Øving 6, Matte 4K

Rendell Cale, gruppe 2

ok

12,1:

3) $u(x,t) = \cos(4t)\sin(2x)$

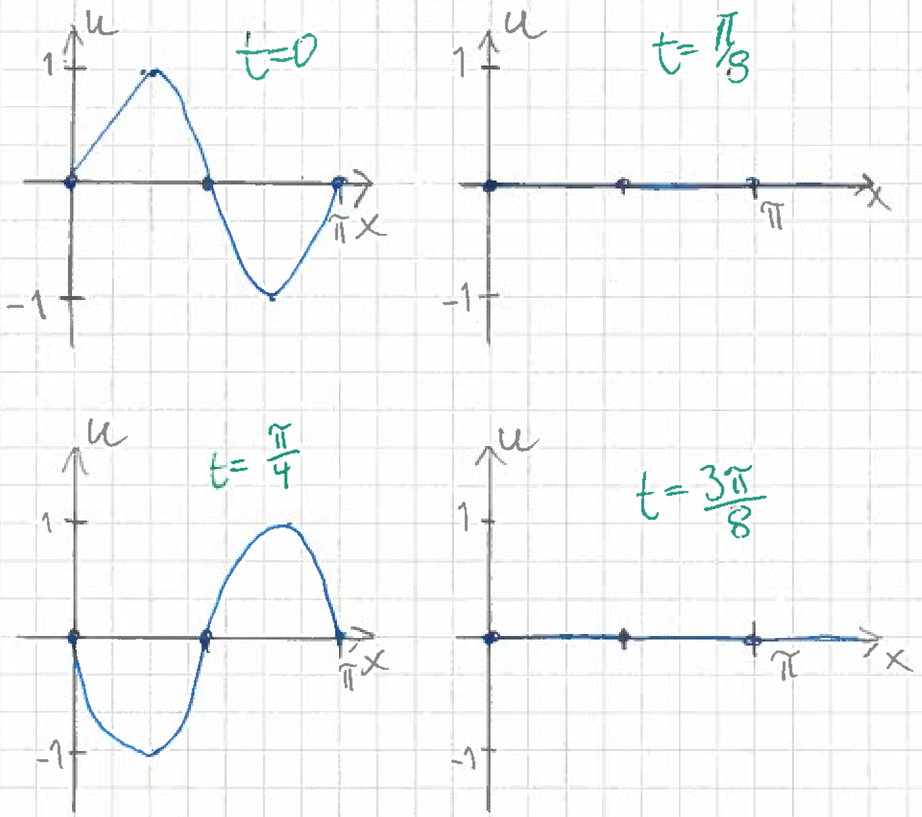
We verify that u solves the wave equation by solving for c.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$-16\cancel{\cos(4t)}\cancel{\sin(2x)} = c^2\left(-4\cancel{\cos(4t)}\cancel{\sin(2x)}\right)$$

$$\Leftrightarrow \quad c^2 = 4$$

$$\Leftrightarrow \quad |c| = 2 \quad \checkmark$$



9) $u(x,t) = e^{-\pi^2 t}\cos(25x)$

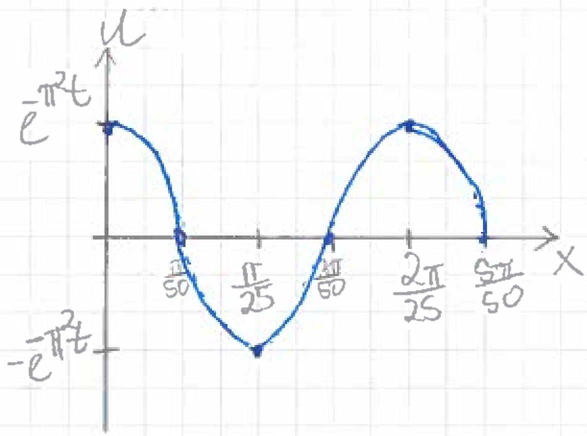
$\frac{\partial u}{\partial t} = -\pi^2 e^{-\pi^2 t}\cos(25x)$

$\frac{\partial^2 u}{\partial x^2} = -25^2 e^{-\pi^2 t}\cos(25x)$

Heat equation:

$-\pi^2 \cancel{e^{-\pi^2 t}}\cancel{\cos(25x)} = -25^2 \cancel{e^{-\pi^2 t}}\cancel{\cos(25x)}\, c^2$

$\Leftrightarrow \quad |c| = \frac{\pi}{25}$ ✓



15) $u(x,y) = a\ln(x^2+y^2) + b$

$$(u_x)_x = \left(\frac{2ax}{x^2+y^2}\right)_x$$

$$= \frac{(x^2+y^2)2a - 2x\,2ax}{(x+y^2)^2}$$

$$= \frac{2ax^2+2ay^2-4ax^2}{(x^2+y^2)^2}$$

$$= 2a\frac{y^2-x^2}{(x^2+y^2)^2}$$

Due to symmetry, we get

$$u_{yy} = 2a\frac{x^2-y^2}{x^2+y^2}$$

$$\Leftarrow \quad = -u_{xx}$$

$$\Leftrightarrow \quad u_{yy} + u_{xx} = 0$$

So u satisfies Laplace's equation.

Want to find a, b such that

$$110 = a\ln(1) + b \quad (1)$$

$$0 = a\ln(100) + b \quad (2)$$

$(1) \Rightarrow \quad b = 110$

$(2) \Leftrightarrow \quad a = \frac{-b}{\ln(100)} = -\frac{110}{\ln(100)} \approx -23{,}89$

$a = -\frac{110}{\ln(100)}$ and $b = 110$ are the desired constants.

12.3:

1) The fundamental mode is given by

$$u_1(x,t) = \left(B_1 \cos(\lambda_1 t) + B_1^* \sin(\lambda_1 t)\right) \sin\left(\frac{\pi}{L} x\right)$$

where $\lambda_1 = \frac{c\pi}{L}$, $c^2 = \frac{T}{\rho}$

The freq. of the fundamental mode decreases when L increases because λ_1 decreases.
When the mass increases (per unit length), c decreases so the frequency also decreases.
If the tension is doubled, then c is increased by a factor of $\sqrt{2}$, and thus the frequency is increased by the same ($\sqrt{2}$) factor.

The contrabass has to produce lower notes (lower freq.) so it has thicker and longer strings.

7) $u(x,0) = Kx(1-x)$

$L=1, \quad c^2 = 1 \quad , \quad u_x(x,0) = 0$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

Since the string is fastened at the end-points, we have

$$u(0,t) = 0 \quad , \quad u(1,t) = 0$$

Using seperation of variables we have

$$u_n(x,t) = F_n(x)\, G_n(t)$$

where $\quad F_n(x) = \sin(n\pi x)$

and $\quad G_n(t) = B_n \cos(n\pi t) + B_n^* \sin(n\pi t)$

where $B_n = 2\int_0^1 Kx(1-x)\sin(\pi n x)\,dx$

and $B_n^* = 2\int_0^1 u_x(x,0)\sin(\pi n x)\,dx = 0$

$$B_n = \cancel{\frac{-2K\cdot(x(1-x))\cos(\pi n x)}{\pi n}\Bigg|_0^1} = 0$$

$$+\frac{2K}{\pi n}\int_0^L (-2x+1)\cos(n\pi x)\,dx$$

$$0 =$$

$$\Leftrightarrow \quad B_n = \left.\cancel{\frac{2K(-2x+1)\sin(\pi n x)}{(\pi n)^2}}\right|_0^1 + \frac{4K}{\pi n^2}\int_0^1 \sin(n\pi x)\,dx$$

$$= -\frac{4K}{n^3\pi^3}\Big[\cos(\pi n x)\Big]_0^1$$

$$= -\frac{4K}{n^3\pi^3}\left((-1)^n - 1\right)$$

n even gives $B_n = 0$

n odd gives $B_n = \dfrac{8K}{n^3\pi^3}$

So $G_{2m+1}(t) = \dfrac{8K}{(2m+1)^3\pi^3}\cos(\pi n t)$, $m = 0, 1, 2, \ldots$

This gives

$$u_{2m+1}(x,t) = F_{2m+1}(x)\, G_{2m+1}(t)$$

$$= \sin((2m+1)\pi x) \cdot \frac{8K}{(2m+1)^3\pi^3}\cos((2m+1)\pi t)$$

which means

$$u(x,t) = \frac{8K}{\pi^3}\left[\frac{\sin(\pi x)\cos(\pi t)}{1^3} + \frac{\sin(3\pi x)\cos(3\pi t)}{3^3} + \frac{\sin(5\pi x)\cos(5\pi x)}{5^3} \ldots\right]$$

15) ~~[illegible]~~

$$\frac{\partial^2 u}{\partial t^2} = -c^2 \frac{\partial^4 u}{\partial x^4} \qquad c^2 = \frac{EI}{\rho A} \qquad (21)$$

Assume $u(x,t) = F(x) \cdot G(t)$, then

$$\frac{\partial^2 u}{\partial t^2} = F(x)\ddot{G}(t)$$

$$\frac{\partial^4 u}{\partial t^4} = F^{(4)}(x)G(t)$$

Subst. into (21) gives

$$F(x)\ddot{G}(t) = -c^2 F^{(4)}(x)\,G(t)$$

$$\Leftrightarrow \quad \frac{F^{(4)}(x)}{F(x)} = -\frac{\ddot{G}(t)}{c^2 G(t)}$$

Since $\frac{F^{(4)}(x)}{F(x)}$ depends only on x and $-\frac{\ddot{G}(t)}{c^2 G(t)}$ depends only on t, both sides must be the same constant, call it k

$$\frac{F^{(4)}(x)}{F(x)} = -\frac{\ddot{G}(t)}{c^2 G(t)} = k.$$

Want to show $k \geq 0$.

If $k < 0$ then $k = -\beta^4$ for some $\beta \in \mathbb{R}$.

$$\Rightarrow \quad \ddot{G}(t) = \beta^4 c^2 G(t) = (\beta^2 c)^2 G(t)$$

$$\Rightarrow \quad G(t) = a e^{\beta^2 c t} + b e^{-\beta^2 c t}$$

$$\Rightarrow \quad |G| \to \infty \text{ as } |t| \to \infty$$

so G is unbounded (above).

$$\frac{F^{(4)}}{F} = -\beta^4$$

$$\iff F^{(4)} = -\beta^4 F$$

$$\Rightarrow \mathcal{L}\{F^{(4)}\} = -\beta^4 \mathcal{L}\{F\}, \quad V(s) = \mathcal{L}\{F\}(s)$$

$$s^4 V - s^3 F(0) - s^2 F'(0) - s^1 F''(0) - F'''(0) = -\beta^4 V$$

Assume $k > 0$, then $k = \beta^4$ for some β.
This gives two equations:

(1) $\ddot{G}(t) = -\beta^4 c^2 G(t)$
(2) $F^{(4)}(x) = \beta^4 F(x)$

(1) has solution $G(t) = a\cos(\beta^2 c t) + b\sin(\beta^2 c t)$

(2) is an ODE of order 4, so we need four linearly independent solutions.
It is quite clear that $\cos\beta x$, $\sin\beta x$, $\cosh(\beta x)$ and $\sinh(\beta x)$ are four candidates and since they are linearly indep. we get

$$F(x) = A\cos(\beta x) + B\sin(\beta x) + C\cosh(\beta x) + D\sinh(\beta x)$$

16) $$\frac{\partial^2 u}{\partial t^2} = -c^2 \frac{\partial^4 u}{\partial x^4} \qquad (*)$$

$$\begin{cases} u_t(x,0) = 0 & (1) \\ u(0,t) = 0 & (2) \\ u(L,t) = 0 & (3) \\ u_{xx}(0,t) = 0 & (4) \\ u_{xx}(L,t) = 0 & (5) \end{cases}$$

$$\begin{cases} u_n = F_n(x)\, G_n(t) \\ F_n(x) = A_n \cos\beta_n x + B_n \sin\beta_n x + C_n \cosh(\beta_n x) + D_n \sinh(\beta_n x) \\ G_n(t) = a_n \cos(c\beta_n^2 t) + b_n \sin(c\beta_n^2 t) \end{cases}$$

(1): $u_t(x,0) = 0 \Leftrightarrow F_n(x)\dot{G}_n(0) = 0$

$\Rightarrow \dot{G}_n(0) = 0$

$\Leftrightarrow \left[-a_n c\beta_n^2 \sin(c\beta_n^2 t) + b_n c\beta_n^2 \cos(c\beta_n^2 t)\right]_{t=0} = 0$

$\Leftrightarrow b_n = 0$

So $G_n(t) = a_n \cos(c\beta_n^2 t)$

(2): $u(0,t) = 0 \Leftrightarrow F_n(0)\,G(t) = 0$

$\Rightarrow F_n(0) = 0$

$\Leftrightarrow A_n + C_n = 0$

(4): $u_{xx}(0,t) = 0 \Leftrightarrow F_n''(0)\,G(t) = 0$

$\Rightarrow F_n''(0) = 0$

$\Leftrightarrow -A_n + C_n = 0$

So $A_n = C_n = 0$

So $F_n(x) = B_n \sin(\beta x) + D_n \sinh(\beta x)$

(3): $u(L,t) = F(L)G(t) = 0$

$\Rightarrow F(L) = 0$

$\Leftrightarrow B_n \sin(\beta_n L) + D_n \sinh(\beta L) = 0 \qquad (3*)$

(5): $u_{xx}(L,t) = F''(L)G(t) = 0$

$\Rightarrow F''(L) = 0$

$\Leftrightarrow -\beta_n^2 B_n \sin(\beta_n L) + \beta_n^2 D_n \sinh(\beta L) = 0$

$\Leftrightarrow -B_n \sin(\beta_n L) + D_n \sinh(\beta_n L) = 0 \qquad (5*)$

Assume $B_n \neq 0$, then

(3*) − (5*):

$2B_n \sin(\beta_n L) = 0$

$\Rightarrow \beta_n = \frac{n\pi}{L} \qquad , n = 0, 1, 2, \ldots$

(3*) + (5*): $D_n = 0$

So $F(x) = B_n \sin\left(\frac{n\pi}{L}x\right)$

and $u_n(x,t) = B_n \sin\left(\frac{n\pi}{L}x\right) a_n \cos\left(c\frac{n\pi}{L}t\right)$

$= c_n \sin\left(\frac{n\pi}{L}x\right) \cos\left(c\frac{n\pi}{L}t\right)$

, $c_n = B_n a_n$

17) $u_n(x,t) = c_n \sin\left(\frac{n\pi}{L}x\right)\cos\left(c\frac{n\pi}{L}t\right)$

$$\Rightarrow \quad u(x,t) = \sum_{n=0}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right)\cos\left(c\frac{n\pi}{L}t\right)$$

$$u(x,0) = f(x) = x(L-x)$$

$$= \sum_{n=0}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right)$$

$$\Rightarrow c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

$$= \frac{2}{L}\int_0^L x(L-x)\sin\left(\frac{n\pi}{L}x\right)dx$$

$$= \frac{2}{L}\cdot\frac{L}{n\pi}\int^L (L-2x)\cos\left(\frac{n\pi}{L}x\right)dx$$

$$= \frac{4L}{n^2\pi^2}\int_0^L \sin\left(\frac{n\pi}{L}x\right)dx$$

$$= \frac{4L^2}{n^3\pi^3}\left(\cos(n\pi)-1\right)$$

$$= \frac{4L^2}{n^3\pi^3}\left((-1)^n-1\right)$$

$$= \frac{-8L^2}{(2m+1)^3\pi^3}, \quad m=0,1,2,\ldots$$

$$\Rightarrow u(x,t) = -\frac{8L^2}{\pi^3}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^3}\cdot\sin\left(\frac{(2m+1)\pi}{L}x\right)\cos\left(c\frac{(2m+1)\pi}{L}t\right)$$

12.6:

5) $L = 10cm$

$A = 1\ cm^2$

$\rho = 10{,}6\ g/cm^3$

$\sigma = 0{,}056\ cal/(g\ °C)$

$K = 1{,}04\ cal/(cm\ sec\ °C)$

$$c^2 = \frac{K}{\sigma\rho} = \frac{1{,}04}{0{,}056 \cdot 10{,}6} \left[\frac{cal/(cm\ sec\ °C)}{cal/(g\ °C)\ g/cm^3}\right]$$

$$= 1{,}75\ \frac{1}{cm^2 sec}$$

$$\Rightarrow c = 1{,}32$$

$$u(x,0) = f(x) = \sin(0{,}1\pi x)$$

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\lambda_n^2 t}$$

$t=0$: $u(x,0) = f(x) \Rightarrow \frac{n}{L} = 0{,}1$

$\Leftrightarrow n = 1$

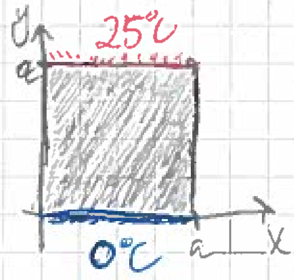
So $B_1 = 1$ and all other $B_n = 0$,

$$\lambda_1^2 = \left(\frac{c\,1\pi}{L}\right)^2 = (0{,}41)^2 = 0{,}17$$

So

$$u(x,t) = \sin(0{,}1\pi x)e^{-0{,}17t}$$

21)



- $u(x,y) = \lim_{t\to\infty} u(x,y,t)$
- $u(x,0,t) = 0$
- $u(x,a,t) = 25$

As $t \to \infty$, the heat distribution should not depend on the x-coordinate, so

$$u(x,y) = G(y)$$

Since $u_{xx} + u_{yy} = 0$, we get

$$\frac{\partial^2 G}{\partial y^2} = 0$$

$$\Rightarrow G(y) = my + b$$

$b = 0$ since $G(0) = b = 0$

$m = \frac{25}{a}$ since $G(a) = ma = 25$

So $u(x,y) = \frac{25}{a}y$, and since we have $a = 24$, we get

$$u(x,y) = \frac{25}{24}y$$

Sup. N

$$|e^{z_0}| = 5$$

$$|e^{2z_0+3i}| = |(e^{z_0})^2 e^{3i}|$$
$$= |e^{z_0}|^2 \quad \text{since } |e^{3i}| = 1$$
$$= 5^2$$
$$= \underline{\underline{25}}$$

Sup. O

a) The superposition principle holds for all linear homogeneous PDE, and since both (*) and (**) are that, it holds for both of them. So u_1+u_2 and u_1-u_2 solves (*) and (**).

b) Let $v(x,t) = u(x,t) - (a+(b-a)x)$ solve (*).
Then $\underline{v(0,t)} = u(0,t) - a \overset{(*)}{=} \underline{0}$
and $\underline{v(1,t)} = u(1,t) - (\cancel{a} + b - \cancel{a}) = u(1,t) - b$
$\overset{(*)}{=} \underline{0}$

So $v(0,t) = 0$ and $v(1,t) = 0$
Since $v_t - v_{xx} \overset{(*)}{=} 0$, $v(x,t)$ satisfies all conditions of (**) so it is a solution.

$v(x,t) = F(x)G(t)$

Then $v_t - v_{xx} = F(x)\dot{G}(t) - F''(x)G(t) = 0$

$\iff \frac{F''}{F} = \frac{\dot{G}}{G} = -p^2$ (proof omitted)

$\iff F'' = -p^2 F$
$\dot{G} = -p^2 G$

$\Rightarrow F(x) = A\cos px + B\sin px$

$\frac{dG}{dt} = -p^2 G$

$\iff \frac{dG}{G} = -p^2 dt$

$\iff \ln|G| = -p^2 t + C$

$G(t) = Ce^{-p^2 t}$

Using $v(0,t) = 0$ we get

$F(0)G(t) = 0$

$\Rightarrow F(0) = A = 0$

$\Rightarrow F(x) = B\sin(px)$

Using $v(1,t) = 0$ we get

$F(1) = 0$

$\Rightarrow B\sin p = 0$

$\Rightarrow p_n = \pi n$, assuming $B \neq 0$
$n = 1, 2, 3, \ldots$

So $v_n(x,t) = B_n \sin(p_n x) e^{-p_n^2 t}$,
were the constant C from G is combined into B_n.

The general solution is then

$$v(x,t) = \sum v_n$$
$$= \sum_{n=1}^{\infty} C_n \sin(p_n x) e^{-p_n^2 t} \quad , \quad p_n = n\pi$$

if $v(x,0) = f(x)$ then

$$\sum_{n=1}^{\infty} C_n \sin(p_n x) = f(x)$$ so the C_n's

are given by the (Sine) Fourier expansion of f.

$$C_n = \int_0^1 f(x) \sin(n\pi x)\,dx$$

c) If $v(x,t)$ is a solution of $(**)$, then
$$u(x,t) = v(x,t) + (a-(b-a)x) \quad , x \in [0,1], t \geq 0$$
is a solution of $(*)$.

We've shown that
$$v(x,t) = \sum_{n=1}^{\infty} C_n \sin(p_n x) e^{-p^2 t} \quad , \quad p_n = n\pi$$
$$C_n = \int_0^1 u(x,0) \sin(n\pi x)\,dx$$

In our case $a = -1$, $b = 1$.

Since $u(x,0) = \sin(\pi x)$, all $C_n = 0$, $n \neq 1$ and
$C_1 = 1$

So $v(x,t) = v_1(x,t) = \sin(\pi x) e^{-\pi^2 t}$

and $\underline{\underline{u(x,t) = \sin(\pi x) e^{-p^2 t} - (1+2x)}}$