Øving 12

4.2

1) $$v = y' \qquad\qquad y' = v$$
$$v' + 2v - 3y = 0 \iff v' = -2v + 3y$$

2) $$v = y' \qquad\qquad y' = v$$
$$v' + 4v + y = 0 \iff v' = -4v - y'$$

5.7

1) $$\vec{x}' = A\vec{x} \quad , \quad \vec{v}_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \text{ with } \lambda_1 = 4$$
$$\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ with } \lambda_2 = 2$$

The general solution is

$$\vec{x}(t) = c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t}$$

We want $\vec{x}(0) = \begin{pmatrix} -6 \\ -1 \end{pmatrix}$

$$\vec{x}(0) = c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} -3 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -6 \\ 1 \end{pmatrix}$$

$$\Rightarrow \left(\begin{array}{cc|c} -3 & -1 & -6 \\ 1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & \frac{1}{3} & 2 \\ 1 & 1 & 1 \end{array}\right)$$
$$\sim \left(\begin{array}{cc|c} 1 & \frac{1}{3} & 2 \\ 0 & \frac{2}{3} & -1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & \frac{1}{3} & 2 \\ 0 & \frac{1}{3} & -\frac{1}{2} \end{array}\right)$$
$$\sim \left(\begin{array}{cc|c} 1 & 0 & \frac{5}{2} \\ 0 & \frac{1}{3} & -\frac{1}{2} \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & \frac{5}{2} \\ 0 & 1 & -\frac{3}{2} \end{array}\right)$$

$$\Rightarrow \vec{x}(t) = \frac{5}{2} \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{4t} - \frac{3}{2} \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t}$$

3) We want to solve $\vec{x}'(t) = A\vec{x}(t)$ with $A = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$

Eigenvalues:

$$\det(A-\lambda I) = 0$$
$$(2-\lambda)(-2-\lambda)+3 = 0$$
$$2(-2)-2\lambda-\lambda(-2)+\lambda^2+3 = 0$$
$$\lambda^2-2\lambda+2\lambda-4+3 = 0$$
$$\lambda^2 = 1$$
$$|\lambda| = 1$$
$$\lambda_1 = 1, \quad \lambda_2 = -1$$

Eigenvectors:

1) $(A-I)\vec{v}_1 = \vec{0}$, solving for $\vec{v}_1$

$$\begin{pmatrix} 2-1 & 3 \\ -1 & -2-1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ -1 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}$$

$x_1 = -3x_2$, x_2 free

$$\vec{v}_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

2) $(A+I)\vec{v}_2 = \vec{0}$, solving for $\vec{v}_2$

$$\begin{pmatrix} 2+1 & 3 \\ -1 & -2+1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$x_1 = -x_2$, x_2 free

$$\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

So the general solution to $\vec{x}' = A\vec{x}$ is

$$\vec{x}(t) = c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$$

We want $\vec{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$$\vec{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix} = c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \left(\begin{array}{cc|c} -3 & -1 & 3 \\ 1 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1/3 & -1 \\ 1 & 1 & 2 \end{array}\right)$$

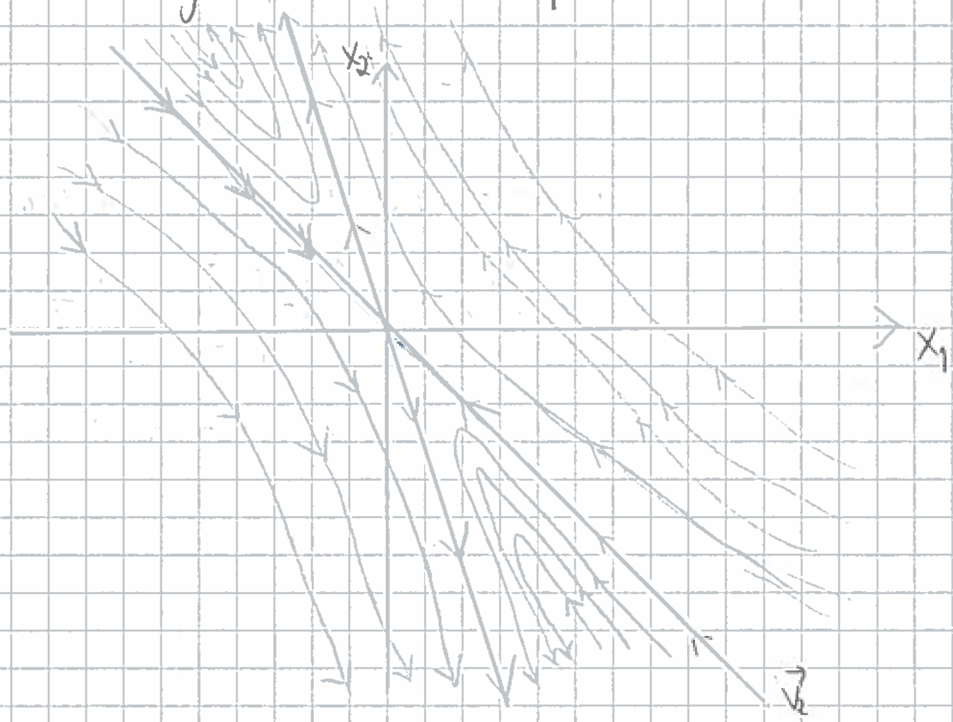
$$\sim \left(\begin{array}{cc|c} 1 & 1/3 & -1 \\ 0 & 2/3 & 3 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1/3 & -1 \\ 0 & 1/3 & 3/2 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 1 & 0 & -5/2 \\ 0 & 1 & 9/2 \end{array}\right)$$

So we get:

$$\vec{x}(t) = -\frac{5}{2} \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{t} + \frac{9}{2} \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$$

The origin is a saddle point.



9) Want to solve $\vec{x}' = A\vec{x}$ for $A = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix}$.

Eigenvalues: $\det(A - \lambda I) = 0$

$$(-3-\lambda)(-1-\lambda) + 2 = 0$$

$$(-1)(3+\lambda)(-1)(1+\lambda) + 2 = 0$$

$$3 + 4\lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 + 4\lambda + 5 = 0$$

$$\Rightarrow \lambda_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 5}}{2}$$

$$= \frac{-2 \pm \sqrt{-4}}{2}$$

$$= \frac{-2 \pm 2i}{2}$$

$$= -2 \pm i$$

$$\lambda_1 = -2+i, \quad \lambda_2 = \overline{\lambda_1} = -2-i$$

Eigenvectors: solve $(A - (-2+i)I)\vec{v}_1 = \vec{0}$ for $\vec{v}_1$.

$$\begin{pmatrix} -3+2-i & 2 \\ -1 & -1+2-i \end{pmatrix} \Leftrightarrow \begin{pmatrix} -1-i & 2 \\ -1 & 1-i \end{pmatrix}$$

Since this matrix acts on $\mathbb{C}^2$ and its rows are linearly dependent we get $\vec{v}_1 = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $(-1-i)v_1 + 2v_2 = 0$

v_2 free

$$\Rightarrow v_1 = -\frac{2}{-1-i} v_2$$

$$= \frac{2}{1+i} \frac{(1-i)}{(1-i)} v_2$$

$$= \left(\frac{2}{1+1} - \frac{2i}{1+1}\right) v_2$$

$$= (1-i) v_2$$

$$\vec{v}_1 = \begin{pmatrix} 1-i \\ 1 \end{pmatrix} \quad \text{corresponds to } \lambda_1 = -2+i$$

$$\vec{v}_2 = \overline{\vec{v}_1} = \begin{pmatrix} 1+i \\ 1 \end{pmatrix} \quad — \text{''} — \quad \lambda_2 = -2-i$$

So the general solution to $\vec{x}' = A\vec{x}$ is:

$$\vec{x}(t) = \underbrace{z_1 \begin{pmatrix} 1-i \\ 1 \end{pmatrix} e^{(-2+i)t}}_{\vec{x}_1(t)} + \underbrace{z_2 \begin{pmatrix} 1+i \\ 1 \end{pmatrix} e^{(-2-i)t}}_{\vec{x}_2(t)}$$

$z_1, z_2 \in \mathbb{C}$

To get the real solution we look at the real and imaginary part of $\vec{x}_1$.

$$\vec{x}_1 = e^{-2t} \begin{pmatrix} 1-i \\ 1 \end{pmatrix} (\cos t + i \sin t)$$

$$= e^{-2t} \begin{pmatrix} \cos t + i \sin t - i \cos t + \sin t \\ \cos t + i \sin t \end{pmatrix}$$

$$\vec{y}_1 = \operatorname{Re} \vec{x}_1(t)$$
$$= e^{-2t} \begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix}$$

$$\vec{y}_2 = \operatorname{Im} \vec{x}_1(t)$$
$$= e^{-2t} \begin{pmatrix} \sin t - \cos t \\ \sin t \end{pmatrix}$$

So the general real solution is

$$\vec{y} = c_1 \begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} \sin t - \cos t \\ \sin t \end{pmatrix} e^{-2t}$$

All trajectories tend to the origin and they will oscilate in some way, often like a spiral, toward the origin.

6.1

6) $\vec{x} = \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} 3 \\ -1 \\ -5 \end{pmatrix}$

$$\frac{\vec{x} \cdot \vec{w}}{\vec{x} \cdot \vec{x}} \vec{x} = \frac{18+2-15}{36+4+9} \vec{x}$$

$$= \frac{5}{49} \vec{x}$$

$$= \begin{pmatrix} \frac{30}{49} \\ \frac{-10}{49} \\ \frac{15}{49} \end{pmatrix}$$

9) Normalize the vector: $\vec{x} = \begin{pmatrix} -30 \\ 40 \end{pmatrix}$

$$||\vec{x}|| = \sqrt{(-30)^2 + 40^2}$$

$$= \sqrt{900 + 1600}$$

$$= 50$$

The unit vector in the same direction is then $\frac{\vec{x}}{||\vec{x}||} = \begin{pmatrix} -3/5 \\ 4/5 \end{pmatrix}$

10) $\vec{x} = \begin{pmatrix} -6 \\ 4 \\ -3 \end{pmatrix}$, $||\vec{x}|| = \sqrt{6^2 + 4^2 + 3^2}$

$$= \sqrt{36 + 16 + 9}$$

$$= \sqrt{61}$$

The unit vector with the same direction is then $\frac{\vec{x}}{||\vec{x}||} = \begin{pmatrix} -6/\sqrt{61} \\ 4/\sqrt{61} \\ -3/\sqrt{61} \end{pmatrix}$

15) $\vec{a} = \begin{pmatrix} 8 \\ -5 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$

$\vec{a} \cdot \vec{b} = -16 + 15 = -1 \neq 0$

So $\vec{a}$ and $\vec{b}$ are not orthogonal

16) $\vec{u} = \begin{pmatrix} 12 \\ 3 \\ -5 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}$

$\vec{u} \cdot \vec{v} = 24 - 9 - 15 = 0$

So $\vec{u}$ and $\vec{v}$ are orthogonal

26) $\vec{u} = \begin{pmatrix} 5 \\ -6 \\ 7 \end{pmatrix}$

$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$\vec{u} \cdot \vec{x} = 5x_1 - 6x_2 + 7x_3 = 0$

This is a plane.

We also know that

$\vec{u} \cdot \vec{x} = \vec{u}^T \vec{x} = 0$

Now the set of all $\vec{x}$ is $Nul(\vec{u}^T)$.

- The nullspace of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$. (Theorem 2, pg. 217)

$\vec{u}^T = \begin{pmatrix} 5 & -6 & 7 \end{pmatrix}$ is a 1×3 matrix so by theorem 2 it is a subspace of $\mathbb{R}^3$.

29) All vectors in W can be written as a linear combination of $\vec{v}_1, \dots, \vec{v}_p$. So any vector $\vec{u} \in W$ is of the form

$$\vec{u} = c_1\vec{v}_1 + \dots + c_p\vec{v}_p$$

$$\Rightarrow \vec{u}\cdot\vec{x} = (c_1\vec{v}_1 + \dots + c_p\vec{v}_p)\vec{x}$$

$$= c_1\vec{v}_1\cdot\vec{x} + \dots + c_p\vec{v}_p\cdot\vec{x}$$

$$= 0 + \dots + 0 \quad \text{since } \vec{x}\cdot\vec{v}_j = 0$$

$$= 0$$

$\vec{u}\cdot\vec{x} = 0$ for any vector $\vec{u} \in W$ so $\vec{x}$ is orthogonal to every vector in W.

Extra: def $\vec{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$

$\vec{w} = \begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix}$

So we want $\vec{w}\cdot(c_1\vec{u} + c_2\vec{v}) = 0$

$\Leftrightarrow \vec{w}^T(c_1\vec{v}_1 + c_2\vec{v}_2) = 0$

So we have to find a vector $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{x}$ in $Nul(\vec{w}^T)$

$\vec{w}^T = (4 \quad -1 \quad -1)$

$x_1 = \frac{1}{4}x_2 + \frac{1}{4}x_3$

x_2, x_3 free

$$\text{Nul}(\vec{w}^T) = \text{span}\left\{\begin{pmatrix}1\\4\\6\end{pmatrix}, \begin{pmatrix}1\\0\\4\end{pmatrix}\right\}$$

Have to find a vector in $\text{Nul}(\vec{w}^T)$, $\vec{x}$, such that $\vec{x} = c_1\begin{pmatrix}1\\0\\0\end{pmatrix} + c_2\begin{pmatrix}1\\2\\4\end{pmatrix}$

$$\frac{1}{2}\begin{pmatrix}1\\4\\6\end{pmatrix} + \begin{pmatrix}1\\0\\4\end{pmatrix} = \begin{pmatrix}3/2\\2\\4\end{pmatrix} \in \text{Nul}(\vec{w}^T)$$

If $c_1 = \frac{1}{2}$ and $c_2 = 1$, $\vec{x} = \begin{pmatrix}1/2\\0\\0\end{pmatrix} + \begin{pmatrix}1\\2\\4\end{pmatrix} = \begin{pmatrix}3/2\\2\\4\end{pmatrix} \in \text{Nul}(\vec{w}^T)$

So $\frac{1}{2}\vec{u} + \vec{v}$ is a linear combination of $\vec{u}$ and $\vec{v}$, which when dotted with $\vec{w}$ gives 0.

6.2

7) $\vec{u}_1 = \begin{pmatrix}2\\3\end{pmatrix}$, $\vec{u}_2 = \begin{pmatrix}6\\4\end{pmatrix}$, $\vec{x} = \begin{pmatrix}9\\-7\end{pmatrix}$

$\vec{u}_1 \cdot \vec{u}_2 = 12 - 12 = 0$

$\vec{u}_1, \vec{u}_2$ are linearly dependent vectors in $\mathbb{R}^2$ so they form a basis for $\mathbb{R}^2$.

$\vec{x} = c_1\vec{u}_1 + c_2\vec{u}_2$

where $c_1 = \dfrac{\vec{u}_1 \cdot \vec{x}}{\vec{u}_1 \cdot \vec{u}_1} = \dfrac{18+21}{4+9} = \dfrac{39}{13} = 3$

$c_2 = \dfrac{\vec{u}_2 \cdot \vec{x}}{\vec{u}_2 \cdot \vec{u}_2} = \dfrac{54-28}{36+16} = \dfrac{26}{52} = \dfrac{1}{2}$

So $\vec{x} = 3\vec{u}_1 + \frac{1}{2}\vec{u}_2$

11) $$\operatorname{proj}_{\binom{-4}{2}} \binom{1}{7} = \frac{\binom{-4}{2}\cdot\binom{1}{7}}{\binom{-4}{2}\cdot\binom{-4}{2}} \binom{-4}{2}$$

$$= \frac{-4+14}{16+4}\binom{-4}{2}$$

$$= \frac{1}{2}\binom{-4}{2}$$

$$= \underline{\binom{-2}{1}}$$

14) $\vec{y} = \binom{2}{6}$, $\vec{u} = \binom{7}{1}$

$\vec{u}^{\perp}$ denotes a vector orthogonal to $\vec{u}$.

$$\vec{y} = c_1\vec{u} + c_2\vec{u}^{\perp}$$

$$c_1 = \frac{\vec{u}\cdot\vec{y}}{\vec{u}\cdot\vec{u}} = \frac{14+6}{49+1} = \frac{20}{50} = \frac{2}{5}$$

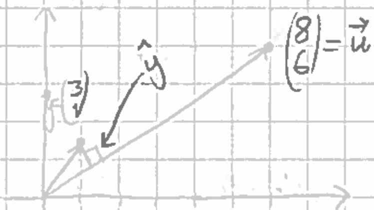
$$c_2\vec{u}^{\perp} = \vec{y} - c_1\vec{u}$$

$$c_2\vec{u}^{\perp} = -\frac{2}{5}\binom{7}{1} + \binom{2}{6} = -\begin{pmatrix} 14/5 - 2 \\ 2/5 - 6 \end{pmatrix}$$

$$c_2\vec{u}^{\perp} = -\begin{pmatrix} 4/5 \\ -28/5 \end{pmatrix} = \underbrace{-\frac{4}{5}}_{c_2}\underbrace{\binom{1}{-7}}_{\vec{u}^{\perp}}$$

So $\vec{y} = \frac{2}{5}\binom{7}{1} - \frac{4}{5}\binom{1}{-7}$

15) Let $\vec{y} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\vec{u} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$



$$\hat{y} = \vec{y} - \text{proj}_{\vec{u}}(\vec{y})$$

$$\text{proj}_{\vec{u}}(\vec{y}) = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}$$

$$= \frac{24+6}{64+36} \vec{u} = \frac{3}{10} \vec{u} = \begin{pmatrix} \frac{24}{10} \\ \frac{18}{10} \end{pmatrix} = \begin{pmatrix} \frac{12}{5} \\ \frac{9}{5} \end{pmatrix}$$

$$\Rightarrow \hat{y} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 12/5 \\ 9/5 \end{pmatrix} = \begin{pmatrix} 3/5 \\ -4/5 \end{pmatrix}$$

The distance from $\vec{y}$ to $\vec{u}$ is $||\hat{y}||$

$$||\hat{y}|| = \frac{1}{5}\sqrt{3^2+4^2} = 1$$

So the distance is $\underline{1.}$

26) W is spanned by a set $\{\vec{v}_1, \dots, \vec{v}_n\}$ of n orthogonal non-zero vectors. Since all the vectors are non-zero the set is linearly independent. A linearly independent set of n vectors spans $\mathbb{R}^n$. So

$$W = \text{span}\{\vec{v}_1, \dots, \vec{v}_n\} = \mathbb{R}^n$$

$$\Rightarrow W = \mathbb{R}^n$$

27) $U = (\vec{v}_1 | \dots | \vec{v}_n)$, $\vec{v}_i \in \mathbb{R}^n$

, $\{\vec{v}_i\}_{i=1}^n$ orthogonal

From (26) we know that the column vectors of U form a basis of $\mathbb{R}^n$.
By the invertible matrix theorem (m, pg.253), U is then invertible. ■