Exercise 4

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Problem 1

a) Convex when the $Q \geq 0$ in $\frac{1}{2}x^T Q x$ in the objective function, and when the feasible set is convex.

b) If we say that $Z^T G Z \geq 0$, then we could have

$$q(x) = \underbrace{\frac{1}{2} u^T Z^T G Z u}_{=0 \text{ for some } u \neq 0} + q(x^*) = q(x^*)$$

for som $x \neq x^*$. Thus the solution would not necessarily be unique.

c)
$$\min_x \quad q(x) = (x_1 - 1)^2 + (x_2 - 2.5)^2$$

$$\text{s.t.} \quad x_1 - 2x_2 + 2 \geq 0$$

$$-x_1 - 2x_2 + 6 \geq 0$$

$$-x_1 + 2x_2 + 2 \geq 0$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

$$q(x) = (x_1 - 1)^2 + (x_2 - 2.5)^2$$

$$= x_1^2 + x_2^2 - 2x_1 - 5x_2 + \underbrace{7.25}_{}$$

independent of x_1, x_2 so doesnt affect solution.

$$\Rightarrow q(x) = x_1^2 + x_2^2 - 2x_1 - 5x_2$$

$$= \frac{1}{2} x^T G x + c^T x \quad , \quad G = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad c = \begin{pmatrix} -2 \\ -5 \end{pmatrix}$$

Constraints can be written as $Ax \geq b$ with

$$A = \begin{pmatrix} 1 & -2 \\ -1 & -2 \\ -1 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} -2 \\ -6 \\ -2 \\ 0 \\ 0 \end{pmatrix}$$

First iteration, $k=0$:

$$W^0 = \{3\}, \quad x^0 = (2 \ \ 0)^T, \quad g^0 = Gx^0 + c = (2 \ \ -5)^T.$$

p^0 given by
$$\min_p \ \frac{1}{2} p^T G p + g^{0T} p$$
$$\text{s.t.} \quad (-1 \ \ 2)^T p = 0$$

$$\Rightarrow p^0 = \begin{pmatrix} 1/5 \\ 1/10 \end{pmatrix}$$

Since $p^0 \neq 0$ we compute α^0.

$$\alpha^0 = \min\left(1 \;,\; \underbrace{\min_{i \notin W^0,\, a_i^T p^0 < 0} \frac{b_i - a_i^T x^0}{a_i^T p^0}}_{0,\; i=5}\right)$$

$$= 0$$

$$\Rightarrow \quad x^1 = x^0, \quad W^1 = \{3, 5\}$$

Second iteration, $K=1$:

$$x^1 = (2 \;\; 0)^T, \quad g^1 = (2 \;\; -5)^T$$

p^1 given by $\min_p \frac{1}{2} p^T G p + g^{1T} p$

s.t. $(1 \;\; -2)^T p = 0$

$(0 \;\; 1)^T p = 0$

$$\Rightarrow p^1 = 0$$

Since $p^1 = 0$ we compute $\hat{\lambda}_3, \hat{\lambda}_5$

$$\begin{pmatrix} -1 \\ 2 \end{pmatrix} \hat{\lambda}_3 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \hat{\lambda}_5 = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$$

$$\Rightarrow \hat{\lambda}_3 = -2, \quad \hat{\lambda}_5 = -1$$

Remove index 3 since $\hat{\lambda}_3 < \hat{\lambda}_5$.

$$W^2 = \{5\}, \quad x^2 = x^1 = (2 \;\; 0)^T$$

Third iteration, $k=2$:

$$g^2 = \begin{pmatrix} 2 \\ -5 \end{pmatrix}, \quad W^2 = \{5\}$$

$$p^2 = \min_p \frac{1}{2} p^T G p + g^{2^T} p$$

$$\text{s.t.} \quad \begin{pmatrix} 0 & 1 \end{pmatrix} p^T = 0$$

$$= \begin{pmatrix} -1 & 0 \end{pmatrix}^T$$

Since $p^2 \neq 0$ we compute α^2:

$$\alpha^2 = \min\left(1, \min_{i \notin W^2, a_i^T p < 0} \frac{b_i - a_i^T x^2}{a_i^T p^2}\right) = 1$$

$$\Rightarrow x^3 = x^2 + p^2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Fourth iteration, $k=3$:

$$W^3 = \{5\}, \quad x^3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad g^3 = Gx^3 + c = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$$

$$p^3 = \min_p \frac{1}{2} p^T G p + g^{3^T} p$$

$$\text{s.t.} \quad \begin{pmatrix} 0 & 1 \end{pmatrix} p^T = 0$$

$$= \begin{pmatrix} 0 & 0 \end{pmatrix}^T$$

Since $p^3 = 0$, compute $\hat{\lambda}_5$.

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \hat{\lambda}_5 = \begin{pmatrix} 0 \\ -5 \end{pmatrix} \Rightarrow \hat{\lambda}_5 = -5$$

$\hat{\lambda}_5 < 0$ so remove 5 from working set.

Fifth iteration, $k=4$:

$$x^4 = (1 \ \ 0)^T, \quad W^4 = \emptyset, \quad g^4 = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$$

$$p^4 = \min_p \tfrac{1}{2} p^T G p + g^{4T} p \quad = \left(0 \quad \tfrac{5}{2}\right)^T$$

Since $p^4 \neq 0$, compute α^4:

$$\alpha^4 = \min\left(1, \ \min_{a_i^T p < 0} \frac{b_i - a_i^T x^4}{a_i^T p}\right)$$

$$= 3/5, \quad i = 1$$

$$\Rightarrow x^5 = x^4 + \frac{3}{5} p^4 = \left(1 \quad \frac{3}{2}\right)^T$$

$$W^5 = \{1\}$$

Sixth iteration, $k=5$:

$$x^5 = (1 \ \ 1.5)^T, \quad W^5 = \{1\}, \quad g^5 = Gx^5 + c = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

$$p^5 = \min_p \tfrac{1}{2} p^T G p + g^{5T} p \quad = (0.4 \quad 0.2)^T$$

$$\text{s.t.} \quad (1 \ -2)^T p = 0$$

Since $p^5 \neq 0$, compute α^5

$$\alpha^5 = \min\left(1, \ \min_{\substack{a_i^T p^5 < 0 \\ i \neq 1}} \frac{b_i - a_i^T x^5}{a_i^T p^5}\right) = 1$$

$$\Rightarrow x^6 = x^5 + p^5 = (1.4 \quad 1.7)^T$$

$$W^6 = W^5 = \{1\}$$

Seventh iteration, $k=6$:

$$x^6 = (1.4 \quad 1.7)^T, \ W^6 = \{1\}, \ g^6 = Gx^6 + c = (0.8 \quad -1.6)^T$$

$$p^6 = \min_p \ \frac{1}{2}p^T G p + g^{6T} p = (0 \quad 0)^T$$
$$\text{s.t.} \ (1 \ -2)^T p = 0$$

Since $p^6 = 0$, we compute $\hat{\lambda}$:

$$\begin{pmatrix} 1 \\ -2 \end{pmatrix} \hat{\lambda}_1 = \begin{pmatrix} 0.8 \\ -1.6 \end{pmatrix} \Rightarrow \hat{\lambda}_1 = 0.8$$

Since all $\hat{\lambda}_i \geq 0$ for $i \in W^6 \cap I$ we stop with solution

$$x^* = (1.4 \quad 1.7)^T$$

d) Let $f(\lambda)$ denote the dual objective

$$f(\lambda) = \inf_x \mathcal{L}(x,\lambda)$$

$$= \inf_\lambda \frac{1}{2} x^T G x + c^T x - \lambda^T (Ax - b)$$

Infimum achived when $\nabla_x \mathcal{L} = 0$

$$\Rightarrow \quad Gx + c - A^T \lambda = 0$$

$$\Leftrightarrow \quad x = G^{-1}(A^T\lambda - c)$$

Substituting in $f(\lambda)$ gives

$$f(\lambda) = \frac{1}{2}\left(G^{-1}(A^T\lambda - c)\right)^T \cancel{G}\left(\cancel{G^{-1}}(A^T\lambda - c)\right) + c^T G^{-1}(A^T\lambda - c)$$

$$- \lambda^T \left(A\, G^{-1}(A^T\lambda - c) - b\right)$$

$$= \frac{1}{2}(A^T\lambda - c)^T G^{-1}(A^T\lambda - c) + c^T G^{-1}(A^T\lambda - c)$$

$$- \lambda^T \left[A G^{-1}(A^T\lambda - c) - b\right]$$

$$= \frac{1}{2} \lambda^T A G^{-1} A^T \lambda + \cancel{2} \cdot \frac{1}{\cancel{2}} \cancel{\lambda^T A G^{-1}(-c)} + \frac{1}{2} c^T G^{-1} c$$

$$+ c^T G^{-1} A^T \lambda - c^T G^{-1} c$$

$$- \lambda^T A G^{-1} A^T \lambda + \cancel{\lambda^T A G^{-1} c} + \lambda^T b$$

$$= -\frac{1}{2} \lambda^T A G^{-1} A^T \lambda + c^T G^{-1} A^T \lambda - \frac{1}{2} c^T G^{-1} c$$

$$+ \lambda^T b$$

$$= -\frac{1}{2}\lambda^T A G^{-1} A^T \lambda + c^T G^{-1} A^T \lambda - \frac{1}{2} c^T G^{-1} c$$
$$+ \lambda^T b$$

$$\Rightarrow f(\lambda) = -\frac{1}{2}\lambda^T A G^{-1} A^T \lambda + \frac{1}{2}\lambda^T A G^{-1} c + \frac{1}{2} c^T G^{-1} A^T \lambda - \frac{1}{2} c^T G^{-1} c + \lambda^T b$$

$$= -\frac{1}{2}(A^T\lambda - c)^T G^{-1} (A^T \lambda - c) + \lambda^T b$$

Since an optimal solution of the primal problem requires $\lambda \geq 0$, we define the dual problem as

$$\max_{\lambda} \; -\frac{1}{2}(A^T\lambda - c)^T G^{-1} (A^T\lambda - c) + \lambda^T b$$
$$\text{s.t.} \quad \lambda \geq 0.$$

e) The optimal solution to the primal problem is the same as the solution to the dual, $c(x^*) = f(\lambda^*)$.

Theorem 12.12 lets us write $q(\bar{x}) \geq f(\bar{\lambda})$ for any feasible pair $(\bar{x}, \bar{\lambda})$.

Since $f(\bar{\lambda}) \leq f(\lambda^*)$ we over-estimate with

$$q(\bar{x}) - q(x^*) = q(\bar{x}) - f(\lambda^*)$$
$$\leq q(\bar{x}) - f(\bar{\lambda})$$

As we converge on a solution $q(\bar{x}) - f(\bar{\lambda}) \to 0$.

Problem 2

a) Profit given by

$$p(x) = (3-0.4x_1)x_1 + (2-0.2x_2)x_2$$
$$= -0.4x_1^2 + 3x_1 - 0.2x_2^2 + 2x_2$$
$$= -\frac{1}{2}x^T\begin{pmatrix}0.8 & 0\\ 0 & 0.4\end{pmatrix}x - \begin{pmatrix}-3\\-2\end{pmatrix}^T x$$

$$G = \begin{pmatrix}0.8 & 0\\ 0 & 0.4\end{pmatrix}, \quad c = \begin{pmatrix}-3 & -2\end{pmatrix}^T$$

Constraints:

$$\left.\begin{matrix} R_I \leq 8 \\ R_{II} \leq 15 \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} 2x_1 + x_2 \leq 8 \\ x_1 + 3x_2 \leq 15 \end{matrix}\right.$$

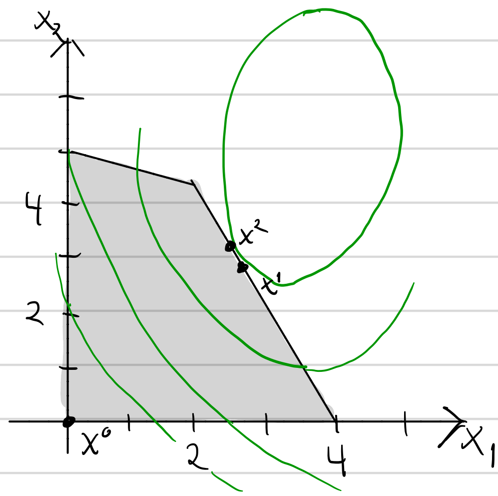
$$A = \begin{pmatrix}2 & 1\\ 1 & 3\end{pmatrix}, \ b = \begin{pmatrix}8\\15\end{pmatrix} \Rightarrow \quad Ax \leq b$$

Formulation as quadratic min. problem:

$$\min_{x\in\mathbb{R}^2} -p(x) = \frac{1}{2}x^TGx + c^Tx$$
$$\text{s.t.} \quad Ax \leq b$$

$$G = \begin{pmatrix}0.8 & 0\\ 0 & 0.4\end{pmatrix}, \ c = \begin{pmatrix}-3 & -2\end{pmatrix}^T, \ A = \begin{pmatrix}2 & 1\\ 1 & 3\end{pmatrix}, \ b \leq \begin{pmatrix}8\\15\end{pmatrix}$$

b)



c) Only the constraint $2x_1 + x_2 \leq 8$ (R_I) is active.

d) The sequence starts at $x^0 = (0 \ 0)$ with working set $W^0 = \emptyset$. It calculates the direction that most decreases the objective function and goes along that direction untill it reaches a blocking constraint.

Once it reaches the blocking constraint it optimizes its position within that constraint.

e) In exercise 3 we found $x = (1.8 \ \ 4.4)^T$ which was in the intersection between the constraints.