NTNU – Trondheim
Norwegian University of
Science and Technology

Department of Engineering Cybernetics

Examination paper for **TTK4115 Linear Systems Theory**

Academic contact during examination: Morten D. Pedersen

Phone: 41602135

Examination date: 12. August 2017

Examination time (from–to): 09:00–13:00

Permitted examination support material: D: No printed or handwritten material allowed. Specific simple calculator allowed.

Other information:
Note that no parts of this problem assume that you have solved any of the previous parts. The given information from previous parts should be sufficient to move on.

You may answer in English or Norwegian.

Language: English

Number of pages: 6

Number pages enclosed: 4

Checked by:

Date Signature

Problem 1 (20 %)

Let a first order low-pass filter and a first order high-pass filter be given respectively by

$$g_l(s) = \frac{1}{\tau s + 1}, \quad g_h(s) = \frac{\tau s}{\tau s + 1}$$

The filters are driven by a common input signal $u(s)$, but produce the two outputs $y_l(s) = g_l(s)u(s)$ and $y_h(s) = g_h(s)u(s)$.

a) (2 %) Find a state-space realization for the system taking $u(t)$ as input and producing $y(t) = y_l(t)$ as output.

b) (3 %) Find a state-space realization for the system taking $u(t)$ as input and producing $y(t) = y_h(t)$ as output.

c) (3 %) Find a state-space realization for the system taking $u(t)$ as input and producing $y(t) = y_l(t) + y_h(t)$ as output.

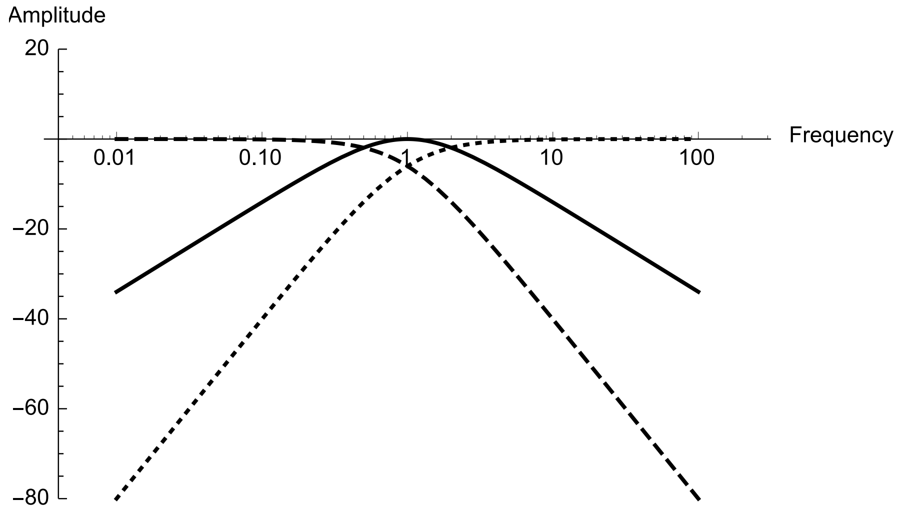
d) (5 %) Find a *minimal* state-space realization for the system taking $u(t)$ as input and producing the *vector* $\mathbf{y}(t) = [y_l(t),\ y_h(t)]^\mathsf{T}$ as output.

e) (7 %) Consider the following state-space model with $\omega_b > 0$.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u, \quad \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{d}u$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\omega_b^2 & -2\omega_b \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \omega_b^2 & 0 \\ 0 & 2\omega_b \\ -\omega_b^2 & -2\omega_b \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Identify the transfer-function $\mathbf{g}(s)$ so that $\mathbf{y}(s) = \mathbf{g}(s)u(s)$. Let $\mathbf{y} = [y_1,\ y_2,\ y_3]^\mathsf{T}$. What kind of filtering action is applied to u on each of the output channels? The image presented on the next page serves as a visual hint!



Problem 2 (30 %)
The shaft velocity of a diesel engine can be modeled notionally by

$$\tau \dot{x} + x = u - d \tag{1}$$

Here, x represents the shaft velocity, u the throttle whereas d represents an applied load. The dynamics are quantified by a dimensionless time-constant τ.

a) (8 %) Suppose that LQR is used to tune a simple controller $u = -kx$ where the feedback proportional to k minimizes the cost function

$$J = \int_0^\infty \left\{ qx^2 + ru^2 \right\} dt$$

Let $x(0) = 1$ with $d = 0$. Draw a sketch comparing the responses in $x(t)$ with the following tunings:

A $q = 1$ and $r = 1$.

B $q = 100$ and $r = 1$.

C $q = 1$ and $r = 100$.

D $q = 100$ and $r = 100$.

Label your graphs with the appropriate letter. You *do not* need to compute $x(t)$ explicitely. Your sketch only needs to capture the qualitative differences between the tunings A through D.

b) (10 %) Let x_r denote a *constant* shaft velocity reference. Stationary deviations from this reference are undesirable, integral effect is therefore to be used. The LQR will be employed to minimize the cost function

$$J = \int_0^\infty \left\{ q_p(x - x_r)^2 + q_i \int_0^t (x - x_r)^2 \, dt' + u^2 \right\} dt \tag{2}$$

In order to find the optimal gain matrix $\mathbf{K}$, the Riccati equation must be solved.

$$\mathbf{A}^\mathsf{T}\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\mathsf{T}\mathbf{P} = \mathbf{0}$$

Find a set of matrices $\mathbf{A},\ \mathbf{B},\ \mathbf{Q},\ \mathbf{R}$, consistent with the model (1) and cost function (2). (Scalars count as 1×1 matrices).

c) (12 %) The engine manufacturer indicates that it is in fact $\dot{u}$ that ought to be limited rather than the input u itself. (The motor prefers to run at a steady speed). Taking this advice leads to the cost function

$$J = \int_0^\infty \left\{ q_p(x - x_r)^2 + \dot{u}^2 \right\} dt \tag{3}$$

As before, the Riccati equation must be solved.

$$\mathbf{A}^\mathsf{T}\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\mathsf{T}\mathbf{P} = \mathbf{0}$$

Find a set of matrices $\mathbf{A},\ \mathbf{B},\ \mathbf{Q},\ \mathbf{R}$, consistent with the model (1) and cost function (3). (Scalars count as 1×1 matrices).

Problem 3 (20 %)

Consider the following state-space model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ \alpha \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & \beta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

a) (3 %) For which values of α and β is this system uncontrollable?

b) (3 %) For which values of α and β is this system unobservable?

c) (5 %) Let $\alpha = 0$. Find the constants k_1 and k_2 in the feedback $u = -(k_1 x_1 + k_2 x_2)$ so that the eigenvalues in *closed loop* are $\lambda_1 = \lambda_2 = -1$.

d) (9 %) Suppose now that $\mathbf{x} = [x_1,\ x_2]^\mathsf{T}$ cannot be measured directly, but only through y.

- Explain how state-feedback can be implemented if you can only measure y and not $\mathbf{x}$.
- Explain why the value $\beta = 1$ is problematic if you can only measure y and not $\mathbf{x}$.
- With $\beta = 1$, identify k so that the *output-feedback* $u = -ky$ results in the closed-loop poles $\lambda_1 = \lambda_2 = -1$.

Problem 4 (30 %)

Consider the measurement $\mathbf{y}$ given by

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{v}$$

The objective is to estimate the unknown vector $\mathbf{x}$, the desired estimate will be denoted $\hat{\mathbf{x}}$. Some information is known; the mean and covariance of $\mathbf{x}$ are given by

$$\mathsf{E}[\mathbf{x}] = \mathbf{m}, \quad \mathsf{E}[(\mathbf{x} - \mathbf{m})(\mathbf{x} - \mathbf{m})^\mathsf{T}] = \mathbf{Q}$$

The measurement noise $\mathbf{v}$ satisfies

$$\mathsf{E}[\mathbf{v}] = \mathbf{0}, \quad \mathsf{E}[\mathbf{v}\mathbf{v}^\mathsf{T}] = \mathbf{R}$$

It is assumed that $\mathbf{x}$ and $\mathbf{v}$ are completely uncorrelated.

Since the problem is linear, it is reasonable to assume that the estimate can be obtained from the general formula

$$\hat{\mathbf{x}} = \mathbf{m} + \mathbf{K}(\mathbf{y} - \mathbf{C}\mathbf{m}) \tag{4}$$

The first term centers the estimate on the mean, the second term allows variations about the mean. The measurement error will be denoted

$$\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$$

Note that the matrix $\mathbf{K}$ is left undetermined for now.

a) (5 %) Show that the estimate $\hat{\mathbf{x}}$ is unbiased, that is

$$\mathsf{E}[\mathbf{e}] = \mathbf{0}$$

Make sure to show your work.

b) (10 %) Show that the covariance matrix of the estimate error $\mathbf{P}$ is given by

$$\mathbf{P} = \mathsf{E}[\mathbf{e}\mathbf{e}^\mathsf{T}] = (\mathbb{I} - \mathbf{KC})\mathbf{Q}(\mathbb{I} - \mathbf{KC})^\mathsf{T} + \mathbf{KRK}^\mathsf{T} \tag{5}$$

Make sure to show your work.

c) (10 %) The *mean-square error* (MSE) is given by the formula

$$\mathsf{MSE} = \sum_i \mathsf{E}[e_i^2] = \mathrm{tr}(\mathbf{P})$$

where e_i is the error associated with the i'th estimate $\hat{x}_i$. The trace operator is used to extract the MSE from $\mathbf{P}$. Now, the MSE is to be *minimized* by selecting the as of yet undetermined matrix $\mathbf{K}$. This leads to an optimal estimate for $\hat{\mathbf{x}}$ when (4) is used.

The following matrix derivatives are supplied

$$\frac{\partial}{\partial \mathbf{X}}\mathrm{tr}(\mathbf{X}) = \mathbb{I}, \quad \frac{\partial}{\partial \mathbf{X}}\mathrm{tr}(\mathbf{A}\mathbf{X}^\mathsf{T}) = \mathbf{A},$$
$$\frac{\partial}{\partial \mathbf{X}}\mathrm{tr}(\mathbf{X}\mathbf{A}) = \mathbf{A}^\mathsf{T}, \quad \frac{\partial}{\partial \mathbf{X}}\mathrm{tr}(\mathbf{X}\mathbf{A}\mathbf{X}^\mathsf{T}) = \mathbf{X}\mathbf{A}^\mathsf{T} + \mathbf{X}\mathbf{A}$$

When the objective is to minimize the MSE, the best choice for $\mathbf{K}$ is found from the relation

$$\frac{\partial \mathsf{MSE}}{\partial \mathbf{K}} = 0$$

Show that the optimal choice is given by

$$\mathbf{K} = \mathbf{QC}^\mathsf{T}(\mathbf{CQC}^\mathsf{T} + \mathbf{R})^{-1} \tag{6}$$

d) (5 %) Let a simple measurement model be given by

$$\mathbf{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

The unknown scalar x satisfies $\mathsf{E}[x] = 0$ and $\mathsf{E}[x^2] = q = \sigma^2$. The zero-mean noise vector $\mathbf{v} = [v_1, v_2]^\mathsf{T}$ is equipped with the covariance matrix

$$\mathbf{R} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \tag{7}$$

It is seen that the second element in $\mathbf{v} = [v_1, v_2]^\mathsf{T}$ is more "noisy" than the first.

Compute the optimal blending matrix $\mathbf{K}$, see (6). This matrix will have two elements $\mathbf{K} = [k_1, k_2]$.

With $\mathbf{m} = 0$ (due to $\mathsf{E}[x] = 0$) the estimate reads as

$$\hat{\mathbf{x}} = \mathbf{K}\mathbf{y}$$

The two elements of $\mathbf{y}$ both measure x, but the elements in $\mathbf{K}$ are not necessarily the same. Noting (7), explain why this could be the case. (Informal insights are sufficient).

Formula sheet

Solutions

$$
\begin{aligned}
\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau \\
\mathbf{x}[k] &= \mathbf{A}^k\mathbf{x}[0] + \sum_{m=0}^{k-1}\mathbf{A}^{k-1-m}\mathbf{B}\mathbf{u}[m]
\end{aligned}
$$

Controllability/Observability

$$
\begin{aligned}
\mathcal{C} &= [\mathbf{B},\ \mathbf{AB},\ \mathbf{A}^2\mathbf{B},\ \cdots,\ \mathbf{A}^{n-1}\mathbf{B}] \\
\mathcal{O} &= \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}
\end{aligned}
$$

Realization

$$
\begin{aligned}
\mathbf{G}(s) &= \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \\
\mathbf{G}(s) &= \mathbf{G}(\infty) + \mathbf{G}_{sp}(s) \\
d(s) &= s^r + \alpha_1 s^{r-1} + \cdots + \alpha_{r-1}s + \alpha_r \\
\mathbf{G}_{sp}(s) &= \frac{1}{d(s)}[\mathbf{N}_1 s^{r-1} + \mathbf{N}_2 s^{r-2} + \cdots + \mathbf{N}_{r-1}s + \mathbf{N}_r] \\
\dot{\mathbf{x}} &= \begin{bmatrix} -\alpha_1\mathbf{I}_p & -\alpha_2\mathbf{I}_p & \cdots & -\alpha_{r-1}\mathbf{I}_p & -\alpha_r\mathbf{I}_p \\ \mathbf{I}_p & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_p & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I}_p & \mathbf{0} \end{bmatrix}\mathbf{x} + \begin{bmatrix} \mathbf{I}_p \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}\mathbf{u} \\
\mathbf{y} &= [\ \mathbf{N}_1 \quad \mathbf{N}_2 \quad \cdots \quad \mathbf{N}_{r-1} \quad \mathbf{N}_r\ ]\mathbf{x} + \mathbf{G}(\infty)\mathbf{u}
\end{aligned}
$$

LQR

$$J = \int_0^\infty \mathbf{x}^\mathsf{T}(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^\mathsf{T}(t)\mathbf{R}\mathbf{u}(t)\,dt$$

$$\mathbf{A}^\mathsf{T}\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\mathsf{T}\mathbf{P} = \mathbf{0}$$

$$\mathbf{u}(t) = -\mathbf{R}^{-1}\mathbf{B}^\mathsf{T}\mathbf{P}\mathbf{x}(t)$$

Lyapunov equation

$$\mathbf{A}^\mathsf{T}\mathbf{M} + \mathbf{M}\mathbf{A} = -\mathbf{N}$$

Kalman filtering (Discrete time)

Process model

$$\mathbf{x}[k+1] = \mathbf{A}_d\mathbf{x}[k] + \mathbf{B}_d\mathbf{u}[k] + \mathbf{w}[k], \quad \mathbf{y}[k] = \mathbf{C}\mathbf{x}[k] + \mathbf{v}[k]$$

The noise and disturbance are unbiased ($\mathsf{E}[\mathbf{v}[k]] = \mathbf{0},\ \mathsf{E}[\mathbf{w}[k]] = \mathbf{0}$) and white

$$\mathsf{E}[\mathbf{v}[k]\mathbf{v}[l]^\mathsf{T}] = \delta[k,l]\mathbf{R}_d, \quad \mathsf{E}[\mathbf{w}[k]\mathbf{w}[l]^\mathsf{T}] = \delta[k,l]\mathbf{Q}_d$$

Algorithm Initialize at $\hat{\mathbf{x}}^-[0] = \mathsf{E}[\mathbf{x}(0)]$ and $\mathbf{P}^-[0] = \mathsf{E}[(\mathbf{x}[0] - \hat{\mathbf{x}}^-[0])(\mathbf{x}[0] - \hat{\mathbf{x}}^-[0])^\mathsf{T}]$. Compute recursively:

1. $\mathbf{L}[k] = \mathbf{P}^-[k]\mathbf{C}^\mathsf{T}(\mathbf{C}\mathbf{P}^-[k]\mathbf{C}^\mathsf{T} + \mathbf{R}_d)^{-1}$
2. $\hat{\mathbf{x}}[k] = \hat{\mathbf{x}}^-[k] + \mathbf{L}[k](\mathbf{y}[k] - \mathbf{C}\hat{\mathbf{x}}^-[k])$
3. $\mathbf{P}[k] = (\mathbb{I} - \mathbf{L}[k]\mathbf{C})\mathbf{P}^-[k](\mathbb{I} - \mathbf{L}[k]\mathbf{C})^\mathsf{T} + \mathbf{L}[k]\mathbf{R}_d\mathbf{L}[k]^\mathsf{T}$
4. $\hat{\mathbf{x}}^-[k+1] = \mathbf{A}_d\hat{\mathbf{x}}[k] + \mathbf{B}_d\mathbf{u}[k], \quad \mathbf{P}^-[k+1] = \mathbf{A}_d\mathbf{P}[k]\mathbf{A}_d^\mathsf{T} + \mathbf{Q}_d$

Kalman filtering (Continuous time)

Process model

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{G}\mathbf{w}, \quad \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{v}$$

The noise and disturbance are unbiased ($\mathsf{E}[\mathbf{v}(t)] = \mathbf{0},\ \mathsf{E}[\mathbf{w}(t)] = \mathbf{0}$) and white

$$\mathsf{E}[\mathbf{v}(t)\mathbf{v}(\tau)^{\mathsf{T}}] = \delta(t-\tau)\mathbf{R}, \quad \mathsf{E}[\mathbf{w}(\tau)\mathbf{w}(t)^{\mathsf{T}}] = \delta(t-\tau)\mathbf{Q}$$

Optimal gain The Kalman gain is given by $\mathbf{L}(t) = \mathbf{P}(t)\mathbf{C}^{\mathsf{T}}\mathbf{R}^{-1}$ where

$$\dot{\mathbf{P}} = \mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^{\mathsf{T}} + \mathbf{G}\mathbf{Q}\mathbf{G}^{\mathsf{T}} - \mathbf{P}\mathbf{C}^{\mathsf{T}}\mathbf{R}^{-1}\mathbf{C}\mathbf{P}$$

Set $\dot{\mathbf{P}} = \mathbf{0}$ to find stationary gain.

Stationary processes

Autocorrelation and power spectral density

$$\mathcal{R}_u(\tau) = \mathsf{E}[u(t)u(t+\tau)], \quad \mathcal{S}_u(\omega) = \mathcal{F}\{\mathcal{R}_u(\tau)\}$$

With $y(s) = H(s)w(s)$ where $\mathsf{E}[w(t)] = 0$ and $\mathcal{R}_w(\tau) = \delta(\tau)q$ it holds that

$$\mathcal{S}_y(\omega) = H(\jmath\omega)H(-\jmath\omega)q$$

Laplace transform pairs

$f(t)$	$\iff$	$F(s)$
$\delta(t)$	$\iff$	1
1	$\iff$	$\frac{1}{s}$
e^{-at}	$\iff$	$\frac{1}{s+a}$
t	$\iff$	$\frac{1}{s^2}$
t^2	$\iff$	$\frac{2}{s^3}$
te^{-at}	$\iff$	$\frac{1}{(s+a)^2}$
$\sin \omega t$	$\iff$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\iff$	$\frac{s}{s^2+\omega^2}$