Øving 11

5.1

6) $$\begin{pmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 - 2\cdot 6 + 1\cdot 7 \\ 3 - 2\cdot 3 + 1\cdot 7 \\ 5 - 2\cdot 6 + 1\cdot 5 \end{pmatrix}$$

$$= \begin{pmatrix} 10 - 12 \\ 10 - 6 \\ 10 - 12 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 4 \\ -2 \end{pmatrix}$$

$$= -2 \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Yes it is an eigenvector and its eigenvalue is -2.

7) Lets compute $\det(A - 4I)$ where A is the matrix. If this is 0 then 4 is an eigenvalue.

$$\det \begin{pmatrix} 3-4 & 0 & -1 \\ 2 & 3-4 & 1 \\ -3 & 4 & 5-4 \end{pmatrix}$$

$$= \det \begin{pmatrix} -1 & 0 & -1 \\ 2 & -1 & 1 \\ -3 & 4 & -1 \end{pmatrix}$$

$$= -1 \cdot \det \begin{pmatrix} -1 & 1 \\ 4 & -1 \end{pmatrix} - 1 \cdot \det \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}$$

$$= -(1-4) - (8-3)$$

$$= 3 - 5$$

$$= -2 \neq 0$$

4 is not an eigenvalue.

9) $A = \begin{pmatrix} 5 & 0 \\ 2 & 1 \end{pmatrix}$, $\lambda_1 = 1$, $\lambda_2 = 5$

The eigenspace of $\lambda_1 = 1$ is

$$\text{Nul}(A - I) = \text{Nul}\begin{pmatrix} 4 & 0 \\ 2 & 0 \end{pmatrix} = \text{Nul}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \text{span}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The eigenspace of $\lambda_2 = 5$ is

$$\text{Nul}(A - 5I) = \text{Nul}\begin{pmatrix} 0 & 0 \\ 2 & -4 \end{pmatrix} = \text{Nul}\begin{pmatrix} 0 & 0 \\ 1 & -2 \end{pmatrix}$$

$$= \text{span}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Basis for $\lambda_1 = 1$: $\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$

Basis for $\lambda_2 = 5$: $\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$

21) a) False as it is always true for $\vec{x} = \vec{0}$, and we need non-trivial solutions.

b) True. If 0 is an eigenvalue then then the column vectors are linearly dependent, in which case A can't be invertible. If 0 is not an eigenvalue then the column vectors are linearly independent, so A would be invertible.

c) True, definition of eigenvalue.

d) True, There is no general formula for quintic polynomials and higher so to solve de $(A-\lambda I)=0$ can be difficult (or impossible), But checking whether $\det(A-\lambda I)=0$ for a given λ is easy (though computationally impossible for large matrices).

e) True, if A is duced to echelon form without aling than the pivots are genvalues.

23) The characteristic equation of an nxn matrix will always be an ~~(at most)~~ n'th degree polyno i l. The funda ental theorem of alge a the tells us that it will have exactly as many solutions for λ as th degree if we allow for complex solutions. Thus a 2x2 matrix will have at most 2 distinct eigenvalues an nxn will have at most n distinct eigenvalues

31) The eigenspace is the set of all points on the line. A doesn't change the size (norm) of the vector so $\lambda = 1$ is an eigenvector.

32) The eigenspace is the given rotation line, beause all points on it will be rotated about themselves which wont change their position. Here, a real eigenvalue is also $\lambda = 1$.

5.2

1) Char. polynomial: $\det\begin{pmatrix} 2-\lambda & 7 \\ 7 & 2-\lambda \end{pmatrix}$.

$$= (2-\lambda)^2 - 7^2$$
$$= \lambda^2 - 4\lambda + 4 - 49$$
$$= \underline{\lambda^2 - 4\lambda - 45}$$

Eigenvalues: $\lambda^2 - 4\lambda - 45 = 0$

$$(2-\lambda)^2 = 7^2$$
$$|2-\lambda| = 7$$
$$\underline{\lambda_1 = -5}$$
$$\underline{\lambda_2 = 9}$$

5) Char. Polynomial: $\det\begin{pmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{pmatrix}$

$$= (2-\lambda)(4-\lambda) + 1$$
$$= 8 - 2\lambda - 4\lambda + \lambda^2 + 1$$
$$= \underline{\lambda^2 - 6\lambda + 9} = (\lambda - 3)^2$$

Eigenvalues: $\lambda^2 - 6\lambda + 9 = 0$

$$(\lambda - 3)^2 = 0$$
$$\underline{\lambda_{1,2} = 3}$$

11) $\det\begin{pmatrix} 4-\lambda & 0 & 0 \\ 5 & 3-\lambda & 2 \\ -2 & 0 & 2-\lambda \end{pmatrix}$

$= (4-\lambda)\cdot\det\begin{pmatrix} 3-\lambda & 2 \\ 0 & 2-\lambda \end{pmatrix} - 0 + 0$

$= (4-\lambda)(3-\lambda)(2-\lambda)$

$= (12-(4+3)\lambda+\lambda^2)(2-\lambda)$

$= (\lambda^2-7\lambda+12)(2-\lambda)$

$= 2\lambda^2-\lambda^3-14\lambda+7\lambda^2+24-12\lambda$

$= \underline{\underline{-\lambda^3+9\lambda^2-26\lambda+24}}$

15) 4, 3, 3, 1

24) If A and B are similar then $A = PBP^{-1}$ for some invertible matrix P. So

$$\begin{aligned} \det(A) &= \det(PBP^{-1}) \\ &= \det(P)\cdot\det B\cdot\det(P^{-1}) \\ &= \det(P)\det(P^{-1})\cdot\det(B) \\ &= \det(P\cdot P^{-1})\det(B) \\ &= \det(I)\cdot\det(B) \\ &= 1\cdot\det(B) \\ \det(A) &= \det B \quad \blacksquare \end{aligned}$$

25) Let $A = \begin{pmatrix} .6 & .3 \\ .4 & .7 \end{pmatrix}$, $\vec{v}_1 = \begin{pmatrix} 3/7 \\ 4/7 \end{pmatrix}$, $\vec{x}_0 = \begin{pmatrix} .5 \\ .5 \end{pmatrix}$

a) $\det \begin{pmatrix} .6-\lambda & .3 \\ .4 & .7-\lambda \end{pmatrix} = 0$

$$(.6-\lambda)(.7-\lambda) - .4 \cdot .3 = 0$$
$$\lambda^2 - 1.3\lambda + 0.42 - 0.12 = 0$$
$$\lambda^2 - 1.3\lambda + 0.3 = 0$$
$$\lambda_{1,2} = \frac{1.3 \pm \sqrt{1.3^2 - 4 \cdot 0.3}}{2}$$
$$= \frac{1.3 \pm 0.7}{2}$$
$$\lambda_1 = 0.3, \quad \lambda_2 = 1$$

Can find eigenvector for λ_2 by solving

$$(A - 1 \cdot I)\vec{x} = \vec{0}$$
$$\begin{pmatrix} 0.6-1 & 0.3 \\ 0.4 & 0.7-1 \end{pmatrix} \sim \begin{pmatrix} -0.4 & 0.3 \\ 0.4 & -0.3 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & -\frac{3}{4} \\ 0 & 0 \end{pmatrix}$$

So $\vec{x} = \begin{pmatrix} 3/4\, x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 3/4 \\ 1 \end{pmatrix}$ is a solution. But $x_2 = \frac{4}{7}$ gives $\vec{x} = \begin{pmatrix} 3/7 \\ 4/7 \end{pmatrix}$ so we already have this vector.

The other eigenvector is then given by

$$(A-0.3I)\vec{x} = \vec{0}$$

$$\begin{pmatrix} 0.6-0.3 & 0.3 \\ 0.4 & 0.7-0.3 \end{pmatrix} \sim \begin{pmatrix} 0.3 & 0.3 \\ 0.4 & 0.4 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

So all solutions are of the form

$$\vec{x} = x_2 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

So a basis of $\mathbb{R}^2$ in terms of these eigenvectors is $\mathcal{B} = \left\{ \begin{pmatrix} 3/7 \\ 4/7 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

b) $\vec{x}_0 = \vec{v}_1 + c \cdot \vec{v}_2$, where $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 3/7 + c \\ 4/7 - c \end{pmatrix} \Rightarrow c = \frac{1}{14}$$

So $\vec{x}_0$ can be written as $\vec{v}_1 + c \cdot \vec{v}_2$.

c)
$$\begin{aligned} A \cdot \vec{x}_0 &= A\vec{v}_1 + A \cdot c \cdot \vec{v}_2 \\ &= 1 \cdot \vec{v}_1 + 0.3 \cdot c\, \vec{v}_2 \\ &= \vec{v}_1 + 0.3 \cdot c \cdot \vec{v}_2 = \vec{x}_1 \end{aligned}$$

$$\begin{aligned} A^2 \cdot \vec{x}_0 &= A(\vec{v}_1 + 0.3 \cdot c\vec{v}_2) \\ &= \vec{v}_1 + 0.3^2 \cdot c \cdot \vec{v}_2 = \vec{x}_2 \end{aligned}$$

$\vdots$ in general

$$A^k \vec{x}_0 = \vec{v}_1 + 0.3^k c\, \vec{v}_2 = \vec{x}_k$$

As k increases, $0.3^k \cdot \vec{v}_2$ tends to $\vec{0}$ so

$$A^k \vec{x}_0 = \vec{x}_k \longrightarrow \vec{v}_1 \quad \text{as } k \to \infty$$

Using formulas above and $c = \frac{1}{14}$, we get

$$\vec{x}_1 = \begin{pmatrix} 9/20 \\ 11/20 \end{pmatrix} = \begin{pmatrix} .45 \\ .55 \end{pmatrix} \text{ and } \vec{x}_2 = \begin{pmatrix} 87/200 \\ 113/200 \end{pmatrix} = \begin{pmatrix} .435 \\ .565 \end{pmatrix}$$

18) The eigenspace for $\lambda = 5$ is

$$\text{Nul}\begin{pmatrix} 0 & -2 & 6 & -1 \\ 0 & -2 & h & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & -2 & 6 & -1 \\ 0 & -2 & h & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & -2 & 6 & -1 \\ 0 & -2 & h & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We want this augmented matrix to have two pivot columns since this would correspond to rank$(A-5I)=2$, which implies that $\dim(\text{Nul}(A-5I)) = \underset{\text{number of columns.}}{4} - \text{rank}(A-5I) = 2$

It is clear that $h=6$ will make the eigenspace 2 dimensional since then row 1 could be reduced, and were left with Row 2 and 3, so let $h=6$

$$\begin{pmatrix} 0 & -2 & 6 & -1 \\ 0 & -2 & 6 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So x_1 and x_3 is free, $x_2 = 3x_3$

$x_4 = 0$

So in general $\vec{x} = x_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \cdot \begin{pmatrix} 0 \\ 3 \\ 1 \\ 0 \end{pmatrix}$

$h = 6$ makes the eigenspace 2 dimensional.

19) We have $\det(A-\lambda I) = (\lambda_1-\lambda)(\lambda_2-\lambda)\dots(\lambda_n-\lambda)$

Let $\lambda = 0$ so $\det(A-\lambda I) = \det A$.

Then, by the above formula,

$$\det A = (\lambda_1 - 0)(\lambda_2-0)\dots(\lambda_n-0)$$
$$= \lambda_1\lambda_2\dots\lambda_n$$

So $\det A$ is the product of the eigenvalues.

27) $A = \begin{pmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{pmatrix}$, $\vec{V}_1 = \begin{pmatrix} .3 \\ .6 \\ .1 \end{pmatrix}$, $\vec{V}_2 = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$, $\vec{V}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

a) $$A\cdot\vec{V}_1 = \begin{pmatrix} 0.5\cdot 0.3 + 0.2\cdot 0.6 + 0.3\cdot 0.1 \\ 0.3\cdot 0.3 + 0.8\cdot 0.6 + 0.3\cdot 0.1 \\ 0.2\cdot 0.3 + 0\cdot 0.6 + 0.4\cdot 0.1 \end{pmatrix}$$

$$= \begin{pmatrix} 0.3 \\ 0.6 \\ 0.1 \end{pmatrix} = \vec{V}_1$$

So $\underline{A\vec{V}_1 = \vec{V}_1}$

$$A\vec{V}_2 = \begin{pmatrix} 0.5\cdot 1 + 0.2\cdot(-3) + 0.3\cdot 2 \\ 0.3\cdot 1 + 0.8\cdot(-3) + 0.3\cdot 2 \\ 0.2\cdot 1 + 0\cdot(-3) + 0.4\cdot 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0.5 \\ -1.5 \\ 1 \end{pmatrix} = \frac{1}{2}\cdot\vec{V}_2$$

So $\underline{A\vec{V}_2 = \frac{1}{2}\vec{V}_2}$

$$A\vec{V}_3 = \begin{pmatrix} -0.5 + 0 + 0.3 \\ -0.3 + 0 + 0.3 \\ -0.2 + 0 + 0.4 \end{pmatrix} = \begin{pmatrix} -0.2 \\ 0 \\ 0.2 \end{pmatrix} = \frac{1}{5}\cdot\vec{V}_3$$

So $A\cdot\vec{V}_3 = \frac{1}{5}\vec{V}_3$

So all the vectors are eigenvectors.

b) The three vectors are linearly independent and each in $\mathbb{R}^3$ so they form a basis for $\mathbb{R}^3$, therefor there must exist constants c_1, c_2, c_3 such that

$$\vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$$

$$\vec{w}^T = (1 \quad 1 \quad 1)$$

$$\vec{w}^T\vec{x} = \vec{w}^T(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3)$$

$$= c_1\vec{w}^T\vec{v}_1 + c_2\vec{w}\vec{v}_2 + c_3\vec{w}^T\vec{v}_3$$

$$\vec{w}^T\cdot\vec{v}_1 = (1 \quad 1 \quad 1)\begin{pmatrix} .3 \\ .6 \\ .1 \end{pmatrix} = (0.3 \quad 0.6 \quad 0.1)$$

$$\vec{w}^T\vec{v}_3 = (1 \quad -3 \quad 2)$$

$$\vec{w}^T\vec{v}_2 = (-1 \quad 0 \quad 1)$$

$$\vec{w}^T\vec{x}_0 = c_1\begin{pmatrix} 0.3 \\ 0.6 \\ 0.1 \end{pmatrix}^T + c_2\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}^T + c_3\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}^T$$

The sum of the entries must be 1 since $\vec{x}_0$ was a probability-vector.

So $1 = c_1(0.3+0.6+0.1) + c_2(1-3+2) + c_3(-1+1)$

$1 = c_1$

So $c_1 = 1$ ✓

c) For $k = 1, 2, \ldots$ define $\vec{x}_k = A^k \vec{x}_0$
with $\vec{x}_0 = \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3$
From (a) we saw that $A\vec{v}_2 = \frac{1}{2}\vec{v}_2$
and $A\vec{v}_3 = \frac{1}{5}\vec{v}_3$.

In general this implies that
$$A^k \vec{v}_2 = \left(\frac{1}{2}\right)^k \vec{v}_2$$
$$A^k \vec{v}_3 = \left(\frac{1}{5}\right)^k \vec{v}_3$$

We also saw that $A\vec{v}_1 = \vec{v}_1 \Rightarrow A^k \vec{v}_1 = \vec{v}_1$.
From this we can see that
$$\vec{x}_k = A^k \vec{x}_0 = A^k \vec{v}_1 + c_2 A^k \vec{v}_2 + c_3 A^k \vec{v}_3$$
$$= \vec{v}_1 + c_2 \cdot \left(\frac{1}{2}\right)^k \vec{v}_2 + c_3 \left(\frac{1}{5}\right)^k \vec{v}_3$$

As $k \to \infty$: $\left(\frac{1}{2}\right)^k \to 0$ and $\left(\frac{1}{5}\right)^k \to 0$ so
$$\vec{x}_k \to \vec{v}_1 + \vec{0} + \vec{0} = \vec{v}_1$$

5.3

1) $P = \begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

$A = PDP^{-1}$

$A^4 = PD^4P^{-1} = P\begin{pmatrix} 2^4 & 0 \\ 0 & 1^4 \end{pmatrix}P^{-1}$

$= P\begin{pmatrix} 16 & 0 \\ 0 & 1 \end{pmatrix}P^{-1}$

P^{-1}: $\det(P) = 15 - 14 = 1$

$P^{-1} = \begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix}$

$A^4 = PD^4P^{-1}$

$PD^4 = \begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 16 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 16\cdot 5 & 7\cdot 1 \\ 16\cdot 2 & 3\cdot 1 \end{pmatrix} = \begin{pmatrix} 80 & 7 \\ 32 & 3 \end{pmatrix}$

$PD^4P^{-1} = \begin{pmatrix} 80 & 7 \\ 32 & 3 \end{pmatrix}\begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} 80\cdot 3 - 2\cdot 7 & -80\cdot 7 + 7\cdot 5 \\ 32\cdot 3 - 3\cdot 2 & -7\cdot 32 + 3\cdot 5 \end{pmatrix}$

$= \begin{pmatrix} 226 & -525 \\ 90 & -209 \end{pmatrix}$

So $A^4 = \begin{pmatrix} 226 & -525 \\ 90 & -209 \end{pmatrix}$

5) Eigen value $\lambda_1 = 5$ has $\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$ as a basis for its eigenspace.

Eigenvalue $\lambda_2 = 1$ has $\left\{\begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}2\\-1\\0\end{pmatrix}\right\}$ as a basis for its eigenspace.

7) $A = \begin{pmatrix}1 & 0\\6 & -1\end{pmatrix}$. A is triangular so it diagonal consists of eigenvalues

$\lambda_1 = 1$:

$$(A-I)\vec{x} = \vec{0}$$

$$\begin{pmatrix}0 & 0\\6 & -2\end{pmatrix} \sim \begin{pmatrix}0 & 0\\1 & -\frac{1}{3}\end{pmatrix} \qquad \begin{matrix}x_1 \text{ free}\\ x_2 = \frac{1}{3}x_1\end{matrix}$$

A corresponding eigenvector is then

$$\vec{v}_1 = \begin{pmatrix}3\\1\end{pmatrix}$$

$\lambda_2 = -1$:

$$(A+I)\vec{x} = \vec{0}$$

$$\begin{pmatrix}2 & 0\\6 & 0\end{pmatrix} \sim \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} \qquad \begin{matrix}x_1 = 0\\ x_2 \text{ free}\end{matrix}$$

A corresponding eigenvector is then

$$\vec{v}_2 = \begin{pmatrix}0\\1\end{pmatrix}$$

So then $P = \begin{pmatrix}3 & 0\\1 & 1\end{pmatrix}$

$$P^{-1} = \frac{1}{\det(P)}\begin{pmatrix}1 & -1\\0 & 3\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & -1\\0 & 3\end{pmatrix}$$

$$= \begin{pmatrix}1/2 & -1/2\\0 & 3/2\end{pmatrix}$$

So then

$$A = \begin{pmatrix}1 & 0\\6 & -1\end{pmatrix} = \begin{pmatrix}3 & 0\\1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1/2 & -1/2\\0 & 3/2\end{pmatrix}$$

12) $A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$

$\lambda_1 = 2 \Rightarrow (A - 2I)\vec{x} = \vec{0}$

$\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

x_2, x_3 free

$x_1 = -x_2 - x_3$

A corresponding eigenvector basis is then $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\lambda_2 = 8 \Rightarrow (A - 8I)\vec{x} = \vec{0}$

$\begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$

$\sim \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{3}{2} & -\frac{3}{2} \\ 0 & -\frac{3}{2} & \frac{3}{2} \end{pmatrix}$

$\sim \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$

$\sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

x_3 free

$x_1 = x_2 = x_3.$

A corresponding eigenvector basis is then $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

The sum of the dimensions of the eigenspaces is 3 (and since A is 3×3) A is infact diagonalizable.

$$P = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

$$P^{-1}: \left(\begin{array}{ccc|ccc} -1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 3 & 1 & 1 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & -2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -2/3 & 1/3 & 1/3 \\ 0 & 1 & 0 & -1/3 & -1/3 & 2/3 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/3 & 2/3 & -1/3 \\ 0 & 1 & 0 & -1/3 & -1/3 & 2/3 \\ 0 & 0 & 1 & 1/3 & 1/3 & 1/3 \end{array}\right)$$

$$P^{-1} = \begin{pmatrix} -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$

$$\text{So} \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

5.5

1) $A = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$

$\det(A - \lambda I) = 0$

$(1-\lambda)(3-\lambda) + 2 = 0$

$3 - \lambda - 3\lambda + \lambda^2 + 2 = 0$

$\lambda^2 - 4\lambda + 5 = 0$

$$\lambda_{1,2} = \frac{4 \pm \sqrt{16 - 4 \cdot 5}}{2} = \frac{4 \pm \sqrt{-4}}{2} = 2 \pm i$$

So $\lambda_1 = 2+i$ and $\lambda_2 = 2-i$

$\text{Nul}\,(A - (2+i)I)$: $\begin{pmatrix} 1-2-i & -2 \\ 1 & 3-2-i \end{pmatrix} \sim \begin{pmatrix} -1-i & -2 \\ 0 & 0 \end{pmatrix}$

x_2 free, $x_1 = x_2 \cdot \frac{+2}{-1-i}$

$= x_2 \frac{-2\,(1-i)}{(1+i)(1-i)}$

$= x_2 \frac{-2+2i}{1+1}$

$= x_2(-1+i)$

$\vec{v}_1 = \begin{pmatrix} -1+i \\ 1 \end{pmatrix}$

Since $\lambda_2 = \overline{\lambda_1}$, $\vec{v}_2 = \overline{\vec{v}_1} = \begin{pmatrix} -1-i \\ 1 \end{pmatrix}$

Basis for e. space of $\lambda_1 = 2+i$: $\begin{pmatrix} -1+i \\ 1 \end{pmatrix}$

—— " —— $\lambda_2 = 2-i$: $\begin{pmatrix} -1-i \\ 1 \end{pmatrix}$

7) $A = \begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$

$$= \begin{pmatrix} r\cos\varphi & -r\sin\varphi \\ r\sin\varphi & r\cos\varphi \end{pmatrix}$$

We have $r\cos\varphi = \sqrt{3}$ (i)

$r\sin\varphi = 1$ (ii)

(i) $\Rightarrow r = \frac{\sqrt{3}}{\cos\varphi}$

(ii) $\Rightarrow \frac{\sqrt{3}}{\cos\varphi}\sin\varphi = 1$

$\tan\varphi = \frac{1}{\sqrt{3}}$

$\varphi = \frac{\pi}{6}$

$\Rightarrow r = \frac{\sqrt{3}}{\cos(\pi/6)} = 2$

So $\varphi = \frac{\pi}{6}$ and $r = 2$.

13) From ex. 1: $\lambda_1 = 2+i$, $\lambda_2 = 2-i$

and $\vec{v}_1 = \begin{pmatrix} -1+i \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -1-i \\ 1 \end{pmatrix}$

$C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ where $\lambda_1 = a + bi$, so $a=2$
$b=1$

$$\Rightarrow C = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

$P = \begin{pmatrix} \mathrm{Re}\,\vec{v}_1 & \mathrm{Im}\,\vec{v}_1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$