Matte 4K, Øving 4

Rendell Cole, gruppe 2 ok

11.3: Ønsker tilbakemelding :)

15) $r(t) = t(\pi^2 - t^2)$, $-\pi < t < \pi$

$r(t+2\pi) = r(t)$

$y'' + cy' + y = r(t)$

We represent $r(t)$ as a Fourier series.

$$r(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$$

Since r is odd, all a_i are zero so

$$r(t) = \sum_{n=1}^{\infty} b_n \sin(nt)$$

where

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t(\pi^2 - t^2)\sin(nt)\,dt$$

$$= \frac{2}{\pi}\int_{0}^{\pi} t(\pi^2 - t^2)\sin(nt)\,dt$$

$$= \cancel{\frac{2t(\pi^2-t^2)(-\cos nt)}{\pi n}\Bigg|_0^{\pi}} + \frac{2}{\pi n}\int_0^{\pi}(\pi^2 - 3t^2)\cos(nt)\,dt$$

$$= \cancel{\frac{2(\pi^2 - 3t^2)\sin(nt)}{\pi n^2}\Bigg|_0^{\pi}} + \frac{12}{\pi n^2}\int_0^{\pi} t\sin(nt)\,dt$$

$$= -\frac{12\, t\cos(nt)}{\pi n^2 \cdot n}\Big|_0^{\pi} + \frac{12}{\pi n^3}\int_0^{\pi} \cos(nt)\,dt$$

$$= -\frac{12(-1)^n}{n^3} + 0$$

$$= -12 \cdot \frac{(-1)^n}{n^3}$$

So $r(t) = -12 \cdot \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin(nt)$ R

We then consider

$$y_n'' + c y_n' + y_n = \underbrace{-12\frac{(-1)^n}{n^3}}_{K} \sin(nt)$$

$$\Leftrightarrow \quad y_n'' + c y_n' + y_n = K \sin(nt)$$

We try to find solutions of the form

$$y_{p_n} = A\cos(nt) + B\sin(nt)$$

$$\Rightarrow y_{p_n}' = -An\sin(nt) + Bn\cos(nt)$$
$$= Bn\cos(nt) - An\sin(nt)$$

$$\Rightarrow y_{p_n}'' = -Bn^2\sin(nt) - An^2\cos(nt)$$
$$= -An^2\cos(nt) - Bn^2\sin(nt)$$

So $y_{p_n}'' + c y_{p_n}' + y_{p_n}$

$$= (-An^2 + cBn + A)\cos(nt) + (-Bn^2 - cAn + B)\sin(nt)$$
$$= K\sin(nt)$$

which gives:

$$(1)\quad -An^2 + cBn + A = 0$$
$$(2)\quad -Bn^2 - cAn + B = K$$

$$(1) \Leftrightarrow A(1-n^2) = -cBn$$
$$\Leftrightarrow A = \frac{-cBn}{1-n^2}$$

$$\overset{(2)}{\Rightarrow} -Bn^2 - c\left(\frac{-cBn}{1-n^2}\right)n + B = K$$

$$\Leftrightarrow B\left(1-n^2+\frac{c^2n^2}{1-n^2}\right) = K$$

$$\Leftrightarrow B\left(\frac{(1-n^2)(1-n^2)+cn^2}{1-n^2}\right) = K$$

$$\Leftrightarrow B\left(\frac{1-2n^2+n^4+cn^2}{1-n^2}\right) = K$$

$$\Leftrightarrow B = \frac{K(1-n^2)}{1+(c-2)n^2+n^4}$$

$$\Leftrightarrow B = \frac{-12(-1)^n}{n^3} \cdot \frac{1-n^2}{1+(c-2)n^2+n^4}$$

$$\overset{(1)}{\Rightarrow} A = \frac{12c(-1)^n}{n^2} \cdot \frac{1}{1+(c-2)n^2+n^4}$$

So $$y_{p_n} = \frac{12c(-1)^n}{n^3(1+(c-2)n^2+n^4)}\cos(nt)$$

$$-\frac{12(1-n^2)(-1)^n}{n^2(1+(c-2)n^2+n^4)}\sin(nt)$$ R

$$= \frac{12(-1)^n}{n^2(1+(c-2)n^2+n^4)} \cdot \left[\frac{c}{n}\cos(nt) - (1-n^2)\sin(nt)\right]$$

So the steady state solution y_p is

$$y_p = 12\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n^2(1+(c-2)n^2+n^4)}\cdot\left(\frac{c}{n}\cos(nt)+(1-n^2)\sin(nt)\right)\right]$$

19) $E(t) = 200t(\pi^2-t^2)$

$R = 10\,\Omega$, $L = 1H$, $C = 10^{-1}F$

$$L\ddot{I}+R\dot{I}+\frac{1}{C}I = E'(t) \qquad (*)$$

$$E'(t) = 200(\pi^2-t^2)+200t(-2t)$$
$$= 200\pi^2-200t^2-400t^2$$
$$= 200(\pi^2-3t^2)$$

$$(*): \quad \ddot{I}+10\dot{I}+10I = 200(\pi^2-3t^2)$$

We first find the Fourier representation of $E'(t)$. Note that $E(t) = 200\cdot r(t)$ from (15), so we can easily compute the Fourier series of $E(t)$ by using $r(t)$.

$$E(t) = 200\cdot r(t)$$
$$= 200\cdot(-12)\cdot\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin(nt)$$
$$= -2400\cdot\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin(nt)$$

Differentiating term by term we get

$$E'(t) = -2400\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nt)$$

So we want to solve

$$\ddot{I}_n + 10\dot{I}_n + 10 I_n = -2400 \frac{(-1)^n}{n^2} \cos(nt)$$

We try $I_{pn} = A\sin(nt) + B\cos(nt)$

$$\Leftrightarrow \dot{I}_{pn} = -Bn\sin(nt) + An\cos(nt)$$

$$\Leftrightarrow \ddot{I}_{pn} = -An^2\sin(nt) - Bn^2\cos(nt)$$

$$= -n^2 I_{pn}$$

So $\ddot{I}_{pn} + 10\dot{I}_{pn} + 10 I_{pn} = 10\dot{I}_{pn} + (10-n^2) I_{pn}$

$$= \left(-10Bn + (10-n^2)A\right)\sin(nt) + \left(10An + (10-n^2)B\right)\cos(nt)$$

$$= -2400 \frac{(-1)^n}{n^2}\cos(nt)$$

$$\Rightarrow \begin{array}{ll} (1) & -10Bn + (10-n^2)A = 0 \\ (2) & 10An + (10-n^2)B = -2400 \frac{(-1)^n}{n^2} \end{array}$$

$$(1) \Leftrightarrow A = \frac{10Bn}{10-n^2}$$

$$\xrightarrow{(2)} 10\left(\frac{10Bn}{10-n^2}\right)n + (10-n^2)B = -2400\frac{(-1)^n}{n^2}$$

$$\Leftrightarrow B\left[\frac{100n^2}{10-n^2} + 10-n^2\right] = -\,\text{''}\,-$$

$$\Leftrightarrow B\left[\frac{100n^2 + (10-n^2)^2}{10-n^2}\right] = -\,\text{''}\,-$$

$$\Leftrightarrow B = \frac{-2400(-1)^n(10-n^2)}{n^2(100n^2 + (10-n^2)^2)}$$

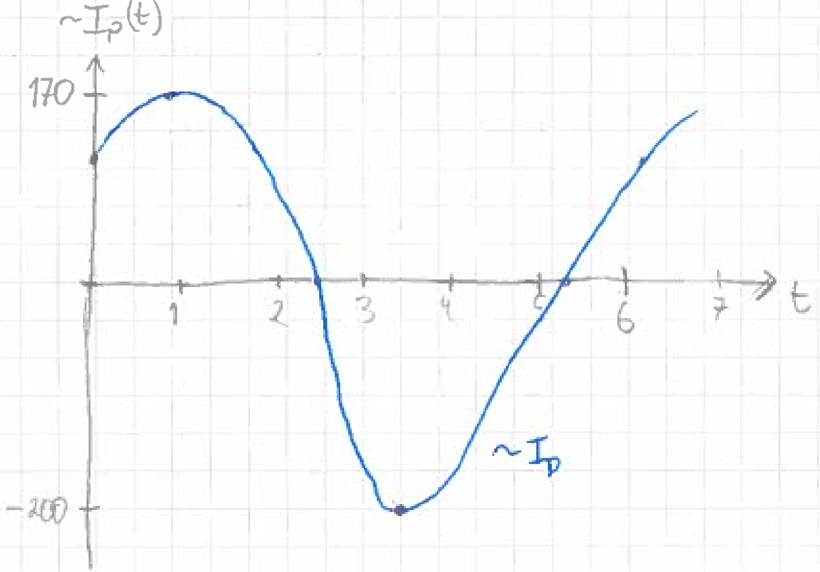
$$\overset{(1)}{\Rightarrow} A = \frac{-24000(-1)^n}{n(100n^2 + (10-n^2)^2)}$$

So the steady state solution I_p is

$$I_p = -2400 \cdot \sum_{n=1}^{\infty} \left[\frac{10(-1)^n}{n(100n^2+(10-n^2)^2)} \cos(nt) + \frac{(-1)^n(10-n^2)}{n^2(100n^2+(10-n^2)^2)} \sin(nt) \right]$$

$$= -2400 \cdot \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{100n^2+(10-n^2)^2} \cdot \left(\frac{10}{n} \cos(nt) + \frac{(10-n^2)}{n^2} \sin(nt) \right) \right]$$

Sketch of first four partial sums:



11.4:

a) $f(x) = x, \quad (-\pi < x < \pi)$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} x e^{-inx}\,dx$

$$= \frac{x e^{-inx}}{-2\pi in}\Big|_{-\pi}^{\pi} + \frac{i}{2\pi n}\int_{-\pi}^{\pi} e^{-inx}\,dx$$

$$= \frac{i}{2\pi n}\left(\pi(-1)^n + \pi(-1)^n\right) + \frac{i}{2\pi n}\cdot\frac{1}{-in}\left(e^{-in\pi} - e^{in\pi}\right)$$

$$= \frac{i(-1)^n}{n} + \frac{-1}{2\pi n^2}\left((-1)^n - (-1)^n\right)$$

$$= \frac{(-1)^n}{n} i, \quad n \neq 0$$

$c_0 = 0$ because f is odd

So $f(x) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n}{n} i e^{inx}$ R

10) Since $c_0 = 0$, $a_0 = 0$.

We then have

$$\frac{1}{2}(a_n - i b_n) = c_n \qquad (1)$$

$$\frac{1}{2}(a_n + i b_n) = c_{-n} \qquad (2)$$

(1)+(2):

$$\begin{aligned} a_n &= c_n + c_{-n} \\ &= \frac{(-1)^n i}{n} + \frac{(-1)^n i}{-n} \\ &= 0 \end{aligned}$$

$$(1) \Leftrightarrow b_n = 2i\, c_n = -2\frac{(-1)^n}{n}$$

So $f(x) = -2 \cdot \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(nx)$

13) $f(x) = x \quad , \quad (0 < x < 2\pi)$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where $c_n = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} x\, e^{-inx}\, dx$

$$= \left. \frac{x \cancel{e^{-inx}}}{-2\pi i n} \right|_0^{2\pi} + \frac{i}{2\pi n}\int_0^{2\pi} e^{-inx}\, dx$$

$$= \frac{2\pi i}{2\pi n} + \cancel{\frac{i}{2\pi i} \cdot \frac{1}{-in}(1-1)}$$

$$= \frac{i}{n} \quad , \quad n \neq 0$$

$$c_0 = a_0 = \frac{1}{2\pi}\int_0^{2\pi} x\, dx$$

$$= \frac{1}{\cancel{4\pi}} \cancel{(2\pi)^2}$$

$$= \pi$$

So $\underline{\underline{f(x) = \pi + i \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n} e^{inx}}}$ R

11.4:

4) $f(x) = x^2, \quad (-\pi < x < \pi)$

f is even so we can write

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$$

where $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx$

$$= \frac{1}{6\pi}(\pi^3 + \pi^3)$$

$$= \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos(nx)\,dx$$

$$= \frac{x^2 \sin(nx)}{\pi n}\Big|_{-\pi}^{\pi} + \frac{1}{\pi n}\int_{-\pi}^{\pi} x \sin(nx)\,dx$$

$$= \frac{-x\cos(nx)}{\pi n^2}\Big|_{-\pi}^{\pi} + \frac{1}{\pi n^2}\int_{-\pi}^{\pi} \cos(nx)\,dx$$

$$= \frac{-\pi(-1)^n - \pi(-1)^n}{\pi n^2}$$

$$= -\frac{2(-1)^n}{n^2}$$

11.4:

4) $f(x) = x^2 \quad , -\pi < x < \pi$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$$

$$a_0 = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} x^2 \, dx = \frac{1}{\pi} \frac{x^3}{3} \Big|_0^{\pi}$$

$$= \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(nx) \, dx$$

$$= \underbrace{\cancel{\frac{x^2 \sin(nx)}{\pi n}\Big|_{-\pi}^{\pi}}}_{=0} - \frac{2}{\pi n} \int_{-\pi}^{\pi} x \sin(nx) \, dx$$

$$= \frac{+2x\cos(nx)}{\pi n^2}\Big|_{-\pi}^{\pi} - \underbrace{\cancel{\frac{2}{\pi n^2} \int_{-\pi}^{\pi} \cos(nx) \, dx}}_{=0}$$

$$= 2 \frac{\cancel{\pi}(-1)^n + \cancel{\pi}(-1)^n}{\cancel{\pi} n^2}$$

$$= \frac{4(-1)^n}{n^2}$$

So $f(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx)$

Let $f_N(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{N} \frac{(-1)^n}{n^2} \cos(nx)$.

Then for any N, the minimum error E_N^* is

$$E_N^* = \int_{-\pi}^{\pi} (x^2)^2 \, dx - \pi \left[2\left(\frac{\pi^2}{3}\right)^2 + 16 \sum_{n=1}^{N} \frac{1}{n^4} \right]$$

$$= \frac{2}{5}\pi^5 - \frac{2\pi^5}{9} - 16\pi \cdot \sum_{n=1}^{N} \frac{1}{n^4} \quad R$$

For $N = 1, \dots, 5$ we get

$E_1^* \approx 4{,}1$

$E_2^* \approx 0{,}996$

$E_3^* \approx 0{,}376$

$E_4^* \approx 0{,}1795$ R

$E_5^* \approx 0.09909$

11) From ex. 1, 11.1:

$$f(x) = \begin{cases} -k, & -\pi < x < 0 \\ k, & 0 < x < \pi \end{cases}$$

$$= \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right)$$

Consider $\int_{0}^{\pi/2} f(x)\,dx$. Integrating termwise we get:

$$\int_0^{\pi/2} f(x)\,dx = \left[\frac{4k}{\pi}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \dots\right)\right]_0^{\pi/2}$$

$$= \frac{4k}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right)$$

From $f(x) = k$ for $0 < x < \pi$, we see that

$$\int_0^{\pi/2} f(x)\,dx = \left(\frac{\pi}{2} - 0\right)k = \frac{\pi}{2}k$$

But these must be identical so

$$\frac{\pi}{2}k = \frac{4k}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right)$$

$$\Leftrightarrow \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$ R

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} \approx 1{,}17$$

11,R:

15) $f(x)=e^x \quad , -5<x<5$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where $c_n = \frac{1}{10}\int_{-5}^{5} e^x e^{-inx}\,dx$

$$= \frac{1}{10}\int_{-5}^{5} e^{(1-in)x}\,dx$$

$$= \frac{1}{10(1-in)} e^{(1-in)x}\Big|_{-5}^{5}$$

$$= \frac{e^5}{10(1-in)}\left(e^{-5in} - e^{5in}\right)$$

$$= \frac{e^5}{10(1-in)}\left(\cancel{\cos 5n} - i\sin 5n - \cancel{\cos 5n} - i\sin 5n\right)$$

$$= \frac{-ie^5 \sin 5n}{10(1-in)}$$

$$= \frac{-i(1+in)e^5\sin 5n}{10(1^2+n^2)}$$

$$= \frac{ne^5\sin 5n}{10(1+n^2)} - \frac{ie^5\sin 5n}{10(1+n^2)}$$

17) $f(x) = \begin{cases} -k & , -\pi < x < 0 \\ k & , 0 < x < \pi \end{cases}$

$$f(x) = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right)$$

Consider $k = \pi/4$ and $x = \frac{\pi}{2}$. We then get

$$f\left(\tfrac{\pi}{2}\right) = 1 - \frac{1}{3} + \frac{1}{5} - \dots$$

$$\parallel$$

$$k = \frac{\pi}{4}$$

So $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \dots$

Sup. E:

$$S(x) = \sum_{m=0}^{\infty} \frac{\sin(2m+1)x}{(2m+1)^3} = \frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \dots$$

$$S\left(\tfrac{\pi}{2}\right) = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \dots$$

We want

$$\frac{1}{1^3} + \frac{1}{3^3} - \frac{1}{5^3} - \frac{1}{7^3} + \dots$$

$$= \sum_{k=0}^{\infty} \frac{-(-1)^k}{(4k+1)^3} + \sum_{k=1}^{\infty} \frac{(-1)^k}{(4k-1)^3}$$

6: $f(x) = x^4$

$$= \frac{\pi^4}{5} + \sum_{n=0}^{\infty} \frac{(-1)^n 8(\pi^2 n^2 - 6)}{n^4} \cos(nx)$$

i) if $x = \pi$ then $\cos nx = (-1)^n$

$$\text{so } f(\pi) = \frac{\pi^4}{5} + 8 \sum_{n=0}^{\infty} \frac{\cancel{((-1)^n)^2}^{=1} \pi^2 n^2 - 6}{n^4}$$

$$\Leftrightarrow \frac{1}{8}\left(\pi^4 - \frac{\pi^4}{5}\right) = \sum_{n=0}^{\infty} \frac{\pi^2 n^2 - 6}{n^4}$$

$$\frac{\pi^4}{10} = - \quad | \ (-$$

$$\text{So } \underline{\sum_{n=0}^{\infty} \frac{\pi^2 n^2 - 6}{n^4} = \frac{\pi^4}{10}} \quad R$$

ii) $$\sum_{n=1}^{\infty} \frac{\pi^4 n^4 - 12 n^2 \pi^2 + 36}{n^8} = \sum_{n=1}^{\infty} \left(\frac{\pi^2 n^2 - 6}{n^4}\right)^2$$

$$= \sum_{n=1}^{\infty} (a_n)^2 \quad , n \geq 1$$

Parseval's identity:

$$2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} f^2 \, dx$$

$$\overset{b_n=0}{\Rightarrow} \sum_{n=1}^{\infty} a_n^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 dx - 2a_0^2$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(x^4)^2\,dx = \frac{1}{\pi}\cdot\frac{1}{9}x^9\Big|_{-\pi}^{\pi} = \frac{2}{9}\pi^8$$

$$2a_0^2 = 2\cdot\left(\frac{\pi^4}{5}\right)^2 = 2\cdot\frac{\pi^8}{25} = \frac{2}{25}\pi^8$$

This gives

$$\sum_{n=1}^{\infty} a_n^2 = \frac{2}{9}\pi^8 - \frac{2}{25}\pi^8$$

$$= \frac{32}{225}\pi^8$$

So

$$\sum_{n=1}^{\infty} \frac{\pi^4 n^4 - 12n^2\pi^2 + 36}{n^8} = \frac{32}{225}\pi^8$$

Sup H:

$$f(x) = \begin{cases} e^{-x} & , x \geq 0 \\ -e^{x} & , x < 0 \end{cases}$$

Note that $f(-x) = -f(x)$.

So we compute the Fourier sine transform,

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{iwx} \, dx$$

$$\overset{(odd\ f)}{=} \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x} \sin(wx) \, dx$$

$$I =: \int_0^{\infty} e^{-x} \sin wx \, dx = \frac{-e^{-x} \cos(wx)}{w} \Big|_0^{\infty} - \frac{1}{w} \int_0^{\infty} e^{-x} \cos(wx)$$

$$= \frac{1}{w} - \frac{1}{w} \Bigg[\underbrace{\frac{e^{-x} \sin wx}{w} \Big|_0^{\infty}}_{=0} + \frac{1}{w} \underbrace{\int_0^{\infty} e^{-x} \sin wx \, dx}_{I} \Bigg]$$

$$\Leftrightarrow \quad I = \frac{1}{w} - \frac{1}{w^2} I$$

$$I\left(1 + \frac{1}{w^2}\right) = \frac{1}{w}$$

$$\Leftrightarrow \quad I\left(\frac{w^2+1}{w^2}\right) = \frac{1}{w}$$

$$\Leftrightarrow \quad I = \frac{w}{w^2+1}$$

$$\Leftrightarrow \quad \int_0^{\infty} e^{-x} \sin wx \, dx = \frac{w}{w^2+1}$$

$$\Rightarrow \hat{f}(w) = \sqrt{\frac{2}{\pi}}\, I$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{w}{w^2+1}$$

We also have

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}(w) \sin(wx)\, dw$$

$$= \sqrt{\frac{2}{\pi}} \cdot \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{w}{w^2+1} \sin(wx)\, dw$$

$$= \frac{2}{\pi} \cdot \int_0^\infty \frac{w \sin(wx)}{w^2+1}\, dw$$

If we let $x=1$ then we get the desired integral (up to a factor) and we know this equals $f(1)$.

$$f(1) = \frac{2}{\pi} \int_0^\infty \frac{w \sin w}{w^2+1}\, dw$$

$$\Leftrightarrow \int_0^\infty \frac{w \sin w}{w^2+1}\, dw = \underline{\underline{\frac{\pi}{2} \cdot e^{-1}}}$$