Exercise 2

Rendell Cale, rendellc@stud.ntnu.no, mttk

Problem 1

a) $f(x) = x_1^3 + 3x_1x_2^2$

Want to solve $f(x+p) = f(x) + \nabla f(x+\alpha p)^T p$ for α

with $x = \begin{pmatrix}0\\0\end{pmatrix}$, $p = \begin{pmatrix}2\\1\end{pmatrix}$,

$$f(x+p) = f\begin{pmatrix}2\\1\end{pmatrix} = 2^3 + 3\cdot 2\cdot 1^2 = 8+6 = 14$$

$$f(x) = 0$$

$$\nabla f(x+\alpha p) = \nabla f(\alpha p)$$

$$= \begin{pmatrix} 3x_1^2 + 3x_2^2 \\ 6x_1x_2 \end{pmatrix}\Bigg|_{x=\alpha p}$$

$$= \begin{pmatrix} 3\alpha^2\cdot 2^2 + 3\cdot\alpha^2 1^2 \\ 6\cdot\alpha\cdot 2\cdot\alpha\cdot 1 \end{pmatrix}$$

$$= \begin{pmatrix} 15\alpha^2 \\ 12\alpha^2 \end{pmatrix}$$

$$\Rightarrow \nabla f(x+\alpha p)^T p = \begin{pmatrix} 15\alpha^2 \\ 12\alpha^2 \end{pmatrix}^T \begin{pmatrix}2\\1\end{pmatrix} = 42\alpha^2$$

From this we see that

$$\alpha = \sqrt{\frac{14}{42}} = \underline{\underline{0.577}} \quad \in (0,1)$$

satisfies the condition.

b) Since $\frac{d}{dx}\sqrt{x} \rightarrow \infty$ as $x \rightarrow 0^+$, the Lipschitz constant would have to be infinite.

More formaly:

$$\| f(x) - f(0) \| \leq L \| x - 0 \|$$

$$\Rightarrow \quad L \geq \frac{\| f(x) - f(0) \|}{\| x \|} \xrightarrow{x \to 0} \left\| \frac{d}{dx} f(x) \right\|_{x=0}$$

When $f(x) = \sqrt{x}$, we have $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$

$$\Rightarrow \quad L \geq \left\| \frac{1}{2\sqrt{x}} \right\| \rightarrow \infty \text{ as } x \rightarrow 0.$$

Problem 2

Say that λ is the Lagrange multipliers associated with $Ax=b$ and s are the ones associated with $x\geq 0$.

Then $\mathcal{L}(x,\lambda,s) = c^Tx - \lambda^T(Ax-b) - s^Tx$

$\nabla_x \mathcal{L} = 0$ gives

$$c - A^T\lambda - s = 0$$

$$\Leftrightarrow \quad A^T\lambda + s = c \qquad \text{(KKT-1)}$$

The equality constraint gives

$$Ax = b \qquad \text{(KKT-2)}$$

Inequality constraint gives

$$x \geq 0 \qquad \text{(KKT-3)}$$

$$s \geq 0 \qquad \text{(KKT-4)}$$

$$s^Tx = 0 \qquad \text{(KKT-5)}$$

Problem 3

a) Let x_R, x_S, x_T by the amount of tonnes produced of R, S, ~~and~~ T.

Then profit is given by

$$P(x) = 100x_R + 75x_S + 55x_T.$$

Let t_A, t_B denote time utilized in stage A and B.

$$t_A = 3x_R + 2x_S + x_T = 7200$$
$$t_B = 2x_R + 2x_S + 3x_T = 6000.$$

As an LP problem:

$$\max \; 100x_R + 75x_S + 55x_T$$
$$\text{s.t.} \quad 3x_R + 2x_S + x_T = 7200$$
$$2x_R + 2x_S + 3x_T = 6000.$$

By writing $x = (x_R \;\; x_S \;\; x_T)^T$

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$$

$$b = (7200 \;\; 6000)^T$$

$$c = (100 \;\; 75 \;\; 55)^T$$

we can reformulate this as

$$\min_{x \in \mathbb{R}^3} -c^T x$$

s.t. $Ax = b$, $x \geq 0$

b) There are three candidates for basic feasible points.

1) $\mathcal{B} = \{1, 2\} \Rightarrow x_3 = x_T = 0$

$$B = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$$

2) $\mathcal{B} = \{1, 3\} \Rightarrow x_2 = x_S = 0$

$$B = \begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix}$$

3) $\mathcal{B} = \{2, 3\} \Rightarrow x_1 = x_R = 0$

$$B = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$$

These B-matrices are all nonsingular so we proceed to find the BFPs.

(1) gives

$$\begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_R \\ x_S \end{pmatrix} = \begin{pmatrix} 7200 \\ 6000 \end{pmatrix}, \quad x_T = 0$$

$$\Rightarrow \quad x = \begin{pmatrix} 1200 \\ 1800 \\ 0 \end{pmatrix}$$

(2) $$\begin{pmatrix} 3 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_R \\ x_T \end{pmatrix} = \begin{pmatrix} 7200 \\ 6000 \end{pmatrix}, \quad x_S = 0$$

$$\Rightarrow \quad x = \begin{pmatrix} \frac{15600}{7} \\ 0 \\ \frac{3600}{7} \end{pmatrix}$$

(3) $$\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_S \\ x_T \end{pmatrix} = \begin{pmatrix} 7200 \\ 6000 \end{pmatrix}, \quad x_R = 0$$

$$\Rightarrow \quad x = \begin{pmatrix} 0 \\ 3900 \\ -600 \end{pmatrix}$$ which is <u>not</u> feasible.

The two basic feasible points are

$$\begin{pmatrix} 1200 \\ 1800 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 15600/7 \\ 0 \\ 3600/7 \end{pmatrix}$$

(1) (2)

c) KKT of (1): $A^T\lambda + s = c$

$$Ax = b$$ ✓

$$x \geq 0$$ ✓

$$s \geq 0$$

$$x_i s_i = 0 \quad \text{for } i = 1, \ldots, 3.$$ ✓

Since $x_1 \neq 0$ and $x_2 \neq 0$, we must have

$$s_1 = s_2 = 0.$$

This gives $A^T\lambda + s = c$

$$\Leftrightarrow \begin{pmatrix} 3 & 2 \\ 2 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ s_3 \end{pmatrix} = \begin{pmatrix} 100 \\ 75 \\ 55 \end{pmatrix}$$

$$\Rightarrow \lambda_1 = 25, \quad \lambda_2 = 25/2$$

$$s_3 = 55 - 1 \cdot 25 - 3 \cdot \frac{25}{2} = -7.5$$

This doesn't satisfy $s \geq 0$ so it fails the KKT conditions.

KKT of (2): $A^T\lambda + s = c$

$$Ax = b \quad \checkmark$$

$$x \geq 0 \quad \checkmark$$

$$s \geq 0$$

$$x_i s_i = 0 \quad \text{for } i = 1, \dots, 3.$$

We must require $s_1 = s_3 = 0$ since $x_1 \neq 0$ and $x_3 \neq 0$.

This gives

$$\begin{pmatrix} 3 & 2 \\ 2 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} + \begin{pmatrix} 0 \\ s_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 100 \\ 75 \\ 55 \end{pmatrix}$$

$$\Rightarrow \quad \lambda_1 = \frac{190}{7}, \quad \lambda_2 = \frac{65}{7}$$

$$s_2 = 75 - 2 \cdot \frac{190}{7} - 2 \cdot \frac{65}{7} = \frac{15}{7}$$

With this we get $\lambda \geq 0$, $s \geq 0$, $A^T\lambda + s = c$

so KKT are satisfied.

Thus the solution is

$$x^* = \begin{pmatrix} 15600/7 \\ 0 \\ 3600/7 \end{pmatrix}$$

d) The dual problem is

$$\max b^T \lambda \quad \text{s.t.} \quad A^T \lambda \leq c$$

$$\Leftrightarrow \max \begin{pmatrix} 7200 & 6000 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

$$\text{s.t.} \quad \begin{pmatrix} 3 & 2 \\ 2 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \leq \begin{pmatrix} 100 \\ 75 \\ 55 \end{pmatrix}$$

e) $$c^T x^* = \begin{pmatrix} 100 & 75 & 55 \end{pmatrix} \begin{pmatrix} 15600/7 \\ 0 \\ 3600/7 \end{pmatrix} = \frac{1758000}{7}$$

$$\shortparallel$$

$$b^T \lambda^* = \begin{pmatrix} 7200 & 6000 \end{pmatrix} \begin{pmatrix} 190/7 \\ 65/7 \end{pmatrix} = \frac{1758000}{7}$$

f) Since $\lambda_1 > \lambda_2$ and λ_1 corresponds to stage A, it would be better to make stage A more available.

Increasing stage A to 7201 makes the solution be given by

$$\begin{pmatrix} 3 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_R \\ x_T \end{pmatrix} = \begin{pmatrix} 7201 \\ 6000 \end{pmatrix} \qquad x_S = 0$$

$$\Rightarrow x^* = \begin{pmatrix} 2229 & 0 & 514 \end{pmatrix}^T$$

$$\Rightarrow c^T x^* = 251170$$

Increasing stage B to 6001 gives

$$x^* = \begin{pmatrix} 3 & 1 \\ 2 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 7200 \\ 6001 \end{pmatrix} = \frac{1758069}{7}$$

Increasing A improved solution by ~27 NOK

Increasing B — || — ~9 NOK

This was expected since $\lambda_1 \approx 3 \cdot \lambda_2$.