Linear Systemteori – 2

Problem 1

a) We want to rewrite the state-space equation

$$\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t)$$

$$y(t) = C\underline{x}(t) + D\underline{u}(t)$$

to a discrete variant with values at $t = kT$. Assuming that the input $u(t)$ can be written as

$$\underline{u}(t) = \underline{u}(kT) = \underline{u}[k]$$

for $k = 0, 1, \dots$ and T = constant, we have

$$x[k] = e^{AkT}\underline{x}[0] + \int_0^{kT} e^{A(kT-t)} B\underline{u}(t)\,dt$$

which gives

$$x[k+1] = e^{A(k+1)T}\underline{x}[0] + \int_0^{(k+1)T} e^{A((k+1)T-t)} B\underline{u}(t)\,dt$$

$$= e^{AT}e^{AkT}\underline{x}[0] + e^{AT}\int_0^{kT} e^{A(kT-t)} B\underline{u}(t)\,dt$$

$$+ \int_{kT}^{kT+T} e^{A(kT+T-t)} B\underline{u}(t)\,dt$$

By our assumption, $u(t) = u[k]$ is constant in the last integral so we get

$$x[k+1] = e^{AT}\left(e^{AkT}\underline{x}[0] + \int_0^{kT} e^{A(kT-t)} B\, \underline{u}(t)\, dt\right) + \left[\int_{kT}^{kT+T} e^{A(kT+T-t)} dt\right] B\, \underline{u}[k]$$

$$= e^{AT}\underline{x}[k] + \underbrace{\int_0^{T} e^{A(T-t)} dt\, B}\, \underline{u}[k]$$

$$= A_d\, \underline{x}[k] + B_d\, \underline{u}[k]$$

For $y[k]$ we get

$$y[k] = y(kT) = C\underline{x}(kT) + D\underline{u}(kT)$$
$$= C_d \underline{x}[k] + D_d \underline{u}[k]$$

So in summary we have

$$A_d = e^{AT}, \quad B_d = \int_0^T e^{A(T-t)} dt\, B$$
$$C_d = C, \quad D_d = D$$

Computing A_d, B_d we get

$$A_d = e^{AT}$$

Eigenvalues of A:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -2 & -2-\lambda \end{vmatrix}$$

$$\Rightarrow \quad 0 = (-\lambda)(-2-\lambda) + 2$$

$$\Leftrightarrow \quad 0 = \lambda^2 + 2\lambda + 2$$

$$\Rightarrow \begin{cases} \lambda_1 = \dfrac{-2 + \sqrt{4 - 4 \cdot 2}}{2} = -1 + i = \lambda \\ \lambda_2 = \bar{\lambda} \end{cases}$$

$$\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix} = \begin{pmatrix} -1+i & 0 \\ 0 & -1-i \end{pmatrix}$$

Eigenvectors:

$$(A - \lambda I)\underline{q} = 0$$

$$\Rightarrow \begin{pmatrix} 1-i & 1 \\ -2 & -1-i \end{pmatrix} \underline{q} = 0$$

Since $\underline{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$, we get that q must solve

$$(1-i)\, q_1 + q_2 = 0$$

$$\Rightarrow \quad q_1 = -\frac{1}{1-i} = -\frac{1}{2}(1+i) \quad , \quad q_2 = 1$$

So $\underline{q} = \begin{pmatrix} -\frac{1}{2}(1+i) \\ 1 \end{pmatrix}$ is the eigenvector for λ

and $\overline{\underline{q}} = \begin{pmatrix} -\frac{1}{2}(1-i) \\ 1 \end{pmatrix}$ is —— || —— for $\overline{\lambda}$.

Then $Q = [\, \underline{q} \mid \overline{\underline{q}} \,]$ lets us diagonalize A.

$$Q = \begin{pmatrix} -\frac{1}{2} - \frac{i}{2} & -\frac{1}{2} + \frac{i}{2} \\ 1 & 1 \end{pmatrix}$$

$$Q^{-1} = \underbrace{\frac{1}{-\frac{1}{2} - \frac{i}{2} + \frac{1}{2} - \frac{i}{2}}}_{=i} \begin{pmatrix} 1 & \frac{1}{2} - \frac{i}{2} \\ -1 & -\frac{1}{2} - \frac{i}{2} \end{pmatrix}$$

$$= \begin{pmatrix} i & \frac{1}{2} + \frac{i}{2} \\ -i & \frac{1}{2} - \frac{i}{2} \end{pmatrix}$$

We can now write

$$A = Q \Lambda Q^{-1}$$

where Λ is diagonal. This gives

$$e^{AT} = Q\, e^{\Lambda T} Q^{-1}$$

$$= Q \begin{pmatrix} e^{\lambda T} & 0 \\ 0 & e^{\bar{\lambda} T} \end{pmatrix} Q^{-1}$$

$$= Q \begin{pmatrix} e^{(-1+i)T} & 0 \\ 0 & e^{(-1-i)T} \end{pmatrix} Q^{-1}$$

$$= \begin{pmatrix} -\frac{1}{2} - \frac{i}{2} & -\frac{1}{2} + \frac{i}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{(-1+i)T} & 0 \\ 0 & e^{(-1-i)T} \end{pmatrix} Q^{-1}$$

$$= \begin{pmatrix} \left(-\frac{1}{2} - \frac{i}{2}\right) e^{(-1+i)T} & \left(-\frac{1}{2} + \frac{i}{2}\right) e^{(-1-i)T} \\ e^{(-1+i)T} & e^{(-1-i)T} \end{pmatrix} \begin{pmatrix} i & \frac{1}{2} + \frac{i}{2} \\ -i & \frac{1}{2} - \frac{i}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}$$

$$\Phi_{11} = \left(-\frac{1}{2} - \frac{i}{2}\right) i\, e^{(-1+i)T} - \left(-\frac{1}{2} + \frac{i}{2}\right) i\, e^{(-1-i)T}$$

$$= \frac{e^{-T}}{2}\left[e^{iT} + e^{-iT} + i\left(-e^{iT} + e^{-iT}\right)\right]$$

$$= e^{-T}\left(\cos T + \sin T\right)$$

$$\Phi_{12} = \left(-\frac{1}{2} - \frac{i}{2}\right)\left(\frac{1}{2} + \frac{i}{2}\right) e^{(-1+i)T} + \left(-\frac{1}{2} + \frac{i}{2}\right)\left(\frac{1}{2} - \frac{i}{2}\right) e^{(-1-i)T}$$

$$= -\frac{e^{-T}}{4}(1+i)^2 e^{iT} - \frac{e^{-T}}{4}(1-i)^2 e^{-iT}$$

$$= e^{-T}\left[-\frac{i}{2} e^{iT} + \frac{i}{2} e^{-iT}\right]$$

$$= e^{-T} \sin T$$

$$\Phi_{21} = i e^{(-1+i)T} - i e^{(-1-i)T}$$

$$= 2e^{-T}\left[\frac{i e^{iT}}{2} - \frac{i e^{-iT}}{2}\right]$$

$$= -2e^{-T}\sin T$$

$$\Phi_{22} = \left(\frac{1}{2} + \frac{i}{2}\right) e^{(-1+i)T} + \left(\frac{1}{2} - \frac{i}{2}\right) e^{(-1-i)T}$$

$$= e^{-T}\left[\cos T - \sin T\right]$$

With $T = \frac{\pi}{2}$ we then get

$$A_d = e^{AT}$$
$$= \begin{pmatrix} e^{-\frac{\pi}{2}} & e^{-\frac{\pi}{2}} \\ -2e^{-\frac{\pi}{2}} & -e^{-\frac{\pi}{2}} \end{pmatrix}$$

For B_d, we have

$$B_d = \int_0^T e^{A(T-t)} dt \, B$$
$$= \int_0^T e^{A(T-t)} B \, dt$$
$$= \int_0^T \begin{pmatrix} 0 & e^{-(T-t)} \sin(T-t) \\ 0 & e^{-(T-t)}(\cos(T-t) - \sin(T-t)) \end{pmatrix} dt$$
$$= \begin{pmatrix} 0 & -\frac{1}{2}e^{-T}\cos T - \frac{1}{2}e^{-T}\sin T + \frac{1}{2} \\ 0 & e^{-T}\sin T \end{pmatrix}$$

For $T = \frac{\pi}{2}$ we get

$$\underline{\underline{B_d = \begin{pmatrix} 0 & -\frac{1}{2}e^{-\frac{\pi}{2}} + \frac{1}{2} \\ 0 & e^{-\frac{\pi}{2}} \end{pmatrix}}}$$

$$\underline{\underline{C_d = C = \begin{pmatrix} 4 & 0 \end{pmatrix}}}$$

$$\underline{\underline{D_d = D = \begin{pmatrix} 0 & 0 \end{pmatrix}}}$$

Problem 2

a) A transfer matrix is realizable if all the transfer functions in the matrix are realizable, e.g. are proper.

So for all $\hat{g}(s)$ in $\hat{G}(s)$

$$\hat{g}(s) = \frac{n(s)}{d(s)}$$

with $\deg(n) \leq d(s)$

b) We verify that all of them are proper.

1) $$\hat{G}_{11} = \frac{s^2 + 4s + 2}{s^2 + 2s}$$

Proper since $\deg(s^2+4s+2) = 2 \leq 2 = \deg(s^2+2s)$

2) $$\hat{G}_{12} = \frac{3}{s+2}$$

Proper since $\deg(3) = 0 \leq 1 = \deg(s+2)$

3) $\hat{G}_{21} = 0 = 0/1$

Proper since $\deg(0) = 0 \leq 1 = \deg(1)$

4) $\hat{G}_{22} = \dfrac{2s^2}{s^2-4}$

Proper since $\deg(2s^2) = 2 \leq 2 = \deg(s^2-4)$

All of them are proper so $\hat{G}(s)$ is realizable.

c) We can write $\hat{G}$ as

$$\hat{G}(s) = \hat{G}_{sp}(s) + G_\infty \quad , \quad G_\infty = \lim_{s \to \infty} G(s)$$

where G_∞ is a constant matrix and $\hat{G}_{sp}$ is strictly proper.

The realization of G to a state-space model will then be $G_\infty = D$ so we can also write

$$\hat{G}(s) = \hat{G}_{sp}(s) + D$$

d) It is quite clear from looking at $\hat{G}$ that

$$\begin{aligned} d(s) &= s(s+2)(s-2) \\ &= s^3 + (2-2)s^2 + (2\cdot(-2))s + 0 \\ &= s^3 - 4s \end{aligned}$$

$\Rightarrow \quad \alpha_3 = \alpha_1 = 0, \ \alpha_2 = -4$

We first find $D = G_\infty$.

$$D = \lim_{s \to \infty} \hat{G}(s)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

This gives

$$\hat{G}_{sp}(s) = \hat{G}(s) - D$$

$$= \begin{pmatrix} \frac{s^2+4s+2}{s^2+2s} - 1 & \frac{3}{s+2} \\ 0 & \frac{2s^2}{s^2-4} - 2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{s^2+4s+2-s^2-2s}{s^2+2s} & \frac{3}{s+2} \\ 0 & \frac{2s^2-2s^2+8}{s^2-4} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2s+2}{s(s+2)} & \frac{3}{s+2} \\ 0 & \frac{8}{s^2-4} \end{pmatrix}$$

Factoring out $d(s)$ from $\hat{G}_{sp}(s)$ we get

$$\hat{G}_{sp}(s) = \frac{1}{s^3-4s}\begin{pmatrix} (2s+2)(s-2) & 3s(s-2) \\ 0 & 8s \end{pmatrix}$$

$$= \frac{1}{s^3-4s}\begin{pmatrix} 2s^2+2s-4 & 3s^2-6s \\ 0 & 8s \end{pmatrix}$$

$$= \frac{1}{s^3-4s}\left(\underbrace{\begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix}}_{N_1} s^2 + \underbrace{\begin{pmatrix} 2 & -6 \\ 0 & 8 \end{pmatrix}}_{N_2} s + \underbrace{\begin{pmatrix} -4 & 0 \\ 0 & 0 \end{pmatrix}}_{N_3}\right)$$

$$N_1 = \begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix},\ N_2 = \begin{pmatrix} 2 & -6 \\ 0 & 8 \end{pmatrix},\ N_3 = \begin{pmatrix} -4 & 0 \\ 0 & 0 \end{pmatrix}$$

e) Inserting values for α and N into the formula, we get

$$A = \left(\begin{array}{cc|cc|cc} 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right)$$

$$B = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$C = \left(\begin{array}{cc|cc|cc} 2 & 3 & -2 & -6 & -4 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 \end{array}\right)$$

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Problem 3

a) We are given $A = \begin{pmatrix} -2 & 4 \\ -1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$, $C = \begin{pmatrix} 1 & -1 \end{pmatrix}$, $D = 2$

$$T = \begin{pmatrix} 0 & -2 \\ 1 & -2 \end{pmatrix}$$

$$\left.\begin{aligned} \dot{\underline{x}} &= A\underline{x} + Bu \\ y &= C\underline{x} + Du \end{aligned}\right\} \quad (*)$$

Substituting the transform $\bar{\underline{x}} = T\underline{x} \Leftrightarrow \underline{x} = T^{-1}\bar{\underline{x}}$ into $(*)$, we get

$$T^{-1}\dot{\bar{\underline{x}}} = AT^{-1}\bar{\underline{x}} + Bu$$

$$y = CT^{-1}\underline{x} + Du$$

This gives

$$\bar{A} = TAT^{-1}, \quad \bar{B} = TB, \quad \bar{C} = CT^{-1}, \quad \bar{D} = D$$

We compute them:

$$\bar{A} = \begin{pmatrix} 0 & -2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -2 & 4 \\ -1 & 3 \end{pmatrix} \frac{1}{2} \begin{pmatrix} -1 & 2 \\ -1 & 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2 & -6 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ -1 & 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 4 & 4 \\ 0 & -2 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} 2 & 2 \\ 0 & -1 \end{pmatrix}}}$$

$$\bar{B} = \begin{pmatrix} 0 & -2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 8 \\ 2 \end{pmatrix} = \underline{\underline{\begin{pmatrix} -4 \\ 6 \end{pmatrix}}}$$

$$\bar{C} = \begin{pmatrix} 1 & -1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} -1 & 2 \\ -1 & 0 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 0 & 2 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} 0 & 1 \end{pmatrix}}}$$

$$\underline{\underline{\bar{D} = D = 2}}$$

b) Yes, since they can be transformed into each other, both from $x \to \bar{x}$ with $\bar{x} = Tx$ and also from $\bar{x} \to x$ with $x = T^{-1}\bar{x}$.

Since they are algebraically equivalent, they must also be zero-state equivalent.

Problem 4

a) The system is controllable if the controllabillity matrix $\mathcal{C} = [B \ \ AB]$ has rank $= 2$

$$\mathcal{C} = \begin{pmatrix} 0 & 0 & -6 & -6 \\ 2 & 2 & -10 & -10 \end{pmatrix}$$

which clearly has rank $\mathcal{C} = 2$.

So the system <u>is</u> controllable.

b) $\Delta(\lambda) = \det(A - \lambda \mathbb{I}) = 0$

$$\Rightarrow (2-\lambda)(-5-\lambda) + 4 \cdot 3 = 0$$

$$\Leftrightarrow \lambda^2 + 3\lambda - 10 + 12 = 0$$

$$\Leftrightarrow \lambda^2 + 3\lambda + 2 = 0$$

$$\Rightarrow \lambda = \frac{-3 \pm \sqrt{9 - 4 \cdot 2}}{2}$$

$$= -\frac{3}{2} \pm \frac{1}{2}$$

$$= -1, -2$$

c) $$(A - \lambda I \ \ B) = \begin{pmatrix} 2-\lambda & -3 & 0 & 0 \\ 4 & -5-\lambda & 2 & 2 \end{pmatrix}$$

$\lambda = -1$:

$$\text{Rank}\begin{pmatrix} 3 & -3 & 0 & 0 \\ 4 & -4 & 2 & 2 \end{pmatrix}$$

$$= \text{Rank}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 4 & 0 & 2 & 0 \end{pmatrix}$$

$$= 2 \quad \checkmark$$

$\lambda = -2$:

$$\text{Rank}\begin{pmatrix} 4 & -3 & 0 & 0 \\ 4 & -3 & 2 & 2 \end{pmatrix}$$

$$= 2 \quad \checkmark$$

c) We want to solve

$$AW + WA^T = -BB^T$$

for W

$$-BB^T = -\begin{pmatrix} 0 & 0 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -8 \end{pmatrix}$$

Writing $W = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ and expanding the Lyapunov-equation we get

$$\begin{pmatrix} 2 & -3 \\ 4 & -5 \end{pmatrix} W + W\begin{pmatrix} 2 & 4 \\ -3 & -5 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -8 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 2a - 3b + 2a - 3b = 0 \\ 4a - 5b + 2b - 3c = 0 \\ 2b - 3c + 4a - 5b = 0 \\ 4b - 5c + 4b - 5c = -8 \end{cases}$$

$$\Leftrightarrow \begin{cases} 4a - 6b = 0 \\ 4a - 3b - 3c = 0 \\ 4a - 3b - 3c = 0 \\ 8b - 10c = -8 \end{cases}$$

$$\Leftrightarrow \left(\begin{array}{ccc|c} 4 & -6 & & \\ 4 & -3 & -3 & \\ 4 & -3 & -3 & \\ & 8 & -10 & -8 \end{array}\right)$$ (Empty cell = 0)

$$\sim \left(\begin{array}{ccc|c} 4 & -6 & & \\ & 1 & -1 & \\ & & & \\ & 4 & -5 & -4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 4 & -6 & & \\ & 1 & -1 & \\ & & & \\ & & 1 & 4 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 4 & -6 & & \\ & 1 & & 4 \\ & & & \\ & & 1 & 4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 4 & \cdot & \cdot & 24 \\ & 1 & & 4 \\ & & & \\ & & & 4 \end{array}\right)$$

$$\Rightarrow \quad a = 6$$
$$b = 4$$
$$c = 4$$

$$\Rightarrow \quad W = \begin{pmatrix} 6 & 4 \\ 4 & 4 \end{pmatrix}$$

Since W is non-singular ($\det W \neq 0$) and the eigenvalues of A are strictly negative ($\lambda = -1, -2$), the system is controllable.