TTK4135, Excercise 1

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Problem 1

a) $f(x) = x_1 + 2x_2$

$c_1(x) = 2 - x_1^2 - x_2^2 \geq 0$

$c_2(x) = x_2 \geq 0$

It is quite clear that $x^* = \begin{pmatrix} -\sqrt{2} \\ 0 \end{pmatrix}$ is the optimal solution to this.

b) $\mathcal{L}(x, \lambda) = f(x) - \lambda_1 c_1(x) - \lambda_2 c_2(x)$

$= x_1 + 2x_2 - \lambda_1 (2 - x_1^2 - x_2^2) - \lambda_2 x_2$

We require that for some $\lambda^* \geq 0$, we have

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = \begin{pmatrix} 1 + 2\lambda_1^* x_1^* \\ 2 + 2\lambda_1^* x_2 - \lambda_2^* \end{pmatrix} = 0$$

when $x^* = \begin{pmatrix} -\sqrt{2} \\ 0 \end{pmatrix}$.

$\Rightarrow \lambda^* = \begin{pmatrix} \frac{1}{2\sqrt{2}} \\ 2 \end{pmatrix}$.

Note that $\lambda_1^* \geq 0$ and $\lambda_2^* \geq 0$ which is requirement 4 of KKT.

Since $\mathcal{E} = \phi$, we say that the condition $c_i(x^*) = 0$ for $i \in \mathcal{E}$ is satisfied.

We then have to show that condition 3 and 5 hold.

(3) $c_i(x^*) \geq 0 \quad i \in \{1,2\}$

$$c_1\left(\begin{bmatrix} -\sqrt{2} \\ 0 \end{bmatrix}\right) = 2 - (-\sqrt{2})^2 - 0^2 = 0 \geq 0 \checkmark$$

$$c_2\left(\begin{bmatrix} -\sqrt{2} \\ 0 \end{bmatrix}\right) = 0 \geq 0 \qquad \checkmark$$

(5) Since $c_i(x^*) = 0$ for all $i \in \mathcal{E} \cup \mathcal{I}$, we have

$$\lambda_i^* c_i(x^*) = 0 \qquad \checkmark$$

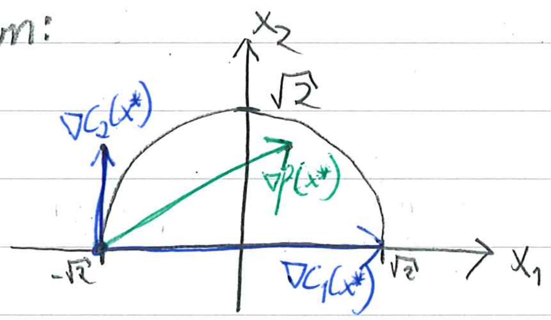
<u>The KKT conditions all hold.</u>

c) Active constraints: c_1 and c_2.

$$\nabla c_1(x^*) = \begin{pmatrix} -2x_1^* \\ -2x_2^* \end{pmatrix} = \begin{pmatrix} 2\sqrt{2} \\ 0 \end{pmatrix}$$

$$\nabla c_2(x^*) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Illustration:



$$\nabla f(x^*) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

d) If any of the Lagrange multipliers had been negative, then we could construct a better solution by following the corresponding gradient into the feasible set.

e) It is convex since $f(x)$ is linear (and thus convex) and Ω is a convex set (half circle).

Problem 2

We then have $f(x) = 2x_1 + x_2$

$$\text{s.t. } c_1(x) = x_1^2 + x_2^2 - 2 = 0 \quad , \mathcal{E} = \{1\}, \mathcal{I} = \emptyset.$$

a) Extreme points given by

$$\nabla f(x) = \lambda_1 \nabla c_1(x)$$

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \lambda_1 \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix}$$

$$2 - 2\lambda_1 x_1 = 0$$
$$1 - 2\lambda_1 x_2 = 0$$
also $x_1^2 + x_2^2 - 2 = 0$

$$\Rightarrow \begin{cases} x_1 = \frac{1}{\lambda_1} \quad , \quad x_2 = \frac{1}{2\lambda_1} \\ \left(\frac{1}{\lambda_1}\right)^2 + \left(\frac{1}{2\lambda_1}\right)^2 - 2 = 0 \end{cases}$$

$$\Rightarrow \quad \frac{5}{4\lambda_1^2} = 2$$

$$\Rightarrow \quad \lambda_1 = \pm\sqrt{\frac{5}{8}}$$

$$\Rightarrow \quad x_1 = \pm\sqrt{\frac{8}{5}} = \pm\frac{2\sqrt{10}}{5}$$

$$x_2 = \pm\frac{\sqrt{10}}{5}$$

Extreme points: $\left(\frac{2\sqrt{10}}{5}, \frac{\sqrt{10}}{5}\right)$

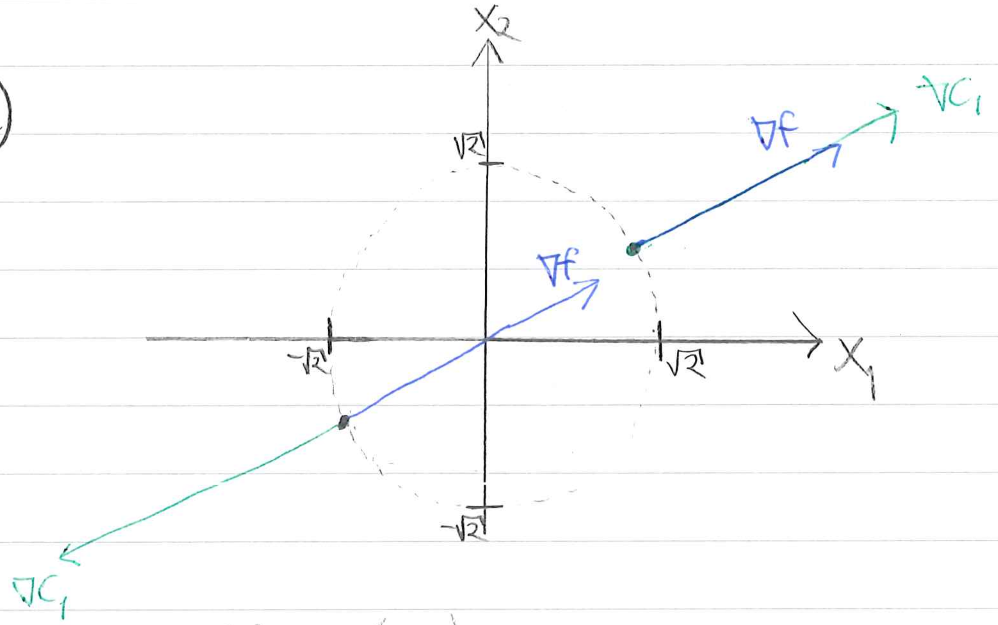
$\left(-\frac{2\sqrt{10}}{5}, \frac{-\sqrt{10}}{5}\right)$

b) By design $\nabla f(x) = \lambda \nabla c_1(x)$ so $\nabla \mathcal{L}(x,\lambda) = 0$ for both the points.
We also have $c_1(x) = 0$ in both the points, and thus also $\lambda_1 c_1(x) = 0$ in both points.

Since $\mathcal{E} = \{1\}$ and $\mathcal{I} = \emptyset$, we allow $\lambda_1 < 0$ so the final KKT condition holds for both $\lambda_1 = +\sqrt{\frac{5}{8}}$ and $\lambda_1 = -\sqrt{\frac{5}{8}}$.

Thus KKT holds for both points.

c)



$$\nabla c_1(x) = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix}$$

$$\nabla c_1\left(\frac{2\sqrt{10}}{5}, \frac{\sqrt{10}}{5}\right) = \begin{pmatrix} \frac{4\sqrt{10}}{5} \\ \frac{2\sqrt{10}}{5} \end{pmatrix} \approx \begin{pmatrix} 2.5 \\ 1.3 \end{pmatrix}$$

$$\nabla c_1\left(-\frac{2\sqrt{10}}{5}, \frac{\sqrt{10}}{5}\right) = \begin{pmatrix} -\frac{4\sqrt{10}}{5} \\ -\frac{2\sqrt{10}}{5} \end{pmatrix}$$

$$\nabla f(x) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

d) The value of the Lagrange multiplier is either $\sqrt{\frac{5}{8}}$ or $-\sqrt{\frac{5}{8}}$. These are both consistent with KKT, since $\lambda_i \geq 0$ only applies to $i \in \mathcal{I}$ and $\mathcal{I} = \emptyset$.

e) $\nabla^2 \mathcal{L}(x^*, \lambda^*) = \nabla^2 \left(2x_1 + x_2 - \lambda_1 (x_1^2 + x_2^2 - 2)\right)$

$$= \begin{pmatrix} \frac{\partial^2}{\partial x_1^2} & \frac{\partial^2}{\partial x_1 \partial x_2} \\ \frac{\partial^2}{\partial x_1 \partial x_2} & \frac{\partial^2}{\partial x_2^2} \end{pmatrix} \mathcal{L}(x^*, \lambda^*)$$

$$= \begin{pmatrix} -2\lambda_1 & 0 \\ 0 & -2\lambda_1 \end{pmatrix}$$

In $\left(-\frac{2\sqrt{10}}{5}, -\frac{\sqrt{10}}{5}\right)$ where $\lambda_1 = -\sqrt{\frac{5}{8}}$ we have

$$\nabla^2 \mathcal{L}(x^*, \lambda^*) = \begin{pmatrix} 2\sqrt{\frac{5}{8}} & 0 \\ 0 & 2\sqrt{\frac{5}{8}} \end{pmatrix} > 0.$$

In $\left(+\frac{2\sqrt{10}}{5}, +\frac{\sqrt{10}}{5}\right)$ where $\lambda_1 = \sqrt{\frac{5}{8}}$ we have

$$\nabla^2 \mathcal{L}(x^*, \lambda^*) = \begin{pmatrix} -2\sqrt{\frac{5}{8}} & 0 \\ 0 & -2\sqrt{\frac{5}{8}} \end{pmatrix} < 0.$$

2nd order condition only holds for $\left(-\frac{2\sqrt{10}}{5}, -\frac{\sqrt{10}}{5}\right)$.
This is sufficient to show that it is an optimal solution.

f) No, since the feasible set is the rim of a circle which is not convex.

Problem 3

$$\min_{x \in \mathbb{R}^2} f(x) = -2x_1 + x_2 \quad \text{s.t.} \quad (1-x_1)^3 - x_2 \geq 0$$
$$x_2 + \frac{1}{4}x_1^2 - 1 \geq 0$$

$$x^* = (0, 1)^T$$

a) At x^* we have $c_1(x^*) = (1-0)^3 - 1 = 0$
and $c_2(x^*) = 1 - 0 - 1 = 0$
so $\mathcal{A}(x^*) = \{1, 2\}$.

$$\nabla c_1(x^*) = \begin{pmatrix} 3(1-x_1^*)^2(-1) \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$$

$$\nabla c_2(x^*) = \begin{pmatrix} \frac{1}{2}x_1^* \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The LICQ holds since $\nabla c_1(x^*)$ and $\nabla c_2(x^*)$ are linearly independent.

b) $\mathcal{L}(x,\lambda) = f(x) - \lambda_1 c_1(x) - \lambda_2 c_2(x)$

$$\nabla \mathcal{L}(x,\lambda) = \begin{pmatrix} -2 \\ 1 \end{pmatrix} - \lambda_1 \begin{pmatrix} -3 \\ -1 \end{pmatrix} - \lambda_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

$$\Rightarrow \lambda_1 = \frac{2}{3}, \quad \lambda_2 = \frac{5}{3}$$

With $\lambda^* = \begin{pmatrix} 2/3 \\ 5/3 \end{pmatrix}$ we have

$$\nabla \mathcal{L}(x^*, \lambda^*) = 0$$

$$c_i(x^*) \geq 0, \quad i \in \mathcal{I}$$

$$\lambda_i^* \geq 0, \quad i \in \mathcal{I}$$

$$\lambda_i^* c_i(x^*) = 0 \quad \text{for all } i \in \mathcal{E} \cup \mathcal{I}$$

$$\left(c_i(x^*) = 0, \quad i \in \mathcal{E} = \emptyset \right)$$

So KKT holds.

d) $\nabla^2 \mathcal{L}(x,\lambda) = \nabla^2 \left[2x_1 + x_2 - \lambda_1 \left((1-x_1)^3 - x_2 \right) - \lambda_2 \left(x_2 + \frac{1}{4} x_1^2 - 1 \right) \right]$

$$= \begin{pmatrix} -6\lambda_1(1-x_1) - \frac{\lambda_2}{2} & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \nabla^2 \mathcal{L}(x^*, \lambda^*) = \begin{pmatrix} 29/6 & 0 \\ 0 & 0 \end{pmatrix} \geq 0$$

The necessary condition holds but not the sufficient since it is not strictly positive definite.

Problem 4

max $f(x) \Leftrightarrow$ min $-f(x)$ so we can write

$$\min_{x \in \mathbb{R}^2} -x_1 x_2 \quad \text{s.t.} \quad c_1(x) = 1 - x_1^2 - x_2^2 \geq 0$$

$$\mathcal{I} = \{1\}, \ \mathcal{E} = \emptyset$$

Solving:

$$\mathcal{L}(x, \lambda) = -x_1 x_2 - \lambda_1 (1 - x_1^2 - x_2^2)$$

$$\Rightarrow \nabla \mathcal{L}(x, \lambda) = \begin{pmatrix} -x_2 + 2\lambda_1 x_1 \\ -x_1 + 2\lambda_1 x_2 \end{pmatrix}.$$

The solution will be when c_1 is active so we also have

$$1 - x_1^2 - x_2^2 = 0$$

The equation $\nabla \mathcal{L}(x^*, \lambda^*) = 0$ gives

$$\cancel{x_1^*} = 2\lambda_1^* x_2 = 2\lambda_1^* 2\lambda_1^* \cancel{x_1^*}$$

$$\Rightarrow \quad \lambda_1^* = \pm \frac{1}{2}$$

$$\Rightarrow \quad \lambda_1^* = + \frac{1}{2} \quad \text{since we want } \lambda_1^* \geq 0.$$

This gives

$$x_2 = x_1$$

$$\Rightarrow \quad 1 - x_1^2 - x_2^2 = 0$$

$$\Leftrightarrow \quad 1 - 2x_1^2 = 0$$

$$\Leftrightarrow \quad x_1 = x_2 = \pm \frac{1}{\sqrt{2}}$$

Secondarder condition:

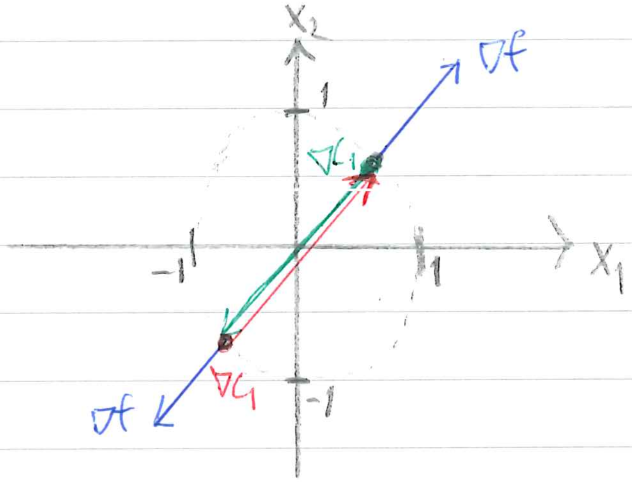
$$\nabla^2 \mathcal{L}(x^*, \lambda^*) = \begin{pmatrix} 2\lambda_1 & -1 \\ -1 & 2\lambda_1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\geq 0$$

The necessary (but not sufficient) condition is fullfilled.

It is clear from the geometry that both $x^* = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $x^* = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ are solutions.



$$\nabla f(x^*) = \begin{pmatrix} x_2^* \\ x_1^* \end{pmatrix} = \pm \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\nabla c_1(x^*) = \begin{pmatrix} -2x_1^* \\ -2x_2^* \end{pmatrix} = \mp \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix}$$