Øving 3

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4.1

linear and homogeneous: 8
linear (and inhogeneous): 1, 2, 4, 6
nonlinear: 3, 5, 7

13) $y_1'' - y_1' - 6y = 0$ (1)

$\Leftrightarrow (e^{3t})'' - (e^{3t})' - 6e^{3t} = 0$

$9e^{3t} - 3e^{3t} - 6e^{3t} = 0$ ✓

$y_2'' - y_2' - 6y_2 = 0$ (2)

$(e^{-2t})'' - (e^{-2t})' - 6e^{-2t} = 0$

$4e^{-2t} + 2e^{-2t} - 6e^{-2t} = 0$ ✓

$(c_1y_1 + c_2y_2)'' - (c_1y_1 + c_2y_2)' - 6(c_1y_1 + c_2y_2) = 0$

$\Leftrightarrow c_1y_1'' + c_2y_2'' - c_1y_1' - c_2y_2' - 6c_1y_1 - 6c_2y_2 = 0$

$\Leftrightarrow c_1\underbrace{(y_1'' - y_1' - 6y_1)}_{=0} + c_2\underbrace{(y_2'' - y_2' - 6y_2)}_{=0} = 0$

as shown in (1) as shown in (2)

$c_1 \cdot 0 + c_2 \cdot 0 = 0$ ✓

14) $y' + 4y = 0 \quad (*)$

$y_1 = \cos(2t) \Rightarrow y_1'' = -4\cos(2t)$
$y_2 = \sin(2t) \Rightarrow y_2'' = -4\sin(2t)$

(1) $y_1'' + 4y_1 = -4\cos 2t + 4\cos 2t$
$= 0$ ✓

(2) $y_2'' + 4y_2 = -4\sin 2t + 4\sin 2t$
$= 0$ ✓

$y = c_1 y_1 + c_2 y_2$

$(*)$: $(c_1y_1 + c_2y_2)'' + 4(c_1y_1 + c_2y_2)$

$= c_1y_1'' + c_2y_2'' + 4c_1y_1 + 4c_2y_2$

$= c_1\underbrace{(y_1'' + 4y_1)}_{=0 \ (1)} + c_2\underbrace{(y_2'' + 4y_2)}_{=0 \ (2)}$

$= 0$ ✓

17) e^{-t} is not a constant multiple of e^{2t}, so they are linearly independent.

$$W(t) = \begin{vmatrix} e^{-t} & e^{2t} \\ -e^{-t} & 2e^{2t} \end{vmatrix}$$

$$= e^{-t} \cdot 2e^{2t} - e^{2t}(-e^{-t})$$

$$= e^{-t}(2e^{2t} - e^{2t})$$

$$= e^{-t}e^{2t}$$

$$= e^{-t+2t}$$

$$= \underline{e^{t} \neq 0}$$

Since $W(t) \neq 0$, y_1 and y_2 are linearly independant.

18) $\cos(3t)$ is a phase-shifted version of $\sin(3t)$, and thus one can't be constant multiple of the other, so they are linearly independent.

$$W(t) = \begin{vmatrix} \cos 3t & \sin 3t \\ -3\sin 3t & 3\cos 3t \end{vmatrix}$$

$$= \cos(3t) \cdot 3\cos(3t) - (\sin(3t)(-3\sin 3t))$$

$$= 3\cos^2 3t + 3\sin^2 3t$$

$$= \underline{3}$$

Since $W(t) \neq 0$, y_1 and y_2 are linearly independant.

19) Same as with (18); y_1 and y_2 contain a common factor of e^{-2t}, but the trig functions are not constant multiples of each other so they are linearly independent.

$$y_1 = e^{-2t}\cos(3t)$$
$$y_1' = -2e^{-2t}\cos(3t) - 3e^{-2t}\sin(3t)$$

$$y_2 = e^{-2t}\sin(3t)$$
$$y_2' = -2e^{-2t}\sin(3t) + 3e^{-2t}\cdot\cos(3t)$$

$$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= e^{-2t}\cos(3t)\left(-2e^{-2t}\sin(3t) + 3e^{-2t}\cos(3t)\right) - e^{-2t}\sin(3t)\left(-2e^{-2t}\cos(3t) - 3e^{-2t}\sin(3t)\right)$$

$$= e^{-2t}\cdot e^{-2t}\left(-2\cos(3t)\sin(3t) + 3\cos^2(3t)\right) + e^{-2t}e^{-2t}\left(2\sin(3t)\cos(3t) + 3\sin^2(3t)\right)$$

$$= e^{-4t}\left(\cancel{-2\cos(3t)\sin(3t)} + \cancel{2\sin(3t)\cos(3t)} + 3\cos^2(3t) + 3\sin^2(3t)\right)$$

$$= 3e^{-4t}$$

Since $W(t) = 3e^{-4t} \neq 0$, y_1 and y_2 are linearly independent.

20) Here $y_2(t) = y_1(t) \cdot t$ so they are obviously linearly independent.

$y_1(t) = e^{-3t}$

$y_1'(t) = -3e^{-3t}$

$y_2(t) = te^{-3t}$

$y_2'(t) = t\cdot(-3)e^{-3t} + e^{-3t} = (1-3t)e^{-3t}$

$$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= e^{-3t}\cdot(1-3t)e^{-3t} - te^{-3t}\cdot(-3)e^{-3t}$$

$$= e^{-6t}(1\cancel{-3t}\cancel{+3t})$$

$$= e^{-6t} \neq 0$$

Since $W(t) = e^{-6t} \neq 0$, y_1 and y_2 are linearly independent.

21) (optional)

$y_1(t) = t^2$, $y_2(t) = t|t|$

$y_1'(t) = 2t$, $y_2'(t) = 2|t|$

$$W(t) = \begin{vmatrix} t^2 & t|t| \\ 2t & 2|t| \end{vmatrix}$$

$$= 2|t|t^2 - 2t \cdot t \cdot |t|$$

$$= 0$$

The proposition says "if dependent then the Wronskian will be 0", but the implication only goes one way.

26) a) $y_1(t) = t^2$

$y_1'(t) = 2t$

$y_1''(t) = 2$

$\Rightarrow t^2 \cdot y_1'' + t \cdot y_1' - 4y_1$

$= 2t^2 + 2t \cdot t - 4 \cdot t^2$

$= 0$ ✓

b) $y_2(t) = v(t) \cdot t^2$

$y_2'(t) = v'(t) t^2 + v(t) \cdot 2t$

$y_2''(t) = v''(t) \cdot t^2 + v'(t) 2t + v'(t) \cdot 2t + v(t) \cdot 2$

$= t^2 v''(t) + 4t v'(t) + 2v(t)$

Inserted into the equation:

$t^2[t^2 v''(t) + 4t v'(t) + 2v(t)] + t[t^2 v'(t) + 2t v(t)] - 4t^2 v(t) = 0$

$t^4 v''(t) + 4t^3 v'(t) + 2t^2 v(t) + t^3 v'(t) + 2t^2 v(t) - 4t^2 v(t) = 0 \quad | \div t^2$

$t^2 v''(t) + 4t v'(t) \cancel{+ 2v(t)} + t v'(t) \cancel{+ 2v(t)} \cancel{- 4v(t)} = 0$

$t^2 v''(t) + 5t v'(t) = 0 \quad | \div t$

$5v'(t) + t v''(t) = 0$ ✓

$$5v' + tv'' = 0$$

Let $x = v'$

$$\Rightarrow 5x + tx' = 0$$

$$\frac{x'}{x} = -\frac{5}{t}$$

$$\frac{\frac{dx}{dt}}{x} = -\frac{5}{t} \quad \Big| \cdot dt$$

$$\frac{dx}{x} = -5 \cdot \frac{1}{t} dt$$

$$\ln|x| = -5 \cdot \ln|t| + C$$

$$|x| = e^{-5\ln|t|} \cdot e^{C}$$

$$x = \frac{C_1}{t^5}$$

$$\Rightarrow v' = \frac{C_1}{t^5}$$

$$dv = C_1 t^{-5} dt$$

$$v = C_1 \cdot \frac{1}{-5+1} \cdot t^{-4} + C_2$$

$$v = -\frac{C_1}{4t^4} + C_2$$

$$\Rightarrow y_2(t) = v(t) \cdot t^2$$

$$y_2(t) = C_2 t^2 - \frac{C_1}{4t^2}$$

The general solution is then:

$$y(t) = c_1 t^2 + c_2 \cdot \left(c_3 t^2 - \frac{c_4}{4t^2}\right)$$

? this looks wrong but I don't see how to remove two of the undetermined coefficients, without loosing the structure $y = y_1 c_1 + y_2 c_2$

4.3

1) $y'' - y' - 2y = 0$

$\Downarrow$

$r^2 - r - 2 = 0$

$\Leftrightarrow r = \frac{1 \pm \sqrt{1+4\cdot 2}}{2}$

$= \frac{1 \pm 3}{2}$

$\underbrace{r_1 = 2}_{y_1 = e^{2x}}, \quad \underbrace{r_2 = -1}_{y_2 = e^{-x}}$

$\Rightarrow \underline{\underline{y(x) = C_1 e^{2x} + C_2 e^{-x}}}$

2) $2y'' - 3y' - 2y = 0$

$\Downarrow$

$2r^2 - 3r - 2 = 0$

$\Leftrightarrow r = \frac{+3 \pm \sqrt{9 + 4\cdot 2\cdot 2}}{2\cdot 2}$

$= \frac{3 \pm 5}{4}$

$\underbrace{r_1 = -\frac{1}{2}}, \quad r_2 = 2$

$y_1 = e^{-\frac{1}{2}x}, \quad y_2 = e^{2x}$

$\Rightarrow \underline{\underline{y(x) = C_1 e^{\frac{1}{2}x} + C_2 e^{2x}}}$

9) $y'' + y = 0$

$\Downarrow$

$r^2 + 1 = 0$

$r = \pm i \Rightarrow \alpha = 0, \beta = 1$

$\Rightarrow \underline{\underline{y(x) = C_1 \cos x + C_2 \sin x}}$

10) $y'' + 4y = 0$

$\Downarrow$

$r^2 + 4 = 0$

$r = \pm 2i \Rightarrow \alpha = 0, \beta = 2$

$\Rightarrow y(x) = C_1 \cos 2x + C_2 \sin 2x$

17) $y'' - 4y' + 4y = 0$

$\Downarrow$

$r^2 - 4r + 4 = 0$

$r = \frac{4 \pm \sqrt{16 - 4 \cdot 4}}{2}$

$r = 2$

$\Rightarrow y_1 = e^{2x}$

$y_2 = x \cdot e^{2x}$

$\Rightarrow \underline{\underline{y = C_1 e^{2x} + C_2 x e^{2x}}}$

1a) $y' - 6y' + 9y = 0$

$$\Downarrow$$

$$r^2 - 6r + 9 = 0$$
$$(r-3)^2 = 0$$
$$r = 3$$

$\Rightarrow$ $y_1 = e^{3x}$

$y_2 = x e^{3x}$

$$\Rightarrow y(x) = C_1 e^{3x} + C_2 x e^{3x}$$

2b) $y'' - y' - 2y = 0, \quad y(0) = -1, \; y'(0) = 2$

$$\Downarrow$$

$$r^2 - r - 2 = 0$$

$$r = \frac{1 \pm \sqrt{1 - 4 \cdot 2}}{2}$$

$$= \frac{1}{2} \pm \frac{\sqrt{7}}{2} i \Rightarrow \alpha = \frac{1}{2}, \; \beta = \frac{\sqrt{7}}{2}$$

$$y(x) = e^{\frac{1}{2}x}\left(C_1 \cos\left(\frac{\sqrt{7}}{2}x\right) + C_2 \sin\left(\frac{\sqrt{7}}{2}x\right)\right)$$

$$y(0) = C_1 = -1$$

$$y'(x) = \frac{1}{2} e^{\frac{1}{2}x} \cdot \left(-\cos\left(\frac{\sqrt{7}}{2}x\right) + C_2 \sin\left(\frac{\sqrt{7}}{2}x\right)\right)$$
$$+ e^{\frac{1}{2}x}\left(\sin\left(\frac{\sqrt{7}}{2}x\right) \cdot \frac{\sqrt{7}}{2} + C_2 \cos\left(\frac{\sqrt{7}}{2}x\right) \frac{\sqrt{7}}{2}\right)$$

$$y'(0) = \frac{1}{2} \cdot (-1) + C_2 \frac{\sqrt{7}}{2} = 2 \quad | \cdot 2$$

$$-1 + C_2 \sqrt{7} = 4$$

$$C_2 = \frac{5}{\sqrt{7}}$$

$$\Rightarrow y(x) = e^{\frac{1}{2}x}\left[-\cos\left(\frac{\sqrt{7}}{2}x\right) + \frac{5}{\sqrt{7}} \sin\left(\frac{\sqrt{7}}{2}x\right)\right]$$

26) $10y'' - y' - 3y = 0, \quad y(0) = 1, \quad y'(0) = 0$

$$\Downarrow$$

$$10r^2 - r - 3 = 0$$

$$r = \frac{1 \pm \sqrt{1 - 4 \cdot 10 \cdot (-3)}}{2 \cdot 10}$$

$$= \frac{1}{20} \pm \frac{11}{20}$$

$$\Rightarrow r_1 = -\frac{1}{2}, \quad r_2 = \frac{3}{5}$$

$$y(x) = C_1 e^{-\frac{1}{2}x} + C_2 e^{\frac{3}{5}x}$$

$$y'(x) = -\frac{1}{2} C_1 e^{-\frac{1}{2}x} + C_2 \cdot \frac{3}{5} e^{\frac{3}{5}x}$$

$$y(0) = C_1 + C_2 = 1 \qquad (i)$$

$$y'(0) = -\frac{1}{2} C_1 + \frac{3}{5} C_2 = 0 \qquad (ii)$$

$$(i) + 2 \cdot (ii) \Rightarrow C_2 + \frac{3}{5} C_2 \cdot 2 = 1$$

$$\frac{11}{5} C_2 = 1$$

$$C_2 = \frac{5}{11}$$

$$\Rightarrow C_1 = 1 - \frac{5}{11} = \frac{6}{11}$$

$$y(x) = \frac{6}{11} e^{-\frac{1}{2}x} + \frac{5}{11} e^{\frac{3}{5}x}$$

27) $y'' - 2y' + 17y = 0,\ y(0) = -2,\ y'(0) = 3$

$$\downarrow$$

$$r^2 - 2r + 17 = 0$$

$$r = \frac{2 \pm \sqrt{4 - 4\cdot 17}}{2}$$

$$= 1 \pm \frac{2}{2}\sqrt{1-17}$$

$$= 1 \pm 4i \Rightarrow \alpha = 1,\ \beta = 4$$

$$\Rightarrow y(x) = e^x \cdot [c_1 \cos 4x + c_2 \sin 4x]$$

$$y'(x) = e^x [c_1 \cos 4x + c_2 \sin 4x]$$
$$+ e^x \cdot [4c_1(-\sin 4x) + 4c_2 \cos 4x]$$

$$y(0) = c_1 = -2$$

$$y'(0) = -2 + 4c_2 = 3$$

$$c_2 = \frac{5}{4}$$

$$\Rightarrow y(x) = e^x \cdot \left[-2\cos 4x + \frac{5}{4}\sin 4x\right]$$

28) $y'' + 25y = 0, \quad y(0) = 1, \quad y'(0) = -1$

$$\Downarrow$$

$$r^2 + 25 = 0$$

$$r = \pm 5i, \quad \alpha = 0; \beta = 5$$

$$y(x) = C_1 \cos 5x + C_2 \sin 5x$$
$$y'(x) = -5C_1 \sin 5x + 5C_2 \cos 5x$$

$$y(0) = C_1 = 1$$
$$y'(0) = -5C_2 = -1$$
$$\Leftrightarrow C_2 = 1/5$$

$$\underline{y(x) = \cos 5x + \frac{1}{5}\sin 5x}$$

29) $y'' + 10y' + 25y = 0, \quad y(0) = 2, \quad y'(0) = -1$

$$\Downarrow$$

$$r^2 + 10r + 25 = 0$$

$$r = \frac{-10 \pm \sqrt{100 - 4 \cdot 25}}{2}$$

$$r = -5$$

$$\Rightarrow y(x) = C_1 e^{-5x} + C_2 x e^{-5x}$$
$$y'(x) = -5C_1 e^{-5x} + C_2(e^{-5x} + -5xe^{-5x})$$

$$y(0) = C_1 = 2$$
$$y'(0) = -5 \cdot 2 + C_2 = -1$$
$$C_2 = 9$$

$$\Rightarrow \underline{\underline{y(x) = 2e^{-5x} + 9xe^{-5x}}}$$

38) let $y = te^{\lambda_1 t}$

then $y' = t\lambda_1 e^{\lambda_1 t} + e^{\lambda_1 t} = (t\lambda_1 + 1)e^{\lambda_1 t}$

$$\begin{aligned} y'' &= (t\lambda_1 + 1)' \cdot e^{\lambda_1 t} + (t\lambda_1 + 1)(e^{\lambda_1 t})' \\ &= \lambda_1 e^{\lambda_1 t} + (t\lambda_1 + 1)\lambda_1 e^{\lambda_1 t} \\ &= (\lambda_1 + t\lambda_1^2 + \lambda_1)e^{\lambda_1 t} \\ &= (\lambda_1^2 t + 2\lambda_1)e^{\lambda_1 t} \\ &= \lambda_1(\lambda_1 t + 2)e^{\lambda_1 t} \end{aligned}$$

inserted into the equation:

$$y'' + py' + qy = 0$$

$$(\lambda_1^2 t + 2\lambda_1)\cancel{e^{\lambda_1 t}} + p(t\lambda_1 + 1)\cancel{e^{\lambda_1 t}} + qt\cancel{e^{\lambda_1 t}} = 0$$

$$\lambda_1^2 t + 2\lambda_1 + p\lambda_1 t + p + qt = 0 \qquad (*)$$

If $\lambda^2 + p\lambda + q = 0$ has a double root, then $p^2 - 4q = 0$ and the solution is

$$\lambda = -\frac{p}{2}$$

$$\text{and } q = \frac{p^2}{4}$$

So $(*)$ becomes

$$\left(-\frac{p}{2}\right)^2 t + 2 \cdot \frac{-p}{2} + p \cdot \left(\frac{-p}{2}\right)t + p + \frac{p^2}{4}t = 0$$

$$\underbrace{\cancel{\frac{p^2}{4}t} - \cancel{p} - \cancel{\frac{p^2}{2}t} + \cancel{p} + \cancel{\frac{p^2}{4}t}}_{=\,0} = 0$$

$$= 0 \checkmark$$

This shows that $y = te^{\lambda_1 t}$ is a solution.

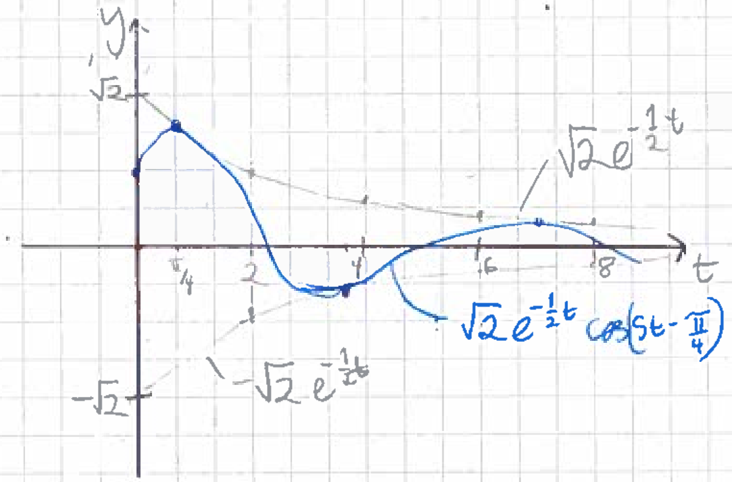
4.4

7) $A = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\varphi = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$

$\omega_0 = 5$

$\Rightarrow \quad y = \sqrt{2}\, e^{-\frac{1}{2}t} \cos\left(5t - \frac{\pi}{4}\right)$

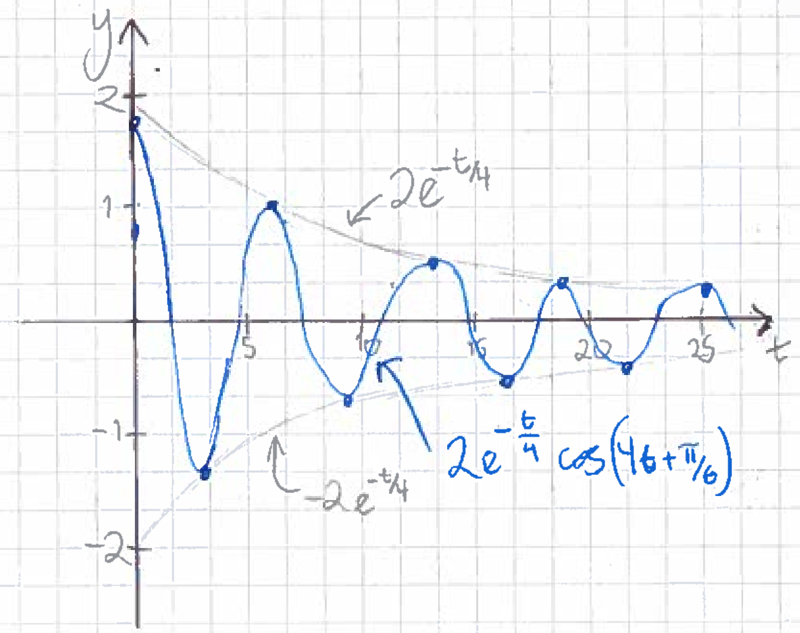


8) $A = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2$

$\varphi = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

$\omega_0 = 4$

$\Rightarrow \quad y = 2e^{-\frac{t}{4}} \cos\left(4t + \frac{\pi}{6}\right)$

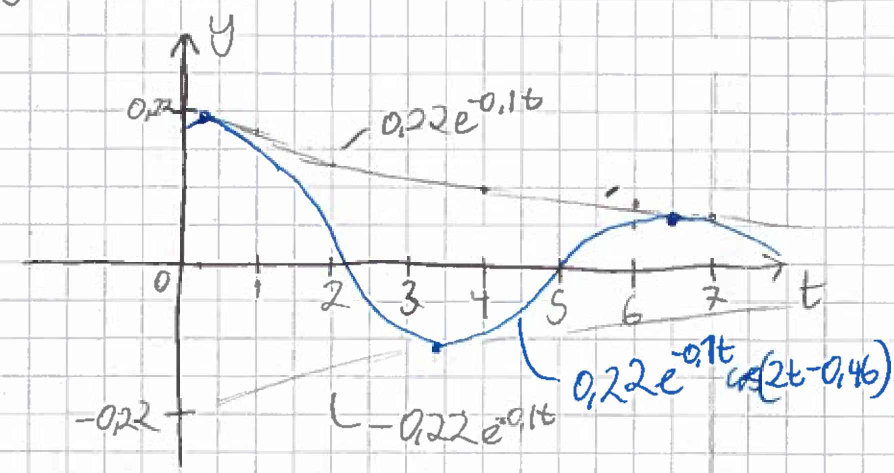


9) $A = \sqrt{0{,}2^2 + 0{,}1^2} = 0{,}22$

$\varphi = \tan^{-1}\left(\frac{0{,}1}{0{,}2}\right) = 0{,}46$

$w_0 = 2$

$\Rightarrow \quad y = 0{,}22 \cdot e^{-0{,}1t} \cdot \cos(2t - 0{,}46)$

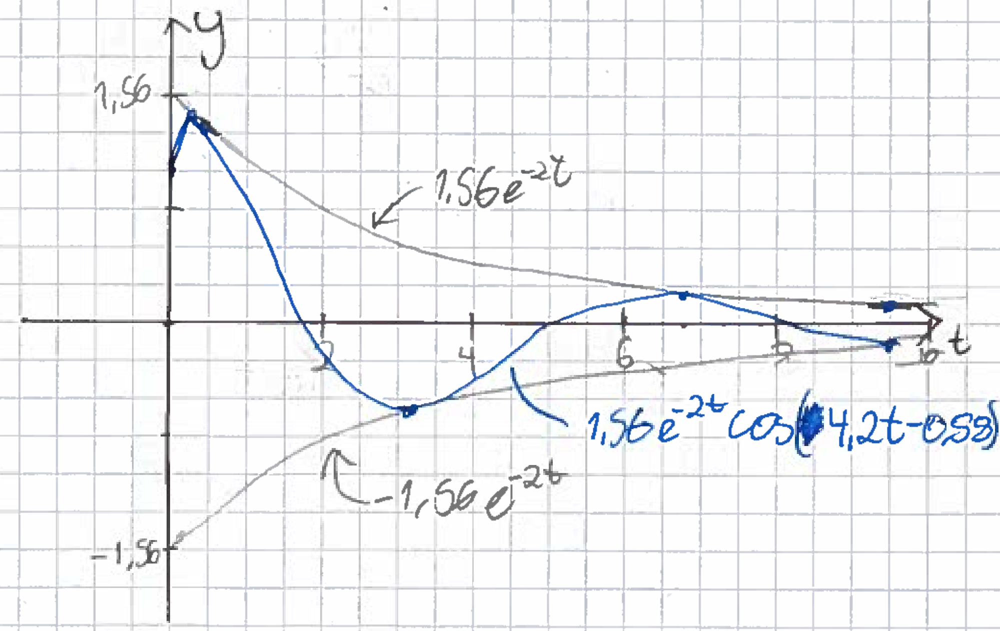


10) $A = \sqrt{1^2 + (4{,}2)^2} = 1{,}56$

$\varphi = \tan^{-1}\left(\frac{-1{,}2}{1}\right) = 0{,}88$

$w_0 = 4{,}2$

$\Rightarrow \quad y = 1{,}56\, e^{-2t} \cos(4{,}2t - 0{,}88)$



13) $\frac{2}{5}x'' + Kx = 0, \quad x(0) = 0, \quad \dot{x}(0) = V_0$

$$\Downarrow$$

$$\frac{2}{5}r^2 + K = 0$$

$$r^2 = -\frac{5}{2}K$$

$$r = \sqrt{-\frac{5}{2}K}$$

$$r = \pm\frac{\sqrt{10}}{2}\sqrt{K}\,i$$

$$\Rightarrow x(t) = C_1 \cos\left(\frac{\sqrt{10K}}{2}t\right) + C_2 \sin\left(\frac{\sqrt{10K}}{2}t\right)$$

$$x'(t) = -C_1 \cdot \frac{\sqrt{10K}}{2}\sin\left(\frac{\sqrt{10K}}{2}t\right) + C_2\frac{\sqrt{10K}}{2}\cos\left(\frac{\sqrt{10K}}{2}t\right)$$

$$x(0) = C_1 = 0$$

$$\dot{x}(0) = C_2\frac{\sqrt{10K}}{2} = V_0$$

$$\Rightarrow C_2 = \frac{2V_0}{\sqrt{10K}}$$

$$\Rightarrow x(t) = \frac{2V_0}{\sqrt{10K}}\sin\left(\frac{\sqrt{10K}}{2}t\right)$$

period: $\frac{\pi}{2} = \frac{2\pi}{\frac{\sqrt{10K}}{2}}$ (1)

amplitude: $2 = \frac{2V_0}{\sqrt{10K}}$ (2)

(1): $\frac{\pi}{2} = \frac{4\pi}{\sqrt{10K}} \quad \Big| \cdot \frac{2}{\pi}\cdot\sqrt{10K}$

$$\sqrt{10K} = 8$$

$$K = \frac{8^2}{10} = \frac{64}{10} = \frac{32}{5}$$

(2) : $2 = \frac{2V_0}{\sqrt{10 \cdot K}}$

$$V_0 = \sqrt{10 \cdot \frac{8^2}{10}} = 8$$

K er $\frac{32}{5}$ og V_0 er 8

14) $m x'' + K x = 0, \; x(0) = x_0, \; x'(0) = V_0$

$$\Downarrow$$

$$x'' + \frac{K}{m} x = 0$$

$$\Downarrow$$

$$r^2 + \frac{K}{m} = 0$$

$$r = \pm \sqrt{\frac{K}{m}}\, i$$

$$\Rightarrow x(t) = C_1 \cos\left(\sqrt{\frac{K}{m}} t\right) + C_2 \sin\left(\sqrt{\frac{K}{m}} t\right)$$

$$\Rightarrow x'(t) = -C_1 \sqrt{\frac{K}{m}} \sin\left(\sqrt{\frac{K}{m}} t\right) + C_2 \sqrt{\frac{K}{m}} \cos\left(\sqrt{\frac{K}{m}} t\right)$$

$$x(0) = C_1 = x_0$$

$$x'(0) = C_2 \cdot \sqrt{\frac{K}{m}} = V_0$$

$$\Rightarrow C_2 = \sqrt{\frac{m}{K}}\, V_0$$

$$\Rightarrow x(t) = x_0 \cos\left(\sqrt{\frac{K}{m}} t\right) + \sqrt{\frac{m}{K}}\, V_0 \sin\left(\sqrt{\frac{K}{m}} t\right)$$

$$A = \sqrt{x_0^2 + \left(\sqrt{\frac{m}{K}}\, V_0\right)^2}$$

$$A = \sqrt{x_0^2 + \frac{m V_0^2}{K}} \qquad |||$$