Solution Suggestion
Exam - TTK4115 Linear System Theory
12. August, 2017

August 8, 2017

Problem 1

A first order low-pass filter and a first order high-pass filter are given respectively by

$$g_l(s) = \frac{1}{\tau s + 1}, \quad g_h(s) = \frac{\tau s}{\tau s + 1}$$

The filters are driven by a common input signal $u(s)$, but produce the two outputs $y_l(s) = g_l(s)u(s)$ and $y_h(s) = g_h(s)u(s)$.

a) A state-space realization for the transfer function $g_l(s)$ is to be found. Since the plant is first order, the realization must be on the general form

$$\dot{x} = ax + bu, \quad y = cx + du \tag{1}$$

A Laplace transform here gives

$$\frac{y}{u}(s) = \frac{ds + (bc - da)}{s - a} \tag{2}$$

Comparison leads to the equations

$$a = -\frac{1}{\tau}, \quad d = 0, \quad bc = \frac{1}{\tau} \tag{3}$$

Setting $c = 1$, a state-space realization follows as

$$\dot{x} = -\frac{1}{\tau}x + \frac{1}{\tau}u, \quad y = x \tag{4}$$

Recall that state-space realizations are not unique!

b) Proceeding in a similar manner, for $g_h(s)$ one must have

$$a = -\frac{1}{\tau}, \quad d = 1, \quad bc - da = 0 \tag{5}$$

Setting $c = -1$, one has $b = 1/\tau$ and hence

$$\dot{x} = -\frac{1}{\tau}x + \frac{1}{\tau}u, \quad y = u - x \tag{6}$$

c) This question asks for a realization of $g_l(s) + g_h(s)$. Now verify that

$$g_l(s) + g_h(s) = \frac{1}{\tau s + 1} + \frac{\tau s}{\tau s + 1} = 1 \tag{7}$$

Our realization is thus

$$y = u \tag{8}$$

d) We are asked to find a <u>minimal</u> state-space realization for the transfer-function

$$\begin{bmatrix} g_l(s) \\ g_h(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{\tau s+1} \\ \frac{\tau s}{\tau s+1} \end{bmatrix} \tag{9}$$

It is seen that this transfer-function is first-order, and one state will suffice. (Using two would render the model non-minimal). A general description for the state-space realization is thus

$$\dot{x} = ax + bu, \quad \mathbf{y} = \mathbf{c}x + \mathbf{d}u \tag{10}$$

where $\mathbf{c}$ and $\mathbf{d}$ are 2×1 vectors. Taking a Laplace-transform leads to

$$\mathbf{y}(s) = \frac{\mathbf{d}s + (\mathbf{c}b - \mathbf{d}a)}{s - a} \tag{11}$$

Clearly, $a = -1/\tau$ and hence

$$\mathbf{d} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{12}$$

Since $\tau(\mathbf{c}b - \mathbf{d}a) = [1, 0]^\mathsf{T}$ it must hold that $\tau\mathbf{c}b = [1, -1]^\mathsf{T}$. Choosing $b = 1/\tau$ leads to $\mathbf{c} = [1, -1]^\mathsf{T}$ and the realization

$$\dot{x} = -\frac{1}{\tau}x + \frac{1}{\tau}u, \quad y = \begin{bmatrix} 1 \\ -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{13}$$

e) This task asks for the transfer-function of the following state-space model

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u, \quad \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{d}u$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\omega_b^2 & -2\omega_b \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \omega_b^2 & 0 \\ 0 & 2\omega_b \\ -\omega_b^2 & -2\omega_b \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

It is well known that the general formula reads as $\mathbf{g}(s) = \mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{b} + \mathbf{d}$. Here, one finds that

$$\begin{aligned}
&\begin{bmatrix} \omega_b^2 & 0 \\ 0 & 2\omega_b \\ -\omega_b^2 & -2\omega_b \end{bmatrix} \begin{bmatrix} s & -1 \\ \omega_b^2 & s + 2\omega_b \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \frac{1}{s^2 + 2\omega_b s + \omega_b^2} \begin{bmatrix} \omega_b^2 & 0 \\ 0 & 2\omega_b \\ -\omega_b^2 & -2\omega_b \end{bmatrix} \begin{bmatrix} s + 2\omega_b & 1 \\ -\omega_b^2 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \frac{1}{s^2 + 2\omega_b s + \omega_b^2} \begin{bmatrix} \omega_b^2 \\ 2\omega_b s \\ -(2\omega_b s + \omega_b^2) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{s^2 + 2\omega_b s + \omega_b^2} \begin{bmatrix} \omega_b^2 \\ 2\omega_b s \\ s^2 \end{bmatrix}
\end{aligned} \tag{14}$$

It is seen that the outputs $\mathbf{y}$ represent, in descending order, a low-pass filter, a band-pass filter and finally a high-pass filter.

Problem 2

A diesel engine model is supplied as

$$\tau\dot{x} + x = u - d \tag{15}$$

a) This task examines the response of the diesel engine under LQR control. The following functional is minimized with state-feedback

$$J = \int_0^\infty \left\{ qx^2 + ru^2 \right\} dt$$

A number of different tunings are considered.

A $q = 1$ and $r = 1$.

B $q = 100$ and $r = 1$.

C $q = 1$ and $r = 100$.

D $q = 100$ and $r = 100$.

The response will be on the form $x(t) = e^{-at}x_0$. It is seen that A and D will result in similar responses. Tuning C will imply a <u>slower</u> decay of x compared to A and D, whereas B will result in a <u>faster</u> decay.

b) In order to use the LQR with the cost objective given below, a state-space description of the error-state $\tilde{x} = x - x_r$ and its integral $\dot{\tilde{x}}_i = x - x_r$ must be found.

$$J = \int_0^\infty \left\{ q_p(x - x_r)^2 + q_i \int_0^t (x - x_r)^2 \, dt' + u^2 \right\} dt \tag{16}$$

One now has

$$J = \int_0^\infty \left\{ q_p\tilde{x}^2 + q_i\tilde{x}_i^2 + u^2 \right\} dt \tag{17}$$

Since x_r is constant, the error state $\tilde{x}$ can be modeled by

$$\tau\dot{\tilde{x}} + \tilde{x} = u - d - x_r \tag{18}$$

The integral effect follows as

$$\dot{\tilde{x}}_i = \tilde{x} \quad \Rightarrow \tilde{x}_i = \int_0^t (x - x_r) \, dt' \tag{19}$$

Gathering definitions leads to an augmented state-space model

$$\begin{bmatrix} \dot{\tilde{x}} \\ \dot{\tilde{x}}_i \end{bmatrix} = \overbrace{\begin{bmatrix} -\frac{1}{\tau} & 0 \\ 1 & 0 \end{bmatrix}}^{\mathbf{A}} \begin{bmatrix} \tilde{x} \\ \tilde{x}_i \end{bmatrix} + \overbrace{\begin{bmatrix} \frac{1}{\tau} \\ 0 \end{bmatrix}}^{\mathbf{B}} u - \frac{1}{\tau} \begin{bmatrix} d + x_r \\ 0 \end{bmatrix} \tag{20}$$

The associated cost-matrices are found as

$$\mathbf{Q} = \begin{bmatrix} q_p & 0 \\ 0 & q_i \end{bmatrix}, \quad R = 1 \tag{21}$$

c) In this task, the trick is to define

$$\dot{u} = v \tag{22}$$

Using the error-state from the previous task leads to the cost-function

$$J = \int_0^\infty \left\{ q_p\tilde{x}^2 + v^2 \right\} dt \tag{23}$$

The associated state-space model reads as

$$\begin{bmatrix} \dot{\tilde{x}} \\ \dot{u} \end{bmatrix} = \overbrace{\begin{bmatrix} -\frac{1}{\tau} & \frac{1}{\tau} \\ 0 & 0 \end{bmatrix}}^{\mathbf{A}} \begin{bmatrix} \tilde{x} \\ u \end{bmatrix} + \overbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}^{\mathbf{B}} v - \frac{1}{\tau} \begin{bmatrix} d + x_r \\ 0 \end{bmatrix} \tag{24}$$

The associated cost-matrices are found as

$$\mathbf{Q} = \begin{bmatrix} q_p & 0 \\ 0 & 0 \end{bmatrix}, \quad R = 1 \tag{25}$$

Problem 3

The following state-space model is to be examined

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ \alpha \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & \beta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

a) The controllability matrix is readily computed as

$$\mathcal{C} = \begin{bmatrix} \mathbf{b} & \mathbf{A}b \end{bmatrix} = \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix} \tag{26}$$

When $\det(\mathcal{C}) = 0$, the matrix ceases to have full rank. Verify that

$$\det(\mathcal{C}) = 1 - \alpha^2 \tag{27}$$

Therefore; the system is controllable for all β and $\alpha \neq \pm 1$.

b) The observability matrix is readily computed as

$$\mathcal{O} = \begin{bmatrix} \mathbf{c} \\ \mathbf{cA} \end{bmatrix} = \begin{bmatrix} 1 & \beta \\ \beta & 1 \end{bmatrix} \tag{28}$$

When $\det(\mathcal{O}) = 0$, the matrix ceases to have full rank. Verify that

$$\det(\mathcal{O}) = 1 - \beta^2 \tag{29}$$

Therefore; the system is observable for all α and $\beta \neq \pm 1$.

c) With $\alpha = 0$ the state dynamics read as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

In order to place poles, define $\mathbf{k} = [k_1, k_2]$. With $u = -\mathbf{kx}$ one has

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{bk})\mathbf{x}, \quad \mathbf{A} - \mathbf{bk} = \begin{bmatrix} -k_1 & 1 - k_2 \\ 1 & 0 \end{bmatrix} \tag{30}$$

The eigenvalues of the closed-loop system matrix may be found as solutions to the characteristic equation

$$|\lambda \mathbf{I} - (\mathbf{A} - \mathbf{bk})| = \begin{vmatrix} \lambda + k_1 & k_2 - 1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 + \lambda k_1 + 1 - k_2 = 0 \tag{31}$$

Comparison to the desired result $(\lambda + 1)^2 = \lambda^2 + 2\lambda + 1 = 0$ implies that $k_1 = 2$ and $k_2 = 2$.

d)
- State-feedback can be implemented with an observer $\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{b}u + \mathbf{l}(y - \hat{y})$ generating an estimate of the state $\hat{\mathbf{x}}$ based on the measurement y. This requires that the plant is observable.
- With $\beta = 1$, the plant is unobservable. This implies that one cannot identify $\mathbf{x}$ using an observer. State-feedback is therefore difficult.
- The plant is now

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ \alpha \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Inserting $u = -ky$ leads to the closed-loop dynamics

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{b}ky = (\mathbf{A} - k\mathbf{b}\mathbf{c})\mathbf{x}, \quad \mathbf{A} - k\mathbf{b}\mathbf{c} = \begin{bmatrix} -k & 1-k \\ 1-k\alpha & -k\alpha \end{bmatrix} \tag{32}$$

The eigenvalues of the matrix $\mathbf{A} - k\mathbf{b}\mathbf{c}$ solve

$$\begin{vmatrix} \lambda + k & k-1 \\ k\alpha - 1 & \lambda + k\alpha \end{vmatrix} = (\lambda+k)(\lambda+k\alpha) - (k-1)(k\alpha-1) = \lambda^2 + \lambda k(\alpha+1) + k(\alpha+1) - 1 = 0 \tag{33}$$

Yet again comparing to the desired polynomial $(\lambda+1)^2 = \lambda^2 + 2\lambda + 1 = 0$ implies

$$k(\alpha+1) = 2 \quad \Rightarrow k = \frac{2}{\alpha+1} \tag{34}$$

Verify that this also gives

$$k(\alpha+1) - 1 = \frac{2(\alpha+1)}{\alpha+1} - 1 = 1 \tag{35}$$

Sometimes, one can proceed without observability!

Problem 4

The measurement problem given below is considered.

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{v}$$

a) The estimate $\hat{\mathbf{x}}$ is indeed unbiased. The estimation error $\mathbf{e}$ is given by

$$\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}} = \mathbf{x} - (\mathbf{m} + \mathbf{K}(\mathbf{y} - \mathbf{C}\mathbf{m})) = (\mathbb{I} - \mathbf{K}\mathbf{C})(\mathbf{x} - \mathbf{m}) - \mathbf{K}\mathbf{v}$$

With $\mathsf{E}[\mathbf{x}] = \mathbf{m}$, one finds that

$$\mathsf{E}[\mathbf{e}] = (\mathbb{I} - \mathbf{K}\mathbf{C})\mathsf{E}[\mathbf{x} - \mathbf{m}] - \mathbf{K}\mathsf{E}[\mathbf{v}] = \mathbf{0} \tag{36}$$

b) The covariance matrix of $\mathbf{e}$ is found as follows: first identify the outer product

$$\begin{aligned}\mathbf{e}\mathbf{e}^\mathsf{T} &= [(\mathbb{I} - \mathbf{K}\mathbf{C})(\mathbf{x} - \mathbf{m}) - \mathbf{K}\mathbf{v}][(\mathbb{I} - \mathbf{K}\mathbf{C})(\mathbf{x} - \mathbf{m}) - \mathbf{K}\mathbf{v}]^\mathsf{T} \\ &= (\mathbb{I} - \mathbf{K}\mathbf{C})(\mathbf{x} - \mathbf{m})(\mathbf{x} - \mathbf{m})^\mathsf{T}(\mathbb{I} - \mathbf{K}\mathbf{C})^\mathsf{T} - \mathbf{K}\mathbf{v}(\mathbf{x} - \mathbf{m})^\mathsf{T}(\mathbb{I} - \mathbf{K}\mathbf{C})^\mathsf{T} - (\mathbb{I} - \mathbf{K}\mathbf{C})(\mathbf{x} - \mathbf{m})\mathbf{v}^\mathsf{T}\mathbf{K}^\mathsf{T} + \mathbf{K}\mathbf{v}\mathbf{v}^\mathsf{T}\mathbf{K}^\mathsf{T}\end{aligned} \tag{37}$$

The expectancy of the outer product follows as

$$\begin{aligned}\mathsf{E}[\mathbf{e}\mathbf{e}^\mathsf{T}] = (\mathbb{I} - \mathbf{K}\mathbf{C})\mathsf{E}[(\mathbf{x} - \mathbf{m})(\mathbf{x} - \mathbf{m})^\mathsf{T}](\mathbb{I} - \mathbf{K}\mathbf{C})^\mathsf{T} + \mathbf{K}\mathsf{E}[\mathbf{v}\mathbf{v}^\mathsf{T}]\mathbf{K}^\mathsf{T} \\ - \mathbf{K}\mathsf{E}[\mathbf{v}(\mathbf{x} - \mathbf{m})^\mathsf{T}](\mathbb{I} - \mathbf{K}\mathbf{C})^\mathsf{T} - (\mathbb{I} - \mathbf{K}\mathbf{C})\mathsf{E}[(\mathbf{x} - \mathbf{m})\mathbf{v}^\mathsf{T}]\mathbf{K}^\mathsf{T}\end{aligned} \tag{38}$$

Since $\mathbf{x}$ and $\mathbf{v}$ are uncorrelated and $\mathsf{E}[\mathbf{v}] = 0$ it follows that

$$\mathsf{E}[\mathbf{v}(\mathbf{x}-\mathbf{m})^{\mathsf{T}}] = \mathbf{0}, \quad \mathsf{E}[(\mathbf{x}-\mathbf{m})\mathbf{v}^{\mathsf{T}}] = \mathbf{0} \tag{39}$$

Using the supplied data, one is left with

$$\mathsf{E}[\mathbf{e}\mathbf{e}^{\mathsf{T}}] = (\mathbb{I} - \mathbf{K}\mathbf{C})\mathbf{Q}(\mathbb{I} - \mathbf{K}\mathbf{C})^{\mathsf{T}} + \mathbf{K}\mathbf{R}\mathbf{K}^{\mathsf{T}} \tag{40}$$

c) The mean-square error is to be minimized. This is achieved by solving

$$\frac{\partial \mathrm{tr}(\mathbf{P})}{\partial \mathbf{K}} = 0 \tag{41}$$

Expanding and rearranging $\mathbf{P}$ gives

$$\mathbf{P} = \mathbf{Q} - (\mathbf{Q}\mathbf{C}^{\mathsf{T}}\mathbf{K}^{\mathsf{T}} + \mathbf{K}\mathbf{C}\mathbf{Q}) + \mathbf{K}(\mathbf{C}\mathbf{Q}\mathbf{C}^{\mathsf{T}} + \mathbf{R})\mathbf{K}^{\mathsf{T}} \tag{42}$$

Differentiating the trace with respect to $\mathbf{K}$ gives

$$\frac{\partial \mathrm{tr}(\mathbf{P})}{\partial \mathbf{K}} = -2\mathbf{Q}\mathbf{C}^{\mathsf{T}} + 2\mathbf{K}(\mathbf{C}\mathbf{Q}\mathbf{C}^{\mathsf{T}} + \mathbf{R}) \tag{43}$$

Setting the derivative equal to zero one finds that

$$\mathbf{K} = \mathbf{Q}\mathbf{C}^{\mathsf{T}}(\mathbf{C}\mathbf{Q}\mathbf{C}^{\mathsf{T}} + \mathbf{R})^{-1} \tag{44}$$

d) A simple measurement model is given by

$$\mathbf{y} = \overbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}^{\mathbf{c}} x + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{45}$$

The blending matrix follows from

$$\mathbf{K} = \mathbf{Q}\mathbf{C}^{\mathsf{T}}(\mathbf{C}\mathbf{Q}\mathbf{C}^{\mathsf{T}} + \mathbf{R})^{-1} \tag{46}$$

In this case,

$$\mathbf{Q} = \sigma^2, \quad \mathbf{c} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \tag{47}$$

Therefore

$$\mathbf{K} = \sigma^2 \begin{bmatrix} 1 & 1 \end{bmatrix} \left(\sigma^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right)^{-1} = \frac{\sigma^2}{2 + 3\sigma^2} \begin{bmatrix} 2 & 1 \end{bmatrix} \tag{48}$$

It is seen that the first element in $\mathbf{y}$ contributes more to the estimate $\hat{x}$. This is natural since the second element is more noisy! The optimal estimator can be said to "trust" the better measurement more.