Øving 9

oppg 4.1

1) $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x \geq 0, y \geq 0 \right\}$

a) let $\vec{u} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$
with $\vec{u}, \vec{v} \in V$.
Then $x_1, y_1 \geq 0$ and $x_2, y_2 \geq 0$

$$\vec{u} + \vec{v} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$$

Since $x_1 \geq 0$ and $x_2 \geq 0$, $x_1 + x_2 \geq 0$.
The same goes for $y_1 + y_2$, thus
$\vec{u} + \vec{v} \in V$.

b) $\vec{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $c = -1$.
Then $c\vec{u} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \notin V$

9) $H = \left\{ \begin{pmatrix} s \\ 3s \\ 2s \end{pmatrix} : s \in \mathbb{R} \right\}$

$$\begin{pmatrix} s \\ 3s \\ 2s \end{pmatrix} = s \cdot \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \Rightarrow H = \underline{\underline{\text{span}\left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \right\}}}$$

This shows that H is a line in $\mathbb{R}^3$
and since H is also a ~~subspace~~ vectorspace,
it must then be a subspace of $\mathbb{R}^3$.

11) $W = \left\{ \begin{pmatrix} 5b+2c \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 : b, c \in \mathbb{R} \right\}$

$$\begin{pmatrix} 5b+2c \\ b \\ c \end{pmatrix} = b \cdot \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} + c \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

If $\vec{u} = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ then $W = \text{Span}\{\vec{u}, \vec{v}\}$

Since this span is a plane in $\mathbb{R}^3$ and a vectorspace, it is also a subspace of $\mathbb{R}^3$.

32) Let H and K be subspaces of a vector space V. We want to show that the intersection of H and K, that is $H \cap K$, is also a subspace of V.

Choose an arbitrary element from $H \cap K$. $a \in H \cap K$. By set laws, $a \in H$ and $a \in K$. Since $a \in H$, a is also in V because H is a subspace/subset of V. The same goes for K. Since a was arbitrary and $a \in V$, the rule of universal generalisation gives that $H \cap K \subseteq V$. ▥

$\text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\} \cup \text{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ is the union of two subspaces of $\mathbb{R}^2$. Both $\vec{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ will be in this union, but $\vec{u} + \vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is not, so it is not a subspace of $\mathbb{R}^2$.

pg 4.2

6) $A = \begin{pmatrix} 1 & 5 & -4 & -3 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$R_1 \leftarrow R_1 - 5R_2 \Rightarrow \begin{pmatrix} 1 & 0 & 6 & -8 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

x_3, x_4, x_5 are free

$x_1 = -6x_3 + 8x_4 - x_5$

$x_2 = 2x_3 - x_4$

We write this in vector parametric form

$$\begin{pmatrix} -6x_3 + 8x_4 - x_5 \\ 2x_3 - x_4 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_3 \cdot \begin{pmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5 \cdot \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

From this we see that

$$Nul(A) = span\left\{ \begin{pmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$= t_1 \cdot \begin{pmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t_2 \cdot \begin{pmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t_3 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

21) $A = \begin{pmatrix} 2 & -6 \\ -1 & 3 \\ -4 & 12 \\ 3 & -9 \end{pmatrix}$ $\left.\begin{matrix} R_2 \leftarrow R_2 + \frac{1}{2} R_1 \\ R_3 \leftarrow R_3 + 2R_1 \\ R_4 \leftarrow R_4 - \frac{3}{2} R_1 \end{matrix}\right\} \Rightarrow \begin{pmatrix} 2 & -6 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\sim \begin{pmatrix} 1 & -3 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

x_2 is free and $x_1 = 3x_2$

$\Rightarrow Nul(A) = Span\left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right\}$

We also get that

$$\operatorname{col}(A) = \operatorname{span}\left\{\begin{pmatrix}2\\-1\\-4\\3\end{pmatrix}\right\}$$

So $\begin{pmatrix}3\\1\end{pmatrix} \in \operatorname{Nul}(A)$ and $\begin{pmatrix}2\\-1\\-4\\3\end{pmatrix} \in \operatorname{col}(A)$

24) Have to reduce the aug. matrix $(A \ \vec{w})$.

$$\left(\begin{array}{ccc|c} -8 & -2 & -9 & 2 \\ 6 & 4 & 8 & 1 \\ 4 & 0 & 4 & -2 \end{array}\right) \qquad \begin{array}{l} R_2 \leftarrow R_2 + \frac{3}{4} R_1 \\ R_3 \leftarrow R_3 + \frac{1}{2} R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} -8 & -2 & -9 & 2 \\ 0 & 5/2 & 5/4 & 5/2 \\ 0 & -1 & -1/2 & -1 \end{array}\right) \qquad \begin{array}{l} R_1 \leftarrow -R_1 \\ R_2 \leftarrow \frac{4}{5} \cdot R_2 \\ R_3 \leftarrow -2 R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 8 & 2 & 9 & -2 \\ 0 & 2 & 1 & 2 \\ 0 & 2 & 1 & 2 \end{array}\right) \qquad \begin{array}{l} R_3 \leftarrow R_3 - R_2 \\ R_1 \leftarrow R_1 - R_2 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 8 & 0 & 8 & -4 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The system is consistent so $\vec{w} = \begin{pmatrix}2\\1\\-2\end{pmatrix}$ is in $\operatorname{col}(A)$.

To check Nul A, we compute $A \cdot \vec{w}$

$$A \cdot \vec{w} = 2 \cdot \begin{pmatrix}-8\\6\\4\end{pmatrix} + 1 \cdot \begin{pmatrix}-2\\4\\0\end{pmatrix} - 2 \cdot \begin{pmatrix}-9\\8\\4\end{pmatrix} = \begin{pmatrix}-16 - 2 + 18\\ 12 + 4 - 16 \\ 8 \quad - 8\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

So $\vec{w}$ is also in $\operatorname{Nul}(A)$

28) If we describe the systems with a matrix $A = \begin{pmatrix} 5 & 1 & -3 \\ -9 & 2 & 5 \\ 4 & 1 & -6 \end{pmatrix}$, then the first system is $A\vec{x} = \begin{pmatrix} 0 \\ 1 \\ 9 \end{pmatrix}$ and the second is $A\vec{x} = \begin{pmatrix} 0 \\ 5 \\ 45 \end{pmatrix}$.

The first system has a solution so $\begin{pmatrix} 0 \\ 1 \\ 9 \end{pmatrix}$ is in $col(A)$.

This gives that $\begin{pmatrix} 0 \\ 1 \\ 9 \end{pmatrix} = t_1 \cdot \begin{pmatrix} 5 \\ -9 \\ 4 \end{pmatrix} + t_2 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + t_3 \cdot \begin{pmatrix} -3 \\ 5 \\ -6 \end{pmatrix}$

Multiply both sides by 5 and let $t_i' = 5\, t_i$, then we get that

$$5\begin{pmatrix} 0 \\ 1 \\ 9 \end{pmatrix} = 5 \cdot \left(t_1 \begin{pmatrix} 5 \\ -9 \\ 4 \end{pmatrix} + t_2 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + t_3 \begin{pmatrix} -3 \\ 5 \\ -6 \end{pmatrix} \right)$$

$$\Leftrightarrow \begin{pmatrix} 0 \\ 5 \\ 45 \end{pmatrix} = t_1' \begin{pmatrix} 5 \\ -9 \\ 4 \end{pmatrix} + t_2' \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + t_3' \begin{pmatrix} -3 \\ 5 \\ -6 \end{pmatrix}$$

It is clear from this that $\begin{pmatrix} 0 \\ 5 \\ 45 \end{pmatrix}$ is also in $col(A)$ so the second system must then also have a solution.

4.3

1) The matrix $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ is in echelon form

and has 3 pivots (3×3)

so its columns are independent

3) $\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix}, \begin{pmatrix} -3 \\ -5 \\ 1 \end{pmatrix}$

Have to reduce the matrix: $A = \begin{pmatrix} 1 & 3 & -3 \\ 0 & 2 & -5 \\ -2 & -4 & 1 \end{pmatrix}$

$R_3 \leftarrow R_3 + 2R_1$ $\qquad$ $R_3 \leftarrow R_3 - R_2$ $\qquad$ $R_2 \leftarrow R_2/2$

$$\sim \begin{pmatrix} 1 & 3 & -3 \\ 0 & 2 & -5 \\ 0 & 2 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & -3 \\ 0 & 2 & -5 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & -3 \\ 0 & 1 & -5/2 \\ 0 & 0 & 0 \end{pmatrix}$$

$R_1 \leftarrow R_1 - 3R_2$

$$\sim \begin{pmatrix} 1 & 0 & 9/2 \\ 0 & 1 & -5/2 \\ 0 & 0 & 0 \end{pmatrix}$$

The set does not form a basis for $\mathbb{R}^3$ since the set is linearly dependent. Using this dependency we can also see that

$$\operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix}, \begin{pmatrix} -3 \\ -5 \\ 1 \end{pmatrix}\right\} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix}\right\}.$$

Two vectors can't span $\mathbb{R}^3$ so the set can't be a basis for $\mathbb{R}^3$.

8) $\left\{ \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} \right\}$

Note: four vectors in $\mathbb{R}^3$ so this a dependent set and thus not a basis.

Lets reduce the matrix

$$\begin{pmatrix} 1 & 0 & 3 & 0 \\ -4 & 3 & -5 & 2 \\ 3 & 1 & 4 & -2 \end{pmatrix} \quad \begin{matrix} R_2 \leftarrow R_2 + 4R_1 \\ R_3 \leftarrow R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 3 & 7 & 2 \\ 0 & 1 & -5 & -2 \end{pmatrix} \quad R_3 \leftarrow R_3 - \frac{1}{3} R_2$$

$$\sim \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 3 & 7 & 2 \\ 0 & 0 & -22/3 & -8/3 \end{pmatrix}$$

So it has 3 pivot columns, and thus it will span $\mathbb{R}^3$.

24) $\mathbb{R}^4 = \text{span}\{\vec{v}_1, \dots, \vec{v}_4\}$

For $\{\vec{v}_1, \dots \vec{v}_4\}$ to be a basis, the set has to be linearly independent and span $\mathbb{R}^4$. We know it spans $\mathbb{R}^4$, so we only need to determine if it is independent.

Let $A = (\vec{v}_1 | \dots | \vec{v}_4)$ be the 4×4 matrix formed the set of vectors. Then

$\text{col}(A) = \text{span}\{\vec{v}_1, \dots, \vec{v}_4\} = \mathbb{R}^4$.

To span $\mathbb{R}^4$, A must have 4 pivot columns. Since A does span $\mathbb{R}^4$ and A only has 4 columns, it must be an independent set of vectors, so $\{\vec{v}_1, \dots, \vec{v}_4\}$ are then independent.

29) Let $S = \{\vec{V}_1, \dots, \vec{V}_k\}$ and look at the $n \times K$ matrix $A = (\vec{V}_1 | \dots | \vec{V}_k)$ with $n > K$. There are n rows and K columns. Since $n > K$, when A is row reduced it will have some empty rows. By theorem 4 of section 1.4, since A does not have a pivot in every row, the ~~equat~~ columns of A does not span $\mathbb{R}^n$ and ~~there for~~ thus S is not a basis for A.

30) Let $S = \{\vec{V}_1, \dots, \vec{V}_K\}$ be a set of K vectors in $\mathbb{R}^n$, with $K > n$.
This is a set with more vectors than there are entries in each vector, so we can use theorem 8 of section 1.7, which gives us that S is a dependent set.
A dependent set can't (by definition) form a basis, so S is not a basis of $\mathbb{R}^n$.

32) Let V, W be vector spaces.
Let $T: V \to W$ be a linear transformation.
Let $\{\vec{v}_1, \dots, \vec{v}_p\} \subseteq V$ be a subset of V.

Suppose T is one-to-one such that
$T(\vec{u}) = T(\vec{v}) \Rightarrow \vec{u} = \vec{v}$

We want to show that if the set
$\{T(\vec{v}_1), \dots, T(\vec{v}_p)\}$ is linearly dependent,
then $\{\vec{v}_1, \dots, \vec{v}_p\}$ is also linearly dependent.

Since $\{T(\vec{v}_1), \dots, T(\vec{v}_p)\}$ is linearly dependent,
at least one of its elements $T(\vec{v}_k)$, $1 \leq k \leq p$
can be written as a linear combination
of the others, so

$$T(\vec{v}_k) = a_1 \cdot T(\vec{v}_1) + a_2 \cdot T(\vec{v}_2) + \dots + a_p T(\vec{v}_p)$$

By linearity, we can write this as

$$T(\vec{v}_k) = T(a_1\vec{v}_1 + \dots + a_p\vec{v}_p)$$

Which implies

$$\vec{v}_k = a_1\vec{v}_1 + \dots + a_p\vec{v}_p$$

Since T is one-to-one. It is clear from
this that $\vec{v}_k$ is also a linear combination
of the set $\{\vec{v}_1, \dots, \vec{v}_p\}$, and thus the
set is linearly dependent. ▯

Extra

1) A is $m \times n$ and such that there exists a C such that $CA = I$

Assume there is an $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \vec{0}$.

Mutiply both sides with C and we get

$$(C \cdot A)\vec{x} = C \cdot \vec{0}$$

$$\vec{x} = \vec{0}$$

This shows that $A\vec{x} = \vec{0}$ ~~...~~ has only the trivial solution, which is equivalent with the collumns of A being independent.

2) A is $m \times n$ and has a right inverse C, so $AC = I$.

Want to show that $\operatorname{col} A = \mathbb{R}^m$.

The equation $AC\vec{x} = \vec{b}$ has a solution for all $\vec{b}$, since $AC = I$ so $AC\vec{x} = \vec{x}$. That is, $\vec{x} = \vec{b}$ is the solution to this equation. This implies that $A\vec{y} = \vec{b}$ has a solution for all $\vec{b} \in \mathbb{R}^m$, just let $\vec{y} = C\vec{b}$.

So from this we can see that ~~...~~ $\operatorname{col} A = \mathbb{R}^m$, which is what we wanted to prove.