Solution Suggestion
Exam - TTK4115 Linear System Theory
December 15, 2014

December 5, 2014

Problem 1

The annotated plant is:

$$\begin{aligned}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} &= \overbrace{\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}}^{\mathbf{A}}\overbrace{\begin{bmatrix}x_1\\ x_2\end{bmatrix}}^{\mathbf{x}} + \overbrace{\begin{bmatrix}0\\ 1\end{bmatrix}}^{\mathbf{B}}u(t)\\ y(t) &= \overbrace{\begin{bmatrix}1 & 0\end{bmatrix}}^{\mathbf{C}}\begin{bmatrix}x_1\\ x_2\end{bmatrix}\end{aligned}$$

a) We check for observability by computing the observability matrix:

$$\mathcal{O} = \begin{bmatrix}\mathbf{C}\\ \mathbf{CA}\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

The rank of this matrix is 2, which is <u>full rank</u>. The plant is observable.

b) We check for controllability by computing the controllability matrix:

$$\mathcal{C} = \begin{bmatrix}\mathbf{B} & \mathbf{AB}\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}$$

The rank of this matrix is 2, which is <u>full rank</u>. The plant is controllable.

c) Proportional feedback is now to be used:

$$u(t) = -\overbrace{\begin{bmatrix}k_1 & k_2\end{bmatrix}}^{\mathbf{K}}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \tag{1}$$

Closing the loop yields:

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x}, \quad \mathbf{A} - \mathbf{BK} = \begin{bmatrix}0 & 1\\ -k_1 & 1-k_2\end{bmatrix} \tag{2}$$

The eigenvalues are found by solving:

$$|\lambda\mathbf{I} - (\mathbf{A} - \mathbf{BK})| = \begin{vmatrix}\lambda & -1\\ k_1 & \lambda + (k_2 - 1)\end{vmatrix} = \lambda^2 + \lambda(k_2 - 1) + k_1 = 0 \tag{3}$$

Comparing to the desired result yields the gains:

$$(\lambda+1)^2 = \lambda^2 + 2\lambda + 1 = \lambda^2 + \lambda(k_2 - 1) + k_1 \quad \Rightarrow k_1 = 1,\, k_2 = 3 \tag{4}$$

Thus:

$$\mathbf{K} = \begin{bmatrix} 1 & 3 \end{bmatrix} \tag{5}$$

d) For this problem we have the closed loop system matrix:

$$\mathbf{A}_{cl} = \mathbf{A} - \mathbf{B}\mathbf{K}' = \begin{bmatrix} 0 & 1 \\ -k_1 & 1-k_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \tag{6}$$

The relevant Lyapunov equation reads as:

$$\mathbf{A}_{cl}^T \mathbf{M} + \mathbf{M}\mathbf{A}_{cl} = -\mathbf{N} \tag{7}$$

If the system is asymptotically stable, then for any p.d. symmetric matrix $\mathbf{N}$ then there will be a p.d. symmetric matrix $\mathbf{M}$ which solves the equation. Let us choose the simple matrix:

$$\mathbf{N} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} > 0 \tag{8}$$

to give the equation:

$$\begin{aligned} &\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} + \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &\Rightarrow \begin{bmatrix} -2m_{12} & m_{11} - m_{12} - m_{22} \\ m_{11} - m_{12} - m_{22} & 2\left(m_{12} - m_{22}\right) \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned} \tag{9}$$

Now, by inspection we have $m_{12} = 1/2$ as the solution to the equation in the first element on the diagonal. The second diagonal equation is thus solved by $m_{22} = 1$. Finally, on the off diagonal we have the solution $m_{11} = 3/2$. Summarizing:

$$\mathbf{M} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \tag{10}$$

In more complicated cases we would be well served by constructing a set linear equations in the elements. All that remains is the check for positive definiteness of $\mathbf{M}$. The principal minors are:

$$\mu_1 = \frac{3}{2} > 0, \quad \mu_2 = \begin{vmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} = \frac{5}{4} > 0 \tag{11}$$

Both are positive: the system is stable.

Problem 2

a) The transfer function of the PID controller can be written as follows:

$$\begin{aligned} \mathbf{G}(s) = \frac{\hat{u}(s)}{\hat{e}(s)} &= K_P + \frac{K_I}{s} + \frac{K_D s}{\tau s + 1} \\ &= K_P + \frac{K_I}{s} + \frac{\frac{K_D}{\tau} s}{s + \frac{1}{\tau}} \\ &= K_P + \frac{K_I}{s} + \frac{K_D}{\tau}\left(1 - \frac{\frac{1}{\tau}}{s + \frac{1}{\tau}}\right) \\ &= \frac{K_I\left(s + \frac{1}{\tau}\right)}{s\left(s + \frac{1}{\tau}\right)} - \frac{\frac{K_D}{\tau^2} s}{s\left(s + \frac{1}{\tau}\right)} + K_P + \frac{K_D}{\tau} \\ &= \frac{\left(K_I - \frac{K_D}{\tau^2}\right) s + \frac{K_I}{\tau}}{s^2 + \frac{1}{\tau} s} + K_P + \frac{K_D}{\tau} \end{aligned}$$

We write the transfer function in the form:

$$\mathbf{G}(s) = \mathbf{G}_{sp}(s) + \mathbf{G}(\infty)$$

with

$$\mathbf{G}_{sp}(s) = \frac{\left(K_I - \frac{K_D}{\tau^2}\right)s + \frac{K_I}{\tau}}{s^2 + \frac{1}{\tau}s} \qquad \text{and} \qquad \mathbf{G}(\infty) = K_P + \frac{K_D}{\tau}$$

The strictly proper part $\mathbf{G}_{sp}(s)$ of the transfer function can be written as:

$$\mathbf{G}_{sp}(s) = \frac{1}{d(s)}\left[N_1 s + N_2\right]$$

with

$$N_1 = K_I - \frac{K_D}{\tau^2} \qquad \text{and} \qquad N_2 = \frac{K_I}{\tau}$$

where the denominator polynomial $d(s)$ is given by:

$$d(s) = s^2 + \alpha_1 s + \alpha_2$$

with

$$\alpha_1 = \frac{1}{\tau} \qquad \text{and} \qquad \alpha_2 = 0$$

Using the formulas in the appendix, we obtain the following realization:

$$\dot{\mathbf{z}}(t) = \underbrace{\begin{bmatrix} -\frac{1}{\tau} & 0 \\ 1 & 0 \end{bmatrix}}_{\mathbf{A}_c}\mathbf{z}(t) + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\mathbf{B}_c} e(t)$$

$$u(t) = \underbrace{\begin{bmatrix} K_I - \frac{K_D}{\tau^2} & \frac{K_I}{\tau} \end{bmatrix}}_{\mathbf{C}_c}\mathbf{z}(t) + \underbrace{\left(K_P + \frac{K_D}{\tau}\right)}_{D_c} e(t)$$

We see that the matrix $\mathbf{A}_c$ has dimensions 2×2, that the matrix $\mathbf{B}_c$ has dimensions 2×1, that the matrix $\mathbf{C}_c$ has dimensions 1×2, and that D_c is a scalar. Because the dimension of the realization $n = 2$ is equal to the order of the denominator of the transfer function (the term in the denominator with the highest power is s^2), we conclude that the realization is minimal, as requested.

b) The realization is <u>not</u> unique. Using the equivalence transformation $\bar{z}(t) = \mathbf{P}\mathbf{z}(t)$, where $\mathbf{P}$ is an invertible matrix, we obtain the realization

$$\dot{\bar{\mathbf{z}}}(t) = \mathbf{P}\mathbf{A}_c\mathbf{P}^{-1}\bar{\mathbf{z}}(t) + \mathbf{P}\mathbf{B}_c e(t)$$
$$u(t) = \mathbf{C}_c\mathbf{P}^{-1}\bar{\mathbf{z}}(t) + D_c e(t)$$

Because $\mathbf{P}$ can be chosen arbitrarily (as long as it is invertible), there exist infinitely many realizations of the PID controller. Therefore, there are infinitely many choices of the matrices $\mathbf{A}_c$, $\mathbf{B}_c$ and $\mathbf{C}_c$. However, the scalar D_c is the same for each realization.

c) The transfer function of the controller is given by:

$$\frac{\hat{u}(s)}{\hat{e}(s)} = K_P + \frac{K_I}{s} + K_D s = \frac{K_D s^2 + K_P s + K_I}{s}$$

The order of the numerator is two (the term with the highest power in the numerator is $K_D s^2$). The order of the denominator is one (the term with the highest power in the denominator is s). Because the order of the numerator is larger than the order of the denominator, the transfer function is not proper. Therefore, it is <u>not</u> possible to find a realization of this controller as in a).

d) By combining the equations of the system and the PID controller, we obtain:

$$\begin{aligned}
\dot{\mathbf{x}}(t) &= \mathbf{A}_s\mathbf{x}(t) + \mathbf{B}_s u(t) \\
&= \mathbf{A}_s\mathbf{x}(t) + \mathbf{B}_s\left(\mathbf{C}_c\mathbf{z}(t) + D_c e(t)\right) \\
&= \mathbf{A}_s\mathbf{x}(t) + \mathbf{B}_s\mathbf{C}_c\mathbf{z}(t) + \mathbf{B}_s D_c(r(t) - y(t)) \\
&= \mathbf{A}_s\mathbf{x}(t) + \mathbf{B}_s\mathbf{C}_c\mathbf{z}(t) + \mathbf{B}_s D_c r(t) - \mathbf{B}_s D_c\mathbf{C}_s\mathbf{x}(t) \\
&= \left(\mathbf{A}_s - D_c\mathbf{B}_s\mathbf{C}_s\right)\mathbf{x}(t) + \mathbf{B}_s\mathbf{C}_c\mathbf{z}(t) + D_c\mathbf{B}_s r(t)
\end{aligned}$$

and

$$\begin{aligned}
\dot{\mathbf{z}}(t) &= \mathbf{A}_c\mathbf{z}(t) + \mathbf{B}_c e(t) \\
\dot{\mathbf{z}}(t) &= \mathbf{A}_c\mathbf{z}(t) + \mathbf{B}_c(r(t) - y(t)) \\
\dot{\mathbf{z}}(t) &= \mathbf{A}_c\mathbf{z}(t) + \mathbf{B}_c r(t) - \mathbf{B}_c\mathbf{C}_s\mathbf{x}(t)
\end{aligned}$$

The system in closed-loop with the controller can therefore be written as

$$\begin{bmatrix}\dot{\mathbf{x}}(t)\\ \dot{\mathbf{z}}(t)\end{bmatrix} = \underbrace{\begin{bmatrix}\mathbf{A}_s - D_c\mathbf{B}_s\mathbf{C}_s & \mathbf{B}_s\mathbf{C}_c\\ -\mathbf{B}_c\mathbf{C}_s & \mathbf{A}_c\end{bmatrix}}_{\bar{\mathbf{A}}}\begin{bmatrix}\mathbf{x}(t)\\ \mathbf{z}(t)\end{bmatrix} + \begin{bmatrix}D_c\mathbf{B}_s\\ \mathbf{B}_c\end{bmatrix} r(t)$$

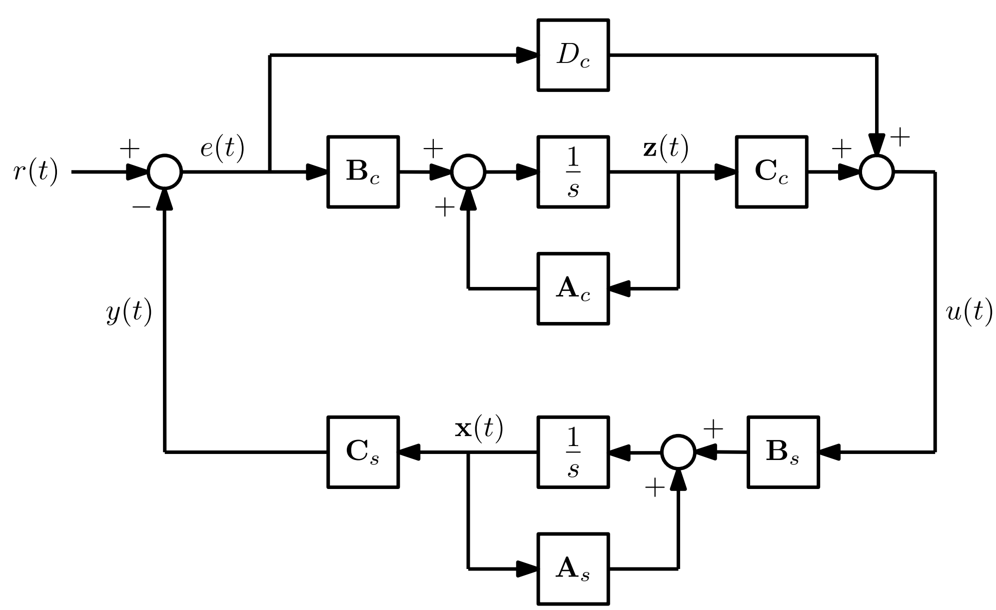
The eigenvalues of the closed-loop system can be obtained by determining the roots of the characteristic polynomial of the system. The characteristic polynomial is given by:

$$\det(\bar{\mathbf{A}} - \lambda\mathbb{I}) = \begin{vmatrix}\mathbf{A}_s - D_c\mathbf{B}_s\mathbf{C}_s - \lambda\mathbb{I} & \mathbf{B}_s\mathbf{C}_c\\ -\mathbf{B}_c\mathbf{C}_s & \mathbf{A}_c - \lambda\mathbb{I}\end{vmatrix}$$

Hence, the eigenvalues of the closed-loop system can be obtained from the equation

$$\begin{vmatrix}\mathbf{A}_s - D_c\mathbf{B}_s\mathbf{C}_s - \lambda\mathbb{I} & \mathbf{B}_s\mathbf{C}_c\\ -\mathbf{B}_c\mathbf{C}_s & \mathbf{A}_c - \lambda\mathbb{I}\end{vmatrix} = 0$$

e) The block diagram is given by:



Problem 3

We consider the differential equation:

$$\ddot{y}(t) + 4y(t) = \ddot{u}(t) + 2u(t) \tag{12}$$

a) The dynamics are to be expressed in terms of the state vector:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \dot{y}(t) - \dot{u}(t) \\ y(t) - u(t) \end{bmatrix} \tag{13}$$

Obviously $\dot{x}_2(t) = x_1(t)$. Rearrange the differential equation to obtain the expression for $\dot{x}_1(t)$:

$$[\ddot{y}(t) - \ddot{u}(t)] = -4y(t) + 2u(t) = -4[y(t) - u(t)] - 2u(t) \quad \Rightarrow \dot{x}_1(t) = -4x_2(t) - 2u(t) \tag{14}$$

The output is obtained by noting that:

$$y(t) = x_2(t) + u(t) \tag{15}$$

Summarizing our findings as a state-space model yields:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix}}^{\mathbf{A}} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \overbrace{-\begin{bmatrix} 2 \\ 0 \end{bmatrix}}^{\mathbf{B}} u(t) \tag{16}$$

$$y(t) = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{\mathbf{C}} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{1}_{D} u(t) \tag{17}$$

b) The impulse response of the system is obtained by:

$$h(t) = \mathcal{L}^{-1}\left\{ \frac{\hat{y}(s)}{\hat{u}(s)} \right\} \tag{18}$$

The transfer function may be computed easily as:

$$s^2\hat{y}(s) + 4\hat{y}(s) = s^2\hat{u}(s) + 2\hat{u}(s) \quad \Rightarrow \frac{\hat{y}(s)}{\hat{u}(s)} = \frac{s^2+2}{s^2+4} = 1 - \frac{2}{s^2+2^2} \tag{19}$$

Now, using the transform table:

$$h(t) = \mathcal{L}^{-1}\left\{ 1 - \frac{2}{s^2+2^2} \right\} = \delta(t) - \sin(2t) \tag{20}$$

c) For BIBO stability all the poles must have real parts in the LHP. In the present case the characteristic equation reads as:

$$s^2 + 4 = (s - 2j)(s + 2j) = 0 \tag{21}$$

The poles are on the imaginary axis; the system is not BIBO-stable.

Problem 4

We consider the system:

$$\begin{aligned} \dot{x}(t) &= -3x(t) + u(t) \\ y(t) &= 2x(t), \end{aligned}$$

with input:

$$u(t) = \begin{cases} 1, & \text{if } 0 \leq t < 1 \\ 0, & \text{if } t \geq 1. \end{cases}$$

The initial condition is given as $x(0) = 0$. The general formulation for scalar systems is:

$$y(t) = ce^{at}x_0 + c\int_0^t e^{a(t-\tau)}bu(\tau)\ \mathrm{d}\tau \tag{22}$$

In our case it will be practical to write the above as:

$$y(t) = ce^{at}x_0 + c\int_0^t e^{a(t-\tau)}bu(\tau)\ \mathrm{d}\tau, \quad \text{if } 0 \leq t < 1 \tag{23}$$

$$y(t) = ce^{a(t-1)}x_1 + c\int_1^t e^{a(t-\tau)}bu(\tau)\ \mathrm{d}\tau, \quad \text{if } t \geq 1 \tag{24}$$

$$\tag{25}$$

Inserting constants and initial conditions yields the integrals:

$$y(t) = 2\int_0^t e^{-3(t-\tau)}\ \mathrm{d}\tau, \quad \text{if } 0 \leq t < 1 \tag{26}$$

$$y(t) = 2e^{-3(t-1)}x_1, \quad \text{if } t \geq 1 \tag{27}$$

$$\tag{28}$$

The former integral may be computed to be:

$$y(t) = \int_0^t 2e^{-3(t-\tau)}u(\tau)\ \mathrm{d}\tau = \frac{2}{3}\left[e^{3(\tau-t)}\right]_0^t = \frac{2}{3}(1-e^{-3t}) \tag{29}$$

The state value at $t = 1$ is thus:

$$x_1 = \frac{y(1)}{2} = \frac{1}{3}(1-e^{-3}) \tag{30}$$

$$y(t) = \frac{2}{3}(1-e^{-3t}), \quad \text{if } 0 \leq t < 1 \tag{31}$$

$$y(t) = \frac{2}{3}(1-e^{-3})e^{-3(t-1)}, \quad \text{if } t \geq 1 \tag{32}$$

$$\tag{33}$$

Now:

$$y(2) = \frac{2}{3}(1-e^{-3})e^{-3(2-1)} \simeq 0.0315 \tag{34}$$

Problem 5

a) The processes are called 'white noise'. They are characterized by having a constant power spectral density. They have a zero mean, an infinite variance (in continuous time) and are stationary.

b) The autocorrelation functions of the white noise processes are simply:

$$R_v(\tau) = \mathcal{F}^{-1}[S_v(j\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_v(j\omega)e^{j\omega\tau}\ \mathrm{d}\omega = 2\delta(\tau) \tag{35}$$

$$R_w(\tau) = \mathcal{F}^{-1}[S_w(j\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_w(j\omega)e^{j\omega\tau}\ \mathrm{d}\omega = \delta(\tau) \tag{36}$$

c) The first step is to compute frequency response of the output $y(t)$. The model is:

$$\dot{x}(t) = -x(t) + 2w(t)$$
$$z(t) = x(t) + v(t),$$

Laplace-transformation yields:

$$\hat{z}(s) = G(s)\hat{w}(s) + \hat{v}(s), \quad G(s) = \frac{2}{1+s} \tag{37}$$

The power spectral density may now be computed as:

$$S_z(j\omega) = G(j\omega)G(-j\omega)S_w(j\omega) + S_v(j\omega) = \frac{4}{\omega^2+1} + 2 \tag{38}$$

(We may take a sum since $v(t)$ and $w(t)$ are uncorrelated processes).

d) The general Riccati differential equation is:

$$\dot{\hat{P}}(t) = F\hat{P}(t) + \hat{P}(t)F^T + G\tilde{Q}G^T - \hat{P}(t)H^T\tilde{R}^{-1}H\hat{P}(t) \tag{39}$$

Examining the model we find that:

$$F = -1, \quad G = 2, \quad H = 1 \tag{40}$$

The covariance parameters $\tilde{Q}$ and $\tilde{R}$ are given by:

$$E[w(t)w(\tau)] = \tilde{Q}\delta(t-\tau) \tag{41}$$
$$E[v(t)v(\tau)] = \tilde{R}\delta(t-\tau) \tag{42}$$

For the present case these are provided as:

$$\tilde{Q} = 1, \quad \tilde{R} = 2 \tag{43}$$

We do however note that the spectral densities from task b) yield the same values. Upon insertion we have:

$$\dot{\hat{P}}(t) = -2\hat{P}(t) - \frac{1}{2}\hat{P}^2(t) + 4 \tag{44}$$

e) The stationary value of the error covariance matrix is obtained by setting $\dot{\hat{P}}(t) = 0$, which yields

$$-\frac{1}{2}\hat{P}^2(t) - 2\hat{P}(t) + 4 = 0 \tag{45}$$

From this equation, we obtain that

$$\hat{P}(t) = \frac{2 \pm \sqrt{2^2 - 4(-\frac{1}{2})4}}{2\left(-\frac{1}{2}\right)} = -2 \mp \sqrt{4+8} = -2 \mp 2\sqrt{3} \tag{46}$$

Because the error covariance matrix is positive semidefinite by definition, $\hat{P}(t)$ must be larger than or equal to zero. Therefore, we have that the stationary value of the error covariance matrix is given by

$$\hat{P}(t) = 2\sqrt{3} - 2 \tag{47}$$

e) The initial condition is $\hat{P}(0) = 2\sqrt{3} - 2$. To find the slope at $t = 0$, insert the above to obtain:

$$\dot{\hat{P}}(0) = -2(2\sqrt{3}-2) - \frac{(2\sqrt{3}-2)^2}{2} + 4 = 0 \tag{48}$$

There is no slope, the equation is hence in equilibrium: $\hat{P}(t) = 2\sqrt{3} - 2$. The Kalman gain is generally:

$$K(t) = \hat{P}(t)H^T\tilde{R}^{-1} \tag{49}$$

In the present case:

$$K(t) = \sqrt{3} - 1 \tag{50}$$

Problem 6

We consider the stochastic process:

$$z_k = x + v_k \tag{51}$$

where x is an unknown constant and v_k is a normally distributed zero-mean white noise process. An estimate of the unknown constant is:

$$\hat{x}_k = \frac{1}{1+k}\sum_{i=0}^{k} z_i \tag{52}$$

a) The expected value of the estimate is:

$$E[\hat{x}_k] = E\left[\frac{1}{1+k}\sum_{i=0}^{k}(x+v_i)\right] = \frac{1}{1+k}\sum_{i=0}^{k}E[x+v_i] = \frac{1}{1+k}\sum_{i=0}^{k}x + E[v_i] = \frac{1}{1+k}\sum_{i=0}^{k}x = x \tag{53}$$

where we have used the fact that v_k is a zero-mean process.

b) This task involves conversion of the infinite sum in the given estimate into a recursion:

$$\hat{x}_{k+1} = \hat{x}_k + K_{k+1}(z_{k+1} - \hat{x}_k) \tag{54}$$

Let us write the estimate (52) for the step $k+1$:

$$\hat{x}_{k+1} = \frac{1}{2+k}\sum_{i=0}^{k+1} z_i \tag{55}$$

Now, the estimate update must be:

$$\begin{aligned}
\hat{x}_{k+1} - \hat{x}_k &= \left(\frac{1}{2+k}\sum_{i=0}^{k+1} z_i\right) - \left(\frac{1}{1+k}\sum_{i=0}^{k} z_i\right) \\
&= \left(\frac{1}{2+k}\sum_{i=0}^{k} z_i + \frac{1}{2+k}z_{k+1}\right) - \left(\frac{1}{1+k}\sum_{i=0}^{k} z_i\right) \\
&= \frac{1}{2+k}z_{k+1} + \left(\frac{1}{2+k} - \frac{1}{1+k}\right)\sum_{i=0}^{k} z_i
\end{aligned} \tag{56}$$

Here:

$$\frac{1}{2+k} - \frac{1}{1+k} = -\frac{1}{(2+k)(1+k)} \tag{57}$$

and so:

$$\hat{x}_{k+1} - \hat{x}_k = \frac{1}{2+k}z_{k+1} - \frac{1}{(2+k)(1+k)}\sum_{i=0}^{k} z_i = \frac{1}{2+k}\left(z_{k+1} - \hat{x}_k\right) \tag{58}$$

Upon comparison, it is easy to see that:

$$K_{k+1} = \frac{1}{2+k} \tag{59}$$

c) The covariance at step k is:

$$\begin{aligned}
\hat{P}_k = E[(x-\hat{x}_k)^2] &= E\left[\left(x - \frac{1}{1+k}\sum_{i=0}^{k}(x+v_i)\right)^2\right] \\
&= E\left[\left(\frac{1}{1+k}\sum_{i=0}^{k} v_i\right)^2\right] = \left(\frac{1}{1+k}\right)^2 E\left[\sum_{i=0}^{k}\sum_{j=0}^{k} v_j v_i\right] = \left(\frac{1}{1+k}\right)^2 \sum_{i=0}^{k}\sum_{j=0}^{k} E\left[v_j v_i\right]
\end{aligned} \tag{60}$$

Consider now the discrete autocorrelation appearing above. Utilizing the given variance we must have:

$$E\left[v_j v_i\right] = R\delta_{ij} \tag{61}$$

Now:

$$\sum_{i=0}^{k}\sum_{j=0}^{k} E\left[v_j v_i\right] = \sum_{i=0}^{k}\sum_{j=0}^{k} R\delta_{ij} = \sum_{i=0}^{k} R = (1+k)R \tag{62}$$

Hence:

$$\hat{P}_k = \left(\frac{1}{1+k}\right)^2 (1+k)R = \frac{R}{1+k} \tag{63}$$

d) A recursion must again be found, this time for $\hat{P}_k$. By reverse-engineering, it is easy to show that:

$$\hat{P}_{k+1} = \left(\frac{k+1}{k+2}\right)^2 \hat{P}_k + \frac{1}{(k+2)^2}R = \frac{k+1}{(k+2)^2}R + \frac{1}{(k+2)^2}R = \frac{1}{k+2}R = \hat{P}_{k+1} \tag{64}$$