NORGES TEKNISK- NATURVITENSKAPELIGE UNIVERSITET
INSTITUTT FOR TEKNISK KYBERNETIKK

Faglig kontakt / *contact person*:

Navn: Morten D. Pedersen
Tlf.: 41602135

Eksamen - TTK 4115 Lineær systemteori
Exam - TTK 4115 Linear systems theory

15. desember 2014, 09:00 – 13:00

Hjelpemidler: D - Ingen trykte eller håndskrevne hjelpemidler tillatt. Bestemt, enkel kalkulator tillatt.

Supporting materials: D - No printed or handwritten material allowed. Specific, simple calculator allowed.

Merk at ingen deloppgave avhenger av at du har greid å løse noen av de andre deloppgavene. Oppgitt informasjon fra tidligere deloppgaver skal være tilstrekkelig for å komme videre.
Note that no parts of this problem assume that you have solved any of the of previous parts. The given information from previous parts should be sufficient to move on.

Oppgave 1 (15 %)

La et system være gitt ved:
Let a system be given by:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

a) (3%)
Er systemet observerbart?
Is the system observable?

b) (3%)
Er systemet styrbart?
Is the system controllable?

c) (4%)
La $u(t) = -\mathbf{K}\mathbf{x}(t)$, hvor $\mathbf{x}(t) = [x_1,\ x_2]^\mathsf{T}$. Velg $\mathbf{K}$ slik at polene til det tilbakekoblede systemet blir:
Let $u(t) = -\mathbf{K}\mathbf{x}(t)$, where $\mathbf{x}(t) = [x_1,\ x_2]^\mathsf{T}$. Choose $\mathbf{K}$ such that the poles of the closed-loop system become:

$$\lambda_1 = -1, \quad \lambda_2 = -1$$

d) (5%)
La nå forsterkningsmatrisen være gitt ved:
Let now the feedback gain be given by:

$$\mathbf{K} = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

Bruk Lyapunovs ligning til å undersøke om denne forsterkningsmatrisen gir et asymptotisk stabilt system[1].
Use the Lyapunov equation to examine if this feedback gain gives an asymptotically stable system[1].

[1] Hint: $\mathbf{N} = \mathbf{I}$

Oppgave 2 (20 %)

PID regulatoren vist under kan brukes til å styre et bredt utvalg av prosesser.
The following PID controller may be used to control a wide variety of systems.

$$\hat{u}(s) = \left(K_P + \frac{K_I}{s} + \frac{K_D s}{\tau s + 1} \right) \hat{e}(s)$$

Her er avviket definert som $e(t) = r(t) - y(t)$, hvor $r(t)$ er referansen og $y(t)$ er utgangen på prosessen. Avviket fungerer som inngangen til regulatoren. Utgangen til PID regulatoren, $u(t)$, er pådraget til prosessen. Anta at du har valgt forsterkningene K_P, K_I og K_D samt tidskontanten τ ved hjelp av frekvensdomeneteknikker.
Here, the error is defined as $e(t) = r(t) - y(t)$, where $r(t)$ is the reference and $y(t)$ is the output of the system. The error acts as the input to the controller. The output $u(t)$ of the PID controller is the input to the system. Suppose that you have selected the gains K_P, K_I and K_D as well as the time-constant τ using frequency-domain techniques.

a) (10%)
Den fysiske implementasjonen av din regulator er i tidsdomenet. La en realisering av regulatoren være gitt ved:
The physical implementation of your regulator will be in the time-domain. Let a realization of your regulator be given by:

$$\begin{aligned} \dot{\mathbf{z}}(t) &= \mathbf{A}_c \mathbf{z}(t) + \mathbf{B}_c e(t) \\ u(t) &= \mathbf{C}_c \mathbf{z}(t) + D_c e(t) \end{aligned}$$

Identifisér $\mathbf{A}_c$, $\mathbf{B}_c$, $\mathbf{C}_c$ og D_c. Sørg for at realiseringen er minimal, og indikér dimensjonene til matrisene i tilstandsrommodellen.
Determine the matrices $\mathbf{A}_c$, $\mathbf{B}_c$, $\mathbf{C}_c$ and D_c. Make sure that the realization is minimal. Moreover, indicate the dimensions of the matrices in the state-space model.

b) (2%)
Finnes det mer enn ett (rett) valg av matriser i oppgave a)? Rettferdiggjør ditt svar.
Are there more than one (correct) choice of matrices in assignment a)? Justify your answer.

c) (3%)
Regulatoren nedenfor blir foreslått som et alternativ. Er det mulig å finne en realisering slik som den i oppgave a) for denne regulatoren? Rettferdiggjør ditt svar.
Suppose now that the controller below is proposed as an alternative. Is it possible to find a realization like the one in a) for this controller? Justify your answer.

$$\hat{u}(s) = \left(K_P + \frac{K_I}{s} + K_D s \right) \hat{e}(s)$$

d) (4%)
La prosessen som skal reguleres være gitt ved:
Let the regulated system be given by:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}_s \mathbf{x}(t) + \mathbf{B}_s u(t) \\ y(t) &= \mathbf{C}_s \mathbf{x}(t) \end{aligned}$$

Vis at egenverdiene til prosessen i lukket sløyfe med regulatoren har egenverdier som løser ligningen:
Show that the eigenvalues of the system in closed-loop with the controller can be found by solving the equation:

$$\begin{vmatrix} \mathbf{A}_s - D_c \mathbf{B}_s \mathbf{C}_s - \lambda \mathbf{I} & \mathbf{B}_s \mathbf{C}_c \\ -\mathbf{B}_c \mathbf{C}_s & \mathbf{A}_c - \lambda \mathbf{I} \end{vmatrix} = 0$$

e) (3%)
Tegn et blokkdiagram av prosessen i lukket sløyfe hvor du bruker matrisene fra regulator og prosessmodeller samt integrator blokker $1/s$. Sørg for å indikere de forskjellige signalene.
Draw a block-diagram of the closed loop plant using the matrices of the regulator and plant models as well as integrator blocks $1/s$. Make sure to denote the various signals.

Oppgave 3 (15 %)

Betrakt nå systemet med inngang $u(t)$ og utgang $y(t)$ beskrevet av differensialligningen:
Consider the system with input $u(t)$ and output $y(t)$ described by the differential equation:

$$\ddot{y}(t) + 4y(t) = \ddot{u}(t) + 2u(t)$$

a) (5%)
Finn en tilstandsrommodell på formen:
Derive a state-space equation of the form:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\ y(t) &= \mathbf{C}\mathbf{x}(t) + Du(t) \end{aligned}$$

med tilstandsvektor
with state vector

$$\mathbf{x}(t) = \begin{bmatrix} \dot{y}(t) - \dot{u}(t) \\ y(t) - u(t) \end{bmatrix}$$

b) (6%)
Finn impulsresponsen til systemet.
Determine the impulse response of the system.

c) (4%)
Er systemet BIBO stabilt?
Is the system BIBO stable?

Oppgave 4 (5 %)

Betrakt systemet
Consider the system

$$\begin{aligned} \dot{x}(t) &= -3x(t) + u(t) \\ y(t) &= 2x(t) \end{aligned}$$

med pådrag
with input

$$u(t) = \begin{cases} 1, & \text{if } 0 \leq t < 1 \\ 0, & \text{if } t \geq 1 \end{cases}$$

La $x(0) = 0$. Hva er utgangen $y(t)$ ved tiden $t = 2$?
Suppose that $x(0) = 0$. What is the output $y(t)$ at time $t = 2$?

Oppgave 5 (25 %)

Betrakt systemet
Consider the system

$$\dot{x}(t) = -x(t) + 2w(t)$$
$$z(t) = x(t) + v(t)$$

hvor $w(t)$ og $v(t)$ er uavhengige stokastiske prosesser med spektraltetthet
where $w(t)$ and $v(t)$ are independent stochastic processes with power spectral densities

$$S_w(j\omega) = 1, \qquad S_v(j\omega) = 2$$

a) (5%)
Hva kalles prosessene $w(t)$ og $v(t)$ og hvilke egenskaper har de?
What are the processes $w(t)$ and $v(t)$ called and what properties do they have?

b) (3%)
Finn autokorrelasjonsfunksjonene $R_w(\tau)$ for prosessen $w(t)$ og $R_v(\tau)$ for prosessen $v(t)$.
Determine the autocorrelation functions $R_w(\tau)$ of the process $w(t)$ and $R_v(\tau)$ of the process $v(t)$.

c) (7%)
Finn den stasjonære spektraltetthetsfunksjonen $S_z(j\omega)$ for utgangen $z(t)$.
Determine the stationary power spectral density $S_z(j\omega)$ of the output $z(t)$.

d) (5%)
Vi skal bruke et Kalman-filter i kontinuerlig tid til å estimere tilstanden $x(t)$. La den assosierte estimeringsfeilens kovariansmatrise være betegnet $\hat{P}(t)$. Vis at $\hat{P}(t)$ tilfredstiller differensialligningen under vha. formlene i appendix.
We use a continuous-time Kalman filter to estimate the state $x(t)$. Let the associated error covariance matrix be denoted by $\hat{P}(t)$. Using the formulas in the appendix, show that $\hat{P}(t)$ satisfies the differential equation

$$\dot{\hat{P}}(t) = -2\hat{P}(t) - \frac{1}{2}\hat{P}^2(t) + 4$$

Bruk $\tilde{Q} = 1$ og $\tilde{R} = 2$ i dine beregninger.
Use $\tilde{Q} = 1$ and $\tilde{R} = 2$ for your calculations.

e) (5%)
Finn stasjonærverdien til estimeringsfeilens kovariansmatrise $\hat{P}(t)$.
Determine the stationary value of the error covariance matrix $\hat{P}(t)$.

f) (5%)
La initialbetingelsen være $\hat{P}(0) = 2\sqrt{3} - 2$. Finn Kalmanforsterkningen $K(t)$.
Let the intial condition be $\hat{P}(0) = 2\sqrt{3} - 2$. *Determine the corresponding Kalman gain* $K(t)$.

Oppgave 6 (20 %)

La v_k være en normalfordelt hvitstøyprosess med forventningsverdi null og varians R. La en måleserie være gitt ved
Let v_k *be a normally distributed zero-mean white-noise process with variance* R. *Consider the measurements*

$$z_k = x + v_k$$

hvor x er en ukjent konstant. Et estimat av x er gitt ved
where x *is an unknown constant. An estimate of* x *is given by*

$$\hat{x}_k = \frac{1}{k+1} \sum_{i=0}^{k} z_i$$

a) (5%)
Regn ut forventningsverdien $E[\hat{x}_k]$ for estimatet $\hat{x}_k$.
Calculate the expected value $E[\hat{x}_k]$ *of the estimate* $\hat{x}_k$.

b) (5%)
Finn forsterkningen K_{k+1} slik at
Determine the gain K_{k+1}, *such that*

$$\hat{x}_{k+1} = \hat{x}_k + K_{k+1}(z_{k+1} - \hat{x}_k)$$

c) (5%)
Vis at estimeringsfeilens kovarians $\hat{P}_k = E[(x - \hat{x}_k)^2]$ er gitt ved
Show that the error covariance $\hat{P}_k = E[(x - \hat{x}_k)^2]$ *is given by*

$$\hat{P}_k = \frac{1}{k+1} R$$

d) (5%)
Vis at
Show that

$$\hat{P}_{k+1} = \left(\frac{k+1}{k+2}\right)^2 \hat{P}_k + \frac{1}{(k+2)^2} R.$$

Vedlegg til eksamen (noen nyttige formler og uttrykk):
Appendix to the exam (some useful formulas and expressions):

Solutions:

$$\begin{aligned}\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau \\ \mathbf{x}[k] &= \mathbf{A}^k\mathbf{x}[0] + \sum_{m=0}^{k-1}\mathbf{A}^{k-1-m}\mathbf{B}\mathbf{u}[m]\end{aligned}$$

Controllability/Observability:

$$\begin{aligned}\mathcal{C} &= [\mathbf{B},\ \mathbf{AB},\ \mathbf{A}^2\mathbf{B},\ \cdots,\ \mathbf{A}^{n-1}\mathbf{B}] \\ \mathcal{O} &= \begin{bmatrix}\mathbf{C}\\ \mathbf{CA}\\ \mathbf{CA}^2\\ \vdots\\ \mathbf{CA}^{n-1}\end{bmatrix}\end{aligned}$$

Realization:

$$\begin{aligned}\mathbf{G}(s) &= \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D} \\ \mathbf{G}(s) &= \mathbf{G}(\infty)+\mathbf{G}_{sp}(s) \\ d(s) &= s^r+\alpha_1 s^{r-1}+\cdots+\alpha_{r-1}s+\alpha_r \\ \mathbf{G}_{sp}(s) &= \frac{1}{d(s)}[\mathbf{N}_1 s^{r-1}+\mathbf{N}_2 s^{r-2}+\cdots+\mathbf{N}_{r-1}s+\mathbf{N}_r] \\ \dot{\mathbf{x}} &= \begin{bmatrix}-\alpha_1\mathbf{I}_p & -\alpha_2\mathbf{I}_p & \cdots & -\alpha_{r-1}\mathbf{I}_p & -\alpha_r\mathbf{I}_p\\ \mathbf{I}_p & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_p & \cdots & \mathbf{0} & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I}_p & \mathbf{0}\end{bmatrix}\mathbf{x}+\begin{bmatrix}\mathbf{I}_p\\ \mathbf{0}\\ \mathbf{0}\\ \vdots\\ \mathbf{0}\end{bmatrix}\mathbf{u} \\ \mathbf{y} &= [\ \mathbf{N}_1\ \ \mathbf{N}_2\ \ \cdots\ \ \mathbf{N}_{r-1}\ \ \mathbf{N}_r\ ]\mathbf{x}+\mathbf{G}(\infty)\mathbf{u}\end{aligned}$$

LQR:

$$J_{\mathsf{LQR}} = \int_0^\infty \mathbf{x}^\mathsf{T}(t)\mathbf{Q}\mathbf{x}(t)+\mathbf{u}^\mathsf{T}(t)\mathbf{R}\mathbf{u}(t)\,dt$$

$$\mathbf{A}^\mathsf{T}\mathbf{P}+\mathbf{PA}+\mathbf{Q}-\mathbf{PBR}^{-1}\mathbf{B}^\mathsf{T}\mathbf{P}=\mathbf{0}$$

$$\mathbf{u}(t) = -\mathbf{R}^{-1}\mathbf{B}^\mathsf{T}\mathbf{P}\mathbf{x}(t)$$

Lyapunov equation:

$$\mathbf{A}^{\mathsf{T}}\mathbf{M} + \mathbf{MA} = -\mathbf{N}$$

Discrete-time Kalman filter:

$$\begin{aligned}
\underline{\bar{x}}_{k+1} &= \Phi \underline{\hat{x}}_k; \quad \underline{\hat{x}}_0 \text{ given;} \\
\bar{P}_{k+1} &= \Phi \hat{P}_k \Phi^T + \Gamma Q \Gamma^T; \quad \hat{P}_0 \text{ given} \\
\underline{\hat{x}}_k &= \underline{\bar{x}}_k + K_k \left(\underline{z}_k - H \underline{\bar{x}}_k \right) \\
\hat{P}_k &= \left(I - K_k H \right) \bar{P}_k \left(I - K_k H \right)^T + K_k R K_k^T \\
K_k &= \bar{P}_k H^T \left(H \bar{P}_k H^T + R \right)^{-1}
\end{aligned}$$

Continuous-time Kalman filter:

$$\begin{aligned}
\dot{\underline{\hat{x}}}(t) &= F \underline{\hat{x}}(t) + K(t) \left(\underline{z}(t) - H \underline{\hat{x}}(t) \right); \quad \underline{\hat{x}}(t_0) \text{ given} \\
\dot{\hat{P}}(t) &= F \hat{P}(t) + \hat{P}(t) F^T + G \tilde{Q} G^T - \hat{P}(t) H^T \tilde{R}^{-1} H \hat{P}(t); \quad \hat{P}(t_0) \text{ given} \\
K(t) &= \hat{P}(t) H^T \tilde{R}^{-1}
\end{aligned}$$

Auto-correlation:

$$\begin{aligned}
R_X(\tau) &= E[X(t)X(t+\tau)] \text{ (Stationary process)} \\
R_X(t_1, t_2) &= E[X(t_1)X(t_2)] \text{ (Non-stationary process)} \\
Y(s) &= G(s)U(s) \Rightarrow \\
R_y(t_1, t_2) &= E[y(t_1)y(t_2)] = \int_0^{t_2} \int_0^{t_1} g(\xi) g(\eta) E\left[u(t_1 - \xi) u(t_2 - \eta) \right] d\xi d\eta
\end{aligned}$$

Laplace transform pairs:

$f(t)$	$\Longleftrightarrow$	$F(s)$
$\delta(t)$	$\Longleftrightarrow$	1
1	$\Longleftrightarrow$	$\frac{1}{s}$
e^{-at}	$\Longleftrightarrow$	$\frac{1}{s+a}$
t	$\Longleftrightarrow$	$\frac{1}{s^2}$
t^2	$\Longleftrightarrow$	$\frac{2}{s^3}$
te^{-at}	$\Longleftrightarrow$	$\frac{1}{(s+a)^2}$
$\sin \omega t$	$\Longleftrightarrow$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\Longleftrightarrow$	$\frac{s}{s^2+\omega^2}$