Norges teknisk- naturvitenskapelige universitet
Institutt for teknisk kybernetikk

Faglig kontakt under eksamen:

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Eksamen
TTK 4115 Lineær systemteori

5. desember 2003
Tid: 0900 – 1400

Hjelpemidler: D - Ingen trykte eller håndskrevne hjelpemidler tillatt. Bestemt, enkel kalkulator tillatt.

Oppgave 1 (10 %)

Anta gitt et system $\dot{x} = Ax + Bu$ der

$$A = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ \alpha \end{pmatrix}$$

der α er en konstant.

a) For hvilke verdier av α er systemet styrbart?

b) Anta $\alpha = -4$. Finn en tilstandstilbakekopling som plasserer egenverdiene til systemet i $\lambda = -2 \pm i$.

Oppgave 2 (25 %)

a) Anta gitt en $n \times n$-matrise A. Vis at ved diagonalisering kan denne uttrykkes på formen $A = Q\overline{A}Q^{-1}$, og angi hvordan matrisene $\overline{A}$ og Q kan bestemmes.

b) Hvilke forutsetninger må A tilfredstille for at den skal være diagonaliserbar?

c) Hvilke betingelser må A tilfredstille for at systemet $\dot{x} = Ax$ skal være stabilt?

d) Hvilke betingelser må A tilfredstille for at systemet $\dot{x} = Ax$ skal være asymptotisk stabilt?

e) Anta i det etterfølgende at A er gitt ved

$$A = \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}$$

Finn en symmetrisk matrise M som er en løsning til Lyapunov-ligningen

$$A^T M + MA = -I$$

og benytt M til å bestemme hvorvidt systemet $\dot{x} = Ax$ er asymptotisk stabilt.

f) Beregn e^{At}, når A er gitt som i punkt e).

Oppgave 3 (15 %)

a) Hvilke egenskaper har en *minimal* realisasjon?

b) Er systemet definert ved transferfunksjonen

$$G(s) = \frac{2(1-s)}{(1+s)(1+4s)}$$

realiserbart? Begrunn.

c) Finn en tilstandsromrealisasjon for følgende transferfunksjonsmatrise

$$G(s) = \left(-\frac{2}{(1+2s)^2}, \quad \frac{1+s}{1+2s} \right)$$

Oppgave 4 (20 %)

Vi har et 1. ordens system på følgende form

$$\dot{x} = ax + bu + v, \qquad y = x + w$$

der x er tilstanden, u er pådraget, y er en måling, v er en ukjent forstyrrelse, og w er målestøy. Parametrene a og b er skalare konstanter.

a) Anta det benyttes en tilstandsestimator på formen

$$\dot{\hat{x}} = a\hat{x} + bu, \qquad \hat{x}(0) = 0$$

Hvilke ulemper har denne tilstandsestimatoren?

b) Gitt tilstandsestimatoren $\hat{x} = y$, diskuter hvilke ulemper denne har.

c) Anta det benyttes en tilstandsestimator på formen

$$\dot{\hat{x}} = a\hat{x} + bu + \ell(y - \hat{x}), \qquad \hat{x}(0) = 0$$

Anta i de etterfølgende oppgaver at $a = -2$, $b = 1$ og $\ell = 8$. Vis at transferfunksjonen fra målestøyen w til tilstandsestimatet $\hat{x}$ er gitt ved

$$\frac{\hat{x}}{w}(s) \quad = \quad \frac{8}{s + 10}$$

d) Sammenlign fordeler og ulemper til estimatoren fra punkt c) med estimatorene fra punktene a) og b).

Oppgave 5 (15 %)

Følgende system er gitt

$$\begin{aligned} \dot{x} &= -0.4x + 0.1u_d + u \\ y &= x + 0.1v \end{aligned}$$

der x er tilstanden, u_d er pådraget, y er en måling, og v og u er hvitstøyprosesser der:

$$\begin{aligned} E[u(t)u(\tau)] &= Q\delta(t - \tau) \\ E[v(t)v(\tau)] &= R\delta(t - \tau) \\ E[u(t)v(\tau)] &= 0 \end{aligned}$$

a) Sett opp ligningene for Kalman-filteret for dette systemet, og forklar hvilken informasjon som er nødvendig for å beregne tilstandsestimatet.

b) Anta at Q økes. Hva skjer med Kalmanforsterkningen?

c) Anta at v får større varians på grunn av endringer i sensoren, men at R holdes konstant i filteret fordi man ikke er klar over dette. Hva skjer nå med Kalmanforsterkningen?

d) Ved nærmere analyse av spekteret til v, viser det seg at v ikke er hvit, men har spektraltettheten:

$$S(\omega) = \frac{1}{\omega^2 + 1}$$

Hva blir ligningene til Kalman-filteret nå?

Oppgave 6 (15 %)

Signalet $v(t)$ er et stasjonært signal definert ved $V(s) = G(s)W(s)$, der $w(t)$ er et normalfordelt hvitstøy-signal med null middelverdi og varians 1. Transferfunksjonen $G(s)$ er definert ved

$$G(s) = \frac{K}{1 + Ts}$$

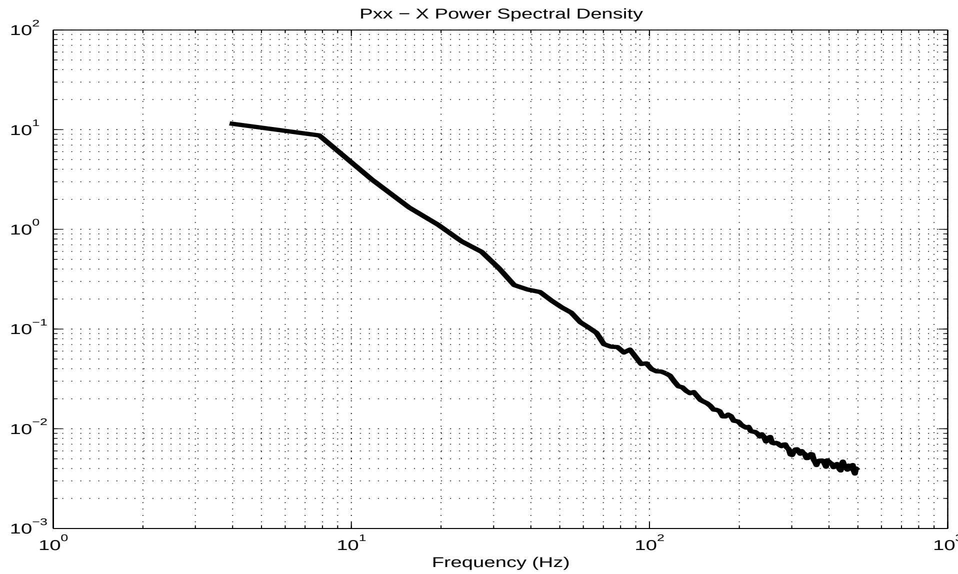
a) Hva er autokorrelasjonsfunksjonen og effektspektret for signalet $w(t)$?

b) Vis at effektspektret for signalet $v(t)$ er

$$S_v(j\omega) = \frac{K^2}{(\omega T)^2 + 1}$$

Hva er autokorrelasjonsfunksjonen?

c) Anslå verdiene til K og T fra følgende estimerte effektspektrum:


Pxx – X Power Spectral Density
10^2
10^1
10^0
10^-1
10^-2
10^-3
10^0
10^1
10^2
10^3
Frequency (Hz)

Vedlegg til eksamen (Noen nyttige formler og uttrykk)

$$
\begin{aligned}
x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \\
x(k) &= A^k x(0) + \sum_{m=0}^{k-1} A^{k-1-m}Bu(m) \\
A^{-1} &= \frac{adj(A)}{det(A)} \\
det(A) &= \sum_{i=1}^{n} a_{ij}c_{ij} \\
adj(A) &= \{c_{ij}\}^T \\
c_{ij} &= (-1)^{i+j}det(A_{ij}) \quad \text{(kofaktor)},\ A_{ij} = \text{undermatrise til } A
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{C} &= (B\ AB\ A^2B\ \cdots\ A^{n-1}B) \\
\mathcal{O} &= \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix} \\
G(s) &= C(sI-A)^{-1}B + D \\
G(z) &= C(zI-A)^{-1}B + D \\
\mathbf{G(s)} &= \mathbf{G}(\infty) + \mathbf{G_{sp}(s)} \\
d(s) &= s^r + \alpha_1 s^{r-1} + \cdots + \alpha_{r-1}s + \alpha_r \\
\mathbf{G_{sp}}(s) &= \frac{1}{d(s)}[\mathbf{N_1}s^{r-1} + \mathbf{N_2}s^{r-2} + \cdots + \mathbf{N_{r-1}}s + \mathbf{N_r}] \\
\dot{\mathbf{x}} &= \begin{bmatrix} -\alpha_1\mathbf{I_p} & -\alpha_2\mathbf{I_p} & \cdots & -\alpha_{r-1}\mathbf{I_p} & -\alpha_r\mathbf{I_p} \\ \mathbf{I_p} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I_p} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I_p} & \mathbf{0} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{I_p} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} \mathbf{u} \\
\mathbf{y} &= [\ \mathbf{N_1}\quad \mathbf{N_2}\quad \cdots\quad \mathbf{N_{r-1}}\quad \mathbf{N_r}\ ]\mathbf{x} + \mathbf{G}(\infty)\mathbf{u}
\end{aligned}
$$

Vedlegg til eksamen.

Diskret Kalman-filter:

$$
\begin{aligned}
\mathbf{x}_{k+1} &= \boldsymbol{\Phi}_k \mathbf{x}_k + \mathbf{w}_k \\
\mathbf{z}_k &= \mathbf{H}_k \mathbf{x}_k + \mathbf{v}_k \\
E[\mathbf{w}_k \mathbf{w}_i^T] &= \begin{cases} \mathbf{Q}_k, & i = k \\ 0, & i \neq k \end{cases} \\
E[\mathbf{v}_k \mathbf{v}_i^T] &= \begin{cases} \mathbf{R}_k, & i = k \\ 0, & i \neq k \end{cases} \\
E[\mathbf{w}_k \mathbf{v}_i^T] &= 0, \forall i, k \\
\mathbf{P}_k^- &= E[\mathbf{e}_k^- \mathbf{e}_k^{-T}] \\
\mathbf{P}_k &= E[\mathbf{e}_k \mathbf{e}_k^T] = (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) \mathbf{P}_k^- (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)^T + \mathbf{K}_k \mathbf{R}_k \mathbf{K}_k^T \\
\mathbf{K}_k &= \mathbf{P}_k^- \mathbf{H}_k^T \left(\mathbf{H}_k \mathbf{P}_k^- \mathbf{H}_k^T + \mathbf{R}_k \right)^{-1} \\
\mathbf{P}_{k+1}^- &= \boldsymbol{\Phi}_k \mathbf{P}_k \boldsymbol{\Phi}_k + \mathbf{Q}_k
\end{aligned}
$$

Kontinuerlig Kalman filter:

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{F}\mathbf{x} + \mathbf{G}\mathbf{u} \\
\mathbf{z} &= \mathbf{H}\mathbf{x} + \mathbf{v} \\
E[\mathbf{u}(t)\mathbf{u}(t)^T] &= \mathbf{Q}\delta(t - \tau) \\
E[\mathbf{v}(t)\mathbf{v}(t)^T] &= \mathbf{R}\delta(t - \tau) \\
E[\mathbf{u}(t)\mathbf{v}(t)^T] &= 0 \\
\mathbf{K} &= \mathbf{P}\mathbf{H}^T \mathbf{R}^{-1} \\
\dot{\mathbf{P}} &= \mathbf{F}\mathbf{P} + \mathbf{P}\mathbf{F}^T - \mathbf{P}\mathbf{H}^T \mathbf{R}^{-1} \mathbf{H}\mathbf{P} + \mathbf{G}\mathbf{Q}\mathbf{G}^T, \quad \mathbf{P}(0) = \mathbf{P}_0
\end{aligned}
$$

Autokorrelasjon:

$$
\begin{aligned}
R_X(\tau) &= E[X(t)X(t+\tau)] \text{ (Stajonær prosess)} \\
R_X(t_1, t_2) &= E[X(t_1)X(t_2)] \text{ (Ikke stajonær prosess)} \\
Y(s) &= G(s)U(s) \Rightarrow \\
R_y(t_1, t_2) &= E[y(t_1)y(t_2)] \\
&= \int_0^{t_1} \int_0^{t_2} g(\xi) g(\eta) E\left[u(t_1 - \xi) u(t_2 - \eta)\right] d\xi d\eta \text{ (Transient analyse)}
\end{aligned}
$$

Minste kvadraters estimering:

$$
\begin{aligned}
V(\bar{\theta}) &= \frac{1}{2}(\mathcal{Y}_N - \Phi_N\bar{\theta})^T(\mathcal{Y}_N - \Phi_N\bar{\theta}) \\
E[\hat{\theta}\hat{\theta}^T] &= \hat{\sigma}_e^2(\Phi_N^T\Phi_N)^{-1} \\
\hat{\sigma}_e^2 &= \frac{2}{N-p}V(\hat{\theta})
\end{aligned}
$$

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Exam
TTK 4115 Linear systems

5. December 2003
Time: 0900 – 1400

Supporting materials: D - No printed or handwritten material allowed. Specific, simple calculator allowed.

Oppgave 1 (10 %)

Given a system $\dot{x} = Ax + Bu$ where

$$A = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ \alpha \end{pmatrix}$$

where α is a constant.

a) For what values of α is the system controllable?

b) Assume that $\alpha = -4$. Find a state feedback control law that places the systems eigenvalues in $\lambda = -2 \pm i$.

Oppgave 2 (25 %)

a) Let A be a given $n \times n$-matrix . Show that by diagonalization the matrix can be represented as $A = Q\overline{A}Q^{-1}$, and state how the matrices $\overline{A}$ and Q can be determined.

b) Which conditions must A satisfy to ensure that the matrix is diagonalizable?

c) Which conditions must A satisfy to ensure that the system $\dot{x} = Ax$ is stable?

d) Which conditions must A satisfy to ensure that the system $\dot{x} = Ax$ is asymptotically stable?

e) From now on let A be given by

$$A = \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}$$

Find a symmetric matrix M that solves the Lyapunov-equation

$$A^T M + MA = -I$$

and use M to determine if the system $\dot{x} = Ax$ is asymptotically stable.

f) Calculate e^{At}, where A is as defined in e).

Oppgave 3 (15 %)

a) What properties does *minimal* realization have?

b) Is the system defined by the following transfer function

$$G(s) = \frac{2(1-s)}{(1+s)(1+4s)}$$

realizable? Justify your answer.

c) Find a state space representation of the following transfer function matrix

$$G(s) = \left(-\frac{2}{(1+2s)^2}, \quad \frac{1+s}{1+2s} \right)$$

Oppgave 4 (20 %)

Given the following first order system

$$\dot{x} = ax + bu + v, \qquad y = x + w$$

where x is the state, u is the input, y is the measurement, v is an unknown disturbance, and w is measurement noise. The parameters a and b er scalar constants.

a) Assume that a state estimator on the form

$$\dot{\hat{x}} = a\hat{x} + bu, \qquad \hat{x}(0) = 0$$

is used. Which disadvantages does this estimator have?

b) Given the state estimator $\hat{x} = y$, discuss the disadvantages with this approach.

c) Assume that a state estimator on the form

$$\dot{\hat{x}} = a\hat{x} + bu + \ell(y - \hat{x}), \qquad \hat{x}(0) = 0$$

is used. From now on, let $a = -2$, $b = 1$ and $\ell = 8$. Show that the transfer function from the measurement noise w to the state estimate $\hat{x}$ is given by

$$\frac{\hat{x}}{w}(s) = \frac{8}{s + 10}$$

d) Compare advantages and disadvantages of the estimator from exercise c) with the estimators from a) and b).

Oppgave 5 (15 %)

Given the following system

$$\begin{aligned} \dot{x} &= -0.4x + 0.1u_d + u \\ y &= x + 0.1v \end{aligned}$$

where x is the state, u_d is the input, y is the measurement, and v and u are white noise processes where:

$$\begin{aligned} E[u(t)u(\tau)] &= Q\delta(t - \tau) \\ E[v(t)v(\tau)] &= R\delta(t - \tau) \\ E[u(t)v(\tau)] &= 0 \end{aligned}$$

a) Define the Kalman filter equations for the system and explain what information is needed to compute the state estimate.

b) Assume that Q is increased. How does this affect the Kalman gain?

c) Assume that the variance in v increases due to changes in the sensor, and since we are not aware of this, R is kept constant in the filter. How does this affect the Kalman gain?

d) Analyzes of the power spectrum of v reveals that v is no white noise process, but has the following power spectral density:

$$S(\omega) = \frac{1}{\omega^2 + 1}$$

Define the Kalman filter equations in this case.

Oppgave 6 (15 %)

The signal $v(t)$ is a stationary signal defined by $V(s) = G(s)W(s)$, where $w(t)$ is a Gaussian distributed white noise signal with zero mean and unit variance. The transfer function $G(s)$ is defined by

$$G(s) = \frac{K}{1 + Ts}$$

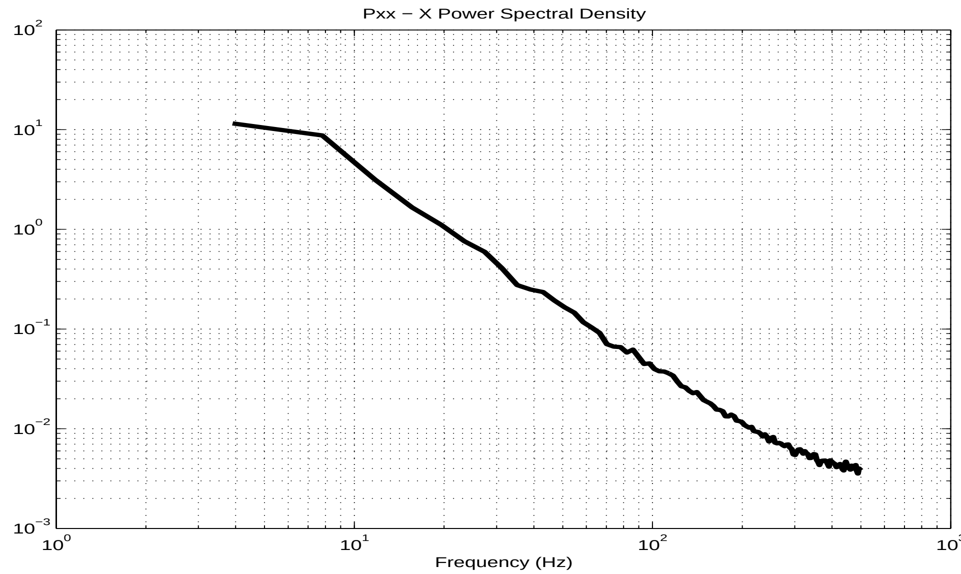
a) Find the power spectral density function and the autocorrelation function for the signal $w(t)$.

b) Show that the power spectral density function for the signal $v(t)$ is

$$S_v(j\omega) = \frac{K^2}{(\omega T)^2 + 1}$$

Find the autocorrelation function.

c) Find approximate values of K and T from the following estimated effect spectrum:



Pxx − X Power Spectral Density
10^2
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
10^0
10^1
10^2
10^3
Frequency (Hz)

Attachment to the exam (Some useful formulas and expressions)

$$
\begin{aligned}
x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \\
x(k) &= A^k x(0) + \sum_{m=0}^{k-1} A^{k-1-m}Bu(m) \\
A^{-1} &= \frac{adj(A)}{det(A)} \\
det(A) &= \sum_{i=1}^{n} a_{ij}c_{ij} \\
adj(A) &= \{c_{ij}\}^T \\
c_{ij} &= (-1)^{i+j}det(A_{ij}) \quad \text{(cofactor)},\ A_{ij} = \text{sub matrix of } A
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{C} &= (B\ AB\ A^2B\ \cdots\ A^{n-1}B) \\
\mathcal{O} &= \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix} \\
G(s) &= C(sI-A)^{-1}B + D \\
G(z) &= C(zI-A)^{-1}B + D \\
\mathbf{G(s)} &= \mathbf{G(\infty)} + \mathbf{G_{sp}(s)} \\
d(s) &= s^r + \alpha_1 s^{r-1} + \cdots + \alpha_{r-1}s + \alpha_r \\
\mathbf{G_{sp}}(s) &= \frac{1}{d(s)}[\mathbf{N_1}s^{r-1} + \mathbf{N_2}s^{r-2} + \cdots + \mathbf{N_{r-1}}s + \mathbf{N_r}] \\
\dot{\mathbf{x}} &= \begin{bmatrix} -\alpha_1\mathbf{I_p} & -\alpha_2\mathbf{I_p} & \cdots & -\alpha_{r-1}\mathbf{I_p} & -\alpha_r\mathbf{I_p} \\ \mathbf{I_p} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I_p} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I_p} & \mathbf{0} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{I_p} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} \mathbf{u} \\
\mathbf{y} &= [\ \mathbf{N_1} \quad \mathbf{N_2} \quad \cdots \quad \mathbf{N_{r-1}} \quad \mathbf{N_r}\ ]\mathbf{x} + \mathbf{G(\infty)u}
\end{aligned}
$$

Attachment to the exam.

Discrete Kalman-filter:

$$
\begin{aligned}
\mathbf{x}_{k+1} &= \mathbf{\Phi}_k \mathbf{x}_k + \mathbf{w}_k \\
\mathbf{z}_k &= \mathbf{H}_k \mathbf{x}_k + \mathbf{v}_k \\
E[\mathbf{w}_k \mathbf{w}_i^T] &= \begin{cases} \mathbf{Q}_k, & i = k \\ 0, & i \neq k \end{cases} \\
E[\mathbf{v}_k \mathbf{v}_i^T] &= \begin{cases} \mathbf{R}_k, & i = k \\ 0, & i \neq k \end{cases} \\
E[\mathbf{w}_k \mathbf{v}_i^T] &= 0, \forall i, k \\
\mathbf{P}_k^- &= E[\mathbf{e}_k^- \mathbf{e}_k^{-T}] \\
\mathbf{P}_k &= E[\mathbf{e}_k \mathbf{e}_k^T] = (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) \mathbf{P}_k^- (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)^T + \mathbf{K}_k \mathbf{R}_k \mathbf{K}_k^T \\
\mathbf{K}_k &= \mathbf{P}_k^- \mathbf{H}_k^T \left(\mathbf{H}_k \mathbf{P}_k^- \mathbf{H}_k^T + \mathbf{R}_k \right)^{-1} \\
\mathbf{P}_{k+1}^- &= \mathbf{\Phi}_k \mathbf{P}_k \mathbf{\Phi}_k + \mathbf{Q}_k
\end{aligned}
$$

Continuous Kalman filter:

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{F}\mathbf{x} + \mathbf{G}\mathbf{u} \\
\mathbf{z} &= \mathbf{H}\mathbf{x} + \mathbf{v} \\
E[\mathbf{u}(t)\mathbf{u}(t)^T] &= \mathbf{Q}\delta(t - \tau) \\
E[\mathbf{v}(t)\mathbf{v}(t)^T] &= \mathbf{R}\delta(t - \tau) \\
E[\mathbf{u}(t)\mathbf{v}(t)^T] &= 0 \\
\mathbf{K} &= \mathbf{P}\mathbf{H}^T \mathbf{R}^{-1} \\
\dot{\mathbf{P}} &= \mathbf{F}\mathbf{P} + \mathbf{P}\mathbf{F}^T - \mathbf{P}\mathbf{H}^T \mathbf{R}^{-1} \mathbf{H}\mathbf{P} + \mathbf{G}\mathbf{Q}\mathbf{G}^T, \quad \mathbf{P}(0) = \mathbf{P}_0
\end{aligned}
$$

Autocorrelation:

$$
\begin{aligned}
R_X(\tau) &= E[X(t)X(t+\tau)] \text{ (Stationary process)} \\
R_X(t_1, t_2) &= E[X(t_1)X(t_2)] \text{ (Non-stationary process)} \\
Y(s) &= G(s)U(s) \Rightarrow \\
R_y(t_1, t_2) &= E[y(t_1)y(t_2)] \\
&= \int_0^{t_1} \int_0^{t_2} g(\xi)g(\eta)E\left[u(t_1 - \xi)u(t_2 - \eta)\right] d\xi d\eta \text{ (Transient analyzes)}
\end{aligned}
$$

Least squares estimation:

$$
\begin{aligned}
V(\bar{\theta}) &= \frac{1}{2}(\mathcal{Y}_N - \Phi_N \bar{\theta})^T (\mathcal{Y}_N - \Phi_N \bar{\theta}) \\
E[\hat{\theta}\hat{\theta}^T] &= \hat{\sigma}_e^2 (\Phi_N^T \Phi_N)^{-1} \\
\hat{\sigma}_e^2 &= \frac{2}{N - p} V(\hat{\theta})
\end{aligned}
$$