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Faculty of Information Technology, Mathematics and Electrical Engineering
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English version

Exam in TTK4135

Optimization and Control

Optimalisering og regulering

Friday June 6, 2014

Time: 09:00 – 13:00

Combination of allowed help remedies:
D — No printed or hand-written notes.
Certifed calculator with empty memory.

In the Appendix potentially useful information is included.
The grades will be available by June 27.

1 Various topics (20 %)

Convexity

a (2 %) Assume that all constraints in (A.1) are linear. Is the feasible set (if it exists) a convex set?

b (4 %) Assume that there is only one equality constraint, c_1, in (A.1), and that c_1 is a nonlinear function. Will the feasible set always be a non-convex set in this case? Please use a sketch to justify your answer.

c (4 %) Why are non-convex optimization problems much harder to solve than convex optimization problems?

Linear program - LP

Consider an LP problem on standard form as in (A.6).

d (4 %) The Simplex algorithm is used for solving LP problems. This algorithm searches for the solution at *basic feasible points*. How does the algorithm test if a basic feasible point is the solution?

e (6 %) The Simplex algorithm has to solve large linear equation sets efficiently. This requires factorization and for instance the use of LU decomposition. Explain why $Ax = b$, where $A \in \mathbb{R}^{n \times n}$, is easy to solve when a LU decomposition of A exists.

2 Nonlinear programming and SQP (45 %)

a (4 %) Problem (A.1) shown in the Appendix presents a nonlinear optimization problem. Explain the meaning of $\mathcal{E}$ and $\mathcal{I}$, and give an example of these sets if there are 2 equality constraints and 3 inequality constraints.

b (4 %) Assume the equality constraint

$$x_1^2 + 4x_2^2 = 4$$

Sketch this constraint and indicate its feasible set. Is the feasible set a convex set?

c (4 %) Assume the inequality constraint

$$x_1^2 + 4x_2^2 \geq 4$$

Sketch this constraint and indicate its feasible set. Is the feasible set a convex set?

d (6 %) The SQP method creates a new subproblem at each iteration k. The QP subproblem at iteration k is given by (the notation equals the notation in the course textbook)

$$\begin{aligned}
\min_{p} \quad & \frac{1}{2} p^\top \nabla_{xx}^2 \mathcal{L}(x_k, \lambda_k) p + \nabla f(x_k)^\top p + f(x_k) \\
\text{s.t.} \quad & \\
& \nabla c_i(x_k)^\top p + c_i(x_k) = 0, \qquad i \in \mathcal{E} \\
& \nabla c_i(x_k)^\top p + c_i(x_k) \geq 0, \qquad i \in \mathcal{I}
\end{aligned}$$

This may lead to an infeasible QP problem even if the original nonlinear problem is feasible.
Assume a nonlinear program with the following inequality constraints (there are no other constraints)

$$\begin{aligned}
x_1^2 &\geq 4 \\
x_1 &\leq 1
\end{aligned}$$

Show that these constraints are inconsistent (no feasible set exists) if we linearize them at $x_1 = 1$.

e (7 %) Show that the objective function $\frac{1}{2} p^\top \nabla_{xx}^2 \mathcal{L}(x_k, \lambda_k) p + \nabla \mathcal{L}(x_k)^\top p + \mathcal{L}(x_k)$ can be re-written as $\frac{1}{2} p^\top \nabla_{xx}^2 \mathcal{L}(x_k, \lambda_k) p + \nabla f(x_k)^\top p + f(x_k)$ in the QP problem above. Assume that there are only equality constraints.

f (7 %) The *merit function* is an important part of SQP algorithms, and it appears as ϕ_1 in Alg.18.3 in the Appendix. A widely used merit function is

$$\phi_1(x;\mu) = f(x) + \mu \sum_{i\in\mathcal{E}} |c_i(x)| + \mu \sum_{i\in\mathcal{I}} [c_i(x)]^-$$

where $[z]^- = \max\{0, -z\}$, and $|z|$ is the absolute value. (This notation is identical to the notation in the course textbook).
What is the purpose of the parameter μ?
How does it normally change during the course of the SQP algorithm, i.e. from one iteration to the next? Please justify your answer.

g (6 %) Explain the concept *exact merit function*.
Is the ϕ_1 function above an exact penalty function? Justify your answer.

h (7 %) What is the Maratos effect?
Sketch, using a figure of a 2-dimensional problem ($x \in \mathbb{R}^2$), a situation that illustrates the Maratos effect.
How can a practical algorithm reduce, or avoid, the Maratos effect?

3 MPC and dynamic optimization (35 %)

For the questions below we consider the dynamic optimization problem (A.9) in the Appendix.

a (3 %) Explain briefly why (A.9) is called an open loop optimization problem.

b (6 %) Assume that $N = 15$, $n_x = 12$ and $n_u = 3$. What is the dimension of z in (A.9l) in this case?
Provide a formula for the dimension of z as a function of N, n_x and n_u.
Does the number of variables grow linearly with the length of the prediction horizon?

c (4 %) Assume that it is important to limit the control moves Δu_t to prevent wear and tear on control actuators. Propose a reformulation of (A.9) that facilitates this effect. (Please refer specifically to which parts that are changed and how they are changed.)

d (6 %) The choice of objective function is important in optimal control. Assume that the linear terms in (A.9a) are zero and that the weight matrices Q and R are time-invariant, and that the origin is the setpoint. Sketch a procedure for selecting the weight matrices Q and R. (The procedure may preferably be presented using bullet points. Extensive text is not expected.)

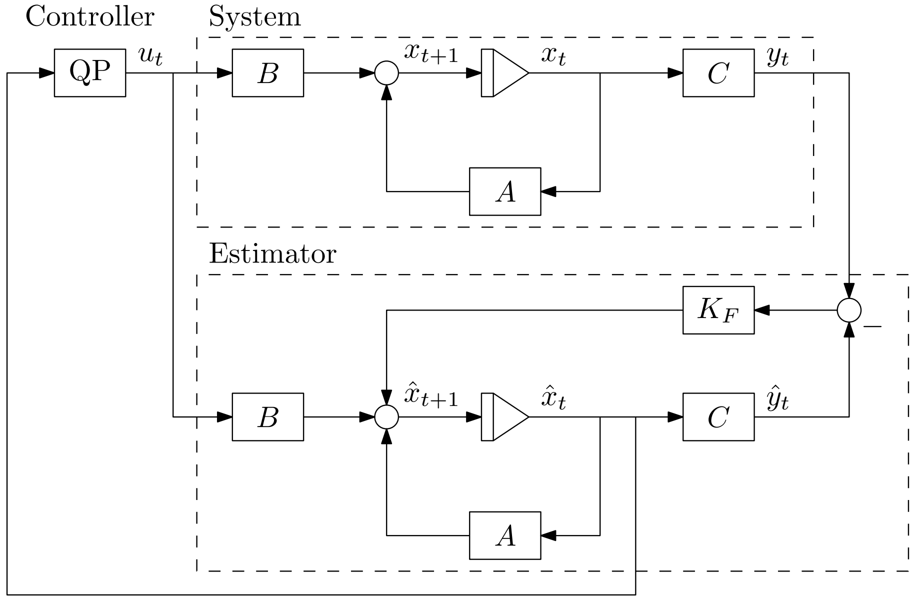

Figure 1: The structure of an output feedback linear MPC.

e (6 %) Linear MPC with output feedback is illustrated in Figure 1.
Formulate (only) the state space equations that describe the estimator part in Figure 1.
The estimator is tuned through the K_F matrix. How would you choose the estimator dynamics compared to the dynamics of the MPC feedback loop?

f (10 %) A control system may be presented in a layered structure where the lower layer includes a process with sensors and actuators, the second layer contains regulatory control, and the top layer advanced process control (APC). Sketch this in a control hierarchy.
Describe the information that is passed between the regulatory layer and the APC layer (both the upwards and the downwards information stream).
The sampling time for the controllers in the regulatory layer and the APC layer will be quite different. How do they differ?
The regulatory layer is usually implemented in a DCS (Distributed Control System). Describe how the APC layer is implemented. Please support your discussion with a figure.

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Utgave/Utgåve: bokmål/nynorsk

Eksamen i TTK4135

Optimalisering og regulering

Optimization and Control

Fredag 6. juni 2014

Tid: 09:00 – 13:00

Tillatte hjelpemidler / Tilletne hjelpemiddel:
D — Ingen trykte eller skrevne hjelpemidler. / Inga trykte eller skrevne hjelpemiddel.
Godkjent kalkulator med tomt minne. / Godkjend kalkulator med tomt minne.

Nyttig informasjon finnes i vedlegg. / Nyttig informasjon finns i vedlegg.
(Denne informasjonen er gitt på engelsk for å samsvare med pensumlitteraturen som den er hentet ifra.)
Sensur faller 27. juni. / Sensur fell 27. juni.

1 Ulike spørsmål (20 %)

Konveksitet

a (2 %) Anta at alle begrensningene i (A.1) er lineære. Er det gyldige området (feasible set), dersom det eksisterer, en konveks mengde?

b (4 %) Anta at at det kun finnes en likhetsbetingelse, c_1, i (A.1), og at c_1 er en lineær funksjon. Er det gyldige området en ikke-konveks mengde i dette tilfellet? Begrunn svaret ved bruk av figur.

c (4 %) Hvorfor er det mye vanskeligere å løse ikke-konvekse problemer enn konvekse problemer?

Lineær programmering - LP

Vi ser nå på LP problemer på standard form, se også (A.6).

d (4 %) Simplex-algoritmen benyttes for å løse LP problemer. Denne algoritmen søker etter løsningen i basis-punkter (*basic feasible points*). Hvordan sjekker algoritmen om et basis-punkt er løsningspunktet?

e (6 %) Simplex-algoritmen må løse linære likningssett på en effektiv måte. Dette betinger faktorisering og bruk av for eksempel LU dekomponering. Forklar hvorfor $Ax = b$, hvor $A \in \mathbb{R}^{n \times n}$, er enkel å løse dersom det finnes en LU dekomponering av A.

2 Ulineær programmering og SQP (45 %)

a (4 %) (A.1) i Appendix er et ulineært optimaliseringsproblem. Forklar betydningen av $\mathcal{E}$ og $\mathcal{I}$, og gi et eksempel på disse når det er 2 likhetsbetingelser og 3 ulikhetsbetingelser.

b (4 %) Anta likhetsbetingelsen

$$x_1^2 + 4x_2^2 = 4$$

Skisser denne likhetsbetingelsen og vis hva som er dens gyldige mengde (feasible set). Er den gyldige mengden en konveks mengde?

c (4 %) Anta ulikhetsbetingelsen

$$x_1^2 + 4x_2^2 \geq 4$$

Skisser denne ulikhetsbetingelsen og vis hva som er dens gyldige mengde (feasible set). Er den gyldige mengden en konveks mengde?

d (6 %) SQP-metoden lager et nytt QP sub-problem ved hver iterasjon k. Dette er vist nedenfor. (Notasjonen sammenfaller med læreboken.)

$$\begin{aligned}
\min_{p} \quad & \frac{1}{2} p^\top \nabla^2_{xx} \mathcal{L}(x_k, \lambda_k) p + \nabla f(x_k)^\top p + f(x_k) \\
\text{s.t.} \quad & \\
& \nabla c_i(x_k)^\top p + c_i(x_k) = 0, \qquad i \in \mathcal{E} \\
& \nabla c_i(x_k)^\top p + c_i(x_k) \geq 0, \qquad i \in \mathcal{I}
\end{aligned}$$

Dette kan vi et 'infeasible' QP problem selv om det opprinnelige ulineære problemet er 'feasible'.
Anta et ulineært problem med følgende ulikhetsbetingelser (det antas ingen andre begrensninger).

$$\begin{aligned}
x_1^2 &\geq 4 \\
x_1 &\leq 1
\end{aligned}$$

Vis at disse betingelsene (constraints) er inkonsistente (infeasible) når vi lineariserer om punktet $x_1 = 1$.

e (7 %) Vis hvordan objektfunksjonen $\frac{1}{2} p^\top \nabla^2_{xx} \mathcal{L}(x_k, \lambda_k) p + \nabla \mathcal{L}(x_k)^\top p + \mathcal{L}(x_k)$ kan reformuleres som $\frac{1}{2} p^\top \nabla^2_{xx} \mathcal{L}(x_k, \lambda_k) p + \nabla f(x_k)^\top p + f(x_k)$ i QP-problemet ovenfor. Anta kun likhetsbetingelser.

f (7 %) *Merit-funksjonen* er viktig i SQP algoritmer, og er definert av ϕ_1 i Alg.18.3 i Appendix. En vanlig merit-funksjon er

$$\phi_1(x;\mu) = f(x) + \mu \sum_{i\in\mathcal{E}} |c_i(x)| + \mu \sum_{i\in\mathcal{I}} [c_i(x)]^-$$

hvor $[z]^- = \max\{0, -z\}$, og $|z|$ er absoluttverdi. (Notasjonen sammenfaller med læreboken.)
Hva er hensikten med parameteren μ?
Hvordan endres denne vanligvis fra en iterasjon til den neste i en SQP algoritme? Begrunn svaret.

g (6 %) Forklar hva som menes med en *eksakt merit-funksjon.*
Er ϕ_1 funksjonen ovenfor en eksakt merit-funksjon? Begrunn svaret.

h (7 %) Hva menes med Maratos-effekten?
Skisser, ved bruk av et 2-dimensjonalt problem ($x \in \mathbb{R}^2$), en situasjon som illustrerer Maratos-effekten.
Hvordan kan en praktisk algoritme redusere eller unngå Maratos-effekten?

3 MPC og dynamisk optimalisering (35 %)

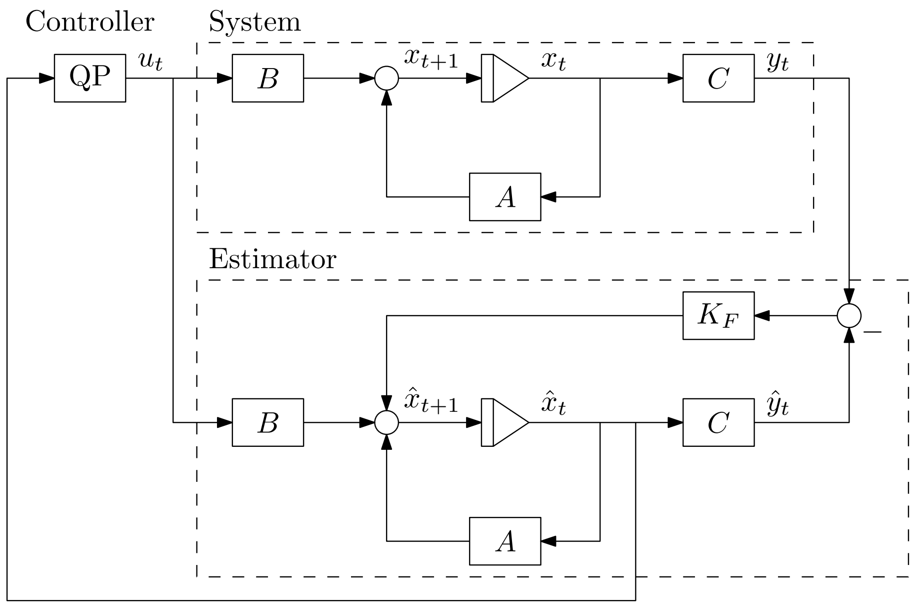
Vi tar nå utgangspunkt i (A.9) fra Appendix.

a (3 %) Forklar kort hvorfor (A.9) kalles et åpen sløyfe optimaliseringsproblem.

b (6 %) Anta at $N = 15$, $n_x = 12$ og $n_u = 3$. Hva er dimensjonen av z i (A.9l) i dette tilfellet?
Utled formelen for dimensjonen av z som funksjon av N, n_x og n_u.
Vokser antall variable lineært med lengden av prediksjonshorisonten?

c (4 %) Anta at det er viktig å begrense endringene i pådraget (Δu_t) for å begrense aktuator-slitasje. Foreslå endringer i (A.9) slik at dette oppnås. (Du må spesifisere hvilke deler av optimaliseringsproblemet som endres og hvordan de endres.)

d (6 %) Objektfunksjonen er viktig i optimalregulering. Anta at lineærleddene i (A.9a) er null, at vektmatrisene Q og R er tidsinvariante, og at origo er settpunktet. Skisser en prosedyre for å velge Q og R. (Bruk gjerne kulepunkter i forklaringen. Lange tekstlige utredninger forventes ikke.)



Figur 2: Struktur for lineær MPC med tilstandsestimering.

e (6 %) Lineær MPC med tilstandsestimering er vist i Figur 2.
Formuler (kun) tilstandslikningene som beskriver estimator-delen i Figur 2.
Estimatoren tunes med K_F matrisen. Hvordan velger du estimatordynamikken sammenliknet med dynamikken for MPC sløyfen som helhet?

f (10 %) Et styringshierarki kan framstilles i en lagdelt struktur hvor nederste lag er prosessen med sensorer og aktuatorer, det neste laget inneholder basisreguleringen og det øverste laget inneholder 'advanced process control (APC)'. Skisser et slikt styringshierarki.
Beskriv informasjonen som sendes mellom basisregulerings-laget og APC-laget (både informasjonen som flyter oppover og nedover).
Samplingstiden for regulatorene i basisregulerings-laget og APC-laget er ganske forskjellige. Hva er forskjellen?
Basisreguleringen implementeres gjerne i en DCS (Distributed Control System). Forklar hvordan APC-laget kan implementeres. Inkluder en figur i forklaringen.

Appendix

Part 1 Optimization Problems and Optimality Conditions

A general formulation for constrained optimization problems is

$$\min_{x \in \mathbb{R}^n} \quad f(x) \tag{A.1a}$$
$$\text{s.t.} \quad c_i(x) = 0, \qquad i \in \mathcal{E} \tag{A.1b}$$
$$c_i(x) \geq 0, \qquad i \in \mathcal{I} \tag{A.1c}$$

where f and the functions c_i are all smooth, differentiable, real-valued functions on a subset of $\mathbb{R}^n$, and $\mathcal{E}$ and $\mathcal{I}$ are two finite sets of indices.

The Lagrangean function for the general problem (A.1) is

$$\mathcal{L}(x, \lambda) = f(x) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i c_i(x) \tag{A.2}$$

The KKT-conditions for (A.1) are given by:

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = 0 \tag{A.3a}$$
$$c_i(x^*) = 0, \qquad i \in \mathcal{E} \tag{A.3b}$$
$$c_i(x^*) \geq 0, \qquad i \in \mathcal{I} \tag{A.3c}$$
$$\lambda_i^* \geq 0, \qquad i \in \mathcal{I} \tag{A.3d}$$
$$\lambda_i^* c_i(x^*) = 0, \qquad i \in \mathcal{E} \cup \mathcal{I} \tag{A.3e}$$

2nd order (sufficient) conditions for (A.1) are given by:

$$w \in \mathcal{C}(x^*, \lambda^*) \Leftrightarrow \begin{cases} \nabla c_i(x^*)^\top w = 0 & \text{for all } i \in \mathcal{E} \\ \nabla c_i(x^*)^\top w = 0 & \text{for all } i \in \mathcal{A}(x^*) \cap \mathcal{I} \text{ with } \lambda_i^* > 0 \\ \nabla c_i(x^*)^\top w \geq 0 & \text{for all } i \in \mathcal{A}(x^*) \cap \mathcal{I} \text{ with } \lambda_i^* = 0 \end{cases} \tag{A.4}$$

Theorem 1: (Second-Order Sufficient Conditions) *Suppose that for some feasible point $x^* \in \mathbb{R}^n$ there is a Lagrange multiplier vector λ^* such that the KKT conditions* (A.3) *are satisfied. Suppose also that*

$$w^\top \nabla^2_{xx} \mathcal{L}(x^*, \lambda^*) w > 0, \qquad \text{for all } w \in \mathcal{C}(x^*, \lambda^*),\ w \neq 0. \tag{A.5}$$

Then x^* is a strict local solution for (A.1).

LP problem in standard form:

$$
\begin{aligned}
\min_{x} \quad & f(x) = c^\top x && \text{(A.6a)} \\
\text{s.t.} \quad & Ax = b && \text{(A.6b)} \\
& x \geq 0 && \text{(A.6c)}
\end{aligned}
$$

where $A \in \mathbb{R}^{m \times n}$ and $\operatorname{rank} A = m$.

QP problem in standard form:

$$
\begin{aligned}
\min_{x} \quad & f(x) = \frac{1}{2} x^\top G x + x^\top c && \text{(A.7a)} \\
\text{s.t.} \quad & a_i^\top x = b_i, \qquad i \in \mathcal{E} && \text{(A.7b)} \\
& a_i^\top x \geq b_i, \qquad i \in \mathcal{I} && \text{(A.7c)}
\end{aligned}
$$

where G is a symmetric $n \times n$ matrix, $\mathcal{E}$ and $\mathcal{I}$ are finite sets of indices and c, x and $\{a_i\}, i \in \mathcal{E} \cup \mathcal{I}$, are vectors in $\mathbb{R}^n$. Alternatively, the equalities can be written $Ax = b$, $A \in \mathbb{R}^{m \times n}$.

Iterative method:

$$
\begin{aligned}
x_{k+1} &= x_k + \alpha_k p_k && \text{(A.8a)} \\
& x_0 \text{ given} && \text{(A.8b)} \\
x_k, p_k &\in \mathbb{R}^n, \;\; \alpha_k \in \mathbb{R} && \text{(A.8c)}
\end{aligned}
$$

p_k is the search direction and α_k is the line search parameter.

Part 2 Optimal Control

A typical open-loop optimal control problem on the time horizon 0 to N is

$$\min_{z\in\mathbb{R}^n} f(z) = \sum_{t=0}^{N-1} \frac{1}{2}x_{t+1}^\top Q_{t+1}x_{t+1} + d_{xt+1}x_{t+1} + \frac{1}{2}u_t^\top R_t u_t + d_{ut}u_t \tag{A.9a}$$

subject to

$$x_{t+1} = A_t x_t + B_t u_t, \qquad t = 0,\dots,N-1 \tag{A.9b}$$
$$x_0 = \text{given} \tag{A.9c}$$
$$x^{\text{low}} \le x_t \le x^{\text{high}}, \qquad t = 1,\dots,N \tag{A.9d}$$
$$u^{\text{low}} \le u_t \le u^{\text{high}}, \qquad t = 0,\dots,N-1 \tag{A.9e}$$
$$-\Delta u^{\text{high}} \le \Delta u_t \le \Delta u^{\text{high}}, \qquad t = 0,\dots,N-1 \tag{A.9f}$$
$$Q_t \succeq 0 \qquad t = 1,\dots,N \tag{A.9g}$$
$$R_t \succeq 0 \qquad t = 0,\dots,N-1 \tag{A.9h}$$

where

$$u_t \in \mathbb{R}^{n_u} \tag{A.9i}$$
$$x_t \in \mathbb{R}^{n_x} \tag{A.9j}$$
$$\Delta u_t = u_t - u_{t-1} \tag{A.9k}$$
$$z^\top = (x_1^\top,\dots,x_N^\top, u_0^\top,\dots,u_{N-1}^\top) \tag{A.9l}$$

The subscript t denotes discrete time sampling instants.

The optimization problem for linear quadratic control of discrete dynamic systems is given by

$$\min_{z\in\mathbb{R}^n} f(z) = \sum_{t=0}^{N-1} \frac{1}{2}x_{t+1}^\top Q_{t+1}x_{t+1} + \frac{1}{2}u_t^\top R_t u_t \tag{A.10a}$$

subject to

$$x_{t+1} = A_t x_t + B_t u_t \tag{A.10b}$$
$$x_0 = \text{given} \tag{A.10c}$$

where

$$u_t \in \mathbb{R}^{n_u} \tag{A.10d}$$
$$x_t \in \mathbb{R}^{n_x} \tag{A.10e}$$
$$z^\top = (x_1^\top,\dots,x_N^\top, u_0^\top \dots, u_{N-1}^\top) \tag{A.10f}$$

Theorem 2: *The solution of (A.10) with $Q_t \succeq 0$ and $R_t \succ 0$ is given by*

$$u_t = -K_t x_t \tag{A.11a}$$

where the feedback gain matrix is derived by

$$K_t = R_t^{-1} B_t^\top P_{t+1}(I + B_t R_t^{-1} B_t^\top P_{t+1})^{-1} A_t, \qquad t = 0, \dots, N-1 \tag{A.11b}$$

$$P_t = Q_t + A_t^\top P_{t+1}(I + B_t R_t^{-1} B_t^\top P_{t+1})^{-1} A_t, \qquad t = 0, \dots, N-1 \tag{A.11c}$$

$$P_N = Q_N \tag{A.11d}$$

Part 3 Sequential quadratic programming (SQP)

Algorithm 18.3 (Line Search SQP Algorithm).

Choose parameters $\eta \in (0, 0.5)$, $\tau \in (0, 1)$, and an initial pair (x_0, λ_0);
Evaluate $f_0, \nabla f_0, c_0, A_0$;
If a quasi-Newton approximation is used, choose an initial $n \times n$ symmetric positive definite Hessian approximation B_0, otherwise compute $\nabla^2_{xx}\mathcal{L}_0$;
repeat until a convergence test is satisfied
    Compute p_k by solving (18.11); let $\hat{\lambda}$ be the corresponding multiplier;
    Set $p_\lambda \leftarrow \hat{\lambda} - \lambda_k$;
    Choose μ_k to satisfy (18.36) with $\sigma = 1$;
    Set $\alpha_k \leftarrow 1$;
    while $\phi_1(x_k + \alpha_k p_k; \mu_k) > \phi_1(x_k; \mu_k) + \eta\alpha_k D_1(\phi(x_k; \mu_k) p_k)$
        Reset $\alpha_k \leftarrow \tau_\alpha \alpha_k$ for some $\tau_\alpha \in (0, \tau]$;
    end (while)
    Set $x_{k+1} \leftarrow x_k + \alpha_k p_k$ and $\lambda_{k+1} \leftarrow \lambda_k + \alpha_k p_\lambda$;
    Evaluate $f_{k+1}, \nabla f_{k+1}, c_{k+1}, A_{k+1}$, (and possibly $\nabla^2_{xx}\mathcal{L}_{k+1}$);
    If a quasi-Newton approximation is used, set
        $s_k \leftarrow \alpha_k p_k$ and $y_k \leftarrow \nabla_x \mathcal{L}(x_{k+1}, \lambda_{k+1}) - \nabla_x \mathcal{L}(x_k, \lambda_{k+1})$,
    and obtain B_{k+1} by updating B_k using a quasi-Newton formula;
end (repeat)