TMA4120, Øving 1

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G1:

1) $\mathcal{L}(2t+8) = 2\cdot\mathcal{L}(t) + 8\,\mathcal{L}(1)$

$$= \frac{2}{s^2} + \frac{8}{s}$$

7) $\mathcal{L}(\cos(\omega t+\theta)) =: F(s)$

Using $\cos(\omega t+\theta) = \cos(\omega t)\cos\theta - \sin(\omega t)\sin(\theta)$ we get

$$F(s) = \mathcal{L}\{\cos(\omega t)\cos\theta - \sin(\omega t)\sin\theta\}$$

$$= \cos\theta\,\mathcal{L}\{\cos(\omega t)\} - \sin\theta\,\mathcal{L}\{\sin(\omega t)\}$$

$$= \frac{s\cdot\cos\theta}{s^2+\omega^2} - \frac{\omega\cdot\sin\theta}{s^2+\omega^2}$$

12) We have the graph $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 1, & 1 \le t < 2 \\ 0, & t \geq 2 \end{cases}$

Using the heaviside function we write this as

$$f(t) = t\cdot(u-u_1) + (u_1-u_2)$$
$$= t\cdot u - t u_1 + u_1 - u_2$$

Taking the Laplace transform of f gives

$$F(s) = \mathcal{L}\{f\}(s)$$
$$= \mathcal{L}\{t \cdot u - t u_1 + u_1 - u_2\}(s)$$

Since $\mathcal{L}\{f(t-a) u_a(t)\} = e^{-as} \mathcal{L}\{f\}$ (t-shift)

we get

$$F(s) = \cancel{e^{-0.5}} \frac{1}{s^2} - e^{-1.5} \frac{1}{s^2} + \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}$$

$$= \frac{1}{s^2} - \frac{e^{-s}}{s^2} + \frac{e^{-s}}{s} - \frac{e^{-2 \cdot s}}{s}$$

19) Since $\sinh x = \frac{e^x - e^{-x}}{2}$

and $\cosh x = \frac{e^x + e^{-x}}{2}$,

we have $e^x = \sinh x + \cosh x$

This means $\mathcal{L}\{e^{at}\} = \mathcal{L}\{\sinh at\} + \mathcal{L}\{\cosh at\}$

From table 6.1 we then get

$$\mathcal{L}\{e^{at}\} = \frac{a}{s^2 - a^2} + \frac{s}{s^2 - a^2}$$
$$= \frac{\cancel{s+a}}{\cancel{(s+a)}(s-a)}$$
$$= \frac{1}{s-a}$$

which is what we wanted to show.

21) a) e^{t^2}

b) $t^t \geq |Me^{kt}|$ for all M and K

(proof below) so by theorem 3 it does not have a Laplace transform.

Proof: $t^t = Me^{kt}$

$\Leftrightarrow t\ln(t) = \ln(M) + Kt$

$\Leftrightarrow \underbrace{t(\ln(t) - K)}_{\text{non-constant}} = \ln M$ ← constant

This a contradiction so $t^t \neq Me^{kt}$ for all M, K

22) $\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} = \mathcal{L}\left\{t^{-\frac{1}{2}}\right\}$

$= \frac{\Gamma(-\frac{1}{2}+1)}{s^{-\frac{1}{2}+1}}$

$= \frac{\Gamma(\frac{1}{2})}{\sqrt{s}}$

$= \frac{\sqrt{\pi}}{\sqrt{s}}$

23) $\mathcal{L}\{f(t)\}(s) = F(s)$

c is a positive constant

We want to compute

$$\mathcal{L}\{f(ct)\}(s)$$

$$= \int_0^\infty f(ct)e^{-st}\,dt$$

Let $\tau = ct$

$$\Rightarrow \quad d\tau = c\,dt, \quad t = \frac{\tau}{c}$$

This gives

$$\mathcal{L}\{f(ct)\}(s) = \int_0^\infty f(\tau)e^{-\frac{s}{c}\tau}\frac{d\tau}{c}$$

$$= \frac{1}{c}F(s/c)$$

which is what we wanted to show.

Since $\mathcal{L}\{\cos t\}(s) = \dfrac{s}{s^2+1}$

we get $\mathcal{L}\{\cos(\omega t)\}(s) = \dfrac{1}{\omega}\dfrac{s/\omega}{(\frac{s}{\omega})^2+1}$

$$= \frac{s}{\omega^2\left(\frac{s^2}{\omega^2}+1\right)} = \frac{s}{s^2+\omega^2}$$

25) $F(s) = \dfrac{0{,}2 \cdot s + 1{,}4}{s^2 + 1{,}96}$

$= 0{,}2 \cdot \dfrac{s}{s^2 + (1{,}4)^2} + \dfrac{1{,}4}{s^2 + (1{,}4)^2}$

We can then see

$f(t) = 0{,}2 \cos(1{,}4t) + \sin(1{,}4t)$

26) $\dfrac{5 \cdot s + 1}{s^2 - 25} = F(s)$

$= 5 \cdot \dfrac{s}{s^2 - 5^2} + \dfrac{1}{5} \cdot \dfrac{5}{s^2 - 5^2}$

We can then see

$f(t) = 5 \cosh(5t) + \frac{1}{5} \sinh(5t)$

41) $F(s) = \dfrac{\pi}{s^2 + 4\pi \cdot s + 3\pi^2}$

Want to factor $s^2 + 4\pi s + 3\pi^2$

$\dfrac{-4\pi \pm \sqrt{16\pi^2 - 4 \cdot 3\pi^2}}{2}$

$= -2\pi \pm \frac{2}{2}\sqrt{4\pi^2 - 3\pi^2}$

$= -2\pi \pm \pi$

So $s^2 + 4\pi s + 3\pi^2 = (s + \pi)(s + 3\pi)$

and $F(s) = \dfrac{\pi}{(s+\pi)(s+3\pi)}$

Want a and b such that

$$\frac{a}{s+\pi} + \frac{b}{s+3\pi} = \frac{\pi}{(s+\pi)(s+3\pi)}$$

$$\Rightarrow a(s+3\pi) + b(s+\pi) = \pi$$

$$\Rightarrow \quad a s + b \cdot s = 0 \quad \Rightarrow a = -b$$

$$a 3\pi + b\pi = \pi$$

$$\Rightarrow (-b)3\pi + b\pi = \pi$$

$$-2b = 1$$

$$b = -1/2$$

$$a = 1/2$$

So $F(s) = \dfrac{1/2}{s+\pi} - \dfrac{1/2}{s+3\pi}$

which gives

$$f(t) = \mathcal{L}^{-1}\{F\} = \underline{\underline{\frac{1}{2}e^{-\pi t} - \frac{1}{2}e^{-3\pi t}}}$$

6.2:

a) $y'' - 3y' + 2y = 4t - 8$, $y(0) = 2$, $y'(0) = 7$

Taking the Laplace transform we get

$$[s^2Y - s\cdot y(0) - y'(0)] - 3[sY - y(0)] + 2Y = \frac{4}{s^2} - \frac{8}{s}$$

$$\Leftrightarrow Y[s^2 - 3s + 2] - 2s - 7 + 6 = \frac{4}{s^2} - \frac{8}{s}$$

$$\Leftrightarrow Y(s-1)(s-2) = \frac{4}{s^2} - \frac{8}{s} + 1 + 2s$$

$$= \frac{4 - 8s + s^2 + 2s^3}{s^2}$$

$$\Leftrightarrow Y = \frac{2\cdot s^3 + s^2 - 8\cdot s + 4}{s^2(s-1)(s-2)}$$

Want to partially factor this so we solve

$$\frac{2s^3 + s^2 - 8\cdot s + 4}{s^2(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s-2}$$

$$\Rightarrow 2s^3 + s^2 - 8s + 4 = As(s-1)(s-2)$$
$$+ B(s-1)(s-2)$$
$$+ Cs^2(s-2)$$
$$+ Ds^2(s-1)$$

(i): $A + C + D = 2$

(ii): $-3A + B - 2C - D = 1$

(iii): $2A - 3B = -8$

(iv): $2B = 4 \Rightarrow B = 2$

$B=2$ + (iii) $\Rightarrow$ $A = -1$

This gives

(i): $C + D = 3$

(ii): $-2C - D = -4$

(i) + (ii) $\Leftrightarrow$ $-C = -1$ $\Leftrightarrow$ $C = 1$

$\Rightarrow$ $D = 2$

So $Y(s) = \frac{-1}{s} + \frac{2}{s^2} + \frac{1}{s-1} + \frac{2}{s-2}$

which means

$y(t) = -1 + 2t + e^t + 2e^{2t}$

27) $\mathcal{L}\left\{\int_0^t g(\tau)d\tau\right\} = \frac{s+8}{s^4+4s^2}$, $f(t) = \int_0^t g(\tau)\,d\tau$

$\Rightarrow \int_0^t g(\tau)d\tau = \mathcal{L}^{-1}\left\{\frac{s+8}{s^4+4s^2}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{s} \cdot \frac{s+8}{s^3+4s}\right\}$

theorem 3 $\Rightarrow$ $g(t) = \mathcal{L}^{-1}\left\{\frac{s+8}{s^3+4s}\right\}$

$s^3 + 4s = s(s^2+4)$

$\frac{s+8}{s(s^2+4)} = \frac{A}{s} + \frac{B\cdot s + C}{s^2+4}$

$\Leftrightarrow s + 8 = A(s^2+4) + Bs^2 + Cs$

$\Rightarrow$ (i) $A + B = 0$

(ii) $C = 1$

(iii) $4A = 8 \Leftrightarrow A = 2$

$B = -2$

$$\frac{s+8}{s(s^2+4)} = \frac{2}{s} + \frac{-2s+1}{s^2+4}$$

$$= \underbrace{\frac{2}{s} - 2\cdot\frac{s}{s^2+2^2} + \frac{1}{2}\cdot\frac{2}{s^2+2^2}}$$

$$g(t) := \mathcal{L}^{-1}\{\cdots\}$$

$$g(t) = 2 - 2\cos 2t + \frac{1}{2}\sin 2t$$

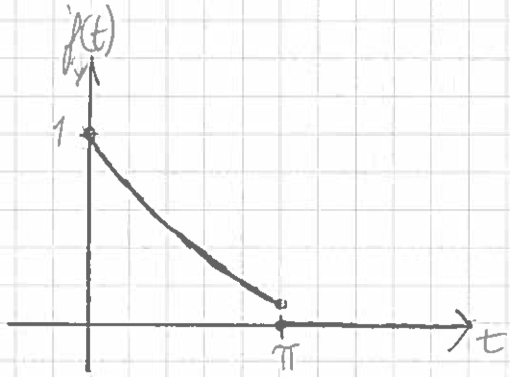
So $f(t) = \int_0^t 2 - 2\cos 2\tau + \frac{1}{2}\sin 2\tau \, d\tau$

$$= \left[2\tau - \sin 2\tau - \frac{1}{4}\cos 2\tau\right]_0^t$$

$$= \underline{\underline{2t - \sin 2t - \frac{1}{4}\cos 2t + \frac{1}{4}}}$$

6.3:

5)



$$\underline{f(t) = e^{-t}\left(1 - u(t-\pi)\right)}$$

$$f(t) = e^{-t} - u(t-\pi)e^{-(t-\pi)}\cdot e^{-\pi}$$

$$\Rightarrow \mathcal{L}\{f\} = \frac{1}{s+1} - \frac{e^{-\pi}\cdot e^{-\pi s}}{s+1}$$

$$= \frac{1}{s+1}\cdot\left(1 - e^{-\pi(s+1)}\right)$$

13) We have

$$\mathcal{L}\{f\} = F(s) = 4\left(1-e^{-\pi s}\right)/(s^2+4)$$

$$= 2\cdot\frac{2}{s^2+2^2} - 2\cdot\frac{2e^{-\pi s}}{s^2+2^2}$$

$$f(t) = 2\cdot\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$-2\,\mathcal{L}^{-1}\left\{\frac{2e^{-\pi s}}{s^2+2^2}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{2e^{-\pi s}}{s^2+2^2}\right\} = u(t-\pi)\,\mathcal{L}^{-1}\left\{\frac{2}{(s-\pi)^2+2^2}\right\}$$

$$= u(t-\pi)\,e^{\pi t}\sin(2t)$$

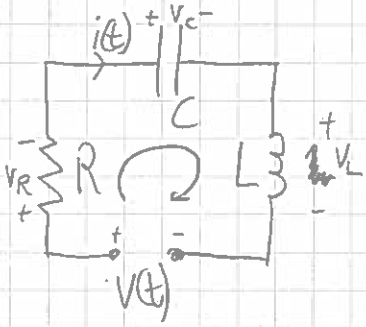
This gives

$$f(t) = 2\sin 2t + e^{\pi t}\sin(2t)\,u(t-\pi)$$

40) $R = 2\,\Omega$, $L = 1\,H$, $C = 0.1\,F$

$V = 255 \sin t \; V$ for $0 < t < 2\pi$

~~$0 < t < 2\pi$~~ and $v = 0$ for $t > 2\pi$



V_R : voltage over R

V_C : voltage over C

V_L : voltage over L

$$V_R + V_C + V_L = v(t)$$

$$\Leftrightarrow \quad R \cdot i(t) + \frac{1}{C}\int_0^t i(\tau)\,d\tau + v_0 + L\frac{di}{dt} = v(t)$$

$$\Leftrightarrow \quad 2i + 10\int_0^t i\,d\tau + \frac{di}{dt} = v(t)$$

$$v(t) = 255 \sin(t)(1 - u(t-2\pi))$$

$$V(s) = \mathcal{L}\{v\}(s) = \frac{255}{s^2+1} - \frac{\cancel{255}e^{-2\pi s}}{s^2+1} = \frac{255}{s^2+1}(1 - e^{-2\pi s})$$

Let $I = \mathcal{L}\{i\}$, then

$$2I + \frac{10I}{s} + sI - \cancel{i(0)} = \frac{255}{s^2+1}(1 - e^{-2\pi s})$$

$$\Leftrightarrow \quad I\left(2 + \frac{10}{s} + s\right) = \frac{255}{s^2+1}(1 - e^{-2\pi s})$$

$$\Leftrightarrow \quad I(s^2 + 2s + 10) = \frac{255 \cdot s}{s^2+1}(1 - e^{-2\pi s})$$

$$I = 255 \cdot \frac{s}{s^2+1} \cdot \frac{1}{s^2+2s+10} \cdot \left(1 - e^{-2\pi s}\right)$$

$$\frac{255 \cdot s}{(s^2+1)(s^2+2s+10)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+10}$$

$$\Rightarrow \quad 255 s = (As+B)(s^2+2s+10) + (Cs+D)(s^2+1)$$

$(s^3 \text{ terms}) \quad A + C = 0 \qquad (i)$

$(s^2 \text{ terms}) \quad 2A + B + D = 0 \qquad (ii)$

$(s^1 \text{ terms}) \quad 10A + 2B + C = 255 \qquad (iii)$

$(\text{constants}) \quad 10B + D = 0 \qquad (iv)$

$(i) \Leftrightarrow A = -C$

$(iv) \Leftrightarrow D = -10B$

$\overset{(ii)}{\Rightarrow} -2C + B - 10B = 0$

$\Leftrightarrow C = -\frac{9}{2}B$

$\overset{(iii)}{\Rightarrow} -10C - \frac{4}{9}C + C = 255$

$\Leftrightarrow C = 255\left(-\frac{9}{85}\right)$

$= -27$

$\Rightarrow \quad B = 6$

$A = 27$

$D = -60$

We get

$$I = \left[\frac{27s+6}{s^2+1} - \frac{27\cdot s+60}{s^2+2s+10}\right]\left(1-e^{-2\pi\cdot s}\right)$$

Want to ~~u~~find a, w such that

$$s^2+2s+10 = (s-a)^2+w^2$$

$$\Leftrightarrow \qquad = s^2-2as+a^2+w^2$$

$a=-1$ and $w=3$ gives

$$(s-a)^2+w^2 = (s+1)^2+3^2 = s^2+2s+10$$

So $$\frac{27\cdot s+60}{s^2+2s+10} = \frac{27\cdot s+60}{(s+1)^2+3^2} = \frac{3}{(s+1)^2+3^2}(9s+20)$$

Expanding I we get

$$I = \left[\frac{27\cdot s}{s^2+1} + \frac{6}{s^2+1} - \frac{27\cdot s}{(s+1)^2+3^2} - \frac{60}{(s+1)^2+3^2}\right]\left(1-e^{-2\pi s}\right)$$

$$= \left[\ \frac{\quad}{\quad}\ \ |\ \ |\ \frac{\quad}{\quad} - 27\,\frac{s+1}{(s+1)^2+3^2} - 11\cdot\frac{3}{(s+1)^2+3^2}\right]$$

$$\cdot\left(1-e^{-2\pi\cdot s}\right)$$

So $$i(t) = \Big[27\cos t + 6\sin t - 27e^{-t}\cos(3t)$$
$$- 11e^{-t}\sin(3t)\Big]$$
$$- u(t-2\pi)\Big[27\cos(t-2\pi) + 6\sin(t-2\pi)$$
$$-27e^{-t+2\pi}\cos(3t-6\pi)$$
$$-11e^{-t+2\pi}\sin(3t-6\pi)\Big]$$

$$i(t) = 27\cos t\,(1 - u(t-2\pi))$$
$$+ 6\sin t\,(1 - u(t-2\pi))$$
$$-27e^{-t}\cos(3t)(1 - e^{2\pi}u(t-2\pi))$$
$$-11e^{-t}\sin(3t)(1 - e^{2\pi}u(t-2\pi))$$

Supl. Prob. A

$y(t)$ solves

$$y'' + y' - 2y = r(t), \; y(0) = 1, \; y'(0) = 1$$

From the graph we see that

$$r(t) = u(t) + u(t-1) - 2u(t-2)$$

This gives $R(s) = \mathcal{L}\{r\}(s)$

$$= \frac{1}{s} + \frac{e^{-s}}{s} - \frac{2e^{-2s}}{s}$$

Taking the Laplace transform we get

$$s^2Y - s\,y(0) - y'(0) + sY - y(0) - 2Y$$
$$= Y(s^2 + s - 2) - (s+2)$$
$$= Y(s-1)(s+2) - (s+2) = R(s)$$

$$\Rightarrow \quad Y = \frac{R(s)}{(s-1)(s+2)} + \frac{\cancel{s+2}}{(s-1)\cancel{(s+2)}}$$

$$= \frac{R(s)}{(s-1)(s+2)} + \frac{1}{s-1}$$

Want A, B, C such that

$$\frac{1}{s(s-1)(s+2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+2}$$

$$\Rightarrow 1 = A(s-1)(s+2) + B\cdot s(s+2) + C s(s-1)$$

$$\begin{aligned} &(s^2 \text{ terms}) & A+B+C &= 0 & &(i)\\ &(s \text{ terms}) & A+2B-C &= 0 & &(ii)\\ &(\text{const. terms}) & -2A &= 1 & &\Rightarrow A = -\tfrac{1}{2} \end{aligned}$$

$$(i)+(ii) \Leftrightarrow 3B = 1 \Rightarrow B = \frac{1}{3}$$

$$\overset{(i)}{\Rightarrow} C = -A - B = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

So $$\frac{R(s)}{(s-1)(s+2)} = \left(\frac{-1/2}{s} + \frac{1/3}{s-1} + \frac{1/6}{s+2}\right)(1 + e^{-s} - 2e^{-2s})$$

$$\mathcal{L}^{-1}\left\{\frac{R(s)}{(s-1)(s+2)}\right\} = -\frac{t}{2} + \frac{1}{3}e^{t} + \frac{1}{6}e^{-2t}$$
$$+ u(t-1)\left[-\frac{1}{2}(t-1) + \frac{1}{3}e^{t-1} + \frac{1}{6}e^{-2t+2}\right]$$
$$-2u(t-2)\left[-\frac{1}{2}(t-2) + \frac{1}{3}e^{t-2} + \frac{1}{6}e^{-2t+4}\right]$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{R(s)}{(s-1)(s+2)}\right\}(t) + e^{t}$$

$$Y(s) = \frac{1}{s(s-1)(s+2)}(1 + e^{-s} - 2e^{-2s}) + \frac{1}{s-1}$$