T} \ddot{x}(t')\ dt' + \int_{kT}^{(k+1)T} \beta(t')\ dt' \tag{28}$$

Using the definition and approximation

$$\bar{\alpha}[k] \triangleq \frac{1}{T} \int_{kT}^{(k+1)T} \alpha(t')\ dt', \quad \beta[k] \simeq \frac{1}{T} \int_{kT}^{(k+1)T} \beta(t')\ dt' \tag{29}$$

allows

$$T\bar{\alpha}[k] = \dot{x}[k+1] - \dot{x}[k] + T\beta[k] \tag{30}$$

Rearranging this relation yields the desired result.

b) The equations describing the discrete dynamics are summarized by

$$\beta[k+1] = \beta[k] + w[k] \tag{31}$$

$$\dot{x}[k+1] = \dot{x}[k] - T\beta[k] + T\bar{\alpha}[k] \tag{32}$$

$$x[k+1] = x[k] + T\dot{x}[k] \tag{33}$$

Here $\bar{\alpha}[k]$ is a deterministic, known, input. With this proviso, the appropriate state-space formulation reads as

$$\mathbf{x}[k+1] = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -T & 1 & 0 \\ 0 & T & 1 \end{bmatrix}}_{\mathbf{A}_d} \mathbf{x}[k] + \underbrace{\begin{bmatrix} 0 \\ T \\ 0 \end{bmatrix}}_{\mathbf{b}_d} \bar{\alpha}[k] + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{g}_d} w[k], \quad \mathbf{x}[k] = \begin{bmatrix} \beta[k] \\ \dot{x}[k] \\ x[k] \end{bmatrix} \tag{34}$$

where the measurement is obtained from

$$y[k] = \underbrace{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}}_{\mathbf{c}_d} \mathbf{x}[k] + v[k] \tag{35}$$

The discrete process is observable, which is verified by full column rank of

$$\mathcal{O}_d = \begin{bmatrix} 0 & 0 & 1 \\ 0 & T & 1 \\ -T^2 & 2T & 1 \end{bmatrix} \tag{36}$$

This implies that the state-vector $\mathbf{x}[k]$ can, in principle, be determined from the measurement.

c) In this problem, the Kalman filter must used to obtain the optimal estimate $\hat{\mathbf{x}}[k]$. The optimal estimate $\hat{\mathbf{x}}[k-1]$ and covariance matrix $\mathbf{P}[k-1]$ from the previous time-step are available.

The a-priori estimate at $t = kT$ is given by

$$\hat{\mathbf{x}}^-[k] = \mathbf{A}_d\hat{\mathbf{x}}[k-1] + \mathbf{b}_d\bar{\alpha}[k-1] \tag{37}$$

whereas the a-priori covariance matrix follows from

$$\mathbf{P}^-[k] = \mathbf{A}_d\mathbf{P}[k-1]\mathbf{A}_d^\mathsf{T} + q\mathbf{g}\mathbf{g}^\mathsf{T} \tag{38}$$

Note here that $\mathbf{Q}_d = q\mathbf{g}\mathbf{g}^\mathsf{T}$.

Since $R_d = \sigma^2$, the Kalman gain follows as

$$\mathbf{L}[k] = \mathbf{P}^-[k]\mathbf{c}^\mathsf{T}(\mathbf{c}\mathbf{P}^-[k]\mathbf{c}^\mathsf{T} + \sigma^2)^{-1} \tag{39}$$

The optimal estimate can now be found through the computation

$$\hat{\mathbf{x}}[k] = \hat{\mathbf{x}}^-[k] + \mathbf{L}[k](\mathbf{y}[k] - \mathbf{c}\hat{\mathbf{x}}^-[k]) \tag{40}$$

Problem 5

a) The system matrices (here scalar) of $\tau\dot{x} + x = u$ are given by

$$A = -\frac{1}{\tau}, \quad B = \frac{1}{\tau} \tag{41}$$

Letting $Q = 1$ and $R = \rho$, the Riccati equation (found in the appendix) reduces, after slight rearrangement, to

$$\frac{1}{\rho}\left(\frac{P}{\tau}\right)^2 + 2\left(\frac{P}{\tau}\right) - 1 = 0 \tag{42}$$

The ABC-formula gives the two solutions

$$\frac{P}{\tau} = \frac{-2 \pm \sqrt{2^2 + 4/\rho}}{2(1/\rho)} = \rho\left(-1 \pm \sqrt{1 + \frac{1}{\rho}}\right) \tag{43}$$

The matrix $\mathbf{P}$ must be positive definite, in the scalar case $P > 0$. Two cases are of importance namely $\tau < 0$ and $\tau > 0$. By picking the solution depending on the sign of τ, one has

$$P = \begin{cases} \tau\rho\left(-1 + \sqrt{1 + \frac{1}{\rho}}\right) & \tau > 0 \\ \tau\rho\left(-1 - \sqrt{1 + \frac{1}{\rho}}\right) & \tau < 0 \end{cases} \tag{44}$$

Simplifying, this is equal to

$$P = \tau\rho\left(-1 + \mathrm{sign}(\tau)\sqrt{1 + \frac{1}{\rho}}\right) > 0 \tag{45}$$

The optimal gain (found in the appendix) can now be furnished by

$$k = \frac{P}{\tau\rho} = -1 + \mathrm{sign}(\tau)\sqrt{1 + \frac{1}{\rho}} \tag{46}$$

The closed loop dynamics follows by inserting the optimal feedback into the dynamic model

$$\tau\dot{x} + x = -kx \Rightarrow |\tau|\dot{x} + \left(\sqrt{1 + \frac{1}{\rho}}\right)x \tag{47}$$

b) When the cost ρ is made vary large $\rho \to \infty$, the dynamics reduce to

$$|\tau|\dot{x} + x = 0 \tag{48}$$

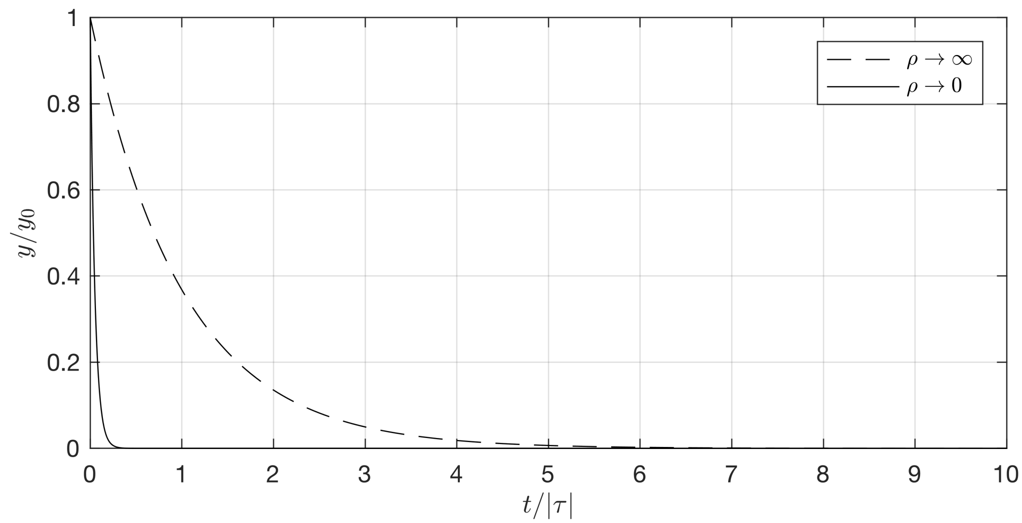
whereas the input becomes

$$u = (1 - \mathrm{sign}(\tau))\,x \tag{49}$$

A stable system yields the expected result $u = 0$ whereas an unstable system will produce the feedback $u = -2x$ capable of stabilizing the plant. The reason is that the cost associated with an unstable plant, whose solution grows to infinity, is far larger than the cost associated with an expensive input. The LQR will always render the closed loop dynamics stable.

c) The solution of the closed loop dynamics are given by

$$y(t) = \exp\left(-\frac{t}{|\tau|}\sqrt{1+\frac{1}{\rho}}\right) y_0 \tag{50}$$



As the input is made cheap, the solution will converge very fast to zero. Expensive control will yield the slower response shown in the figure.

d) Integral effect can be included by augmenting the state-space. Let an additional state be equipped with the dynamics

$$\dot{x}_a(t) = x(t) \quad \Rightarrow x_a(t) = \int_0^t x(t')\ dt' + x_a(0) \tag{51}$$

An augmented state-space model reads as

$$\begin{bmatrix} \dot{x} \\ \dot{x}_a \end{bmatrix} = \overbrace{\begin{bmatrix} -1/\tau & 0 \\ 1 & 0 \end{bmatrix}}^{\mathbf{A}} \begin{bmatrix} x \\ x_a \end{bmatrix} + \overbrace{\begin{bmatrix} 1/\tau \\ 0 \end{bmatrix}}^{\mathbf{b}} u \tag{52}$$

Since the integral represented by x_a must be weighted to produce a nonzero integral gain, the cost function could read as

$$J_{\mathrm{PI}} = \int_0^\infty x^2 + q_a x_a^2 + \rho u^2\ dt \tag{53}$$

The appropriate $\mathbf{Q}$ matrix would read as diag$[1,\ q_a]$.