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Problem 1

a) $$F(s) = \frac{s(s+2)}{s^3+s^2+s+1}$$

$$= \frac{s(s+2)}{(s^2+1)(s+1)}$$

Want to write

$$\frac{s(s+2)}{(s^2+1)(s+1)} = \frac{As+B}{s^2+1} + \frac{C}{s+1}$$

$$\Rightarrow s^2+2s = (As+B)(s+1)+C(s^2+1)$$

$$\Rightarrow s^2: \quad 1 = A+C \qquad (i)$$

$$s^1: \quad 2 = A+B \qquad (ii)$$

$$s^0: \quad 0 = B+C \qquad (iii)$$

(iii) $\Rightarrow B = -C$

(ii) $\Rightarrow A = 2-B = 2+C$

(i) $\Rightarrow 1 = A+C = 2+2C$

$\Rightarrow C = -\frac{1}{2}$

$B = \frac{1}{2}$

$A = \frac{3}{2}$

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So

$$F(s) = \frac{\frac{3}{2}s + \frac{1}{2}}{s^2+1} - \frac{1}{2}\cdot\frac{1}{s+1}$$

$$= \frac{3}{2}\cdot\frac{s}{s^2+1} + \frac{1}{2}\cdot\frac{1}{s^2+1} - \frac{1}{2}\frac{1}{s+1}$$

so the inverse Laplace transform is

$$\mathcal{L}^{-1}\{F\}(t) = \frac{3}{2}\cos(t) + \frac{1}{2}\sin t - \frac{1}{2}e^{-t}$$ ✓

b) We have

$$f(t) = \cos t + e^{-2t}\int_0^t f(\tau)e^{2\tau}\,d\tau$$

$$= \cos(t) + \int_0^t f(\tau)e^{2(\tau-t)}\,d\tau$$

$$= \cos t + \int_0^t f(\tau)e^{-2(t-\tau)}\,d\tau$$

$$= \cos t + f(t) * e^{-2t}(t)$$

Let $F(s) = \mathcal{L}\{f(t)\}$. Then we have

$$F(s) = \frac{s}{s^2+1} + F(s)\cdot\frac{1}{s+2}$$

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Rearranging we get

$$F(s)\left(\cancel{1} - \frac{1}{s+2}\right) = \frac{s}{s+1}$$

$$\Leftrightarrow \quad F(s)\left(\frac{s+1}{s+2}\right) = \frac{s}{s+1}$$

$$\Rightarrow \quad F(s) = \frac{s(s+2)}{(s+1)(s+1)} \quad \checkmark$$

Which is the same F as in (a), i.e.

$$f(t) = \frac{1}{2}\cos t + \frac{1}{2}\sin t - \frac{1}{2}e^{-t}$$

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Problem 2

Let $f(x)$ be 2π-periodic $f(x) = 1 - |x|$ for $|x| < 1$

a) $L = 1$.

Note that $f(-x) = 1 - |-x| = 1 - |x| = f(x)$

so f is even. We thus compute the Fourier cosine series.

$$a_0 = \frac{1}{2}\int_{-1}^{1} 1 - |x| \, dx$$

$$= \int_0^2 1 - x \, dx \quad (2\text{-period}$$

$$= \frac{1}{2}\int_{-1}^{0} 1 + x \, dx + \frac{1}{2}\int_0^1 1 - x \, dx$$

$$= \frac{1}{2}\left(\left[x + \frac{1}{2}x^2\right]_{-1}^{0} + \left[x - \frac{x^2}{2}\right]_0^1\right)$$

$$\Rightarrow a_0 = \frac{1}{2}$$

$$a_n = \int_{-1}^{1} \underbrace{(1-|x|)\cos(n\pi x)}_{\text{even}} \, dx$$

$$= 2\int_0^1 (1-x)\cos(n\pi x) \, dx$$

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$$\Rightarrow a_n = \underbrace{\left.\frac{2(1-x)\sin(n\pi x)}{n\pi}\right|_0^1}_{=0} + \frac{2}{n\pi}\int_0^1 \sin(n\pi x)\,dx$$

$$= \frac{-2}{(n\pi)^2}\left[\cos n\pi x\right]_0^1$$

$$= -\frac{2((-1)^n-1)}{(n\pi)^2}$$

$$= \frac{2(1-(-1)^n)}{(n\pi)^2}$$

$$= \begin{cases} 0 & , n \text{ even} \\ \frac{4}{(n\pi)^2} & , n \text{ odd} \end{cases}$$

Write $n = 2k+1$ to only get odd numbered (non-zero) coefficients. Then we get

$$f(x) \sim \frac{1}{2} + \sum_{k=0}^{\infty} \frac{4}{\pi^2(2k+1)^2}\cos((2k+1)\pi x)$$ ✓

b) $$y'' + 9y = f(x) \quad (*)$$

From (a) we have

$$f(x) = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{4}{\pi^2(2k+1)^2} \cos((2k+1)\pi x)$$

We thus split up (*) into several ODEs.

$$y_0'' + 9y_0 = a_0 = \frac{1}{2}$$

$$y_{2k+1}'' + 9y_{2k+1} = a_{2k+1} = \frac{4}{\pi^2(2k+1)^2}$$

$$y_{2k}'' + 9y_{2k} = a_{2k} = 0$$

For y_0 we see that

$y_0 = \frac{1}{18}$ is a particular solution,

since

$$\left(\frac{1}{18}\right)'' + 9 \cdot \frac{1}{18} = \frac{1}{2}$$

For y_{2k+1} we have

$$y_{2k+1}'' + 9y_{2k+1} = \frac{4}{\pi^2(2k+1)^2} \cos((2k+1)\pi x)$$

Write $n = 2k+1$ to simplify.

$$\Rightarrow y_n'' + 9y_n = \frac{4}{\pi^2 n^2} \cos(n\pi x)$$

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The particular solution should be of the form

$$y_n = A_n \cos(n\pi x)$$

$$\Rightarrow y_n'' + 9y_n = -A_n n^2\pi^2 \cos(n\pi x) + 9A_n \cos(n\pi x)$$

which means (from ODE)

$$\frac{4}{\pi^2 n^2} = -A_n n^2\pi^2 + 9A_n$$

$$\Leftrightarrow \frac{4}{\pi^2 n^2} = A_n(-n^2\pi^2 + 9)$$

$$\Rightarrow A_n = \frac{4}{\pi^2 n^2(-n^2\pi^2 + 9)}$$

So

$$y_n = \frac{4}{\pi^2 n^2(-n^2\pi^2+9)} \cos(n\pi x)$$

Thus particular solution of y is

$$y_p = y_0 + \sum y_n$$

$$= \frac{1}{15} + \sum_{k=0}^{\infty} \frac{4}{\pi^2(2k+1)^2(-(2k+1)^2\pi^2+9)} \cos((2k+1)\pi x)$$

✓

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Problem 3

$$f(x) = \begin{cases} e^{-|x|} - e^{-1}, & |x|<1 \\ 0, & |x|\geq 1 \end{cases}$$

$$\hat{f}(\omega) = \mathcal{F}\{f\}(\omega)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (e^{-|x|} - e^{-1}) e^{-i\omega x} dx$$

$$\int_{-1}^{1} e^{-1} e^{-i\omega x} dx = \frac{e^{-1}}{-i\omega} \left[e^{-i\omega x} \right]_{-1}^{1}$$

$$= -\frac{e^{-1}}{i\omega} (e^{-i\omega} - e^{i\omega})$$

$$= \frac{2e^{-1}}{\omega} \frac{(e^{i\omega} - e^{-i\omega})}{2i}$$

$$= e^{-1} \frac{2}{\omega} \sin(\omega)$$

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$$\int_{-1}^{1} e^{-|x|} e^{-i\omega x} dx = \int_{-1}^{0} e^{(1-i\omega)x} dx + \int_{0}^{1} e^{-(1+i\omega)x} dx$$

$$= \frac{1}{1-i\omega}\left(1 - e^{-1}e^{i\omega}\right) + \frac{-1}{1+i\omega}\left(e^{-1}e^{-i\omega} - 1\right)$$

$$= \frac{(1+i\omega)(1-e^{-1}e^{i\omega}) + (1-i\omega)(1-e^{-1}e^{-i\omega})}{1+\omega^2}$$

Expanding the numerator we get

$$1 - e^{-1}e^{i\omega} + \cancel{i\omega} - i\omega e^{-1}e^{i\omega} + 1 - e^{-1}e^{-i\omega} - \cancel{i\omega} + i\omega e^{-1}e^{-i\omega}$$

$$= 2 - e^{-1}\left(e^{i\omega} + e^{-i\omega}\right) + e^{-1}i\omega\left(e^{-i\omega} - e^{i\omega}\right)$$

$$= 2 - 2e^{-1}\frac{\left(e^{i\omega} + e^{-i\omega}\right)}{2} + 2e^{-1}\omega\frac{\left(e^{i\omega} - e^{-i\omega}\right)}{2i}$$

$$= 2 - 2e^{-1}\cos\omega + 2e^{-1}\omega\sin\omega$$

which gives

$$\int_{-1}^{1} e^{-|x|} e^{-i\omega x} dx = \frac{2(1 - e^{-1}(\cos\omega + e^{-1}\omega\sin\omega)}{1+\omega^2}$$

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$\frac{w\sin w}{1+w^2} - \frac{\sin w}{w}$

$= \frac{w^2 \cancel{\sin w} - (1+w^2)\cancel{\sin w}}{w(1+w^2)}$

$= -\frac{\sin w}{w(1+w^2)}$

Collecting everything we get

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\left[\frac{2(1-e^{-1}\cos w + e^{-1}w\sin w)}{1+w^2} - e^{-1}2\frac{\sin w}{w}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{1-e^{-1}\cos w + e^{-1}w\sin w}{1+w^2} - e^{-1}\frac{\sin w}{w}\right]$$

Want to solve the equation

$$u_t = u_{xx}, \quad -\infty < x < \infty, \quad t > 0$$
$$u(x,0) = f(x)$$

~~Take~~ If we take the Fourier transform with respect to x, we get

$$\hat{u}_t = u_{xx}$$

$$\Leftrightarrow \widehat{u_t} = \widehat{u_{xx}}$$

$$\Leftrightarrow \hat{u}_t = -w^2\hat{u}$$

$$\Rightarrow \hat{u}(w,t) = A(w)e^{-w^2t}$$

This ~~g~~ means

The initial condition gives

$$\hat{u}(w,0) = \hat{f}(w)$$

From our calculations we have that

$$\hat{u}(w,0) = A(w)\cdot e^{0} = A(w)$$

So $A(w) = \hat{f}(w)$

Therefore the solution in the ~~the~~ w-domain is

$$\hat{u}(w,t) = \hat{f}(w)\cdot e^{-w^2 t}$$

and in the t-domain the integral form of the solution is

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(w)\, e^{-w^2 t}\, e^{iwx}\, dw, \quad \hat{f}(w) \text{ given in (a).}$$

where $\hat{f}(w)$ is given ~~in~~ in (a).

Problem 4

$w = e^z = e^{x+iy}$.

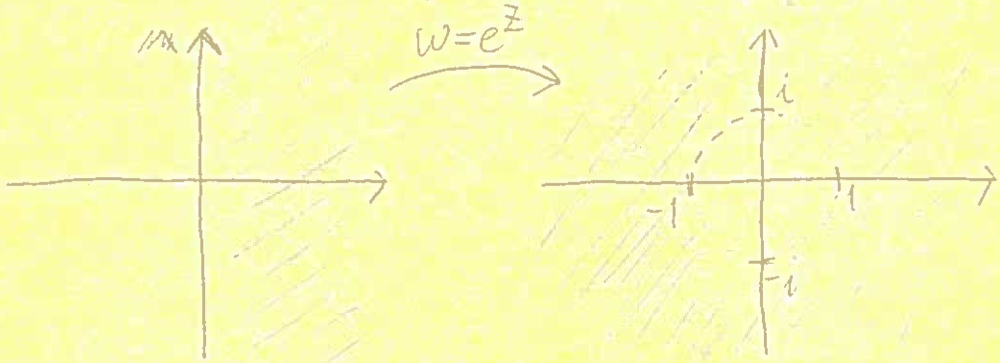
In the half plane $D = \{ Re(z) > 0 \}$,

$$|w| = |e^z| = |e^x| > 1 \quad \text{for } x = Re(z) > 0$$

since $x > 0$.

Since $y = Im(z)$ can take ~~a~~ any value, $\arg(w) = y$ can take ~~a~~ any value.

From this we get the mapping



More precisely, $w = e^z$ maps $\{Re\, z > 0\}$ to ~~every~~ $\{w : |w| > 1\}$

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Problem 5

$$f(z) = \sum_{n=1}^{\infty} \frac{3^n}{2n} z^{2n}$$

a)

$$L = \limsup_{n\to\infty} \sqrt[n]{a_n} \quad \text{where} \quad a_n = \begin{cases} \frac{3^{n/2}}{n}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

$$= \lim_{n\to\infty} \sqrt[n]{\frac{3^{n/2}}{2n}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{3}}{\sqrt[n]{2n}}$$

$$= \sqrt{3}$$

Since this limit exists, $R = \frac{1}{L} = \frac{1}{\sqrt{3}}$.
The radius of convergence is

$$\underline{R = \frac{1}{\sqrt{3}}}$$

b) Note that f is given by a Taylor-series, which we know to be uniformly convergent. We can thus differentiate term-wise and get

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$$f(z) = \sum_{n=1}^{\infty} \left(\frac{3^n}{2n} z^{2n} \right)'$$

$$= \sum_{n=1}^{\infty} \frac{3^n}{2n} \cdot 2n\, z^{2n-1}$$

$$= \sum_{n=1}^{\infty} 3^n z^{2n-1}$$

$$= \sum_{n=0}^{\infty} 3^{n+1} z^{2n+1}$$

~~This gives~~

Using the formula for the geometric series ~~(given above)~~ and assuming z is in the radius of convergence ($|z| < \frac{1}{\sqrt{3}}$), we can write

$$f'(z) = \sum_{n=0}^{\infty} 3^{n+1} z^{2n+1}$$

$$= 3z \sum_{n=0}^{\infty} 3^n z^{2n}$$

$$= 3z \sum_{n=0}^{\infty} \left((\sqrt{3}z)^2\right)^n$$

$$= 3z \cdot \frac{1}{1-(\sqrt{3}z)^2}$$

So

$$f'(z) = \frac{3z}{1-3z^2}$$

c) Since we are given, $f(z) = -\frac{1}{2}\operatorname{Ln}(1-3z^2)$, we will just differentiate this and verify that we get $f'(z)$ as above.

$$\frac{d}{dz}\left(-\frac{1}{2}\operatorname{Ln}(1-3z^2)\right)$$

$$= -\frac{1}{2} \cdot \frac{-6z}{1-3z^2}$$

$$= \frac{3z}{1-3z^2}$$

which is what we had above.

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~~The ~~Maclaurin~~ series for $f'(z)$ converges in the disk $|z|<\frac{1}{\sqrt{3}}$, and integrating this ~~want~~ will not change the radius of convergence, so~~

~~$$f(z)=\sum_{n=1}^{\infty}\frac{3^n}{2n}z^{2n} \quad \text{also converges for } |z|<\frac{1}{\sqrt{3}}.$$~~

~~(in a disk around the origin)~~

(a) shows that $\sum_{n=1}^{\infty}\frac{3^n}{2n}z^{2n}$ converges in a disk around the origin.

Problem 6

Want to evaluate

$$I=\int_{-\pi}^{\pi}\frac{d\theta}{1+(\sin\theta)^2}$$

Let $z=e^{i\theta}$, then $dz=i\cancel{e^{i\theta}}\,d\cancel{z}= iz\,d\theta$

and

$$\sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i}=\frac{z-z^{-1}}{2i}$$

So
$$\frac{d\theta}{1+\sin^2\theta}=\frac{dz}{iz\left(1+\left(\frac{z-z^{-1}}{2i}\right)^2\right)}$$

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$$iz\left(1+\left(\frac{z-z^{-1}}{2i}\right)^2\right) = iz\left(1-\frac{1}{4}\left(z^2-2+z^{-2}\right)\right)$$

$$= \frac{i}{z}\left(z^2-\frac{1}{4}\left(z^4-2z^2+1\right)\right)$$

$$= \frac{i}{z}\left(z^2-\frac{z^4}{4}+\frac{z^2}{2}-\frac{1}{4}\right)$$

$$= \frac{i}{4z}\left(-z^4+6z^2-1\right)$$

$$= \frac{-i}{2z}\left(z^4-6z^2+1\right)$$

$$= \frac{-i}{2z}\left(z^2-\text{[illegible]}\right)\left(z^2-\text{[illegible]}\right)$$

So

$$\frac{d\theta}{1+\sin^2\theta} = \frac{2z\,dz}{-i\left(z^2-\text{[illegible]}\right)\left(z^2-\text{[illegible]}\right)}$$

with $z=e^{i\theta}$, $\theta\in[-\pi,\pi]$ we are integrating over $C=$ "unit circle" clockwise.

$$I = \oint_C \frac{2zi}{\left(z^2-\text{[illegible]}\right)\left(z^2-\text{[illegible]}\right)}\,dz$$

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Using residue integration we get

$$I = 2\pi i \sum_{\substack{\text{poles inside} \\ C}} \operatorname{Res} f(z).$$

where $f(z) = \dfrac{2zi}{(z^2 - [illegible])(z^2 - [illegible])}$

We have two set of poles

$$z^2 = [illegible] \rightarrow z = \pm [illegible] \quad (\text{outside } C)$$

$$z^2 = [illegible] \rightarrow z \approx \pm [illegible] \quad (\text{inside } C)$$

So we have to compute residues at

$$z_1 = +\sqrt{[illegible]}$$

$$z_2 = -\sqrt{[illegible]}$$

Since $f(z) = \dfrac{p(z)}{q(z)}$ and all poles are simple (proof not given as it should be clear), we can compute residues with

$$\operatorname{Res}_{z_j} f = \frac{p(z_j)}{q'(z_j)} = \frac{2z_j i}{4z_j^3 - \cancel{X}12 z_j}$$

$$= \frac{2 i}{4z_j^2 - \cancel{X}12}$$

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For both z_1 and z_2 we have

$$z_1^2 = z_2^2 = \text{[illegible]}$$

So

$$\operatorname{Res}_{z_1} f = \operatorname{Res}_{z_2} f = \frac{2i}{4(\text{[illegible]}) - 12}$$

$$= \frac{2i}{\text{[illegible]}}$$

$$= \frac{i}{\text{[illegible]}} = -\frac{i}{4\sqrt{2}-6}$$

which means

$$\sum_{\substack{\text{inside}\\ C}} \operatorname{Res} f = 2\left(\frac{-i}{\text{[illegible]}}\right) = -\frac{i}{\text{[illegible]}}$$

and thus

$$I = 2\pi i \sum \operatorname{Res} f$$

$$= \frac{2\pi}{\text{[illegible]}}.$$

So

$$\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta} = \frac{2\pi}{\text{[illegible]}}$$