NORGES TEKNISK-NATURVITSKAPLEGE UNIVERSITET
INSTITUTT FOR TELETEKNIKK
Signalbehandling

Suggested solutions for
TTT4120 Digital Signal Processing,
August 2012

Problem 1

(a) The system must be *linear* and *time-invariant* (LTI) if it is to be described by the unit impulse response $h(n)$.

Definitions of stability and causality in terms of $h(n)$:

- stability: $\sum_n |h(n)| < \infty$
- causality: $h(n) = 0,\ n < 0$

(b) The Z-transform of $h(n)$ is

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}.$$

The region of convergence (ROC) is the set of values of z for which $H(z)$ attains a finite value, that is, $z \in \text{ROC}$ iff $|H(z)| < \infty$.

Figure 1 shows the ROCs for a causal and an anti-causal system.

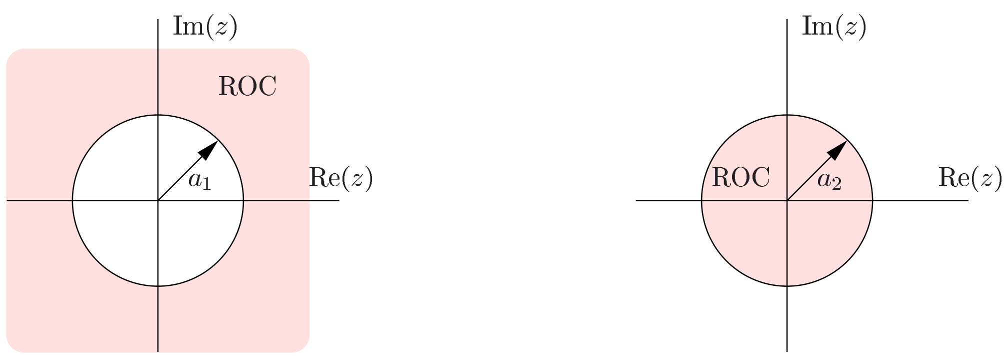


Figure 1: The shaded areas are the ROCs for a causal system (left) and an anti-causal system (right).

The unit circle, $z = e^{j\omega}$ ($|z| = 1$), must be included in the ROC for the system to be stable. This is because the Fourier transform for a stable system always has to exist.

(c) When the system is stable we must have $0 \leq b < 1$. If the system is also minimum phase we must have $0 \leq a < 1$. For the system to be causal, we must then have a ROC given by $|z| > b$.

Problem 2

(a) The up-sampled sequence $v(m)$ can be written

$$v(m) = \begin{cases} x(m/U), & m = 0, \pm U \pm 2U, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

The sequence $w(m)$ is

$$\begin{aligned} w(m) &= v(m) * h_u(m) \\ &= \sum_{k=-\infty}^{\infty} v(k) h_u(m-k) \\ &= \sum_{p=-\infty}^{\infty} v(pU) h_u(m-pU) \\ &= \sum_{p=-\infty}^{\infty} x(p) h_u(m-pU). \end{aligned}$$

The sequence $z(m)$ is given by

$$z(m) = w(m) * h_d(m) = \sum_{k=-\infty}^{\infty} w(k) h_d(m-k).$$

The decimated sequence is

$$y(l) = z(lD) = \sum_{k=-\infty}^{\infty} w(k) h_d(lD-k).$$

The sequence $z(m)$ is

$$\begin{aligned} z(m) &= w(m) * h_d(m) \\ &= v(m) * h_u(m) * h_d(m) \\ &= v(m) * h(m) \\ &= \sum_{p=-\infty}^{\infty} x(p) h(m-pU). \end{aligned}$$

Changing variable from p to k and substituting into the expression for $y(l)$ above gives

$$y(l) = z(lD) = \sum_{k=-\infty}^{\infty} x(k) h(lD-kU).$$

(b) We can write the input sequence as

$$x(n) = \sin(2\pi f_1 n) + 2\sin(2\pi f_2 n),$$

where $f_1 = F_1/F_x = 0.1$ and $f_2 = F_2/F_x = 0.2$. In order to find the DTFT of $x(n)$, recall that the discrete-time Fourier transform (DTFT) of $\sin(2\pi f_0 n)$ is $-j/2[\delta(f - f_0) - \delta(f + f_0)]$ (This is most easily checked by taking the inverse DTFT of the latter). Thus, we get

$$\begin{aligned} X(f) &= -\frac{j}{2}\Big[\delta(f-f_1) - \delta(f+f_1)\Big] - j\Big[\delta(f-f_2) - \delta(f+f_2)\Big] \\ &= -\frac{j}{2}\Big[\delta(f-f_1) - \delta(f+f_1) + 2\delta(f-f_2) - 2\delta(f+f_2)\Big] \end{aligned}$$

and

$$|X(f)| = \frac{1}{2}\delta(f-f_1) + \frac{1}{2}\delta(f+f_1) + \delta(f-f_2) + \delta(f+f_2),$$

because for each f, only one of the four terms in $X(f)$ is nonzero, implying that all the four terms in $|X(f)|$ are positive.

Plots of $|X(f)|$, $|V(f)|$, $|H(f)|$, $|Z(f)|$ and $|Y(f)|$ for $U/D = 1/2$ and $U/D = 2$ are shown in Figure 2 and Figure 3, respectively. In order to get the plots right, it is important to be aware of the fact that the DTFT of a sequence is periodic with period equal to the sampling period. When the DTFT is vied as a function of normalized frequency f, the period is 1.

The cutoff frequency of the filter $H(f)$ is

$$f_c = \min\left\{\frac{1}{2U}, \frac{1}{2D}\right\},$$

or, equivalently, since $f_c = F/(UF_x)$,

$$F_c = \min\left\{\frac{F_x}{2}, \frac{UF_x}{2D}\right\}.$$

Thus, for $U/D = 1/2$ the cutoff frequency is $F_c = F_x/4 = 250$ Hz, and for $U/D = 2$ the cutoff frequency is $F_c = F_x/2 = 500$ Hz.

(c) Figure 4 and Figure 5 show the DTFT of the sequences involved in the sample-rate converter for $U/D = 2/3$ and $U/D = 3/2$, respectively. The cutoff frequency is $F_c = F_x/3 \approx 333$ Hz for the former and $F_c = F_x/2 = 500$ Hz for the latter.

(d) Figure 6 shows the DTFT of the sequences involved in the sample-rate converter for $U/D = 1/3$. The cutoff frequency is $F_c = F_x/6 \approx 167$ Hz. Thus the sinusoid with frequency $F_2 = 200$ Hz is removed by the filter. If the filter were not present, there would be aliasing in the output.

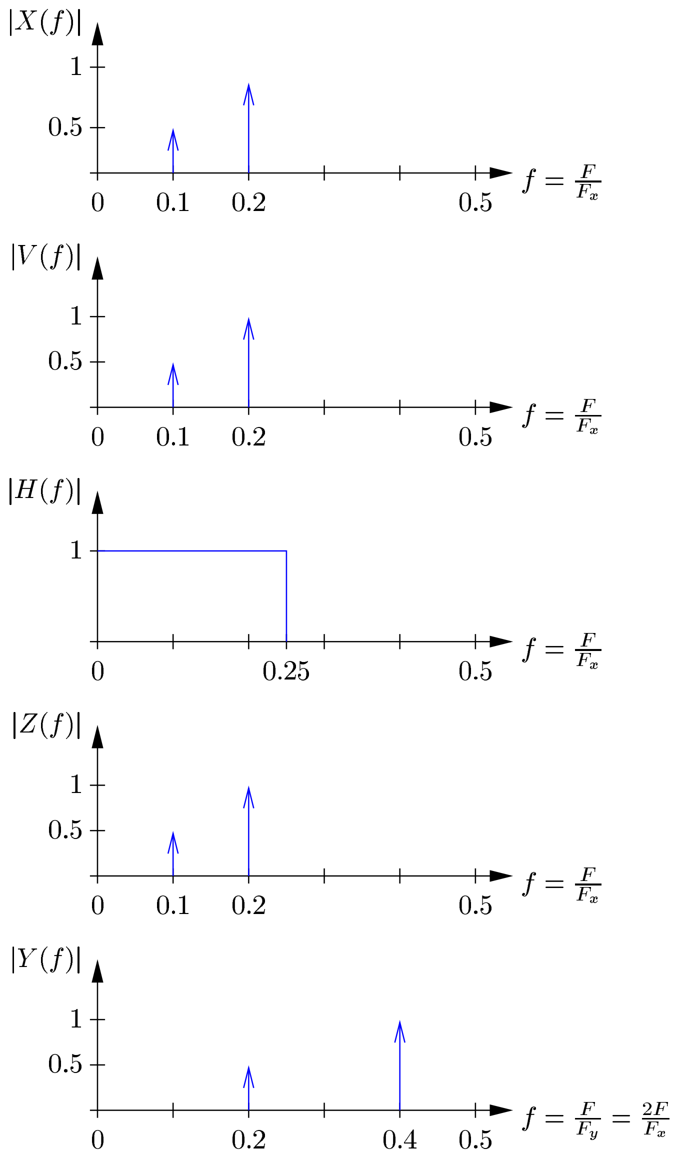


Figure 2: Sampling-rate conversion with factor $\frac{U}{D} = \frac{1}{2}$.

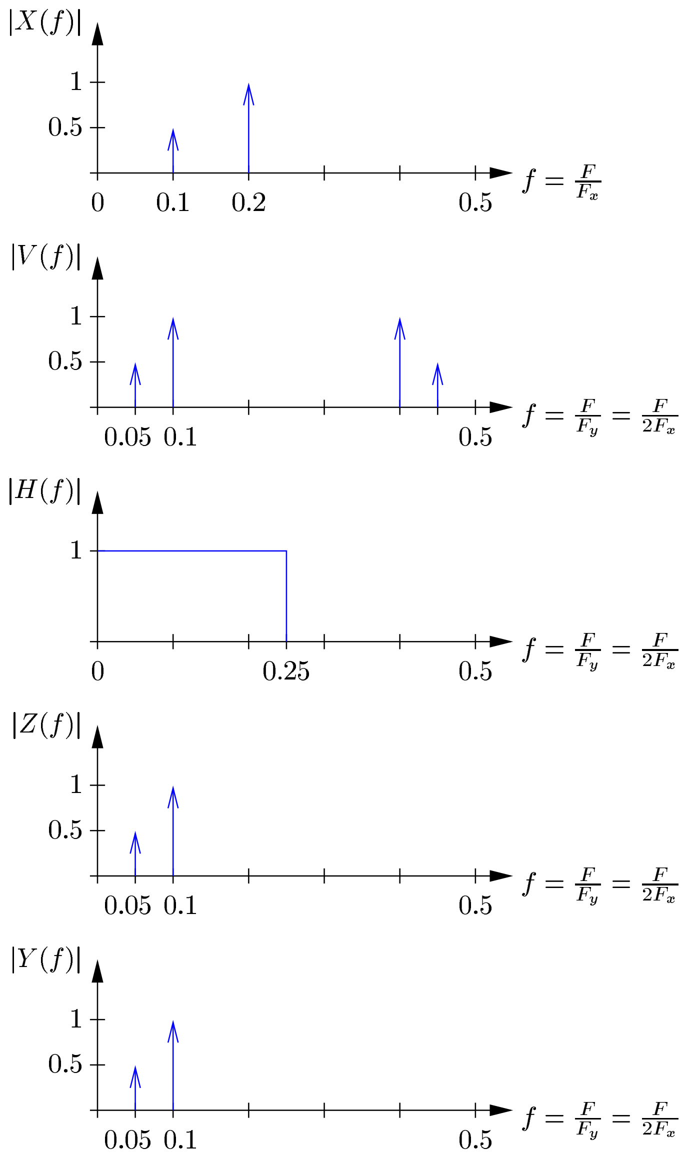


Figure 3: Sampling-rate conversion with factor $\frac{U}{D} = 2$.

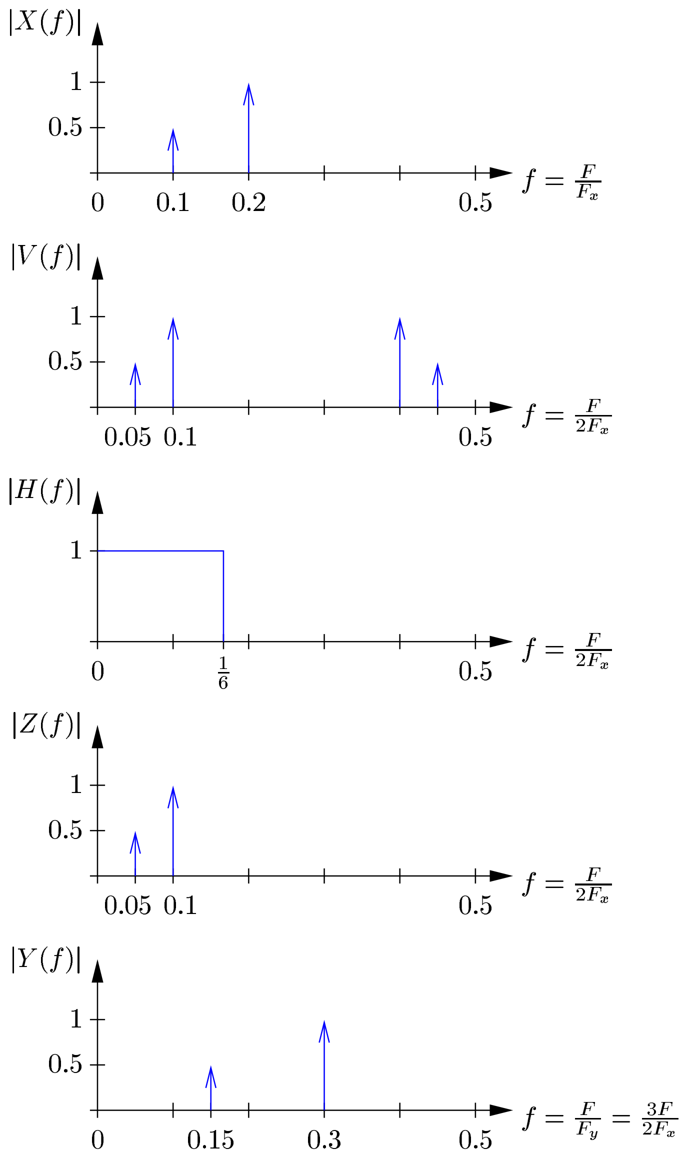


Figure 4: Sampling-rate conversion with factor $\frac{U}{D} = \frac{2}{3}$.

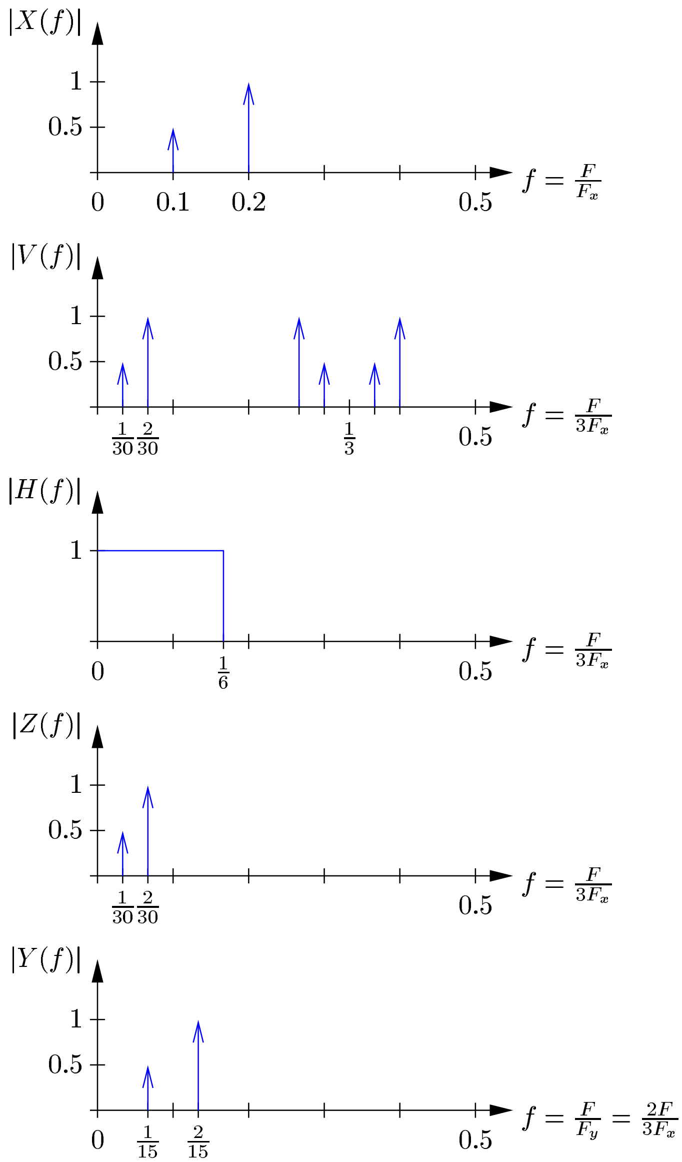


Figure 5: Sampling-rate conversion with factor $\frac{U}{D} = \frac{3}{2}$.

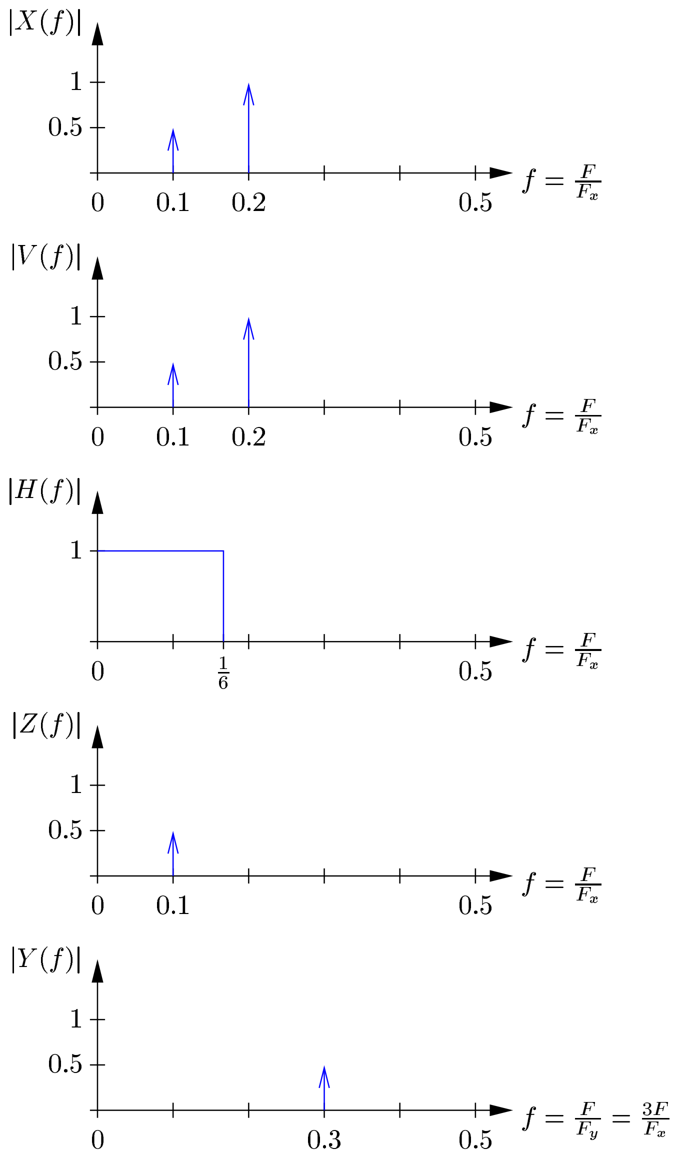


Figure 6: Sampling-rate conversion with factor $\frac{U}{D} = \frac{1}{3}$.

Problem 3

(a) The filter $H_1(z)$ has a single pole at $z = a$ and a single zero at $z = 1/a$. Since the pole and the zero form a reciprocal pair, $H_1(z)$ is an all-pass filter. To prove this, we must show that $|H_1(e^{j\omega})| = 1$. We have

$$|H_1(e^{j\omega})|^2 = H_1(e^{j\omega})H_1^*(e^{j\omega}) = \frac{e^{-j\omega} - a}{1 - ae^{-j\omega}} \frac{e^{j\omega} - a}{1 - ae^{j\omega}} = \frac{1 - ae^{-j\omega} - ae^{j\omega} + a^2}{1 - ae^{j\omega} - ae^{-j\omega} + a^2} = 1,$$

and hence $|H_1(e^{j\omega})| = 1$, which was to be proved.

For the filter $H_1(z)$ to be stable and causal, it must have a region of convergence (ROC) given by $|z| > |a|$, where $|a| < 1$.

A filter is minimum phase if all its poles and zeros are inside the unit circle. Clearly, since the pole and the zero of $H_1(z)$ form a reciprocal pair, they cannot both be inside the unit circle. The filter $H_1(z)$ can therefore not be minimum phase.

(b) The filter can be decomposed as

$$H(z) = H_1(z)H_2(z) = \frac{z^{-1} + \frac{1}{2}}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{2}z^{-1})} = \frac{A}{1 - \frac{1}{2}z^{-1}} + \frac{B}{1 + \frac{1}{2}z^{-1}},$$

where

$$A = \left.\frac{z^{-1} + \frac{1}{2}}{1 + \frac{1}{2}z^{-1}}\right|_{z=\frac{1}{2}} = \frac{\cancel{3}\,5}{4} \quad \text{and} \quad B = \left.\frac{z^{-1} + \frac{1}{2}}{1 - \frac{1}{2}z^{-1}}\right|_{z=-\frac{1}{2}} = -\frac{\cancel{5}\,3}{4}.$$

Hence,

$$H(z) = \frac{\frac{\cancel{3}\,5}{4}}{1 - \frac{1}{2}z^{-1}} + \frac{-\frac{\cancel{5}\,3}{4}}{1 + \frac{1}{2}z^{-1}}.$$

(c) Recall that a stable and causal filter on the form

$$G(z) = \frac{1}{1 - az^{-1}}$$

has inverse z-transform

$$g(n) = a^n u(n).$$

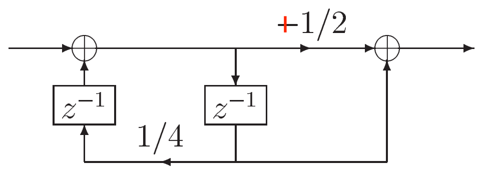
From this result we can easily find the unit impulse response of $H(z)$, that is,

$$h(n) = \frac{5\cancel{3}}{4}\left(\frac{1}{2}\right)^n u(n) - \frac{3\cancel{5}}{4}\left(-\frac{1}{2}\right)^n u(n).$$

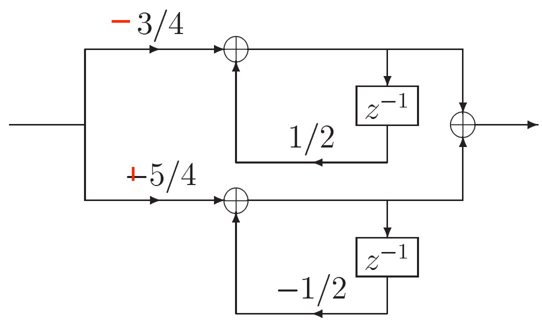
(d) In order to sketch the DF2 structure of $H(z)$ the following calculation is useful. We have

$$\begin{aligned}
H(z) = \frac{Y(z)}{X(z)} &= \frac{z^{-1} + \frac{1}{2}}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{2}z^{-1})} \\
Y(z) - \frac{1}{4}z^{-2}Y(z) &= z^{-1}X(z) + \frac{1}{2}X(z) \\
y(n) - \frac{1}{4}y(n-2) &= x(n-1) + \frac{1}{2}x(n).
\end{aligned}$$

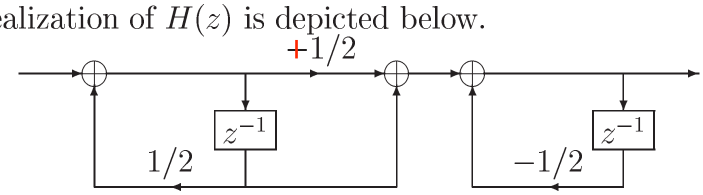
The DF2 structure of the filter is depicted below.



A parallel realization of $H(z)$ is depicted below.



A cascade realization of $H(z)$ is depicted below.



Problem 4

(a) The unit impulse response of $G(z) = H_3(z)$ is

$$g(n) = \frac{5\,\cancel{3}}{4}\left(\frac{1}{2}\right)^n u(n).$$

Substituting $g(n)$ into Eq. (7) in the problem set gives, for $m \geq 0$,

$$\begin{aligned}
\gamma_{yy}(m) &= \sigma_w^2 \sum_{n=0}^{\infty} \frac{3}{4}\left(\frac{1}{2}\right)^n u(n) \cdot \frac{3}{4}\left(\frac{1}{2}\right)^{n+m} u(n+m) \\
&= \sigma_w^2 \sum_{n=0}^{\infty} \frac{3}{4}\left(\frac{1}{2}\right)^n \cdot \frac{3}{4}\left(\frac{1}{2}\right)^{n+m} \\
&= \sigma_w^2 \left(\frac{3}{4}\right)^2 \left(\frac{1}{2}\right)^m \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{2n} \\
&= \sigma_w^2 \left(\frac{3}{4}\right)^2 \left(\frac{1}{2}\right)^m \frac{1}{1-1/4} \\
&= \sigma_w^2 \left(\frac{3}{4}\right)^2 \left(\frac{1}{2}\right)^m \frac{4}{3} \\
&= \frac{3}{4}\left(\frac{1}{2}\right)^m \sigma_w^2.
\end{aligned}$$

3/4 -> 5/4

Since $\gamma_{yy}(m) = \gamma_{yy}(-m)$ for $m < 0$, we get

$$\gamma_{yy}(m) = \frac{5\,\cancel{3}}{4}\left(\frac{1}{2}\right)^{|m|} \sigma_w^2 \quad \text{for all } m.$$

(b) We know from Problem 3a) that $G(z) = H_1(z)$ is an all-pass filter and thus $G(z)G(z^{-1}) = 1$. From Eq. (8) in the problem set we then get that $\Gamma_{yy}(z) = \sigma_w^2$. The autocorrelation function of $y(n)$ is the inverse z-transform of $\Gamma_{yy}(z)$. Thus, $\gamma_{yy}(m) = \sigma_w^2 \delta(m)$, or, equivalently, $\gamma_{yy}(0) = \sigma_w^2$ and $\gamma_{yy}(m) = 0$ for $m \neq 0$.

(c) We know that $H_1(z)$ is an all-pass filter. The output process of $H_1(z)$ is therefore equal to the input process which is white noise with variance σ_w^2. The filter $H_3(z)$ is an AR[1]-model since it has only one pole. The output of $H_3(z)$ is therefore an AR[1] process. ~~The filter $H(z)$ has a single zero at $z = 2$ and two poles at $z = \pm 1/2$. The output of $H(z)$ is therefore an ARMA[1,2] process.~~

(d) The general form of the filter of an AR[1] model is $A(z) = b/(1 + az^{-1})$. To find the AR[1] model of $H_1(z)$ we can use the fact that the output process of $H_1(z)$ is white noise with variance σ_w^2. We will therefore get a perfect model by choosing $a = 0$ and $b = \sigma_w$, yielding the filter $A(z) = \sigma_w$. Alternatively, we could have gotten the same result by substituting the autocorrelation function found in 4b) into the Yule-Walker equations,

$$\hat{\gamma}_{yy}(0) + a\hat{\gamma}_{yy}(-1) = b^2,$$
$$\hat{\gamma}_{yy}(1) + a\hat{\gamma}_{yy}(0) = 0.$$

To find the process parameters of $H_3(z)$, insert $\hat{\gamma}_{yy}(0) = \overset{25/12}{\cancel{\tfrac{3}{4}}}\sigma_w^2$ and $\hat{\gamma}_{yy}(1) = \hat{\gamma}_{yy}(-1) =$ 25/24$\cancel{\tfrac{3}{8}}\sigma_w^2$ into the Yule-Walker equations given above. This ~~gives~~

$$\cancel{\frac{3}{4}\sigma_w^2 + a\frac{3}{8}\sigma_w^2 = b^2},$$
$$\cancel{\frac{3}{8}\sigma_w^2 + a\frac{3}{4}\sigma_w^2 = 0},$$

~~which~~ has the solution $a = -1/2$ and $b^2 = \overset{25}{\cancel{9}}\sigma_w^2/\overset{8}{\cancel{16}}$. These parameters give the filter

$$A(z) = \frac{\overset{\text{5/2sqrt(2)}}{\cancel{\tfrac{3}{4}}}\sigma_w}{1 - \frac{1}{2}z^{-1}}.$$