Innlevering 3

Rendell Cale

Oppgave 1

a) Siden lysene er seriekoblet vil

$$U = \min \{ X_i \}_{i=1}^{36}$$

Dette gir

$$P(U<t) = 1 - P(U \geq t)$$
$$= 1 - P(X_1 \geq t \cap \ldots \cap X_{36} \geq t)$$

X_i uavhengige

$$= 1 - P(X_1 \geq t) \cdot \ldots \cdot P(X_{36} \geq t)$$
$$= 1 - \left(P(X_1 \geq t)\right)^{36}$$
$$= 1 - \left(1 - P(X_1 < t)\right)^{36}$$

Siden $X_i \sim \exp(\beta = \mu)$ har vi

$$P(X_1 < t) = 1 - e^{-t/\mu}$$

Dette gir

$$P(U<t) = 1 - \left(1 - \left(1 - e^{-t/\mu}\right)\right)^{36}$$
$$= 1 - e^{-36t/\mu}$$
$$= 1 - e^{-t/\mu/36}$$

Så $U \sim \exp\left(\mu/36\right)$ og forventningsverdien til U er da

$$\underline{\underline{E(U) = \frac{\mu}{36}}}$$

$\mu = 5000$ timer $\Rightarrow$

$$\underline{\underline{E(U) = \frac{5000}{36} \overset{\text{timer}}{=} 139 \text{ timer}}}$$

b) Siden $Y = 0{,}072 \cdot U$ vil Y være eksponentialfordelt

$$E(Y) = E(0{,}072 \cdot U)$$
$$= 0{,}072 \cdot E(U)$$
$$= 0{,}072 \cdot \frac{\mu}{36}$$

$\mu = 5000$ timer gir

$$E(Y) = 0{,}072 \cdot \frac{5000}{36} = \underline{10 \text{ KWh}}$$

Oppgave 2

a) $\hat{p}$:

$$E(\hat{p}) = \frac{1}{2n_1} E(X_1) + \frac{1}{2n_2} E(X_2)$$

$$= \frac{1}{2n_1} n_1 p + \frac{1}{2n_2} n_2 p$$

$$= \underline{\underline{p}}$$

$$Var(\hat{p}) = \left(\frac{1}{2n_1}\right)^2 Var(X_1) + \left(\frac{1}{2n_2}\right)^2 Var(X_2)$$

$$= \frac{1}{4}\left(\frac{n_1 p(1-p)}{n_1^2} + \frac{n_2 p(1-p)}{n_2^2}\right)$$

$$= \underline{\underline{\frac{1}{4}\left(\frac{p(1-p)}{n_1} + \frac{p(1-p)}{n_2}\right)}}$$

p^*:

$$E(p^*) = \frac{1}{n_1+n_2}\left(E(X_1) + E(X_2)\right)$$

$$= \underline{\underline{\frac{n_1 p + n_2 p}{n_1+n_2}}}$$

$$Var(p^*) = \frac{1}{(n_1+n_2)^2}\left(Var(X_1) + Var(X_2)\right)$$

$$= \frac{1}{(n_1+n_2)^2}\left(n_1 p(1-p) + n_2 p(1-p)\right)$$

$$\Rightarrow Var(p^*) = \frac{p(1-p}{n_1+n_2}$$

Ønsker at størrelsen skal ha innvirkning så ville valgt p^*.

$$\hat{p} = p^* = \frac{X_1 + X_2}{2n}$$

$$Z = \frac{\hat{P} - p}{\sqrt{\frac{1}{2n}\hat{p}(1-\hat{p})}} \approx Z' \sim N(0,1).$$

$$P(-Z_{\frac{\alpha}{2}} < Z < Z_{\frac{\alpha}{2}}) = 1-\alpha$$

$$\Leftrightarrow P\left(-Z_{\frac{\alpha}{2}} < \frac{\hat{P} - p}{\sqrt{\frac{1}{2n}\hat{p}(1-\hat{p})}} < Z_{\frac{\alpha}{2}}\right) = 1-\alpha$$

$$\Leftrightarrow P\left(-Z_{\frac{\alpha}{2}}\sqrt{\frac{1}{2n}\hat{p}(1-\hat{p})} - \hat{P} < -p < Z_{\frac{\alpha}{2}}\sqrt{\frac{1}{2n}\hat{p}(1-\hat{p})} - \hat{P}\right)$$
$$= 1-\alpha$$

Et $100 \cdot (1-\alpha)\%$ konfidensintervall er dermed:

$$\left(\hat{P} - Z_{\frac{\alpha}{2}}\sqrt{\frac{1}{2n}\hat{p}(1-\hat{p})}, \hat{P} + Z_{\frac{\alpha}{2}}\sqrt{\frac{1}{2n}\hat{p}(1-\hat{p})}\right)$$

$\alpha = 0{,}05$ gir det vi ønsker men får ikke regnet ut uten $\hat{P}$.

b) Siden $n = 1000$ er vesentlig større enn 30 er det rimelig at X_1, X_2 og X_3 er normalfordelte.

$$Y = X_3 - n\left(\frac{X_1 + X_2}{2n}\right) = X_3 - \frac{1}{2}(X_1 + X_2)$$

Y er summen av tre tilnærmet normalfordelte variabler, så Y er tilnærmet normalfordelt.

$$Var(Y) = \underbrace{Var(X_3)}_{\text{antar } p_3 = p} + \frac{1}{(-2)^2}\left(Var(X_1) + Var(X_2)\right)$$

$$= np(1-p) + \frac{1}{4}\left(np(1-p) + np(1-p)\right)$$

$$= \underline{\underline{\frac{3}{2}np(1-p)}}$$

Siden $E(X_3) = np$ vil

$$Y = X_3 - n\hat{p} \sim N(0, Var(Y))$$

$$Z = \frac{Y}{\sqrt{Var(Y)}} \sim N(0,1)$$

$$P\left(-Z_{\frac{\alpha}{2}} < Z < Z_{\frac{\alpha}{2}}\right) = 1-\alpha$$

$$\Leftrightarrow P\left(-Z_{\frac{\alpha}{2}}\sqrt{Var(Y)} < Y < Z_{\frac{\alpha}{2}}\sqrt{Var(Y)}\right) = 1-\alpha$$

$$\overset{\sigma_Y^2 = Var(Y)}{\Leftrightarrow} P\left(-Z_{\frac{\alpha}{2}}\sigma_Y < X_3 - n\hat{p} < Z_{\frac{\alpha}{2}}\sigma_Y\right) = 1-\alpha$$

Dette gir et $100 \cdot (1-\alpha)\%$ konfidensintervall for X_3 som følger:

$$\left(n\hat{p} - z_{\frac{\alpha}{2}} \sigma_Y \,,\; n\hat{p} + z_{\frac{\alpha}{2}} \sigma_Y \right)$$

Vi har

$$\begin{cases} \hat{p} = \dfrac{x_1 + x_2}{2 \cdot n} = \dfrac{645 + 692}{2 \cdot 1000} = 0{,}6685 \\ z_{\frac{\alpha}{2}} = z_{0{,}025} = 1{,}960 \\ \sigma_Y = \sqrt{\dfrac{3}{2} \cdot n \cdot \hat{p} \cdot (1-\hat{p})} = \sqrt{\dfrac{3}{2} \cdot 1000 \cdot 0{,}6685 \cdot (1 - 0{,}6685)} \\ \quad = 18{,}232 \end{cases}$$

$$\Rightarrow \left(1000 \cdot 0{,}6685 - 1{,}960 \cdot 18{,}232 \,,\; 1000 \cdot 0{,}6685 + 1{,}960 \cdot 18{,}232\right)$$

$$\Leftrightarrow \underline{\underline{(633 \,,\; 704)}}$$

Oppgave 3

a) $P(M=m) = P(m < Z < m+1)$

$$= F(m+1;\lambda) - F(m;\lambda)$$

$$= 1 - e^{-\lambda(m+1)} - (1 - e^{-\lambda m})$$

$$= e^{-\lambda m} - e^{-\lambda m} e^{-\lambda}$$

$$\underline{\underline{= (1-e^{-\lambda}) e^{-\lambda m}}}$$

b) $L(M_1, \dots, M_n; \lambda) = \prod_{i=1}^{n} (1-e^{-\lambda}) e^{-\lambda M_i}$

$$= (1-e^{-\lambda})^n e^{-\lambda \sum_{i=1}^{n} M_i}$$

$$\ln(L) = n \ln(1-e^{-\lambda}) - \lambda \sum M_i$$

$$\frac{\partial \ln(L)}{\partial \lambda} = 0$$

$$\Rightarrow \quad \frac{n e^{-\hat{\lambda}}}{1-e^{-\hat{\lambda}}} - \sum M_i = 0$$

$$\Leftrightarrow \quad \underbrace{\frac{1}{n} \sum M_i}_{\bar{M}} = \frac{1}{e^{\hat{\lambda}} - 1}$$

$$\Leftrightarrow \quad \frac{1}{\bar{M}} = e^{\hat{\lambda}} - 1$$

$$\Rightarrow \hat{\lambda} = \ln\left(\frac{1}{\bar{m}} + 1\right)$$

Oppgave 4

a) En god estimator for μ er $\bar{X}$ siden

$$E(\bar{X}) = E(X_1) = \mu.$$

Punktestimatet blir

$$\hat{\mu} = \bar{x} = 13{,}18$$

Lager en "normalisert" variabel $T = \frac{\bar{X} - \hat{\mu}}{S_x / \sqrt{n}} \sim t(n-1)$

Et $100(1-\alpha)\%$ konfidensinterval for $\hat{\mu}$ er da gitt av

$$\left(-t_{\frac{\alpha}{2}},\ t_{\frac{\alpha}{2}}\right)$$

$$\Leftrightarrow \left(\bar{X} - t_{\frac{\alpha}{2}} \frac{S_x}{\sqrt{n}},\ \bar{X} + t_{\frac{\alpha}{2}} \frac{S_x}{\sqrt{n}}\right)$$

Med $\alpha = 0{,}1$ og data som gitt får vi

$$\left(13.18 - 1.833 \cdot \frac{4{,}44}{\sqrt{10}},\ 13.18 + 1.833 \cdot \frac{4{,}44}{\sqrt{10}}\right)$$

$$\Leftrightarrow \underline{\underline{(10.61,\ 15.75)}}$$

b) Går rett til resultatene

$$\hat{\eta} = \bar{y} = 8.00$$

$\alpha = 0.01$:

$$\left(8.00 - 3{,}250 \cdot \frac{2.77}{\sqrt{10}},\ 8.00 + 3{,}250 \cdot \frac{2.77}{\sqrt{10}}\right)$$

$$\Leftrightarrow \quad \underline{\underline{(5.15,\ 10.85)}}$$

Det vil være smalere siden for å være sikker på intervallet må man gjøre intervallet større.

c) $\delta = \mu - \eta$

En god estimator er

$$\hat{\delta} = \hat{\mu} - \hat{\eta} = \bar{X} - \bar{Y}$$

Punktestimatet vårt blir

$$\hat{\delta} = \bar{x} - \bar{y} = 13.18 - 8.00$$

$$= \underline{\underline{5.18}}$$

La $D = \bar{X} - \bar{Y}$ slik at $E(D) = \delta$ og

$$Var(D) = Var(\bar{X}) + Var(\bar{Y})$$

$$= \frac{S_X^2}{n_1} + \frac{S_Y^2}{n_1}$$

$$\Rightarrow \quad S_D = \sqrt{\operatorname{Var} D} = \sqrt{\frac{S_X^2}{n_1} + \frac{S_Y^2}{n_2}}$$

Konfidensintervallet blir da

$$\left(\bar{X} - \bar{Y} - {}_{\frac{\alpha}{2}} S_D,\ \bar{X} - \bar{Y} + {}_{\frac{\alpha}{2}} S_D\right)$$

$$S_D = \sqrt{\frac{4{,}44^2}{10} + \frac{2{,}77^2}{10}} = 1{,}65$$

Vårt 95% konf. int. blir da

$$\left(\bar{x} - \bar{y} - t_{0{,}025} \cdot S_D,\ \bar{x} - \bar{y} + t_{0{,}025} S_D\right)$$

$$\Leftrightarrow \quad (5.18 - 2.262 \cdot 1{,}65,\ 5.18 + 2.262 \cdot 1{,}65)$$

$$\Leftrightarrow \quad \underline{\underline{(1.45\ ,\ 8.91)}}$$

Ja. Dersom de hadde hatt lik effekt burde vi ihvertfall sett 0 i konf. intervallet over, og generelt sett et mer symmetrisk interval om 0.

Oppgave 5

a) La $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} x_i$

$$S_x^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

$$n = 1$$

$$\alpha = 0{,}1$$

Da vil konfidensintervallet for μ være

$$\left(\bar{X} - t_{\frac{\alpha}{2}} \frac{S_x}{\sqrt{n}},\ \bar{X} + t_{\frac{\alpha}{2}} \frac{S_x}{\sqrt{n}} \right)$$

b) 90% prosent av konfidensintervallene forventes å inneholde den samme verdien for μ, altså $\underline{\underline{270 \text{ stk}}}$

Kjører mer eller mindre samme kode og får at antall som er innenfor i en serie er:

269, 272, 264, 262, 266, 267, 273

så det ovenfor stemmer godt.

Utvidet programmet til å kjøre i 30 000 dager og fikk da

26 979 (89,93%)

som stemmer bra.