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Examination paper for **TTK4115 Linear Systems Theory**

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Permitted examination support material: D: No printed or handwritten material allowed. Specific simple calculator allowed.

Other information:
Note that no parts of this problem assume that you have solved any of the previous parts. The given information from previous parts should be sufficient to move on.

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Problem 1 (30 %)
A linear plant is modeled by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{1}$$

a) (5 %) Demonstrate that plant is controllable and design a state-feedback control

$$u(t) = -\mathbf{kx}(t), \quad \mathbf{x}(t) = [x_1(t),\ x_2(t)]^{\mathsf{T}} \tag{2}$$

that places the poles of the closed loop plant at $\lambda_1 = -1$ and $\lambda_2 = -1$.

b) (5 %) Suppose now that an unspecified state-feedback $u = -\mathbf{k}'\mathbf{x}$ has been used to bring forth the closed loop system matrix

$$\mathbf{A}_{\text{cl}} = \mathbf{A} - \mathbf{bk}' = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \tag{3}$$

Utilize Lyapunov's theorem $\mathbf{A}_{\text{cl}}^{\mathsf{T}}\mathbf{M} + \mathbf{M}\mathbf{A}_{\text{cl}} + \mathbf{N} = \mathbf{0}$ to demonstrate that the unspecified feedback results in stable operation.

c) (5 %) First demonstrate that the plant is observable. This implies that a Luenberger observer can be utilized. Proceed to give the general form of the Luenberger observer equation

$$\dot{\hat{\mathbf{x}}} = \dots \tag{4}$$

The right hand side consists of several terms. Give an informal and brief account of their function. Would it be possible to utilize an *open loop* observer with the plant (1)? Write down the general form of this observer type $\dot{\hat{\mathbf{x}}}_{\text{ol}} = \dots$ along with your answer.

d) (10 %) Output feedback control can be accomplished using state feedback in series with a Luenberger estimator. Let a very simple first order process be described by

$$\dot{x}(t) = u(t), \quad y(t) = x(t) \tag{5}$$

Sometimes it is not possible measure all the states in the model (in this example you can). In this case output feedback can be used. An output feedback controller utilizing feedback from *estimated* states can be given by

$$\dot{\hat{x}}(t) = -(k+l)\hat{x}(t) + ly(t) \tag{6a}$$
$$u(t) = -k\hat{x}(t) \tag{6b}$$

where $k > 0$ is the feedback gain and $l > 0$ is the observer gain. *Demonstrate that $k > 0$ can be chosen independently of $l > 0$ when the aim is overall stability.* Does your result generalize to state feedback from estimated states in general?

e) (5 %) Clearly, an output feedback

$$u(t) = -ky(t) \tag{7}$$

will stabilize the simple process (5) as long as $k > 0$. Suppose now the measurement is polluted by *high-frequency* noise so that $y(t) = x(t) + n(t)$. Derive the transfer functions from $y(s)$ to $u(s)$ for (7) and (6). Comparing the two, which will tend to give a smoother input $u(t)$? Explain your reasoning.

Problem 2 (20 %)
PID regulators, exemplified by

$$u(s) = \left(K_p + K_d s + \frac{K_i}{s} \right) e(s) \tag{8}$$

are ubiquitous in process control. The preceding equation is given in the Laplace-domain. For implementation, a time-domain formulation must be used.

a) (5 %) Explain why the following formulation is preferable for time-domain implemenation

$$u(s) = \left(K_p + \frac{K_d s}{\tau s + 1} + \frac{K_i}{s} \right) e(s) \tag{9}$$

Here $\tau > 0$ is a small time-constant. In particular, give reasons why (8) *cannot* be implemented as a state-space model and why (9) does not suffer from this problem.

b) (10 %) Find the matrices $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$ and the scalar d in the plant

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}e(t) \tag{10}$$
$$u(t) = \mathbf{c}\mathbf{x}(t) + de(t) \tag{11}$$

such that the system *realizes* the PID regulator. That is

$$\mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + d = K_p + \frac{K_d s}{\tau s + 1} + \frac{K_i}{s} \tag{12}$$

Is there more than one correct answer to this problem? Explain.

c) (5 %) At times, it is sufficient to employ a PI-regulator. If the PI-regulator is to be implemented on a computer, a discrete formulation is desirable. Show that the state-space implementation

$$\dot{x}(t) = e(t), \quad u(t) = K_i x(t) + K_p e(t) \tag{13}$$

gives rise to the transfer function

$$K(s) = \frac{u}{e}(s) = K_p + \frac{K_i}{s} \tag{14}$$

Assume now that the error $e(t)$ is approximately constant over a time interval $t \in [kT, (k+1)T]$ where T represents the sampling time and $k = 0, 1, 2 \ldots n$. Derive a discretization of (13) on the form

$$x[k+1] = a_d x[k] + b_d e[k], \quad u[k] = c_d x[k] + d_d e[k] \tag{15}$$

You are here to produce the scalar model coefficients a_d, b_d, c_d, d_d taking care to show your work.

Problem 3 (25 %)
Modern cellular phones are equipped with accelerometers and GPS. Much improved position estimates are possible through combination of sensor data, known as sensor fusion. For simplicity, only the horizontal position $x(t)$ along one direction is considered here.

GPS measurements of the true position are subject to noise caused by a variety of effects. This is modeled by

$$y[k] = x[k] + v[k], \quad \mathsf{E}[v[k]] = 0, \quad \mathsf{E}[v[k]v[l]] = \delta[k,l]\sigma^2 \tag{16}$$

where σ describes the standard-deviation of the position measurement error. The accelerometer produces a signal $\alpha(t)$ that measures $\ddot{x}(t)$, but the reading is often influenced by a slowly varying unknown *bias* $\beta(t)$, so that

$$\alpha(t) = \ddot{x}(t) + \beta(t) \tag{17}$$

a) (5 %) The accelerometer produces continuous readings whereas the GPS measurements arrives at discrete intervals. In order to fuse the two sensors it is necessary to discretize (17). *Assume that bias varies so slowly that it can be considered constant over a sampling interval.* This permits the simplification

$$\beta[k] \simeq \frac{1}{T}\int_{kT}^{(k+1)T} \beta(t')\ dt' \tag{18}$$

Define also an averaged accelerometer reading as

$$\bar{\alpha}[k] \triangleq \frac{1}{T}\int_{kT}^{(k+1)T} \alpha(t')\ dt' \tag{19}$$

Show that (17) can be used to produce the exact discretization

$$\dot{x}[k+1] = \dot{x}[k] + T\bar{\alpha}[k] - T\beta[k] \tag{20}$$

b) (8 %) It will be assumed that the sample interval is small enough ($T^2 \sim 0$) to permit the simplification

$$x[k+1] \simeq x[k] + T\dot{x}[k] \tag{21}$$

Suppose now that the bias is well modeled by a random walk

$$\beta[k+1] = \beta[k] + w[k], \quad \mathsf{E}[w[k]] = 0, \quad \mathsf{E}[w[k]w[l]] = \delta[k,l]q \tag{22}$$

where q represents the intensity of the white noise driving the random walk, (typically a tuning parameter). Let the random process describing the sensor fusion problem be represented by the discrete-time state-space model

$$\mathbf{x}[k+1] = \mathbf{A}_d\mathbf{x}[k] + \mathbf{b}_d\bar{\alpha}[k] + \mathbf{g}_d w[k], \quad y[k] = \mathbf{c}\mathbf{x}[k] + v[k] \tag{23}$$

where the state vector reads as

$$\mathbf{x}[k] = \begin{bmatrix} \beta[k] \\ \dot{x}[k] \\ x[k] \end{bmatrix} \tag{24}$$

Identify the system matrices $\mathbf{A}_d, \mathbf{b}_d, \mathbf{g}_d$ and $\mathbf{c}$ using the equations (16,20,21,22).

Observability in discrete time can be established if the following matrix has full column rank

$$\mathcal{O}_d = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA}_d \\ \mathbf{CA}_d^2 \\ \vdots \\ \mathbf{CA}_d^{n-1} \end{bmatrix} \tag{25}$$

Is it, in principle, possible to determine the accelerometer bias $\beta[k]$, velocity $\dot{x}[k]$ and position $x[k]$ from the measurement $y[k]$?

c) (12 %) Let $\hat{\mathbf{x}}[k]$ denote an estimate minimizing the mean-square error

$$\text{MSE} = \text{tr}(\mathbf{P}[k]), \quad \mathbf{P}[k] = \mathsf{E}[(\mathbf{x}[k] - \hat{\mathbf{x}}[k])(\mathbf{x}[k] - \hat{\mathbf{x}}[k])^{\mathsf{T}}] \tag{26}$$

Given the optimal estimate $\hat{\mathbf{x}}[k-1]$ and covariance matrix $\mathbf{P}[k-1]$ from the previous time-step. *Produce a sequence of operations giving the optimal estimate* $\hat{\mathbf{x}}[k]$ *using the process model given in* [1] (23) *and the known sequence of accelerometer readings* $\bar{\alpha}[k]$. Make sure to illustrate where the standard deviation σ enters the computation. (Do not use the explicit expressions for $\mathbf{A}_d, \mathbf{b}_d, \mathbf{g}_d$ and $\mathbf{c}$ found above).

[1] Hint: $\mathbf{Q}_d = q\mathbf{g}\mathbf{g}^{\mathsf{T}}$.

Problem 4 (25 %)
The Linear Quadratic Regulator is a powerful tool in control design. Let a simple plant be given by the low-pass filter

$$\tau \dot{x}(t) + x(t) = u(t) \tag{27}$$

Here, $\tau \neq 0$ represents the time constant of the model, not necessarily positive!

a) (10 %) Suppose that the following cost function is to be minimized using $u(t)$.

$$J = \int_0^\infty x(t)^2 + \rho u(t)^2 \; dt \tag{28}$$

Here ρ describes the *input cost*. Show that the optimal feedback gain k, to be used as $u(t) = -kx(t)$, is given by

$$k = -1 + \operatorname{sign}(\tau)\sqrt{1 + \frac{1}{\rho}} \tag{29}$$

Show furthermore that the closed loop dynamics read as

$$|\tau|\dot{x}(t) + \left(\sqrt{1 + \frac{1}{\rho}}\right) x(t) = 0 \tag{30}$$

b) (5 %) Suppose that $\tau < 0$, rendering (27) *unstable*. One might imagine that an infinitely large input cost $\rho \to \infty$ would hinder the LQR-regulator from producing an input that is sufficiently large to stabilize the system. Based on (30,29), give an account of what actually happens! Explain your finding using (28).

c) (5 %) Let $x(0) = x_0$. Find $x(t)$ from (30). What happens to the solution $x(t)$ as the control is made *cheap* $\rho \to 0$? Please supply a small sketch comparing $x(t)$ using cheap control vs. $x(t)$ using expensive control $\rho \to \infty$.

d) (5 %) The LQR regulator designed for (27) relied solely on the proportional effect. Explain how a PI feedback regulator

$$u(t) = -kx(t) - k_i \int_0^t x(t') \; dt' \tag{31}$$

can be designed using the LQR method. Give an example of an appropriate $\mathbf{A}$ and $\mathbf{B}$ matrix to be used with the method. Also indicate any modifications to the cost functions in terms of the weighting matrix $\mathbf{Q}$, (given the value $Q = 1$ above).

Formula sheet

Solutions

$$
\begin{aligned}
\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau \\
\mathbf{x}[k] &= \mathbf{A}^k\mathbf{x}[0] + \sum_{m=0}^{k-1} \mathbf{A}^{k-1-m}\mathbf{B}\mathbf{u}[m]
\end{aligned}
$$

Controllability/Observability

$$
\begin{aligned}
\mathcal{C} &= [\mathbf{B},\ \mathbf{AB},\ \mathbf{A}^2\mathbf{B},\ \cdots,\ \mathbf{A}^{n-1}\mathbf{B}] \\
\mathcal{O} &= \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}
\end{aligned}
$$

Realization

$$
\begin{aligned}
\mathbf{G}(s) &= \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \\
\mathbf{G}(s) &= \mathbf{G}(\infty) + \mathbf{G}_{sp}(s) \\
d(s) &= s^r + \alpha_1 s^{r-1} + \cdots + \alpha_{r-1}s + \alpha_r \\
\mathbf{G}_{sp}(s) &= \frac{1}{d(s)}[\mathbf{N}_1 s^{r-1} + \mathbf{N}_2 s^{r-2} + \cdots + \mathbf{N}_{r-1}s + \mathbf{N}_r] \\
\dot{\mathbf{x}} &= \begin{bmatrix} -\alpha_1\mathbf{I}_p & -\alpha_2\mathbf{I}_p & \cdots & -\alpha_{r-1}\mathbf{I}_p & -\alpha_r\mathbf{I}_p \\ \mathbf{I}_p & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_p & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I}_p & \mathbf{0} \end{bmatrix}\mathbf{x} + \begin{bmatrix} \mathbf{I}_p \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}\mathbf{u} \\
\mathbf{y} &= [\ \mathbf{N}_1 \quad \mathbf{N}_2 \quad \cdots \quad \mathbf{N}_{r-1} \quad \mathbf{N}_r\ ]\mathbf{x} + \mathbf{G}(\infty)\mathbf{u}
\end{aligned}
$$

LQR

$$J = \int_0^\infty \mathbf{x}^\mathsf{T}(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^\mathsf{T}(t)\mathbf{R}\mathbf{u}(t)\,dt$$

$$\mathbf{A}^\mathsf{T}\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\mathsf{T}\mathbf{P} = \mathbf{0}$$

$$\mathbf{u}(t) = -\mathbf{R}^{-1}\mathbf{B}^\mathsf{T}\mathbf{P}\mathbf{x}(t)$$

Lyapunov equation

$$\mathbf{A}^\mathsf{T}\mathbf{M} + \mathbf{M}\mathbf{A} = -\mathbf{N}$$

Kalman filtering (Discrete time)

Process model

$$\mathbf{x}[k+1] = \mathbf{A}_d\mathbf{x}[k] + \mathbf{B}_d\mathbf{u}[k] + \mathbf{w}[k], \quad \mathbf{y}[k] = \mathbf{C}\mathbf{x}[k] + \mathbf{v}[k]$$

The noise and disturbance are unbiased ($\mathsf{E}[\mathbf{v}[k]] = \mathbf{0},\ \mathsf{E}[\mathbf{w}[k]] = \mathbf{0}$) and white

$$\mathsf{E}[\mathbf{v}[k]\mathbf{v}[l]^\mathsf{T}] = \delta[k,l]\mathbf{R}_d, \quad \mathsf{E}[\mathbf{w}[k]\mathbf{w}[l]^\mathsf{T}] = \delta[k,l]\mathbf{Q}_d$$

Algorithm Initialize at $\hat{\mathbf{x}}^-[0] = \mathsf{E}[\mathbf{x}(0)]$ and $\mathbf{P}^-[0] = \mathsf{E}[(\mathbf{x}[0] - \hat{\mathbf{x}}^-[0])(\mathbf{x}[0] - \hat{\mathbf{x}}^-[0])^\mathsf{T}]$. Compute recursively:

1. $\mathbf{L}[k] = \mathbf{P}^-[k]\mathbf{C}^\mathsf{T}(\mathbf{C}\mathbf{P}^-[k]\mathbf{C}^\mathsf{T} + \mathbf{R}_d)^{-1}$
2. $\hat{\mathbf{x}}[k] = \hat{\mathbf{x}}^-[k] + \mathbf{L}[k](\mathbf{y}[k] - \mathbf{C}\hat{\mathbf{x}}^-[k])$
3. $\mathbf{P}[k] = (\mathbb{I} - \mathbf{L}[k]\mathbf{C})\mathbf{P}^-[k](\mathbb{I} - \mathbf{L}[k]\mathbf{C})^\mathsf{T} + \mathbf{L}[k]\mathbf{R}_d\mathbf{L}[k]^\mathsf{T}$
4. $\hat{\mathbf{x}}^-[k+1] = \mathbf{A}_d\hat{\mathbf{x}}[k] + \mathbf{B}_d\mathbf{u}[k], \quad \mathbf{P}^-[k+1] = \mathbf{A}_d\mathbf{P}[k]\mathbf{A}_d^\mathsf{T} + \mathbf{Q}_d$

Kalman filtering (Continuous time)

Process model

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{G}\mathbf{w}, \quad \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{v}$$

The noise and disturbance are unbiased ($\mathsf{E}[\mathbf{v}(t)] = \mathbf{0},\ \mathsf{E}[\mathbf{w}(t)] = \mathbf{0}$) and white

$$\mathsf{E}[\mathbf{v}(t)\mathbf{v}(\tau)^{\mathsf{T}}] = \delta(t-\tau)\mathbf{R}, \quad \mathsf{E}[\mathbf{w}(\tau)\mathbf{w}(t)^{\mathsf{T}}] = \delta(t-\tau)\mathbf{Q}$$

Optimal gain The Kalman gain is given by $\mathbf{L}(t) = \mathbf{P}(t)\mathbf{C}^{\mathsf{T}}\mathbf{R}^{-1}$ where

$$\dot{\mathbf{P}} = \mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^{\mathsf{T}} + \mathbf{G}\mathbf{Q}\mathbf{G}^{\mathsf{T}} - \mathbf{P}\mathbf{C}^{\mathsf{T}}\mathbf{R}^{-1}\mathbf{C}\mathbf{P}$$

Set $\dot{\mathbf{P}} = \mathbf{0}$ to find stationary gain.

Stationary processes

Autocorrelation and power spectral density

$$\mathcal{R}_u(\tau) = \mathsf{E}[u(t)u(t+\tau)], \quad \mathcal{S}_u(\omega) = \mathcal{F}\{\mathcal{R}_u(\tau)\}$$

With $y(s) = H(s)w(s)$ where $\mathsf{E}[w(t)] = 0$ and $\mathcal{R}_w(\tau) = \delta(\tau)q$ it holds that

$$\mathcal{S}_y(\omega) = H(\jmath\omega)H(-\jmath\omega)q$$

Laplace transform pairs

$f(t)$	$\iff$	$F(s)$
$\delta(t)$	$\iff$	1
1	$\iff$	$\frac{1}{s}$
e^{-at}	$\iff$	$\frac{1}{s+a}$
t	$\iff$	$\frac{1}{s^2}$
t^2	$\iff$	$\frac{2}{s^3}$
te^{-at}	$\iff$	$\frac{1}{(s+a)^2}$
$\sin \omega t$	$\iff$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\iff$	$\frac{s}{s^2+\omega^2}$