Exercise 7

TTK4130 Modeling and Simulation

Problem 1 (Kinematic modeling of a quadrotor)
In this problem, and the next, we will develop a model for all degrees of freedom for a quadrotor, modeling the quadrotor as a rigid body. See Figure 1 for definition of coordinate systems, and a "free body diagram" with forces and moments acting on the quadrotor.

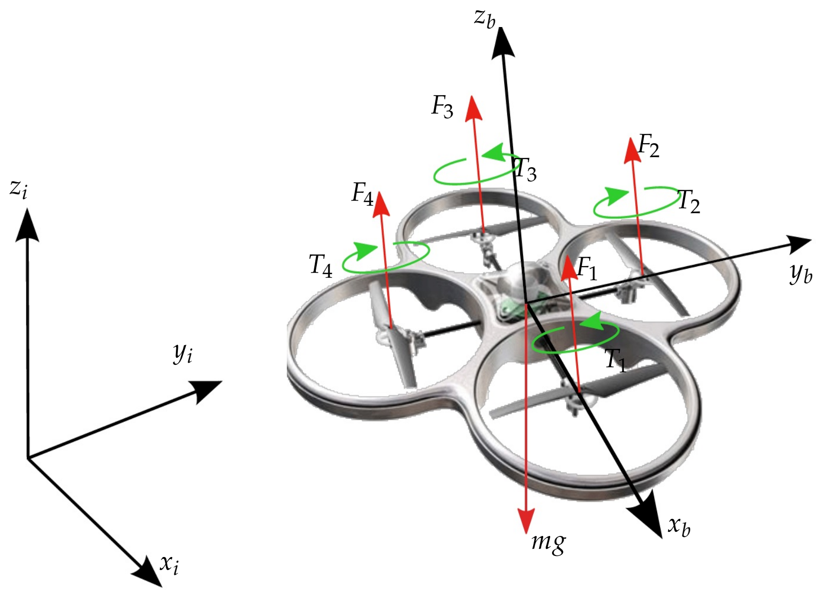

Figure 1: Coordinate systems and forces/moments.

(a) To specify the orientation of the quadrotor, the Z-X-Y Euler angles are sometimes used. These are specified by first a rotation α about the (inertial) z-axis, then β about the intermediate (rotated) x-axis, and finally γ about the body y-axis. Write up an expression for the rotation matrix $\mathbf{R}_b^i = \mathbf{R}_b^i(\boldsymbol{\phi})$ as a function of the Euler angles $\boldsymbol{\phi} = (\alpha, \beta, \gamma)^\mathsf{T}$.

Solution: The rotation matrix is found by writing up the simple rotations in the order they appear:

$$
\begin{aligned}
\mathbf{R}_b^i &= \mathbf{R}_{z,\alpha}\mathbf{R}_{x,\beta}\mathbf{R}_{y,\gamma} \\
&= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{pmatrix} \\
&= \begin{pmatrix} \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma & -\cos\beta\sin\alpha & \cos\alpha\sin\gamma + \cos\gamma\sin\alpha\sin\beta \\ \cos\gamma\sin\alpha + \cos\alpha\sin\beta\sin\gamma & \cos\alpha\cos\beta & \sin\alpha\sin\gamma - \cos\alpha\cos\gamma\sin\beta \\ -\cos\beta\sin\gamma & \sin\beta & \cos\beta\cos\gamma \end{pmatrix}
\end{aligned}
$$

These Euler angles are used for instance in Vijay Kumar's lab at University of Pennsylvania, and Raffaello D'Andrea's lab at ETH. Other groups use other conventions.

(b) Find the kinematic differential equations for this choice of Euler angles. Assume that the angular velocity is given in body-frame. (It is not necessary to perform a matrix inversion for full score.)

Solution: The answer depends on whether the angular velocity is given in the inertial or body system. The latter is more natural, and is assumed in this problem.

The total angular velocity is the sum of the angular velocities of each rotation (6.269), but we need to transform the angular velocities to a common coordinate system when summing. In this case this common system is the body system:

$$\begin{aligned}
\boldsymbol{\omega}_{ib}^{b} &= \mathbf{R}_{y,-\gamma}\mathbf{R}_{x,-\beta}\begin{pmatrix}0\\0\\\dot{\alpha}\end{pmatrix} + \mathbf{R}_{y,-\gamma}\begin{pmatrix}\dot{\beta}\\0\\0\end{pmatrix} + \begin{pmatrix}0\\\dot{\gamma}\\0\end{pmatrix} \\
&= \begin{pmatrix}-\sin\gamma\cos\beta\dot{\alpha} + \cos\gamma\dot{\beta}\\ \sin\beta\dot{\alpha} + \dot{\gamma}\\ \cos\gamma\cos\beta\dot{\alpha} + \sin\gamma\dot{\beta}\end{pmatrix} \\
&= \begin{pmatrix}-\sin\gamma\cos\beta & \cos\gamma & 0\\ \sin\beta & 0 & 1\\ \cos\gamma\cos\beta & \sin\gamma & 0\end{pmatrix}\begin{pmatrix}\dot{\alpha}\\\dot{\beta}\\\dot{\gamma}\end{pmatrix} \\
&= \mathbf{E}_b(\phi)\dot{\boldsymbol{\phi}}
\end{aligned}$$

where $\dot{\boldsymbol{\phi}} = (\dot{\alpha}, \dot{\beta}, \dot{\gamma})^{\mathsf{T}}$ and

$$\mathbf{E}_b(\boldsymbol{\phi}) = \begin{pmatrix}-\sin\gamma\cos\beta & \cos\gamma & 0\\ \sin\beta & 0 & 1\\ \cos\gamma\cos\beta & \sin\gamma & 0\end{pmatrix}.$$

The kinematic differential equations are then

$$\dot{\boldsymbol{\phi}} = \mathbf{E}_b^{-1}(\boldsymbol{\phi})\boldsymbol{\omega}_{ib}^{b}.$$

Compare (6.316) for the choice of Euler angles used in the book (roll-pitch-yaw Euler angles).

Not asked for: The inverse of $\mathbf{E}_b(\boldsymbol{\phi})$ is

$$\mathbf{E}_b^{-1}(\boldsymbol{\phi}) = \frac{1}{\cos\beta}\begin{pmatrix}-\sin\gamma & 0 & \cos\gamma\\ \cos\gamma\cos\beta & 0 & \sin\gamma\cos\beta\\ \sin\beta\sin\gamma & \cos\beta & -\cos\gamma\sin\beta\end{pmatrix}$$

and we see that we have a singularity for $\beta = \pi/2 + k\pi$, $k = 0, \pm1, \pm2, \ldots$.

Problem 2 (Complete dynamic model of a quadrotor)

In this problem, we will continue to develop the complete dynamic model of the quadrotor by modeling the kinetics. The forces and moments acting on the quadrotor are illustrated in Figure 1. The body system has origin in the center of mass, and the quadrotor has mass m and an inertia matrix $\mathbf{M}_{b/c}^{b}$. Note that the moments T_i due to rotation of the rotors give moments acting about the z_b-axis, and that the rotor forces F_i will give cause to moments about the x_b and y_b axis, with "arm" (distance from center of mass to rotor) L for all rotors. Note also that T_i has a "sign" defined in the figure, due to the default direction of rotation of the rotors.

(a) Why is it natural to use the Newton-Euler equations of motions as starting point, rather than the Lagrange equations of motion?

Solution: We will model the quadcopter in all ("six") degrees of freedom, therefore there are no "forces of constraints" to eliminate.

(b) Write up expressions for the force and torque vectors acting on the center of mass, $\mathbf{F}_{bc}^{b}$ and $\mathbf{T}_{bc}^{b}$, decomposed in the body system, as function of the forces and torques defined in Figure 1.

Solution:

$$\mathbf{F}^b_{bc} = \mathbf{R}^b_i(\boldsymbol{\phi}) \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \sum_{i=1}^4 F_i \end{pmatrix}, \quad \mathbf{T}^b_{bc} = \begin{pmatrix} L(F_2 - F_4) \\ L(F_3 - F_1) \\ T_1 - T_2 + T_3 - T_4 \end{pmatrix}$$

(The very observant will have noticed that the sign of T_i in the figure is wrong when you consider the propeller configuration used on this particular quadcopter.)

(c) What are the equations of motion of the quadrotor, on vector form? The components of vectors equations in the answer should amount to 12 first-order differential equations, including the answer from Problem 2(b).

Solution: There are (at least) two different correct answers here, depending on whether the velocity is expressed in body-fixed or inertial coordinates (and whether angular velocity is expressed in body-fixed or inertial coordinates, see Problem 2(b)).

First, the force balance. With velocity in inertial coordinates (which perhaps is simplest and most natural in this case), we can write down

$$\begin{aligned} m\dot{\mathbf{v}}^i_c &= \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix} + \mathbf{R}^i_b(\boldsymbol{\phi}) \begin{pmatrix} 0 \\ 0 \\ \sum_{i=1}^4 F_i \end{pmatrix} \\ \dot{\mathbf{r}}^i_c &= \mathbf{v}^i_c \end{aligned}$$

where the latter equation is a kinematic differential equation.

Alternatively, with velocity in body-fixed coordinates, we get (since $\dot{\mathbf{v}}^i_c = \mathbf{R}^i_b(\boldsymbol{\phi}) \left(\dot{\mathbf{v}}^b_c + (\boldsymbol{\omega}^b_{ib})^\times \mathbf{v}^b_c \right)$)

$$\begin{aligned} m\dot{\mathbf{v}}^b_c &= -m(\boldsymbol{\omega}^b_{ib})^\times \mathbf{v}^b_c + \mathbf{R}^b_i(\boldsymbol{\phi}) \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \sum_{i=1}^4 F_i \end{pmatrix} \\ \dot{\mathbf{r}}^i_c &= \mathbf{R}^i_b(\boldsymbol{\phi})\mathbf{v}^b_c. \end{aligned}$$

Second, the torque balance (Euler's equation) is

$$\mathbf{M}^b_{b/c}\dot{\boldsymbol{\omega}}^b_{ib} = \mathbf{T}^b_{bc} - \boldsymbol{\omega}^b_{ib} \times \left(\mathbf{M}^b_{b/c}\boldsymbol{\omega}^b_{ib} \right).$$

Together with

$$\dot{\boldsymbol{\phi}} = \mathbf{E}_b^{-1}(\boldsymbol{\phi})\boldsymbol{\omega}^b_{ib}$$

from Problem 2(b), this specifies 12 ODEs.

(Problems 3 and 4 are based on the article: Daniel Mellinger, Nathan Michael and Vijay Kumar, *Trajectory generation and control for precise aggressive maneuvers with quadrotors*, The International Journal of Robotics Research 31(5):664–674, 2012.)

Problem 3 (Kinematic modeling of a ladder of a fire truck)

A simulation model for an autonomous fire truck should be found. For test purposes a small scale model of the fire truck was build (Fig. 2).

Figure 2: Fire truck - Small scale model.

The project has several groups working on different parts of the model. Your task is to find the equation of motion of the ladder on the fire truck to precisely control the position of the hose outlet. The only subtask we will consider here is to find the connection between inertial coordinate frame, which is aligned with the truck, and the body frame of the hose.

The movement can be described by three rotation and one translation. Two rotations are in the attachment of the ladder to the truck. One of these rotations rotates the ladder around θ in the horizontal plane, the other rotates the ladder around ψ up or down. Afterward a translation $l(t)$ has to be performed from the attachment to the joint of the hose. The hose joint can be rotated around ϕ up or down.

(a) In the first step the coordinate systems (inertial and body frames) have to be defined. Draw an abstraction (body diagram) of the process that shows the inertial coordinate system with the axes ($\{x_i, y_i, z_i\}$). The origin is in the attachment of the ladder and one axis is aligned with the vehicle. Furthermore, draw the two body coordinate systems with the axes ($\{x_a, y_a, z_a\}$) and ($\{x_b, y_b, z_b\}$). The first body coordinate system is the one after the first rotation with its origin in the attachment. The second has its origin in the joint of the hose at the end of the ladder. Include in your drawing also the angular velocities $\dot{\theta}$, $\dot{\psi}$ and $\dot{\phi}$ and the translation $l(t)$.

Solution: One of the possible configuration is shown in Fig. 3. *Other configuration can also be correct. Important is to have a right-hand coordinate system and describe the rotation correctly in this frame.*

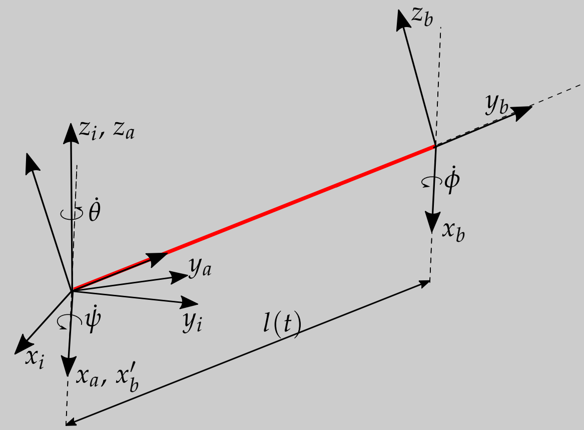

Figure 3: Abstraction of fire truck

(b) Define homogenous transformation matrices to calculate the position of the hose outlet in the inertial frame. Show how the overall transformation matrix can be found. You do not need to do the matrix multiplication.

Solution: To come from the attachment to the hose outlet in the chosen abstraction (task (a)), we have to rotate the system an angle θ about the z_i-axis. Hereafter, we have to rotate an angle ψ about the x_a-axis. Then we have to do a translation $l(t)$ along the y-axis. Finally, we have to rotate an angle ϕ about the x_b-axis. The overall transformation matrix can be calculated with the following homogenous transformation matrices:

$$\mathbf{T}_c^i = \mathbf{T}_a^i \mathbf{T}_{b'}^a \mathbf{T}_b^{b'} \mathbf{T}_c^b, \tag{1}$$

where

$$\mathbf{T}_a^i = \begin{bmatrix} c\theta & -s\theta & 0 & 0 \\ s\theta & c\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{T}_{b'}^a = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\psi & -s\psi & 0 \\ 0 & s\psi & c\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{T}_b^{b'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l(t) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{T}_c^b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\phi & -s\phi & 0 \\ 0 & s\phi & c\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The overall transformation matrix is on the form

$$\mathbf{T}_c^i = \begin{pmatrix} \mathbf{R}_c^i & \mathbf{r}_c^i \\ 0 & 1 \end{pmatrix}, \tag{2}$$

where r_c^i gives the position of the hose outlet in the inertia frame.

(c) Define the position of the attachment relative to the hose outlet. Matrix multiplications are not required.

Solution: To find the position of the attachment relative to the hose outlet the homogenous transformation matrix can be used that describes the hose outlet relative to the inertial frame:

$$\mathbf{r}_i^c = -(\mathbf{R}_c^i)^T \mathbf{r}_c^i. \tag{3}$$