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Eksamen - TTK 4115 Lineær systemteori
Exam - TTK 4115 Linear system theory

11. desember 2009, 09:00 – 13:00

Hjelpemidler: D - Ingen trykte eller håndskrevne hjelpemidler tillatt. Bestemt, enkel kalkulator tillatt.

Supporting materials: D - No printed or handwritten material allowed. Specific, simple calculator allowed.

Oppgave 1 (20 %)

Gitt
Given:

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= Cx + Du \end{aligned}$$

a)
La (A, B) ha styrbarhetsmatrisa:
Let (A, B) have controllability matrix:

$$\mathcal{C} = \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix}$$

Anta at $\mathcal{C}$ har full rang. Vis at dette impliserer at styrbarhetsmatrisa $\overline{\mathcal{C}}$ til det ekvivalente systemet ($\overline{A} = PAP^{-1}, \overline{B} = PB$), hvor P er similaritetstransformasjonen, også har full rang.

Assume that $\mathcal{C}$ has full rank. Show that this implies that the controllability matrix $\overline{\mathcal{C}}$ of the equivalent system ($\overline{A} = PAP^{-1}, \overline{B} = PB$), with P being the equivalence transformation, also has full rank.

b)
La:
Let:

$$A = \begin{pmatrix} -1 & -1 \\ -4 & 2 \end{pmatrix}$$

Finn egenverdiene og tilhørende egenvektorer til A.
Find the eigenvalues and the corresponding eigenvectors of A.

c)
La:
Let:

$$B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \; D = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

Transformer systemet til diagonalform, ved hjelp av similaritetstransformasjonen $\overline{x} = Px$.
Transform the system to diagonal form, by using the equivalence transformation $\overline{x} = Px$.

d)
Anta konstant pådrag $u \equiv 1$. Hva blir løsningen $y(t)$ for $t = 1$ når $x(0) = x_0$? Unngå integral i svaret.
Assume a constant input $u \equiv 1$. What is the solution $y(t)$ at $t = 1$ when $x(0) = x_0$? Avoid using integrals in the answer.

Oppgave 2 (20 %)

Gitt følgende system:
Given the following system:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

a)
Er systemet BIBO-stabilt? Er systemet styrbart? Forklar!
Is the system BIBO-stable? Is the system controllable? Explain!

b)
La α, β, γ være positive konstanter. Vi ønsker å minimere kostfunksjonen:
Let α, β, γ be positive constants. We wish to minimize the cost function:

$$J = \int_0^{t_e} \alpha x_1^2 + \beta x_2^2 + \gamma u^2 dt, \quad t_e > 0 \tag{1}$$

La $x = (x_1, x_2)^T$. Skriv kostfunksjonen på formen:
Let $x = (x_1, x_2)^T$. Write the cost function on the form:

$$J = \int_0^{t_e} x^T Q x + u^T R u dt, \quad t_e > 0 \tag{2}$$

Hva blir Q? Hva blir R? Hvilken effekt har det å øke henholdsvis α, β og γ?
What is Q? What is R? What is the effect of increasing α, β and γ, respectively?

c)
La t_e i (2) være endelig. Blir den resulterende regulatoren fra å løse optimaliseringsproblemet tidsvarierende eller tidsinvariant? Forklar!
Let t_e in (2) be finite. Will the resulting controller from solving the optimization problem be timevarying or timeinvariant? Explain!

d)
La $\alpha = \beta = \gamma = 1$ i (1). Gitt en positiv semidefinitt matrise:
Let $\alpha = \beta = \gamma = 1$ in (1). Given a positive semidefinite matrix:

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}.$$

Finn den stasjonære verdien av P ved å løse Ricatti-likningen:
Find the stationary value of P by solving the Ricatti equation:

$$0 = PA + A^T P + Q - PBR^{-1}B^T P$$

e)
Bruk P fra forrige oppgave til å vise at systemet med tilbakekobling $u = -R^{-1}B^T P x$ er asymptotisk stabilt.
Use P from the previous exercise to show that the system with feedback $u = -R^{-1}B^T P x$ is asymptotically stable.

Oppgave 3 (35 %)

Gitt et signal med autokorrelasjonsfunksjon:
Consider a signal with autocorrelation function:

$$R_s(\tau) = \frac{8}{3} \exp^{-\frac{3}{4}|\tau|}$$

La målinga av signalet være addert med støy med autokorrelasjonsfunksjon:
Let the measurement of the signal be corrupted with additive noise with autocorrelation function:

$$R_n(\tau) = 4\delta(\tau)$$

a)
Vis at systemet kan skrives som:
Show that the system can be written as:

$$\begin{aligned} \dot{x} &= -\frac{3}{4}x + 2u \\ z &= x + v, \end{aligned}$$

hvor u er enhets hvit støy og v er hvit støy med kovariansparameter $R = 4$.
where u is unity white noise and v is white noise with covariance parameter $R = 4$.

b)
Et kontinuerlig Kalman filter skal designes for systemet. Vis at kovariansfeilmatrisen tilfredsstiller:
A continuous Kalman filter is to be designed for the system. Show that the error covariance matrix satisfies:

$$\dot{P} = -\frac{1}{4}P^2 - \frac{3}{2}P + 4$$

c)
La $P(0) = 0$. Bestem $P(t)$. (Avhenging av hvordan du velger å løse oppgaven, kan du finne det nyttig å vite at $a^2 + 6a - 16 = (a + 8)(a - 2)$ eller at $\cos ix = \cosh x = 1/2(e^x + e^{-x})$ og at $-i \sin ix = \sinh x = 1/2(e^x - e^{-x})$).
Let $P(0) = 0$. Find $P(t)$. (Depending on your solution method, you might find it useful to know that $a^2 + 6a - 16 = (a + 8)(a - 2)$ or that $\cos ix = \cosh x = 1/2(e^x + e^{-x})$ and that $-i \sin ix = \sinh x = 1/2(e^x - e^{-x})$).

d)
Hva blir Kalman-forsterkninga K i det stasjonære tilfellet?
What is the Kalman gain K in the stationary case?

Oppgave 4 (25 %)

Gitt følgende system:
Given the following system:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -4 & -4 & 0 \\ 2 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

a)
Finn overføringsfunksjonsmatrisa $G(s)$ til systemet.
Find the transfer function matrix $G(s)$ *of the system.*

b)
Bruk likningene i vedlegget til å finne en annen realisering (A_m, B_m, C_m, D_m) av $G(s)$.
Use the equations in the appendix to find another realization (A_m, B_m, C_m, D_m) *of* $G(s)$.

c)
Gitt:
Given:

$$A_{m1} = \begin{pmatrix} 0 & 1 \\ -4 & -4 \end{pmatrix}, \; B_{m1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; C_{m1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; D_{m1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Er $(A_{m1}, B_{m1}, C_{m1}, D_{m1})$ også en realisering av $G(s)$? Hvorfor/Hvorfor ikke?
Is $(A_{m1}, B_{m1}, C_{m1}, D_{m1})$ *also a realization of* $G(s)$*? Why/Why not?*

Vedlegg til eksamen (noen nyttige formler og uttrykk):
Appendix to the exam (some useful formulas and expressions):

$$
\begin{aligned}
x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \\
x(k) &= A^k x(0) + \sum_{m=0}^{k-1} A^{k-1-m}Bu(m) \\
A^{-1} &= \frac{adj(A)}{det(A)} \\
det(A) &= \sum_{i=1}^{n} a_{ij}c_{ij} \\
adj(A) &= \{c_{ij}\}^T \\
c_{ij} &= (-1)^{i+j}det(A_{ij}) \quad \text{(kofaktor)},\ A_{ij} = \text{submatrix to } A
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{C} &= (B\ AB\ A^2B\ \cdots\ A^{n-1}B) \\
\mathcal{O} &= \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix} \\
G(s) &= C(sI - A)^{-1}B + D \\
G(z) &= C(zI - A)^{-1}B + D \\
\mathbf{G(s)} &= \mathbf{G}(\infty) + \mathbf{G_{sp}(s)} \\
d(s) &= s^r + \alpha_1 s^{r-1} + \cdots + \alpha_{r-1}s + \alpha_r \\
\mathbf{G_{sp}}(s) &= \frac{1}{d(s)}[\mathbf{N_1}s^{r-1} + \mathbf{N_2}s^{r-2} + \cdots + \mathbf{N_{r-1}}s + \mathbf{N_r}] \\
\dot{\mathbf{x}} &= \begin{bmatrix} -\alpha_1\mathbf{I_p} & -\alpha_2\mathbf{I_p} & \cdots & -\alpha_{r-1}\mathbf{I_p} & -\alpha_r\mathbf{I_p} \\ \mathbf{I_p} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I_p} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I_p} & \mathbf{0} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{I_p} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} \mathbf{u} \\
\mathbf{y} &= [\ \mathbf{N_1}\quad \mathbf{N_2}\quad \cdots\quad \mathbf{N_{r-1}}\quad \mathbf{N_r}\ ]\mathbf{x} + \mathbf{G}(\infty)\mathbf{u}
\end{aligned}
$$

Discrete-time Kalman filter:

$$
\begin{aligned}
\mathbf{x}_{k+1} &= \mathbf{\Phi}_k \mathbf{x}_k + \mathbf{w}_k \\
\mathbf{z}_k &= \mathbf{H}_k \mathbf{x}_k + \mathbf{v}_k \\
E[\mathbf{w}_k \mathbf{w}_i^T] &= \begin{cases} \mathbf{Q}_k, & i = k \\ 0, & i \neq k \end{cases} \\
E[\mathbf{v}_k \mathbf{v}_i^T] &= \begin{cases} \mathbf{R}_k, & i = k \\ 0, & i \neq k \end{cases} \\
E[\mathbf{w}_k \mathbf{v}_i^T] &= 0, \forall i, k \\
\mathbf{P}_k^- &= E[\mathbf{e}_k^- \mathbf{e}_k^{-T}] \\
\mathbf{P}_k &= E[\mathbf{e}_k \mathbf{e}_k^T] = (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) \mathbf{P}_k^- (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)^T + \mathbf{K}_k \mathbf{R}_k \mathbf{K}_k^T \\
\mathbf{K}_k &= \mathbf{P}_k^- \mathbf{H}_k^T \left(\mathbf{H}_k \mathbf{P}_k^- \mathbf{H}_k^T + \mathbf{R}_k \right)^{-1} \\
\mathbf{P}_{k+1}^- &= \mathbf{\Phi}_k \mathbf{P}_k \mathbf{\Phi}_k^T + \mathbf{Q}_k
\end{aligned}
$$

Continuous-time Kalman filter:

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{F}\mathbf{x} + \mathbf{G}\mathbf{u} \\
\mathbf{z} &= \mathbf{H}\mathbf{x} + \mathbf{v} \\
E[\mathbf{u}(t)\mathbf{u}(\tau)^T] &= \mathbf{Q}\delta(t - \tau) \\
E[\mathbf{v}(t)\mathbf{v}(\tau)^T] &= \mathbf{R}\delta(t - \tau) \\
E[\mathbf{u}(t)\mathbf{v}(\tau)^T] &= 0 \\
\mathbf{K} &= \mathbf{P}\mathbf{H}^T \mathbf{R}^{-1} \\
\dot{\mathbf{P}} &= \mathbf{F}\mathbf{P} + \mathbf{P}\mathbf{F}^T - \mathbf{P}\mathbf{H}^T \mathbf{R}^{-1} \mathbf{H}\mathbf{P} + \mathbf{G}\mathbf{Q}\mathbf{G}^T, \quad \mathbf{P}(0) = \mathbf{P}_0
\end{aligned}
$$

Auto-correlation:

$$
\begin{aligned}
R_X(\tau) &= E[X(t)X(t+\tau)] \text{ (Stationary process)} \\
R_X(t_1, t_2) &= E[X(t_1)X(t_2)] \text{ (Non-stationary process)} \\
Y(s) &= G(s)U(s) \Rightarrow \\
R_y(t_1, t_2) &= E[y(t_1)y(t_2)] \\
&= \int_0^{t_2} \int_0^{t_1} g(\xi) g(\eta) E\left[u(t_1 - \xi) u(t_2 - \eta)\right] d\xi d\eta \text{ (Transient analysis)}
\end{aligned}
$$

Laplace transform pairs:

$f(t)$	$\Longleftrightarrow$	$F(s)$
1	$\Longleftrightarrow$	$\frac{1}{s}$
e^{-at}	$\Longleftrightarrow$	$\frac{1}{s+a}$
t	$\Longleftrightarrow$	$\frac{1}{s^2}$
t^2	$\Longleftrightarrow$	$\frac{2}{s^3}$
te^{-at}	$\Longleftrightarrow$	$\frac{1}{(s+a)^2}$
$\sin \omega t$	$\Longleftrightarrow$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\Longleftrightarrow$	$\frac{s}{s^2+\omega^2}$