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English version

Exam in TTK4135

Optimization and Control

Optimalisering og regulering

Friday May 29, 2015

Time: 09:00 – 13:00

Combination of allowed help remedies:
D — No printed or hand-written notes.
Certified calculator with empty memory.

In the Appendix potentially useful information is included.

1 Various topics (32 %)

Problem classification

a (8 %) For each of the optimization problems below:

- Classify the optimization problem.
- Is the problem convex?
- Suggest a suitable optimization algorithm to solve the problem.

a-1

$$\begin{aligned} \min \ & 3x_1 + x_2 + x_1^2 \\ \text{s.t.} \ & 4x_1 - x_2 \leq 5 \\ & 3x_1 + x_2 \geq 0 \end{aligned}$$

a-2

$$\begin{aligned} \min \ & -15x_1 - 4x_2 \\ \text{s.t.} \ & -4x_1 - 16x_2 \leq 25 \\ & \frac{5}{4}x_1 + \frac{1}{3}x_2 = 1 \end{aligned}$$

a-3

$$\min \ 100\left(x_2 - x_1^2\right)^2 + (1 - x_1)^2$$

a-4

$$\begin{aligned} \min \ & (x_1 - x_2)^2 + 2x_1 - x_2 \\ \text{s.t.} \ & 4x_1^2 + x_2^2 = 16 \\ & x_1 + \frac{4}{5}x_2 \geq 1 \end{aligned}$$

b (3 %) What is meant by an active-set method? Give two examples of active-set methods.

c (3 %) Which challenge is related to starting an active-set method? A short verbal explanation suffices.

d (5 %) Formulate the linear program (LP)

$$\begin{aligned} \max \ & 2x_1 - x_2 + 5x_3 \\ \text{s.t.} \ & x_1 + 2x_2 \geq 5 \\ & 3x_1 + 7x_2 - 2x_3 \leq 25 \\ & 5x_1 + 6x_3 = 40 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

in standard form (A.6) given in the appendix. Specify A, b and c explicitly.

e (7 %) Derive the KKT conditions (first-order necessary conditions) for the following quadratic program (QP):

$$\begin{aligned} \min_{x,y} \;& \frac{1}{2}(x-y)^T G(x-y) + c^T x + d^T y \\ \text{s.t.} \;& Ax = b \\ & Ey = h \end{aligned}$$

where $G = G^T$ is a symmetric matrix, and where x and y are vectors.

f (6 %) Consider the problem

$$\begin{aligned} \min \;& -x_1 - x_2 \\ \text{s.t.} \;& x_1^2 + x_2^2 \leq 2 \\ & x_2 - x_1^2 \geq 0 \end{aligned} \tag{1}$$

with solution $x^* = (1,1)^T$. Suppose that you could perturb the right-hand side of one of the constraints in (1) appropriately in order to decrease the objective value. Which constraint would you choose? Justify your answer. (Hint: Draw the contours and feasible region of (1).)

2 MPC and optimal control (32 %)

a (6 %) Explain the principle of model predictive control (MPC). Please include a figure to support your answer.

b (6 %) When using the dynamic optimization problem (A.9) in a linear MPC controller, which considerations must be made in the choice of the prediction horizon N?

c (8 %) A common problem when solving problem (A.9) in MPC applications is that it may give an infeasible solution. Explain which of the constraints in (A.9) this concerns, why this is often an issue in MPC applications, and suggest an approach to address the problem. Present a reformulation of problem (A.9) that fixes this issue.

d (6 %) Assume that there are no bounds (inequality constraints) on the state x_t or input u_t, that our system is described by a linear time-invariant (LTI) model, that all states are measured, and that we choose $Q_t := Q \succeq 0$, and $R_t := R \succ 0$ as constant matrices. Furthermore, we consider an infinite horizon ($N = \infty$), in which our optimal-control problem reduces to

$$\begin{aligned} \min \ & \sum_{t=0}^{\infty} \frac{1}{2} x_{t+1}^T Q x_{t+1} + \frac{1}{2} u_t^T R u_t \\ \text{s.t.} \ & x_{t+1} = A x_t + B u_t \\ & x_0 = \text{given} \end{aligned} \tag{2}$$

What type of controller does this optimization problem translate to? Use the fact that $P_t = P_{t+1} = P$ when considering an infinite horizon N, and state the equations for computing the resulting controller. What is the equation for computing P called, and why must we include the additional requirement $P = P^T \succeq 0$?

e (6 %) Suppose that some, but not all of the states x_t can be measured, and that a stationary Kalman filter therefore is applied to estimate the states. When using this estimated state $\hat{x}_t$ together with the controller derived from the optimization problem (2) in question **2d**, what is the final control structure called? State the requirements for stability of this closed-loop system.

3 Nonlinear programming and SQP (36 %)

Consider the following nonlinear program (NLP)

$$
\begin{aligned}
\min\ & x_2 \\
\text{s.t. } & x_1^3 - 2x_1^2 + x_2 \geq 0 \\
& (1-x_1)^3 - 2(1-x_1)^2 + x_2 \geq 0
\end{aligned}
\tag{3}
$$

with the global solution $x^* = \left(\frac{1}{2}, \frac{3}{8}\right)$.

a (8 %) Derive the KKT conditions for the NLP (3).

b (7 %) Compute the Hessian $\nabla^2_{xx}\mathcal{L}(x,\lambda)$. Which problems may this exact Hessian cause if used in the SQP Algorithm 18.3 given in the appendix? Suggest a method to circumvent the problem.

c (8 %) In each iteration of a line-search SQP algorithm, we compute the search direction p by solving a quadratic subproblem of the form

$$
\begin{aligned}
\min\ & f_k + \nabla f_k^T p + \frac{1}{2} p^T \nabla^2_{xx}\mathcal{L}_k p \\
\text{s.t. } & \nabla c_i(x_k)^T p + c_i(x_k) = 0, \quad i \in \mathcal{E} \\
& \nabla c_i(x_k)^T p + c_i(x_k) \geq 0, \quad i \in \mathcal{I}
\end{aligned}
\tag{4}
$$

Assume that we use a quasi-Newton approximation, in which we replace the initial Hessian $\nabla^2_{xx}\mathcal{L}_0$ with $B_0 = I$. Let $x_0 = (\frac{1}{2}, 0)$ be the chosen starting point. Show that the QP subproblem (4) in this case is given by

$$
\min\ p_2 + \frac{1}{2}(p_1^2 + p_2^2) \tag{5a}
$$

$$
\text{s.t. } -\frac{5}{4}p_1 + p_2 - \frac{3}{8} \geq 0 \tag{5b}
$$

$$
\frac{5}{4}p_1 + p_2 - \frac{3}{8} \geq 0 \tag{5c}
$$

d (4 %) Draw the feasible region for the QP problem (5), and the contours of the objective function (5a). By inspecting your plot, what is the solution to (5)? (You do not need to use an iterative method to solve the problem.)

e (6 %) Why do we use a merit-function in Algorithm 18.3? Suggest and formulate *explicitly* a suitable merit-function for the NLP (3).

f (3 %) Let $\alpha_0 = 1$, and show that Algorithm 18.3 converges to x^* in one iteration with the provided starting point $x_0 = (\frac{1}{2}, 0)$ and the search direction p computed from the QP (5).

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Utgave/Utgåve: bokmål/nynorsk

Eksamen i TTK4135

Optimalisering og regulering

Optimization and Control

Fredag 29. mai 2015

Tid: 09:00 – 13:00

Tillatte hjelpemidler / Tilletne hjelpemiddel:
D — Ingen trykte eller skrevne hjelpemidler. / Inga trykte eller skrevne hjelpemiddel.
Godkjent kalkulator med tomt minne. / Godkjend kalkulator med tomt minne.

Nyttig informasjon finnes i vedlegg. / Nyttig informasjon finns i vedlegg.
(Denne informasjonen er gitt på engelsk for å samsvare med pensumlitteraturen som den er hentet ifra.)

1 Forskjellige emner (32 %)

Problem klassifisering

a (8 %) For hvert av optimeringsproblemene under:

- Klassifiser optimeringsproblemet.
- Er problemet konvekst?
- Foreslå en egnet algoritme for å løse optimeringsproblemet.

a-1

$$\begin{aligned} \min\ & 3x_1 + x_2 + x_1^2 \\ \text{s.t.}\ & 4x_1 - x_2 \leq 5 \\ & 3x_1 + x_2 \geq 0 \end{aligned}$$

a-2

$$\begin{aligned} \min\ & -15x_1 - 4x_2 \\ \text{s.t.}\ & -4x_1 - 16x_2 \leq 25 \\ & \frac{5}{4}x_1 + \frac{1}{3}x_2 = 1 \end{aligned}$$

a-3

$$\min\ 100\left(x_2 - x_1^2\right)^2 + (1 - x_1)^2$$

a-4

$$\begin{aligned} \min\ & (x_1 - x_2)^2 + 2x_1 - x_2 \\ \text{s.t.}\ & 4x_1^2 + x_2^2 = 16 \\ & x_1 + \frac{4}{5}x_2 \geq 1 \end{aligned}$$

b (3 %) Hva menes med en aktiv-sett metode ("active-set method")? Gi to eksempler på aktiv-sett metoder.

c (3 %) Hvilken utfordring er relatert til å starte en aktiv-sett metode? En kort verbal forklaring er tilstrekkelig.

d (5 %) Formuler det lineære programmet (LP)

$$\begin{aligned} \max\ & 2x_1 - x_2 + 5x_3 \\ \text{s.t.}\ & x_1 + 2x_2 \geq 5 \\ & 3x_1 + 7x_2 - 2x_3 \leq 25 \\ & 5x_1 + 6x_3 = 40 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

på standard form (A.6) gitt i appendikset. Spesifiser A, b og c eksplisitt.

e (7 %) Utled KKT betingelsene (førsteordens nødvendige betingleser) for følgende kvadratiske program (QP):

$$\begin{aligned} \min_{x,y} \quad & \frac{1}{2}(x-y)^T G(x-y) + c^T x + d^T y \\ \text{s.t.} \quad & Ax = b \\ & Ey = h \end{aligned}$$

hvor $G = G^T$ er en symmetrisk matrise, og hvor x og y er vektorer.

f (6 %) Gitt optimeringsproblemet

$$\begin{aligned} \min \quad & -x_1 - x_2 \\ \text{s.t.} \quad & x_1^2 + x_2^2 \leq 2 \\ & x_2 - x_1^2 \geq 0 \end{aligned} \tag{1}$$

med løsning $x^* = (1,1)^T$. Anta at du kan gjøre en passende pertubasjon av høyre side av en av begrensningene ("constraints") i (1) for å minke objektverdien. Hvilken begrensning ville du valgt? Begrunn svaret ditt. (Hint: tegn konturene og gyldig område ("feasible region") for (1).)

2 MPC og optimalregulering (32 %)

a (6 %) Forklar prinsippet bak modell prediktiv regulering (MPC). Vennligst inkluder en figur i forklaringen din.

b (6 %) Når det dynamiske optimeringsproblemet (A.9) benyttes i en lineær MPC regulator, hvilke betrakninger må man gjøre i valg av prediksjonshorisont N?

c (8 %) Et vanlig problem når man løser optimeringsproblemet (A.9) i MPC applikasjoner er at man kan få ugyldige løsninger ("infeasible solutions"). Forklar hvilke av begresningene i (A.9) dette gjelder, hvorfor dette ofte er en utfordring i MPC applikasjoner, og foreslå en måte for å håndtere problemet. Formuler en reformulering av optimeringsproblemet (A.9) som løser dette problemet.

d (6 %) Anta at det er ingen ulikhetsbetingelser ("inequality constraints") på tilstandene x_t eller inngangene u_t, at systemet vårt er beskrevet av en lineær tidsinvariant (LTI) modell, at alle tilstander kan måles, og at vi velger $Q_t := Q \succeq 0$, og $R_t := R \succ 0$ som konstante matriser. Videre, så bekrakter vi en uendelig horisont ($N = \infty$), hvorav optimalregulingsproblemet vårt reduseres til

$$\begin{aligned} \min \quad & \sum_{t=0}^{\infty} \frac{1}{2} x_{t+1}^T Q x_{t+1} + \frac{1}{2} u_t^T R u_t \\ \text{s.t.} \quad & x_{t+1} = A x_t + B u_t \\ & x_0 = \text{gitt} \end{aligned} \tag{2}$$

Hvilken type regulator gir dette optimeringsproblemet? Bruk at $P_t = P_{t+1} = P$ når man betrakter en uendelig horisont N, og uttrykk likningene for å beregne den resulterende regulatoren. Hva kalles likningen for å beregne P, og hvorfor må vi i tillegg kreve at $P = P^T \succeq 0$?

e (6 %) Anta at noen, men ikke alle av tilstandene x_t kan måles, og at et stasjonært Kalman filter derfor benyttes for å estimere tilstandene. Når man benytter denne estimerte tilstanden $\hat{x}_t$ sammen med regulatoren utledet fra optimeringsproblemet (2) i spørsmål **2d**, hva kalles den endelige regulatorstrukturen? Oppgi kravene for stabilitet av dette lukket-sløyfe systemet.

3 Ulineær optimering og SQP (36 %)

Betrakt det følgende ulineære optimeringsproblemet (NLP)

$$\begin{aligned}\min\ & x_2 \\ \text{s.t.}\ & x_1^3 - 2x_1^2 + x_2 \geq 0 \\ & (1-x_1)^3 - 2(1-x_1)^2 + x_2 \geq 0\end{aligned} \tag{3}$$

med global optimal løsning $x^* = \left(\frac{1}{2}, \frac{3}{8}\right)$.

a (8 %) Utled KKT betingelsene for NLP (3).

b (7 %) Beregn Hessian $\nabla^2_{xx}\mathcal{L}(x,\lambda)$. Hvilke problem kan denne eksakte Hessian gi dersom den benyttes i SQP Algoritme 18.3 gitt i appendikset? Foreslå en metode for å håndtere problemet.

c (8 %) I hver iterasjon av en linje-søk SQP algoritme, så beregnes søkeretning p ved å løse et kvadratisk subproblem på formen

$$\begin{aligned}\min\ & f_k + \nabla f_k^T p + \frac{1}{2} p^T \nabla^2_{xx}\mathcal{L}_k p \\ \text{s.t.}\ & \nabla c_i(x_k)^T p + c_i(x_k) = 0, \qquad i \in \mathcal{E} \\ & \nabla c_i(x_k)^T p + c_i(x_k) \geq 0, \qquad i \in \mathcal{I}\end{aligned} \tag{4}$$

Anta at vi benytter en quasi-Newton tilnærming, hvor vi erstatter initiell Hessian $\nabla^2_{xx}\mathcal{L}_0$ med $B_0 = I$. La $x_0 = (\frac{1}{2}, 0)$ være valgt startpunkt. Vis at QP subproblemet (4) i dette tilfellet er gitt av

$$\min\ p_2 + \frac{1}{2}(p_1^2 + p_2^2) \tag{5a}$$

$$\text{s.t.}\ -\frac{5}{4}p_1 + p_2 - \frac{3}{8} \geq 0 \tag{5b}$$

$$\frac{5}{4}p_1 + p_2 - \frac{3}{8} \geq 0 \tag{5c}$$

d (4 %) Tegn gyldig område ("feasible region") for QP problemet (5), og konturene til objektfunksjonen (5a). Ved å betrakte plottet dit, hva er løsningen til (5)? (Du trenger ikke å benytte en iterativ metode for å løse problemet.)

e (6 %) Hvorfor benyttes en merit-funksjon i Algoritme 18.3? Foreslå og formuler *eksplisitt* en egnet merit-funksjon for det ulineære optimeringsproblemet (3).

f (3 %) La $\alpha_0 = 1$, og vis at Algoritme 18.3 konvergerer til x^* i én iterasjon med gitt startpunkt $x_0 = (\frac{1}{2}, 0)$ og søkeretning p beregnet av QP (5).

Appendix

Part 1 Optimization Problems and Optimality Conditions

A general formulation for constrained optimization problems is

$$\min_{x \in \mathbb{R}^n} \quad f(x) \tag{A.1a}$$
$$\text{s.t.} \quad c_i(x) = 0, \qquad i \in \mathcal{E} \tag{A.1b}$$
$$c_i(x) \geq 0, \qquad i \in \mathcal{I} \tag{A.1c}$$

where f and the functions c_i are all smooth, differentiable, real-valued functions on a subset of $\mathbb{R}^n$, and $\mathcal{E}$ and $\mathcal{I}$ are two finite sets of indices.

The Lagrangean function for the general problem (A.1) is

$$\mathcal{L}(x, \lambda) = f(x) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i c_i(x) \tag{A.2}$$

The KKT-conditions for (A.1) are given by:

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = 0 \tag{A.3a}$$
$$c_i(x^*) = 0, \qquad i \in \mathcal{E} \tag{A.3b}$$
$$c_i(x^*) \geq 0, \qquad i \in \mathcal{I} \tag{A.3c}$$
$$\lambda_i^* \geq 0, \qquad i \in \mathcal{I} \tag{A.3d}$$
$$\lambda_i^* c_i(x^*) = 0, \qquad i \in \mathcal{E} \cup \mathcal{I} \tag{A.3e}$$

2nd order (sufficient) conditions for (A.1) are given by:

$$w \in \mathcal{C}(x^*, \lambda^*) \Leftrightarrow \begin{cases} \nabla c_i(x^*)^\top w = 0 & \text{for all } i \in \mathcal{E} \\ \nabla c_i(x^*)^\top w = 0 & \text{for all } i \in \mathcal{A}(x^*) \cap \mathcal{I} \text{ with } \lambda_i^* > 0 \\ \nabla c_i(x^*)^\top w \geq 0 & \text{for all } i \in \mathcal{A}(x^*) \cap \mathcal{I} \text{ with } \lambda_i^* = 0 \end{cases} \tag{A.4}$$

Theorem 1: (Second-Order Sufficient Conditions) *Suppose that for some feasible point $x^* \in \mathbb{R}^n$ there is a Lagrange multiplier vector λ^* such that the KKT conditions* (A.3) *are satisfied. Suppose also that*

$$w^\top \nabla^2_{xx} \mathcal{L}(x^*, \lambda^*) w > 0, \qquad \text{for all } w \in \mathcal{C}(x^*, \lambda^*),\ w \neq 0. \tag{A.5}$$

Then x^* is a strict local solution for (A.1).

LP problem in standard form:

$$\begin{aligned} \min_{x} \quad & f(x) = c^\top x && \text{(A.6a)} \\ \text{s.t.} \quad & Ax = b && \text{(A.6b)} \\ & x \geq 0 && \text{(A.6c)} \end{aligned}$$

where $A \in \mathbb{R}^{m \times n}$ and rank $A = m$.

QP problem in standard form:

$$\begin{aligned} \min_{x} \quad & f(x) = \frac{1}{2} x^\top G x + x^\top c && \text{(A.7a)} \\ \text{s.t.} \quad & a_i^\top x = b_i, \qquad i \in \mathcal{E} && \text{(A.7b)} \\ & a_i^\top x \geq b_i, \qquad i \in \mathcal{I} && \text{(A.7c)} \end{aligned}$$

where G is a symmetric $n \times n$ matrix, $\mathcal{E}$ and $\mathcal{I}$ are finite sets of indices and c, x and $\{a_i\}, i \in \mathcal{E} \cup \mathcal{I}$, are vectors in $\mathbb{R}^n$. Alternatively, the equalities can be written $Ax = b$, $A \in \mathbb{R}^{m \times n}$.

Iterative method:

$$\begin{aligned} x_{k+1} &= x_k + \alpha_k p_k && \text{(A.8a)} \\ & x_0 \text{ given} && \text{(A.8b)} \\ & x_k, p_k \in \mathbb{R}^n, \ \alpha_k \in \mathbb{R} && \text{(A.8c)} \end{aligned}$$

p_k is the search direction and α_k is the line search parameter.

Part 2 Optimal Control

A typical open-loop optimal control problem on the time horizon 0 to N is

$$\min_{z\in\mathbb{R}^n} f(z) = \sum_{t=0}^{N-1} \frac{1}{2}x_{t+1}^\top Q_{t+1}x_{t+1} + d_{xt+1}x_{t+1} + \frac{1}{2}u_t^\top R_t u_t + d_{ut}u_t \tag{A.9a}$$

subject to

$$\begin{aligned}
x_{t+1} &= A_t x_t + B_t u_t, && t = 0,\ldots,N-1 && \text{(A.9b)}\\
x_0 &= \text{given} && && \text{(A.9c)}\\
x^{\text{low}} &\le x_t \le x^{\text{high}}, && t = 1,\ldots,N && \text{(A.9d)}\\
u^{\text{low}} &\le u_t \le u^{\text{high}}, && t = 0,\ldots,N-1 && \text{(A.9e)}\\
-\Delta u^{\text{high}} &\le \Delta u_t \le \Delta u^{\text{high}}, && t = 0,\ldots,N-1 && \text{(A.9f)}\\
Q_t &\succeq 0 && t = 1,\ldots,N && \text{(A.9g)}\\
R_t &\succeq 0 && t = 0,\ldots,N-1 && \text{(A.9h)}
\end{aligned}$$

where

$$\begin{aligned}
u_t &\in \mathbb{R}^{n_u} && \text{(A.9i)}\\
x_t &\in \mathbb{R}^{n_x} && \text{(A.9j)}\\
\Delta u_t &= u_t - u_{t-1} && \text{(A.9k)}\\
z^\top &= (x_1^\top,\ldots,x_N^\top, u_0^\top,\ldots,u_{N-1}^\top) && \text{(A.9l)}
\end{aligned}$$

The subscript t denotes discrete time sampling instants.

The optimization problem for linear quadratic control of discrete dynamic systems is given by

$$\min_{z\in\mathbb{R}^n} f(z) = \sum_{t=0}^{N-1} \frac{1}{2}x_{t+1}^\top Q_{t+1}x_{t+1} + \frac{1}{2}u_t^\top R_t u_t \tag{A.10a}$$

subject to

$$\begin{aligned}
x_{t+1} &= A_t x_t + B_t u_t && \text{(A.10b)}\\
x_0 &= \text{given} && \text{(A.10c)}
\end{aligned}$$

where

$$\begin{aligned}
u_t &\in \mathbb{R}^{n_u} && \text{(A.10d)}\\
x_t &\in \mathbb{R}^{n_x} && \text{(A.10e)}\\
z^\top &= (x_1^\top,\ldots,x_N^\top, u_0^\top \ldots, u_{N-1}^\top) && \text{(A.10f)}
\end{aligned}$$

Theorem 2: *The solution of (A.10) with $Q_t \succeq 0$ and $R_t \succ 0$ is given by*

$$u_t = -K_t x_t \tag{A.11a}$$

where the feedback gain matrix is derived by

$$K_t = R_t^{-1} B_t^\top P_{t+1}(I + B_t R_t^{-1} B_t^\top P_{t+1})^{-1} A_t, \qquad t = 0, \dots, N-1 \tag{A.11b}$$

$$P_t = Q_t + A_t^\top P_{t+1}(I + B_t R_t^{-1} B_t^\top P_{t+1})^{-1} A_t, \qquad t = 0, \dots, N-1 \tag{A.11c}$$

$$P_N = Q_N \tag{A.11d}$$

Part 3 Sequential quadratic programming (SQP)

Algorithm 18.3 (Line Search SQP Algorithm).

Choose parameters $\eta \in (0, 0.5)$, $\tau \in (0, 1)$, and an initial pair (x_0, λ_0);
Evaluate f_0, ∇f_0, c_0, A_0;
If a quasi-Newton approximation is used, choose an initial $n \times n$ symmetric positive definite Hessian approximation B_0, otherwise compute $\nabla^2_{xx}\mathcal{L}_0$;
repeat until a convergence test is satisfied
    Compute p_k by solving (18.11); let $\hat{\lambda}$ be the corresponding multiplier;
    Set $p_\lambda \leftarrow \hat{\lambda} - \lambda_k$;
    Choose μ_k to satisfy (18.36) with $\sigma = 1$;
    Set $\alpha_k \leftarrow 1$;
    while $\phi_1(x_k + \alpha_k p_k; \mu_k) > \phi_1(x_k; \mu_k) + \eta \alpha_k D_1(\phi(x_k; \mu_k) p_k)$
        Reset $\alpha_k \leftarrow \tau_\alpha \alpha_k$ for some $\tau_\alpha \in (0, \tau]$;
    end (while)
    Set $x_{k+1} \leftarrow x_k + \alpha_k p_k$ and $\lambda_{k+1} \leftarrow \lambda_k + \alpha_k p_\lambda$;
    Evaluate f_{k+1}, ∇f_{k+1}, c_{k+1}, A_{k+1}, (and possibly $\nabla^2_{xx}\mathcal{L}_{k+1}$);
    If a quasi-Newton approximation is used, set
        $s_k \leftarrow \alpha_k p_k$ and $y_k \leftarrow \nabla_x \mathcal{L}(x_{k+1}, \lambda_{k+1}) - \nabla_x \mathcal{L}(x_k, \lambda_{k+1})$,
    and obtain B_{k+1} by updating B_k using a quasi-Newton formula;
end (repeat)