DEPARTMENT OF ELECTRONIC SYSTEMS • NTNU

TFE 4130

BOUNDARY CONDITIONS

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Contents

1 Introduction and background

Based on the discussion in the book (pp. 334-335) assume we‘re interested in the propagation of an electromagnetic wave in a linear, isotropic, homogeneous, nonconducting ($\sigma = 0$) medium characterized by ε and μ. Maxwell‘s equations, under these approximations, are

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \tag{7-79a}$$

$$\nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t} \tag{7-79b}$$

$$\nabla \cdot \mathbf{E} = 0 \tag{7-79c}$$

$$\nabla \cdot \mathbf{H} = 0 \tag{7-79d}$$

Following the book and taking the curl of both sides of equation (7-79a) and proceeding accordingly we end up with the wave equation

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \tag{7-81}$$

with $c = 1/\sqrt{\mu\varepsilon}$. The wave equation as stated in equation (7-81) has infinitely many solutions. To find a single solution we need a specific problem. This is achieved by stating 2 boundary conditions (BC) and 2 initial conditions (IC). These conditions represent our best best attempt to mathematically describe the physical situation.

So, with a properly formulated boundary condition we will now have a uniquely defined vector field. The hard part is figuring out how to describe the BCs mathematically. A boundary will be defined as the interface between two regions characterized by different values of a physical variable which is used for describing the material properties. Examples of such variables are, heat conductivity κ $[W/m\,K]$, viscosity ν $[kg/s\,m]$, electrical permittivity ϵ $[F/m]$, magnetic permeability μ $[H/m]$, mass density ρ $[kg/m^3]$, etc.

The first observation we make is that the boundary is approximated to be "*infinitely*" sharp. This is obviously not true, figure 1, where we show how the electrical permittivity is *gradually* changing from a higher to a lower value as we are moving from one material to another.

However, since this transition region is typically very short (a few atomic distances) compared to the wavelengths we are interested in, we can safely approximate the transition region as an infinitely sharp discontinuity occurring at a specific location, the *boundary*.

To keep the derivations of the BCs general we use the letter $\tilde{\mathbf{V}}$ to represent an arbitrary vector field. $\tilde{\mathbf{V}}$ can be substituted for any of the electric and magnetic fields that we are concerned with in this course; e.g. $\mathbf{E}$, $\mathbf{D}$, $\mathbf{P}$, $\mathbf{B}$, $\mathbf{H}$, $\mathbf{M}$, $\mathbf{J}$, etc. . Or any other vector field for that matter !

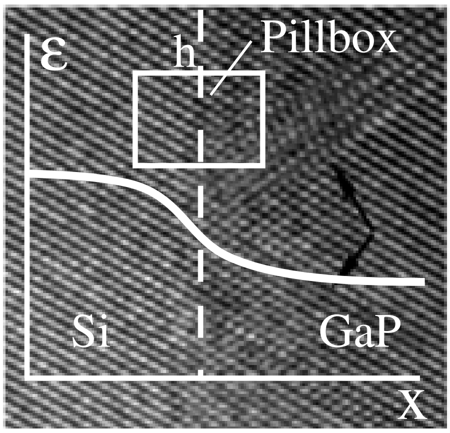

Figure 1: Qualitative change in ϵ going from one material to another.

2 Normal Components

To investigate how the vector field behaves at a boundary we will again make use of the control surface method (Gauss Law). Our control surface (often referred to as the "*pillbox*" in boundary condition calculations) is positioned across the boundary, figures 1 and 2.

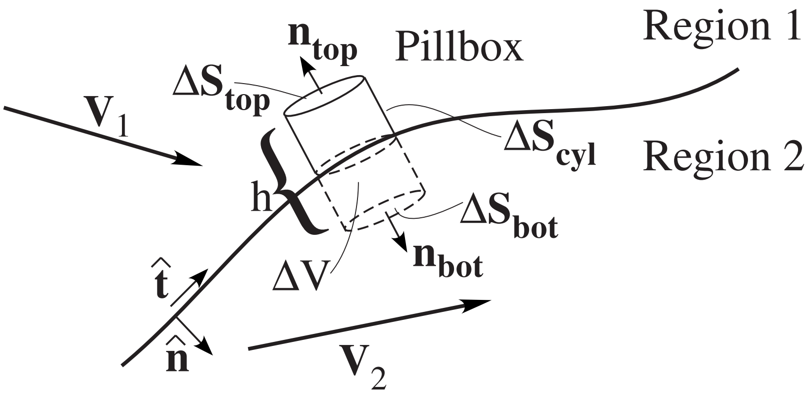

Figure 2: Geometry for calculating boundary conditions.

The boundary surface can be arbitrary, our only requirement is that the surface is flat with constant physical parameters and a constant vector field over the *infinitesimal* surface area ΔS (which again is an order of magnitude larger than atomic distances but smaller than the wavelength). The control surface is conveniently chosen to be that of a soda can with top and bottom having circular areas of $\Delta S_{top} = \Delta S_{bot} = \Delta S$, respectively, and a cylindrical body with area ΔS_{cyl} of height h.

The flux through the pillbox is

$$\begin{aligned}\oint_S \tilde{\mathbf{V}} \cdot \mathrm{d}\mathbf{S} = \tilde{\mathbf{V}}_1 \cdot \Delta S_{top} + \tilde{\mathbf{V}}_2 \cdot \Delta S_{bot} \\ + \tilde{\mathbf{V}}_1 \cdot \Delta S_{cyl}^{tophalf} + \tilde{\mathbf{V}}_2 \cdot \Delta S_{cyl}^{bottomhalf}\end{aligned} \tag{1}$$

To proceed we need to introduce the unit vectors for the various surfaces involved in this analysis. We have $\hat{\mathbf{n}}_{bot}, \hat{\mathbf{n}}_{top}$, for the bottom and top of the pillbox, respectively. $\hat{\mathbf{t}}$ is the tangential vector for the boundary as well as the normal vector for the cylindrical portion of the pillbox, and finally $\hat{\mathbf{n}}$ is the normal vector of the boundary, figure 2. Note that,

$$\hat{\mathbf{n}}_{bot} = -\hat{\mathbf{n}}_{top}. \tag{2}$$

We also need to decompose our vector field into a normal and a tangential component,

$$\tilde{\mathbf{V}} = \tilde{V}_n\hat{\mathbf{n}} + \tilde{V}_t\hat{\mathbf{t}} \tag{3}$$

So, finally

$$\begin{aligned}\oint_S \tilde{\mathbf{V}} \cdot \mathrm{d}\mathbf{S} = &\tilde{V}_{1t}\underbrace{\hat{\mathbf{t}}\cdot\hat{\mathbf{n}}}_{=0}\Delta S + \tilde{V}_{1n}\underbrace{\hat{\mathbf{n}}\cdot(-\hat{\mathbf{n}})}_{=-1}\Delta S \\ &+ \tilde{V}_{2t}\underbrace{\hat{\mathbf{t}}\cdot\hat{\mathbf{n}}}_{=0}\Delta S + \tilde{V}_{2n}\underbrace{\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}}_{=1}\Delta S \\ &+ \tilde{V}_{1t}\underbrace{\hat{\mathbf{t}}\cdot\hat{\mathbf{t}}}_{=1}\Delta S_{cyl}^{tophalf} + \tilde{V}_{1n}\underbrace{\hat{\mathbf{n}}\cdot\hat{\mathbf{t}}}_{=0}\Delta S_{cyl}^{tophalf} \\ &+ \tilde{V}_{2t}\underbrace{\hat{\mathbf{t}}\cdot\hat{\mathbf{t}}}_{=1}\Delta S_{cyl}^{bottomhalf} + \tilde{V}_{2n}\underbrace{\hat{\mathbf{n}}\cdot\hat{\mathbf{t}}}_{=0}\Delta S_{cyl}^{bottomhalf}\end{aligned} \tag{4}$$

From the divergence theorem we know that

$$\oint_S \tilde{\mathbf{V}} \cdot \mathrm{d}\mathbf{S} = \int\int_{\mathbf{V}}\int \tilde{\mathbf{f}}\ \mathrm{d}\mathbf{V}$$

So, for our "*tiny*" pillbox we can assume a constant source density $\tilde{f}$, therefore

$$\int\int_V\int \tilde{f}\ \mathrm{d}V \cong \tilde{f} \cdot h\Delta S \tag{5}$$

Finally, collecting all the terms and eliminating the zero ones we have

$$\begin{aligned}\tilde{f} \cdot h\Delta S = &\tilde{V}_{1n}\underbrace{\hat{\mathbf{n}}\cdot(-\hat{\mathbf{n}})}_{=-1}\Delta S + \tilde{V}_{2n}\underbrace{\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}}_{=1}\Delta S \\ &+ \tilde{V}_{1t}\underbrace{\hat{\mathbf{t}}\cdot\hat{\mathbf{t}}}_{=1}\Delta S_{cyl}^{tophalf} + \tilde{V}_{2t}\underbrace{\hat{\mathbf{t}}\cdot\hat{\mathbf{t}}}_{=1}\Delta S_{cyl}^{bottomhalf}\end{aligned} \tag{6}$$

Since we are only interested in what is happening at the boundary between region 1 and 2, we let $h \to 0$. By doing so, the contributions to the flux from the cylindrical surfaces go to zero (since $\Delta S_{cyl} \propto h$) and we are left with

$$\hat{\mathbf{n}} \cdot (\tilde{\mathbf{V}}_2 - \tilde{\mathbf{V}}_1) = \lim_{h\to 0}(h\tilde{f}) = \lim_{h\to 0}(h\nabla\cdot\tilde{\mathbf{V}}) \tag{7}$$

Equation 7 means that only surface flux sources present on the boundary can alter the normal component of the vector field. The tangential component is **not** affected by flux sources (it is only affected by circulation sources as we will see in the next chapter). It is also obvious that flux source strengths are directly proportional to the divergence, that if the divergence is zero at a boundary, there is no "*jump*" in the normal component of the field.

3 Tangential Components

In the previous section we showed that flux sources at a boundary will change the value of the field's *normal component* when moving from one region to another, figure [3a].

As we will show in this section, circulation sources at a boundary will alter the *tangential component* of the vector field.

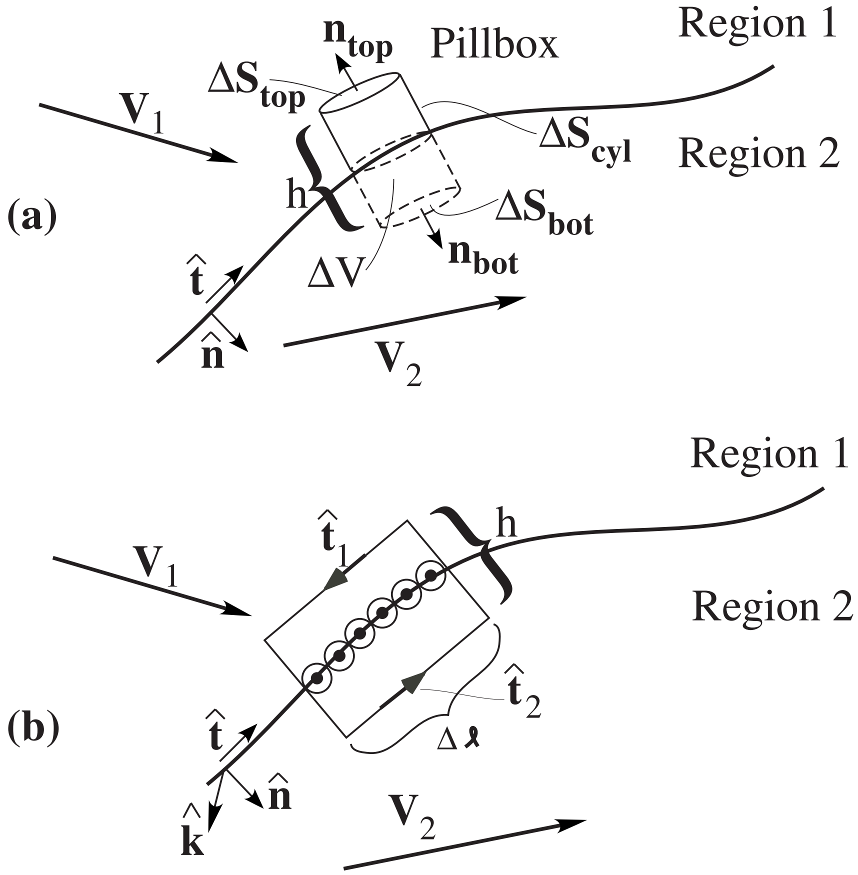

Figure 3: Geometry at a boundary.

To show how the normal component changes for flux sources at a boundary, figure [3a], we used an infinitesimal control surface together with the divergence theorem. Along the same line, to show the relationship between circulation sources on the boundary and the subsequent *jump* in the field's tangential component we will use an infinitesimal control loop, figure [3b], and Stokes' theorem.

Similarly to our approach in the previous section we *assume* that the boundary can be adequately modelled as being infinitely sharp. To investigate how the vector field behaves at the boundary we will make use of the control loop method. Our control loop is positioned across the boundary, figure [3b]. The boundary surface can be arbitrary, our only requirement is that the surface is flat with constant physical parameters and a constant vector field [1] over the *infinitesimal* length $\Delta\ell$ (which again is an order of magnitude larger than atomic distances but much smaller than the bulk dimensions). The control loop is conveniently chosen to be that of a rectangle with height h and length $\Delta\ell$.

The circulation through the loop is

$$\oint_L \tilde{\mathbf{V}} \cdot \mathrm{d}\ell = \tilde{\mathbf{V}}_1 \cdot \hat{\mathbf{t}}_1 \Delta\ell + \tilde{\mathbf{V}}_2 \cdot \hat{\mathbf{t}}_2 \Delta\ell + H \tag{8}$$

where H is the contribution to the line integral from the ends of the path. To proceed we need to introduce the unit vectors for the various surfaces involved in this analysis. We have $\hat{\mathbf{t}}_2, \hat{\mathbf{t}}_1$, for the bottom and top of the rectangle, respectively. $\hat{\mathbf{t}}$ is the tangential vector for the boundary, $\hat{\mathbf{n}}$ is the normal vector of the boundary, and $\hat{\mathbf{k}}$ is the normal vector to the loop, pointing out of the paper, figure 3. Note that,

$$\hat{\mathbf{k}} = \hat{\mathbf{n}} \times \hat{\mathbf{t}}, \qquad \hat{\mathbf{t}} = \hat{\mathbf{k}} \times \hat{\mathbf{n}}, \qquad \hat{\mathbf{n}} = \hat{\mathbf{t}} \times \hat{\mathbf{k}} \tag{9}$$

Also note that the vector area for the rectangle is $\hat{\mathbf{k}}\, h\, \Delta\ell$. From Stokes' theorem we have

$$\oint_L \tilde{\mathbf{V}} \cdot \mathrm{d}\ell = \int\!\!\int_S \tilde{\boldsymbol{\gamma}} \cdot \mathrm{d}\mathbf{s} = \tilde{\boldsymbol{\gamma}} \cdot \hat{\mathbf{k}}\, h\, \Delta\ell \tag{10}$$

Collecting everything that we have so far

$$\begin{aligned}\oint_L \tilde{\mathbf{V}} \cdot \mathrm{d}\ell &= \tilde{\mathbf{V}}_2 \cdot \hat{\mathbf{t}}_2 \Delta\ell + \tilde{\mathbf{V}}_1 \cdot \hat{\mathbf{t}}_1 \Delta\ell + H = \\ &= \hat{\mathbf{t}} \cdot (\tilde{\mathbf{V}}_2 - \tilde{\mathbf{V}}_1) + H = \tilde{\boldsymbol{\gamma}} \cdot \hat{\mathbf{k}}\, h\, \Delta\ell\end{aligned} \tag{11}$$

If we replace $\hat{\mathbf{t}}$ with $\hat{\mathbf{k}} \times \hat{\mathbf{n}}$ we can rewrite equation [11] as

$$\hat{\mathbf{k}} \cdot [\hat{\mathbf{n}} \times (\tilde{\mathbf{V}}_2 - \tilde{\mathbf{V}}_1) - h\,\tilde{\boldsymbol{\gamma}}]\Delta\ell + H = 0 \tag{12}$$

Using the same procedure as for flux sources in chapter 4, we let the loop height $h \to 0$ while we keep $\delta\ell$ constant. Since H is obviously proportional to h its contribution will disappear and we are left with

$$\hat{\mathbf{k}} \cdot [\hat{\mathbf{n}} \times (\tilde{\mathbf{V}}_2 - \tilde{\mathbf{V}}_1) - \lim_{h\to 0} h\,\tilde{\boldsymbol{\gamma}}] = 0 \tag{13}$$

This equation *has* to be zero for **any** $\hat{\mathbf{k}}$, i.e. equation [13] can only be true *iff* the expression in the bracket is zero. This leads to

$$\hat{\mathbf{n}} \times (\tilde{\mathbf{V}}_2 - \tilde{\mathbf{V}}_1) = \lim_{h\to 0} h\,\tilde{\boldsymbol{\gamma}} = \lim_{h\to 0} h\,(\nabla \times \tilde{\mathbf{V}}) \tag{14}$$

To proceed we decompose our vector field into a normal and a tangential component,

[1] or more accurately, a constant *average* vector field

$$\tilde{\mathbf{V}} = \tilde{V}_n \hat{\mathbf{n}} + \tilde{V}_t \hat{\mathbf{t}} \tag{15}$$

Consequently,

$$\hat{\mathbf{n}} \times \tilde{\mathbf{V}} = \tilde{\mathbf{V}}_n \hat{\mathbf{n}} \times \hat{\mathbf{n}} + \hat{\mathbf{n}} \times \tilde{\mathbf{V}}_t = \hat{\mathbf{n}} \times \tilde{\mathbf{V}}_t \tag{16}$$

Inserting this into our previous equation we obtain

$$\hat{\mathbf{n}} \times (\tilde{\mathbf{V}}_{2t} - \tilde{\mathbf{V}}_{1t}) = \lim_{h \to 0} h \, (\nabla \times \tilde{\mathbf{V}}) \tag{17}$$

Using our vector algebra skills from chapter 2 we can further refine equation [17] by noticing that

$$(\hat{\mathbf{n}} \times \tilde{\mathbf{V}}) \times \hat{\mathbf{n}} = \tilde{\mathbf{V}} - \tilde{\mathbf{V}}_n \hat{\mathbf{n}} = \tilde{\mathbf{V}}_t \tag{18}$$

Finally, we arrive at

$$\tilde{\mathbf{V}}_{2t} - \tilde{\mathbf{V}}_{1t} = \lim_{h \to 0} h \, [(\nabla \times \tilde{\mathbf{V}}) \times \hat{\mathbf{n}}] \tag{19}$$

In summary,

$$\boxed{\begin{aligned} \tilde{\mathbf{V}}_{2n} - \tilde{\mathbf{V}}_{1n} &= \lim_{h \to 0} h \, (\nabla \cdot \tilde{\mathbf{V}}) && \text{(20a)} \\ \tilde{\mathbf{V}}_{2t} - \tilde{\mathbf{V}}_{1t} &= \lim_{h \to 0} h \, [(\nabla \times \tilde{\mathbf{V}}) \times \hat{\mathbf{n}}] && \text{(20b)} \end{aligned}}$$