LinSys 4

Rendell Cale, rendellc@stud.ntnu.no, mttk

Problem 1

$$A = \begin{pmatrix} 4 & 2 \\ -1 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

a) With $u(t) = -K_p y(t)$ we have

$$u(t) = -K_p C x(t)$$

$\Rightarrow$

$$\begin{aligned} \dot{x} &= Ax + B(-K_p C x) \\ &= (A - K_p BC) x \\ &= A_p x \end{aligned}$$

with

$$\begin{aligned} A_p &= \begin{pmatrix} 4 & 2 \\ -1 & -2 \end{pmatrix} - K_p \begin{pmatrix} 0 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 2 \\ -1 & -2 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 2K_p & 0 \end{pmatrix} \\ &= \underline{\underline{\begin{pmatrix} 4 & 2 \\ -1-2K_p & -2 \end{pmatrix}}} \end{aligned}$$

The eigenvalues of A_p are given by

$$\det(\mathbb{I}\lambda - A_p) = 0$$

$$\Rightarrow \quad 0 = \begin{vmatrix} \lambda-4 & 2 \\ -1-2K_p & \lambda+2 \end{vmatrix}$$

$$= (\lambda-4)(\lambda+2) + (1+2K_p)2$$

$$= \lambda^2 - 2\lambda - 8 + 2 + 4K_p$$

$$= \lambda^2 - 2\lambda - 6 + 4K_p$$

This gives

$$\lambda = \frac{-(-2) \pm \sqrt{(-2)^2 - 4\cdot(-6+4K_p)}}{2}$$

$$= 1 \pm \frac{1}{2}\sqrt{4-4(-6+4K_p)}$$

$$= 1 \pm \sqrt{1+6-4K_p}$$

$$= 1 \pm \sqrt{7-4K_p}$$

This is symmetric over 1 so <u>no</u> K_p will make both eigenvalues negative.

<u>Impossible!</u>

b) $u(t) = -k_p y(t) - k_d \dot{y}(t)$

$$\dot{x} = Ax - k_p BC x(t) - k_d BC \dot{x}(t)$$

$$\underbrace{(\mathbb{I} + k_d BC)}_{A_d^{-1}} \dot{x} = (A - k_p BC) x$$

$$= A_p x$$

$$A_d = (A_d^{-1})^{-1} = (\mathbb{I} + k_d BC)^{-1}$$

$$= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + k_d \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \right)^{-1}$$

$$= \begin{pmatrix} 1 & 0 \\ 2k_d & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & 0 \\ -2k_d & 1 \end{pmatrix}$$

So $\dot{x} = A_d A_p x$

$$= \begin{pmatrix} 1 & 0 \\ -2k_d & 1 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ -1-2k_p & -2 \end{pmatrix} x$$

$$= \underbrace{\begin{pmatrix} 4 & 2 \\ -8k_d - 1 - 2k_p & -4k_d - 2 \end{pmatrix}}_{A_{pd}} x$$

d) Want $|\lambda \mathbb{I} - A_{pd}| = (\lambda+1+i)(\lambda+1-i)$

$$= \lambda^2 + 2\lambda + 2$$

Computation:

$$|\lambda \mathbb{I} - A_{pd}| = (\lambda-4)(\lambda+2+4K_d)$$
$$- (-1-2K_p-8K_d)\cdot 2$$
$$= \lambda^2 + (-4+2+4K_d)\lambda - 4(2+4K_d)$$
$$+ 2 + 4K_p + 16K_d$$
$$= \lambda^2 + (4K_d-2)\lambda - 8 - \cancel{16K_d} + 2 + 4K_p + \cancel{16K_d}$$
$$= \lambda^2 + (4K_d-2)\lambda - 6 + 4K_p$$

$\Rightarrow$ $4K_d - 2 = 2 \Rightarrow \underline{\underline{K_d = 1}}$

$-6+4K_p = 2 \Rightarrow \underline{\underline{K_p = 2}}$

Problem 2

a)
$$\underline{\dot{x}} = A\underline{x} + B\underline{u}$$
$$= A\underline{x} + B(-K\underline{\hat{x}})$$
$$= A\underline{x} - BK(\underline{x} + \underline{e})$$
$$= (A - BK)\underline{x} - BK\underline{e}$$

This confirms the top row of H.

$$\underline{\dot{e}} = \underline{\dot{\hat{x}}} - \underline{\dot{x}}$$
$$= A\underline{\hat{x}} + \cancel{B\underline{u}} + L(\underline{y} - (C\underline{\hat{x}} - D\underline{u}) - A\underline{x} - \cancel{B\underline{u}})$$
$$= A\underline{e} + L[C\underline{x} + \cancel{D\underline{u}} - C\underline{\hat{x}} - \cancel{D\underline{u}}]$$
$$= (A - LC)\underline{e}$$

This confirms row two of H so we can write

$$\begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix} = \begin{bmatrix} A-BK & -BK \\ 0 & A-LC \end{bmatrix} \begin{pmatrix} x \\ e \end{pmatrix}$$

b) Assume that λ is an eigenvalue of H.

Then $\det(\lambda \mathbb{I} - H) = 0$, but since $\lambda \mathbb{I} - H$ is upper triangular we then have

$$\det(\lambda \mathbb{I} - (A - BK))\det(\mathbb{I}\lambda - (A - LC)) = 0.$$

This shows that λ is an eigenvalue of $A-BK$ or $A-LC$ (or both).

Since λ was arbitrarily chosen this must apply to all λ.

c) Since the eigenvalues of H are determined by $A-BK$ and $A-LC$, we have to show that these can can be placed arbitrarily.

Assuming that the system is controllable and observable, then (A, B) is controllable and (A, C) is observable.

Due to these properties we can control the poles of $A-BK$ and $A-LC$ by choosing K and L appropriately.

Since the poles of H are the union of the poles of $A-BK$ and $A-LC$, we can then choose the poles of H as we wish.

Problem 3

a) $\underline{\dot{x}} = A\underline{x} + Bu \quad , \quad \underline{y} = C\underline{x}$

$$A = \begin{pmatrix} -4 & -4 & -10 \\ 0 & -2 & 5 \\ 0 & 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 4 \\ -2 \\ -1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\hat{G}(s) = C(s\mathbb{I} - A)^{-1} B$$

$$= \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{s+4} & \frac{-4}{(s+2)(s+4)} & \frac{-10}{(s+2)(s-3)} \\ 0 & \frac{1}{s+2} & \frac{5}{(s+2)(s-3)} \\ 0 & 0 & \frac{1}{s-3} \end{pmatrix} \begin{pmatrix} 4 \\ -2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{4}{s+4} + \frac{8}{(s+2)(s+4)} + \frac{10}{(s+2)(s-3)} \\ -\frac{2}{s+2} - \frac{5}{(s+2)(s-3)} \\ -\frac{1}{s-3} \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} 0 \\ \frac{1}{s-3} \end{pmatrix}}}$$

b) Since A is 3×3 we have that the controllability matrix is given by

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 & 26 \\ -2 & -1 & -13 \\ -1 & -3 & -9 \end{bmatrix}$$

Note that $\det(\mathcal{C}) = 4\begin{vmatrix} -1 & -13 \\ -3 & -9 \end{vmatrix} - 2\begin{vmatrix} -2 & -13 \\ -1 & -9 \end{vmatrix} + 26\begin{vmatrix} -2 & -1 \\ -1 & -3 \end{vmatrix}$

$$= -120 - 10 + 130$$

$$= 0$$

$$\Rightarrow \text{Rank } \mathcal{C} < 3$$

The system does not have full rank so it is uncontrollable.

c) $P = \begin{pmatrix} 4 & 2 & 1 \\ -2 & -1 & 0 \\ -1 & -3 & 0 \end{pmatrix}$

Using $x = P\hat{x}$ we have

$$\begin{cases} (P\dot{\hat{\underline{x}}}) = AP\hat{\underline{x}} + Bu \\ \underline{y} = CP\hat{\underline{x}} \end{cases}$$

$$\Leftrightarrow \begin{cases} \dot{\hat{\underline{x}}} = P^{-1}AP\hat{\underline{x}} + P^{-1}Bu \\ \underline{y} = CP\hat{\underline{x}} \end{cases}$$

$$\Rightarrow \hat{A} = P^{-1}AP, \quad \hat{B} = P^{-1}B, \quad \hat{C} = CP$$

Calculating we get:

$$P^{-1} = \begin{pmatrix} 4 & 2 & 1 \\ -2 & -1 & 0 \\ -1 & -3 & 0 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 0 & -3/5 & 1/5 \\ 0 & 1/5 & -2/5 \\ 1 & 2 & 0 \end{pmatrix}$$

$$\hat{A} = P^{-1}AP$$

$$= \begin{pmatrix} 0 & -3/5 & 1/5 \\ 0 & 1/5 & -2/5 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} -4 & -4 & -10 \\ 0 & -2 & 5 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 4 & 2 & 1 \\ -2 & -1 & 0 \\ -1 & -3 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 6/5 & -3+3/5 \\ 0 & -2/5 & 1-6/5 \\ -4 & -8 & 0 \end{pmatrix} \begin{pmatrix} 4 & 2 & 1 \\ -2 & -1 & 0 \\ -1 & -3 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 6 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$

$$\hat{B} = P^{-1}B$$

$$= \begin{pmatrix} 0 & -3/5 & 1/5 \\ 0 & 1/5 & -2/5 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ -2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} +\frac{6}{5} - \frac{1}{5} \\ -\frac{2}{5} + \frac{2}{5} \\ 4 - 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\hat{C} = CP$$

$$= \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 4 & 2 & 1 \\ -2 & -1 & 0 \\ -1 & -3 & 0 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} 0 & 0 & 1 \\ 1 & 3 & 0 \end{pmatrix}}}$$

d) We calculate $\hat{C}_c(s\mathbb{I} - \hat{A}_c)^{-1}\hat{B}_c$:

$$\begin{pmatrix} s & -6 \\ -1 & s-1 \end{pmatrix}^{-1} = \frac{1}{s^2-s-6}\begin{pmatrix} s-1 & 6 \\ 1 & s \end{pmatrix}$$

$$= \frac{1}{(s+2)(s-3)}\begin{pmatrix} s-1 & 6 \\ 1 & s \end{pmatrix}$$

$$\Rightarrow \hat{C}_c(s\mathbb{I} - \hat{A}_c)^{-1}\hat{B}_c = \begin{pmatrix} 0 & 0 \\ 1 & 3 \end{pmatrix} \frac{1}{(s+2)(s-3)} \begin{pmatrix} s-1 & 6 \\ 1 & s \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{(s+2)(s-3)} \begin{pmatrix} 0 & 0 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} s-1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \frac{\cancel{s-1+3}}{\cancel{(s+2)}(s-3)} \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} 0 \\ \frac{1}{s-3} \end{pmatrix}}}$$

Problem 4

$$A = \begin{pmatrix} -5 & 2 \\ 5 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad C = \begin{pmatrix} -5 & 1 \\ 10 & -2 \end{pmatrix}, \quad D = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

a) $$\Delta(\lambda) = |\lambda \mathbb{I} - A| = (\lambda+5)(\lambda-4) - 5 \cdot 2$$
$$= \lambda^2 + \lambda - 20 - 10$$
$$= \lambda^2 + \lambda - 30$$

Solving $\Delta(\lambda) = 0$ we get

$$\lambda = \frac{-1 \pm \sqrt{1+4 \cdot 30}}{2}$$

$$= \frac{-1 \pm 11}{2}$$

$$= -6, 5$$

So the eigenvalues of the system are

$$\lambda_1 = -6$$

$$\lambda_2 = 5$$

b) $\lambda = \lambda_1 = -6$:

$$\begin{pmatrix} A + 6I & B \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 1 \\ 5 & 10 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 5 & 0 & -2 \end{pmatrix}$$

which has $\underline{\text{rank } 2}$

$\lambda = \lambda_2 = 5$:

$$\begin{pmatrix} A - 5I & B \end{pmatrix}$$

$$= \begin{pmatrix} -10 & 2 & 1 \\ 5 & -1 & -2 \end{pmatrix} \sim \begin{pmatrix} -10 & 0 & 1 \\ 5 & 0 & -2 \end{pmatrix}$$

which has $\underline{\text{rank } 2}$

The system is then by the Popov-Belevitch-Hautus test <u>controllable</u>.

c) $\lambda = \lambda_1 = -6$:

$$\text{Rank}\begin{pmatrix} 1 & 2 \\ 5 & 10 \\ -5 & 1 \\ 10 & -2 \end{pmatrix} = \text{Rank}\begin{pmatrix} 1 & 2 \\ 0 & 0 \\ -5 & 1 \\ 0 & 0 \end{pmatrix} = \underline{2}$$

$\lambda = \lambda_2 = 5$:

$$\text{Rank}\begin{pmatrix} -10 & 2 \\ 5 & -1 \\ -5 & 1 \\ 10 & -2 \end{pmatrix} = \text{Rank}\begin{pmatrix} -10 & 2 \\ 0 & 0 \\ -5 & 1 \\ 0 & 0 \end{pmatrix}$$

$$= \text{Rank}\begin{pmatrix} -10 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \underline{1}$$

It is <u>not</u> observable since

$$\text{Rank}\begin{pmatrix} A - \lambda \mathbb{I} \\ C \end{pmatrix} < n = 2$$

for $\lambda = 5$.

d) The system is not a minimal realization since the pair (A, C) is not observable.

e) $$\hat{G}(s) = C(sI - A)^{-1}B + D$$

$$(sI - A)^{-1} = \begin{pmatrix} s+5 & -2 \\ -5 & s-4 \end{pmatrix}^{-1}$$

$$= \frac{1}{s^2+s-30} \begin{pmatrix} s-4 & 2 \\ 5 & s+5 \end{pmatrix}$$

$$= \frac{1}{(s+6)(s-5)} \begin{pmatrix} s-4 & 2 \\ 5 & s+5 \end{pmatrix}$$

$$C(sI-A)^{-1}B = \begin{pmatrix} -5 & 1 \\ 10 & -2 \end{pmatrix} \frac{1}{(s+6)(s-5)} \begin{pmatrix} s-4 & 2 \\ 5 & s+5 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} -5 & 1 \\ 10 & -2 \end{pmatrix} \frac{1}{(s+6)(s-5)} \begin{pmatrix} s-4-4 \\ 5-2s-10 \end{pmatrix}$$

$$= \frac{1}{(s+6)(s-5)} \begin{pmatrix} -5 & 1 \\ 10 & -2 \end{pmatrix} \begin{pmatrix} s-8 \\ -2s-5 \end{pmatrix}$$

$$= \frac{1}{(s+6)(s-5)} \begin{pmatrix} -5s+40-2s-5 \\ 10s-80+4s+10 \end{pmatrix}$$

$$= \frac{1}{(s+6)(s-5)} \begin{pmatrix} -7(s-5) \\ 14(s-5) \end{pmatrix}$$

$$\Rightarrow \hat{G}(s) = \frac{1}{s+6}\begin{pmatrix} -7 \\ 14 \end{pmatrix} + \overbrace{\begin{pmatrix} 1 \\ -2 \end{pmatrix}}^{D}$$

$$= \frac{1}{s+6}\begin{pmatrix} -7 + s+6 \\ 14 - 2s - 12 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{s-1}{s+6} \\ \frac{-2(s-1)}{s+6} \end{pmatrix}$$

f) The degree of $\hat{G}(s)$ is 1 (1 pole at $s=-6$)
but $\dim A = 2$.
Since

$$\dim A \neq \deg \hat{G}$$

the system is not a minimal realization.