Problem Set 5 - digsig

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Problem 1

a) $$x[n] = \begin{cases} a^n & , n \geq 1, \ |a| < 1 \\ 0 & , n < 0 \end{cases}$$

The energy density spectrum of x is given by

$$S_{xx}(\omega) = X(\omega)X(-\omega)$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(ae^{-j\omega}\right)^n$$

$$= \frac{1}{1-ae^{-j\omega}}$$

$$\Rightarrow S_{xx}(\omega) = \frac{1}{1-ae^{-j\omega}} \cdot \frac{1}{1-ae^{j\omega}}$$

$$= \frac{1}{1-a\cos\omega + j\sin(\omega)a} \cdot \frac{1}{1-a\cos\omega - j\sin(\omega)a}$$

$$\Rightarrow S_{xx}(\omega) = \frac{1}{(1-a\cos\omega)^2 + (a\sin\omega)^2}$$

$$= \frac{1}{1 - 2a\cos\omega + a^2\cos^2\omega + a^2\sin^2\omega}$$

$$= \frac{1}{a^2 - 2a\cos\omega + 1}$$

Or equivalently:

$$\underline{\underline{S_{xx}(f) = \frac{1}{a^2 - 2a\cos(2\pi f) + 1}}}$$

b) $$r_{xx}[l] = x[l] * x[-l]$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, x[n-l]$$

$$= \sum_{n=l}^{\infty} a^n a^{n-l}$$

$$= a^{-l} \sum_{n=l}^{\infty} a^{2n}$$

$$= a^{-l} \sum_{n=0}^{\infty} a^{2(n+l)}$$

$$= a^{l} \sum_{n=0}^{\infty} (a^2)^n \quad , \quad |a^2| < 1 \text{ since } |a| < 1$$

$$\Rightarrow r_{xx}[\ell] = \frac{a^{\ell}}{1-a^2}, \quad \ell \geq 0, \quad r_{xx}[-\ell] = r_{xx}[\ell]$$

We should have $\mathcal{F}\{r_{xx}[\ell]\} = S_{xx}(\omega)$

$$\mathcal{F}\{r_{xx}\} = \sum_{\ell=-\infty}^{\infty} r_{xx}[\ell] e^{-j\omega\ell}$$

$$= \sum_{\ell=-\infty}^{-1} r_{xx}[\ell] e^{-j\omega\ell} + r_{xx}[0] + \sum_{\ell=1}^{\infty} r_{xx}[\ell] e^{-j\omega\ell}$$

$$= r_{xx}[0] + \sum_{\ell=1}^{\infty} \left(r_{xx}[-\ell] e^{j\omega\ell} + r_{xx}[\ell] e^{-j\omega\ell} \right)$$

$$= r_{xx}[0] + 2\sum_{\ell=1}^{\infty} r_{xx}[\ell] \cos(\omega\ell)$$

$$= \frac{1}{1-a^2} + \frac{2}{1-a^2} \sum_{\ell=0}^{\infty} a^{\ell+1} \cos(\omega(\ell+1))$$

$$= \frac{1}{1-a^2} \left[1 + 2\,\mathrm{Re}\left\{ \sum_{\ell=0}^{\infty} \left(a e^{j\omega}\right)^{\ell+1} \right\} \right]$$

$$= \frac{1}{1-a^2} \left[1 + 2\,\mathrm{Re}\left\{ \frac{a e^{j\omega}}{1 - a e^{j\omega}} \right\} \right]$$

$$\Rightarrow \mathcal{F}\{r_{xx}\} = \frac{1}{1-a^2}\left(1 + 2\,\mathrm{Re}\left(\frac{1}{a^{-1}e^{-j\omega}-1}\right)\right)$$

$$\mathrm{Re}\left\{\frac{1}{a^{-1}e^{-j\omega}-1}\right\} = \mathrm{Re}\left\{\frac{1}{a^{-1}\cos\omega - 1 - j a^{-1}\sin\omega}\right\}$$

$$= \mathrm{Re}\left\{\frac{a\left((\cos\omega - a) + j\sin\omega\right)}{(\cos\omega - a)^2 + \sin^2\omega}\right\}$$

$$= \frac{a\cos\omega - a^2}{\cos^2\omega - 2a\cos\omega + a^2 + \sin^2\omega}$$

$$= \frac{a(\cos\omega - a)}{a^2 - 2a\cos\omega + 1}$$

$$\Rightarrow 1 + 2\,\mathrm{Re}\{\ldots\}$$

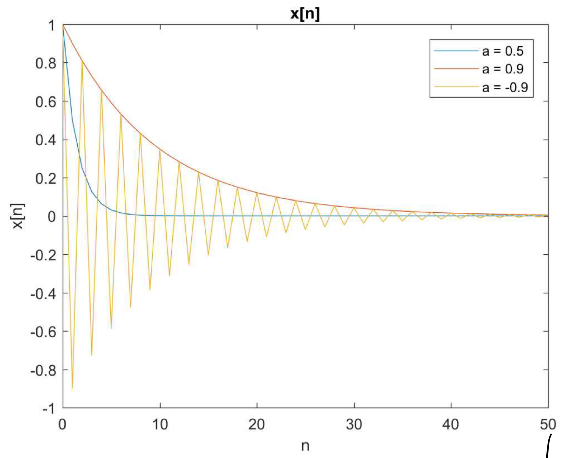
$$= \frac{a^2 - \cancel{2a\cos\omega} + 1 + \cancel{2a\cos\omega} - 2a^2}{a^2 - 2a\cos\omega + 1}$$

$$= \frac{1-a^2}{a^2 - 2a\cos\omega + 1}$$

$$\Rightarrow F\{r_{xx}\} = \frac{1}{1-a^2} \cdot \frac{1-a^2}{a^2 - 2a\cos\omega + 1}$$

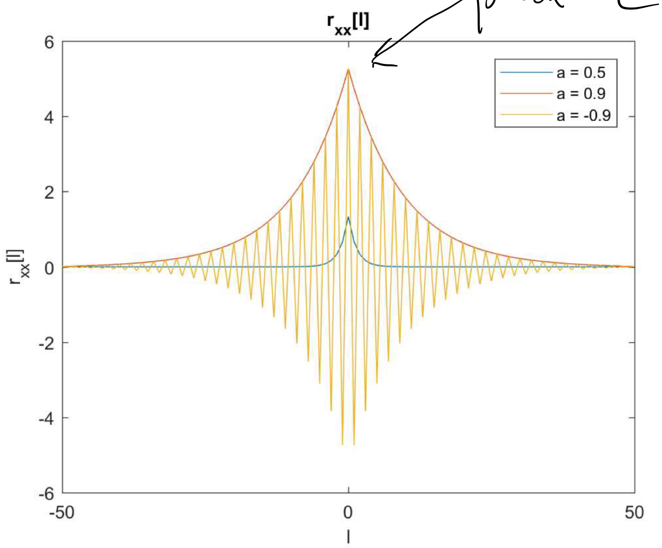
$$= \underline{\underline{\frac{1}{a^2 - 2a\cos\omega + 1}}} = S_{xx}(\omega)$$

c) The most striking observation we can make from plot 1, 2 and 3 is that when a is negative, r_{xx} oscillates and S_{xx} has a peak at $f=0$, instead of $f = \pm \frac{1}{2}$

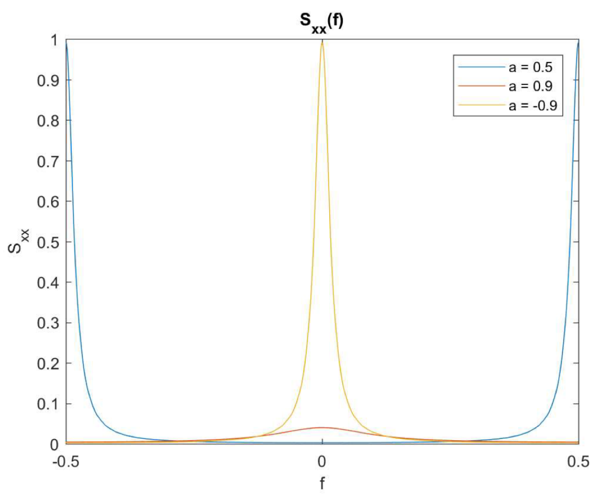


1

peak @ $l=0$ for all $r_{xx}[l]$



2



3

d) The energy of the signal is given by

$$E_x = r_{xx}[0]$$

$$= \frac{a^0}{1-a^2}$$

$$= \frac{1}{1-a^2}$$

e) $y_1[n] = x[n] * h_1[n]$

$$\Rightarrow Y_1(\omega) = X(\omega) H_1(\omega)$$

$$H_1(\omega) = \mathcal{F}\{\delta(n) - a\,\delta(n-1)\}$$

$$= 1 - ae^{-j\omega}$$

$$Y_1(\omega) = \frac{1}{1-ae^{-j\omega}} \cdot (1-ae^{-j\omega}) = 1$$

$$Y(\omega) = Y_1(\omega) \cdot H_2(\omega)$$

$$= H_2(\omega)$$

$$= \begin{cases} \cos(\omega), & |\omega| < \frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$S_{yy}(\omega) = Y(\omega)\,Y(-\omega)$$

$$= \begin{cases} \cos^2\omega, & |\omega| < \pi/2 \\ 0, & \text{otherwise} \end{cases}$$

$$\Rightarrow S_{yy}(f) = \begin{cases} \cos^2(2\pi f), & |f| \leq 1/4 \\ 0, & \frac{1}{4} \leq |f| \leq \frac{1}{2} \end{cases}$$

The total energy of the signal is

$$E_y = \frac{1}{2\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} S_{yy}(f)\, df$$

$$= \frac{1}{2\pi} \int_{-\frac{1}{4}}^{1/4} \cos^2(2\pi f)\, df$$

$$\Rightarrow E_y = \frac{1}{2\pi}\cdot\frac{1}{2} \int_{-\frac{1}{4}}^{\frac{1}{4}} (1 + \cos 4\pi f)\, df$$

$$= \frac{1}{4\pi} \left[f + \frac{1}{4\pi} \sin(4\pi f) \right]_{-\frac{1}{4}}^{1/4}$$

$$= \frac{1}{4\pi} \left(\frac{1}{4} + \frac{1}{4} + 0 - 0 \right)$$

$$= \frac{1}{8\pi}$$

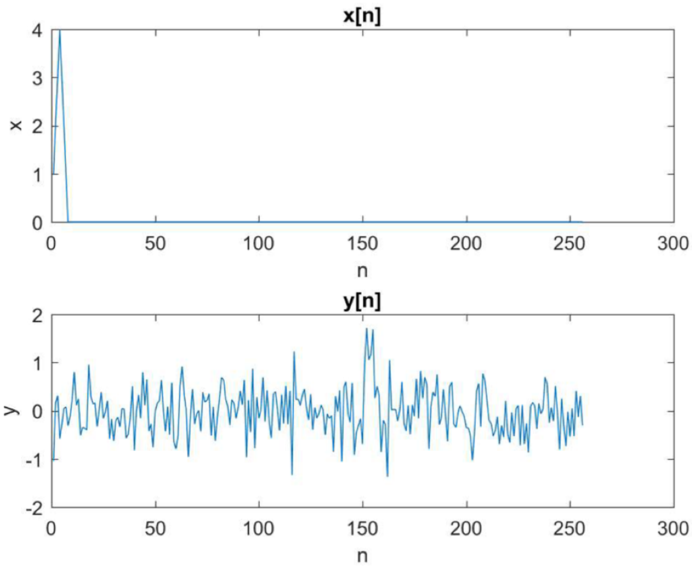
Contents

Problem 1

```
close all
clear
clc

n = 0:50;
N_FFT = power(2, 2*ceil(log2(size(n,2)))); %2 times smallest power of 2 greater than siz
e(n,2)
l = -n(end):n(end);
as = [0.5 0.9 -0.9]';


%computation
%x(n)
M = size(as,1);
ns = repmat(n,M,1);
xs = as.^n;

%rxx(l)
%rxxs = abs(ifft(fft(xs, N_FFT,2).*fft(fliplr(xs), N_FFT,2)));
%rxxs = conv2(fliplr(xs),xs,'same');
rxxs = zeros(M,size(l,2));
for i=1:M
    rxxs(i,:) = conv(xs(i,:), fliplr(xs(i,:)));
end

%Sxx(w)
L = size(l,2)-1;
Sxxs = abs(fftshift(fft(rxxs, N_FFT, 2))/L);

%plotting
as_legend = "a = " + string(as');
f = linspace(-0.5, 0.5, size(Sxxs,2));

figure();
plot(n, xs);
title("x[n]"); xlabel("n"); ylabel("x[n]"); legend(as_legend);

figure();
plot(l, rxxs);
title("r_{xx}[l]"); xlabel("l"); ylabel("r_{xx}[l]"); legend(as_legend);

figure();
plot(f, Sxxs);
title("S_{xx}(f)"); xlabel("f"); ylabel("S_{xx}"); legend(as_legend);
```

Problem 2

a) Plots:

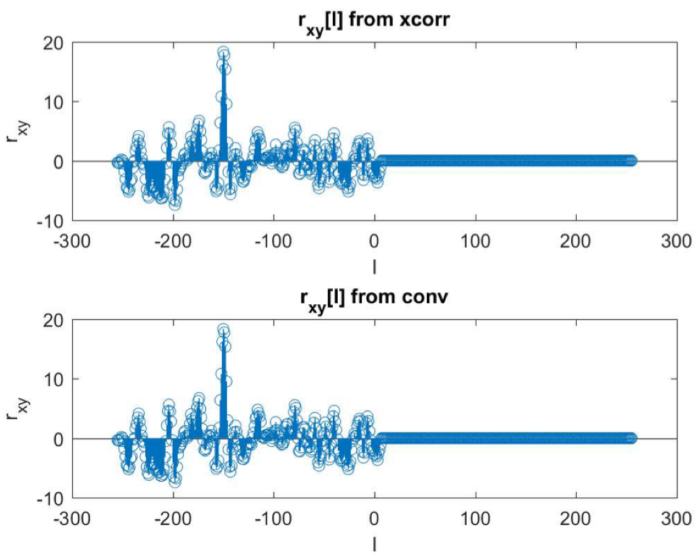


4

5

The signal $y[n]$ is really noisy. There might be a peak at $n \approx 150$ but it is barely above the noise floor.
Thus i would guess one cannot reliably determine whether an object has been hit.

c)



6

7

Problem 2

```
clear; close all; clc;
load("Signals.mat");

n = 1:size(x,2);


figure();
subplot(2,1,1);
plot(n,x); title("x[n]"); ylabel("x"); xlabel("n");
subplot(2,1,2);
plot(n,y); title("y[n]"); ylabel("y"); xlabel("n");

l = (-size(x,2)+1):(size(x,2)-1);
rxy = xcorr(x,y);
rxy_conv = conv(x,fliplr(y));

figure();
subplot(2,1,1);
stem(l,rxy); title("r_{xy}[l] from xcorr"); ylabel("r_{xy}"); xlabel("l");
subplot(2,1,2);
stem(l, rxy_conv); title("r_{xy}[l] from conv"); ylabel("r_{xy}"); xlabel("l");

[m, i] = max(rxy);
D = -l(i) % index of maximum
```

```
D =

   150
```

d) There is a clear peak around $\ell \simeq -150$.
This means that
$$D \simeq 150$$

Problem 3

Finding $H(z)$:

$$y[n] = x[n] + \alpha z^{-R} x[n]$$
$$= (1 + \alpha z^{-R}) x[n]$$

$$\Rightarrow H(z) = \frac{y[n]}{x[n]}(z)$$
$$= 1 + \alpha z^{-R}$$

Delay:

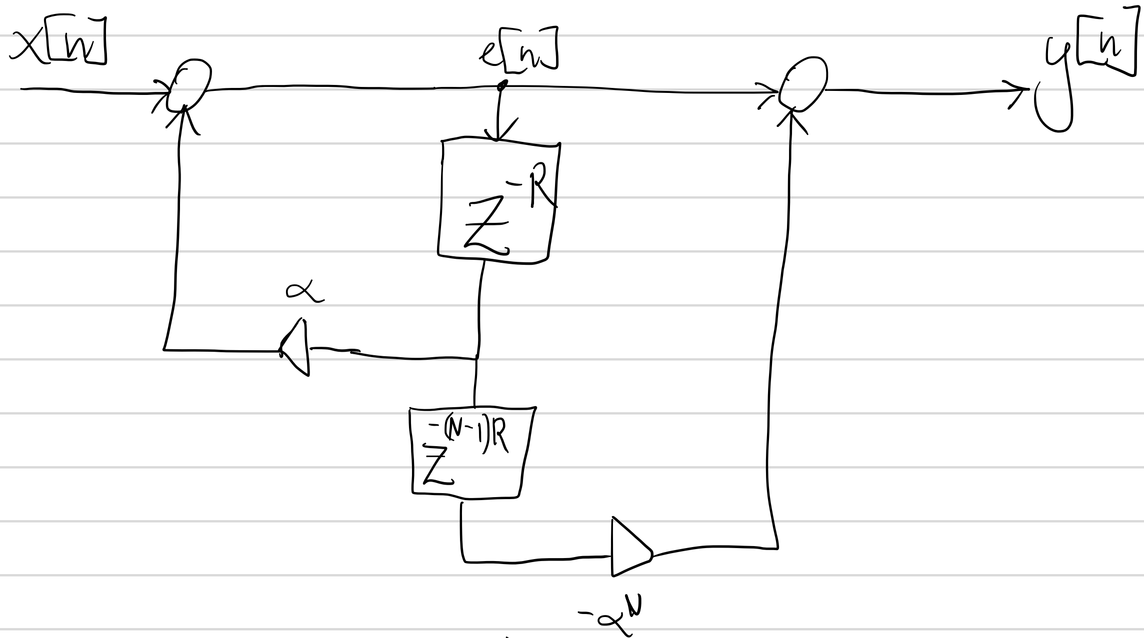
R means a delay after R samples, so with sample freq. F_s, we get a delay of

$$\text{delay} = \frac{R}{F_s}.$$

$$\Rightarrow \text{delay} @ 22050\,\text{Hz} = \frac{R}{22050\,\text{Hz}}$$

After loading "piano.wav" and playing around with different parameters our intuition is confirmed.
R controls the time between the original and the echo, and α controls the magnitude of the echo.

New filter



$$y[n] = -\alpha^N z^{-(N-1)R} \cdot z^{-R} e[n] + e[n]$$

$$e[n] = x[n] + \alpha z^{-R} e[n]$$

$$\Rightarrow \quad e[n] = \frac{1}{1-\alpha z^{-R}} x[n]$$

$$\Rightarrow y[n] = -\alpha^N z^{-NR} e[n] + e[n]$$

$$= (1 - \alpha^N z^{-N}) e[n]$$

$$= \frac{1 - \alpha^N z^{-NR}}{1 - \alpha z^{-R}} x[n]$$

$$\Rightarrow b_i = \begin{cases} 1 & , \ i = 1 \\ -\alpha^N & , \ i = NR \\ 0 & , \ \text{otherwise} \end{cases}$$

$$a_i = \begin{cases} 1 & , \ i = 1 \\ -\alpha & , \ i = R \\ 0 & , \ \text{otherwise} \end{cases}$$

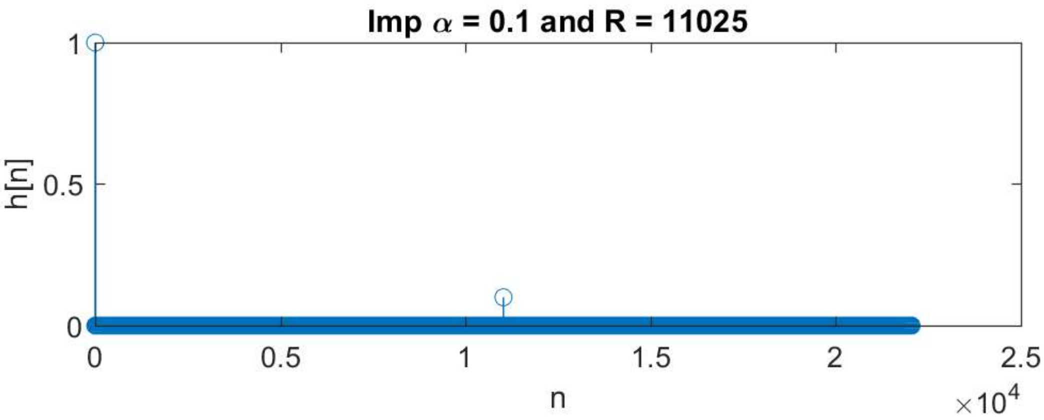
Comments:

- Impulse response has 5 peaks in addition to the first peak $(n = 0)$.
- The filtered sound sounds bigger as if played in a small hall.
- The one with more echoes sounds more natural since we can hear the echoes softly decaying, whereas the filter with one echo was a simple delay which sounded more mechanical.

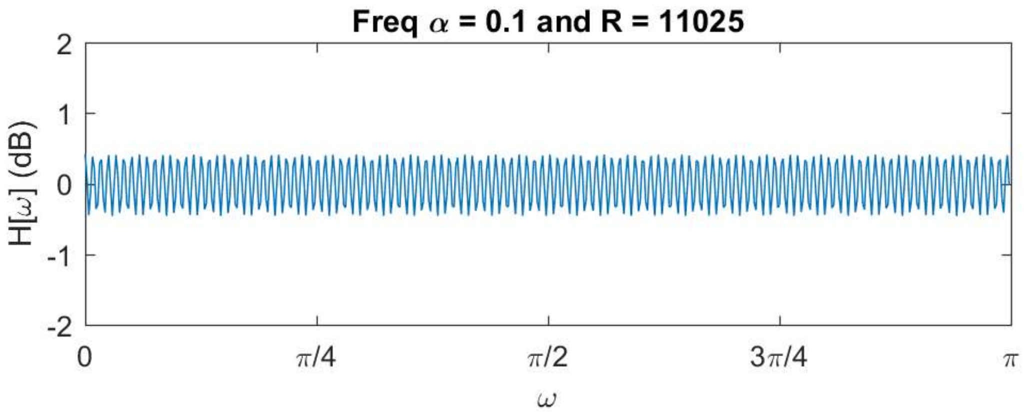
Problem 3

```
clear; close all; clc;
filename = "piano.wav";
[x, Fs] = audioread(filename{1});
```

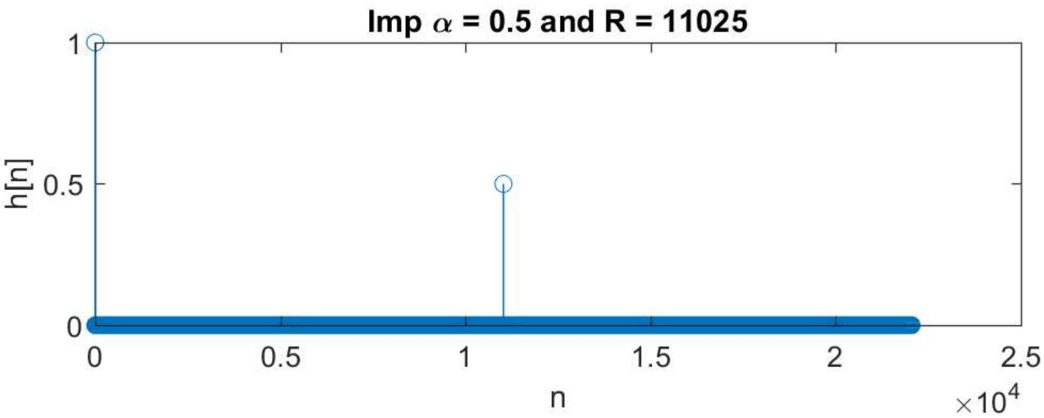
```
Rs = [11025, 22050, 33075];
alphas = [0.1 0.5 1];

for R=Rs
   for alpha=alphas
       figure();
       B = zeros(1,R+1);
       B(1) = 1; B(end) = alpha;
       A = 1;

       [H,T] = impz(B,A, 2*R);
       subplot(2,1,1);
       stem(T,H);
       title("Imp \alpha = " + string(alpha) + " and R = " + string(R));
       xlabel("n"); ylabel("h[n]");

       [H,W] = freqz(B, A);
       subplot(2,1,2);
       log_H = 10*log10(abs(H));
       plot(W,log_H);
       title("Freq \alpha = " + string(alpha) + " and R = " + string(R));
       xticks([0 pi/4 pi/2 3*pi/4 pi 5*pi/4 3*pi/2 7*pi/4 2*pi]);
       xticklabels({'0' '\pi/4' '\pi/2' '3\pi/4' '\pi' '5\pi/4' '3\pi/2' '7\pi/4' '2\pi'
});
       xlim([0 pi]);    xlabel("\omega");
       ylim([floor(min(log_H))-1, ceil(max(log_H))+1]);   ylabel("H[\omega] (dB)");

       %disp("Filtered sound: alpha=" + string(alpha) + "R=" + string(R));
       %y = filter(B,A,x);
       %soundsc(y,Fs);
       %input('---press enter to continue---', 's');
   end
end
```


Imp α = 0.1 and R = 11025
h[n]
n
$\times 10^4$
1
0.5
0
0
0.5
1
1.5
2
2.5

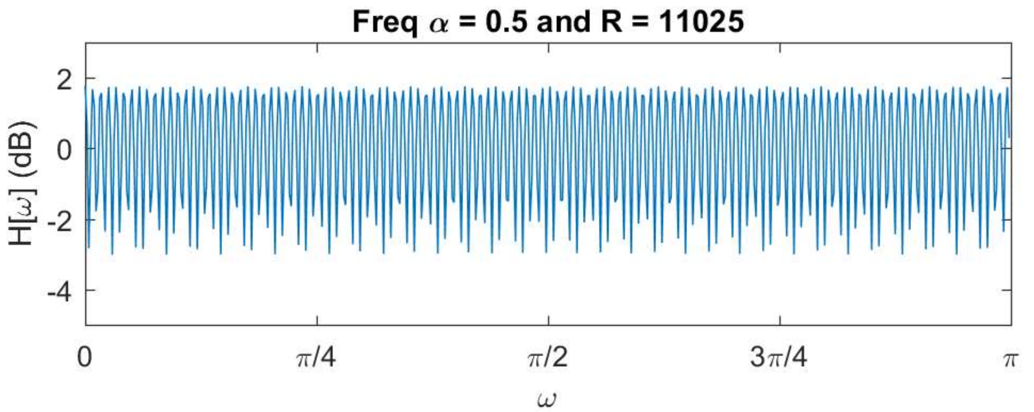
8


Freq α = 0.1 and R = 11025
H[ω] (dB)
ω
2
1
0
-1
-2
0
$\pi/4$
$\pi/2$
$3\pi/4$
π

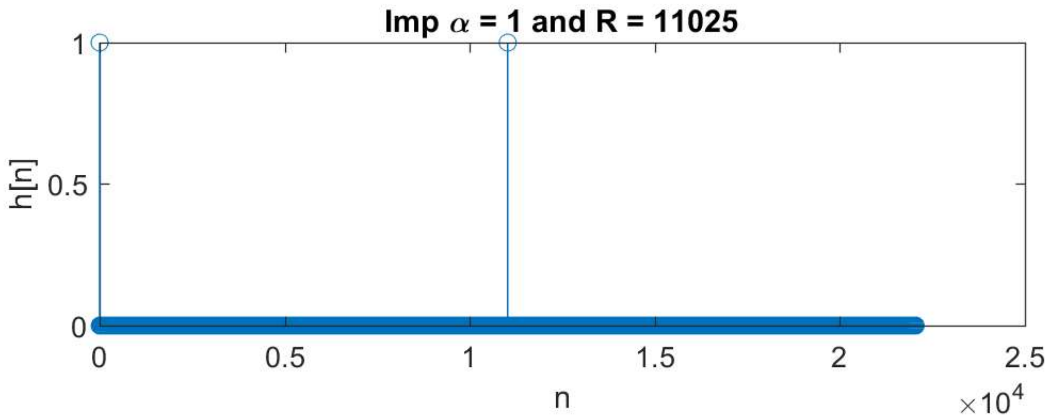
9


Imp α = 0.5 and R = 11025
h[n]
n
$\times 10^4$
1
0.5
0
0
0.5
1
1.5
2
2.5

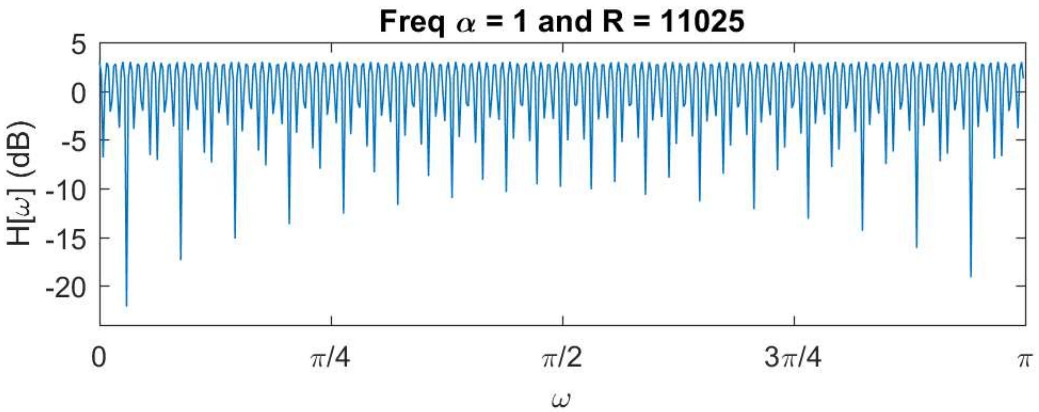
10


Freq α = 0.5 and R = 11025
H[ω] (dB)
ω
2
0
-2
-4
0
$\pi/4$
$\pi/2$
$3\pi/4$
π

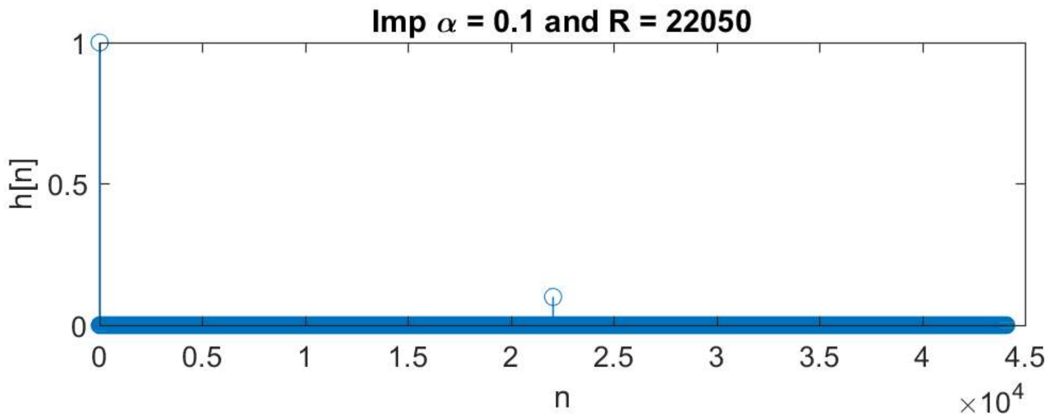
11


Imp α = 1 and R = 11025
h[n]
1
0.5
0
0
0.5
1
1.5
2
2.5
n
×10^4

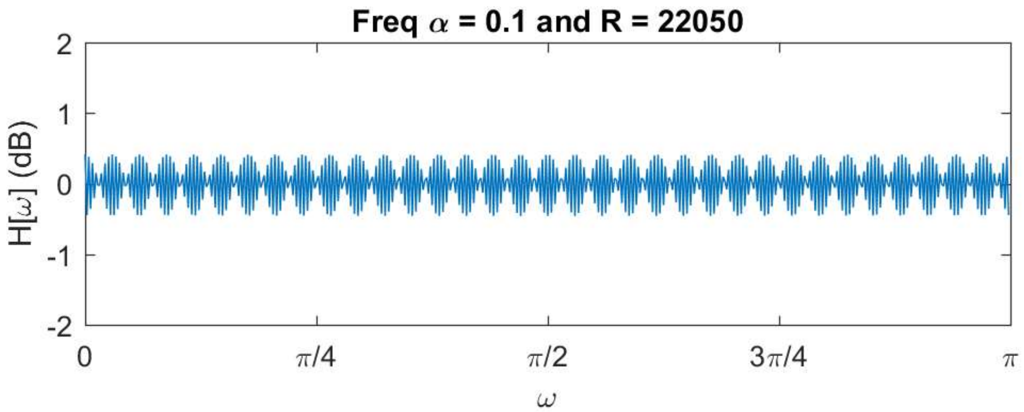
12


Freq α = 1 and R = 11025
H[ω] (dB)
5
0
-5
-10
-15
-20
0
π/4
π/2
3π/4
π
ω

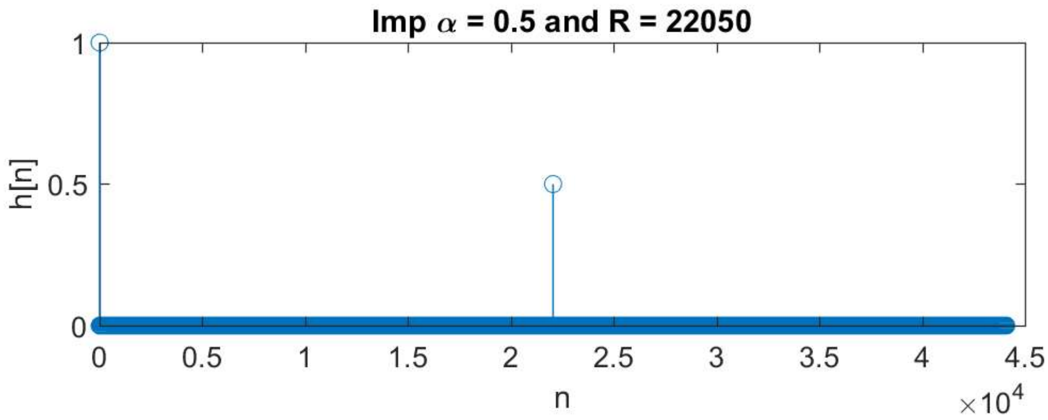
13


Imp α = 0.1 and R = 22050
h[n]
1
0.5
0
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
n
×10^4

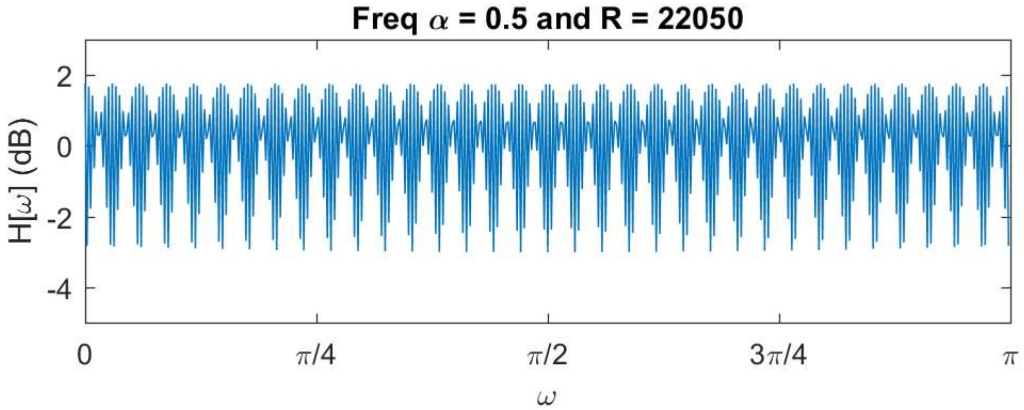
14


Freq α = 0.1 and R = 22050
H[ω] (dB)
2
1
0
-1
-2
0
π/4
π/2
3π/4
π
ω

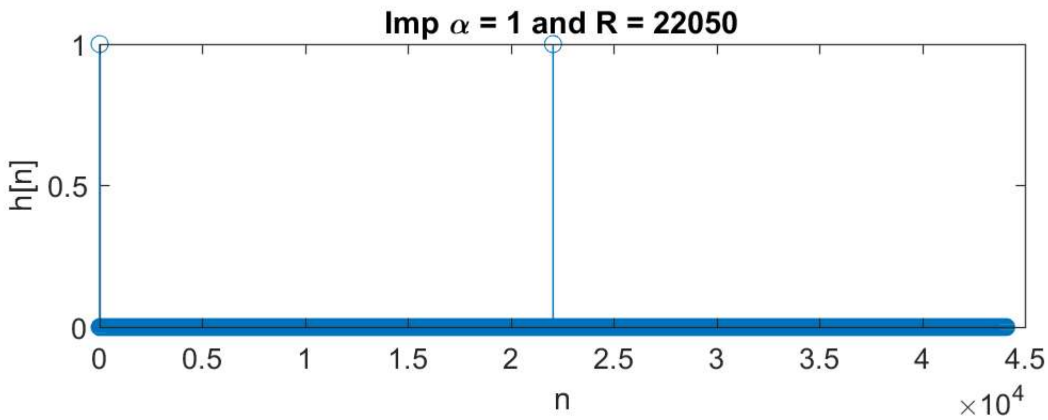
15


Imp α = 0.5 and R = 22050
h[n]
1
0.5
0
0 0.5 1 1.5 2 2.5 3 3.5 4 4.5
n
$\times 10^4$

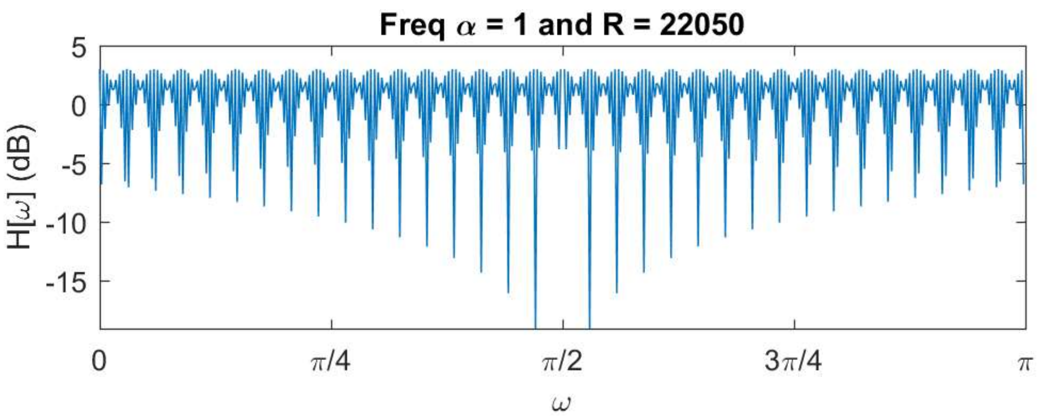
16


Freq α = 0.5 and R = 22050
H[ω] (dB)
2
0
-2
-4
0 $\pi/4$ $\pi/2$ $3\pi/4$ π
ω

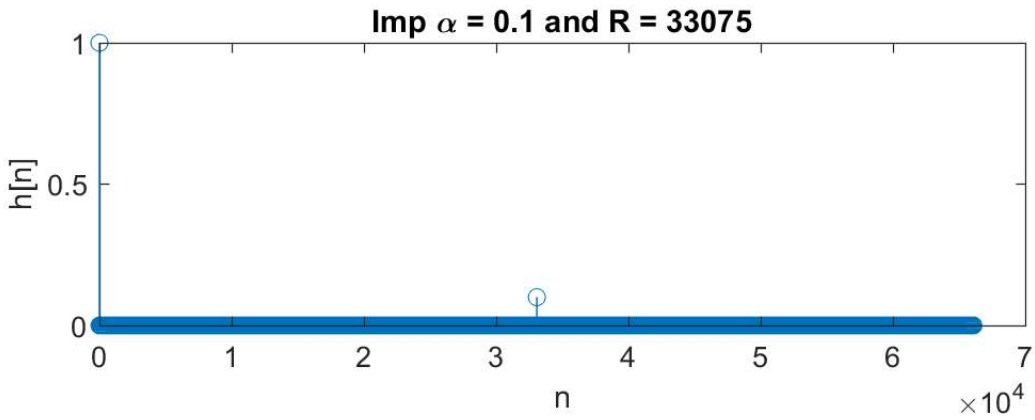
17


Imp α = 1 and R = 22050
h[n]
1
0.5
0
0 0.5 1 1.5 2 2.5 3 3.5 4 4.5
n
$\times 10^4$

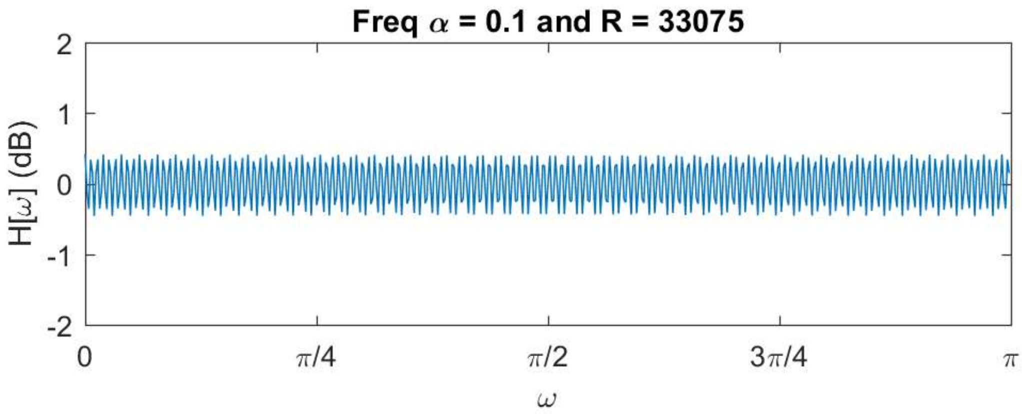
18


Freq α = 1 and R = 22050
H[ω] (dB)
5
0
-5
-10
-15
0 $\pi/4$ $\pi/2$ $3\pi/4$ π
ω

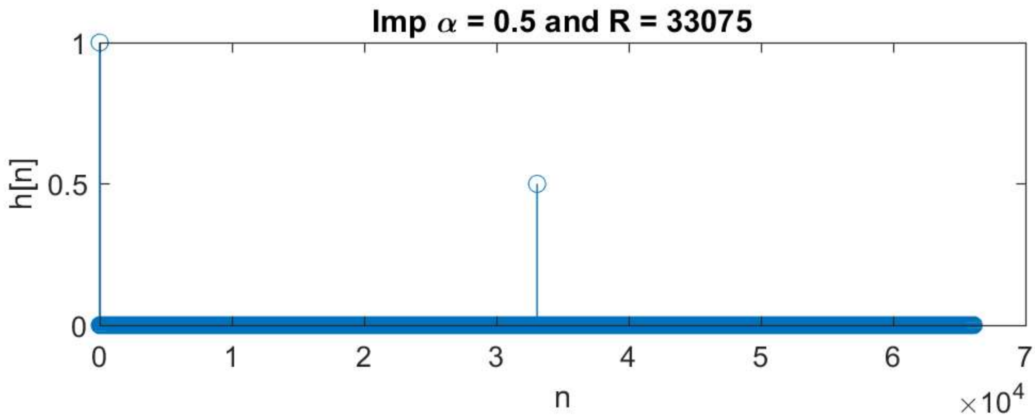
19


Imp α = 0.1 and R = 33075
h[n]
n
$\times 10^4$

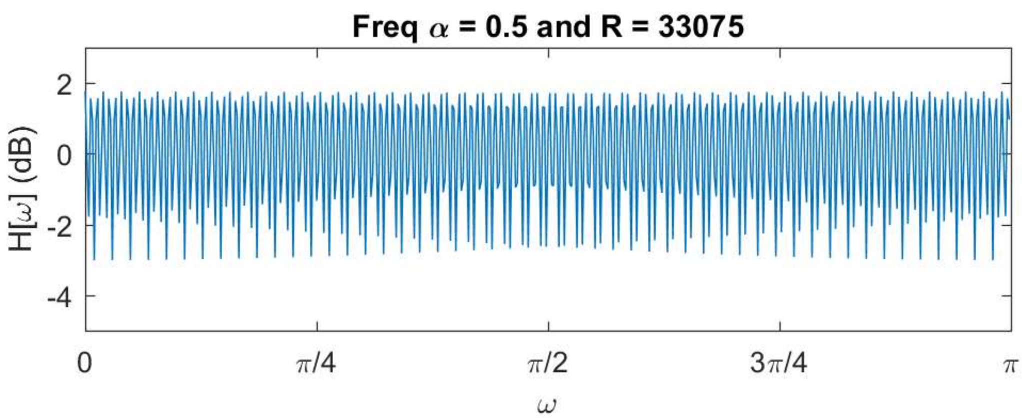
20


Freq α = 0.1 and R = 33075
H[ω] (dB)
ω

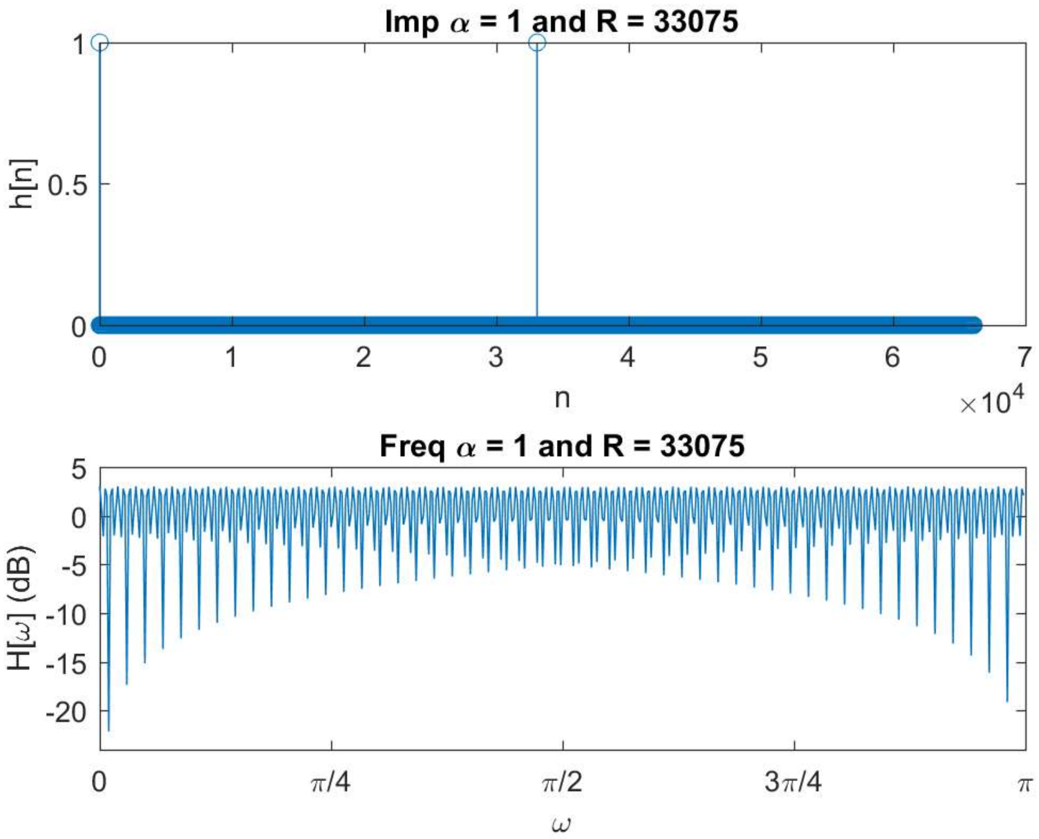
21


Imp α = 0.5 and R = 33075
h[n]
n
$\times 10^4$

22


Freq α = 0.5 and R = 33075
H[ω] (dB)
ω

23

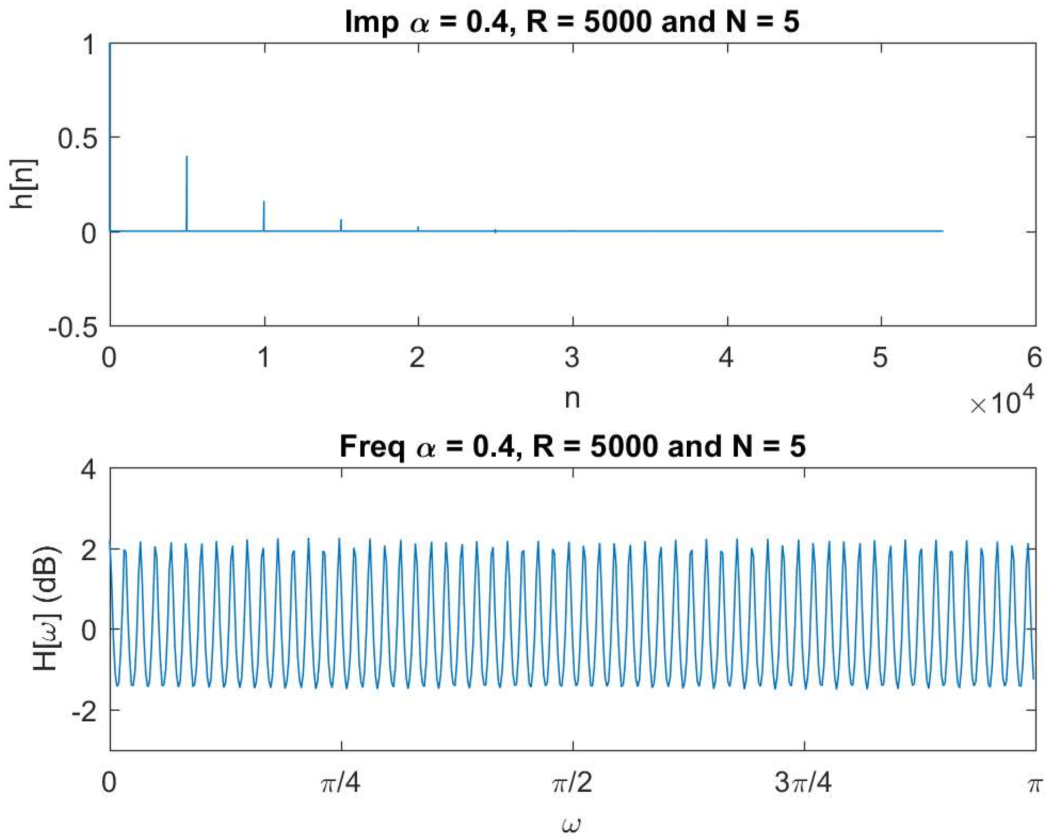


24

25

Problem 3 - new filter

```
clear; close all; clc;
filename = "piano.wav";
[x, Fs] = audioread(filename{1});

R=5000; alpha = 0.4; N=5;

figure();
B = zeros(1,N*R); A = zeros(1,R);

B(1) = 1; B(end) = -alpha^N;
A(1) = 1; A(end) = -alpha;

[H,T] = impz(B,A);
subplot(2,1,1);
plot(T,H);
title("Imp \alpha = " + string(alpha) + ", R = " + string(R)+ " and N = " + string(N));
xlabel("n"); ylabel("h[n]");

[H,W] = freqz(B, A);
subplot(2,1,2);
log_H = 10*log10(abs(H));
plot(W,log_H);
title("Freq \alpha = " + string(alpha) + ", R = " + string(R) + " and N = " + string(N))
;
xticks([0 pi/4 pi/2 3*pi/4 pi 5*pi/4 3*pi/2 7*pi/4 2*pi]);
xticklabels({'0' '\pi/4' '\pi/2' '3\pi/4' '\pi' '5\pi/4' '3\pi/2' '7\pi/4' '2\pi'});
xlim([0 pi]);    xlabel("\omega");
ylim([floor(min(log_H))-1, ceil(max(log_H))+1]);   ylabel("H[\omega] (dB)");

%y = filter(B,A,x);
```



26

27

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