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Problem 1

a) $y[n] = ay[n-1] - ax[n] + x[n-1]$

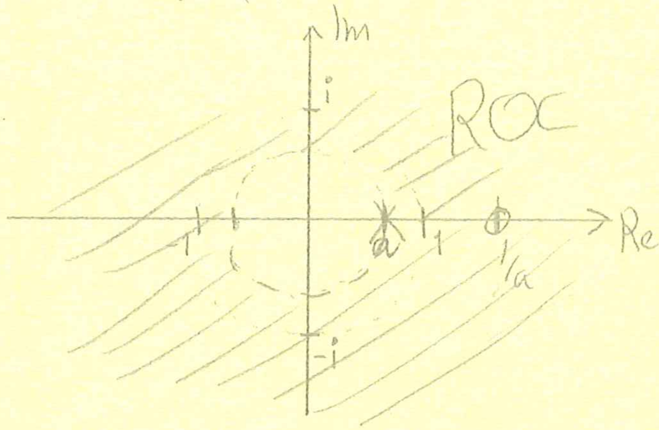
$\Rightarrow Y(z) = az^{-1}Y(z) - aX(z) + z^{-1}X(z)$

$\Leftrightarrow Y(z)(1-az^{-1}) = X(z)(-a+z^{-1})$

$$\Leftrightarrow H_1(z) = \frac{Y(z)}{X(z)} = \frac{-a+z^{-1}}{1-az^{-1}}$$

IIR filter since it has poles.

b) The filter has pole at $z=a$ and zero at $z=\frac{1}{a}$.
Drawn with $|a|<1$.



Since it is causal we have ROC: $|z|>|a|$
It is stable when $|a|<1$ since then the unit circle is in ROC.
Can't have minimum phase if we require stability since then the pole of $H_I(z) = H_1^{-1}(z)$ will be outside the unit circle.

c) With $a = \frac{1}{2}$ we have

$$H_1(z) = \frac{-\frac{1}{2} + z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

$$\Rightarrow H_1(f) = \frac{-\frac{1}{2} + e^{-j2\pi f}}{1 - \frac{1}{2}}$$

$$\Rightarrow |H_1(f)|^2 = \frac{\left|-\frac{1}{2} + \cos(2\pi f) - j\sin(2\pi f)\right|^2}{\left|1 - \frac{1}{2}\cos(2\pi f) + \frac{j}{2}\sin(2\pi f)\right|^2}$$

$$= \frac{\left(-\frac{1}{2} + \cos(2\pi f)\right)^2 + \sin(2\pi f)^2}{\left(1 - \frac{1}{2}\cos(2\pi f)\right)^2 + \left(\frac{1}{2}\sin(2\pi f)\right)^2}$$

$$= \frac{\frac{1}{4} - \cos(2\pi f) + 1}{1 - \cos(2\pi f) + \frac{1}{4}}$$

$$\underline{= 1}$$

$H_1(f)$ with $a = 0.5$ passes all frequencies so it is an all pass.

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d) $H(z) = H_1(z)\,H_2(z) = H_3(z) + H_4(z)$

$$= \frac{-\frac{1}{2} + z^{-1}}{1 - \frac{1}{2}z^{-1}} \cdot \frac{1}{1+\frac{1}{2}z^{-1}} = \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1+\frac{1}{2}z^{-1}}$$

$$\Rightarrow -\frac{1}{2} + z^{-1} = A\left(1 + \frac{1}{2}z^{-1}\right) + B\left(1 - \frac{1}{2}z^{-1}\right)$$

$$\Rightarrow A + B = -\frac{1}{2} \quad , \quad \frac{A}{2} - \frac{B}{2} = 1$$

$$\Rightarrow A = \frac{3}{4} \quad , \quad B = -\frac{5}{4}$$

This gives

$$h[n] = \frac{3}{4} Z^{-1}\left\{\frac{1}{1-\frac{1}{2}z^{-1}}\right\} - \frac{5}{4} Z^{-1}\left\{\frac{1}{1-(-\frac{1}{2})z^{-1}}\right\}$$

$$= \underline{\underline{\frac{3}{4}\left(\frac{1}{2}\right)^n u[n] - \frac{5}{4}\left(-\frac{1}{2}\right)^n u[n]}}$$

e) Stability:

If $x_{max} = \max\{x[n]\} < \infty$ then $y[n] < \infty$ for all n.

Causality:

$h[n] = 0$ for all $n < 0$.

$y[n]$ doesn't depend on $x[n+k]$ for any $k > 0$.

(or $y[n+k-1]$)

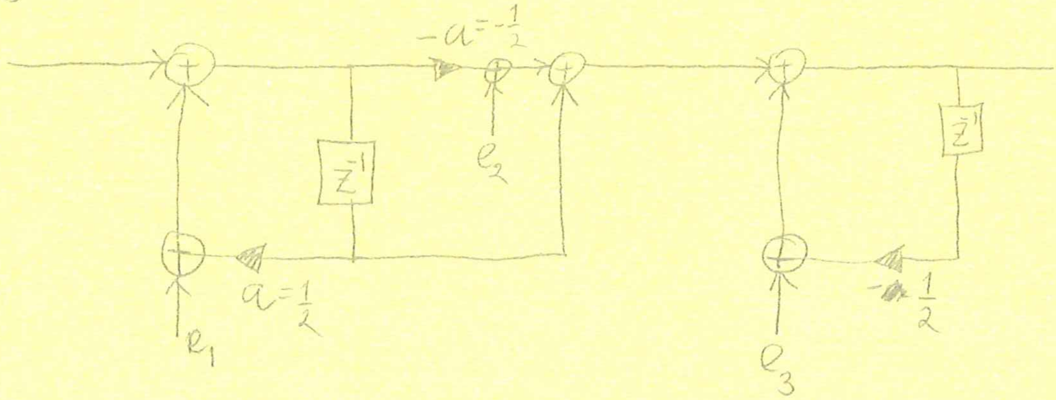
Time invariance:

$$y[n+k] = \underline{h[n]} * x[n+k] \quad \text{for all } K$$

no K here.

Problem 2

a)



Since $e_1[n]$ is passed through the entirity of $H_1(z)$, it wont be affected by it. All noise sources are thus only affected by $H_2(z)$.

$$\sigma_z^2 = 3 \cdot \sigma_e^2 \cdot \sum h_2[n]^2$$

$$= 3\sigma_e^2 \sum \left(-\frac{5}{4}\left(-\frac{1}{2}\right)^n u[n]\right)^2$$

$$= \frac{75}{16}\sigma_e^2 \sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n$$

$$= \frac{75}{16}\cdot\frac{1}{1-\frac{1}{4}}\cdot\sigma_e^2$$

$$= \underline{\underline{\frac{25}{4}\sigma_e^2}}$$

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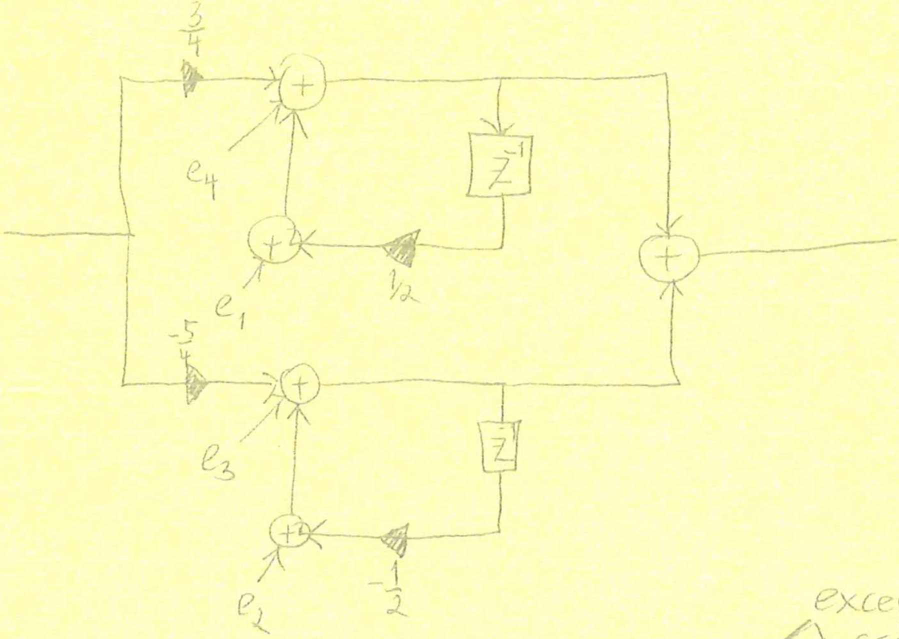
b)



e_1 and e_4 go through all of H_3; while e_2 and e_3 go through all of H_4.

except scaling factor!

From e_1, e_4 we get:

$$\sigma^2_{q,14} = 2\sigma_e^2 \cdot \sum h_3[n]^2$$

$$= 2\sigma_e^2 \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n$$

$$= 2 \cdot \frac{1}{1-\frac{1}{4}} \sigma_e^2$$

$$= \frac{8}{3} \sigma_e^2$$

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From e_2 and e_3 we get:

$$\sigma_{q,23}^2 = 2 \cdot \sigma_e^2 \sum_n h_4[n]^2$$

$$= 2\, \sigma_e^2 \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^{2n}$$

$$= 2 \cdot \frac{1}{1-\frac{1}{4}}\, \sigma_e^2$$

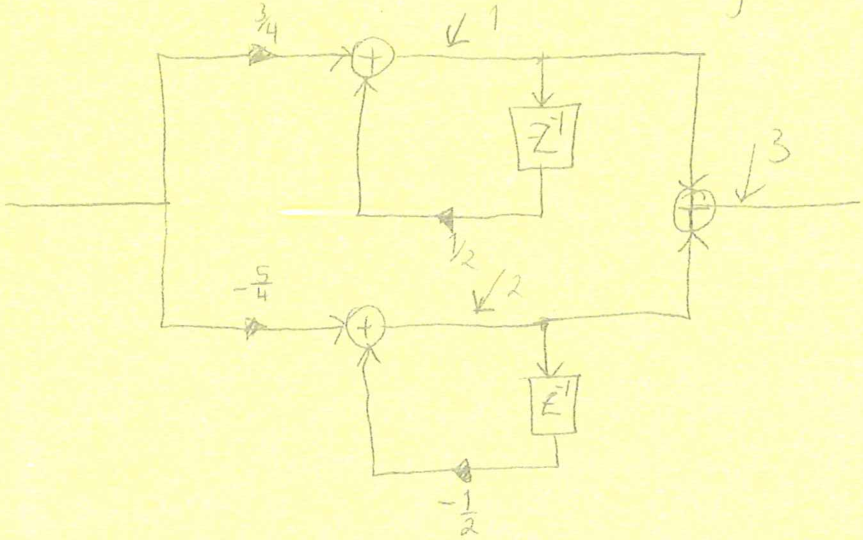
$$= \frac{8}{3} \sigma_e^2$$

So the total noise power at output is

$$\sigma_z^2 = \sigma_{q,14}^2 + \sigma_{q,23}^2$$

$$= \underline{\underline{\frac{16}{3} \sigma_e^2}}$$

c) Overflow can occour after each addition, so we have three locations to investigate.



Let $h_i'[n]$ denote the unit impulse from the start (input) to the position marked i.

1: $\sum |h_1'[n]| = \sum_{n=0}^{\infty} \frac{3}{4}\left(\frac{1}{2}\right)^n = \frac{3}{2}$

2: $\sum |h_2'[n]| = \sum_{n=0}^{\infty} \frac{5}{4}\left(\frac{1}{2}\right)^n = \frac{5}{2}$

3: $\sum |h_3'[n]| = \sum |h[n]|$

$$= \sum_{n=0}^{\infty} \left|\frac{3}{4}\left(\frac{1}{2}\right)^n - \frac{5}{4}\left(-\frac{1}{2}\right)^n\right|$$

$$= \sum_{n=0}^{\infty} \left|\frac{3}{4}\left(\frac{1}{2}\right)^n - \frac{5}{4}(-1)^n\left(\frac{1}{2}\right)^n\right|$$

$$= \sum_{n=0}^{\infty} \left|\left(\frac{1}{2}\right)^n \left(\frac{3}{4} - (-1)^n \frac{5}{4}\right)\right|$$

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$$\Rightarrow \sum |h_3'[n]| = \frac{1}{2}\sum_{\substack{n \text{ even} \\ n=0}}^{\infty} \left(\frac{1}{2}\right)^n + 2\sum_{\substack{n \text{ odd} \\ n=1}}^{\infty} \left(\frac{1}{2}\right)^n$$

$$= \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^k + \sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^k$$

$$= \left(\frac{1}{2}+1\right)\frac{1}{1-\frac{1}{4}}$$

$$= 2$$

Since $\frac{5}{2} > 2 > \frac{3}{2}$, we will avoid overflow by scaling the signal by a factor of $\frac{5}{2}$, ie. multiplying by $\underline{\underline{\frac{2}{5}}}$.

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Problem 3

a) Since $H_1(z)$ is an allpass, we can model $x[n]$ as an AR(1) process (order = 1).

The spectrum is given by

$$\Gamma_{xx}(f) = |H_2(f)|^2 \, \Gamma_{ww}(f)$$

$$= \sigma_w^2 \, |H_2(f)|^2$$

$$|H_2(f)|^2 = \frac{1}{\left|1 + \frac{1}{2} e^{-j2\pi f}\right|^2}$$

$$= \frac{1}{\left|1 + \frac{1}{2}\cos(2\pi f) - \frac{1}{2} j \sin(2\pi f)\right|^2}$$

$$= \frac{1}{\left(1 + \frac{1}{2}\cos(2\pi f)\right)^2 + \left(\frac{1}{2}\sin(2\pi f)\right)^2}$$

$$= \frac{1}{1 + \cos(2\pi f) + \frac{1}{4}\underbrace{\left(\cancel{\cos^2(2\pi f) + \sin^2(2\pi f)}\right)}_{=1}}$$

$$= \frac{1}{\frac{5}{4} + \cos(2\pi f)}$$

$$\Rightarrow \quad \Gamma_{xx}(f) = \frac{\sigma_w^2}{\frac{5}{4} + \cos(2\pi f)}$$

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b) Want to model $x[n]$ as

$$x[n] = a_1 x[n-1] + w[n].$$

Yule-Walker equations give optimal solution:

$$\begin{pmatrix} \gamma_{xx}[0] & \gamma_{xx}[-1] \\ \gamma_{xx}[1] & \gamma_{xx}[0] \end{pmatrix} \begin{pmatrix} 1 \\ a_1 \end{pmatrix} = \begin{pmatrix} \sigma_f^2 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} a_1 = -\dfrac{\gamma_{xx}[1]}{\gamma_{xx}[0]} \\ \sigma_f^2 = \gamma_{xx}[0] + a_1 \gamma_{xx}[-1] \end{cases}$$

With $\gamma_{xx}[l] = \sigma_w^2 \dfrac{(-0.5)^{|l|}}{1-0.5^2}$ and $\sigma_w^2 = 1$

we get

$$a_1 = -\frac{2/3}{4/3} = -\frac{1}{2}$$

$$\sigma_f^2 = \left(\frac{4}{3} - \frac{1}{2} \cdot \frac{2}{3}\right)\sigma_w^2 = 1 \cdot \sigma_w^2 = \underline{1}$$

Best AR(1) model:

$$\underline{\underline{x[n] = -\frac{1}{2} x[n-1] + w[n]}}$$

prediction error power: $\underline{\underline{\sigma_f^2 = 1}}$

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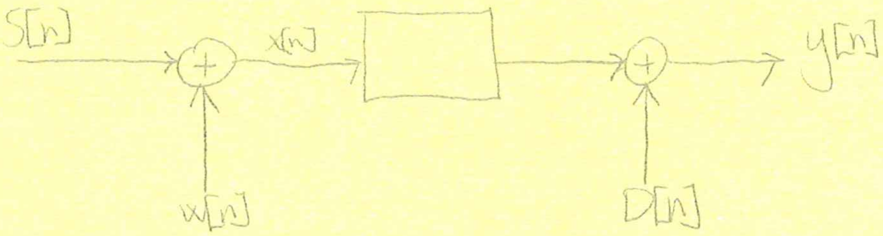
The optimal order for the case when ~~a~~ $a = \frac{1}{2}$ is 1 since it is the simplest model given that $H_1(z)$ is an all pass.

c) The problem that Wiener filtering solves is illustrated below:



We have some signal of interest ~~s~~ S[n] which is assumed to have zero mean. This gets polluted by noise w[n] and passed through a system.

On the other side we have an idea of what the desired signal should look like D[n], and Wiener filtering provides general approach to estimating S[n] from x[n] and D[n] with minimal square error, by using a linear filter.

Mai 2017

Svanemerket trykksak fra Skipnes Kommunikasjon Lisensnr. 241 731

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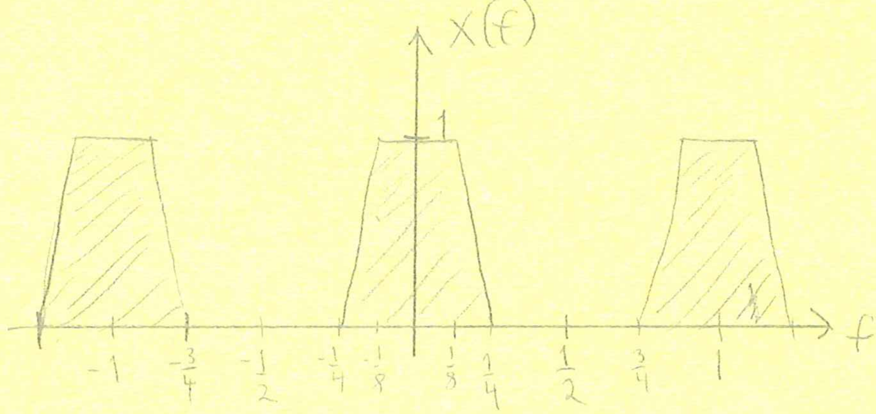
Problem 4

a) Sampling frequency $F_x = \frac{1}{T_x} = 8\text{ kHz}$

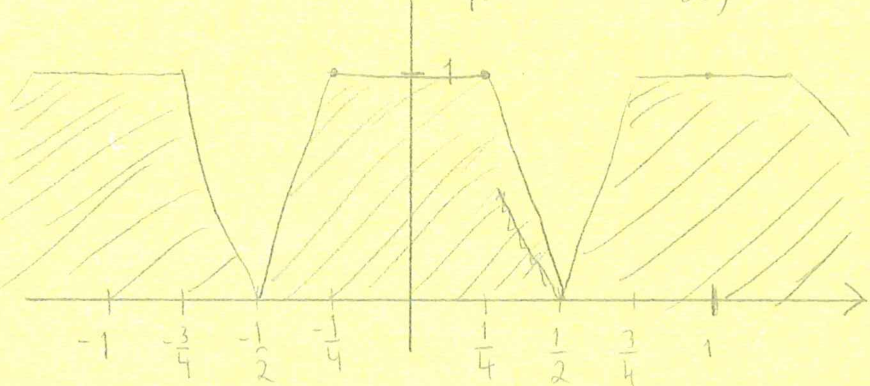
$$\Rightarrow \begin{cases} 2\text{ kHz} \Leftrightarrow f = \frac{1}{4} \\ 1\text{ kHz} \Leftrightarrow f = \frac{1}{8} \end{cases}$$

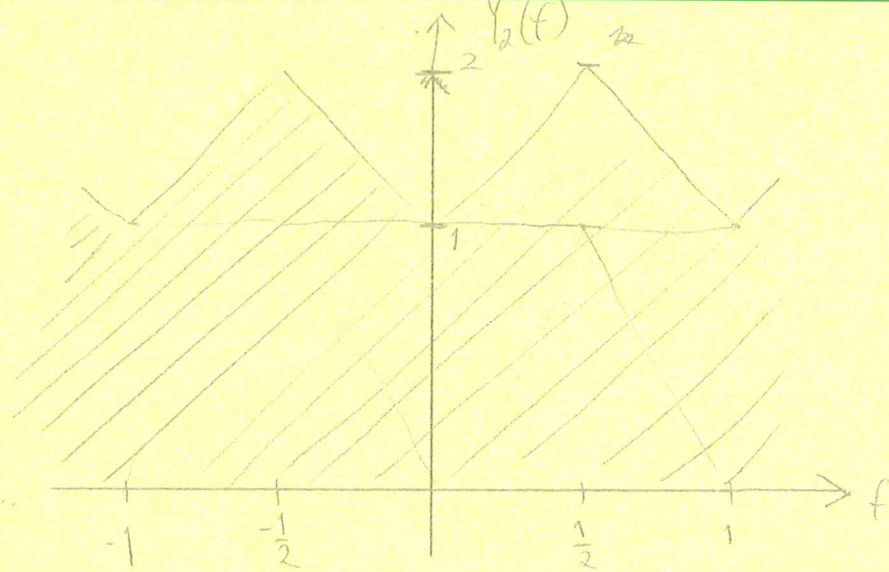


b) $y_1[m] = x[2m]$ corresponds to decimation by factor of 2,

$y_2[m] = x[4m]$ ——— " ——— of 4.

$Y_1(f) = X\left(\frac{f}{2}\right)$,





Spectrum completely changed for y_2 so can't serve as a replacement.

Spectrum of $y_1[n]$ is quite similar and no aliasing troubles occoured there so we should send $y_1[n]$ to our email account.

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c) Yes it is possible since Nyquists criteria was still obeyed for y_1.

~~We first filter away the~~

We first interpolate to restore the original frequency of 8 kHz. Then we filter away the parts of the spectrum that are irrelevant using an LPF, ideally

$$H(f) = \begin{cases} 1, & |f| \leq \frac{1}{2 \cdot 2} = \frac{1}{4} \\ 0, & \text{otherwise} \end{cases}$$

$$y_1[m] \rightarrow \underset{I=2}{\boxed{I\uparrow}} \rightarrow \boxed{H(f)} \rightarrow x[n]$$

d) Minimum size for $x[n]$ is $\underline{M = 1090}$

Minimum size for $h[n]$ is $\underline{L = 52}$

Minimum size for $y[n]$ is $M+L-1 = \underline{1141}$

Most suitable length is smallest power 2 larger than $M+L-1$

$$\underline{N_{FFT} = 2^{11} = 2048}$$