Bra!
NB

Øving 7, Ønsker tilbakemelding :)

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5.2

1)

b) $\{(x,y) \mid (x,y) \in \mathbb{R},\ y^2 = x\}$

This is not a function since x would have to be negative in order to cover the entire domain, but this is impossible in the real numbers.

c) $\{(x,y) \mid x,y \in \mathbb{R},\ y = 3x+1\}$

This is a function as it is defined for all $x \in \mathbb{R}$, and its range is all of $\mathbb{R}$.

8)

a) $\lfloor a \rfloor = \lceil a \rceil$ for all $a \in \mathbb{Z}$

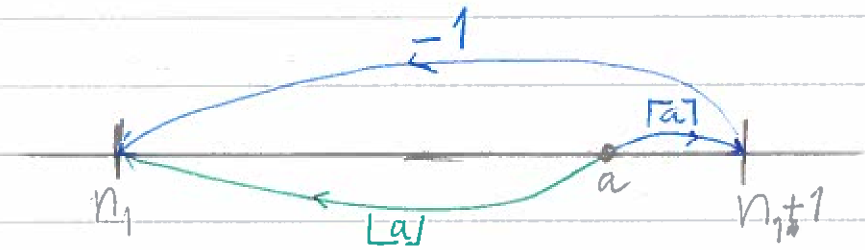
True because $\lfloor a \rfloor = a$ and $\lceil a \rceil = a$ for all $a \in \mathbb{Z}$

b) $\lfloor a \rfloor = \lceil a \rceil$ for all $a \in \mathbb{R}$

False. if $a = 1.5$ then $\lfloor a \rfloor = 1$ and $\lceil a \rceil = 2$ $(\neq 1)$

c) $\lfloor a \rfloor = \lceil a \rceil - 1$ for all $a \in \mathbb{R} \setminus \mathbb{Z}$

True.
For all $a \in \mathbb{R} \setminus \mathbb{Z}$ $\lfloor a \rfloor$ is the integer to the left of a, and $\lceil a \rceil$ is the integer to the right of a. Thus $\lceil a \rceil - 1$ will be the integer to the right, shifted 1 to the left, which will be the integer to the left of a. $(= \lfloor a \rfloor)$.



d) $-\lceil a \rceil = \lceil -a \rceil$ for all $a \in \mathbb{R}$

False
Consider $a = 1.5$
then $-\lceil a \rceil = -2$ and $\lceil -1.5 \rceil = -1$
So $-\lceil a \rceil \neq \lceil -a \rceil$ for all $a \in \mathbb{R}$

15) a) $f: \mathbb{Z} \to \mathbb{Z}$, $f(x) = 2x+1$

$$f(x_1) = f(x_2)$$

$$\Leftrightarrow 2x_1 + 1 = 2x_2 + 1$$

$$\Leftrightarrow x_1 = x_2$$

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

So f is an injection.

The range of f is all odd integers because all odd numbers are of the form $2n+1$

b) $f: \mathbb{Q} \to \mathbb{Q}$, $f(x) = 2x+1$

The exact same calculation as in (a) will show that this f is also injective.

The range of f is all of $\mathbb{Q}$

16) Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

a) $A = \{2,3\}$

$f(A) = \{2^2, 3^2\} = \{4, 9\}$ R

d) $A = (-3, 2]$

$f(A) = [0, 9)$ R

f) $A = (-4,-3] \cup [5,6]$

$f(A) = [9,16) \cup [25,36]$ R

5.3

2) $f: \mathbb{Z} \to \mathbb{Z}$

a) $f(x) = x+7$

$f(x_1) = f(x_2) \Rightarrow x_1 + 7 = x_2 + 7$
$\Rightarrow x_1 = x_2$

This shows that f is injective.

f is also surjective because for any integer x, $f(x-7) = (x-7)+7 = x$

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c) $f(x) = -x+5$

$f(x_1) = f(x_2) \Rightarrow -x_1+5 = -x_2+5 \Rightarrow x_1 = x_2$

So f is injective

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For any integer x, $f(5-x) = -(5-x)+5$
$= -5+x-5$
$= x$

Onto?

Since $5-x$ is also an integer this shows that the range of f is all of $\mathbb{Z}$

b) $f(x) = 2x-3$

f is injective because

$$\begin{aligned} & f(x_1) = f(x_2) \\ \Leftrightarrow\ & 2x_1 - 3 = 2x_2 - 3 \\ \Leftrightarrow\ & x_1 = x_2 \end{aligned}$$

The range of f is all the odd numbers, so it is not surjective.

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d) $f(x) = x^2$

$$f(x_1) = f(x_2)$$
$$\Leftrightarrow \quad x_1^2 = x_2^2$$
$$\Leftrightarrow \quad |x_1| = |x_2| \not\Rightarrow x_1 = x_2$$

f is not injective

f is not surjective either because x^2 cant be negative and it can only reach perfect squares. So the range of f is

$$f(\mathbb{Z}) = \{y \in \mathbb{Z} \mid x \in \mathbb{Z}, y = x^2\}$$
$$= \{0, 1, 4, 9, 16, \ldots\}$$

e) $f(x) = x^2 + x$

$$f(x_1) = f(x_2)$$
$$\Leftrightarrow \quad x_1^2 + x_1 = x_2^2 + x_2 \quad \Big| + \tfrac{1}{4}$$
$$\Leftrightarrow \quad x_1^2 + x_1 + 1/4 = x_2^2 + x_2 + 1/4$$
$$\Leftrightarrow \quad \left(x_1 + \tfrac{1}{2}\right)^2 = \left(x_2 + 1/2\right)^2$$

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$$\Leftrightarrow \quad \left|x_1 + \tfrac{1}{2}\right| = \left|x_2 + \tfrac{1}{2}\right|$$
$$\not\Rightarrow \quad x_1 = x_2$$

f is not injective

f is not surjective because $f(\mathbb{Z}) \neq \mathbb{Z}$

Proof: $f(x) = 7$

$\Leftrightarrow x^2 + x = 7 \quad | + \frac{1}{4}$

$\Leftrightarrow x^2 + x + \frac{1}{4} = 7 + \frac{1}{4}$

$\Leftrightarrow \left(x + \frac{1}{2}\right)^2 = \frac{29}{4}$

$\Rightarrow x + \frac{1}{2} = \frac{\sqrt{29}}{2}$

$\Leftrightarrow x = \frac{\sqrt{29} - 1}{2}$ which is not an integer, so 7 is not in the range of f.

$f(\mathbb{Z}) = \{ y \mid y = x^2 + x,\ x \in \mathbb{Z} \}$

f) $f(x) = x^3$

$f(x_1) = f(x_2)$

$\Leftrightarrow x_1^3 = x_2^3$

$\Leftrightarrow x_1 = x_2$

So f is injective

$f(x) = 2$

$x^3 = 2$

$x = \sqrt[3]{2}$

So f has no integer solution to $f(x) = 2$, and thus it is not surjective

The range of f is all cubes.

$$f(\mathbb{Z}) = \{y \mid y = x^3, x \in \mathbb{Z}\}$$

5.6

4) $g: \mathbb{N} \to \mathbb{N}$, $g(n) = 2 \cdot n$

$f: A \to \mathbb{N}$, $A = \{1, 2, 3, 4\}$

$f = \{(1,2), (2,3), (3,5), (4,7)\}$

$g \circ f: A \to \mathbb{N}$

$g \circ f = \{(1,4), (2,6), (3,10), (4,14)\}$

5) Let $f: A \to B$ and $g: B \to C$

a) If $g \circ f$ is onto, then by definition $g(f)$ is onto.
This means that for all $c \in C$ there is atleast one $a \in A$ such that $g(f(a)) = c$.
Since $f(a) = b \in B$, we can instead write the statement as "for all $c \in C$, there exist $b \in B$ such that $g(b) = c$".
So g is by def. onto.

b) Assume $g \circ f : A \to C$ is injective.

This means that for all $x_1, x_2 \in A$, if $(g \circ f)(x_1) = (g \circ f)(x_2)$ then $x_1 = x_2$

Want to prove that f must also be injective, so we'll assume the opposite. So assume f is <u>not</u> injective. This means that there is $x_1, x_2 \in A$, $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$
But then $(g \circ f)(x_1) = g(f(x_1)) = g(f(x_2)) = (g \circ f)(x_2)$, so $g \circ f$ is not injective, which is a contradiction, so f must be injective for $g \circ f$ to be injective.

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10) $f : \mathbb{R} \to \mathbb{R}$

a) $f = \{(x, y) \mid 2x + 3y = 7\}$

f is clearly invertible, but for the sake of rigour we'll show it.
Some algebra shows that f is equivalently defined by

$$y = f(x) = \frac{7}{3} - \frac{2}{3}x$$

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$f(x_1) = f(x_2) \Rightarrow \frac{7}{3} - \frac{2}{3}x_1 = \frac{7}{3} - \frac{2}{3}x_2 \Rightarrow x_1 = x_2$
So f is injective

f is also surjective because for any y_0, pick $x_0 = \frac{7}{2} - \frac{3}{2}y_0$ to get $f(x_0) = y_0$

f is a bijection so it is invertible

$$f^{-1} = \{(x,y) \mid 2y + 3x = 7\}$$

d) $f = \{(x,y) \mid y = x^4 + x\}$

f is neither injective nor surjective. We can see this by solving for x when $y = 0$.

$$0 = x^4 + x = x(x^3+1)$$

$$\Rightarrow x = 0 \text{ and } x = -1$$

$$\Rightarrow f \text{ is not injective.}$$

Therefore f is not invertible either

12) $A = \{1, 2, 3, 4, 5, 6, 7\}$
$B = \{2, 4, 6, 8, 10, 12\}$
$f : A \to B$, $f = \{(1,2), (2,6), (3,6), (4,8), (5,6), (6,8), (7,12)\}$

a) $f(X) = \{2\} \Rightarrow X = \{1\}$
b) $f(X) = \{6\} \Rightarrow X = \{2, 3, 5\}$

c) $f(X) = \{6, 8\} \Rightarrow X = \{2, 3, 4, 5, 6\}$

d) $f(X) = \{6, 8, 10\} \Rightarrow X = \{2, 3, 4, 5, 6\}$

e) $f(X) = \{6, 8, 10, 12\} \Rightarrow X = \{2, 3, 4, 5, 6, 7\}$

f) $f(X) = \{10, 12\} \Rightarrow X = \{7\}$

21) Let $f: \mathbb{Z} \to \mathbb{N}$

$$f(x) = \begin{cases} 2x - 1, & x > 0 \\ -2x, & x \leq 0 \end{cases}$$

First note that when $x > 0$, $f(x)$ is odd, and when $x \leq 0$, $f(x)$ is even.
Using this we can see that f is surjective

$x > 0: f = \{1, 3, 5, \ldots\}$

$x \leq 0: f = \{0, 2, 4, \ldots\}$

The union of the odd and even positive integers is $\mathbb{N}$, so f is surjective.

To check if f is injective, we have two cases to consider, $f(x)$ is even, $f(x)$ is odd.

Even: $f(x_1) = f(x_2)$

$\Rightarrow -2x_1 = -2x_2$

$\Rightarrow x_1 = x_2$ ✓

Odd: $f(x_1) = f(x_2)$

$\Rightarrow 2x_1 - 1 = 2x_2 - 1$

$\Rightarrow x_1 = x_2$ ✓

So f is injective.

Since f is both injective and surjective, f is invertible.

b)

$$f(x) = \begin{cases} 2x-1, & x>0 \\ -2x, & x \leq 0 \end{cases}$$

$f^{-1}: \mathbb{N} \to \mathbb{Z}$

Like with f, there are two cases for f^{-1}, one for even inputs and one for odd inputs.

Even: $x = -2 \cdot y \Rightarrow y = -\frac{x}{2}$

Odd: $x = 2y-1 \Rightarrow y = \frac{x+1}{2}$

So f^{-1} will be

$$f^{-1}(x) = \begin{cases} \frac{x+1}{2}, & x \text{ odd} \\ -\frac{x}{2}, & x \text{ even} \end{cases}$$