Øving 6

Ønsker tilbakemelding :)

Rendell Cale, gruppe 2, mttk

Bra!

S.1

9)

b) $A, B, C \subseteq \mathcal{U}$

Part (b): $A \times (B \cup C) = (A \times B) \cup (A \times C)$

For all pairs $(a,b) \in A \times (B \cup C)$
$a \in A$, and $b \in (B \cup C)$
Since $b \in (B \cup C)$, $b \in B$ or $b \in C$ (or both).
The pair (a,b) will then be an element of $A \times B$ or $A \times C$, which is the same as saying $(a,b) \in (A \times B) \cup (A \times C)$. Since a and b were arbitrary, this implies that

$$A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$$

For all pairs $(a,b) \in (A \times B) \cup (A \times C)$
we have $(a,b) \in (A \times B)$ or $(a,b) \in (A \times C)$.
So $a \in A$ and $b \in B$ or $b \in C$. Which is the same as saying $b \in (B \cup C)$
So $(a,b) \in A \times (B \cup C)$, which implies that $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$, since a, b were arbitrary.

Bra!

Since $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$
and $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$

We get $A \times (B \cup C) = (A \times B) \cup (A \times C)$ ▯

12) $A, B \subset \mathcal{U}$, $|B| = 3$

$$2^{|A|\cdot|B|} = 4096$$
$$2^{3|A|} = 4096$$

$$3|A| = 12 \quad (= \log_2(4096))$$

$$\underline{|A| = 4}$$

7.1

5)

e) xRy iff $x+y$ is odd

$\Leftrightarrow xRy$ iff $x+y = 2n+1$ for some $n \in \mathbb{Z}$

If $x+y$ is odd then $y+x$ is also odd
so $xRy \Leftrightarrow yRx$, so it is symmetric

It is not reflexive because
$x + x = 2x \neq 2n+1$

If xRy and yRz
then $x+y = 2n+1$ and $y+z = 2m+1$
$\Rightarrow y = 2m+1-z$
$\Rightarrow x + 2m+1 - z = 2n+1$
$x - z = 2(n-m) \quad | +2z$
$\Rightarrow x+z = \underbrace{2(n-m) + 2z}_{\text{even}}$ = not odd

$\Rightarrow$ not transitive

If $x=3$, $y=2$, then
$x+y =$ odd number
$y+x =$ — " —
So xRy, yRx, $x \neq y$
So it is not antisymmetric.

So R is symmetric, but not reflexive, antisymmetric nor transitive.

f) xRy if $x-y=2n$

xRx because $x-x=0=2\cdot 0$
So it is reflexive

If we have xRy then $x-y=2n$.
This means we also have $y-x$ since
$y-x=-(x-y)=-2n$ which is also even,
So it is symmetric.

To show that R is not antisymmetric
let $x=1$, $y=3$
xRy because $x-y=-2\ (=2\cdot n)$
yRx because $y-x=2\ (=2n)$
but $x \neq y$, so it cant be antisymmetric.

If xRy and yRz then
$x-y=2n$ and $y-z=2m$
$\Rightarrow y=2m+z$
$\Rightarrow x-2m-z=2n$
$\Leftrightarrow x-z=2(n+m)$
So xRz and R is transitive. R ✓

So R is reflexive, symmetric and transitive
and not antisymmetric.

h) $(a,b)R(c,d)$ iff $a\leq c$

$(a,b)R(a,b)$ because $a\leq a$
So it is reflexive.

R is not symmetric. Let $a<c$ (strictly less than) then $(a,b)R(c,d)$ but $(c,d)\not R(a,b)$.

It is transitive:
If $(a,b)R(c,d)$ and $(c,d)R(e,f)$,
then $a\leq c$ and $c\leq e$
$\Rightarrow a\leq e$
$\Rightarrow (a,b)R(e,f)$

It is also antisymmetric since $a\leq c$ and $c\leq a$ is only possible when $a=c$. ✗ Ebs. $(2,4),(2,5)\in R$
men $(2,4)\neq(2,5)$

So R is reflexive, transitive and antisymmetric
and not symmetric.

8) Assume R_1 and R_2 are relations on A

a) Reflixive: R_1 and R_2 reflexive

Proof by contradiction:
Assume that $R_1 \cup R_2$ is not reflexive
This implies there is an element
$(x,y) \in R_1 \cup R_2$, $x \neq y$, such that
$(x,x) \notin R_1 \cup R_2$.
$(x,x) \notin R_1 \cup R_2$ implies that either
$(x,x) \notin R_1$ or $(x,x) \notin R_2$.
But neither of these can be
true since R_1 and R_2 are reflexive.
Our assumption must then be wrong
and $R_1 \cup R_2$ is also reflexive. ■

R

b) (i) Symmetric: R_1 and R_2 symmetric

proof by contradiction:
Assume that $R_1 \cup R_2$ is not
symmetric. Then there is an element
$(x,y) \in R_1 \cup R_2$, $x \neq y$ such that $(y,x) \notin R_1 \cup R_2$

So either $\exists (y,x) \in R_1 [y \neq x, (x,y) \notin R_1]$
or $\exists (y,x) \in R_2 [y \neq x, (x,y) \notin R_2]$

But both of these are impossible
because R_1 and R_2 are symmetric.

This proves that $R_1 \cup R_2$ is symmetric. ▯

(ii) Antisymmetric: R_1 and R_2 antisymmetric

If $R_1 = \{(1,3)\}$, $R_2 = \{(3,1)\}$
then $R_1 \cup R_2 = \{(1,3), (3,1)\}$
which is not antisymmetric, but
R_1 and R_2 were antisymmetric
so the statement

$$R_1 \text{ antisym.} \wedge R_2 \text{ antisym.} \Rightarrow R_1 \cup R_2 \text{ antisym.}$$

is false (by counterexample). ▯

(iii) Transitive: R_1 and R_2 transitive.

Let $R_1 = \{(1,2)\}$, $R_2 = \{(2,3)\}$
so that R_1 and R_2 are transitive.
$R_1 \cup R_2 = \{(1,2), (2,3)\}$
The union $R_1 \cup R_2$ is not transitive
because $(1,3) \notin R_1 \cup R_2$

Thus R_1 trans. and R_2 trans.

$$\not\Rightarrow R_1 \cup R_2 \text{ trans.}$$

7.2 4) $A = \{1,2,3\}$, $B = \{w,x,y,z\}$, $C = \{4,5,6\}$

$R_1 \subseteq A \times B$, $R_1 = \{(1,w),(3,w),(2,x),(1,y)\}$

$R_2 \subseteq B \times C$, $R_2 = \{(w,5),(x,6),(y,4),(y,6)\}$

$R_3 \subseteq B \times C$, $R_3 = \{(w,4),(w,5),(y,5)\}$

a) $R_1 \circ (R_2 \cup R_3) \subseteq A \times C$

$R_2 \cup R_3 = \{(w,4),(w,5),(x,6),(y,4),(y,5),(y,6)\}$

$R_1 \circ (R_2 \cup R_3) = \{(1,4),(1,5),(3,4),(3,5),(2,6),(1,4),(1,5),(1,6)\}$

$= \{(1,4),(1,5),(1,6),(2,6),(3,4),(3,5)\}$

$(R_1 \circ R_2) = \{(1,5),(3,5),(2,6),(1,4),(1,6)\}$

$= \{(1,4),(1,5),(1,6),(2,6),(3,5)\}$

$(R_1 \circ R_3) = \{(1,4),(1,5),(3,4),(3,5),(1,5)\}$

$= \{(1,4),(1,5),(3,4),(3,5)\}$

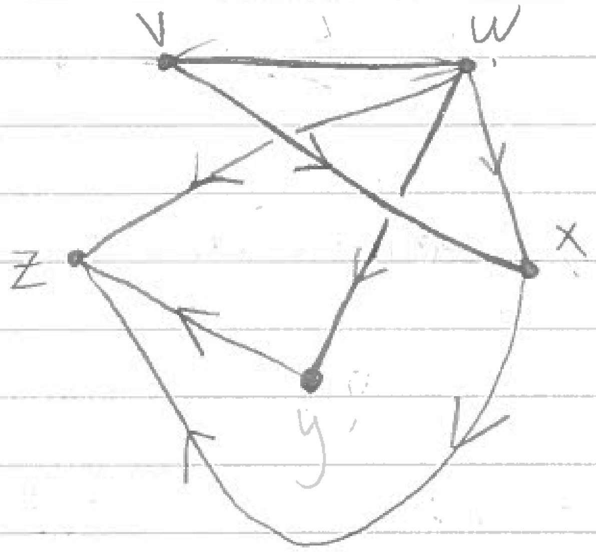
$(R_1 \circ R_2) \cup (R_1 \circ R_3) = \{(1,4),(1,5),(1,6),(2,6),(3,4),(3,5)\}$

b) $R_2 \cap R_3 = \{(w,5)\}$

R $\quad \underline{\underline{R_1 \circ (R_2 \cap R_3) = \{(1,5),(3,5)\}}}$

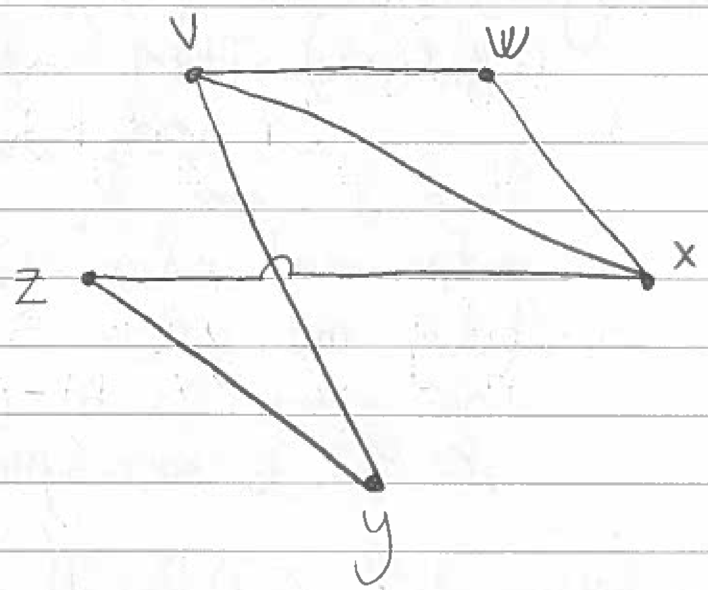
R $\quad \underline{\underline{(R_1 \circ R_2) \cap (R_1 \circ R_3) = \{(1,4),(1,5),(3,5)\}}}$

18) a) $R = \{(v,w),(v,x),(w,v),(w,x),(w,y),(w,z),(x,z),(y,z)\}$



R

b) $R = \{(v,w),(v,x),(v,y),(w,v),(w,x),(x,v),(x,w),(x,z),(y,v),(y,z),(z,x),(z,y)\}$



7.3 3) (A, R_1) and (B, R_2) are posets

R: $(a,b) R (x,y)$ if $a R_1 x$ and $b R_2 y$.

Need to prove that R is a poset,

$\Rightarrow$ R is reflexive, antisymmetric and transitive.

Reflexive: Let $(a,b) R (x,y)$

Then $a R_1 x$ and $b R_2 y$

Since R_1 and R_2 are posets they are reflexive, so $x R_1 a$ and $y R_2 b$.

This means that $(x,y) R (a,b)$

so R is reflexive. ✓

Antisymmetric: Let $(a,b)\mathcal{R}(x,y)$ and $(x,y)\mathcal{R}(a,b)$. Then aR_1x, xR_1a and bR_2y, yR_2b

Since R_1 and R_2 are antisymmetric

aR_1x and $xR_1a \Rightarrow a=x$

bR_2y and $yR_2b \Rightarrow b=y$

This gives $(a,b)=(x,y)$

so $\mathcal{R}$ is antisymmetric. ✓

Transitive: Let $(a,b)\mathcal{R}(c,d)$ and $(c,d)\mathcal{R}(x,y)$

Then aR_1c, cR_1x and bR_2d and dR_2y. R_1 and R_2 are transitive

so aR_1c and $cR_1x \Rightarrow aR_1x$

and bR_2d and $dR_2y \Rightarrow bR_2y$

This means ~~that~~ $(a,b)\mathcal{R}(x,y)$

so $\mathcal{R}$ is transitive. ✓

$\mathcal{R}$ is reflexive, antisymmetric and transitive so $\mathcal{R}$ is a poset.

4) R_1 and R_2 are total orders

By definition of total orders:

- For all $a_1, a_2 \in A$ $a_1 R_1 a_2$ or $a_2 R_1 a_1$
- For all $b_1, b_2 \in B$ $b_1 R_2 b_2$ or $b_2 R_2 b_1$

For all $a_1, a_2 \in A$, $b_1, b_2 \in B$

$[a_1 R_1 a_2$ or $a_2 R_1 a_1]$
and $[b_1 R_2 b_2$ or $b_2 R_2 b_1]$

By the definition of R this is equivalent to

For all $a_1, a_2 \in A$, $b_1, b_2 \in B$

$(a_1, b_1) R (a_2, b_2)$ or $(a_2, b_2) R (a_1, b_1)$

Thus R is also a total order.

× se c.f

6) $A = \{a, b, c, d, e\}$

a) $R = \{(a,a), (a,b), (a,c), (b,b), (c,c), (b,d), (c,d), (a,d), (d,d), (d,e), (b,e), (c,e), (a,e), (e,e)\}$

b) $e > d > c > b > a$

R

We need to add (1,2) and (4,2)
so two more edges are needed

7.4 7) $A = \{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4, 5\}$
$(x_1, y_1) \mathcal{R} (x_2, y_2)$ if $x_1 + y_1 = x_2 + y_2$

a) It is symmetric because $x_1+y_1 = x_2+y_2$
$\Leftrightarrow x_2+y_2 = x_1+y_1$

It is reflexive because $x_1+y_1 = x_1+y_1$

Assume $(x_2, y_2) \mathcal{R} (x_3, y_3)$, then $x_2+y_2 = x_3+y_3$
Since $x_1+y_1 = x_2+y_2$, $x_1+y_1 = x_3+y_3$
So $(x_1, y_1) \mathcal{R} (x_3, y_3)$
$\mathcal{R}$ is thus transitive ✓

$\mathcal{R}$ is reflexive, symmetric and transitive
so $\mathcal{R}$ is an equivalence relation. R

b) $[(1,3)] = \{(x,y) \mid (x,y) \in A, x+y = 1+3\}$
$= \{(1,3), (2,2), (3,1)\}$

$[(2,4)] = \{(x,y) \mid (x,y) \in A, x+y = 2+4\}$
$= \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$

$[(1,1)] = \{(x,y) \mid (x,y) \in A, x+y = 1+1\}$
$= \{(1,1)\}$

R

c) A will be the union of all the different equivalance classes, and they determine the partition induced by $\mathcal{R}$.

$A = [(1,1)] \cup [(1,2)] \cup [(1,3)] \cup [(1,4)] \cup [(1,5)] \cup [(2,5)] \cup [(3,5)]$
$\cup [(4,5)] \cup [(5,5)]$

$A_1 = [(1,1)] = \{(1,1)\}$
$A_2 = [(1,2)] = \{(1,2),(2,1)\}$
$A_3 = [(1,3)] = \{(1,3),(2,2),(3,1)\}$
$A_4 = [(1,4)] = \{(1,4),(2,3),(3,2),(4,1)\}$
$A_5 = [(1,5)] = [(2,4)]$ "listed in (b)"
$A_6 = [(2,5)] = \{(2,5),(3,4),(4,3),(5,2)\}$
$A_7 = [(3,5)] = \{(3,5),(4,4),(5,3)\}$
$A_8 = [(4,5)] = \{(4,5),(5,4)\}$
$A_9 = [(5,5)] = \{(5,5)\}$

10) $X \mathcal{R} Y$ if $B \cap X = B \cap Y$

$B \subseteq A$, $X, Y \subseteq A$

a) Reflexive: $X \mathcal{R} X$ because $B \cap X = B \cap X$. ✓
Symmetric: $X \mathcal{R} Y \Leftrightarrow Y \mathcal{R} X$ because
$B \cap X = B \cap Y \Leftrightarrow B \cap Y = B \cap X$ ✓
Transitive: If $X \mathcal{R} Y$ and $Y \mathcal{R} Z$, $Z \subseteq A$
then $B \cap X = B \cap Y$, $B \cap Y = B \cap Z$
So $B \cap X = B \cap Z$
So $X \mathcal{R} Z$, so it is transitive ✓

b) $A = \{1, 2, 3\}$, $B = \{1, 2\}$

$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

$\mathcal{P}(A) = \{\emptyset, \{3\}\} \cup \{\{1\}, \{1,3\}\} \cup \{\{2\}, \{2,3\}\} \cup \{\{1,2\}, \{1,2,3\}\}$ R

$A_0 = \{\emptyset, \{3\}\}$, $\emptyset \cap B = \{3\} \cap B = \emptyset$
$A_1 = \{\{1\}, \{1,3\}\}$, $\{1\} \cap B = \{1,3\} \cap B = \{1\}$
$A_2 = \{\{2\}, \{2,3\}\}$, $\{2\} \cap B = \{2,3\} \cap B = \{2\}$
$A_3 = \{\{1,2\}, \{1,2,3\}\}$, $\{1,2\} \cap B = \{1,2,3\} \cap B = \{1,2\}$

c) $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 2, 3\}$
$X = \{1, 3, 5\}$
$[X] = \{\{1, 3\}, \{1, 3, 4\}, \{1, 3, 4, 5\}\}$ R

$\{1,3,5\}$

d) If $X \subseteq A$, then $X \cap B$ can be any combination of elements from B. The number of equivalence classes is then $|\mathcal{P}(B)| = 2^{|B|} = 2^3 = \underline{\underline{8}}$

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