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TTK 4115 Linear System Theory

Exam Fall 2010 – Solution Suggestion

Question 1 (30%)

Given the second-order system:

$$\begin{aligned} \ddot{q} + \omega_0^2 q &= u && (1a)\\ y &= q, && (1b)\end{aligned}$$

where q is the position of a mass in motion, and ω_0 is a constant.

(a) Use $\mathbf{x} = (x_1, x_2)^\top = (q, \dot{q}/\omega_0)^\top$, and write the system on state space form:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{b}u,\\ y &= \mathbf{c}\mathbf{x}.\end{aligned}$$

What is $\mathbf{A}$, $\mathbf{b}$ and $\mathbf{c}$?

Solution: Using $\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}^\top = \begin{pmatrix} q & \dot{q}/\omega_0 \end{pmatrix}^\top$, the system (1) can be written as

$$\begin{aligned} \dot{x}_1 &= \omega_0 x_2\\ \dot{x}_2 &= -\omega_0 x_1 + \frac{u}{\omega_0}\\ y &= x_1 \end{aligned}$$

or equivalently

$$\begin{aligned} \dot{x} &= \overbrace{\begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix}}^{\mathbf{A}} \mathbf{x} + \overbrace{\begin{pmatrix} 0 \\ \frac{1}{\omega_0} \end{pmatrix}}^{\mathbf{b}} u && (3a)\\ y &= \underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{\mathbf{c}} \mathbf{x} && (3b)\end{aligned}$$

(b) Find the eigenvalues and the corresponding eigenvectors of $\mathbf{A}$.

Solution: To calculate the eigenvalues of $\mathbf{A}$ the following equation has to be solved:

$$\det(\mathbf{A} - \mathbf{I}\lambda) \stackrel{!}{=} 0 \qquad (4)$$

This yields

$$\begin{aligned} \det(\mathbf{A} - \mathbf{I}\lambda) &= \det \begin{pmatrix} -\lambda & \omega_0 \\ -\omega_0 & -\lambda \end{pmatrix}\\ &= \lambda^2 + \omega_0^2 \stackrel{!}{=} 0 && (5)\end{aligned}$$

Solving the quadratic equation (5) to compute the eigenvalues of $\mathbf{A}$ results in

$$\begin{aligned} \lambda_1 &= \jmath\omega_0 && (6a)\\ \lambda_2 &= -\jmath\omega_0 && (6b)\end{aligned}$$

The eigenvectors of $\mathbf{A}$ are determined by solving

$$(\mathbf{A} - \mathbf{I}\lambda)\,\mathbf{x} \overset{!}{=} \mathbf{0} \tag{7}$$

for the two eigenvalues in (6). Using the first eigenvalue (6a), (7) is used to compute the corresponding eigenvector

$$(\mathbf{A} - \mathbf{I}\lambda_1)\,\mathbf{x} = \begin{pmatrix} -\jmath\omega_0 & \omega_0 \\ -\omega_0 & -\jmath\omega_0 \end{pmatrix} \mathbf{x} \overset{!}{=} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{8}$$

This results in the first eigenvector

$$\mathbf{q}_1 = \begin{pmatrix} -\jmath \\ 1 \end{pmatrix} \tag{9}$$

Similarly for the second eigenvalue, inserting (6b) in (7) results in

$$(\mathbf{A} - \mathbf{I}\lambda_2)\,\mathbf{x} = \begin{pmatrix} \jmath\omega_0 & \omega_0 \\ -\omega_0 & \jmath\omega_0 \end{pmatrix} \mathbf{x} \overset{!}{=} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{10}$$

Solving the above equation gives the second eigenvector

$$\mathbf{q}_2 = \begin{pmatrix} \jmath \\ 1 \end{pmatrix} \tag{11}$$

(c) Explain what methods you have learned in order to find the matrix exponential function $e^{\mathbf{A}}$.

Solution:

1. Definition of $\mathrm{e}^{\mathbf{A}t}$:
$$\mathrm{e}^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{1}{k!} t^k \mathbf{A}^k$$
2. Use the fact, that $\mathrm{e}^{\mathbf{A}t} = \mathcal{L}^{-1}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}$
3. Theorem 3.5 in Chen
4. Jordan form:
$$\mathrm{e}^{\mathbf{A}t} = \mathbf{Q}\mathrm{e}^{\hat{\mathbf{A}}t}\mathbf{Q}^{-1},$$
where $\hat{\mathbf{A}} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}$ is the Jordan (diagonal) form of $\mathbf{A}$
5. Method based on Cayley-Hamiltons theorem.

(d) Use *one* of the methods $e^{\mathbf{A}t}$. Depending on the method you choose, you might find the following formulas useful:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

Solution:

1. Using $\mathrm{e}^{\mathbf{A}t} = \mathcal{L}^{-1}\left(s\mathbf{I}-\mathbf{A}\right)^{-1}$

$$\left(s\mathbf{I}-\mathbf{A}\right)^{-1} = \begin{pmatrix} \frac{s}{s^2+\omega_0^2} & \frac{\omega_0}{s^2+\omega_0^2} \\ \frac{-\omega_0}{s^2+\omega_0^2} & \frac{s}{s^2+\omega_0^2} \end{pmatrix} \tag{12}$$

   Consequently:

$$\begin{aligned} \mathrm{e}^{\mathbf{A}t} &= \mathcal{L}^{-1}\left(s\mathbf{I}-\mathbf{A}\right)^{-1} \\ &= \mathcal{L}^{-1}\begin{pmatrix} \frac{s}{s^2+\omega_0^2} & \frac{\omega_0}{s^2+\omega_0^2} \\ \frac{-\omega_0}{s^2+\omega_0^2} & \frac{s}{s^2+\omega_0^2} \end{pmatrix} \\ &= \begin{pmatrix} \cos\left(\omega_0 t\right) & \sin\left(\omega_0 t\right) \\ -\sin\left(\omega_0 t\right) & \cos\left(\omega_0 t\right) \end{pmatrix} \end{aligned} \tag{13}$$

2. Using Theorem 3.5 in Chen

   The characteristic polynomial is given by

$$\begin{aligned} \Delta\left(\lambda\right) &= \left(\lambda-\lambda_1\right)\left(\lambda-\lambda_2\right) \\ &= \left(\lambda-\jmath\omega_0\right)\left(\lambda+\jmath\omega_0\right) \end{aligned} \tag{14}$$

   Define

$$h\left(\lambda\right) := \beta_0 + \beta_1\lambda \tag{15}$$

   The coefficients β_i are determined by the following equations

$$f\left(\lambda_1\right) = h\left(\lambda_1\right) \tag{16a}$$

$$f\left(\lambda_2\right) = h\left(\lambda_2\right) \tag{16b}$$

   where $f\left(\lambda\right) = \mathrm{e}^{\lambda t}$. This yields two equations with two unknowns

$$\mathrm{e}^{\jmath\omega_0 t} = \beta_0 + \jmath\omega_0\beta_1 \tag{17a}$$

$$\mathrm{e}^{-\jmath\omega_0 t} = \beta_0 - \jmath\omega_0\beta_1 \tag{17b}$$

   which can then be solved for the unknowns

$$\beta_1 = \frac{\mathrm{e}^{\jmath\omega_0 t} - \mathrm{e}^{-\jmath\omega_0 t}}{2\jmath\omega_0} \tag{18a}$$

$$\beta_0 = \mathrm{e}^{\jmath\omega_0 t} - \jmath\omega_0\beta_1 = \frac{\mathrm{e}^{\jmath\omega_0 t} + \mathrm{e}^{-\jmath\omega_0 t}}{2} \tag{18b}$$

   By using the Euler's formula the expression for β_i can be simplifed to

$$\beta_1 = \frac{1}{\omega_0}\sin\left(\omega_0 t\right) \tag{19a}$$

$$\beta_0 = \cos\left(\omega_0 t\right) \tag{19b}$$

   Theorem 3.5 states the $h\left(\lambda\right)$ is equal $f\left(\lambda\right)$ on the spectrum of $\mathbf{A}$, i.e. $f\left(\mathbf{A}\right) = h\left(\mathbf{A}\right)$. Thus $\mathrm{e}^{\mathbf{A}t}$ can be computed to

$$\begin{aligned} \mathrm{e}^{\mathbf{A}t} &= h\left(\mathbf{A}\right) = \cos\left(\omega_0 t\right)\mathbf{I} + \frac{1}{\omega_0}\sin\left(\omega_0 t\right)\mathbf{A} \\ &= \begin{pmatrix} \cos\left(\omega_0 t\right) & 0 \\ 0 & \cos\left(\omega_0 t\right) \end{pmatrix} + \begin{pmatrix} 0 & \sin\left(\omega_0 t\right) \\ -\sin\left(\omega_0 t\right) & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos\left(\omega_0 t\right) & \sin\left(\omega_0 t\right) \\ -\sin\left(\omega_0 t\right) & \cos\left(\omega_0 t\right) \end{pmatrix} \end{aligned} \tag{20}$$

3. Jordan form

The Jordan (diagonal) form of $\mathbf{A}$ is $\hat{\mathbf{A}} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}$. $\mathbf{Q}$ is given by

$$\mathbf{Q} = \begin{pmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{pmatrix} \tag{21}$$

where $\mathbf{q}_1$ and $\mathbf{q}_2$ are the eigenvectors as computed in (9) and (11). Thus, $\hat{\mathbf{A}}$ is calculated to

$$\begin{aligned} \hat{\mathbf{A}} &= \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} \\ &= \begin{pmatrix} -\frac{1}{2\jmath} & \frac{1}{2} \\ \frac{1}{2\jmath} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & \omega_0 \\ \omega_0 & 0 \end{pmatrix} \begin{pmatrix} -\jmath & \jmath \\ 1 & 1 \end{pmatrix} \\ &= \cdots = \begin{pmatrix} \jmath\omega_0 & 0 \\ 0 & -\jmath\omega_0 \end{pmatrix} \end{aligned} \tag{22}$$

Then, $\mathrm{e}^{\mathbf{A}t}$ can be computed:

$$\begin{aligned} \mathrm{e}^{\mathbf{A}t} &= \mathbf{Q}\mathrm{e}^{\hat{\mathbf{A}}t}\mathbf{Q}^{-1} \\ &= \begin{pmatrix} -\jmath & \jmath \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\jmath\omega_0 t} & 0 \\ 0 & \mathrm{e}^{-\jmath\omega_0 t} \end{pmatrix} \begin{pmatrix} -\frac{1}{2\jmath} & \frac{1}{2} \\ \frac{1}{2\jmath} & \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}\mathrm{e}^{\jmath\omega_0 t} + \frac{1}{2}\mathrm{e}^{-\jmath\omega_0 t} & \frac{1}{2\jmath}\mathrm{e}^{\jmath\omega_0 t} - \frac{1}{2\jmath}\mathrm{e}^{-\jmath\omega_0 t} \\ -\frac{1}{2\jmath}\mathrm{e}^{\jmath\omega_0 t} + \frac{1}{2\jmath}\mathrm{e}^{-\jmath\omega_0 t} & \frac{1}{2}\mathrm{e}^{\jmath\omega_0 t} + \frac{1}{2}\mathrm{e}^{-\jmath\omega_0 t} \end{pmatrix} \\ &= \begin{pmatrix} \cos(\omega_0 t) & \sin(\omega_0 t) \\ -\sin(\omega_0 t) & \cos(\omega_0 t) \end{pmatrix} \end{aligned} \tag{23}$$

(e) Use

$$e^{\mathbf{A}t} = \mathbf{I} + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \ldots,$$

to show that:

$$\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t}. \tag{24}$$

Solution: Note that

$$\mathrm{e}^{At} = \mathbf{I} + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!} t^k \mathbf{A}^k \tag{25}$$

Differentiating (25) w.r.t. time yields:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\mathbf{A}t} &= \sum_{k=1}^{\infty} \frac{k}{k!} t^{k-1} \mathbf{A}^k \\
&= \sum_{k=1}^{\infty} \frac{1}{(k-1)!} t^{k-1} \mathbf{A}^k \\
&= \mathbf{A} \left(\sum_{k=1}^{\infty} \frac{1}{(k-1)!} t^{k-1} \mathbf{A}^{k-1} \right) \\
&= \mathbf{A} \left(\sum_{k=0}^{\infty} \frac{1}{k!} t^{k} \mathbf{A}^{k} \right) \\
&= \mathbf{A}\mathrm{e}^{\mathbf{A}t}
\end{aligned} \tag{26}
$$

(f) Show that your answer for $e^{\mathbf{A}t}$ in exercise (d) is correct by using the expression in (24).

Solution: In exercise (d) $\mathrm{e}^{\mathbf{A}t}$ was calculated to

$$
\mathrm{e}^{\mathbf{A}t} = \begin{pmatrix} \cos(\omega_0 t) & \sin(\omega_0 t) \\ -\sin(\omega_0 t) & \cos(\omega_0 t) \end{pmatrix}
$$

Differentiating w.r.t. time yields

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\mathbf{A}t} &= \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \cos(\omega_0 t) & \sin(\omega_0 t) \\ -\sin(\omega_0 t) & \cos(\omega_0 t) \end{pmatrix} \\
&= \begin{pmatrix} -\omega_0 \sin(\omega_0 t) & \omega_0 \cos(\omega_0 t) \\ -\omega_0 \cos(\omega_0 t) & -\omega_0 \sin(\omega_0 t) \end{pmatrix}
\end{aligned} \tag{27}
$$

Using the expression in (24) results in

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{At} &= A\mathrm{e}^{At} \\
&= \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix} \begin{pmatrix} \cos(\omega_0 t) & \sin(\omega_0 t) \\ -\sin(\omega_0 t) & \cos(\omega_0 t) \end{pmatrix} \\
&= \begin{pmatrix} -\omega_0 \sin(\omega_0 t) & \omega_0 \cos(\omega_0 t) \\ -\omega_0 \cos(\omega_0 t) & -\omega_0 \sin(\omega_0 t) \end{pmatrix}
\end{aligned} \tag{28}
$$

It is easy to see that (27) and (28) are equivalent.

(g) Assume that $\omega_0 = 1$, $\mathbf{x}(0) = (1,1)^\top$, and that $u(t) = 1$ for all $t \geq 0$. What is $y(t = 1)$?

Solution:

$$
\begin{aligned}
y(t) &= \mathbf{c}\mathrm{e}^{\mathbf{A}t}x(0) + \mathbf{c}\int_0^t \mathrm{e}^{\mathbf{A}(t-\tau)}\mathbf{b}u(\tau)\,\mathrm{d}\tau \\
&= \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}\cos(\omega_0 t) & \sin(\omega_0 t) \\ -\sin(\omega_0 t) & \cos(\omega_0 t)\end{pmatrix} x(0) \\
&\quad + \begin{pmatrix}1 & 0\end{pmatrix}\int_0^t \begin{pmatrix}\cos(\omega_0(t-\tau)) & \sin(\omega_0(t-\tau)) \\ -\sin(\omega_0(t-\tau)) & \cos(\omega_0(t-\tau))\end{pmatrix}\begin{pmatrix}0 \\ \frac{1}{\omega_0}\end{pmatrix} u(\tau)\,\mathrm{d}\tau \\
&= \begin{pmatrix}\cos(\omega_0 t) & \sin(\omega_0 t)\end{pmatrix} x(0) + \frac{1}{\omega_0}\int_0^t \sin(\omega_0(t-\tau))\,u(\tau)\,\mathrm{d}\tau
\end{aligned} \tag{29}
$$

With $\omega_0 = 1$, $\mathbf{x}(0) = \begin{pmatrix}1 & 1\end{pmatrix}^\top$, and $u(t) = 1$ for all $t \geq 0$, $y(t=1)$ is computed to

$$
\begin{aligned}
y(t=1) &= \begin{pmatrix}\cos(1) & \sin(1)\end{pmatrix}\begin{pmatrix}1 \\ 1\end{pmatrix} + \int_0^1 \sin((1-\tau))\,\mathrm{d}\tau \\
&= \cos(1) + \sin(1) + \cos(0) - \cos(1) \\
&= \sin(1) + 1 \\
&\approx 1.84147
\end{aligned} \tag{30}
$$

(h) Assume that $u(t) = 0$ for all t. Is $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ unstable, marginally stable or asymptotically stable? Explain!

Solution: It is assumed that $u(t) = 0\ \forall t$. The eigenvalues of $\mathbf{A}$ are given by (6):

$$
\begin{aligned}
\lambda_1 &= \jmath\omega_0 \\
\lambda_2 &= -\jmath\omega_0
\end{aligned}
$$

Theorem 5.4 in Chen states that the system given by $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is marginally stable if and only if all eigenvalues of $\mathbf{A}$ have zero or negative real parts and those with zero real parts are simple roots of the minimal polynomial of $\mathbf{A}$. It can therefore be concluded that the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is marginally stable.

Question 2 (20%)

(a) Given the transferfunction:

$$
\hat{g}(s) = \frac{s-1}{(s^2-1)(s+2)}
$$

Find a three-dimensional controllable realization by using the equations in the appendix.

Solution: The transfer function is given by

$$
\hat{g} = \frac{s-1}{(s^2-1)(s+2)} \tag{31}
$$

A realization is found by the equations in the appendix:

$$
\hat{g}(s) = \hat{g}(\infty) + \hat{g}_{\mathrm{sp}}(s) = \hat{g}_{\mathrm{sp}}(s) \tag{32}
$$

The transfer function (31) is strictly proper (deg(num) < deg(den)) and consequently $\hat{g}(\infty) = 0$. Furthermore

$$\hat{g}_{\mathrm{sp}}(s) = \frac{1}{d(s)}\left[N_1 s^{r-1} + N_2 s^{r-2} + \cdots + N_{r-1}s + N_r\right] \tag{33a}$$

$$d(s) = s^r + \alpha_1 s^{r-1} + \cdots + \alpha_{r-1}s + \alpha_r \tag{33b}$$

where $d(s)$ is the least common denominator of all entries of $\hat{g}_{\mathrm{sp}}(s) = \hat{g}(s)$. It follows that for the transfer function (31)

$$d(s) = s^3 + 2s^2 - s - 2 \tag{34}$$

and thus

$$\alpha_1 = 2 \tag{35a}$$

$$\alpha_2 = -1 \tag{35b}$$

$$\alpha_3 = -2 \tag{35c}$$

Consequently

$$\hat{g}_{\mathrm{sp}}(s) = \frac{1}{d(s)}\left[N_1 s^2 + N_2 s + N_3\right] = \hat{g}(s) = \frac{s-1}{(s^2-1)(s+2)} \tag{36}$$

This yields

$$N_1 = 0 \tag{37a}$$

$$N_2 = 1 \tag{37b}$$

$$N_3 = -1 \tag{37c}$$

Then, a realization of (31) is given by

$$\dot{\mathbf{x}} = \begin{pmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u = \overbrace{\begin{pmatrix} -2 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}^{\mathbf{A}}\mathbf{x} + \overbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}^{\mathbf{b}} u \tag{38a}$$

$$y = \begin{pmatrix} N_1 & N_2 & N_3 \end{pmatrix}\mathbf{x} = \underbrace{\begin{pmatrix} 0 & 1 & -1 \end{pmatrix}}_{\mathbf{c}}\mathbf{x} \tag{38b}$$

(b) What characterizes a minimal realization? Is the realization you found in (a) minimal? Why/ Why not?

Solution: According to Chen, the following statements are equivalent for minimal realizations:

1. A minimal realization is a realization with the smallest possible dimension.
2. A realization of $\hat{g}(s) = \frac{N(s)}{D(s)}$ is a minimal realization if and only if it is controllable and observable.
3. A realization of $\hat{g}(s) = \frac{N(s)}{D(s)}$ is a minimal realization if and only if its dimension equals the degree of $\hat{g}(s)$. The degree of $\hat{g}(s)$ is defined as the degree of

$D(s)$ if the two polynomials $D(s)$ and $N(s)$ have no common factors or are coprime, i.e. they have no common factor of degree at least 1. If they are not coprime, then the degree is defined as the degree of its coprime fraction.

There are several possiblities to check whether the realization (38) is minimal:

- To check the controllability and the observability of the realization (38) it is necessary to compute the controllability and the observability matrix, respectively. Using $\mathbf{A}$, $\mathbf{b}$, and $\mathbf{c}$ as computed in (38) yields

$$\mathcal{C} = \begin{pmatrix} \mathbf{b} & \mathbf{Ab} & \mathbf{A}^2\mathbf{b} \end{pmatrix} = \begin{pmatrix} 1 & -2 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \tag{39}$$

$$\mathcal{O} = \begin{pmatrix} \mathbf{c} \\ \mathbf{cA} \\ \mathbf{cA}^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ -3 & 1 & 2 \end{pmatrix} \tag{40}$$

  The controllability matrix (39) has rank three, whereas the observability matrix (40) has rank two. Therefore, the realization (38) is not minimal.

- To determine the degree of $\hat{g}(s)$, it has to be reduced to its coprime fraction:

$$\hat{g} = \frac{s-1}{(s^2-1)(s+2)} = \frac{1}{(s+1)(s+2)} = \frac{1}{s^2+3s+2} \tag{41}$$

  The degree is therefore two and since the degree of $\mathbf{A}$ is three, it follows that the realization (38) is not minimal.

(c) Given a transferfunction:

$$\hat{g}(s) = \frac{\hat{y}(s)}{\hat{u}(s)} = \frac{N(s)}{D(s)}.$$

We seek a realization on standard form: $\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{b}u$, $y = \mathbf{cx}$. Under what condition/conditions can we choose $y(t)$ and its derivatives as the state variables?

Solution: Given the transferfunction

$$\hat{g}(s) = \frac{\hat{y}(s)}{\hat{u}(s)} = \frac{N(s)}{D(s)} \tag{42}$$

We can choose $y(t)$ and its derivatives as the state variables if $N(s)$ is not a polynomial of degree 1 or higher (see Chen, page 186).

Question 3 (25%)

Consider an experiment where an object with mass $m = 1$ kg is released from a certain height. Consider only the gravitational force, and assume that the gravitational constant g is perfectly known. Assume that the initial position and velocity are random variables described by $\mathcal{N}(0, \rho_p^2)$, and $\mathcal{N}(0, \rho_v^2)$. Let the state variables x_1 and x_2 be position and velocity in the downward direction, and let the measurements of the position of the object take place at uniform intervals Δt beginning at $t = 0$. The standard deviation of the measurement error is ξ.

(a) What are the (continuous time) state equations describing the falling object?

Solution: With the state variables x_1 and x_2 being the the position and the velocity downwards of the falling object and under the assumption that only the gravitational force is acting on the object, the state equations can be written as

$$\dot{x}_1 = x_2 \tag{43a}$$
$$\dot{x}_2 = \frac{g}{m} = g \tag{43b}$$
$$y = x_1 + v \tag{43c}$$

where the mass of the object is $m = 1$, and v denotes the measurement error. This is equivalent to

$$\dot{\mathbf{x}} = \overbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}^{\mathbf{A}} \mathbf{x} + \overbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}^{\mathbf{b}} g \tag{44}$$
$$y = \underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{\mathbf{c}} \mathbf{x} + v \tag{45}$$

(b) Discretize the state equations from a) using zero-order hold element on the input (exact discretization).

Solution: The discrete-time state-space equation is

$$\mathbf{x}[k+1] = \mathbf{A}_d \mathbf{x}[k] + \mathbf{b}_d g \tag{46a}$$
$$y[k] = \mathbf{c}_d \mathbf{x}[k] + v[k] \tag{46b}$$

where

$$\mathbf{A}_d = \mathrm{e}^{\mathbf{A}\Delta t} \tag{47a}$$
$$\mathbf{b}_d = \left(\int_0^{\Delta t} \mathrm{e}^{\mathbf{A}\tau} \mathrm{d}\tau \right) \mathbf{b} \tag{47b}$$
$$\mathbf{c}_d = \mathbf{c} \tag{47c}$$

Here, Δt denotes the sampling time. The computation of $\mathbf{A}_d$ in (47a) can be done in different ways. Here, it is calculated as

$$\begin{aligned} \mathbf{A}_d &= \mathrm{e}^{\mathbf{A}\Delta t} = \mathcal{L}^{-1} (s\mathbf{I} - \mathbf{A})^{-1} \\ &= \mathcal{L}^{-1} \begin{pmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{pmatrix} = \begin{pmatrix} 1 & \Delta t \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{48}$$

$\mathbf{b}_d$ in (47b) then becomes

$$\begin{aligned} \mathbf{b}_d &= \left(\int_0^{\Delta t} \mathrm{e}^{\mathbf{A}\tau} \mathrm{d}\tau \right) \mathbf{b} = \left(\int_0^{\Delta t} \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} \mathrm{d}\tau \right) \mathbf{b} \\ &= \begin{bmatrix} \Delta t & \frac{1}{2}\Delta t^2 \\ 0 & \Delta t \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\Delta t^2 \\ \Delta t \end{bmatrix} \end{aligned} \tag{49}$$

(c) Find the key parameters for the (discrete time) Kalman filter in the appendix, that is, find $\mathbf{\Phi}_k$, $\mathbf{Q}_k$, $\mathbf{H}_k$, $\mathbf{R}_k$ and the initial $\hat{\mathbf{x}}_0^-$ and $\mathbf{P}_0^-$. Justify any assumptions!

Solution: The parameters of the Kalman filter are

- $\mathbf{\Phi}_k = \mathbf{A}_d$: State transition matrix
- $\mathbf{Q}_k = \mathbf{0}$: Since there is no process noise.
- $\mathbf{H}_k = \mathbf{c}_d = \mathbf{c}$
- $R_k = \xi^2$: Measurement error variance

The initial values are

- $\hat{\mathbf{x}}_0^- = \begin{pmatrix} 0 & 0 \end{pmatrix}^\top$: Initial estimates of the position and velocity are taken as their respective mean.
- $\mathbf{P}_0^- = \begin{bmatrix} \rho_p^2 & 0 \\ 0 & \rho_v^2 \end{bmatrix}$: Initial error covariance matrix. Since the mean values of the position and the velocity are zero, the diagonal elements can be taken as their respective variances.

Question 4 (15%)

The power spectral density (and corresponding autocorrelation) of a Gauss-Markov process is given by:

$$S_x(s) = \frac{6}{-s^2+1}, \quad (R_x(\tau) = 3e^{-|\tau|}). \tag{50}$$

(a) Find the filter (shaping filter) $S_x^+(s)$ for the process, and based on the filter, find a realization on the form $\dot{x} = Fx + Gu$, where u is unity white noise.

Solution: For unity white noise, it holds that

$$S_x(s) = S_x^+(s)\, S_x^+(-s) \tag{51}$$

To find the appropriate $S_x^+(s)$ such that (51) becomes (50) the spectral factorization of $S_x(s)$ has to be found

$$S_x(s) = \frac{6}{-s^2+1} = \frac{\sqrt{6}}{s+1}\frac{\sqrt{6}}{-s+1} \tag{52}$$

Consequently, the shaping filter $S_x^+(s)$ is given by

$$S_x^+(s) = \frac{\sqrt{6}}{s+1} \tag{53}$$

A realization can then be found as

$$\dot{x} = Fx + Gu = -1x + \sqrt{6}u \tag{54}$$

(b) What is the mean and the variance of the process?

Solution: Since $R_x(\infty) = 0$ the mean of the process is zero. Furthermore it is easy to see that the variance of the process is $\sigma^2 = 3$.

Question 5 (10%)

The cost function of a linear quadratic regulator is given by

$$J(u) = \int_0^\infty z^\top(t)Qz(t) + u^\top(t)Ru(t)dt,$$

where $z = Mx$.

(a) Explain briefly what Q and R do, what properties they must have, and how we can inverstigate these properties.

Solution: Q and R are the weight matrices for the state variables and the control input, respectively. Q is used to keep the state errors small, whereas R corresponds to a penalty of large control inputs. Q has to be real, symmetric and positive-semidefinite and R real, symmetric and positive-definite.

Real: All the matrix elements consist of real numbers.

Symmetric: $Q = Q^\top$ and $R = R^\top$

Positive-semidefinite: $z^\top Qz \geq 0$ for all non-zero vectors z with real entries. Test for example that all eigenvalues are non-negative.

Positive-definite: $z^\top Rz > 0$ for all non-zero vectors z with real entries. Test for example that all eigenvalues are positive.

(b) Discuss briefly possible pros and cons for controller design based on the cost function above and controller design based on pole placement.

Solution: There is more than one correct answer to this question. Below just some possible comments:

- The linear quadratic regulator (LQR) is an optimal controller aimed at minimizing a defined cost function. It allows to weight the state errors and the control input with respect to each other. A controller based on pole placement (PP) on the other hand does not optimize a cost function.
- LQR is also straight-forward to use for (MIMO) systems. It allows the specification of the dynamical properties of the different outputs whereas using PP only the pole locations can be specified.
- In the LQR design, the feedback matrix is computed by solving the Ricatti equation. This may be computational expensive and normally numerical approximations are used.
- It is not trivial to choose the weight matrices for the LQR controller and LQR design often requires fine tuning of the weight matrices to achieve the desired performance.