Matte 3 Øving 10

4.4

3) $\vec{x} = 3\cdot\begin{pmatrix}1\\-4\\3\end{pmatrix} + 0\cdot\begin{pmatrix}5\\2\\-2\end{pmatrix} - 1\cdot\begin{pmatrix}4\\-7\\0\end{pmatrix}$

$= \begin{pmatrix}3-4\\-12+7\\9\end{pmatrix} = \begin{pmatrix}-1\\-5\\9\end{pmatrix}$

4) $\vec{x} = -4\cdot\begin{pmatrix}-1\\2\\0\end{pmatrix} + 8\cdot\begin{pmatrix}3\\-5\\2\end{pmatrix} - 7\cdot\begin{pmatrix}4\\-7\\3\end{pmatrix}$

$= \begin{pmatrix}4+24-28\\-8-40+49\\0+16-21\end{pmatrix} = \begin{pmatrix}0\\1\\-5\end{pmatrix}$

7) Have to solve $\left(\begin{array}{ccc|c}1&-3&2&8\\-1&4&-2&-9\\-3&9&4&6\end{array}\right)$

$\sim \left(\begin{array}{ccc|c}1&-3&2&8\\0&1&0&-1\\0&0&10&30\end{array}\right)$

$\sim \left(\begin{array}{ccc|c}1&-3&2&8\\0&1&0&-1\\0&0&1&3\end{array}\right)$

$\sim \left(\begin{array}{ccc|c}1&0&0&8-2\cdot3-3\\0&1&0&-1\\0&0&1&3\end{array}\right)$

$\sim \left(\begin{array}{ccc|c}1&0&0&-1\\0&1&0&-1\\0&0&1&3\end{array}\right)$

So $[x]_{\mathcal{B}} = \begin{pmatrix}-1\\-1\\-3\end{pmatrix}$

21) $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ -4 \end{pmatrix}, \begin{pmatrix} -2 \\ 9 \end{pmatrix} \right\}$

We know that $\begin{pmatrix} 1 \\ -4 \end{pmatrix} x_{b_1} + \begin{pmatrix} -2 \\ 9 \end{pmatrix} x_{b_2} = \vec{x}$

where $[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} x_{b_1} \\ x_{b_2} \end{pmatrix}$.

Simplifying this into a matrix equation we get $\begin{pmatrix} 1 & -2 \\ -4 & 9 \end{pmatrix} \begin{pmatrix} x_{b_1} \\ x_{b_2} \end{pmatrix} = \vec{x}$

$$\Leftrightarrow \begin{pmatrix} 1 & -2 \\ -4 & 9 \end{pmatrix} [\vec{x}]_{\mathcal{B}} = \vec{x}$$

The inverse of the matrix is:

$$\frac{1}{9-8} \begin{pmatrix} 9 & 2 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 9 & 2 \\ 4 & 1 \end{pmatrix}$$

Multiplying by the iverse we get

$$[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} 9 & 2 \\ 4 & 1 \end{pmatrix} \vec{x}$$

which was the desired transformation.

$$A = \begin{pmatrix} 9 & 2 \\ 4 & 1 \end{pmatrix}$$

4.5

3)
$$\begin{pmatrix} 2c \\ a-b \\ b-3c \\ a+2b \end{pmatrix} = a\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + b\begin{pmatrix} 0 \\ -1 \\ 1 \\ 2 \end{pmatrix} + c\begin{pmatrix} 2 \\ 0 \\ -3 \\ 0 \end{pmatrix}$$

So if the vectors above are linearly independent the will form a basis. We'll check by solving an equation $A\vec{x} = \vec{0}$.

$$\left(\begin{array}{ccc|c} 0 & 0 & 2 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 1 & 2 & 0 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 0 & 0 & 2 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 3 & 0 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 0 & 0 & 2 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 9 & 0 \end{array}\right) \Longleftrightarrow \begin{array}{l} 2x_3 = 0 \\ x_1 - x_2 \\ x_2 - 3x_3 = 0 \\ 9x_3 = 0 \end{array}$$

$$\Rightarrow x_3 = 0$$
$$\Rightarrow x_2 = 0$$
$$\Rightarrow x_1 = 0$$

So it has only the trivial solution which means the column vectors are independent.

A basis $\mathcal{B}$ is then given by:

$$\mathcal{B} = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -3 \\ 0 \end{pmatrix} \right\}$$

which has dimension 3.

9) We have a subspace: $\left\{ \begin{pmatrix} a \\ b \\ a \end{pmatrix} : a, b \in \mathbb{R} \right\}$

$$\begin{pmatrix} a \\ b \\ a \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Dimension is clearly 2.

12) Let's row reduce and find the number of pivot columns because that will be the same as the number of dimensions.

$$\begin{pmatrix} 1 & 3 & -8 & -3 \\ -2 & -4 & 6 & 0 \\ 0 & 1 & 5 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & -8 & -3 \\ 0 & 2 & -10 & -6 \\ 0 & 1 & 5 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & -8 & -3 \\ 0 & 1 & -5 & -3 \\ 0 & 1 & 5 & 7 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 3 & -8 & -3 \\ 0 & 1 & -5 & 3 \\ 0 & 0 & 10 & 10 \end{pmatrix}.$$

So there is at least 3 pivot columns and since the vectors are in $\mathbb{R}^3$ they can't have higher dimension than 3, so the dimension is 3.

13) A has 3 pivot columns so $\dim \operatorname{col} A = 3$.
We can also see that the equation $A\vec{x} = \vec{0}$ will have 2 free variables, namely x_3 and x_5, so $\dim \operatorname{Nul} A = 2$.

21) $1,\ 2t,\ -2+4t^2,\ -12t+8t^3$

$\mathbb{P}_3 = \{a+bt+ct^2+dt^3 : a,b,c,d \in \mathbb{R}\}$

The four polynomials written in vector form are

$$\vec{b}_1 = \begin{pmatrix}1\\2\\0\\0\end{pmatrix},\ \vec{b}_2 = \begin{pmatrix}0\\2\\0\\0\end{pmatrix},\ \vec{b}_3 = \begin{pmatrix}-2\\0\\4\\0\end{pmatrix},\ \vec{b}_4 = \begin{pmatrix}0\\-12\\0\\8\end{pmatrix}$$

We want to show that $\text{span}\{\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4\} = \mathbb{R}^4$ because then all polynomials in $\mathbb{P}_3$ can be expressed in terms of these vectors.

Let $A = (\vec{b}_1 | \dots | \vec{b}_4)$. If $A\vec{x} = \vec{0}$ has a unique solution the column vectors are independent. Using row reduction (and omitting right hand side).

$$\begin{pmatrix}1&0&-2&0\\2&2&0&-12\\0&0&4&0\\0&0&0&8\end{pmatrix} \sim \begin{pmatrix}1&0&-2&0\\0&2&-4&-12\\0&0&4&0\\0&0&0&8\end{pmatrix}$$

4 pivot columns span of columns is $\mathbb{R}^4$ so they form a basis of $\mathbb{P}_3$.

26) Let H be an n-dimensional subspace of an n-dimensional space V.

H has a basis, $\mathcal{B}$, of exactly n elements since H is n-dimensional.

Since $H \subseteq V$, the span $\mathcal{B}$ must be a subset (not proper) of V, but since $\mathcal{B}$ has exactly n elements and V is n-dimensional, $\mathcal{B}$ must also be a basis for V.

Since H and V can be expressed with the same basis, and $H \subseteq V$, we conclude $H = V$. ▥

4.6 $\text{rank } A = \dim \text{col} A = \dim \text{col} B = 3$

$\Rightarrow \text{rank } A = 3$

$\dim \text{Nul } A = 2$

Basis for col A:

$$\left\{ \begin{pmatrix} 2 \\ -2 \\ 4 \\ -2 \end{pmatrix}, \begin{pmatrix} 6 \\ -3 \\ 9 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 5 \\ -4 \end{pmatrix} \right\}$$

Basis for row A:

$$\left\{ \begin{array}{l} (2, -3, 6, 2, 5), \\ (0, 0, 3, -1, 1), \\ (0, 0, 0, 1, 3) \end{array} \right\}$$

Basis for Nul A: have to reduce B to RREF

$$\begin{pmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 2 & -3 & 6 & 0 & -1 \\ 0 & 0 & 3 & 0 & 4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 2 & -3 & 0 & 0 & 7 \\ 0 & 0 & 3 & 0 & 4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

From this we can see that

$$\begin{aligned} 2x_1 \qquad\qquad &= -7x_5 + 3x_2 \\ +3x_3 \qquad &= -4x_5 \\ +x_4 &= -3x_5 \end{aligned}$$

$$\Longleftrightarrow \quad \begin{aligned} x_1 &= \tfrac{3}{2}x_2 - \tfrac{7}{2}x_5 \\ x_3 &= -\tfrac{4}{3}x_5 \\ x_4 &= -3x_5 \\ x_2, x_5 &\text{ free} \end{aligned}$$

So we can write

$$\text{Nul } A = \left\{ x_2 \begin{pmatrix} 3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -7/2 \\ 0 \\ -4/3 \\ -3 \\ 1 \end{pmatrix} : x_2, x_5 \in \mathbb{R} \right\}$$

From this we see that $\left\{ \begin{pmatrix} 3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -7/2 \\ 0 \\ -4/3 \\ -3 \\ 1 \end{pmatrix} \right\}$

is a basis of Nul A.

5) A is 3×8, rank $A = 3$.

$8 = \text{rank} A + \dim \text{Nul } A$

$\Rightarrow \dim \text{Nul } A = 8 - 3 = 5$

$\dim \text{Row } A = \text{rank } A = 3$

$\text{rank } A^T = \text{rank } A = 3$

7) • A is 7×5 then A consists of 5 column vectors in $\mathbb{R}^7$. If they are all independent, the rank of A is 5 which is the largest rank.

• If A is 5×7 then A consists of 7 column vectors in $\mathbb{R}^5$. At most 5 of the 7 vector can be independent, so the largest rank of A is 5.

14) • If A is 4×3 then the row vectors are in $\mathbb{R}^3$. 4 independent vectors in $\mathbb{R}^3$ can't span more than $\mathbb{R}^3$ so the largest row space dimension is 3.

• If A is 3×4 and all the 3 row vectors are independent, then $\dim \text{Row} A = 3$, which is the largest possible.

17) A is $m\times n$

a) True, because the row vectors of A are exactly the column vectors of A^T.

b) False, imagine A as a matrix where the first three rows are all zero, (and the remaining row vectors are independent) Then the first three rows do not form a basis of Row A

c) True, this dimension is just the rank of A.

d) False, it equals number of columns

e) True, this happens due to the inexact nature of digital storage and representation (round-off error, etc)

4.9

3) a)

$$M = \begin{array}{cc} & \text{From:} \\ \text{Healthy} & \text{Ill} \end{array}$$

$$M = \begin{pmatrix} .95 & .45 \\ .05 & .55 \end{pmatrix} \begin{array}{l} \text{To:} \\ \text{Healthy} \\ \text{Ill} \end{array}$$

b) $\vec{x}_0 = \begin{pmatrix} .80 \\ .20 \end{pmatrix} \begin{array}{l} \text{healthy} \\ \text{ill} \end{array}$ Monday

$$\vec{x}_1 = \begin{pmatrix} .95 & .45 \\ .05 & .55 \end{pmatrix} \begin{pmatrix} .80 \\ .20 \end{pmatrix} = \begin{pmatrix} .76 + .09 \\ .04 + .11 \end{pmatrix} = \begin{pmatrix} .85 \\ .15 \end{pmatrix}$$

Tuesday: 85% healthy
15% ill

$$\vec{x}_2 = \begin{pmatrix} .95 & .45 \\ .05 & .55 \end{pmatrix} \begin{pmatrix} .85 \\ .15 \end{pmatrix} = \begin{pmatrix} .8075 + .0675 \\ .0425 + .0825 \end{pmatrix} = \begin{pmatrix} .875 \\ .125 \end{pmatrix}$$

Wednesday: 87,5% healthy
12,5% ill

c) $(.95)^2 = 0.9025$

The probability of being healthy two days in a row is 90,25% (given the we start of healthy)

7) Let $M = \begin{pmatrix} .7 & .1 & .1 \\ .2 & .8 & .2 \\ .1 & .1 & .7 \end{pmatrix}$

Have to solve $M\vec{x} = \vec{x} \iff (M - I)\vec{x} = \vec{0}$

So we have $\begin{pmatrix} -.3 & .1 & .1 \\ .2 & -.2 & .2 \\ .1 & .1 & -.3 \end{pmatrix}$

$$\sim \begin{pmatrix} -.3 & .1 & .1 \\ .6 & -.6 & .6 \\ .3 & .3 & -.9 \end{pmatrix}$$

$$\sim \begin{pmatrix} -.3 & .1 & .1 \\ 0 & -.4 & .8 \\ 0 & .4 & -.8 \end{pmatrix} \sim \begin{pmatrix} -.3 & .1 & .1 \\ 0 & .1 & -.2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} -3 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} -3 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

x_3 is free $\vec{x} = x_3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

The sum of the entries is $1+2+1=4$

So the steady state vector is

$$\begin{pmatrix} 1/4 \\ 1/2 \\ 1/4 \end{pmatrix}$$