NORWEGIAN UNIVERSITY
OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF TELECOMMUNICATIONS

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EXAMINATION IN COURSE TTT4120 DIGITAL SIGNAL PROCESSING

Date: yyyday x August 2011
Time: 09.00 - 13.00

Permitted aids: D–No calculators allowed.
No printed or handwritten materials allowed.

INFORMATION

- The examination includes 4 problems, each of which has 4 subsections.
- Problem 1 deals with basic properties of systems/filters.
- Problem 2 deals with filter structures.
- Problem 3 deals with stationary processes and parametric estimation.
- Problem 4 deals with filtering in the frequency domain
- The weight of each subproblem is given in parenthesis at problem start
- The course responsible will visit you twice, the first time around 10.00 o'clock and the second time between 12.00 - 12.30.

Problem 1 (3+5+4+4)

1a) Which properties have to be fulfilled in order to describe a system by its unit pulse response $h(n)$? (1 p)
Given that the above properties are fulfilled, define the two properties stability and causality in terms of $h(n)$. (2 p)

Answers :
$h(n) \Leftrightarrow$ LTI-system
Stability $\Leftrightarrow \sum_n |h(n)| < \infty$
Causality $\Leftrightarrow h(n) = 0$ for $n < 0$

1b) Define the z-transform $H(z)$ in terms of $h(n)$, $n = -\infty, \infty$. (1 p)
What is meant by the term "region of convergence" (ROC) of the transfer function $H(z)$? (1 p)
Sketch ROC in the z-plane for respectively a causal and anti-causal system. (2 p)
What area in the z-plane must be included in ROC if the system is to be stable? State the reason for your answer. (1 p)

Answers :
$H(z) = \sum_n h(n) z^{-n}$
$z \in$ ROC $\Leftrightarrow |H(z)| < \infty$
Causal+ROC $\Leftrightarrow |z| > a$ where $0 < a < 1$)
Anticausal+ROC $\Leftrightarrow |z| < a$ where $a > 1$
Stability $\Leftrightarrow |z| = 1 \in$ ROC due to that $H(f)$ must exist

1c) Given the following stable filter $H_1(z)$.

$$H_1(z) = \frac{z^{-1} - a}{1 - az^{-1}} \tag{1}$$

Show that the filter is allpass. (3 p)
For which values of the filter coeffisient a is the filter causal? (1 p)

Answers :
$H_1(z)H_1(z^{-1})|_{z=e^{j\omega}} = H_1(j\omega)H_1^*(j\omega) = |H_1(j\omega)|^2 = Constant \Rightarrow$

$$H_1(z)H_1(z^{-1}) = \frac{z^{-1} - a}{1 - az^{-1}} \frac{z - a}{1 - az} = \frac{z^{-1} - a}{1 - az^{-1}} \frac{z(1 - az^{-1})}{z(z^{-1} - a)} = 1 \Leftrightarrow \text{qed}$$

Causality $\Leftrightarrow |a| < 1$

1d) Define the autocorrelation sequence $r_{hh}(m), \quad m = -\infty, \infty$ of a general, stable filter $h(n)$. (1 p)

Explain why the autocorrelation sequence of the allpass filter in subtask 1c has the form (3 p)

$$r_{h_1h_1}(m) = \delta(m), \quad m = -\infty, \infty \tag{2}$$

Answer :

$$r_{h_1h_1}(m) = \sum_n h_1(n)h_1(n+m)$$

$$r_{h_1h_1}(m) = IFT[R_{h_1h_1}(j\omega)] = IFT[|H_1(j\omega)|^2] = IFT[1] = \delta(m), \quad m = -\infty, \infty$$

where IFT means Inverse Fourier Transform

Problem 2 (4+3+6+3)

Given a stable, causal filter $H(z)$ on the form

$$H(z) = H_1(z)H_2(z) = \frac{z^{-1} - \frac{2}{3}}{1 - \frac{2}{3}z^{-1}} \frac{1}{1 - \frac{1}{2}z^{-1}} = \frac{z^{-1} - \frac{2}{3}}{1 - \frac{5}{6}z^{-1} + \frac{1}{3}z^{-2}} \tag{3}$$

i.e. $H_1(z)$ is given by the allpass filter in subtask 1c (using $a = \frac{2}{3}$) and $H_2(z)$ is given by

$$H_2(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} \tag{4}$$

2a) Show that $H(z)$ can be written on the following parallel form (4 p)

$$H(z) = H_3(z) + H_4(z) = \frac{\frac{10}{3}}{1 - \frac{2}{3}z^{-1}} + \frac{-4}{1 - \frac{1}{2}z^{-1}} \tag{5}$$

Answer :
Putting eq 5 on common denominator gives

$$\frac{\frac{10}{3}(1 - \frac{1}{2}z^{-1}) - 4(1 - \frac{2}{3}z^{-1})}{(1 - \frac{1}{2}z^{-1})(1 - \frac{2}{3}z^{-1})} = \frac{\frac{10}{3} - 4 - \frac{5}{3}z^{-1} + \frac{8}{3}z^{-1}}{1 - \frac{5}{6}z^{-1} + \frac{1}{3}z^{-2}} = \frac{-\frac{2}{3} + z^{-1}}{1 - \frac{5}{6}z^{-1} + \frac{1}{3}z^{-2}} \Leftrightarrow \text{q}ed$$

2b) Derive the unit impulse response $h(n)$ of the filter $H(z)$ (3 p)

Answer :
From eq 5 we easily se that the two first order terms directly give

$$h(n) = h_3(n) + h_4(n) = \frac{10}{3}(\frac{2}{3})^n - 4(\frac{1}{2})^n \quad n \geq 0$$

and of course $h(n) = 0 \quad n < 0$ due to causality

2c) Sketch the following structures for $H(z)$:

- Direct form 2 (DF2) (2 p)
- Parallel (2 p)
- Cascade (2 p)

Answer :

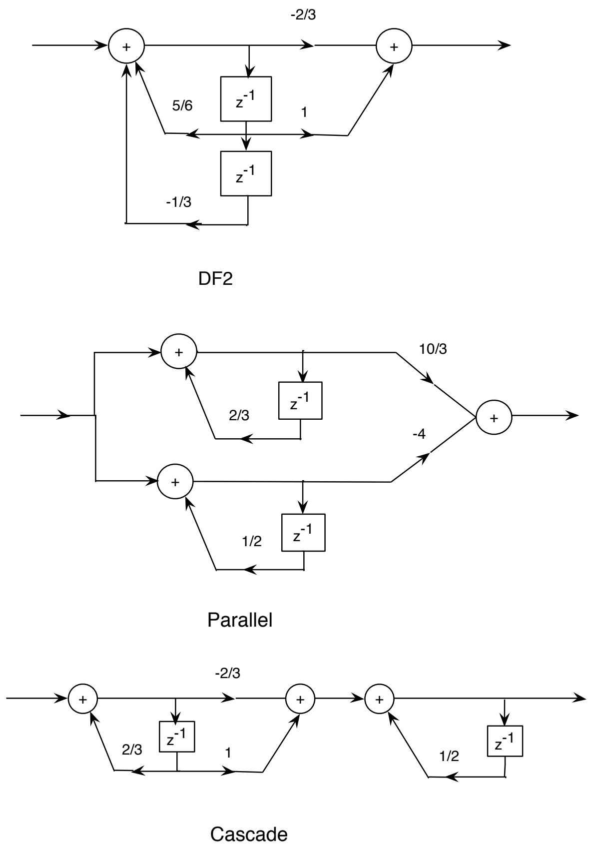

Figure 1: Three different filter structures

2d) Explain why the autocorrelation sequences of $H(z)$ and $H_2(z)$ are identical. (3 p)

Answer :

$$r_{hh}(m) = IFT[R_{hh}(j\omega)] = IFT[|H(j\omega)|^2] = IFT[|H_1(j\omega)H_2(j\omega)|^2] \Rightarrow$$
$$r_{hh}(l) = IFT[|H_1(j\omega)|^2|H_2(j\omega)|^2] = IFT[|H_2(j\omega)|^2] = r_{h_2h_2}(m), \quad m = -\infty, \infty$$

which obviously is true as $|H_1(j\omega)|^2 = 1$

Problem 3 (6+4+4+4)

Given a causal, stable filter with unit pulse response $g(n), \quad n = 0, \infty$. White noise $w(n)$ with power σ_w^2 is input to the filter.
The autocorrelation function $\gamma_{yy}(m), \quad m = -\infty, \infty$, and the power spectrum $\Gamma_{yy}(z)$ of the resulting output signal $y(n)$ are given by

$$\gamma_{yy}(m) = \begin{cases} \sigma_w^2 \displaystyle\sum_{n=0}^{\infty} g(n)g(n+m) = \sigma_w^2 \; r_{gg}(m) & m \geq 0 \\ \gamma_{yy}(-m) & m < 0 \end{cases} \tag{6}$$

$$\Gamma_{yy}(z) = \sigma_w^2 \; G(z)G(z^{-1}) \tag{7}$$

3a) Define respectively an ARMA, AR and MA process. (2+1+1 p)

What is the principial difference between a physical process and a process model? (2 p)

Answer :
All three processes are stationary and ergodic and are derived by inputing white noise to a filter. For the ARMA process we use a general filter of the form $H(z) = B(z)/A(z)$ where $B(z)$ and $A(z)$ are polynominals of some order M and N respectively. The AR process is the special case where we use an allpole filter, i.e. $H(z) = 1/A(z)$ while the MA case is given by a FIR-filter, i.e. $H(z) = B(z)$
A physical process is usually nonstationary and has no exact mathematical description. Thus we often use parametric processes as models (approximations) to the physical processes. We must then of course also approximate the latter (nonstationarity) to be short time stationary processes.

3b) Explain which type of parametric process we will find at the filter output $y(n)$ when white noise with power σ_w^2 is input to respectively :

- $H_1(z)$ (2 p)
- $H(z)$ (2 p)

where the filters are defined in task 2.

Answer :
We refer to equations 6 and 7 for the general descriptions of parametric processes

- $H_1(z)$ gives by first look an ARMA[1,1] process (eq. 1). However subtask 1d showed that $r_{h_1h_1}(m) = \delta(m), \quad m = -\infty, \infty$. Thus $\gamma_{yy}(m) = \sigma_w^2\delta(m), \quad m = -\infty, \infty$; i.e. the filter output is *white noise*
- $H(z)$ gives by first look an ARMA[1,2] process (eq. 3). However subtask 2d showed that we get the same process as when using $H_2(z)$. Thus from eq. 4 we see that the filter output is an *AR[1] process.*

3c) Find the autocorrelation sequence of the output $y(n)$ when white noise with power σ_w^2 is input to $H(z)$. (4 p)

Answer :
We must find the autocorrelation sequence of $H(z)$. However from subtask 2d this is identical to the autocorrelation sequence of $H_2(z)$.

$$r_{h_2h_2}(m) = \sum_n h_2(n)h_2(n+m) = \sum_n (\frac{1}{2})^n(\frac{1}{2})^{(n+m)}$$

Assuming $m \geq 0$ without losing generality (the autocorrelation is symmetric) we get :

$$r_{h_2h_2}(m) = (\frac{1}{2})^m \sum_{n=0}^{\infty}(\frac{1}{4})^n = (\frac{1}{2})^m \frac{1}{1-\frac{1}{4}} = \frac{4}{3}(\frac{1}{2})^m \quad m \geq 0$$

Thus we have

$$\gamma_{yy}(|m|) = \sigma_w^2 \, \frac{4}{3}(\frac{1}{2})^{|m|} \quad m = -\infty, \infty$$

3d) Give the process parameters of the best AR[1] model for each of the two output signals $y(n)$ in subtask 3b. (2+2 p)

State your reason for the answers.

Answer :
The model parameters are the noise power $\sigma_f^2 = \hat{\sigma}_w^2$ and the filter coefficient a_1

- According to subtask 3b $H_1(z)$ gives white noise at the output. Thus the best AR[1] model will have $\sigma_f^2 = \sigma_w^2$ and $a_1 = 0$ (degenerate AR process)
- According to subtask 3b $H(z)$ gives an AR[1] process at the output which is identical to if one used $H_2(z)$ instead of $H(z)$. Thus the best AR[1] model will have $a_1 = 1/2$ and $\sigma_f^2 = \sigma_w^2$.

Problem 4 (3+6+5+3)

4a) Set up the formulas for a N-point Diskret Fourier Transform (DFT) and its inverse (IDFT) for a sequence $x(n)$ of finite length M (1+1 p)

How must N be chosen if one wishes reproduce $x(n)$ from the DFT values? (1 p)

Answer :

$$X(k) = \sum_{n=0}^{M-1} x(n) e^{-2j\pi nk/N} \quad k = 0, N-1$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{2j\pi nk/N} \quad n = 0, M-1$$

Formula for $x(n)$ is only correct if $N \geq M$ (to reproduce $x(n)$ from $X(k)$)

4b) One wants to filter an infinitely long sequence $x(n)$, $n = -\infty, \infty$ by a FIR-filter $h(n)$ of length L.

Explain how the filtering can be performed in the frequency domain by using the so called "overlap-add" method. (2+1+2 p)

Compare the"overlap-add" method to standard time domain filtrering with respect to the number of multiplications and addition per output sample (1 p)

Answer :
Points : 2 for splitting up input and summing at output, 1 for choice of segment lengths, 2 for 4-stage stepwise algorithm, 1 for m+a

One splits the input sequence into consecutive segments $x_i(n)$ of lengths M

$$x_i(n) = x(n + iM) \text{ for } n = 0, \ldots, M-1 \text{ and } i = -\infty, \infty$$

This leads to :

$$y(n) = h(n) * x(n) = h(n) * \sum_i x_i(n) = \sum_i h(n) * x_i(n) = \sum_i y_i(n)$$

where the output segments $y_i(n)$ have lengths $M+L-1$. Thus two consecutive output segments overlap by L samples, but are easily summed to achieve $y(n)$.
The calculation of $y_i(n)$ can be done in the frequency domain as both the input segment and the filter have finite lengths. We choose $N = 2^R \geq M + L - 1$ (in order to use the FFT and be able to reproduce $y_i(n)$). Further given that $H(k)$ $k = 0, N-1$ is calculated only once, i.e precalculated, the algorithm is as follows : For each segment $i = -\infty, \infty$

- Calculate $X_i(k)$ $k = 0, N-1$ from $x_i(n)$
- Calculate $Y_i(k) = H(k)X_i(k)$ $k = 0, N-1$
- Calculate $y_i(n)$ $n = 0, M+L-1$ from $Y_i(k)$
- Calculate $y(n)$ from two consecutive segments $y_i(n)$

To produce N output values of $y(n)$ by time domain filtering we need M*N mults+adds. Using the frequency domain method and FFT we need $N*R+N+N*R = N*(2R+1)$ where $R = log_2(N)$. Thus for any N where $(2log_2(N)+1) < M$ the overlap-add technique with FFT should be used.

4c) One wants to use DFT to perform a frequency analysis of an infinitely long sequence $x(n),\ n = -\infty, \infty$. In real one has to base the analysis of a finite segment of length K of the sequence.

Discuss the problems regarding frequency resolution and frequency "leackage" (sidelobes) as a function of the segment length K. (2+2 p)

How can one manage to achieve a compromise with respect to the two nonidealities in the frequency domain? (1 p)

Answer :
Just using a segment of length K is mathematically equivalent to using a rectangular window of length K. The two nonidealities can be explained by looking at the frequency content of a segment of a harmonic (sinus). Ideally a harmonic is a dirac pulse but using only a segment (i.e. a window) gives a sinc-like function; i.e. a bandwidth and sidelobes. The bandwidth is proportional to $C*(1/K)$ (where C is a constant) and obviously gives us the frequency resolution. The sidelobes give us the frequency leackage. For a rectangular window the constant C is relatively small, i.e the frequency resolution is high. However the sidelobes are relatively high and do only converge towards a finite level (-26 dB relatively to the main lobe) for $K \to \infty$.
A tapered window will decrease the frequency resolution (larger value of C) but also increase the sidelobe attenuation. Different tapering form will give minor differences in this compromize.

4d) The radix-2 Fast Fourier Transform (FFT) is a fast algorithm for calculating the DFT of a sequence when the length N is a power of 2, i.e. $N = 2^R$

Explain shortly the *principle* of the radix-2 FFT algorithm. (3 p)

Answer :
The main principle for the FFT is that one can implement a $N = 2^R$ point DFT by using 2 $N/2$ point DFTs and N multiplications. And it is easily shown than the latter leads to fewer mults+adds (m+a). A general N point DFT use N^2 m+a. Thus $N^2 \geq 2 * (N/2)^2) + N = N^2/2 + N$ for all values $N > 2$! The difference gets large for typical values of N, i.e. $N = 64, 128, 256, 512, 1024, ...$
The radix-2 FFT successively splits the DFTs into smaller such that one ends up with a structure consisting of $(N/2) * log_2(N)$ 2-point DFTs (so called butterflies), which each requires maximum 2 m+a.