# DigSig 9

Rendell Cale, rendellc@stud.ntnu.no, mttk

## Problem 1

a) $$H_d(f) = \begin{cases} 1 & , |f| < f_c \\ 0 & , f_c \leq f \leq 0.5 \end{cases}$$

$$h_d(n) = \int_{-0.5}^{5} H_d(f) e^{j2\pi f n} df$$

$$= \int_{-f_c}^{f_c} e^{j2\pi f n} df$$

For $n \neq 0$ we get:

$$h_d(n) = \frac{1}{j2\pi n} \left[ e^{j2\pi f n} \right]_{f=-f_c}^{f=f_c}$$

$$= \frac{1}{\pi n} \frac{1}{2j} \left( e^{j2\pi f_c n} - e^{j2\pi f_c n} \right)$$

$$= \frac{\sin(2\pi f_c n)}{\pi n}$$

For $n = 0$ we get:

$$h_d(n) = \int_{-f_c}^{f_c} 1 \cdot df = 2f_c$$

So in total we get

$$h_d(n) = \begin{cases} \frac{\sin(2\pi f_c n)}{\pi n} &, n \neq 0 \\ 2f_c &, n = 0 \end{cases}$$

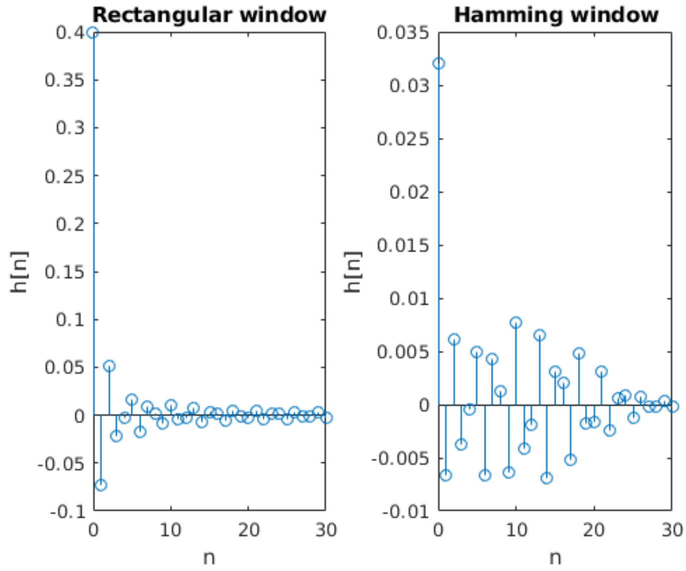
$$= 2f_c \, \text{sinc}(2\pi f_c n)$$

b) The unit sample response is given by

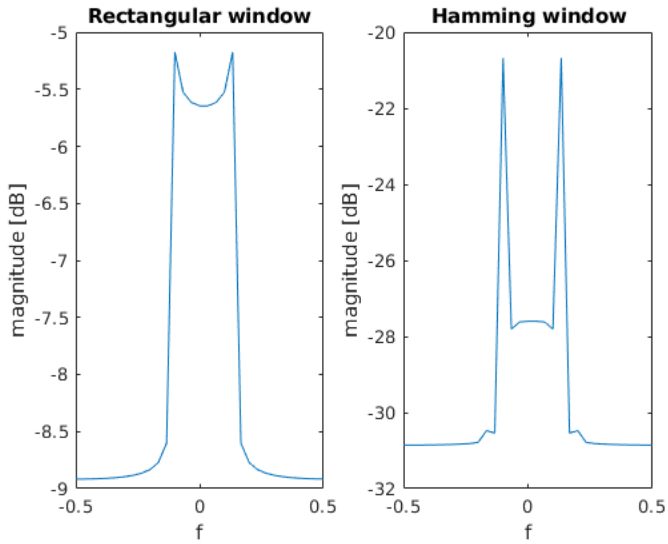
$$h(n) = h_d(n) w(n)$$
$$= 2f_c \, \text{sinc}(2\pi f_c n) \, w(n)$$

c) The filter coefficients for the FIR filter is then given by $h(0)w(0), h(1)w(1), \dots, h(N-1)w(N-1)$

d) Impulse responses:



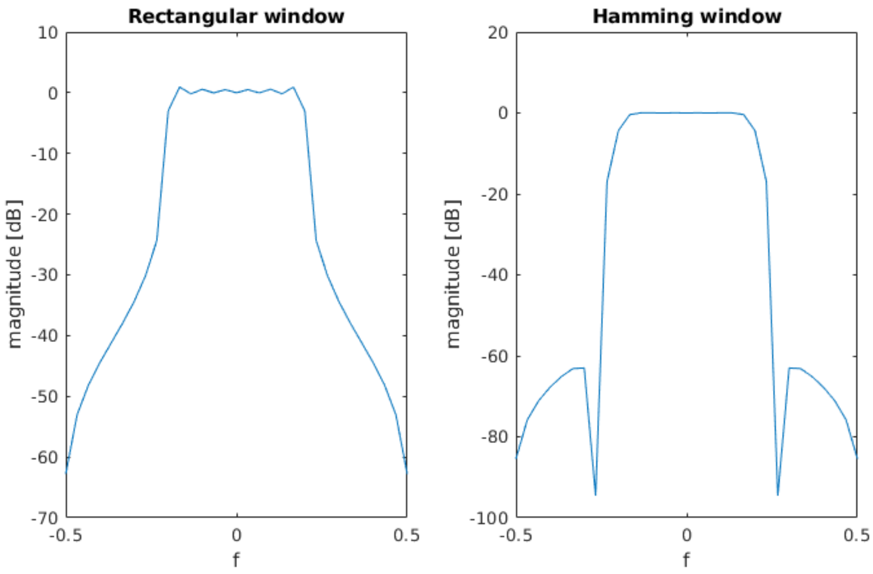

The Hamming window causes a more gradual decrease in impulse response.




The Hamming window seems to give an additional heavy attenuation around $f = 0$.

This is probably a mistake but I can't find it.

e)




The Hamming window provides a steeper drop at cut off frequency, and it also gives more attenuation.
At the desired frequencies it also has a flatter spectrum.

The plots in (d) look completely different so I suspect I've made a mistake somewhere on the way.

## Problem 2

a) The cutoff is found by setting

$$|H_a(j\Omega_c)| = \frac{1}{\sqrt{2}}$$

$$\Leftrightarrow |H_a(j\Omega_c)|^2 = \frac{1}{2}$$

$$\Leftrightarrow \quad \frac{1}{2} = \frac{(1/RC)^2}{\Omega_c^2 + \left(\frac{1}{RC}\right)^2}$$

$$\Leftrightarrow \quad \frac{\Omega_c^2}{2} + \frac{1}{2}\left(\frac{1}{RC}\right)^2 = \left(\frac{1}{RC}\right)^2$$

$$\Leftrightarrow \quad \Omega_c^2 = \left(\frac{1}{RC}\right)^2$$

$$\Leftrightarrow \quad \underline{\underline{\Omega_c = (\pm)\ 1/RC}}$$

b) On the unit circle we have $s = j\Omega$ and $z = e^{j\omega}$

$$j\Omega = \frac{2}{T} \frac{1 - e^{-j\omega}}{1 + e^{-j\omega}}$$

$$\Leftrightarrow j\Omega T \left(1 + e^{-j\omega}\right) = 2\left(1 - e^{-j\omega}\right)$$

$$\Leftrightarrow j\Omega T\left(1 + \cos\omega - j\sin\omega\right) = 2 - 2\cos\omega + 2j\sin\omega$$

$$\Leftrightarrow \Omega T \sin\omega + j\Omega T(1 + \cos\omega) = 2(1 - \cos\omega) + 2j\sin\omega$$

Equating the imaginary parts we get:

$$\Omega T(1 + \cos\omega) = 2\sin\omega$$

$$\Leftrightarrow \quad \Omega = \frac{2}{T} \frac{\sin\omega}{1 + \cos\omega}$$

$$= \frac{2}{T} \tan\left(\frac{\omega}{2}\right)$$

c) A cutoff of $\omega_c = 0.2\pi$ corresponds to

$$\Omega_c = \frac{2}{T}\tan\left(\frac{0.2\pi}{2}\right) = \frac{0.65}{T}$$

Using this in our analog filter we have

$$H_a(s) = \frac{\frac{1}{RC}}{s + \frac{1}{RC}},$$

$$= \frac{\Omega_c}{s+\Omega_c}$$

$$= \frac{0.65}{sT + 0.65}$$

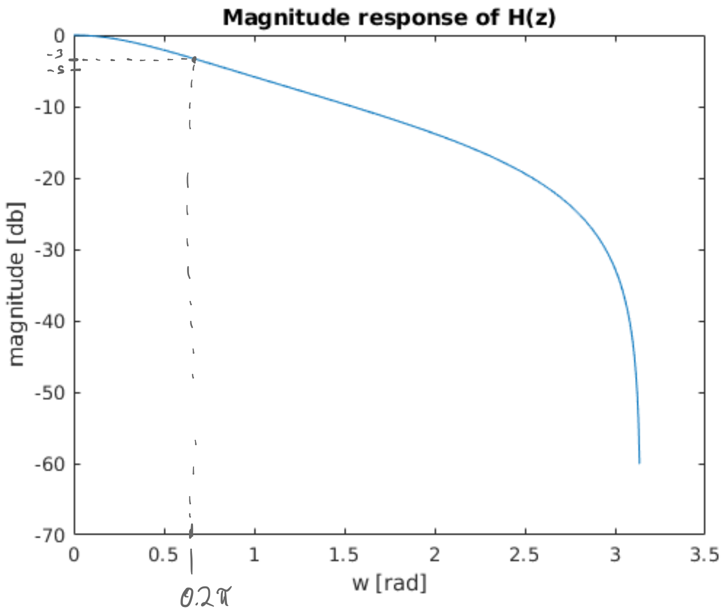
Applying the bilinear transformation then gives

$$H(z) = \frac{0.65}{\frac{2}{\cancel{T}}\frac{1-z^{-1}}{1+z^{-1}}\cancel{T} + 0.65}$$

$$= \frac{0.65\,(1+z^{-1})}{2+0.65 + (0.65-2)z^{-1}}$$

$$= \frac{0.245\,(1+z^{-1})}{1 - 0.51\,z^{-1}}$$

Magnitude response of filter




The filter gives 3dB attenuation at $w_c = 0.2\pi$ as illustrated above.

## Problem 3

$$H_a(s) = \frac{1}{s^2 + \sqrt{2}\,s + 1}$$

a) Computing the magnitude squared response of $H_a(s)$, we get:

$$|H_a(\Omega)|^2 = \frac{1}{|j\sqrt{2}\,\Omega + (1-\Omega^2)|^2}$$

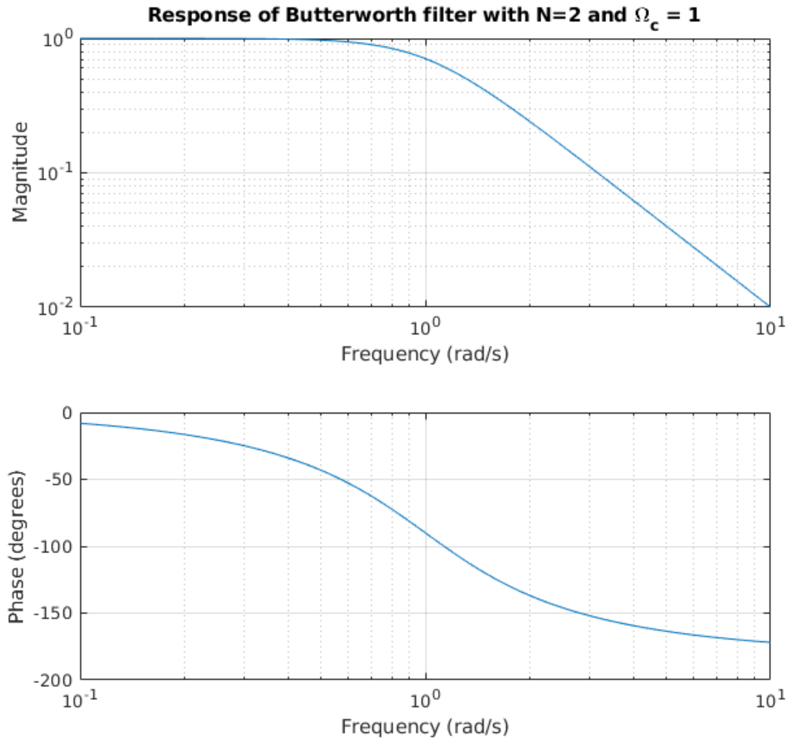
$$= \frac{1}{(1-\Omega^2)^2 + (\sqrt{2}\,\Omega)^2}$$

$$= \frac{1}{1 \cancel{-2\Omega^2} + \Omega^4 \cancel{+2\Omega^2}}$$

$$= \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2\cdot N}}, \quad N = 2, \ \Omega_c = 1$$

Which is the same as a Butterworth filter of order $N=2$ and with cutoff $\Omega_c = 1$.

b)




c) Poles are given by

$$s^2 + \sqrt{2}\,s + 1 = 0$$

$$\Rightarrow \quad s = \frac{-\sqrt{2} \pm \sqrt{2-4}}{2}$$

$$= -\frac{\sqrt{2}}{2} \pm \frac{i\sqrt{2}}{2}$$

d) We have $H_a(s) = \dfrac{1}{(s-p_k)(s-p_k^*)}$

where $p_k = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}j$

$= e^{\frac{3\pi}{4}j}$

and $p_k^* = e^{-\frac{3\pi}{4}j}$

This gives

$$H_a(s) = \frac{A}{s-p_k} + \frac{B}{s-p_k^*}$$

$$\Rightarrow \quad 1 = A(s-p_k^*) + B(s-p_k)$$

$$\Rightarrow \begin{cases} A = \dfrac{1}{p_k-p_k^*} = \dfrac{1}{2\,\mathrm{Im}\{p_k\}} = \dfrac{1}{\sqrt{2}} \\[2ex] B = \dfrac{1}{p_k^*-p_k} = -\dfrac{1}{2\,\mathrm{Im}\{p_k\}} = -\dfrac{1}{\sqrt{2}} \end{cases}$$

So we can write

$$H_a(s) = \frac{1/\sqrt{2}}{s-p_k} - \frac{1/\sqrt{2}}{s-p_k^*}$$

Say we sample the analog filter with period $T$.

Then $h[n] = h_a(nT)$.

$$h_a(t) = \mathcal{L}^{-1}\{H_a(s)\}$$

$$= \frac{1}{\sqrt{2}}\left(e^{p_k t} - e^{p_k^* t}\right)$$

$$= \frac{1}{\sqrt{2}}\left(e^{\frac{3\pi}{4}jt} - e^{-\frac{3\pi}{4}jt}\right)$$

$$= \frac{2}{\sqrt{2}} \sin\left(\frac{3\pi}{4}t\right) = \sqrt{2}\sin\left(\frac{3\pi}{4}t\right)$$

We sample $h_a(t)$ and then we get

$$h[n] = \sqrt{2}\sin\left(\frac{3\pi}{4}Tn\right)$$

We get a cutoff frequency $w_c$ by choosing

$$\Omega_c = \frac{2}{T}\tan\left(\frac{w_c}{2}\right)$$

Since $\Omega_c = 1$, this means that

$$T = 2\tan\left(\frac{w_c}{2}\right)$$

and we can write the discrete IIR as

$$h[n] = 2\sin\left(\frac{3\pi}{2}\tan\left(\frac{w_c}{2}\right)n\right)$$

For $w_{c_1} = 0.25$ we have

$$\underline{\underline{h_1[n] = 2\sin(0.59\,n).}}$$

For $w_{c_2} = 1.4$ we have

$$\underline{\underline{h_2[n] = 2\sin(3.97\,n).}}$$

Using eq. 3.31 in (Proakis) we find that

$$H(z) = \frac{1/\sqrt{2}}{1-e^{p_k T}z^{-1}} + \frac{1/\sqrt{2}}{1-e^{p_k^* T}z^{-1}}$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1-e^{p_k^* T}z^{-1} + 1 - e^{p_k^* T}z^{-1}}{(1-e^{p_k T})(1-e^{p_k^* T})}\right)$$

$$\Rightarrow H(z) = \frac{1}{\sqrt{2}} \left( \frac{2 - \left(e^{p_k T} + e^{p_k^* T}\right) z^{-1}}{e^{(p_k^* + p_k)T} z^{-2} - \left(e^{p_k T} + e^{p_k^* T}\right) z^{-1} + 1} \right)$$

$$= \frac{1}{\sqrt{2}} \frac{2 - 2\cos\left(\mathrm{Im}\{p_k\} T\right) z^{-1}}{e^{2\,\mathrm{Re}\{p_k\} T} z^{-2} - 2\cos\left(\mathrm{Im}\{p_k\} T\right) z^{-1} + 1}$$

We have $p_k = e^{\frac{3\pi}{4} j}$ so we simplify further and get:

$$H(z) = \frac{\sqrt{2}\left(1 - \cos\left(\frac{3\pi}{4} T\right) z^{-1}\right)}{z^{-2} - 2\cos\left(\frac{3\pi}{4} T\right) z^{-1} + 1}$$

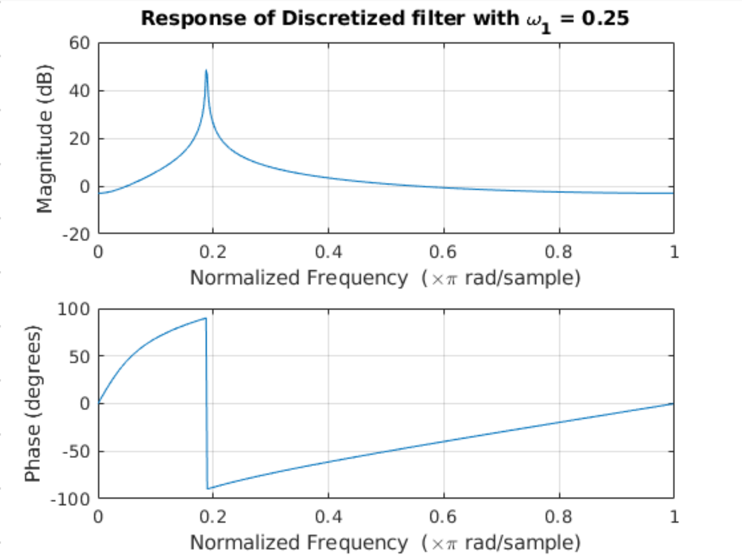
Thus for $w_1$ we have:

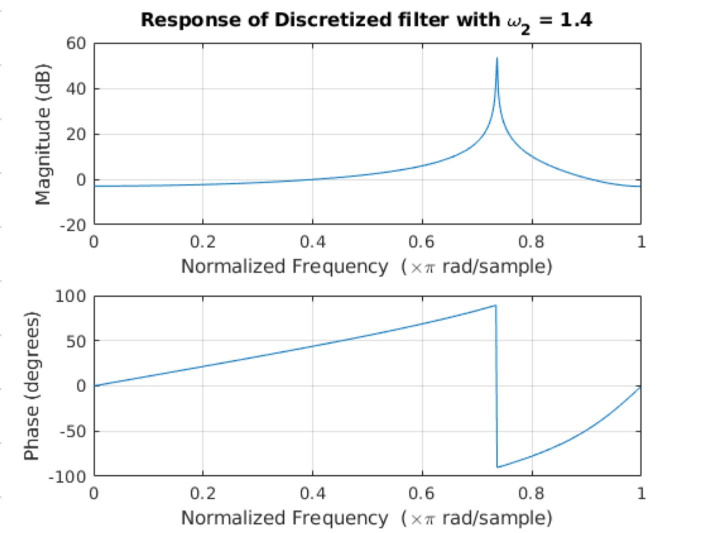
$$H_1(z) = \frac{\sqrt{2}\left(1 - 0.83 z^{-1}\right)}{z^{-2} - 1.66 z^{-1} + 1}$$

and for $w_2$ we have:

$$H_2(z) = \frac{\sqrt{2}\left(1 + 0.68 z^{-1}\right)}{z^{-2} + 1.35 z^{-1} + 1}$$

e)



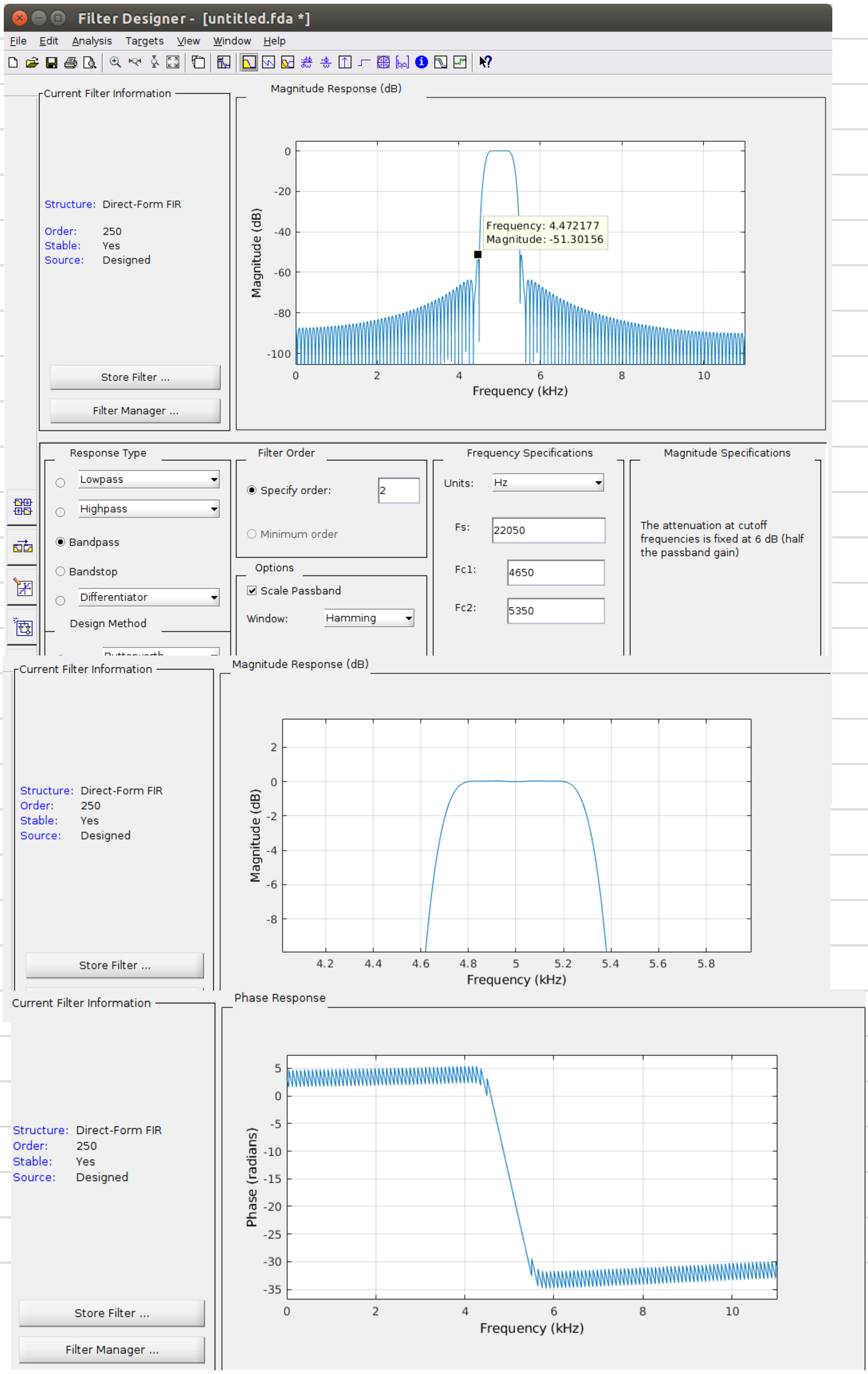




Both the digital filters have a resonance frequency whereas the analog doesn't.

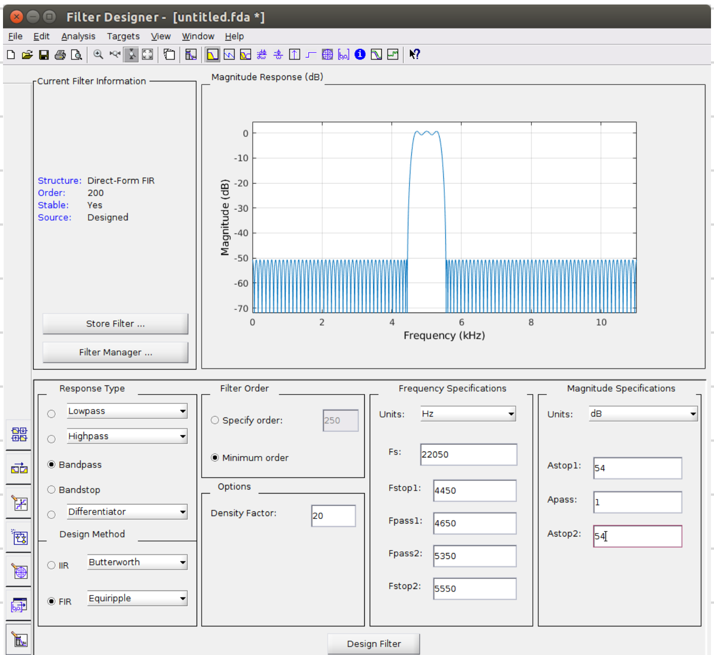
Note: I messed up and did this problem with bandpass instead of bandstop. The results should be quite similar (except oposite) though.

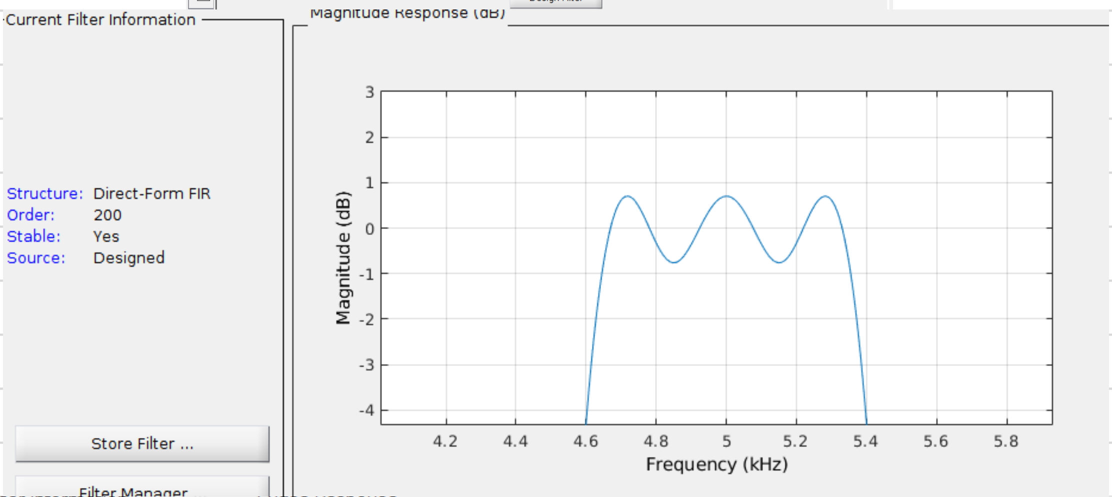
## Problem 4

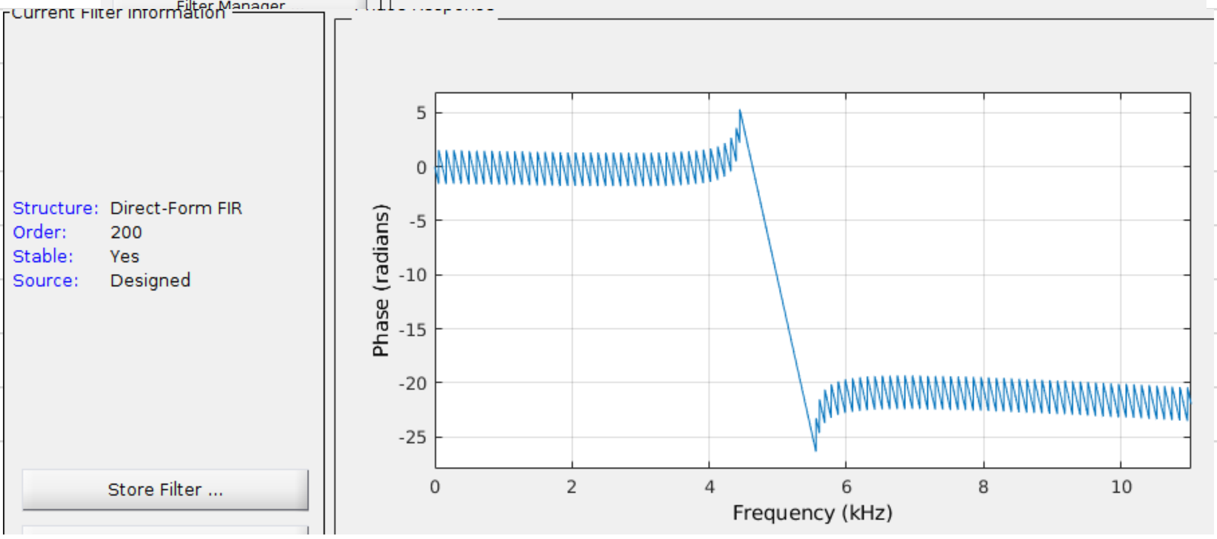
a) Lowest filter lenth is about 250 which gives side lobe level of −52 dB.



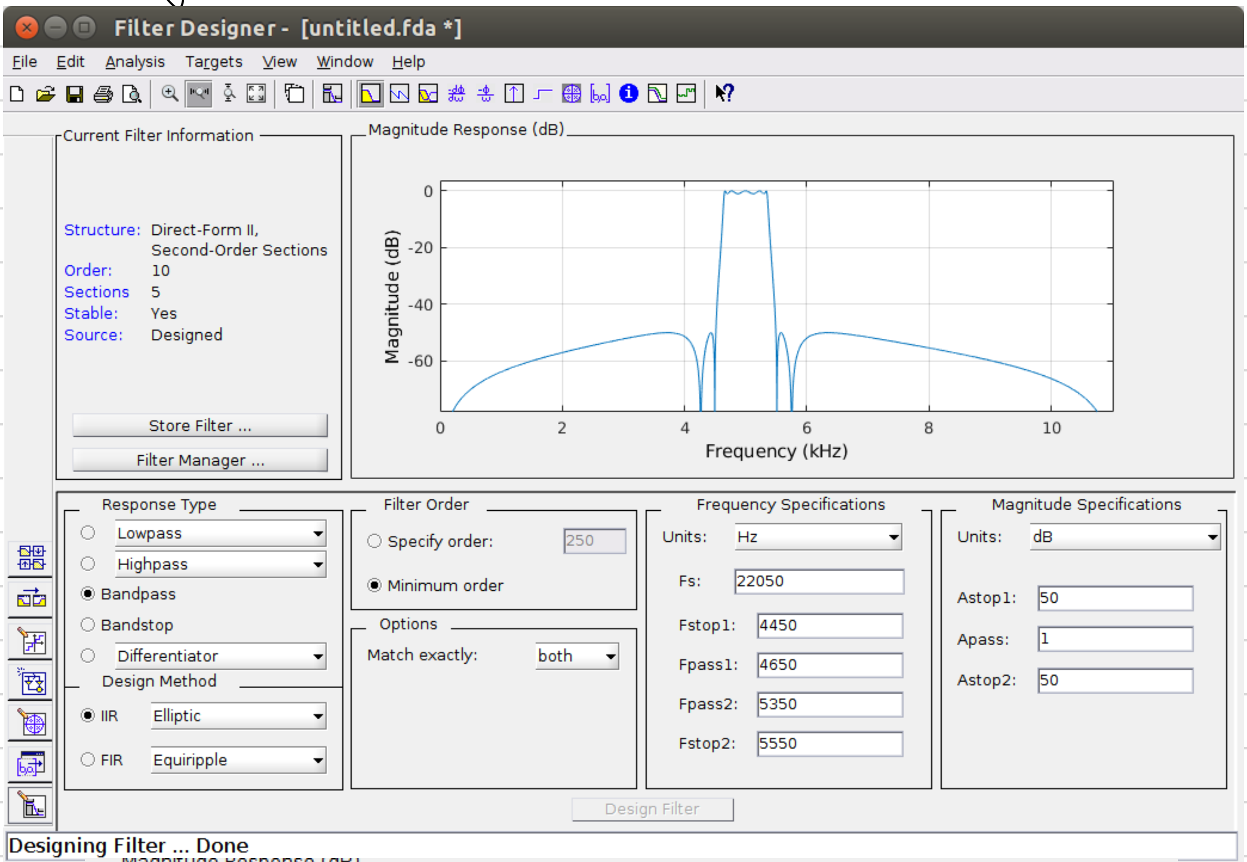
b) Using "minum order" and selecting $A_{stop} = 54$ dB we get a filter of order 200 that satisfies the specs.

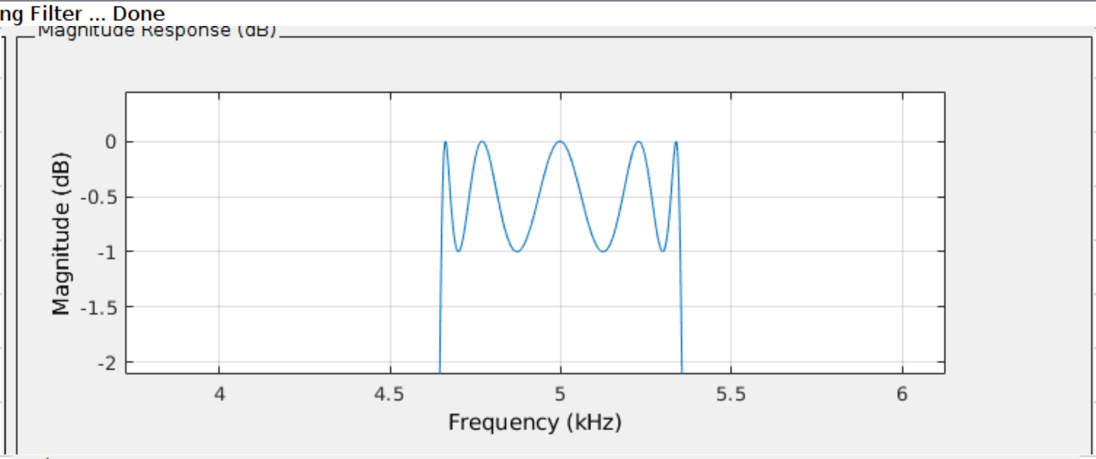


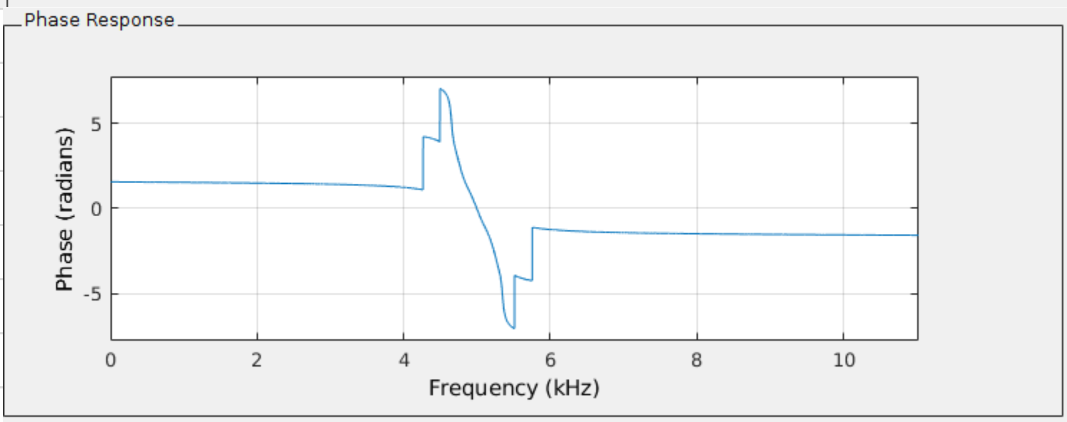



c) Using $A_{stop} = 50$ dB and minimum order we get a filter of order 10.



d) The FIR filters, even when optimized for low order, require a much higher order and thus more memory.

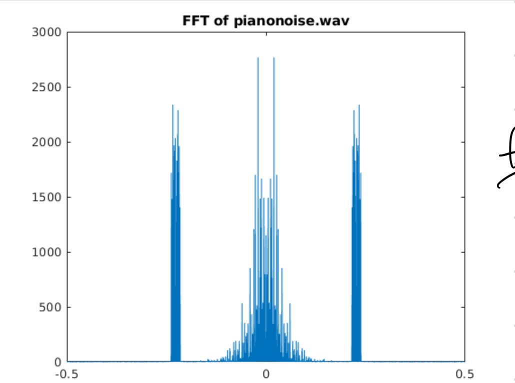
IIR uses only 10 terms and gives a really steep transition band.

One disadvantage of this IIR is that the phase response is quite wild in the transition band.

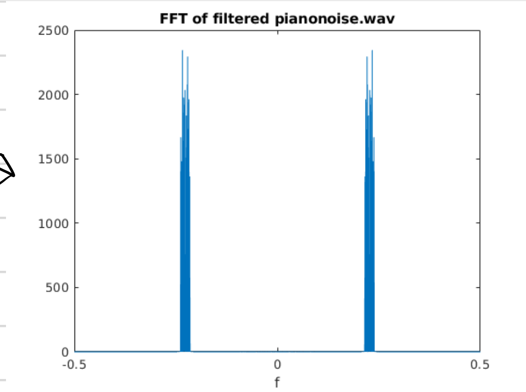
It seems that if we can guarantee that the IIR is stable, then it is superior both in performance and memory.

e) In the unfiltered track, there is both a piano playing and an annoying clicking track.

After filtering with the FIR of order 250, the piano is gone. Taking the fft, its clear the something was removed by the filter.




filter




The filter order affects the transition width. When the order is low, the filter will have a "wide" frequency response and wont reach the stopband fast enough.

The piano is removed perfectly.

If I had used a stopband, only the piano would remain since the piano and noise are at completely different frequencies.