Øving 10, Matte 4K

Rendell Cale, gruppe 2

Godkjent

Ønsker tilbakemelding :)

14,3:

3) $C = \{z : |z-(-i)| = 1{,}41\}$

Note that $1{,}41 < \sqrt{2}$ so $z = \pm 1 \notin C$

$$\oint_C \frac{z^2/(z-1)}{z+1}\,dz = 0$$

Since $z_0 = -1$ is outside of the domain enclosed by C.

18) Let $f(z) = \dfrac{\sin z}{4z}$

then

$$\oint_C \frac{\sin z}{4z^2 - 8iz}\,dz = \oint_C \frac{f(z)}{z-2i}\,dz$$

$$= \oint_{C_{outer}} \frac{f(z)}{z-2i}\,dz + \oint_{C_{inner}} \frac{f(z)}{z-2i}\,dz$$

$\dfrac{\sin z}{z}$ is analytic since the singularity at at $z=0$ is removable. This gives

$$\oint_{C_{outer}} \frac{f(z)}{z-2i}\,dz = 2\pi i f(2i)$$

$$\oint_{C_{inner}} \frac{f(z)}{z-2i}\,dz = 0$$

So in total we get

$$\oint_C \frac{\sin z}{4z^2-8iz}\,dz = \cancel{2}\pi \cancel{i}\,\frac{\sin(2i)}{4\cdot\cancel{2}\cancel{i}}$$

$$= \underline{\underline{\frac{\pi}{4}\sin(2i)}}$$

14,4:

3) $$\oint_C \frac{e^{-z}}{z^n}\,dz \quad , \quad n=1,2,3,\ldots$$

$$= \frac{2\pi i}{(n-1)!}\,\frac{d^n}{dz^n}\left(e^{-z}\right)\Big|_{z=0}$$

$$= \frac{2\pi i}{(n-1)!}(-1)^n$$

4) $$\oint_C \frac{e^z\cos z}{(z-\pi/4)^{2+1}}\,dz = \frac{2\pi i}{2!}\,\frac{d^2}{dz^2}\left(e^z\cos z\right)\Big|_{z=\pi/4}$$

$$\frac{d^2}{dz^2}\left(e^z\cos z\right) = \frac{d}{dz}\left(e^z\cos z - e^z\sin z\right)$$

$$= e^z\cos z - e^z\sin z - e^z\cos z - e^z\sin z$$

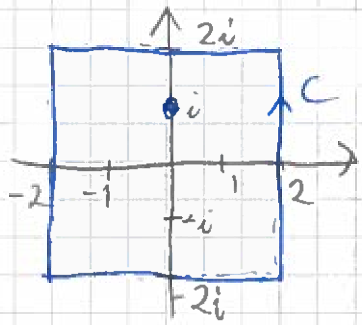
$$= -2e^z\sin z$$

$$\frac{d^2}{dz^2}(\ldots)\Big|_{z=\pi/4} = -2e^{\pi/4}\,\frac{\sqrt{2}}{2} = -\sqrt{2}\,e^{\pi/4}$$

So we have

$$\oint_C \frac{e^z \cos z}{(z-\pi/4)^3}\,dz = \frac{\cancel{2}\pi i(-\sqrt{2}e^{\pi/4})}{\cancel{2}}$$
$$= -\pi\sqrt{2}e^{\pi/4} i$$

8)



$$I = \oint_C \frac{z^3+\sin z}{(z-i)^3}\,dz = \frac{2\pi i}{2!} f^{(2)}(i)$$

$$f(z) = z^3 + \sin z$$

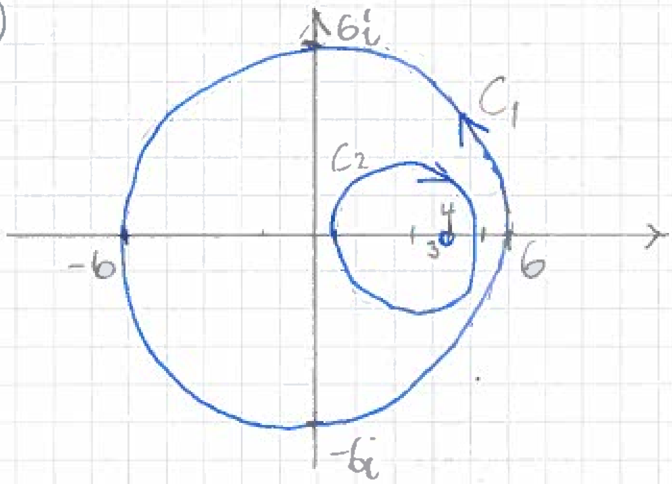
$$f'(z) = 3z^2 + \cos z$$
$$f''(z) = 6z - \sin z$$

$$\Rightarrow f''(i) = 6i - \sin(i)$$

$$\Rightarrow I = \pi i(6i + \sin i)$$
$$= -6\pi + \pi i \sin i$$
$$= -6\pi + \frac{\pi}{2}(e^{-1} - e^{1})$$

So $$\oint_C \frac{z^3+\sin z}{(z-i)^3}\,dz = -6\pi + \frac{\pi}{2}(e^{-1} - e^{1})$$

15)



$$\oint_{C_1 \cup C_2} \frac{\cosh(4z)}{(z-4)^3}\,dz = \oint_{C_1} \frac{\cosh(4z)}{(z-4)^3}\,dz + \oint_{C_2} \frac{\cosh(4z)}{(z-4)^3}\,dz$$

$$= \frac{2\pi i}{2!}\cosh(4\cdot 4) - \frac{2\pi i}{2!}\cosh(4\cdot 4)$$

$$= \underline{\underline{0}}$$

15.1:

1) $z_n = \frac{(1+i)^{2n}}{2^n} = \left(\frac{(1+i)^2}{2}\right)^n$

$(1+i)^2 = \left(\sqrt{2}\, e^{i\pi/4}\right)^2 = 2(i) = 2i$

$\Rightarrow z_n = \left(\frac{2i}{2}\right)^n = i^n$

$\{z_n\}$ is bounded but not convergent, as $|z_n| = 1$ for all n.

2) $z_n = \frac{(1+2i)^n}{n!}$

$|z_n| = \frac{|1+2i|^n}{n!}$

$= \frac{\sqrt{5}^n}{n!} \to 0$ as $n \to \infty$

It is also clearly bounded.

16) $\sum_{n=0}^{\infty} \frac{(20+30i)^n}{n!}$

$z_n = \frac{(20+30i)^n}{n!}$

$L = \lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| \overset{\lim_{n\to\infty}}{=} \frac{n!}{(n+1)!} \frac{(20+30i)^{n+1}}{(20+30i)^n}$

$= \frac{1}{n+1} \cdot (20+30i) \to 0$ as $n \to \infty$.

Since $L=0$ the series converges

17) $$\sum_{n=2}^{\infty} \frac{(-i)^n}{\ln(n)} = \frac{(-i)^2}{\ln 2} + \frac{(-i)^3}{\ln 3} + \frac{(-i)^4}{\ln(4)} + \frac{(-i)^5}{\ln 5} + \dots$$

$$= \frac{-1}{\ln 2} + \frac{i}{\ln 3} + \frac{1}{\ln(4)} - \frac{i}{\ln(5)} + \dots$$

The series is the sum of two alternating series (one real and one complex). Since the $|z_n| \to 0$ as $n \to \infty$, z_n = n'th term the sum must converge.

19) $$\sum_{n=0}^{\infty} \frac{i^n}{n^2 - i} = (*)$$

$|n^2 - i| \geq |n^2|$ so $\sum_{n=1}^{\infty} \frac{1}{|n^2-i|} \leq \sum_{n=1}^{\infty} \frac{1}{|n^2|}$

We know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent and since each term of $(*)$ is smaller in absolute size, $(*)$ must also converge.

30) Let $\left|\frac{z_{n+1}}{z_n}\right| \leq q < 1$

Want to estimate $R_n = z_{n+1} + z_{n+2} + \dots$

Since $\left|\frac{z_{n+1}}{z_n}\right| \leq q < 1$, we must have

$$|z_{n+1}| \leq q \cdot |z_n| < |z_n| \qquad (*)$$

We have

$$|R_n| \leq |z_{n+1}| + |z_{n+2}| + |z_{n+3}|$$

but since $|z_{n+2}| \overset{(*)}{\leq} q^2 |z_{n+1}|$

$$\Rightarrow |z_{n+3}| \overset{(*)}{\leq} q^3 |z_{n+1}|$$

and so on, we get

$$|R_n| \leq |z_{n+1}| + q|z_{n+1}| + q^2|z_{n+2}| + \dots$$

$$= |z_{n+1}|(1 + q + q^2 + \dots)$$

$$= |z_{n+1}| / (1-q) \qquad \text{(geometric series)}$$

since $|q| < 1$.

So we have

$$|R_n| \leq \frac{|z_{n+1}|}{1-q}$$

Want $|R_n| \leq 0{,}05$ so

$$\frac{|z_{n+1}|}{1-q} \leq 0{,}05 \qquad (*)$$

$$\frac{|z_{n+1}|}{1-\frac{|z_{n+1}|}{|z_n|}} = \frac{|z_{n+1}|\cdot|z_n|}{|z_n|-|z_{n+1}|} \leq 0{,}05$$

$$|z_n| = \left|\frac{n+i}{2^n n}\right| = \frac{\sqrt{n^2+1}}{2^n n} = \frac{\sqrt{1+1/n^2}}{2^n}$$

$$|z_{n+1}| = \left|\frac{n+1+i}{2^{n+1}(n+1)}\right| = \frac{\sqrt{(n+1)^2+1}}{2^{n+1}(n+1)} = \frac{\sqrt{1+1/(n+1)^2}}{2^{n+1}}$$

$$q \geq \frac{|z_{n+1}|}{|z_n|} = \frac{2^n}{2^{n+1}}\cdot\sqrt{\frac{1+\frac{1}{(n+1)^2}}{1+\frac{1}{n^2}}}$$

$$= \frac{1}{2}\cdot\sqrt{\frac{1+\frac{1}{(n+1)^2}}{1+\frac{1}{n^2}}}$$

Inserted into $(*)$ we get

$$\underbrace{|z_{n+1}|}_{=\frac{\sqrt{1+\frac{1}{(n+1)^2}}}{2^{n+1}}} \leq 0{,}05\left[1-\frac{1}{2}\underbrace{\sqrt{\frac{1+\frac{1}{(n+1)^2}}{1+\frac{1}{n^2}}}}_{}\right]$$

$$\Longleftrightarrow \frac{\sqrt{1+\frac{1}{(n+1)^2}}}{2^{n+1}}+\frac{0{,}05}{2}\sqrt{\cdots} \leq 0{,}05$$

By plotting the left side we see that $n=5$ is the lowest n that satisfies the inequality.

So we compute

$$\sum_{n=1}^{5} \frac{n+i}{2^n n} = \frac{1+i}{2} + \frac{2+i}{2^2 \cdot 2} + \frac{3+i}{2^3 \cdot 3} + \frac{4+i}{2^4 \cdot 4} + \frac{5+i}{2^5 \cdot 5}$$

$$= \frac{31}{32} + \frac{661}{960} i$$

$$\approx 0.97 + 0.69 i$$

15.2:

5) Assume $\sum a_n z^n$ has a radius of convergence R. That is

$$\lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = R$$

Want to find the radius of convergence for

$$\sum a_n z^{2n} = \sum a_n (z^2)^n$$

We use a change of variable $w = z^2$, then $\sum a_n z^{2n} = \sum a_n w^n$.

We know that $\sum a_n w^n$ has rad. of. conv. of R.

To get back from the w-plane to the z-plane equivalent, we take the root and find that $\sqrt{R}$ is the rad. of. conv. in the z-plane.

6) $\sum_{n=0}^{\infty} 2^n (z-1)^n$

Center of convergence is $z_0 = 1$

Radis is given by

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

$$= \lim_{n\to\infty} \frac{2^n}{2^{n+1}}$$

$$= \frac{1}{2}$$

9) $\sum_{n=0}^{\infty} n(n-1)\left(\frac{(z-i)^2}{2}\right)^n$

this has the same radius of convergence as $\sum_{n=0}^{\infty}\left(\frac{(z-i)^2}{2}\right)^n$ which convergeres when

$$\left|\frac{(z-i)^2}{2}\right| < 1$$

$$\Rightarrow \quad |z|^2+1 < 2$$

$$\Leftrightarrow \quad |z| < 1$$

So the radius of convergence is $R=1$ and the center is $z_0 = i$