NORGES TEKNISK- NATURVITENSKAPELIGE UNIVERSITET
INSTITUTT FOR TEKNISK KYBERNETIKK

Faglig kontakt under eksamen:

Navn: Johannes Tjønnås
Tlf.: 408 25 940

Eksamen
TTK 4115 Lineær systemteori

18. desember 2004
Tid: 0900 – 1300

Hjelpemidler: D - Ingen trykte eller håndskrevne hjelpemidler tillatt. Bestemt, enkel kalkulator tillatt.

Oppgave 1 (10 %)

Anta gitt følgende system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$$

a) Vis at transisjonsmatrisen er

$$\Phi(t) = \begin{pmatrix} e^t & 0 \\ \frac{1}{4}(e^{-3t} - e^t) & e^{-3t} \end{pmatrix}$$

b) Finn en eksakt diskretisering av dette systemet med nullteordens holdeelement på inngangen (ZOH) for samplingintervallet $T = 0.25$ s.

Oppgave 2 (25 %)

a) Definer begrepet *Lyapunov-stabilitet* (ekvivalent med intern stabilitet) for et lineært system.

b) Angi kriterier for å teste hvorvidt et gitt lineært system er Lyapunov-stabilt. Kortfattet beskrivelse er tilstrekkelig.

c) Definer begrepet *BIBO-stabilitet* for et lineært system.

d) Angi kriterier for å teste hvorvidt et gitt lineært system er BIBO-stabilt. Kortfattet beskrivelse er tilstrekkelig.

e) Anta gitt følgende system

$$
\begin{aligned}
\dot{x} &= \begin{pmatrix} -1 & 0 \\ 0.5 & 2 \end{pmatrix} x + \begin{pmatrix} 1 \\ -1 \end{pmatrix} u \\
y &= x_1
\end{aligned}
$$

Undersøk stabilitetsegenskapene (både Lyapunov-stabilitet og BIBO-stabilitet) til dette systemet.

Oppgave 3 (40 %)

Gitt følgende system

$$
\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{1a}
$$

$$
y = x_1 + w, \tag{1b}
$$

hvor w er målestøy og v_1, v_2 er ukjente forstyrrelser.

a) Definer begrepet observerbarhet for et lineært system, og vis at systemet ovenfor er observerbart.

b) Finn en observer for dette systemet som kan benyttes for å beregne et tilstandsestimat $\hat{x}$. Forsterkningsmatrisen L skal velges slik at polene skal plasseres i -5 og -8.

Vi definerer estimeringsfeilen $\tilde{x} = \hat{x} - x$ og får følgende system:

$$
\begin{pmatrix} \dot{x} \\ \dot{\tilde{x}} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A - LC \end{pmatrix} \begin{pmatrix} x \\ \tilde{x} \end{pmatrix} + \begin{pmatrix} I \\ -I \end{pmatrix} v + \begin{pmatrix} 0 \\ L \end{pmatrix} w
$$

c) Hvilken effekt vil en konstant forstyrrelse på v_1 ha på estimeringsfeilen $\tilde{x}_1$ og $\tilde{x}_2$? Hva kan gjøres for å redusere effekten av en slik forstyrrelse?

d) Transferfunksjonene fra målestøyen w til estimeringsfeilen $\tilde{x}_1$ og $\tilde{x}_2$ er gitt ved:

$$\frac{\tilde{x}_1}{w}(s) = \frac{11s + 39}{s^2 + 13s + 40}, \quad \frac{\tilde{x}_2}{w}(s) = \frac{17s - 11}{s^2 + 13s + 40}.$$

Diskuter på grunnlag av dette hvilken effekt målestøyen har. Hva kan gjøres for å redusere effekten av målestøy.

Anta i det etterfølgende at målestøyen w og forstyrrelsen v er begge normalfordelt hvit støy med null middelverdi. Kovariansmatrisen til målestøyen $w(t)$ er W = 35, og kovariansmatrisen til forstyrrelsesvektoren $v(t)$ er

$$V = \begin{pmatrix} 10 & 0 \\ 0 & 2 \end{pmatrix}$$

e) Sett opp det kontinuerlige Kalman-filteret for (1a)-(1b).

f) Stasjonær løsning til Riccati-ligninga oppgis til å være

$$P = \begin{pmatrix} 10.21 & -3.51 \\ -3.51 & 2.17 \end{pmatrix}$$

Beregn stasjonært Kalman-filter, dvs. stasjonær verdi for forsterkningsmatrisen, og sammenlign effekten av målestøy for dette filteret med observeren beregnet ovenfor.

Oppgave 4 (15 %)

La $u(t)$ være en hvitstøy prosess med null middelverdi og varians σ_u^2.

a) Hva er autokorreasjonsfunksjonen og effektspektret til denne stokastiske prosessen? Begrunn hvorvidt dette er en stasjonær prosess eller ikke.

b) La $\dot{x}(t) = u(t)$. Finn forventningsverdi og varians til $x(t)$. Begrunn hvorvidt dette er en stasjonær prosess eller ikke.

c) Finn en spektralfaktorisering av følgende effektspektrum

$$S_x(j\omega) = \frac{\omega^2 + 1}{\omega^2 + 100}.$$

Oppgave 5 (10 %)

Anta du har en prosess med følgende transferfunksjon

$$G(s) = \frac{Ke^{-\tau s}}{1 + Ts}$$

der parametrene er ukjente.

a) Anta du kan bestemme inngangsignalet selv. Beskriv en metode for å identifisere tidsforsinkelsen τ fra et eksperiment.

b) Anta du *ikke* kan bestemme inngangsignalet selv, men at dette kan ses på som et tilfeldig stasjonært signal. Beskriv en metode for å identifisere tidsforsinkelsen τ fra et eksperiment.

Vedlegg til eksamen (Noen nyttige formler og uttrykk)

$$
\begin{aligned}
x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \\
x(k) &= A^k x(0) + \sum_{m=0}^{k-1} A^{k-1-m}Bu(m) \\
A^{-1} &= \frac{adj(A)}{det(A)} \\
det(A) &= \sum_{i=1}^{n} a_{ij}c_{ij} \\
adj(A) &= \{c_{ij}\}^T \\
c_{ij} &= (-1)^{i+j}det(A_{ij}) \quad \text{(kofaktor)},\ A_{ij} = \text{undermatrise til } A
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{C} &= (B\ AB\ A^2B\ \cdots\ A^{n-1}B) \\
\mathcal{O} &= \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix} \\
G(s) &= C(sI-A)^{-1}B + D \\
G(z) &= C(zI-A)^{-1}B + D \\
\mathbf{G(s)} &= \mathbf{G}(\infty) + \mathbf{G_{sp}(s)} \\
d(s) &= s^r + \alpha_1 s^{r-1} + \cdots + \alpha_{r-1}s + \alpha_r \\
\mathbf{G_{sp}}(s) &= \frac{1}{d(s)}[\mathbf{N_1}s^{r-1} + \mathbf{N_2}s^{r-2} + \cdots + \mathbf{N_{r-1}}s + \mathbf{N_r}] \\
\dot{\mathbf{x}} &= \begin{bmatrix} -\alpha_1\mathbf{I_p} & -\alpha_2\mathbf{I_p} & \cdots & -\alpha_{r-1}\mathbf{I_p} & -\alpha_r\mathbf{I_p} \\ \mathbf{I_p} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I_p} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I_p} & \mathbf{0} \end{bmatrix}\mathbf{x} + \begin{bmatrix} \mathbf{I_p} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}\mathbf{u} \\
\mathbf{y} &= [\ \mathbf{N_1}\quad \mathbf{N_2}\quad \cdots\quad \mathbf{N_{r-1}}\quad \mathbf{N_r}\ ]\mathbf{x} + \mathbf{G}(\infty)\mathbf{u}
\end{aligned}
$$

Diskret Kalman-filter:

$$
\begin{aligned}
\mathbf{x}_{k+1} &= \mathbf{\Phi}_k \mathbf{x}_k + \mathbf{w}_k \\
\mathbf{z}_k &= \mathbf{H}_k \mathbf{x}_k + \mathbf{v}_k \\
E[\mathbf{w}_k \mathbf{w}_i^T] &= \begin{cases} \mathbf{Q}_k, & i = k \\ 0, & i \neq k \end{cases} \\
E[\mathbf{v}_k \mathbf{v}_i^T] &= \begin{cases} \mathbf{R}_k, & i = k \\ 0, & i \neq k \end{cases} \\
E[\mathbf{w}_k \mathbf{v}_i^T] &= 0, \forall i, k \\
\mathbf{P}_k^- &= E[\mathbf{e}_k^- \mathbf{e}_k^{-T}] \\
\mathbf{P}_k &= E[\mathbf{e}_k \mathbf{e}_k^T] = (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) \mathbf{P}_k^- (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)^T + \mathbf{K}_k \mathbf{R}_k \mathbf{K}_k^T \\
\mathbf{K}_k &= \mathbf{P}_k^- \mathbf{H}_k^T \left(\mathbf{H}_k \mathbf{P}_k^- \mathbf{H}_k^T + \mathbf{R}_k \right)^{-1} \\
\mathbf{P}_{k+1}^- &= \mathbf{\Phi}_k \mathbf{P}_k \mathbf{\Phi}_k + \mathbf{Q}_k
\end{aligned}
$$

Kontinuerlig Kalman filter:

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{F}\mathbf{x} + \mathbf{G}\mathbf{u} \\
\mathbf{z} &= \mathbf{H}\mathbf{x} + \mathbf{v} \\
E[\mathbf{u}(t)\mathbf{u}(\tau)^T] &= \mathbf{Q}\delta(t - \tau) \\
E[\mathbf{v}(t)\mathbf{v}(\tau)^T] &= \mathbf{R}\delta(t - \tau) \\
E[\mathbf{u}(t)\mathbf{v}(\tau)^T] &= 0 \\
\mathbf{K} &= \mathbf{P}\mathbf{H}^T\mathbf{R}^{-1} \\
\dot{\mathbf{P}} &= \mathbf{F}\mathbf{P} + \mathbf{P}\mathbf{F}^T - \mathbf{P}\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\mathbf{P} + \mathbf{G}\mathbf{Q}\mathbf{G}^T, \quad \mathbf{P}(0) = \mathbf{P}_0
\end{aligned}
$$

Autokorrelasjon:

$$
\begin{aligned}
R_X(\tau) &= E[X(t)X(t+\tau)] \text{ (Stajonær prosess)} \\
R_X(t_1, t_2) &= E[X(t_1)X(t_2)] \text{ (Ikke stajonær prosess)} \\
Y(s) &= G(s)U(s) \Rightarrow \\
R_y(t_1, t_2) &= E[y(t_1)y(t_2)] \\
&= \int_0^{t_2} \int_0^{t_1} g(\xi) g(\eta) E\left[u(t_1 - \xi) u(t_2 - \eta) \right] d\xi d\eta \text{ (Transient analyse)}
\end{aligned}
$$

Minste kvadraters estimering:

$$
\begin{aligned}
\text{Objektfunksjonsverdi i minste.k.estim.} \;: &\quad V(\bar{\theta}) = \frac{1}{2}(\mathcal{Y}_N - \Phi_N \bar{\theta})^T (\mathcal{Y}_N - \Phi_N \bar{\theta}) \\
\text{Kovariansmatrise for } \hat{\theta} : &\quad \sigma_e^2 (\Phi_N^T \Phi_N)^{-1} \\
\text{Estimat av } \sigma_e^2 : &\quad \hat{\sigma}_e^2 = \frac{2}{N - p} V(\hat{\theta})
\end{aligned}
$$

Noen enkle Laplace-transformer:

$f(t)$	$\Longleftrightarrow$	$F(s)$
1	$\Longleftrightarrow$	$\dfrac{1}{s}$
e^{-at}	$\Longleftrightarrow$	$\dfrac{1}{s+a}$
t	$\Longleftrightarrow$	$\dfrac{1}{s^2}$
t^2	$\Longleftrightarrow$	$\dfrac{2}{s^3}$
te^{-at}	$\Longleftrightarrow$	$\dfrac{1}{(s+a)^2}$
$\sin \omega t$	$\Longleftrightarrow$	$\dfrac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\Longleftrightarrow$	$\dfrac{s}{s^2+\omega^2}$

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Contact under the exam:

Name: Johannes Tjønnås
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Exam
TTK 4115 Lineær systems

18. December 2004
Time: 0900 – 1300

Supporting materials: D - No printed or handwritten material allowed. Specific, simple calculator allowed.

Exercise 1 (10 %)

Given the following system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$$

a) Show that the transition matrix is given by

$$\Phi(t) = \begin{pmatrix} e^t & 0 \\ \frac{1}{4}(e^{-3t} - e^t) & e^{-3t} \end{pmatrix}$$

b) Find an exact discretization of the system when a zero hold is used on the input. Let the sampling interval be $T = 0.25$ s.

Exercise 2 (25 %)

a) Define the term *Lyapunov-stability* (equivalent with internal stability) for a linear system.

b) State methods to test if a given linear system is Lyapunov-stable. Brief description is enough.

c) Define the term *BIBO-stability* for a linear system.

d) State methods to test if a given linear system is BIBO-stable. Brief description is enough.

e) Given the following system

$$
\begin{aligned}
\dot{x} &= \begin{pmatrix} -1 & 0 \\ 0.5 & 2 \end{pmatrix} x + \begin{pmatrix} 1 \\ -1 \end{pmatrix} u \\
y &= x_1
\end{aligned}
$$

Is the system Lyapunov-stable? Is the system BIBO-stable?

Exercise 3 (40 %)

Given the following system:

$$
\begin{aligned}
\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \qquad (1a) \\
y &= x_1 + w, \qquad (1b)
\end{aligned}
$$

where w is measurement noise and v_1, v_2 are unknown disturbances.

a) Define the term observability for a linear system, and show that the system above is observable.

b) Find an observer that can be used to calculate a state estimate $\hat{x}$ for this system. Choose the gain matrix L such that the two poles are placed in -5 and -8.

We define the estimation error $\tilde{x} = \hat{x} - x$ and get the following system:

$$
\begin{pmatrix} \dot{x} \\ \dot{\tilde{x}} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A - LC \end{pmatrix} \begin{pmatrix} x \\ \tilde{x} \end{pmatrix} + \begin{pmatrix} I \\ -I \end{pmatrix} v + \begin{pmatrix} 0 \\ L \end{pmatrix} w
$$

c) What effect will a constant disturbance on v_1 have on the estimation error $\tilde{x}_1$ and $\tilde{x}_2$? What can be done to reduce the effect of such a disturbance?

d) The transfer functions from the measurement noise w to the estimations error $\tilde{x}_1$ and $\tilde{x}_2$ is given by:

$$\frac{\tilde{x}_1}{w}(s) = \frac{11s + 39}{s^2 + 13s + 40}, \quad \frac{\tilde{x}_2}{w}(s) = \frac{17s - 11}{s^2 + 13s + 40}.$$

Based on this, discuss the effect the measurement noise have on the system. What can be done to reduce the effect of measurement noise?

In the following, let the measurement noise w and the disturbance v be Gaussian white noise with zero mean. The covariance matrix for the measurement noise $w(t)$ is W = 35, and the covariance matrix for the disturbance $v(t)$ is

$$V = \begin{pmatrix} 10 & 0 \\ 0 & 2 \end{pmatrix}$$

e) Write down the continuous Kalman-filter for (1a)-(1b).

f) The stationary solution to the Riccati equation is

$$P = \begin{pmatrix} 10.21 & -3.51 \\ -3.51 & 2.17 \end{pmatrix}.$$

Calculate the stationary Kalman-filter, that is, the stationary matrix gain. Compare the effect of measurement noise for the Kalman filter with the observer above.

Exercise 4 (15 %)

Let $u(t)$ be a white noise process with zero mean and variance σ_u^2.

a) What is the autocorrelation function and the power spectral density function for the process? Is this a stationary process? Justify your answer.

b) Let $\dot{x}(t) = u(t)$. Find the mean and the variance for $x(t)$. Is this a stationary process? Justify your answer.

c) Find a spectral factorization of the following power spectral density function:

$$S_x(j\omega) = \frac{\omega^2 + 1}{\omega^2 + 100}.$$

Exercise 5 (10 %)

Assume that a process has the following transfer function:

$$G(s) = \frac{Ke^{-\tau s}}{1 + Ts}$$

where the parameters are unknown.

a) Assume that you can freely choose the input signal. Describe a method to determine the time delay τ from an experiment.

b) Assume that you *cannot* choose the input signal, but that the input can be viewed as a random stationary signal. Describe a method to determine the time delay τ from an experiment.

Attachments to the exam (Some useful formulas and expressions)

$$
\begin{aligned}
x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \\
x(k) &= A^k x(0) + \sum_{m=0}^{k-1} A^{k-1-m}Bu(m) \\
A^{-1} &= \frac{adj(A)}{det(A)} \\
det(A) &= \sum_{i=1}^{n} a_{ij}c_{ij} \\
adj(A) &= \{c_{ij}\}^T \\
c_{ij} &= (-1)^{i+j}det(A_{ij}) \quad \text{(co-factor)}, \; A_{ij} = \text{sub-matrix of } \; A
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{C} &= (B \; AB \; A^2B \; \cdots \; A^{n-1}B) \\
\mathcal{O} &= \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix} \\
G(s) &= C(sI - A)^{-1}B + D \\
G(z) &= C(zI - A)^{-1}B + D \\
\mathbf{G(s)} &= \mathbf{G}(\infty) + \mathbf{G_{sp}(s)} \\
d(s) &= s^r + \alpha_1 s^{r-1} + \cdots + \alpha_{r-1}s + \alpha_r \\
\mathbf{G_{sp}}(s) &= \frac{1}{d(s)}[\mathbf{N_1}s^{r-1} + \mathbf{N_2}s^{r-2} + \cdots + \mathbf{N_{r-1}}s + \mathbf{N_r}] \\
\dot{\mathbf{x}} &= \begin{bmatrix} -\alpha_1\mathbf{I_p} & -\alpha_2\mathbf{I_p} & \cdots & -\alpha_{r-1}\mathbf{I_p} & -\alpha_r\mathbf{I_p} \\ \mathbf{I_p} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I_p} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I_p} & \mathbf{0} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{I_p} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} \mathbf{u} \\
\mathbf{y} &= [\; \mathbf{N_1} \;\; \mathbf{N_2} \;\; \cdots \;\; \mathbf{N_{r-1}} \;\; \mathbf{N_r} \;]\mathbf{x} + \mathbf{G}(\infty)\mathbf{u}
\end{aligned}
$$

Discrete Kalman-filter:

$$
\begin{aligned}
\mathbf{x}_{k+1} &= \mathbf{\Phi}_k \mathbf{x}_k + \mathbf{w}_k \\
\mathbf{z}_k &= \mathbf{H}_k \mathbf{x}_k + \mathbf{v}_k \\
E[\mathbf{w}_k \mathbf{w}_i^T] &= \begin{cases} \mathbf{Q}_k, & i = k \\ 0, & i \neq k \end{cases} \\
E[\mathbf{v}_k \mathbf{v}_i^T] &= \begin{cases} \mathbf{R}_k, & i = k \\ 0, & i \neq k \end{cases} \\
E[\mathbf{w}_k \mathbf{v}_i^T] &= 0, \forall i, k \\
\mathbf{P}_k^- &= E[\mathbf{e}_k^- \mathbf{e}_k^{-T}] \\
\mathbf{P}_k &= E[\mathbf{e}_k \mathbf{e}_k^T] = (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)\mathbf{P}_k^- (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)^T + \mathbf{K}_k \mathbf{R}_k \mathbf{K}_k^T \\
\mathbf{K}_k &= \mathbf{P}_k^- \mathbf{H}_k^T \left(\mathbf{H}_k \mathbf{P}_k^- \mathbf{H}_k^T + \mathbf{R}_k \right)^{-1} \\
\mathbf{P}_{k+1}^- &= \mathbf{\Phi}_k \mathbf{P}_k \mathbf{\Phi}_k + \mathbf{Q}_k
\end{aligned}
$$

Continuous Kalman filter:

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{F}\mathbf{x} + \mathbf{G}\mathbf{u} \\
\mathbf{z} &= \mathbf{H}\mathbf{x} + \mathbf{v} \\
E[\mathbf{u}(t)\mathbf{u}(\tau)^T] &= \mathbf{Q}\delta(t - \tau) \\
E[\mathbf{v}(t)\mathbf{v}(\tau)^T] &= \mathbf{R}\delta(t - \tau) \\
E[\mathbf{u}(t)\mathbf{v}(\tau)^T] &= 0 \\
\mathbf{K} &= \mathbf{P}\mathbf{H}^T\mathbf{R}^{-1} \\
\dot{\mathbf{P}} &= \mathbf{F}\mathbf{P} + \mathbf{P}\mathbf{F}^T - \mathbf{P}\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\mathbf{P} + \mathbf{G}\mathbf{Q}\mathbf{G}^T, \quad \mathbf{P}(0) = \mathbf{P}_0
\end{aligned}
$$

Autocorrelation:

$$
\begin{aligned}
R_X(\tau) &= E[X(t)X(t+\tau)] \text{ (Stationary process)} \\
R_X(t_1, t_2) &= E[X(t_1)X(t_2)] \text{ (Non-stationary process)} \\
Y(s) &= G(s)U(s) \Rightarrow \\
R_y(t_1, t_2) &= E[y(t_1)y(t_2)] \\
&= \int_0^{t_2} \int_0^{t_1} g(\xi)g(\eta)E\left[u(t_1 - \xi)u(t_2 - \eta)\right] d\xi d\eta \text{ (Transient analyzes)}
\end{aligned}
$$

Least squares estimation:

$$
\begin{aligned}
\text{Objective function in least squares :} &\quad V(\bar{\theta}) = \frac{1}{2}(\mathcal{Y}_N - \Phi_N\bar{\theta})^T(\mathcal{Y}_N - \Phi_N\bar{\theta}) \\
\text{Covariance matrix for } \hat{\theta} \text{ :} &\quad \sigma_e^2(\Phi_N^T\Phi_N)^{-1} \\
\text{Estimate of } \sigma_e^2 \text{ :} &\quad \hat{\sigma}_e^2 = \frac{2}{N-p}V(\hat{\theta})
\end{aligned}
$$

Some simple Laplace transforms:

$f(t)$	$\iff$	$F(s)$
1	$\iff$	$\frac{1}{s}$
e^{-at}	$\iff$	$\frac{1}{s+a}$
t	$\iff$	$\frac{1}{s^2}$
t^2	$\iff$	$\frac{2}{s^3}$
te^{-at}	$\iff$	$\frac{1}{(s+a)^2}$
$\sin \omega t$	$\iff$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\iff$	$\frac{s}{s^2+\omega^2}$