LinSys - 1

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Problem 1

We have

$$\ddot{y} + 2\dot{y} = \dot{u} + 4u$$

and want to write

$$\dot{\underline{x}} = A\underline{x} + Bu$$

$$y = C\underline{x} + Du$$

with

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y \\ \dot{y} - u \end{pmatrix}$$

This gives

$$x_2 = \dot{x}_1 - u \iff \dot{x}_1 = x_2 + u \qquad (1)$$

We also get

$$\dot{x}_2 = \ddot{y} - \dot{u} = 4u - 2\dot{y}$$
$$= -2(\dot{y} - u) + 2u$$
$$= -2x_2 + 2u \qquad (2)$$

(1) and (2) give

$$A = \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

We have $C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$, $D =$ constant so we have to solve

$$\begin{aligned} y &= C\underline{x} + Du \\ &= c_1 x_1 + c_2 x_2 + Du \\ &= c_1 y + c_2(\dot{y} - u) + Du \\ &= c_1 y + c_2 \dot{y} + (-c_2 + D) u \end{aligned}$$

This gives $c_1 = 1$, $c_2 = D = 0$ so in total we have

$$A = \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\underline{\underline{C = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad D = 0}}$$

b) $\hat{g}(s) = C(sI - A)^{-1}B + D$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} s & -1 \\ 0 & s+2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 0$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{1}{s(s+2)} \begin{pmatrix} s+2 & 1 \\ 0 & s \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \frac{1}{s(s+2)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} s+2+2 \\ 0+2s \end{pmatrix}$$

$$= \frac{1}{s(s+2)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} s+4 \\ 2s \end{pmatrix}$$

$$= \underline{\underline{\frac{s+4}{s(s+2)}}}$$

c) Assuming zero-initial conditions we get

$$\mathcal{L}\{\ddot{y} + 2\dot{y}\} = \mathcal{L}\{\dot{u} + 4u\}$$

$$\Leftrightarrow \quad s^2\hat{y} + 2s\hat{y} = s\hat{u} + 4\hat{u}$$

$$\Leftrightarrow \quad \hat{y}\, s(s+2) = \hat{u}(s+4)$$

$$\Leftrightarrow \quad \hat{g} = \frac{\hat{y}}{\hat{u}} = \underline{\underline{\frac{s+4}{s(s+2)}}}$$

d) $\hat{g} = \frac{\alpha_1}{s} + \frac{\alpha_2}{s+2} = \frac{s+4}{s(s+2)}$ gives

$$(s+2)\alpha_1 + \alpha_2 s = s+4$$

Which is solved by $\alpha_1 = 2$ and $\alpha_2 = -1$

So we have

$$\hat{g} = \frac{2}{s} - \frac{1}{s+2}$$

which means

$$g(t) = \mathcal{L}^{-1}\{\hat{g}\}$$
$$= \begin{cases} 2 - e^{-2t} & , \quad t \geq 0 \\ 0 & , \quad t < 0 \end{cases}$$

Problem 2

a) $e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$

$$(sI - A)^{-1} = \begin{pmatrix} s & -1 \\ 0 & s+3 \end{pmatrix}^{-1} = \frac{1}{s(s+3)} \begin{pmatrix} s+3 & 1 \\ 0 & s \end{pmatrix}$$

$$= \begin{pmatrix} 1/s & \frac{1}{s(s+3)} \\ 0 & \frac{1}{s+3} \end{pmatrix}$$

We can write

$$\frac{1}{s(s+3)} = \frac{\alpha_1}{s} + \frac{\alpha_2}{s+3}$$

$$\Rightarrow \quad 1 = \alpha_1(s+3) + \alpha_2 s$$

$$\Rightarrow \quad \alpha_1 = \frac{1}{3}, \quad \alpha_2 = -\frac{1}{3}$$

So we compute e^{At} as

$$e^{At} = \mathcal{L}^{-1}\left\{\begin{pmatrix} \frac{1}{s} & \frac{1}{3}\frac{1}{s} - \frac{1}{3}\frac{1}{s+3} \\ 0 & \frac{1}{s+3} \end{pmatrix}\right\}$$

$$= \underline{\underline{\begin{pmatrix} 1 & \frac{1}{3}(1-e^{-3t}) \\ 0 & e^{-3t} \end{pmatrix}}}$$

b) Let λ_1, λ_2 be eigenvalues with $\underline{v}_1, \underline{v}_2$ corresponding eigenvectors.
Then
$$\det(A - \lambda \mathbb{I}) = 0$$
$$\Rightarrow \begin{vmatrix} -\lambda & 1 \\ 0 & -3-\lambda \end{vmatrix} = (-\lambda)(-3-\lambda) = 0$$
$$\Rightarrow \underline{\underline{\lambda_1 = 0}}, \quad \underline{\underline{\lambda_2 = -3}}$$

For $\lambda_1 = 0$: $\underline{\underline{\underline{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}}$ is an eigenvector.

For $\lambda_2 = -3$:
$$A \underline{v}_1 = -3 \underline{v}_1$$
$$\Leftrightarrow \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix} \underline{v}_1 = 0$$
$$\Rightarrow \underline{\underline{\underline{v}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}}}$$

c) Using $\hat{A} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -3 \end{pmatrix}$

and $Q = (\underline{v}_1 \mid \underline{v}_2) = \begin{pmatrix} 1 & 1 \\ 0 & -3 \end{pmatrix}$

We have
$$A = Q \hat{A} Q^{-1}$$

d) Since $\hat{A}$ is diagonal we get

$$e^{\hat{A}t} = \begin{pmatrix} e^{0t} & 0 \\ 0 & e^{-3t} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-3t} \end{pmatrix}$$

This gives

$$e^{At} = Q e^{\hat{A}t} Q^{-1}$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{-3t} \end{pmatrix} \frac{1}{-3} \cdot \begin{pmatrix} -3 & -1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & e^{-3t} \\ 0 & -3e^{-3t} \end{pmatrix} \begin{pmatrix} -3 & -1 \\ 0 & 1 \end{pmatrix} \cdot \frac{-1}{3}$$

$$= -\frac{1}{3} \begin{pmatrix} -3 & -1+e^{-3t} \\ 0 & -3e^{-3t} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \frac{1}{3}(1-e^{-3t}) \\ 0 & e^{-3t} \end{pmatrix}$$

e) The solution is given by

$$y(t) = C e^{At} \underline{x}(0) + C \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau + D u(t)$$

- $$C e^{At} \underline{x}(0) = \begin{pmatrix} 3 & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{3}(1-e^{-3t}) \\ 0 & e^{-3t} \end{pmatrix} \underline{x}(0)$$

$$= \begin{pmatrix} 3 & 1-e^{-3t} \end{pmatrix} \underline{x}(0)$$

- $$C \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau = \int_0^t C e^{A(t-\tau)} B u(\tau)\, d\tau$$

$$= \int_0^t \begin{pmatrix} 3 & 1-e^{-3(t-\tau)} \end{pmatrix} \begin{pmatrix} -2 \\ 6 \end{pmatrix} d\tau$$

$$= \int_0^t -6 + 6 - 6e^{-3(t-\tau)}\, d\tau$$

$$= -6e^{-3t} \int_0^t e^{3\tau}\, d\tau$$

$$= -6\, e^{-3t} \frac{1}{3} \left(e^{3t} - 1\right)$$

$$= 2 e^{-3t} \left(1 - e^{3t}\right)$$

$$= 2\left(e^{-3t} - 1\right)$$

- $$D u(t) = 5 \cdot 1 = 5$$

So
$$y(t) = \begin{pmatrix} 3 & 1-e^{-3t} \end{pmatrix} \underline{x}(0) + 2\left(e^{-3t} - 1\right) + 5$$
$$= \underline{\underline{\begin{pmatrix} 3 & 1-e^{-3t} \end{pmatrix} \underline{x}(0) + 2e^{-3t} + 3}}$$

f) Assuming $y(1) = y(2) = 4$ and writing $\underline{x}(0) = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}$, we get
$$y(t) = 3x_{10} + \left(1 - e^{-3t}\right)x_{20} + 2e^{-3t} + 3$$

The assumptions give:
$$y(1) = 3x_{10} + \left(1-e^{-3}\right)x_{20} + 2e^{-3} + 3 = 4 + 0 \cdot e^{-3}$$
$$\Rightarrow \begin{cases} x_{20} = 2 \\ 3x_{10} + x_{20} + 3 = 4 \end{cases}$$
$$\Rightarrow \quad x_{10} = \frac{4-3-2}{3} = -\frac{1}{3}$$

So $\underline{\underline{\underline{x}(0) = \left(-\frac{1}{3}, 2\right)^T}}$

Problem 3

a) $$\ddot{r} + \frac{2u_1}{\sqrt{r^2+9}}\dot{r} + 3r + 4 = \frac{r}{\sqrt{r^2+9}}u_1 + u_2$$

$$y = 5\sqrt{r^2+9}$$

$$\underline{x} = (x_1, x_2)^T, \quad x_1 = r, \quad x_2 = \dot{r}$$

Note that $y = 5\sqrt{r^2+9} = 5\sqrt{x_1^2+9}$ is already in the form

$$y = f(\underline{x}, \underline{u})$$

with $f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \underline{u}\right) = 5\sqrt{x_1^2+9}$

The equation of motion gives

$$\ddot{r} = \dot{x}_2 = \frac{x_1}{\sqrt{x_1^2+9}}u_1 + u_2 - 4 - 3x_1 - \frac{2u_1}{\sqrt{x_1^2+9}}x_2$$

$$\dot{x}_1 = x_2$$

The two eqs above show that we can write

$$\dot{\underline{x}} = \underline{h}(\underline{x}, \underline{u})$$

b) $A = \frac{\partial h}{\partial \underline{x}}(\underline{x}_0, \underline{u}_0)$

$$\frac{\partial h_1}{\partial x_1} = 0$$

$$\frac{\partial h_1}{\partial x_2} = 1$$

$$\frac{\partial h_2}{\partial x_1} = \frac{\partial}{\partial x_1}\left(\frac{x_1}{\sqrt{x_1^2+9}}u_1 + u_2 - 4 - 3x_1 \quad \frac{-2u_1}{\sqrt{x_1^2+9}}x_2\right)$$

$$= \left(\frac{u_1}{\sqrt{x_1^2+9}} - \frac{x_1^2 u_1}{(x_1^2+9)^{3/2}} - 3 + \frac{2x_1x_2u_1}{(x_1^2+9)^{3/2}}\right)\Bigg|_{(\underline{x}_0, \underline{u}_0)}$$

$$= \frac{5}{5} - \cancel{\frac{16\sqrt{25}}{625}}\cdot 5 - 3 + \frac{16\sqrt{25}}{\cancel{625}}\cdot 5$$

$$= -2$$

$$\frac{\partial h_2}{\partial x_2} = \frac{-2u_1(6)}{\sqrt{x_1^2+9}}\Bigg|_{(\underline{x}_0, \underline{u}_0)}$$

$$= -\frac{2}{\sqrt{4^2+9}}$$

$$= -2$$

This gives

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix}$$

For B we get:

- $\frac{\partial h_1}{\partial u_1}(\underline{x}_0, \underline{u}_0) = 0$
- $\frac{\partial h_1}{\partial u_2}(\underline{x}_0, \underline{u}_0) = 0$
- $\frac{\partial h_2}{\partial u_1}(\underline{x}_0, \underline{u}_0) = \left(\frac{x_1}{\sqrt{x_1^2+9}} - \frac{2x_2}{\sqrt{x_1^2+9}}\right)\Bigg|_{(\underline{x}_0, \underline{u}_0)}$

$$= \frac{4}{5} - \frac{2 \cdot 2}{5}$$

$$= 0$$

- $\frac{\partial h_2}{\partial u_2}(\underline{x}_0, \underline{u}_0) = 1$

$$\Rightarrow \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

For C we get

$$\frac{\partial f}{\partial x_1}(\underline{x}_0, \underline{u}_0) = \frac{\partial}{\partial x_1}\left(5\sqrt{x_1^2+9}\right)\Bigg|_{(\underline{x}_0, \underline{u}_0)}$$

$$= \frac{5\,\cancel{2}x_1}{\cancel{2}\sqrt{x_1^2+9}}$$

$$= 5 \cdot \frac{4}{5}$$

$$= 4$$

$$\frac{\partial f}{\partial x_2}(\underline{x}_0, \underline{u}_0) = 0$$

$$\Rightarrow \underline{\underline{C = \begin{pmatrix} 4 & 0 \end{pmatrix}}}$$

For D we get

$$\frac{\partial f}{\partial u_1} = \frac{\partial f}{\partial u_2} = 0$$

$$\Rightarrow \underline{\underline{D = \begin{pmatrix} 0 & 0 \end{pmatrix}}}$$

Problem 4

a) $\det(A - \lambda \mathbb{I}) = \begin{vmatrix} -\lambda & -9 \\ 1 & -6-\lambda \end{vmatrix} = 0$

$$\iff (-\lambda)(-6-\lambda) - 1 \cdot (-9) = 0$$

$$\iff \lambda(\lambda+6) + 9 = 0$$

$$\iff \lambda^2 + 6\lambda + 9 = 0$$

$$\iff (\lambda+3)^2 = 0$$

$$\implies \lambda_1 = \lambda_2 = \underline{\underline{\lambda = -3}}$$

Eigenvectors:

$$A\underline{v} = -3\underline{v}$$

$$\iff (A + 3\mathbb{I})\underline{v} = 0$$

$$\iff \begin{pmatrix} 3 & -9 \\ 1 & -3 \end{pmatrix} \underline{v} = 0$$

Which gives $\underline{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Since the algebraic multiplicity is 2, but the geometric multiplicity is 1, we also have to solve

$$(A + 3\mathbb{I})\underline{v}_2 = \underline{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \implies \underline{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

b) A has repeated eigenvalues and only one eigenvector, thus it does not have a diagonalization

c) A is similar to the Jordan matrix $J = \begin{pmatrix} -3 & 1 \\ 0 & -3 \end{pmatrix}$.

under the similarity transformation

$$J = Q^{-1} A Q$$

Q is given partially by the eigenvector $\underline{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, but since the multiplicity of $\lambda = -3$ is 2 and we only have 1 eigenvector, we also have to solve

$$(A - \lambda I) v_2 = v_1$$

$$\Rightarrow \begin{pmatrix} 3 & -9 \\ 1 & -3 \end{pmatrix} v_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\Rightarrow v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow Q = \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}$$

To transform to jordan form we then use

$$\underline{\hat{x}} = Q^{-1} x \iff \underline{x} = Q \hat{x}$$

Substituting this into the state eq. we get

$$\begin{cases} Q\dot{\hat{\underline{x}}} = AQ\hat{\underline{x}} + Bu \\ y = CQ\hat{\underline{x}} + Du \end{cases}$$

$$\Leftrightarrow \begin{cases} \dot{\hat{\underline{x}}} = Q^{-1}AQ\hat{\underline{x}} + Q^{-1}Bu \\ y = CQ\hat{\underline{x}} + Du \end{cases}$$

Since $A = Q \cdot J \cdot Q^{-1} \Leftrightarrow J = Q^{-1}AQ$ we get

$$\begin{cases} \dot{\hat{\underline{x}}} = J\hat{\underline{x}} + Q^{-1}Bu \\ y = CQ\hat{\underline{x}} + Du \end{cases}$$

$$Q^{-1}B = \begin{pmatrix} 0 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$$

$$CQ = (1, -1) \cdot \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix} = (2, 1)$$

So in summary we get

$$\begin{cases} \dot{\hat{\underline{x}}} = \begin{pmatrix} -3 & 1 \\ 0 & -3 \end{pmatrix}\hat{\underline{x}} + \begin{pmatrix} 1 \\ -5 \end{pmatrix}u \\ y = \begin{pmatrix} 2 & 1 \end{pmatrix}\hat{\underline{x}} + 2u \end{cases}$$