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Exercise 3
TTK4130 Modeling and Simulation

Problem 1 (Euler's method)

In this problem we will study the system

$$\ddot{x} + c\dot{x} + g\left[1 - \left(\frac{1}{x}\right)^{\kappa}\right] = 0, \tag{1}$$

where $\kappa = 1.40$ and $g = 9.81$.

(a) Write the system on state-space form.

Solution: Defining $y_1 = x$ and $y_2 = \dot{x}$, we can write

$$\dot{y}_1 = y_2, \tag{2a}$$

$$\dot{y}_2 = -cy_2 - g\left[1 - \left(\frac{1}{y_1}\right)^{\kappa}\right]. \tag{2b}$$

(b) Set up Euler's method for the system.

Solution: Euler's method for this system becomes

$$\begin{aligned}
y_{1,n+1} &= y_{1,n} + hy_{2,n}, \\
y_{2,n+1} &= y_{2,n} + h\left[-cy_{2,n} - g\left(1 - \left(\frac{1}{y_{1,n}}\right)^{\kappa}\right)\right].
\end{aligned}$$

(c) Set up modified Euler's method for the system.

Solution: We use (2) to find $\mathbf{k}_1$ and $\mathbf{k}_2$ for modified Euler's method

$$\begin{aligned}
\mathbf{k}_1 &= \begin{pmatrix} k_{1,1} \\ k_{1,2} \end{pmatrix} = \begin{pmatrix} y_{2,n} \\ -cy_{2,n} - g\left(1 - \left(\frac{1}{y_{1,n}}\right)^{\kappa}\right) \end{pmatrix}, \\
\mathbf{k}_2 &= \begin{pmatrix} k_{2,1} \\ k_{2,2} \end{pmatrix} = \begin{pmatrix} y_{2,n} + \frac{h}{2}k_{1,2} \\ -c\left(y_{2,n} + \frac{h}{2}k_{1,2}\right) - g\left(1 - \left(\frac{1}{y_{1,n} + \frac{h}{2}k_{1,1}}\right)^{\kappa}\right) \end{pmatrix}.
\end{aligned}$$

Modified Euler's method is then

$$\begin{aligned}
y_{1,n+1} &= y_{1,n} + hk_{2,1}, \\
y_{2,n+1} &= y_{2,n} + hk_{2,2}.
\end{aligned}$$

(d) Show that $x = 1$ is a stationary point, and linearize the system about this point.

Solution: The point $x = 1$ corresponds to $y = (x, \dot{x})^{\mathsf{T}} = (1, 0)$. At this point, (2) evaluates to zero, which means that this point is an equilibrium (a stationary point).
The linear model of the system (in this stationary point) is

$$\begin{aligned}
\Delta\dot{y}_1 &= \left.\frac{\partial f_1}{\partial y_1}\right|_{y_1=1, y_2=0} \Delta y_1 + \left.\frac{\partial f_1}{\partial y_2}\right|_{y_1=1, y_2=0} \Delta y_2 \\
\Delta\dot{y}_2 &= \left.\frac{\partial f_2}{\partial y_1}\right|_{y_1=1, y_2=0} \Delta y_1 + \left.\frac{\partial f_2}{\partial y_2}\right|_{y_1=1, y_2=0} \Delta y_2
\end{aligned}$$

that is, the system matrix in this point is

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{pmatrix}_{(y_1=1, y_2=0)} = \begin{pmatrix} 0 & 1 \\ -\kappa g \left(\frac{1}{y_1}\right)^{\kappa+1} & -c \end{pmatrix}_{(y_1=1, y_2=0)} = \begin{pmatrix} 0 & 1 \\ -\kappa g & -c \end{pmatrix}.$$

The eigenvalues of the linearized system is given from

$$\det\left(A - \lambda I\right) = \det \begin{pmatrix} -\lambda & 1 \\ -\kappa g & -c-\lambda \end{pmatrix} = \lambda^2 + \lambda c + \kappa g = 0.$$

Solving this, we find

$$\lambda_{1,2} = -\frac{1}{2}c \pm \frac{1}{2}\sqrt{c^2 - 4\kappa g}$$

(e) For which steplengths h will Euler's method be stable, if

i) $c = 0$?

ii) $c = 2\sqrt{g\kappa} = 7.412$?

Solution:

i) For $c = 0$ the eigenvalues are

$$\lambda_{1,2} = \pm 3.7059i,$$

that is, purely imaginary (physically: when $c = 0$, there is no damping in the system).

Euler's method is stable when

$$|1 + h\lambda| \leq 1$$

which is impossible to fulfill with purely imaginary eigenvalues for any h. In other words, Euler's method is unstable for all systems with purely imaginary eigenvalues.

Direct computation gives of course the same result:

$$|1 + h\left(\pm 3.7059i\right)| = \sqrt{\left(1 + 13.73h^2\right)} \geq 1, \quad \text{for all } h.$$

ii) For $c = 2\sqrt{g\kappa} = 2\sqrt{(9.81)\,1.4} = 7.412$ the eigenvalues are

$$\lambda_{1,2} = -\frac{c}{2} = -3.7060.$$

Euler's method is stable when

$$|1 + h\lambda| \leq 1.$$

Insertion of the eigenvalues gives

$$\begin{aligned} |1 + h\lambda_{1,2}| &\leq 1 \\ |1 + h\left(-3.7060\right)| &\leq 1 \\ 2 \geq 3.7060h &\geq 0 \\ 0.5397 \geq h &\geq 0 \end{aligned}$$

The steplength h must be less than 0.5397 for Euler's method to be stable.

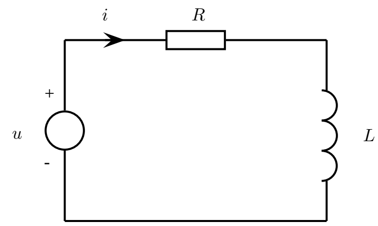

Figure 1: Electrical circuit

Problem 2 (Modeling, control and simulation of RL-circuit)

Figure 1 shows an electrical RL circuit where $L > 0$ is inductance, $R > 0$ is resistance, i is current and i_0 is initial current. The voltage u of the voltage source can be adjusted (is an input). The circuit is modeled by

$$L\frac{\mathrm{d}i}{\mathrm{d}t} + Ri = u, \quad t > 0 \tag{3a}$$

$$i(0) = i_0 \tag{3b}$$

(a) Let $u = 0$, and show that the system is (asymptotically) stable about the equilibrium $i^* = 0$, using energy-based methods. What happens with the current $i(t)$ when $t \to \infty$? Make a sketch.

Solution: We define a positive definite energy function

$$E(t) = \frac{1}{2}Li^2(t) > 0.$$

We differentiate this along the solutions of (3a):

$$\begin{aligned}\dot{E}(t) &= \frac{\partial E}{\partial i}\frac{\mathrm{d}i}{\mathrm{d}t} \\ &= Li(t)\left(-\frac{R}{L}i(t)\right) \\ &= -Ri^2(t) < 0.\end{aligned}$$

A positive definite energy function with negative definite derivative, implies that the system is (asymptotically, exponentially) stable. In this particular case, it is easy to confirm that

$$\dot{E}(t) = -\frac{2R}{L}E(t)$$

which implies that $E(t) = E(t_0)\mathrm{e}^{\frac{-2R}{L}(t-t_0)} \to 0$ as $t \to \infty$, which again implies by definition of $E(t)$ that $i(t) \to 0$.

(Of course, it is also very easy to check asymptotic stability for this linear system by confirming that the eigenvalue $-R/L$ is negative).

(b) Now, assume we desire a given, constant current $i_{\text{ref}} > 0$. Design u such that $i(t) \to i_{\text{ref}}$ when $t \to \infty$, by considering the dynamics of $e(t) = i(t) - i_{\text{ref}}$ (what is $\frac{\mathrm{d}e}{\mathrm{d}t}$?), and use the energy function

$$E(t) = \frac{1}{2}Le^2(t) > 0.$$

Solution: We find that

$$L\frac{\mathrm{d}e}{\mathrm{d}t} + Re = u - Ri_{\mathrm{ref}}.$$

Differentiating the energy function, we obtain

$$\begin{aligned}\dot{E} &= Le\dot{e} \\ &= -Re^2 + e\left(u - Ri_{\mathrm{ref}}\right).\end{aligned}$$

If we let

$$u = Ri_{\mathrm{ref}} - K_p e,$$

for $K_p > 0$, then

$$\dot{E} = -Re^2 - K_p e^2$$

which shows that $e(t) \to 0$, which again means that $i(t) \to i_{\mathrm{ref}}$.

(c) Let

$$\begin{aligned}i_0 &= 1 \text{ A} \\ L &= 1 \text{ H} \\ R &= 2\ \Omega\end{aligned}$$

Now, we want to simulate (3) from $t = 0$ to $t = 5$ by using the (classical) fourth order ERK-method, with constant step length $h = 0.01$ s.

Write a Matlab script that does the job, and enclose a code printout.

Solution:

```
% Simulate RL circuit
t = [0 5]; h = 0.01;
i0 = 1; L = 1; R = 2; Kp = 1; iref = 2;

% Number of iterations
N = t(2)/h;

% Define Butcher-array (book p. 528)
A = diag([0.5 0.5 1]);
b = [1/6 2/6 2/6 1/6]';
c = [0 0.5 0.5 1]';

% Order of ERK method
sigma = size(A,1) + 1;

% RL-circuit model
RL = @(i) ( -R/L * i + 1/L * ( R*iref - Kp*(i - iref)) );

% Initialize storage
y = zeros(N+1,1); y(1) = i0;

% We use numerical scheme from book p. 526
k = zeros(sigma,1);
```

```
for n = 1:N,
    for j = 1:sigma,
        A_diag = diag(A);
        k(j) = RL(y(n) + h * sum(A_diag(1:j - 1) .* k(1:j - 1)));
    end
    y(n + 1) = y(n) + h * sum(b .* k);
end

plot(t(1):h:t(2),y); xlabel('Time [s]'); ylabel('Current [A]');
```

(d) Implement the model in Dymola. This could be done using components from the Modelica Standard Library, but instead write in the parameter definitions and equations. When you simulate, choose the solver Rkfix4 ('Simulation' → 'Setup'), which is the same solver you implemented in task (c). Set the step length ('Fixed integrator step') to 0.01. Compare with the solution of the Matlab implementation.

Solution: A solution may look like

```
model test
 import SI = Modelica.SIunits;
 SI.Current i(start = 1);
 SI.Voltage u;
 parameter SI.Resistance R = 2;
 parameter SI.Inductance L = 1;
 parameter Real Kp = 1;
 parameter SI.Current iref = 2;
equation
  // Controller
  u = R*iref - Kp*(i - iref);
  // Model of RL circuit
  L*der(i) + R*i = u;
end test;
```

The simulation should be identical with the Matlab implementation.

Problem 3 (Use of Matlab/Simulink's ODE solvers)

Download the file orbit.mdl from Blackboard, and experiment with it in MATLAB SIMULINK. This file models/simulates the restricted three-body problem from Ch. 14.1.3 in the book. The parameters in the file are for *Orbit 1* in Table 14.1.

(a) Choose the solver ode45 (this is probably the default solver), and check what *Relative Tolerance* is needed to simulate one, two and three rounds (simulate more rounds by increasing the stop time).

Solution:

- One round: relative tolerance less than 10^{-3}.
- Two rounds: relative tolerance less than 10^{-6}.
- Three rounds: Not possible.

(b) Find the step length that gives an accurate solution for the five-stage explicit Runge-Kutta method ode5. Select the method by choosing *Simulation* → *Configuration parameters* → *Fixed step* and *ode5*.

Solution: The fixed-step size 0.001 gives an accurate solution.