NTNU – Trondheim
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Exercise 6
TTK4130 Modeling and Simulation

Problem 1 (Rotation matrices)
Consider the vectors

$$\mathbf{u}^b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{w}^a = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

and the matrix

$$\mathbf{R}_b^a = \begin{pmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Show that $\mathbf{R}_b^a$ is a rotation matrix by showing that it is part of SO(3).

(b) What (simple) rotation does $\mathbf{R}_b^a$ represent?

(c) What is $\mathbf{R}_a^b$?

(d) Compute $\mathbf{u}^a$ and $\mathbf{w}^b$.

(e) Show that

i) $(\mathbf{u}^a)^\mathsf{T} \mathbf{w}^a = \left(\mathbf{u}^b\right)^\mathsf{T} \mathbf{w}^b$

ii) $(\mathbf{u}^a)^\times \mathbf{w}^a = \mathbf{R}_b^a \left(\mathbf{u}^b\right)^\times \mathbf{w}^b$, where $(\mathbf{u}^a)^\times$ is the skew-symmetric form of $\mathbf{u}^a$. Try both a geometric proof (by drawing a figure and argue based on that) and an algebraic proof (using the coordinate transformation rule for matrix representations of a dyadic).

(f) Let $\mathbf{R}_b^a$ be given by

$$\mathbf{R}_b^a = \mathbf{R}_{z,\psi}\mathbf{R}_{y,\theta}\mathbf{R}_{x,\phi} \tag{1}$$

where $\mathbf{R}_{z,\psi}$, $\mathbf{R}_{y,\theta}$ and $\mathbf{R}_{x,\phi}$ are the simple rotations (defined by (6.101)–(6.103) in the book). Calculate the elements in $\mathbf{R}_b^a$ as a function of the Euler angles ψ, θ and ϕ. (You may abbreviate $\cos\psi$ to $c\psi$, $\sin\psi$ to $s\psi$, etc.)

(g) Given the following three rotation matrices:

$$\mathbf{R}_1 = \begin{pmatrix} * & * & * \\ * & -1 & * \\ * & * & 1 \end{pmatrix}, \quad \mathbf{R}_2 = \begin{pmatrix} * & 1 & * \\ * & * & * \\ * & * & 1 \end{pmatrix}, \quad \mathbf{R}_3 = \begin{pmatrix} * & \frac{1}{\sqrt{3}} & * \\ * & * & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & * & * \end{pmatrix}.$$

Use the fact that these are rotation matrices ($\mathbf{R}_i \in SO(3)$, that is, the matrices are orthogonal and $\det(R_i) = 1$) to find the elements marked with $*$. (It is enough to outline a procedure for $\mathbf{R}_3$, actually finding $\mathbf{R}_3$ is optional.)

Problem 2 (Homogeneous transformation matrices, Denavit-Hartenberg convention)
The Denavit-Hartenberg (D-H) convention is used to specify the relations between the different coordinate systems used in robotic manipulators. In this convention, each homogeneous transformation $A_i = T_{i+1}^i$ is given as a product of four basic transformations

$$A_i = \text{Rot}_{z,\theta_i}\text{Trans}_{z,d_i}\text{Trans}_{x,a_i}\text{Rot}_{x,\alpha_i}, \tag{3}$$

where θ_i, d_i, a_i and α_i are parameters related to joint i, and in addition

$$\begin{aligned} \text{Rot}_{z,\theta_i} &: \text{Rotation } \theta_i \text{ about } z\text{-axis} \\ \text{Trans}_{z,d_i} &: \text{Translation } d_i \text{ along } z\text{-axis} \\ \text{Trans}_{x,a_i} &: \text{Translation } a_i \text{ along } x\text{-axis} \\ \text{Rot}_{x,\alpha_i} &: \text{Rotation } \alpha_i \text{ about } x\text{-axis} \end{aligned}$$

See Figure 2.

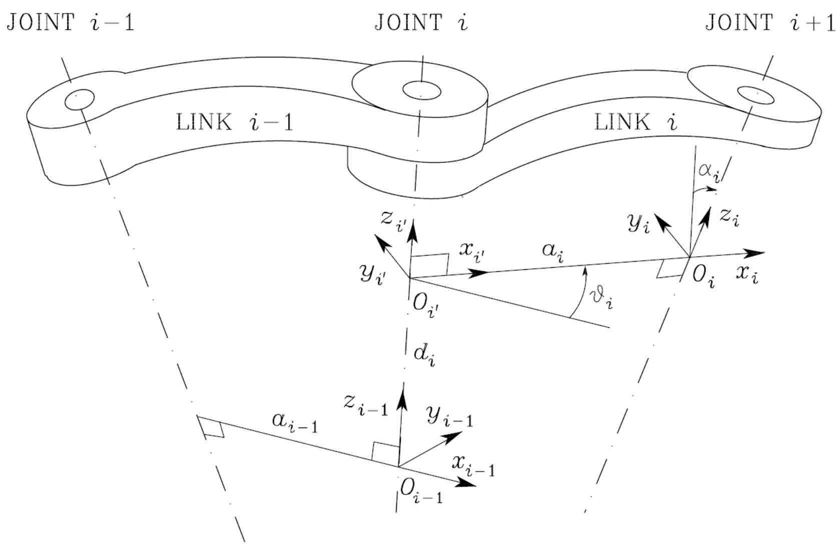

Figure 2: Illustration of transformations involved in the Denavit-Hartenberg convention (taken from Sciavicco, Siciliano, "Modeling and control of robotic manipulators).

We now want to describe the kinematics of the two manipulators in Figure 3:

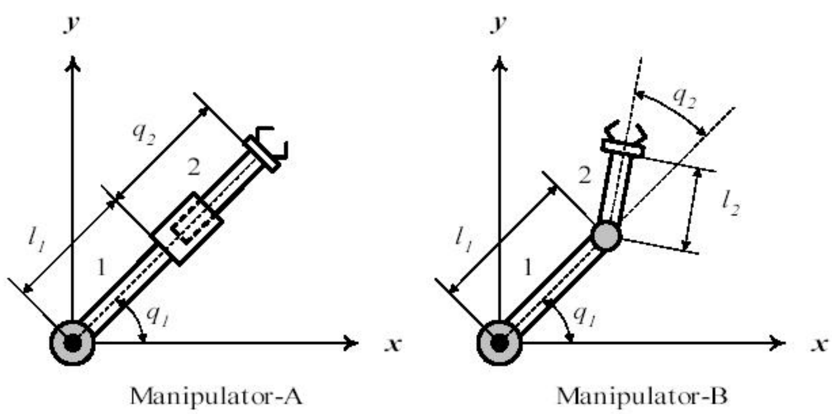

Figure 3: Two robotic manipulators

The D-H parameters for these manipulators can be tabulated as follows:

- Manipulator A, with one rotational joint and one translational (prismatic) joint. The variables q_1 and q_2 are the joint variables (degrees of freedom), while l_1 is constant:

Joint	θ_i	d_i	a_i	α_i
1	q_1	0	l_1	0
2	0	0	q_2	0

- Manipulator B, with two rotational joints. The variables q_1 og q_2 are the joint variables, while l_1

and l_2 are constants:

Joint	θ_i	d_i	a_i	α_i
1	q_1	0	l_1	0
2	q_2	0	l_2	0

(a) Find a general expression for A_i as a function of θ_i, d_i, a_i and α_i.

(b) Find the homogenous transformation matrices for each joint, (A_1 and A_2) for each of the manipulators.

(c) Find the overall transformation matrix T_2^0 for both manipulators.

(d) Assume we have a vector

$$g = g^2 = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}^{\mathsf{T}} \tag{4}$$

given in the tool frame. What is this in the base frame for the manipulators? The base frame (coordinate system) is shown in Figure 3, while the tool frame (coordinate system) is the system given by transforming the base frame with T_2^0.

Problem 3 (From rotation matrix to Euler parameters)

The rotation (or orientation) specified by a rotation matrix

$$\mathbf{R} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

can be represented by an angle θ about an axis $\mathbf{k}$ (angle-axis representation), and may then be written

$$\mathbf{R} = \mathbf{R}_{\mathbf{k},\theta} = \mathbf{e}^{\times} + \cos\theta\mathbf{I} + \mathbf{k}\mathbf{k}^{\mathsf{T}}(1 - \cos\theta) \tag{5}$$

where $\mathbf{e} = \mathbf{k}\sin\theta$ is the Euler rotation vector (equation (6.231) in book). When $\mathbf{R} = \mathbf{R}_b^a = \mathbf{R}_{\mathbf{k},\theta}$, then $\mathbf{k}$ is the same in both system a and b, $\mathbf{k} = \mathbf{k}^a = \mathbf{k}^b$ (why?).

When implementing control systems involving rotations (for instance for robotic manipulators or satellites), it is often desirable to find $\mathbf{k}$ and θ (or Euler parameters) from a given rotation matrix.

A good procedure for this is Shepperd's method (book 6.7.6). In this problem we will derive a procedure that is slightly simpler, but not as general as it cannot be used for $\sin\theta = 0$, and is inaccurate when $\sin\theta$ is small.

(a) Show that

$$\cos\theta = \frac{r_{11} + r_{22} + r_{33} - 1}{2},$$

using (5).

(b) Show that

$$\mathbf{e} = \frac{1}{2}\begin{pmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{pmatrix}$$

Hint: See 6.7.7 in the book.

(c) We can now calculate $\mathbf{k}$ from

$$\mathbf{k} = \frac{\mathbf{e}}{\sin\theta},$$

where θ is as given in (a).

In (a) one can choose if $\theta \in [0, \pi]$ or $\theta \in [\pi, 2\pi]$, and this choice influences $\mathbf{k}$. Explain why $\mathbf{k}$ and θ from both choices represent the same rotation.

(d) Given the matrix

$$\mathbf{R} = \begin{pmatrix} 0.2133 & -0.2915 & 0.9325 \\ 0.9209 & -0.2588 & -0.2915 \\ 0.3263 & 0.9209 & 0.2133 \end{pmatrix}$$

Implement an algorithm in Matlab to find $\mathbf{k}$ and θ by using the formulas above. Enclose a printout of the code, and the results.

(e) (Optional) Implement Shepperd's method (Section 6.7.6 in book) to check your answer.