TTK4135 Optimization and Control
Final exam solution — Spring 2017

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1 Quadratic programming (QP) (34 %)

a (6 %) First, define the Lagrangian

$$\begin{aligned}\mathcal{L}(x,\lambda) &= f(x) - \sum_{i\in\mathcal{I}\cup\mathcal{E}} \lambda_i c_i(x) \\ &= \frac{1}{2}x^T G x + x^T d - \textstyle\sum_{i\in\mathcal{I}\cup\mathcal{E}} \lambda_i \big(a_i^T x - b_i\big),\end{aligned} \tag{1}$$

and taking the partial derivatives with respect to x, to get

$$\nabla_x \mathcal{L}(x,\lambda) = Gx + d - \sum_{i\in\mathcal{I}\cup\mathcal{E}} \lambda_i a_i = Gx + d - A^T\lambda \tag{2}$$

where

$$A^T = [a_i]_{i\in\mathcal{I}\cup\mathcal{E}}. \tag{3}$$

Now assuming the LICQ hold at x^*, the first order necessary conditions (the KKT conditions) can be stated as

$$\begin{aligned} Gx^* + d - A^T\lambda^* &= 0, && \text{(4a)}\\ a_i^T x^* - b_i &= 0, \quad \forall i \in \mathcal{E}, && \text{(4b)}\\ a_i^T x^* - b_i &\geq 0, \quad \forall i \in \mathcal{I}, && \text{(4c)}\\ \lambda_i^* &\geq 0, \quad \forall i \in \mathcal{I}, && \text{(4d)}\\ \lambda_i^* \left(a_i^T x^* - b_i\right) &= 0, \quad \forall i \in \mathcal{E}\cup\mathcal{I}. && \text{(4e)} \end{aligned}$$

b (6 %) See definition 12.5, p.321 in the textbook. This is desirable because it can make convergence faster, since the determination of the active set is easier when this condition is satisfied.

c (4 %) The second order Taylor expansion around x_k is given by

$$m_k(x_k + p) \approx f_k + \nabla f_k^T p + \frac{1}{2} p^T \nabla^2 f_k p. \tag{5}$$

The Newton direction is defined as the step p that minimizes this second order approximation. Differentiating with respect to p and setting this equal to zero yields

$$\begin{aligned} \nabla_p m_k(x_k + p) &= \nabla f_k + \nabla^2 f_k p = 0, \\ p &= -\left(\nabla^2 f_k\right)^{-1} \nabla f_k = -G^{-1}\left(Gx_k + d\right) \end{aligned} \tag{6}$$

This is a descent direction if

$$p^T \nabla f_k < 0 \Leftrightarrow -\nabla f_k^T G^{-1} \nabla f_k \quad < 0. \tag{7}$$

This is satisfied, since $G \prec 0$ by definition, which implies that $G^{-1} \prec 0$. And as long as $\nabla f_k \neq 0$, i.e. x_k is not a stationary point.

d (6 %) The LICQ holds if $A \in \mathbb{R}^{3\times 4}$ matrix has full rank. In this case it should have rank 3.

Examples:

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{8}$$

here A_1 defines a set of constraints where the LICQ hold ($\text{rank}(A_1) = 3$), and A_2 one where it does not ($\text{rank}(A_2) = 2$).

e (12 %) Units are in million kroner. Define the following decision variables:

$$x_1 = \text{loan from Bank 1,} \tag{9a}$$
$$x_2 = \text{loan from Bank 2,} \tag{9b}$$
$$x_3 = \text{loan from Bank 3,} \tag{9c}$$
$$x_4 = \text{average yearly payment.} \tag{9d}$$

The corporation wants to loan 60 million kroner, this can be formulated as an equality constraint:

$$x_1 + x_2 + x_3 = 60. \tag{10}$$

All loans have to be positive:

$$x_i \geq 0,\ \forall i \in \{1, 2, 3\} \tag{11}$$

The mean yearly payment can be defined with the following equality constraint:

$$x_4 = \frac{1}{6}\left(1.75x_1 + 1.65x_2 + 1.45x_3\right) \tag{12}$$

Finally, the corporation wishes the yearly payments to be as similar as possible. This can be formulated with the following cost function:

$$\begin{aligned} f(x_1, x_2, x_3, x_4) = &\left(0.05x_2 + 0.4x_3 - x_4\right)^2 \\ &+ \left(0.15x_2 + 0.4x_3 - x_4\right)^2 \\ &+ \left(0.3x_1 + 0.25x_2 - x_4\right)^2 \\ &+ \left(0.4x_1 + 0.35x_2 + 0.35x_3 - x_4\right)^2 \\ &+ \left(0.5x_1 + 0.4x_2 + 0.15x_3 - x_4\right)^2 \\ &+ \left(0.55x_1 + 0.45x_2 + 0.15x_3 - x_4\right)^2 \end{aligned} \tag{13}$$

Since all the constraints are linear and the objective function f is quadratic, this is a standard QP problem.

2 KKT conditions (12 %)

a (6 %) Rewrite the first inequality to standard form:

$$-x_1^2 - x_2^2 + 4 \geq 0, \tag{14}$$

and define the Lagrangian:

$$\mathcal{L}(x, \lambda) = 2x_1 + x_2 - \lambda_1 \left(-x_1^2 - x_2^2 + 4\right) - \lambda_2 x_2. \tag{15}$$

Taking the derivative with respect to x, inserting $x^* = (-2.0, 0.0)$:

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = \begin{bmatrix} 2 - 4\lambda_1^* \\ 1 - \lambda_2^* \end{bmatrix} = 0. \tag{16}$$

Solving this with respect to λ^* gives $(\lambda_1^*, \lambda_2^*) = (\frac{1}{2}, 1)$. Clearly both Lagrange multipliers are positive, it can be verified that both constraints hold at x^* and that the rest of the KKT conditions hold at this point. Since the problem is convex (linear objective function and convex feasible set), the solution is a global solution.

b (6 %) By inspection, e.g. by drawing a figure, it can be seen that the optimal solution is $x^* = (1, 0)$. Defining the Lagrangian as in the previous task, and calculating the derivative yields:

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = \begin{bmatrix} 2 + 2x_1^* \lambda_1^* - \lambda_3^* \\ 1 + 2x_2^* \lambda_1^* - \lambda_2^* \end{bmatrix}. \tag{17}$$

Since we are not on the edge of the closed circle defined by the first constraint, this constraint is inactive. This implies that $\lambda_1^* = 0$ by the complimentary condition. Solving for λ_2^* and λ_3^* yields

$$\lambda_1^* = 0,\ \lambda_2^* = 1,\ \lambda_3^* = 2. \tag{18}$$

3 MPC and optimal control (24 %)

a (6 %) Make the following changes to the formulation

$$Q_t = Q,\ d_{xt} = d_x,\ d_{ut} = d_u,\ R_t = R,\ A_t = A,\ B_t = B. \tag{19}$$

b (12 %) The main idea here is to realize that with the given assumptions, the recursive Ricatti equation becomes the algebraic Ricatti equation since $P_t = P_{t+1}$ when $N \to \infty$:

$$P = Q + A^T P \big(I + BR^{-1}B^T P\big)^{-1} A \tag{20}$$

This equation can be solved explicitly with the additional constraint that $P \succ 0$. The control solution is then the infinite horizon LQ controller:

$$u_t = -Kx_t, \quad K = R^{-1}B^T P \big(I + BR^{-1}B^T P\big)^{-1} A. \tag{21}$$

c (6 %) Some pros and cons are listed here, others may also be valid.

Pros:

- The ability to include constraints.
- Flexibility in defining the cost function. This can allow fairly complex controllers to be implemented.
- The predictive capabilities of an MPC; the ability to calculate future inputs with respect to a time varying reference signal or model.
- Model based.
- Intuitive.
- Multivariable.

Cons:

- Increased implementation complexity.
- More complex stability proofs (if any).
- Model based (increased modelling effort).

4 Sequential quadratic programming (30 %)

a (6 %) The conditional in the while loop ensures that the next step will decrease the merit function value, analogous to the Armijo condition. See p. 540–541 in the textbook.

b (6 %) The merit function is designed to trade off objection function value decrease with constraint violation. It is important to include constraint violation in the line search, otherwise we may never converge to a feasible point.

c (6 %) This is defined as (see p. 435):

$$|z|^- = \max(\{0, -z\}). \tag{22}$$

Thus, the values at the given points are:

$$|3.5|^- = 0 \quad |-2.7|^- = 2.7. \tag{23}$$

d (6 %) The parameter μ will typically increase with iteration number k. This is to put more emphasis on the constraints as the algorithm convergence, to ensure that the solution is feasible.

e (6 %) Since there is no general way to solve non-convex optimization problems globally, we cannot guarantee that we will find a global solution using SQP. A simple way to improve our chances is to simply run the SQP algorithm with several different initializations. This will likely lead to different local solutions, which we can compare and use the best one.