NORGES TEKNISK- NATURVITENSKAPELIGE UNIVERSITET
INSTITUTT FOR ENERGI- OG PROSESSTEKNIKK

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EKSAMEN I FAG TEP4100 FLUIDMEKANIKK

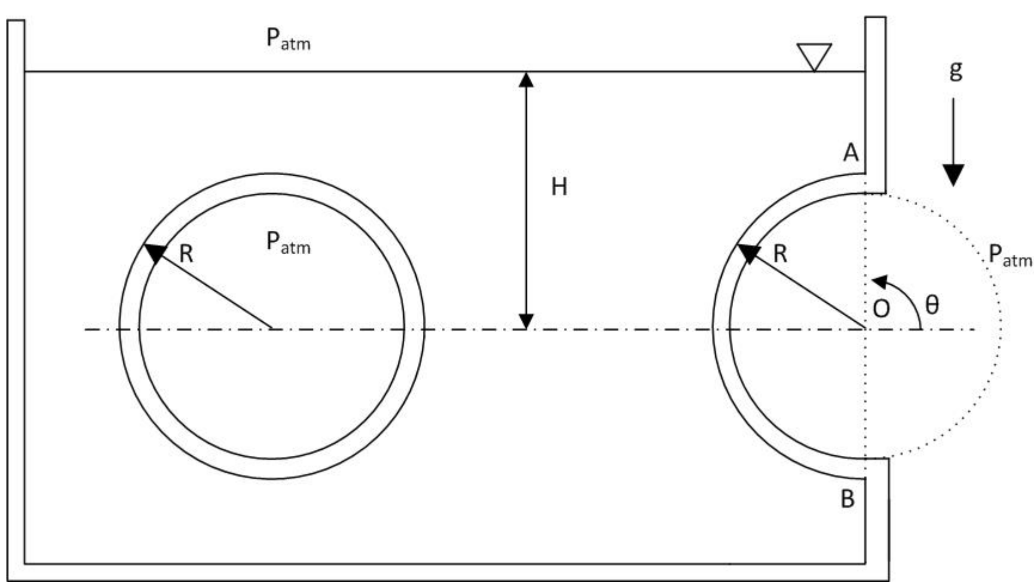
Bokmål

Onsdag 29. mai 2013
Tid: kl. 09.00 - 13.00

Hjelpemidler C: Bestemt enkel kalkulator
K. Rottmann: "Matematisk formelsamling"

Sensuren faller i uke nr 25

Oppgave 1



Figur 1

Akvariet i Trondheim skal bygge en ny saltvannstank og ønsker å tilby de besøkende en spektakulær opplevelse. De ønsker å bygge en gjennomsiktig, sirkulær sylinder som er åpen mot atmosfæren i begge endene, som de besøkende kan gå gjennom, se skisse til venstre på Figur 1.

Ytre radius er R og lengden inn i papirplanet er L. For spesifikk tyngde (= relativ tetthet) på saltvannet i tanken brukes symbol SG, for tettheten til rent vann brukes symbol ρ.

a) • Finn et uttrykk for oppdriftskraften som virker fra vannet i tanken på sylinderen

På grunn av høye kostnader for alternativet med en hel sylinder undersøkes også alternativet med en halvsylinder (begrenset av linjen AB) i veggen av tanken som til høyre i Figur 1.

b) Halvsylinderen skal lages av to kvartsylindere.

• Finn uttrykk for nettokraft i vertikalretning som virker fra vannet på den øvre kvartsylinderen, på den nedre kvartsylinderen og på halvsylinderen.

c) • Finn et uttrykk for resultantkraften som virker fra vannet på halvsylinderen.

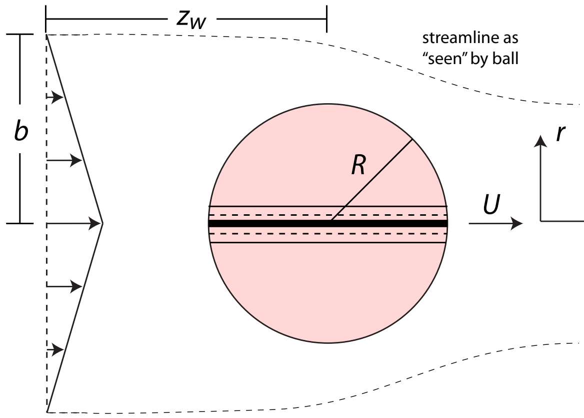
• Finn også et uttrykk for vinkelen resultantkraften har med horisontalplanet, og til slutt vinkelen θ_{angrep} som angrepspunktet på halvsylinderen kan beskrives med. Angrepspunktet er punktet som resultantkraften virker gjennom på halvsirkelens overflate med θ som definert i Figur 1 (*Hint*: Det er ikke nødvendig med noen integrasjon for å finne θ_{angrep}).

d) Følgende Matlab-kode er gitt, og du kan anta at konstantene R og H er definert slik at man ikke får en feilmelding når koden kjøres:

```
%%
theta=0:12:360;
for i=1:length(theta)

    x_sirkel(i)=R*cosd(theta(i));
    y_sirkel(i)=R*sind(theta(i));

    trykkhoyde(i)=H-y_sirkel(i);

    Pillengdex(i)=trykkhoyde(i)*cosd(theta(i));
    Pillengdey(i)=trykkhoyde(i)*sind(theta(i));

    Pilstartx(i)=x_sirkel(i)+Pillengdex(i);
    Pilstarty(i)=y_sirkel(i)+Pillengdey(i);

end
plot(x_sirkel,y_sirkel)
hold on
quiver(Pilstartx,Pilstarty,-Pillengdex,-Pillengdey,0)
%%
```

• Forklar hva denne figuren vil vise, og tegn en enkel skisse av hva som vises.

Oppgave 2



Figur 2 a) Venstre: geometri sett ovenfra. Høyre: Ellyse Perry (Australia), verdens beste kvinnelige "fast-bowler".

I sporten cricket brukes en lærdekket treball med radius R. Denne "bowles" (dvs. kastes, men med rett albue) mot en "batsman"(spiller med balltre fra motlaget) som slår ballen. Langs ballens ekvator er det en søm som forbinder de to halvdelene av lær. Gode bowlere kan få søm-planet til å stå parallelt med bevegelsesretningen. Noen spillere på hvert cricketlag er spesialister på å bowle ballen raskt, og de raskeste bowlerne i verden bowler i rundt 150 km/h. Se bort fra gravitasjon i hele oppgaven.

a) Anta at ballen beveger seg i en rett linje med sømplanet langs bevegelsesretningen og normalt på bakken. Figuren 2 a) viser ballen sett ovenfra. Vaken en avstand z_w bak ballen kan modelleres som en lineær funksjon av radius (i sylinderkoordinater) som illustrert.

$$u(r) = \begin{cases} U(1 - r/b), & r \leq b, \\ 0, & r > b \end{cases}.$$

Her er b vakebredden. Vi antar at hastigheten U er konstant, og vil beregne krefter på ballen ved å bruke et kontrollvolum (CV).

- Tegn en figur for å vise hvordan du velger kontrollvolum og hvorfor.
- Bruk koordinatsystemet som følger ballen og beregn trykket i vaken som funksjon av r. Du kan se bort fra friksjon i denne deloppgaven.

b) Netto trykk-kraft på kontrollvolumet oppgis å være for den virkelige strømningen med friksjon

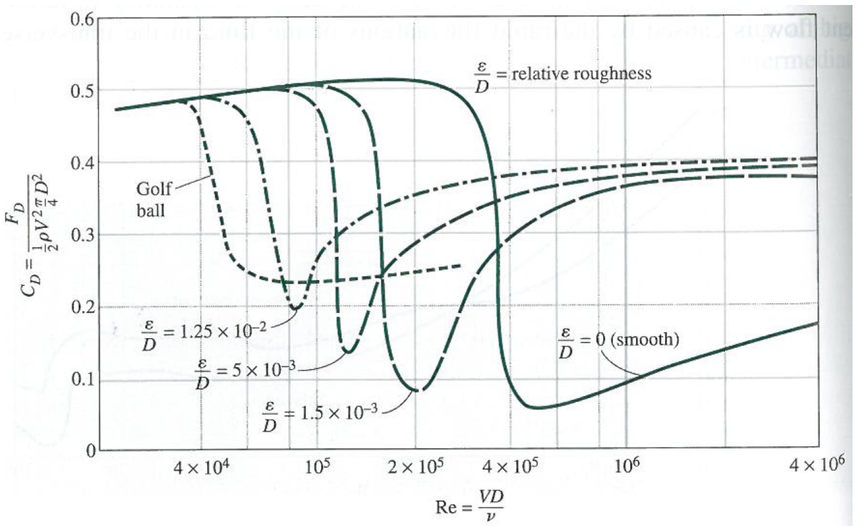
$$\vec{F}_p = \frac{\pi}{12}\rho U^2 b^2 \vec{e}_z.$$

- Finn drag-kraften på ballen ved å bruke kontrollvolumet fra oppgave 2 a) (*Hint*: Det er enklest ikke å la koordinatsystemet følge ballen, men ligge i ro).
- Uttrykk b ved hjelp av dragkoeffisienten C_D og R.

c) Når kampen har vart en stund har ballen blitt slitt og ru. Dragkoeffisienten for en kule varierer som funksjon av Re_D og overflateruhet som vist i grafen, se figur 2 c). Ved temperatur 15°C (på Lord's cricket ground i London) er dynamisk viskositet og tetthet for luft henholdsvis $\mu = 1.87 \times 10^{-5}$kg/(ms) og $\rho = 1.23$kg/m^3. Ballens hastighet er $U = 150$km/h og radius $R = 36$mm.

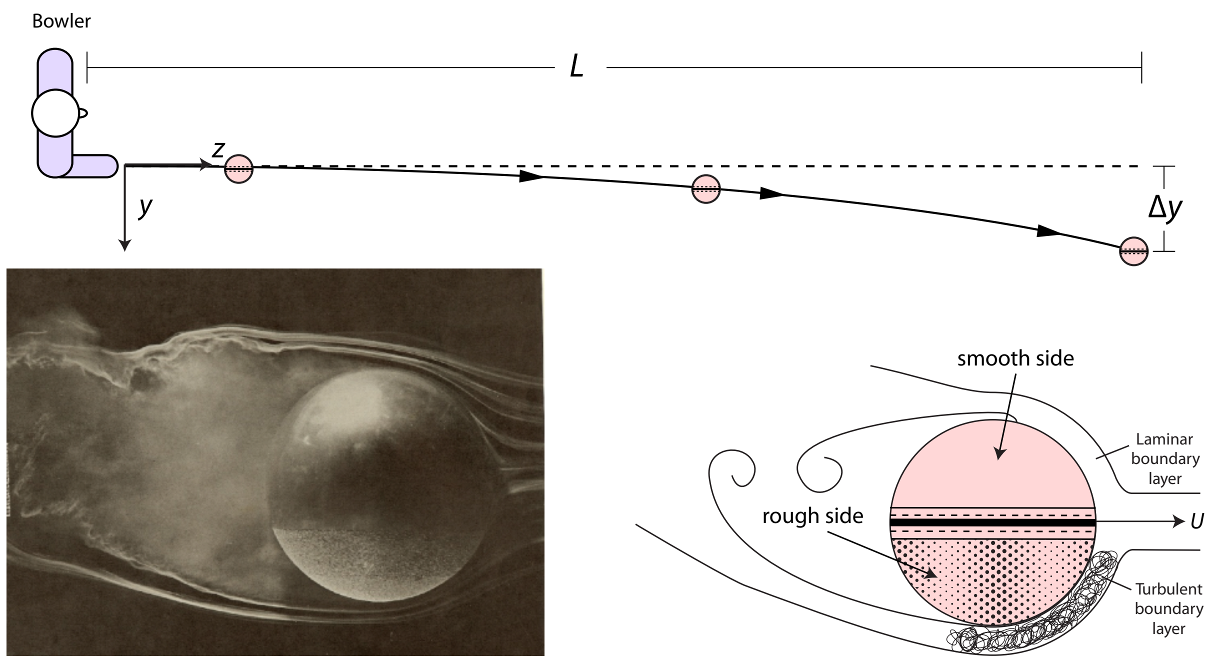
• Bruk grafen til å estimere dragkoeffisient C_D og vakebredde b for (i) en glatt ball og (ii) en ru ball med $\varepsilon/D = 0.0015$. Hvis du ikke har løst oppgave 2 b), kan du bruke $b = 2C_DR$ istedet for den korrekte relasjonen.

• Forklar kort hvorfor vakene bak ru og glatt ball har forskjellig bredde, helst ved hjelp av en figur.



Figur 2 c) Dragkoeffisient for en kule.

d)



Figur 2 d) Øvre: Banelinje med sving sett ovenfra. Nedre venstre: Vindtunnelrøykvisualisering sett ovenfra. Nedre høyre: Skisse av "kontrastsving" for en ball med én ru og én glatt side sett ovenfra. Illustrasjon tilpasset fra R.D. Mehta "An overview of cricket ball swing" *Sports Engineering* **8**, 181 (2005).

"Fast-bowlere" i cricket bruker forskjellige teknikker for å få ballen til å svinge (dvs. bevege seg sidelengs) mens den er i lufta. Én måte er å hele tiden polere den ene siden av ballen slik at etter en stund er halve ballen ru mens den andre halvdelen er glatt. Ballen vil nå kunne svinge mot den ru sida.

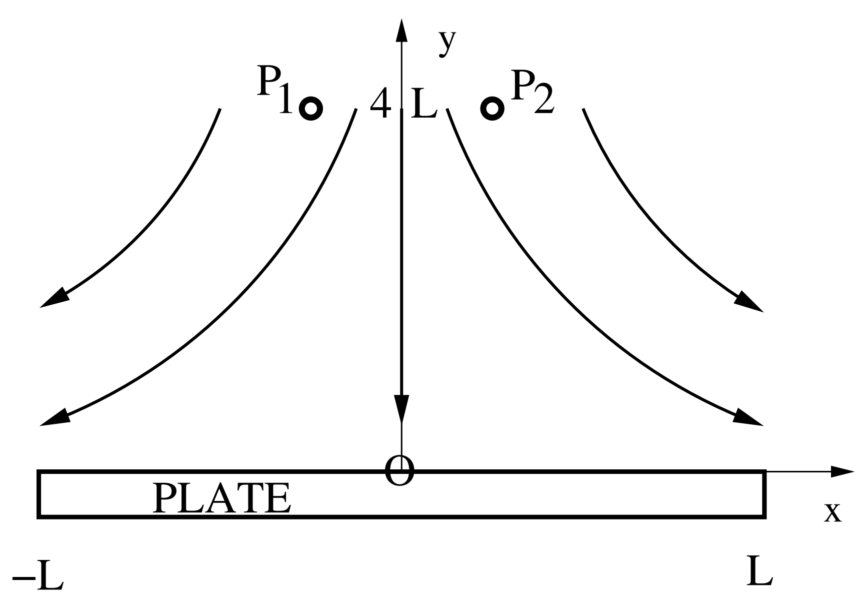
Forskjellen mellom grensesjiktene rundt ballen gir en trykkdifferanse så ballen virker som en flyvinge. Figur 2 d) viser en visualisering fra et vindtunneleksperiment der løftkoeffisienten (for sidelengs "løft") C_L ble målt.

- Anta at hastigheten i z-retning er konstant og utled et algebraisk uttrykk for hvor langt ballen har beveget seg sidelengs, Δy, se figur 2 d), innen ballen kommer fram til batsmannen når du kjenner C_L. Avstanden fra bowler til batsmann er L og ballens masse er m.

- Finn den numeriske verdien for Δy når $L = 20\text{m}$, $C_L = 0.07$, $m = 160\text{g}$ og $\rho = 1.23\text{kg/m}^3$.

Oppgave 3

Betrakt plan vindstrømning mot taket av Trondheim Spektrum modellert som en flat plate av lengde $2L$ og bredde b inn i papiret, se figur 3. Lengden og bredden av platen er så store at endeeffekter kan neglisjeres og strømningen kan anses som todimensjonal. Tetthet ρ og dynamisk viskositet μ av luft er konstante. Trykket P_0 i origo er gitt. Effekter fra gravitasjon kan neglisjeres, dvs. $\vec{g} = 0$.



Figur 3

DEL I

a) Anta at hastighetsfeltet av vindstrømningen er gitt i kartesiske koordinater ved

$$u = ax\,, \qquad v = -ay\,, \tag{1}$$

der a er en positiv konstant med dimensjon av frekvens.

- Eksisterer det en strømfunksjon for denne strømningen? Hvis ja, bestem den.
- Bestem trykkfeltet. Begrunn dine antakelser.

b) • Bestem strømlinje 1 gjennom punkt P_1 ved $(x_1, y_1) = (-\frac{L}{4}, 4L)$ og strømlinje 2 gjennom punkt P_2 ved $(x_2, y_2) = (\frac{L}{4}, 4L)$ som funksjoner $y(x)$.

- Beregn volumstrøm per enhetsbredde mellom strømlinjene 1 og 2.

DEL II

c) Anta at et annet hastighetsfelt av vindstrømningen er gitt i kartesiske koordinater ved

$$u = xf'(y)\,, \qquad v = -f(y)\,, \tag{2}$$

der $f(y)$ er en glatt funksjon og $f'(y)$ dens første deriverte. Trykkfeltet antas er gitt av

$$P = P_0 - \frac{\rho}{2}a^2[x^2 + h(y)]\,, \tag{3}$$

der a er den samme positive konstanten som i ligning (1) og $h(y)$ er en glatt funksjon. Ved å bruke x-komponenten av Navier-Stokes-ligningen får vi følgende differensialligning for $f(y)$:

$$[f'(y)]^2 - f(y)f''(y) = a^2 + \nu f'''(y)\,.$$

Dette skal ikke vises!

- Er kontinuitetsligningen oppfylt? Begrunn svaret.
- Differensialligningen for $f(y)$ er av tredje orden, så vi trenger tre grensebetingelser for $f(y)$. Langt fra taket krever vi at $u(x, y \to \infty) = ax$ og på taket $y = 0$ har vi den vanlige heftbetingelsen. Hva er de tre grensebetingelsene for $f(y)$?

d) Anta at differensialligningen for funksjonen $f(y)$ i oppgave 3 c) ovenfor har blitt løst. Anta at trykket på undersiden av den flate platen som modellerer taket av Trondheim Spektrum er lik trykket P_0.

- Bestem kraften som vindstrømningen utøver på taket av Trondheim Spektrum, nemlig på den venstre delen av taket $[-L, 0]$, den høyre delen av taket $[0, L]$ og hele taket $[-L, L]$. Både x- oy y-komponentene av kraften forlanges. Dine svar skal uttrykkes ved hjelp av funksjonen $f(y)$ eller dens deriverte og andre strømningsparametere.

Hint: Ligning (3) og betingelsen $P(0, 0) = P_0$ medfører $h(0) = 0$.
Veggskjærspenningen er definert ved $\tau_w = \mu \frac{\partial u}{\partial y}_{|y=0}$.
$f''(0)$ er en positiv konstant.

Lykke til!

Formulae, TEP4100 Fluidmekanikk

Ideal gas law (2-4):

$$P = \rho RT; \quad R_{\text{air}} = 287\, Pa\, m^3/kg\, K$$

Kinematic viscosity

$$\nu = \frac{\mu}{\rho}.$$

Surface tension (soap bubble)(2-41)

$$\Delta P = \frac{4\sigma_s}{R}$$

Hydrostatics, constant density (3-6)

$$P_2 - P_1 = -\gamma \Delta z; \quad \gamma = \rho g.$$

Lakes and oceans (3-8)

$$P = P_{atm} + \rho g h.$$

Hydrostatic pressure distribution (3-9)

$$\frac{dP}{dz} = -\rho g.$$

Hydrostatic force on plane submerged surface (3-19)

$$F_R = (P_0 + \rho g h_{\text{C}})A = P_{\text{C}} A.$$

Centre of pressure (3-22 a,b)

$$y_P = y_C + \frac{I_{xx,C}}{[y_C + P_0/(\rho g \sin\theta)]\, A}; \quad y_P = y_c + \frac{I_{xx,C}}{y_c A}$$

Pressure distribution in rigid body motion (3-41)

$$\vec{\nabla} P + \rho g \vec{k} = -\rho \vec{a}.$$

Pressure distribution in rigid body rotation (3-64)

$$P = P_0 + \frac{1}{2}\rho\omega^2 r^2 - \rho g z.$$

Acceleration (4-9,4-11) (Cartesian)

$$\frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$$

On a streamline (4-15)

$$\frac{dr}{V} = \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}.$$

Vorticity (4-28,29)

$$\vec{\zeta} = 2\vec{\omega} = \vec{\nabla} \times \vec{V}$$

Reynolds transport theorem (4-41)

$$\frac{d}{dt}B_{\text{sys}} = \frac{d}{dt}\left(\int_{\text{CV}} \rho b\, d\mathcal{V}\right) + \int_{\text{CS}} \rho b(\vec{V}\cdot\vec{n})dA.$$

Volume flow rate though cross section A_c (5-8)

$$\dot{\mathcal{V}} = \int_{A_c} V_n dA; \quad V_n = \vec{V}\cdot\vec{n}$$

Conservation of mass, fixed CV (5-17)

$$\frac{d}{dt}\int_{\text{CV}} \rho d\mathcal{V} + \int_{\text{CS}} \rho(\vec{V}\cdot\vec{n})dA = 0.$$

Bernoulli equation along streamline, unsteady compressible flow (5-44)

$$\int \frac{dP}{\rho} + \int \frac{\partial V}{\partial t} ds + \frac{V^2}{2} + gz = \text{constant}$$

Bernoulli equation along streamline, steady incompressible flow (5-48)

$$\frac{P}{\rho g} + \frac{V^2}{2g} + z = \text{constant}$$

Energy equation, fixed CV (5-60)

$$\dot{Q}_{\text{net in}} + \dot{W}_{\text{shaft, net in}} = \frac{d}{dt}\int_{\text{CV}} e\rho d\mathcal{V} + \int_{\text{CS}} (\frac{P}{\rho} + e)\rho(\vec{V}\cdot\vec{n})dA$$

where total energy per unit mass is (5-50)

$$e = u + \frac{V^2}{2} + gz.$$

Energy equation for steady flow with one inlet and one outlet(5-74)

$$\frac{P_1}{\rho_1 g} + \frac{V_1^2}{2g} + z_1 + h_{\text{pump,u}} = \frac{P_2}{\rho_2 g} + \frac{V_2^2}{2g} + z_2 + h_{\text{turbine,e}} + h_{\text{L}}$$

Linear momentum equation, fixed CV (6-7)

$$\sum \vec{F} = \frac{d}{dt}\left(\int_{\text{CV}} \rho\vec{V} d\mathcal{V}\right) + \int_{\text{CS}} \rho\vec{V}(\vec{V}\cdot\vec{n})dA.$$

Net pressure force on closed CS

$$\vec{F}_{press} = -\int_{CS} P_{gage}\vec{n}\, dA$$

Angular momentum equation, fixed CV (6-48)

$$\sum \vec{M} = \frac{d}{dt}\int_{\text{CV}} (\vec{r}\times\vec{V})\rho d\mathcal{V} + \int_{\text{CS}} (\vec{r}\times\vec{V})\rho(\vec{V}\cdot\vec{n})dA.$$

Critical Reynolds number, pipe flow (p.340)

$$\text{Re}_{\text{crit}} \approx 2300$$

Entry length (8-6,7)

$$\text{laminar} \qquad \frac{L_{h,\text{ laminar}}}{D} \approx 0.05\text{Re}$$

$$\text{turbulent} \qquad \frac{L_{h,\text{ turbulent}}}{D} \approx 1.359\text{Re}^{1/4}$$

Laminar friction factor (8-23)

$$f = \frac{64}{\text{Re}}.$$

Pipe head loss (f is Darcy friction factor) (8-24)

$$h_L = f\frac{L}{D}\frac{V_{avg}^2}{2g}.$$

Colebrook's formula (8-50)

$$\frac{1}{\sqrt{f}} = -2.0\log\left[\frac{\varepsilon/D}{3.7} + \frac{2.51}{\mathrm{Re}\sqrt{f}}\right].$$

Haaland's formula (8-51)

$$\frac{1}{\sqrt{f}} \cong -1.8\log\left[\frac{6.9}{\mathrm{Re}} + \left(\frac{\varepsilon/D}{3.7}\right)^{1.11}\right]$$

Prandtl's approximation, turbulent pipe flow (p.358)

$$\frac{1}{\sqrt{f}} = 2.0\log(\mathrm{Re}\sqrt{f}) - 0.8.$$

Total head loss (8-58,59)

$$h_{\mathrm{L,\ total}} = h_{\mathrm{L,\ major}} + h_{\mathrm{L,\ minor}} = \left(f\frac{L}{d} + \sum K_L\right)\frac{V^2}{2g}$$

Continuity equation (9-5)

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\left(\rho\vec{V}\right) = 0$$

Continuity equation in cylindrical coordinates (9-12)

$$\frac{\partial\rho}{\partial t} + \frac{1}{r}\frac{\partial(r\rho u_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho u_\theta)}{\partial\theta} + \frac{\partial(\rho u_z)}{\partial z} = 0$$

Incompressible continuity equation (9-16)

$$\vec{\nabla}\cdot\vec{V} = 0.$$

Incompressible stream function $\psi(x,y)$ (9-20,27,29)

$$\begin{aligned}
\text{(Cartesian) } u &= \frac{\partial\psi}{\partial y}; \quad v = -\frac{\partial\psi}{\partial x}.\\
\text{(Cylindrical) } u_r &= \frac{1}{r}\frac{\partial\psi}{\partial\theta}; \quad u_\theta = -\frac{\partial\psi}{\partial r}\\
\text{(Axisymmetric flow) } u_r &= -\frac{1}{r}\frac{\partial\psi}{\partial z}; \quad u_z = \frac{1}{r}\frac{\partial\psi}{\partial r}.
\end{aligned}$$

Incompressible Navier-Stokes equation (9-60)

$$\rho\frac{d\vec{V}}{dt} = -\vec{\nabla}P + \rho\vec{g} + \mu\nabla^2\vec{V}.$$

Fully developed Poiseuille flow (example 9-18 eq. 9)

$$u = \frac{1}{4\mu}\frac{dp}{dz}\left(r^2 - R^2\right).$$

Velocity potential (10-20)

$$\text{If } \vec{\zeta} = 0 \text{ then } \vec{V} = \vec{\nabla}\phi.$$

2D irrotational flow (10-28,30)

$$\nabla^2\phi = \nabla^2\psi = 0.$$

Plane flow (10-36,37,43,46,50,51)

$$\begin{aligned}
&\text{(uniform stream)} && \psi = Vy && \phi = Vx\\
&\text{(source/sink)} && \psi = \frac{\dot{\mathcal{V}}/L}{2\pi}\theta && \phi = \frac{\dot{\mathcal{V}}/L}{2\pi}\ln r\\
&\text{(line vortex)} && \psi = -\frac{\Gamma}{2\pi}\ln r && \phi = \frac{\Gamma}{2\pi}\theta\\
&\text{(doublet)} && \psi = -K\frac{\sin\theta}{r} && \phi = K\frac{\cos\theta}{r}
\end{aligned}$$

Flow past cylinder (10-55)

$$\psi = V_\infty\sin\theta(r - a^2/r)$$

Displacement thickness (10-72)

$$\delta^* = \int_0^\infty\left(1 - \frac{u}{U}\right)dy.$$

Momentum thickness (10-80)

$$\theta = \int_0^\infty\frac{u}{U}\left(1 - \frac{u}{U}\right)dy.$$

Flat plate boundary layer thickness (Table 10-4,p.550)

$$\frac{\delta}{x} \approx \begin{cases}\frac{4.91}{\mathrm{Re}_x^{1/2}}, & \text{laminar}\\[1ex] \frac{0.16}{\mathrm{Re}_x^{1/7}}, & \text{turbulent}\end{cases}$$

Log Law ($\kappa \approx 0.41, B \approx 5.0$) (10-83)

$$\frac{u}{u_*} = \frac{1}{\kappa}\ln\frac{yu_*}{\nu} + B$$

Friction velocity (10-84)

$$u_* = \sqrt{\frac{\tau_w}{\rho}}.$$

Local skin friction coefficient (10-96)

$$C_{\mathrm{f,\ x}} = \frac{\tau_w}{\frac{1}{2}\rho U^2}$$

Karman integral equation for flat plate boundary layer (10-97)

$$C_{\mathrm{f,\ x}} = 2\frac{d\theta}{dx}$$

Drag and lift coefficients (11-5,6)

$$C_D = \frac{F_{\mathrm{D}}}{\frac{1}{2}\rho V^2 A}; \quad C_L = \frac{F_{\mathrm{L}}}{\frac{1}{2}\rho V^2 A}.$$

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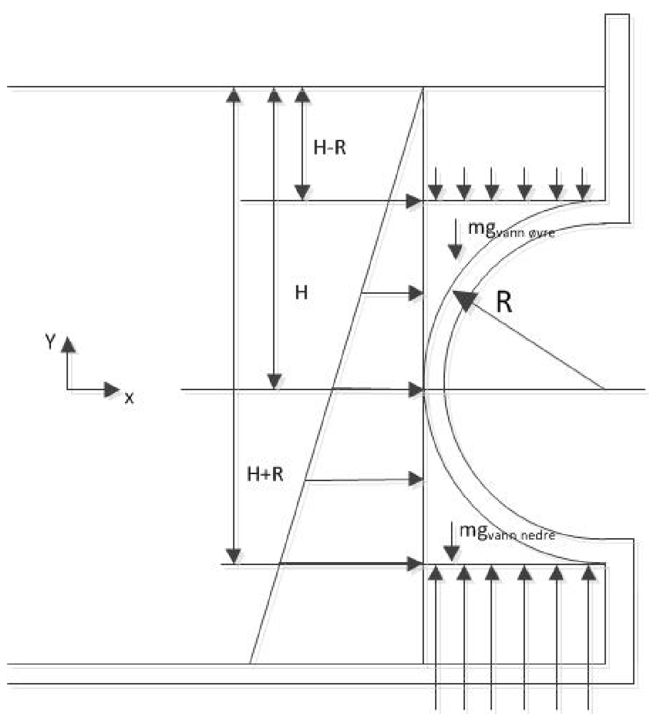
SOLUTION TO EXAM IN TEP4100 FLUID MECHANICS
29.05.2013

Problem 1

a) The buoyancy force exerted on the cylinder by the water is equal to the weight of the displaced water volume and is directed straight up (vector notation is therefore not shown here).

$$\underline{\underline{F_{\text{buoyancy}}}} = \rho \cdot SG \cdot g \cdot \mathcal{V} = \underline{\underline{\rho \cdot SG \cdot g \cdot \pi \cdot R^2 \cdot L}}$$

b) Split the half cylinder into an upper and lower quarter cylinder, remember that net force means that we won't need to account for the atmospheric pressure since it is present on both sides, with positive direction defined in the positive y-direction. The vertical force on



Figur 1

the upper quarter cylinder is determined as the sum of the pressure force on its horizontal projection and the weight of the water between the surface of the upper quarter cylinder and its horizontal projection:

$$\vec{F}_{\text{V,upper}} = \underbrace{\rho \cdot SG \cdot g \cdot (H-R)}_{\text{Pressure on upper surface}} \cdot \underbrace{R \cdot L}_{\substack{\text{Upper}\\\text{horizontal}\\\text{area}}} \cdot \underbrace{\left(-\vec{j}\right)}_{\substack{\text{points}\\\text{downwards}}} + \underbrace{\rho \cdot SG \cdot \underbrace{\left(R^2 - \frac{\pi R^2}{4}\right)}_{\substack{\text{Volume of water=}\\\text{volume square } (R^2)\\\text{- volume quarter circle}}} \cdot L}_{m_{\text{water, upper}}} \cdot g \cdot \underbrace{\left(-\vec{j}\right)}_{\substack{\text{points}\\\text{downwards}}}$$

After a bit of cleaning up:

$$\vec{F}_{\text{V,upper}} = -\rho \cdot SG \cdot g \cdot R \cdot L \cdot \vec{j}\left((H-R) + \left(R - \frac{\pi R}{4}\right)\right) = \rho \cdot SG \cdot g \cdot R \cdot L\left(\frac{\pi R}{4} - H\right) \cdot \vec{j}$$

For the lower quarter circle the expression is as shown below.

$$\vec{F}_{\text{V,lower}} = \underbrace{\rho \cdot SG \cdot g \cdot (H+R)}_{\text{Pressure on lower surface}} \cdot \underbrace{R \cdot L}_{\substack{\text{Lower horizontal}\\\text{area}}} \cdot \underbrace{\vec{j}}_{\substack{\text{points}\\\text{upwards}}} + \underbrace{\rho \cdot SG \cdot \underbrace{\left(R^2 - \frac{\pi R^2}{4}\right)}_{\substack{\text{Volume of water=}\\\text{volume of square } (R^2)\\\text{- volume of quarter circle}}} \cdot L}_{m_{\text{water, lower}}} \cdot g \cdot \underbrace{\left(-\vec{j}\right)}_{\substack{\text{points}\\\text{downwards}}}$$

After a bit of cleaning up:

$$\vec{F}_{\text{V,lower}} = \rho \cdot SG \cdot g \cdot R \cdot L\left(\frac{\pi R}{4} + H\right) \cdot \vec{j}$$

Combining the two vectors:

$$\begin{aligned}
\underline{\underline{\vec{F}_{\text{V}}}} &= \vec{F}_{\text{V,upper}} + \vec{F}_{\text{V,lower}} \\
&= \rho \cdot SG \cdot g \cdot R \cdot L\left(\frac{\pi R}{4} - H\right) \cdot \vec{j} + \rho \cdot SG \cdot g \cdot R \cdot L\left(\frac{\pi R}{4} + H\right) \cdot \vec{j} \\
&= \rho \cdot SG \cdot g \cdot R \cdot L\left(\left(\frac{\pi R}{4} - H\right) + \left(\frac{\pi R}{4} + H\right)\right) \cdot \vec{j} \\
&\underline{\underline{= \rho \cdot SG \cdot g \cdot L \cdot \frac{\pi R^2}{2} \cdot \vec{j}}}
\end{aligned}$$

As we see the answer is half of the answer we found in a), which is logical since the half cylinder only displaces half the volume that the cylinder with the same radius did. Here we have also found the upwards direction of the force mathematically.

c) In the previous task we found the vertical component of the total force, hence we now need to find the horizontal component. Equation 3-19 on the formula sheet states that:

$$F_R = P_C \cdot A$$

This equation describes that the resultant force on a plane surface (that in our case is the rectangular projection of the half cylinder in the vertical plane) is equal to the pressure at the centroid of the surface (which is at the depth H) multiplied by the area of the surface, which is $2RL$. We can neglect the atmospheric pressure here as well since it is present on both sides, and we easily see that the force acts in the positive x-direction.

$$\vec{F}_{\mathrm{H}} = P_C \cdot A \cdot \vec{i} = \rho \cdot SG \cdot g \cdot H \cdot 2 \cdot R \cdot L \cdot \vec{i}$$

Now we utilize the Pythagorean theorem to find the hypotenuse of the right-angled triangle of forces:

$$\begin{aligned}
\underline{\underline{F_{\text{total}}}} &= \sqrt{F_{\mathrm{H}}^2 + F_{\mathrm{V}}^2} \\
&= \sqrt{(\rho \cdot SG \cdot g \cdot H \cdot 2 \cdot R \cdot L)^2 + \left(\rho \cdot SG \cdot g \cdot L \cdot \frac{\pi R^2}{2}\right)^2} \\
&= \rho \cdot SG \cdot g \cdot R \cdot L \sqrt{(2H)^2 + \left(\frac{\pi R}{2}\right)^2}
\end{aligned}$$

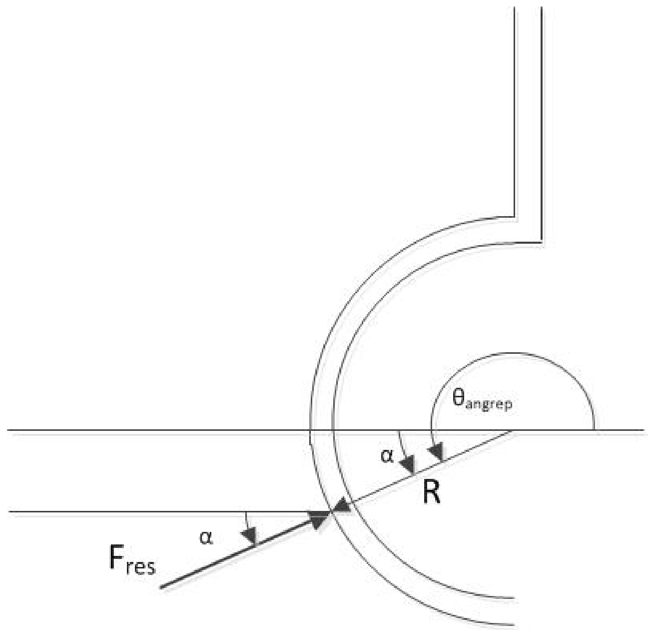
The angle that the total force has with the horizontal plane is given by

$$\begin{aligned}
\tan\alpha &= \frac{F_{\mathrm{V}}}{F_{\mathrm{H}}} \\
\Rightarrow \alpha &= \tan^{-1}\left(\frac{F_{\mathrm{V}}}{F_{\mathrm{H}}}\right) \\
&= \tan^{-1}\left(\frac{\rho \cdot SG \cdot g \cdot L \cdot \frac{\pi R^2}{2}}{\rho \cdot SG \cdot g \cdot H \cdot 2 \cdot R \cdot L}\right) \\
&= \tan^{-1}\left(\frac{\pi R}{4H}\right)
\end{aligned}$$

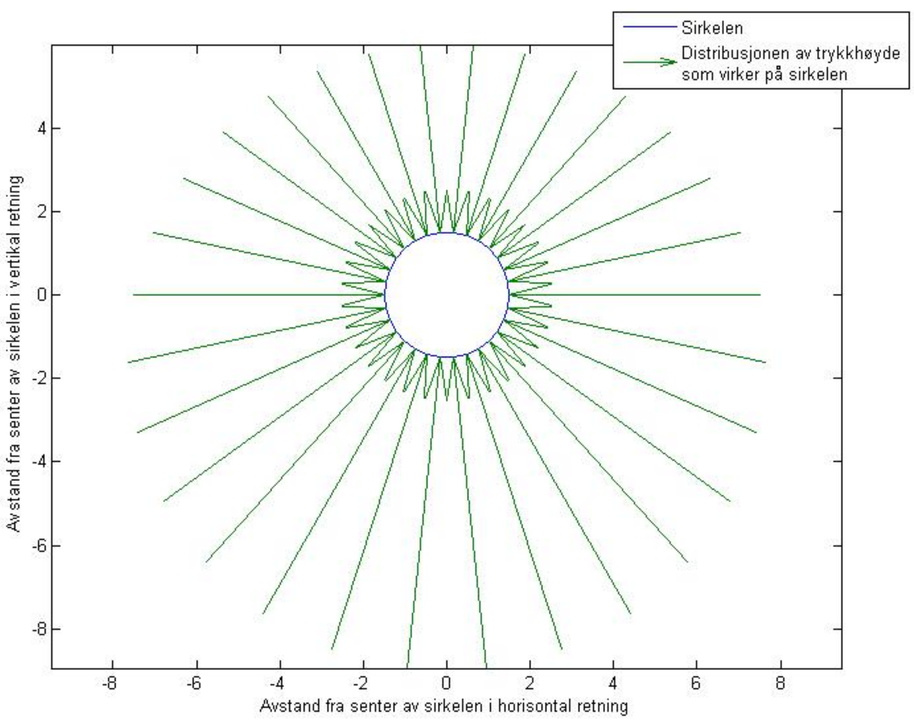
Hydrostatic forces always act normal to the surface. For a circular surface the normal always points in the radial direction. The force on the circular surface therefore always acts in the radial direction, and hence the total force also acts in this direction. We have found the angle the total force forms with the horizontal plane, and this angle is therefore also the angle that describes the point of action. With θ defined as in Figure 1, θ_{action} is equal to the angle we found plus a half rotation as shown in Figure 2.

$$\underline{\underline{\theta_{\text{action}} = \pi + \alpha = \pi + \tan^{-1}\left(\frac{\pi R}{4H}\right)}}$$

d) This MatLab code will draw a circle and then add arrows, pointing inwards on the surface of the circle, evenly around the circumference. The arrows will point in the radial direction and represent the static pressure height exerted on the circle by the fluid it is submerged in. This is shown in Figure 3.



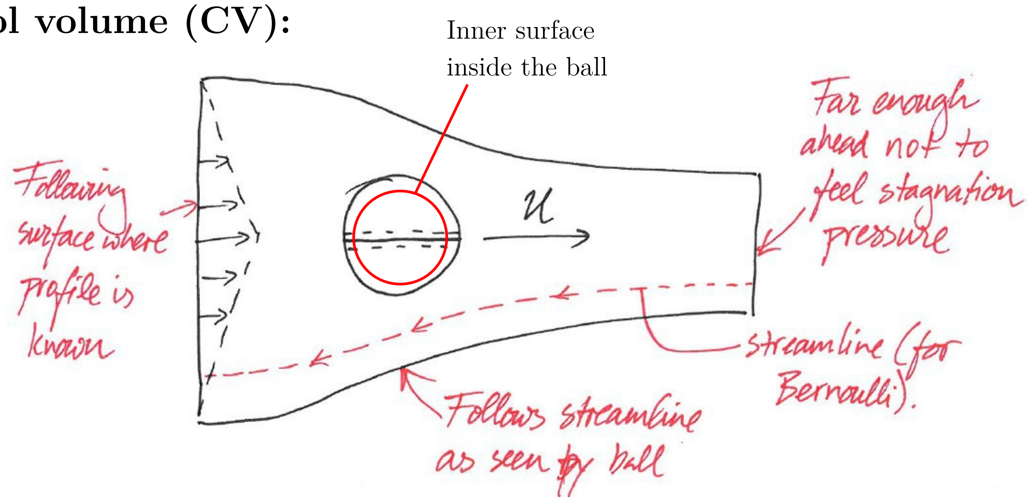
Figur 2



Figur 3

Problem 2 a)

Control volume (CV):



- We choose a CV following the ball with velocity $\vec{U}$ so that the flow field within the CV persists in time.
- Make the outer CS either normal or parallel to the flow; the cylindrical part of the CS follows the streamlines as seen by the ball so that $\vec{u} \perp \vec{n}$. On the left and right side we have $\vec{u} \parallel \vec{n}$.
- Let the right front be far enough ahead of the ball so as not to feel any pressure disturbance from the ball.
- A final minor point is that all external forces exerted on our CV should act on the CS. We will later wish to represent the drag force from the ball as an external force and so we make an internal sphere in our CV, around the ball. However, we do not wish to compute any internal shear and pressure fields around the ball itself, and so we let the internal spherical part of the CS penetrate slightly into the ball surface.

Pressure forces:

Assumptions

(i) The pressure on the right and cylindrical surfaces of the CV is atmospheric.

(ii) Changes in ball velocity are negligible.

(iii) Viscous effects outside the boundary layer are negligible.

(iv) We may follow a streamline backwards anywhere from the left flat surface of the CV to the right flat surface.

Discussion

- We seek to determine the left side CV surface pressure using the Bernoulli equation along a streamline, which will here a be valid approach under the above assumptions. Any streamline will lead to a point at the right CV surface where the pressure is atmospheric and the relative velocity is known.

- Using the steady state Bernoulli equation requires a stationary flow field. To achieve this we let the coordinate system move with the CV. This makes the relative velocity on the right side $-U\vec{e}_z$ and $(u(r) - U)\,\vec{e}_z = -U\frac{r}{b}\vec{e}_z$ on the left side.

Bernoulli:

$$p(r) + \frac{\rho\,[u(r) - U]^2}{2} = p_{\text{atm}} + \frac{\rho\,(-U)^2}{2},$$

or

$$p(r) = p_{\text{atm}} + \frac{\rho}{2}\left(U^2 - \left(U\frac{r}{b}\right)^2\right) = \underline{\underline{p_{\text{atm}} + U^2\frac{\rho}{2}\left(1 - \frac{r^2}{b^2}\right)}}. \tag{1}$$

2 b)

The drag force

Discussion

- We do note have enough information about the flow to compute the total drag directly as an integral over the surface of the ball or the CV. Rather, we evaluate the other terms in the CV linear momentum equation and from this balance extract the drag force.

- With an exact flow field we could compute the pressure force $\vec{F}_p$ as the CS integral using the pressure in all surface points from the Bernoulli equation. Here however, we simply take it as the stated expression:

$$\vec{F}_p = \frac{\pi}{12}\rho U^2 b^2\,\vec{e}_z.$$

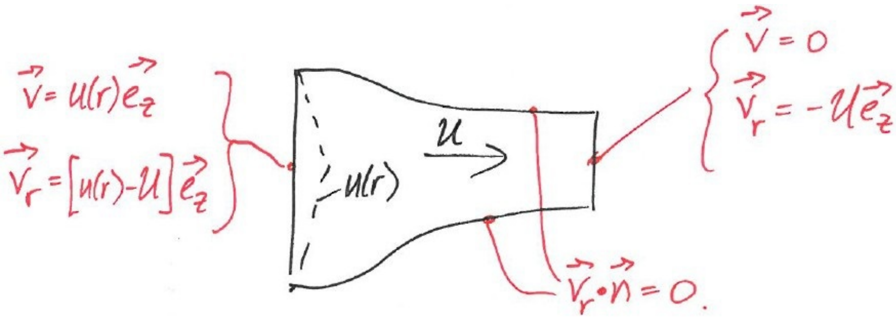
In the general case when the coordinate system does not follow the CV, we must modify the linear momentum equation slightly from the static case when the CV is fixed. The term $(\vec{V} \cdot \vec{n})$ measuring the flux of something across a fixed CS must be modified to $(\vec{V}_r \cdot \vec{n})$ to measure the flux across a CS moving with velocity $\vec{V}_{\text{CS}}$, where $\vec{V}_r = \vec{V} - \vec{V}_{\text{CS}}$ is the velocity of the fluid relative to CS:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\text{CV}} \rho\vec{V}\,\mathrm{d}\mathcal{V} + \int_{\text{CS}} \rho\vec{V}\left(\vec{V}_r \cdot \vec{n}\right)\,\mathrm{d}A = \vec{F}_p + \vec{F}_B \tag{2}$$

where $\vec{F}_p$ and $\vec{F}_B$ are pressure and body forces acting on the fluid CV, respectively. Whilst $\vec{F}_p$ is the external pressure forces acting from outside the CV, $\vec{F}_B$ is the body force provided form the ball within, pushing at the CV fluid.

Again we let the CV follow the ball as it moves. The flow field around the CV is then steady and the leftmost term in (2) vanishes.

We may choose whether to let the coordinate system be relative to the ball's motion as in 3 a) or whether the coordinate system is at rest, not following the ball. Here we will use the latter approach, which is considerably simpler. We preform the calculation with a moving coordinate system in the appendix.



Since the relative flow through the right CS carries with it no momentum, and because there is no flow through the curved cylindrical part of the CS, only the left surface contributes to a non-zero convection integral:

$$\int_{\mathrm{CS}} \rho \vec{V} \left(\vec{V}_r \cdot \vec{n}\right) \mathrm{d}A = \int_0^b \rho \underbrace{u(r)\vec{e}_z}_{\vec{v}} \left[\underbrace{(u(r) - U)\,\vec{e}_z}_{\vec{V}_r} \cdot \underbrace{(-\vec{e}_z)}_{\vec{n}} \right] \underbrace{2\pi r\,\mathrm{d}r}_{\mathrm{d}A}$$

$$= 2\pi\rho U^2 \vec{e}_z \int_0^b \left(1 - \frac{r}{b}\right) \frac{r}{b} r \,\mathrm{d}r = \underline{\frac{1}{6}\pi\rho U^2 b^2 \vec{e}_z}$$

From (2) we then have that

$$\vec{F}_B = \int_{\mathrm{CS}} \rho \vec{V} \left(\vec{V}_r \cdot \vec{n}\right) \mathrm{d}A - \vec{F}_p = \left(\frac{1}{6} - \frac{1}{12}\right) \pi\rho U^2 b^2 \vec{e}_z = \underline{\underline{\frac{1}{12}\pi\rho U^2 b^2 \vec{e}_z}}$$

Comment: Remember that the force $\vec{F}_B$ is the body force *from* the ball *onto* the fluid in the surrounding CV. As expected, this is positive since the ball will continuously accelerate the air around it in its forward movement. The drag force $\vec{F}_D$ *on the ball* is of course equal in magnitude, but oppositely directed.

Drag coefficient

By definition (with the drag taken positive as in a coordinate system with flow from the left):

$$C_D \overset{\mathrm{def}}{=} \frac{|F_D|}{\frac{1}{2}\rho U^2 A_{\mathrm{normal}}}$$

where A_{normal} is the transverse obstacle area as seen by the flow, here πR^2:

$$C_D = \frac{\frac{1}{12}\pi\rho U^2 b^2}{\frac{1}{2}\rho U^2 \pi R^2} \quad \Rightarrow \quad \underline{\underline{d = \sqrt{6C_d}R}}$$

2 c)

With given numbers:

$$U = 150\,\mathrm{km/h}, \approx 41.7\,\mathrm{m/s}, \qquad R = 0.036\,\mathrm{m}, \qquad \mu = 1.87 \cdot 10^{-5}\,\mathrm{kg/ms}, \qquad \rho = 1.23\,\mathrm{kg/m^3}$$

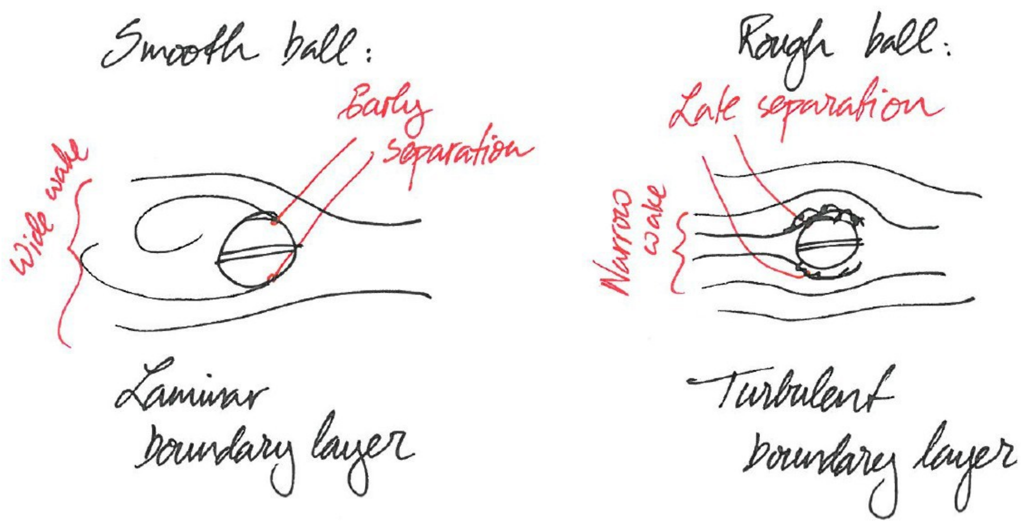
$$Re = \frac{\rho U(2R)}{\mu} \approx \underline{1.97 \cdot 10^5}$$

From graph:

$$\underline{\underline{C_{\text{D, smooth}} \approx 0.51}} \qquad\qquad \underline{\underline{C_{\text{D, rough}} \approx 0.08}}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$b_{\text{smooth}} \approx 1.75R \approx \underline{\underline{6.30\,\text{cm}}} \qquad\qquad b_{\text{rough}} \approx 0.69R \approx \underline{\underline{2.49\,\text{cm}}}$$



Figur 4: Sketch of streamlines for smooth and rough surfaced ball

Figure 4 illustrates the difference in wake width for a polished and an unpolished ball. The surface roughness of the unpolished ball trips the flow, promoting turbulence and a thinner boundary layer. This in turn increases the velocity close to the ball surface, causing delayed flow detachment.

2 d)

Assumptions

(i) The lift coefficient C_L and ball velocity U are approximately constant during the time the ball is in flight.

(ii) The ball seam is aligned with the z-axis during the flight, and the rough side is always pointed in the lateral Cartesian y-direction.

(iii) As the transversal "lift" influences the ball's trajectory, the lift force would develop a small (negative) component in the z-direction as part of the rotational swing motion in the horizontal plain. This force component may of course be neglected, an we assume that the lift force is always normal to the z- and vertical axes.

The time of flight, until the ball has traveled a distance $L = 20\,\text{m}$ is

$$t_{\text{end}} = L/U.$$

By definition of C_L, the lateral "lift" force is given by

$$\vec{F}_L = \frac{\pi}{2}\rho U^2 R^2 C_L \vec{e}_y.$$

By Newton's 2nd law and the above assumptions, the ball's acceleration will then be

$$\vec{a} = \ddot{y}\vec{e}_y = \frac{\vec{F}_L}{m}.$$

Integrating twice in time from $t = 0$ yields

$$y(t) = y(0) + \cancel{\dot{y}(0)}t + \frac{1}{2}at^2,$$

which, when inserting for arrival time and lift force, gives

$$\Delta y = y\left(t_{\text{end}}\right) - y(0) = \frac{1}{2}\frac{\vec{F}_L}{m}\frac{L^2}{U^2} = \frac{\pi\rho L^2 R^2}{4m}C_L.$$

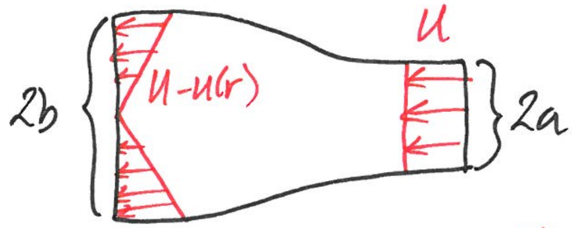
Inserting the numerical values

$$L = 20\,\text{m}, \qquad R = 3.6\,\text{cm}, \qquad \rho = 1.23\,{}^{\text{kg}}/_{\text{m}^3}, \qquad m = 160\,\text{g}, \qquad C_L = 0.07:$$

$$\Delta y \approx \underline{\underline{21.9\,\text{cm}}}$$

APPENDIX

Alternative solution for b) where the coordinate system also follows ball. Now we need to know the width of right hand side of CV:



Mass flux: $$\dot{m} = \rho \int_0^b (U-u(r))\,\overbrace{2\pi r\,dr}^{dA} = 2\pi\rho U \int_0^b \frac{r^2 dr}{b}$$

mass conservation!

$$= \frac{2\pi}{3}\rho b^2 U = \pi a^2 \rho U^2$$

$$\Rightarrow \quad \underline{a = \sqrt{\tfrac{2}{3}}\,b.}$$

Now consider ($\vec{V} = \vec{V}_r$ now)

$$\oint_{CS} \rho \vec{V}_z(\vec{V}_r \cdot \vec{n})\,dA = -\rho\int_0^b [U-u(r)]^2\, 2\pi r\,dr + \rho\int_0^a U^2 \cdot 2\pi r\,dr$$

$$= -2\pi U^2\rho \int_0^b \frac{r^2}{b^2} r\,dr + 2\pi\rho U^2 \int_0^a r\,dr$$

$$= -2\pi U^2\rho^2\underbrace{\left[\tfrac{1}{4}b^2 - \tfrac{1}{2}a^2\right]}_{a^2 = 2b^2/3} = \underline{\frac{\pi}{6}\rho U^2 b^2}$$

same as before.

Problem 3

PART I

a)

- The velocity field given in Cartesian coordinates reads:

$$\vec{V} = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} ax \\ -ay \end{pmatrix}.$$

A stream function exists if the continuity equation is satisfied, i.e. if $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$. Here,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = a - a = 0$$

Therefore a stream function exists and is defined, for an incompressible and two-dimensional flow, as:

$$u = \frac{\partial \psi}{\partial y}$$
$$v = -\frac{\partial \psi}{\partial x}$$

We start with the y-derivative of ψ:

$$\frac{\partial \psi}{\partial y} = u = ax.$$

And we integrate with respect to y:

$$\psi = axy + g(x).$$

The x-derivative of ψ leads to a second expression of v:

$$v = -\frac{\partial \psi}{\partial x} = -ay - g'(x).$$

By equating both expression of v we get $g(x)$ and finally the stream function, ψ.

$$\begin{aligned} v &= -ay - g'(x) = -ay \\ g(x) &= C = \text{constant} \\ \underline{\underline{\psi}} &= \underline{\underline{axy + C}}. \end{aligned}$$

- Applying the <u>curl</u> operator to the velocity vector we get for this 2D flow:

$$\vec{\nabla} \times \vec{V} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k} = \left(-\frac{\partial}{\partial x}ay - \frac{\partial}{\partial y}ax\right)\vec{k} = 0.$$

Thus, the flow is irrotational. Irrotational flow is inviscid, because $\mu\nabla^2\vec{V} = \mu\nabla^2(\vec{\nabla}\phi) = \mu\vec{\nabla}(\nabla^2\phi) = 0$. Here we used that for irrotational flow a velocity potential ϕ with $\vec{V} = \vec{\nabla}\phi$ exists satisfying the continuity equation $\vec{\nabla}\cdot\vec{V} = \nabla^2\phi = 0$.
Alternatively, $\mu\nabla^2\vec{V} = 0$ can be shown by simply inserting the present velocity vector and getting $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ for the x- and y-components of $\nabla^2\vec{V} = 0$, respectively.
Since the present flow is inviscid, steady and incompressible, the Bernoulli equation is valid on streamlines. As the flow is also irrotational due to $\vec{\nabla}\times\vec{V} = 0$, the Bernoulli constant is the same for all streamlines, thus $P + \frac{1}{2}\rho V^2 =$ constant everywhere. The constant can be determined at the origin, i.e. constant $= P(0,0) + \frac{1}{2}\rho(V(0,0))^2 = P_0$. Thus

$$P + \tfrac{1}{2}\rho\left((ax)^2 + (-ay)^2\right) = P_0$$
$$\underline{\underline{P = P_0 - \tfrac{1}{2}\rho a^2\left(x^2 + y^2\right)}}$$

b)

To determine the stream function passing through point $P_1(x_1, y_1)$, we can use the stream function derived in question a), $\psi = axy + C$. Since C is an arbitrary constant, we set it to zero.
This means that $y(x) = \frac{\psi}{ax}$.

$$\begin{aligned}
\psi_1 &= a\left(-\frac{L}{4}\right)(4L) = -aL^2 \\
\psi_2 &= a\left(\frac{L}{4}\right)(4L) = aL^2 \\
\underline{\underline{y_1(x)}} &= \frac{\psi_1}{ax} = \frac{-aL^2}{ax} = \underline{\underline{\frac{-L^2}{x}}} \\
\underline{\underline{y_2(x)}} &= \frac{\psi_2}{ax} = \frac{aL^2}{ax} = \underline{\underline{\frac{L^2}{x}}}
\end{aligned}$$

The volume flow rate per unit width $\dot{\mathcal{V}}_{/b}$ between streamline 1 and streamline 2 is the difference between ψ_2 and ψ_1, i.e.

$$\underline{\underline{\dot{\mathcal{V}}_{/b} = \psi_2 - \psi_1 = 2aL^2}}$$

PART II

c)

- The continuity equation is satisfied if $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$. Applied to the flow field given here, we get:

$$\underline{\underline{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = f'(y) - f'(y) = 0}}$$

- The first condition $u(x, y \to \infty) = xf'(y \to \infty) = ax$ informs us about the far field velocity. It means that $\underline{\underline{f'(y \to \infty) = a}}$.
  The roof being windproof, the vertical velocity component is zero when the wind hits the roof, i.e. at $y = 0$. This means that $v(x, 0) = 0$ and thus $\underline{\underline{f(0) = 0}}$.
  The no-sleep boundary condition implies that $u(x, 0) = 0$ and thus $\underline{\underline{f'(0) = 0}}$.

d)

- The wall shear stress is defined by: $\tau_w = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} = \mu x f''(0)$. This shear stress acts horizontally and will contribute to the x-component of the force, F_x.

$$
\begin{aligned}
\underline{\underline{F_{x+}}} = \int_0^L \tau_w b dx &= \mu b f''(0) \int_0^L x dx = \underline{\underline{\mu b \frac{L^2}{2} f''(0)}} \\
\underline{\underline{F_{x-}}} = \int_{-L}^0 \tau_w b dx &= \mu b f''(0) \int_{-L}^0 x dx = \underline{\underline{-\mu b \frac{L^2}{2} f''(0)}} \\
\underline{\underline{F_{x\text{tot}}}} &= F_{x+} + F_{x-} = \underline{\underline{0}}
\end{aligned}
$$

where F_{x+} is the horizontal force acting to the right on the right part of the roof [0,L], F_{x-} is the horizontal force acting to the left on the left part of the roof [-L,0], and $F_{x\text{tot}}$ is the total horizontal force being zero, because F_{x+} and F_{x-} neutralize each other.

- The y-component of the force, F_y, is due to pressure force on the roof.

$$
\begin{aligned}
F_{y+} &= \int_0^L (P_0 - P(x,0))\, b dx \\
F_{y-} &= \int_{-L}^0 (P_0 - P(x,0))\, b dx \\
\underline{\underline{F_{y+}}} &= \int_0^L \left(P_0 - \left(P_0 - \frac{\rho}{2} a^2 x^2\right)\right) b dx = \underline{\underline{b \frac{\rho}{2} a^2 \frac{L^3}{3}}} \\
\underline{\underline{F_{y-}}} &= \int_{-L}^0 \left(P_0 - \left(P_0 - \frac{\rho}{2} a^2 x^2\right)\right) b dx = \underline{\underline{b \frac{\rho}{2} a^2 \frac{L^3}{3}}} \\
\underline{\underline{F_{y\text{tot}}}} &= F_{y+} + F_{y-} = \underline{\underline{b \rho a^2 \frac{L^3}{3}}}
\end{aligned}
$$

Note that the pressure forces act upwards. Thus, the roof of Trondheim Spektrum has to be bolted well enough to prevent it from being lifted by the wind toward it.