Problem 1

Let $x_1 = \varphi$, $x_2 = \dot{\varphi} = \dot{x}_1$

Assuming $|\dot{\varphi}|$ is small, we approximate

$$|\dot{\varphi}|\dot{\varphi} = \text{sign}(\dot{\varphi})|\dot{\varphi}|^2 \approx 0$$

and get

$$\dot{x}_2 = \ddot{\varphi} = -\frac{d}{j}|\dot{\varphi}|\dot{\varphi} - \frac{k}{j}\varphi + \frac{V}{j}u^2$$

$$\simeq -\frac{k}{j}x_1 + \frac{V}{j}u^2$$

Linearize: ~~[illegible]~~

$$\frac{\partial \dot{x}_2}{\partial u} = \frac{2V}{j}u$$

Which means

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{j} & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{2V}{j} \end{pmatrix} u$$

$$a = k/j, \quad b = \frac{2V}{j}$$

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b) Controllable:

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix}$$

$$= \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix}$$

$\text{Rank}(\mathcal{C}) = 2$

The system is controllable.

Stable:

$$\det(\lambda I - A) = \begin{vmatrix} +\lambda & -1 \\ +a & +\lambda \end{vmatrix}$$

$$= \lambda^2 + a$$

$$= 0$$

$$\Rightarrow \lambda = \pm i\sqrt{a}$$

It is marginally stable, but not asymptotically since no eigenvalues are negative (strictly).

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c) It is possible iff it is observable.

Gyro:

$$\mathcal{O}_{gyro} = \begin{pmatrix} 0 & 1 \\ -a & 0 \end{pmatrix}$$

Rank is 2 so it is in principle possible with gyro.

Incl.:

$$\mathcal{O}_{incl.} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Rank is 2 so it is also possible with inclinometer.

d) Using state feedback $u = -K\underline{x}$

$$\dot{\underline{x}} = \begin{pmatrix} 0 & 1 \\ -a & 0 \end{pmatrix} \underline{x} - \begin{pmatrix} 0 \\ b \end{pmatrix}$$

$$= \left(\begin{pmatrix} 0 & 1 \\ -a & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ bk_1 & bk_2 \end{pmatrix} \right) \underline{x}$$

$$= \begin{pmatrix} 0 & 1 \\ -bk_1 - a & -bk_2 \end{pmatrix} \underline{x}$$

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With $a = b = 1$ we have:

$$\dot{\underline{x}} = \begin{pmatrix} 0 & 1 \\ -k_1 - 1 & -k_2 \end{pmatrix} \underline{x}$$

Finding k_1, k_2:

$$\det\left(\lambda \mathbb{I} - \begin{pmatrix} 0 & 1 \\ -k_1 - 1 & -k_2 \end{pmatrix} \right) = (\lambda - \lambda_1)(\lambda - \lambda_2)$$

$$\Leftrightarrow \quad \lambda(\lambda + k_2) + 1 + k_1 = (\lambda + 1)(\lambda + 2)$$

$$\Leftrightarrow \quad \lambda^2 + k_2 \lambda + 1 + k_1 = \lambda^2 + 3\lambda + 2$$

$$\Rightarrow \quad k_2 = 3 \quad , \quad k_1 = 1$$

The input signal $\underline{\underline{u = -(x_1 + 3x_2)}}$ places the poles at $-1, -2$.

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e) Luenberger observer:

$$\dot{\hat{\underline{x}}} = A\hat{\underline{x}} + Bu + \underline{l}\left(y - C_{gyro}\hat{\underline{x}}\right)$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\hat{\underline{x}} + \begin{pmatrix} 0 \\ b \end{pmatrix}u + \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}\left(y - \hat{x}_2\right)$$

Dynamics of observer can be found from the system $\underline{e} = \underline{x} - \hat{\underline{x}}$

$$\dot{\underline{e}} = A\underline{x} + \cancel{Bu} - A\hat{\underline{x}} - \cancel{Bu} - L(y - C\hat{\underline{x}})$$

$$= A\underline{e} - LC\underline{e}$$

$$= (A - LC)\underline{e}$$

$$A - CL = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1-l_1 \\ -1 & -l_2 \end{pmatrix}$$

$$\det(\lambda I - (A - CL)) = (\lambda - \lambda_1')(\lambda - \lambda_2') = (\lambda + 2)^2$$

$$\Leftrightarrow \begin{vmatrix} \lambda & -1+l_1 \\ 1 & \lambda + l_2 \end{vmatrix} = \lambda^2 + 4\lambda + 4$$

$$\Leftrightarrow \lambda^2 + l_2\lambda + (l_1 + 1) = \lambda^2 + 4\lambda + 4$$

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So $\underline{L} = \underline{l} = \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$

gives observer poles of $\lambda_1 = \lambda_2 = -2$.

f) No, since using a stable observer in the loop does not affect the stability of the plant (in theory).

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Problem 2

$$g(s) = \frac{1}{\tau s+1}\begin{pmatrix} 1 \\ \tau s \end{pmatrix}$$

$$= \frac{1}{s+1/\tau}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} s + \begin{pmatrix} 1/\tau \\ 0 \end{pmatrix}\right)$$

$$\Rightarrow A = -1/\tau, \quad B = 1$$

$$C = \begin{pmatrix} 0 & 1/\tau \\ 1 & 0 \end{pmatrix}, \quad D = \lim_{s\to\infty} g(s) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

This gives a minimal realization since the degree of the denominator of $g(s)$, is the same as the number of states,

$$\dot{x} = -\frac{1}{\tau}x + u$$

$$y = \begin{pmatrix} 0 & 1/\tau \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

With $y'(t) = y_1 + y_2$, we have

$$y'(s) = y_1(s) + y_2(s) = \left(\frac{1}{\tau s+1} + \frac{\tau s}{\tau s+1}\right) u(s)$$

$$= u(s).$$

This clearly isn't observable since $C = 0$

Problem 3

The system is described in normal state space as

$$\dot{x} = -x + \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

so

$$A = -1 \,, \quad B = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

In the cost function we have $Q = 1$, $R = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$.

Computing optimal gain: (note that P is a scalar)

$$A^T P + PA + Q - PBR^{-1}B^T P = 0$$

$$-P - P + 1 - P^2 \begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{r} & 0 \\ 0 & \frac{1}{r} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$$

$$\Rightarrow P^2 \frac{2}{r} + 2P - 1 = 0$$

$$\Leftrightarrow P^2 + rP - \frac{r}{2} = 0$$

$$\Rightarrow P = \frac{-r \pm \sqrt{r^2 + 4\frac{r}{2}}}{2} = -\frac{r}{2} \pm \frac{1}{2}\sqrt{r^2 + 2r}$$

$$K = R^{-1} B^T P$$

$$= \begin{pmatrix} \frac{1}{r} & 0 \\ 0 & \frac{1}{r} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} P = \begin{pmatrix} \frac{1}{r} \\ \frac{1}{r} \end{pmatrix} P$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix}\left(-\frac{1}{2} \pm \frac{\text{sign}(r)}{2}\sqrt{1 + \frac{2}{r}}\right)$$

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Since $r > 0$ and the open loop system is stable we simplify to

$$K = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \left(-\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{2}{r}} \right)$$

The inputs are weigthed equally since they feed into the system in the same way.

If $r_2 >> r_1 > 0$ then most of the feedback would go through u_1. K would look like

$$~~K=~~ \quad \begin{matrix} \cancel{\text{small number}} \\ \cancel{\text{large number}} \end{matrix} \quad K = \begin{pmatrix} \text{large}(r) \\ \text{very small} \end{pmatrix}$$

This is because using input u_2 would be expensive.

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Problem 4

a) Equilibrium when $\dot{x}_1 = \dot{x}_2 = 0$

$$\begin{cases} 0 = \alpha \bar{x}_1 - \beta \bar{x}_1 \bar{x}_2 \\ 0 = \delta \bar{x}_1 \bar{x}_2 - \gamma \bar{x}_2 \end{cases}$$

$$\Rightarrow \begin{cases} \bar{x}_1 = \gamma / \delta \\ \bar{x}_2 = \alpha / \beta \end{cases}$$

Non-linear shifted model: $x_1 = \tilde{x}_1 + \frac{\gamma}{\delta}$, $x_2 = \tilde{x}_2 + \frac{\alpha}{\beta}$

$$\begin{aligned} \dot{\tilde{x}}_1 &= \dot{x}_1 \\ &= \alpha\left(\tilde{x}_1 + \frac{\gamma}{\beta}\right) - \beta\left(\tilde{x}_1 + \frac{\gamma}{\delta}\right)\left(\tilde{x}_2 + \frac{\alpha}{\beta}\right) \\ &= \cancel{\alpha \tilde{x}_1} + \cancel{\frac{\alpha\gamma}{\beta}} - \cancel{\alpha \tilde{x}_1} - \frac{\beta\gamma}{\delta}\tilde{x}_2 - \beta \tilde{x}_1 \tilde{x}_2 - \cancel{\frac{\alpha\gamma}{\delta}} \\ &= -\frac{\beta\gamma}{\delta}\tilde{x}_2 - \beta \tilde{x}_1 \tilde{x}_2 \end{aligned}$$

$$\begin{aligned} \dot{\tilde{x}}_2 &= \dot{x}_2 \\ &= \delta\left(\tilde{x}_1 + \frac{\gamma}{\delta}\right)\left(\tilde{x}_2 + \frac{\alpha}{\beta}\right) - \gamma\left(\tilde{x}_2 + \frac{\alpha}{\beta}\right) \\ &= \frac{\delta\alpha}{\beta}\tilde{x}_1 + \cancel{\gamma \tilde{x}_2} + \delta \tilde{x}_1 \tilde{x}_2 + \cancel{\frac{\alpha\gamma}{\beta}} - \cancel{\frac{\alpha\gamma}{\beta}} - \cancel{\gamma \tilde{x}_2} \\ &= \frac{\alpha\delta}{\beta}\tilde{x}_1 + \delta \tilde{x}_1 \tilde{x}_2 \end{aligned}$$

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Linearizing around equilibrium:

$$\left.\frac{\partial \dot{\tilde{x}}_1}{\partial \tilde{x}_1}\right|_{equi.} = \left.-\beta \tilde{x}_2\right|_{\tilde{x}_2=0} = 0$$

$$\left.\frac{\partial \dot{\tilde{x}}_1}{\partial \tilde{x}_2}\right|_{equilib.} = -\frac{\beta\gamma}{\delta} - \cancel{\beta \tilde{x}_1}^{=0} = -\frac{\beta\gamma}{\delta}$$

$$\left.\frac{\partial \dot{\tilde{x}}_2}{\partial x_1}\right|_{equi.} = \left.\frac{\alpha\delta}{\beta} + \delta \tilde{x}_2\right|_{\tilde{x}_2=0} = \frac{\alpha\delta}{\beta}$$

$$\left.\frac{\partial \dot{\tilde{x}}_2}{\partial \tilde{x}_2}\right|_{equi.} = \left.\delta \tilde{x}_1\right|_{\tilde{x}_1=0} = 0$$

Linearized model is then:

$$\dot{\tilde{x}}_1 = -\frac{\beta\gamma}{\delta}\tilde{x}_2$$

$$\dot{\tilde{x}}_2 = \frac{\alpha\delta}{\beta}\tilde{x}_1$$

Fagtrykk Trondheim AS 05.2015

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b) The system is given by

$$\begin{pmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{pmatrix} = A \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -\frac{\beta\gamma}{\delta} \\ \frac{\alpha\delta}{\beta} & 0 \end{pmatrix}$$

Eigenvalues:

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & \frac{\beta\gamma}{\delta} \\ -\frac{\alpha\delta}{\beta} & \lambda \end{vmatrix}$$

$$= \lambda^2 + \alpha\gamma = 0$$

$$\Rightarrow \lambda = \pm i\sqrt{\alpha\gamma}$$

The system is marginally stable since it has no positive eigenvalues.

With $\alpha = \beta = \gamma = \delta = 1$ we have

$$\begin{pmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}$$

$$\Leftrightarrow \ddot{\tilde{x}}_1 = \dot{\tilde{x}}_2 = -\tilde{x}_1$$

$$\Leftrightarrow \ddot{\tilde{x}}_1 + \tilde{x}_1 = 0$$

This has solutions

$$\tilde{x}_1 = C_1 \cos t + C_2 \sin t$$

$$\tilde{x}_2 = C_2 \cancel{\sin}\, t - C_1 \sin t$$

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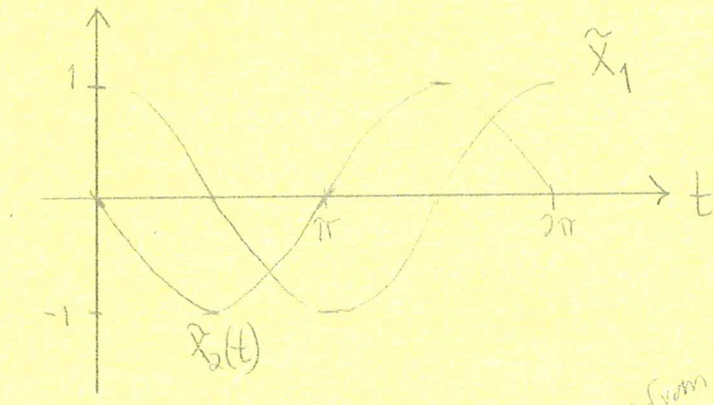
Sketch: $c_1 = 1,\ c_2 = 0$



c) Let $\tilde{y}$ = number of predators away from equilibrium, and $\tilde{\underline{x}} = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}$.

Then

$$\tilde{y} = \begin{pmatrix} 0 & 1 \end{pmatrix} \tilde{\underline{x}}$$

$$= C \tilde{\underline{x}}$$

Observability matrix is then

$$\mathcal{O} = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{\alpha\delta}{\beta} & 0 \end{pmatrix}$$

which has full rank.

Yes, it is possible since the system is observable.

Mai 2017

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d) We now have essentially

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Controllability:

$$\mathcal{C} = \begin{pmatrix} B & AB \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

<u>It is possible</u> since the system is controllable.

Problem 5

a) An antenna that is standing perfectly still wont move and shouldn't be affected by disturbances, so

$$\underline{x}[k+1] = \underline{x}[k]$$

$$A = I$$

$$\underline{w} = 0$$

Will use induction on k to show that (8) holds.

Base case, k=0.

$$L[0] = P^-[0]\, C^T (C P^-[0] C^T + R)^{-1}$$

$$C = I \Rightarrow L[0] = \sigma_v^2 I \cdot I^T (I \sigma_v^2 I \cdot I^T + \sigma_v^2 I)^{-1}$$

$$= \frac{I}{2}$$

$$= \frac{I}{0+2} \quad \checkmark$$

$$P[0] = (\mathbb{I} - L[0]C)\, P^-[0]\, (\mathbb{I} - L[0]C)^T + L[0]RL[0]^T$$

$$= \frac{\mathbb{I}}{2} \cdot \sigma_v^2 \cdot \frac{\mathbb{I}^T}{2} + \frac{\mathbb{I}}{2}\sigma_v^2 \frac{\mathbb{I}^T}{2}$$

$$= \frac{R}{4} + \frac{R}{4}, \quad (R = \sigma_v^2 \mathbb{I})$$

$$= \frac{R}{0+2} \quad \checkmark$$

$$P^-[0+1] = A P[0] A^T + \cancel{Q_d}, \quad Q_d = 0$$

$$= P[0]$$

$$= \frac{R}{0+2} \quad \checkmark$$

Assuming (8) holds for $K-1$, we check if it holds for K.

$$L[K] = P^-[K] C^T (C P^-[K-1] C^T + R)^{-1}$$

$$= \frac{R}{K+1}\left(\frac{R}{K+1} + R\right)^{-1}$$

$$= \frac{R}{\cancel{K+1}}\left(\frac{R + (K+1)R}{\cancel{K+1}}\right)^{-1}$$

$$= \cancel{R}\,(K\cancel{R} + 2\cancel{R})^{-1}$$

$$= \frac{\mathbb{I}}{K+2} \quad \checkmark$$

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$$P[k] = (\mathbb{I} - L[k]C)P^-[k](\mathbb{I} - L[k]C)^T + L[k]RL[k]^T$$

$$= \left(\mathbb{I} - \frac{\mathbb{I}}{k+2}\right)\frac{R}{k+1}\left(\mathbb{I} - \frac{\mathbb{I}}{k+2}\right)^T + \frac{\mathbb{I}}{k+2} R \frac{\mathbb{I}}{k+2}^T$$

$$= \mathbb{I}\,\frac{k+1}{k+2} \cdot \frac{R}{\cancel{k+1}} \cdot \mathbb{I}\,\frac{\cancel{k+1}}{k+2} + \frac{R}{(k+2)^2}$$

$$= R\left(\frac{k+1}{(k+2)^2} + \frac{1}{(k+2)^2}\right)$$

$$= \frac{R}{k+2} \quad \checkmark$$

$$P^-[k+1] = A\,P[k]\,A^T + \cancel{Q_d}^{\,=0}$$

$$= P[k]$$

$$= \frac{R}{k+2} \quad \checkmark$$

This proves that (8) holds for all $k \geq 0$. $\square$

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b) The original std deviation is given by $\bar{p}[0] = R$. At $k=1$ we have halved it to so $P^{-}[1] = \frac{R}{2}$.

It takes 1 measurement.

c)
$$\begin{cases} \hat{x}[k] = \hat{x}^{-}[k] + \frac{I}{k+2}\left(y[k] - \hat{x}^{-}[k]\right) \\ \qquad = \frac{k+1}{k+2}\hat{x}^{-}[k] + \frac{1}{k+2} y[k] \\ \hat{x}^{-}[k+1] = \hat{x}[k] \end{cases}$$

Based on these we compute $\hat{x}$, $\hat{x}^{-}$.

~~$\hat{x}[0] =$~~

k	0	1	2	3	4	5
y_1	10	6	1	1	11	-3
y_2	4	0	1	16	8	6
$\hat{x}^{-}$	$\begin{pmatrix} 8 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 5 \\ 2 \end{pmatrix}$	$\begin{pmatrix} \cancel{5.33} \\ \cancel{1.33} \end{pmatrix}$	$\begin{pmatrix} 4.25 \\ 1.25 \end{pmatrix}$	$\begin{pmatrix} 3.6 \\ 4.2 \end{pmatrix}$	$\begin{pmatrix} 4.8 \\ 4.8 \end{pmatrix}$
$\hat{x}$	$\begin{pmatrix} 5 \\ 2 \end{pmatrix}$	$\begin{pmatrix} \cancel{5.33} \\ \cancel{1.33} \end{pmatrix}$	$\begin{pmatrix} 4.25 \\ 1.25 \end{pmatrix}$	$\begin{pmatrix} 3.6 \\ 4.2 \end{pmatrix}$	$\begin{pmatrix} 4.8 \\ 4.8 \end{pmatrix}$	$\begin{pmatrix} 3.7 \\ 5.3 \end{pmatrix}$

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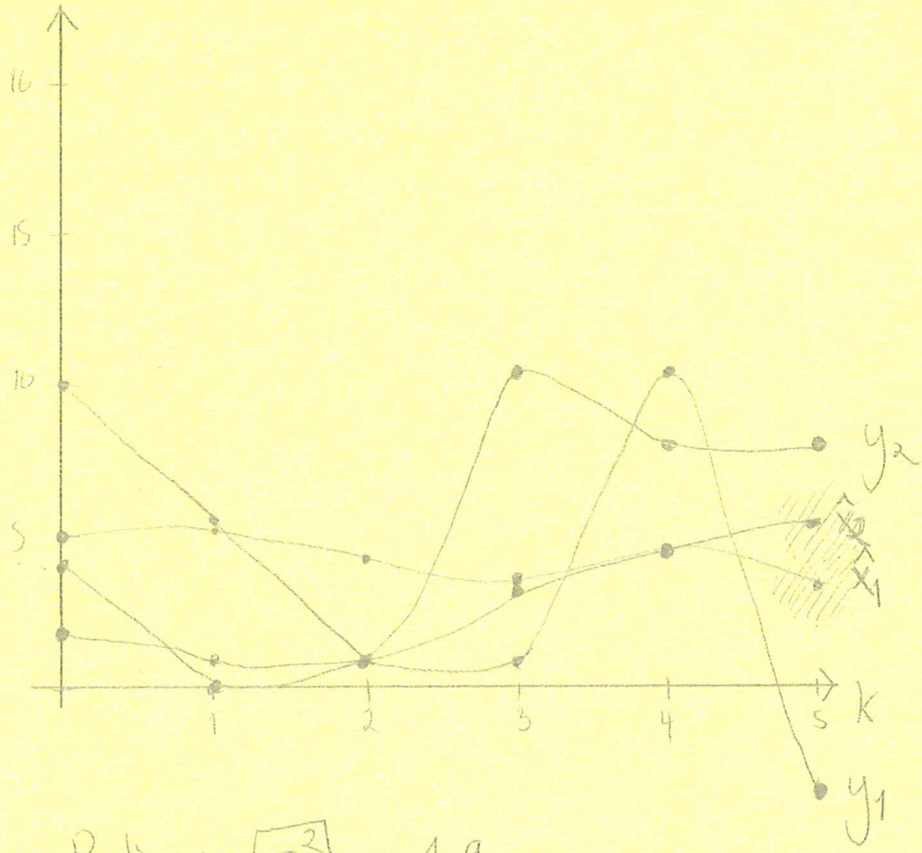

Radius: $\sqrt{\frac{\sigma_v^2}{5+2}} = 1.9$