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Exercise 5
TTK4130 Modeling and Simulation

Problem 1 (Lotka-Volterra predator-prey)

An example of the Volterra-Lotka predator-prey model is

$$\dot{u} = u\,(v-3) \tag{1a}$$
$$\dot{v} = v(2-u) \tag{1b}$$

The equations can be used to describe the dynamics of biological systems in which two species interact, one a predator and one its prey. Here, u may represent the number of predators (in thousands), for instance foxes, and v number of preys in thousands, for instance rabbits.

Consider an island where foxes are the only predator, and rabbits the only prey. When the number of rabbits gets large ($v > 3$), the number of foxes will grow. But when the number of foxes becomes large ($u > 2$), the number of rabbits will decrease, which in turn make the number of foxes decrease. This leads to large, periodic variations in the two populations, with a "phase shift" between the number of foxes and rabbits.

Consider the "energy-like" function[1]

$$V = u - 2\ln u + v - 3\ln v \tag{2}$$

(a) Calculate
$$\dot{V} = \frac{\partial V}{\partial u}\dot{u} + \frac{\partial V}{\partial v}\dot{v} \tag{3}$$
and show that V is constant for solutions of the Lotka-Volterra system (1). Make an attempt at an interpretation.

(b) Implement the system in Dymola, and simulate over 20 time units. Use $u = 1$ and $v = 4$ as initial conditions. Comment on the trajectories of u and v, and verify that V is constant. (That is, in addition to implementation of the two Lotka-Volterra equations, implement the equation for V.)

(c) Linearize the system about the equilibrium $\{u = u^* = 2, v = v^* = 3\}$, and calculate eigenvalues. Simulate the system at this equilibrium, and with small deviations from it, and comment on connection between eigenvalues and periods/frequencies.

(d) Simulate this model (initial conditions and horizon as in (b)) in Matlab using Euler, implicit Euler, and the implicit midpoint rule. Re-use code from Exercise 4. Make plots of u and v (for instance in phase plots) and V for the different numerical solutions. Use $h = 0.1$ (for instance, but experiment). Comment!

Problem 2 (Lobatto IIIA)

(a) Show that Lobatto IIIA is A-stable, by showing that the conditions for A-stability is fulfilled.

(b) Is Lobatto IIIA L-stable? Calculations are required.

Problem 3 (Linear stability of Runge-Kutta methods)

Given the Runge-Kutta method

$$\begin{array}{c|cc} 0 & \frac{1}{4} & -\frac{1}{4} \\ \frac{2}{3} & \frac{1}{4} & \frac{5}{12} \\ \hline & \frac{1}{4} & \frac{3}{4} \end{array}$$

[1] Strictly speaking this function is not positive definite. However, we can show that the states of the Lotka-Volterra system will remain positive if initialized positive, and for positive u and v, V will be positive (in general, after addition of a constant). Therefore, think of V as measure of "energy" (total population) in the system.

(a) Is this an explicit or implicit method? Why?

(b) Find the stability function for the method. Is this a Padé-approximation?

(c) Is this method A-stable? L-stable? Justify your answers, not necessarily with computations.

Problem 4 (Index analysis of DAE-systems)

Consider the following DAE-system

$$\dot{x}_1 = x_3 \tag{5}$$
$$\dot{x}_2 = x_1 \tag{6}$$
$$0 = x_1 - u, \tag{7}$$

where u is the input to the system.

(a) Determine the differential and algebraic variables.

(b) Find the degrees of freedom of the considered system.

(c) Determine the index of the DAE-system.

(d) Set up the system such that it has index 1 by using the maximum amount of algebraic equations and minimal amount of differential equations.