Matte 4K, Øving 3

Gruppe 2
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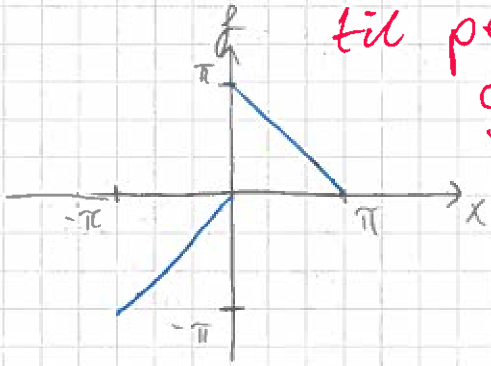
(Ønsker tilbakemelding :)

Bra øving! Virker som om du forstår egenskapene til periodiske funksjoner godt.

11.1:

9) $f(x) = \begin{cases} x & , -\pi < x < 0 \\ \pi - x & , 0 < x < \pi \end{cases}$



14) $f(x) = x^2$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{2\pi}\cdot\frac{1}{3}\cdot\left(\pi^3 - (-\pi)^3\right)$$

$$= \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)\,dx \quad \Big|\cdot\pi$$

$$\pi a_n = \frac{x^2\sin(nx)}{n}\Bigg|_{-\pi}^{\pi} - \frac{2}{n}\int_{-\pi}^{\pi} x\sin(nx)\,dx$$

$$= \frac{2x\cos(nx)}{n^2}\Bigg|_{-\pi}^{\pi} - \cancel{\frac{1}{n^2}\int_{-\pi}^{\pi}\cos(nx)\,dx} = 0$$

Merk at $\cos(n\pi) = (-1)^n$ for $n = 0, \pm 1, \pm 2, \ldots$
så

$$\pi a_n = \frac{2\pi(-1)^n + 2\pi(-1)^n}{n^2}$$

$$\Leftrightarrow a_n = \frac{4(-1)^n}{n^2}$$

Siden $\sin(nx)$ er en odde funksjon og $f(x) = x^2$ er en like,

$$\Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin(nx)\,dx = 0$$

Så

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nx) \quad R$$

Som gir at de første 5 leddene er

$$S_5(x) = \frac{\pi^2}{3} - 4\cos x + \cos(2x) - \frac{4}{9}\cos(3x)$$

$$+ \frac{1}{4}\cos(4x) - \frac{4}{25}\cos(5x)$$

19)

$$f(x) = \begin{cases} 0 & , -\pi < x < 0 \\ x & , 0 < x < \pi \end{cases}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{0}^{\pi} x\,dx = \frac{1}{4\pi}(x^2)\Big|_0^{\pi}$$

$$= \frac{\pi}{4}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} x\cos(nx)\,dx$$

$$= \frac{x\sin(nx)}{\pi n}\Big|_0^{\pi} - \frac{1}{\pi n}\int_0^{\pi}\sin(nx)\,dx$$

$$= \frac{1}{\pi n^2}\cos(nx)\Big|_0^{\pi}$$

$$= \frac{(-1)^n}{\pi n^2} - \frac{1}{\pi n^2} = \frac{(-1)^n - 1}{\pi n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} x\sin(nx)\,dx$$

$$= \frac{-x\cos(nx)}{\pi n}\Big|_0^{\pi} + \frac{1}{\pi n}\int_0^{\pi}\cos(nx)\,dx$$

$$= \frac{-\pi(-1)^n}{\pi n}$$

$$= \frac{(-1)^1(-1)^n}{n} = \frac{(-1)^{n+1}}{n}$$

So $f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{\pi n^2}\cos(nx) - \frac{(-1)^n}{n}\sin(nx)\right]$

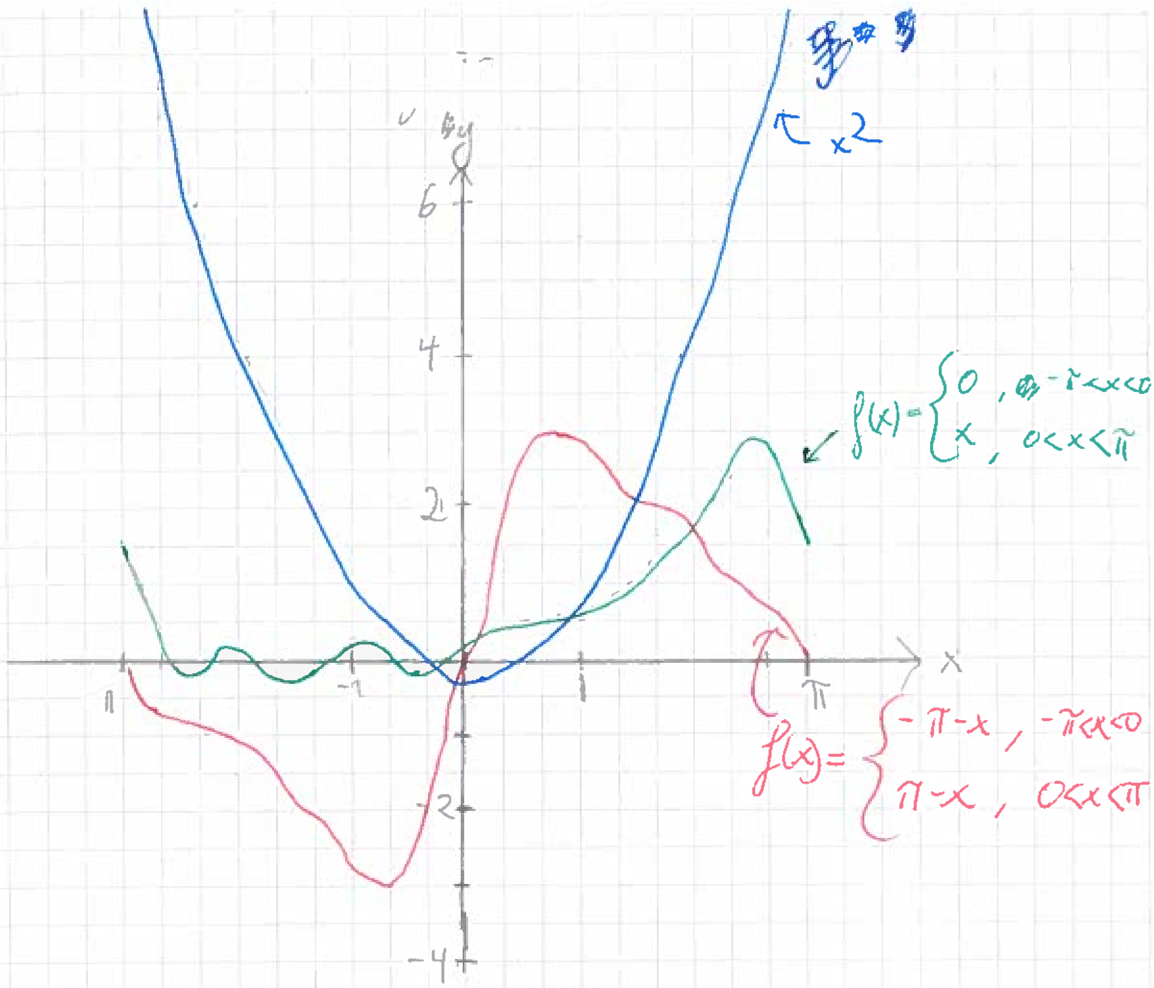
R

$\Rightarrow$

$$S_5(x) = \frac{\pi}{4} - \frac{2}{\pi}\cos(x) + \sin x - \frac{1}{2}\sin(2x)$$

$$- \frac{2}{9\pi}\cos(3x) + \frac{1}{3}\sin(3x) - \frac{1}{4}\sin(4x)$$

$$- \frac{2}{25\pi}\cos(5x) + \frac{1}{5}\sin(5x)$$



21)

$$f(x)=\begin{cases}-\pi-x, & -\pi<x<0\\ \pi-x, & 0<x<\pi\end{cases}$$

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx=0$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{0}(\pi+x)\cos(nx)\,dx+\frac{1}{\pi}\int_{0}^{\pi}(\pi-x)\cos(nx)\,dx$$

$$=\left.\frac{-(\pi+x)\sin(nx)}{\pi n}\right|_{-\pi}^{0}+\frac{1}{\pi n}\int_{-\pi}^{0}\sin(nx)\,dx$$

$$+\left.\frac{(\pi-x)\sin(nx)}{\pi n}\right|_{0}^{\pi}\ \frac{1}{\pi n}\int_{0}^{\pi}\sin(nx)\,dx$$

$$=0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$= -\frac{1}{\pi}\int_{-\pi}^{0}(\pi+x)\sin(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi}(\pi-x)\sin(nx)\,dx$$

$$= -\int_{-\pi}^{0}\sin(nx)\,dx - \frac{1}{\pi}\int_{-\pi}^{0} x\sin(nx)\,dx + \int_{0}^{\pi}\sin(nx)\,dx$$

$$-\frac{1}{\pi}\int_{0}^{\pi} x\sin(nx)\,dx$$

Siden $\sin(nx)$ er odde vil $-\int_{-\pi}^{0}\sin(nx)\,dx = \int_{0}^{\pi}\sin(nx)\,dx$

$$\Rightarrow b_n = 2\cdot\int_{0}^{\pi}\sin(nx)\,dx - \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx$$

$$= -\frac{2}{n}\cos(nx)\Big|_{0}^{\pi} + \frac{x\cos(nx)}{\pi n}\Big|_{-\pi}^{\pi} - \frac{1}{\pi n}\int_{-\pi}^{\pi}\cos(nx)\,dx$$

$$= -\frac{2}{n}(-1)^n + \frac{2}{n} + \frac{\pi(-1)^n}{\pi n} - \frac{(-\pi)(-1)^n}{\pi n}$$

$$= -\frac{2}{n}(-1)^n + \frac{2}{n} + \frac{(-1)^n}{n} + \frac{(-1)^n}{n}$$

$$= \frac{2}{n}$$ R

og svaret: $f(x) = \sum_{n=1}^{\infty} \frac{2}{n}\sin(nx)$

11.2:

11) $f(x) = x^2$, $(-1 < x < 1)$, $p = 2$

f is even so it has a Fourier cosine series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{\pi n}{L} x\right)$$

where $L = p/2 = 1$

and $a_n = \frac{1}{L} \int_{-L}^{L} x^2 \cos\left(\frac{\pi n}{L} x\right) dx$

$$= \int_{-1}^{1} x^2 \cos(\pi n x) dx$$

$$= \left. \frac{x^2 \sin(\pi n x)}{\pi n} \right|_{-1}^{1} - \frac{1}{\pi n} \int_{-1}^{1} x \sin(\pi n x) dx$$

$$= \left. \frac{x \cos(\pi n x)}{\pi^2 n^2} \right|_{-1}^{1} - \frac{1}{\pi^2 n^2} \int_{-1}^{1} \cos(\pi n x) dx$$

$$= \frac{1 \cdot (-1)^n + 1 \cdot (-1)^n}{\pi^2 \cdot n^2}$$

$$= \frac{2(-1)^n}{\pi^2 n^2}$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$= \frac{1}{2}\int_{-1}^{1} x^2\,dx$$

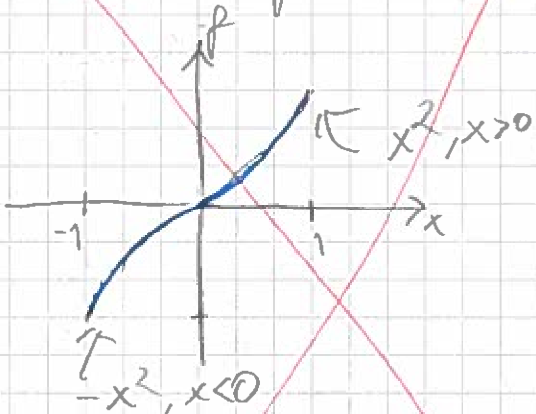
$$= \int_{0}^{1} x^2\,dx$$

$$= \frac{1}{3}$$

So $f(x) = \frac{1}{3} + \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nx)$ R

16) $f(x) = x|x|$, $(-1 < x < 1)$, $p = 2 \iff L = 1$

We'll first sketch f



$f(-x) = (-x)|-x| = -f(x)$ so f is odd.

Since f is odd it has a Fourier sine series.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(\pi n x)$$

17) $f(x) = 1 - |x| \quad , (-1 < x < 1) \ , p = 2 \ , L = 1$

$f(-x) = 1 - |-x| = 1 - |x| = f(x)$, f is even

$$\Rightarrow \quad f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\pi n x)$$

$$a_0 = \int_0^1 1 - |x| \, dx = \int_0^1 1 - x \, dx$$

$$= \left[x - \frac{x^2}{2} \right]_0^1 = 1/2$$

$$a_n = 2 \int_0^1 (1 - x) \cos(\pi n x) \, dx$$

$$= -2 \int_0^1 x \cos(\pi n x) \, dx$$

$$= \left. \frac{-2x \sin(\pi n x)}{\pi n} \right|_0^1 + \frac{2}{\pi n} \int_0^1 \sin(\pi n x) dx$$

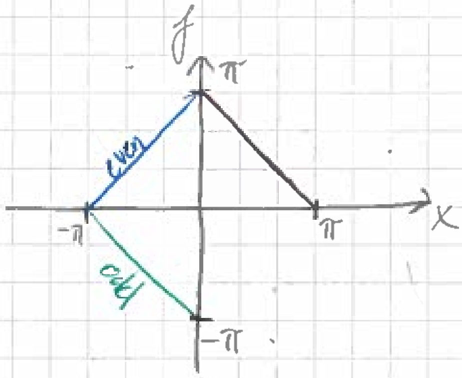
$$= \left. -\frac{2}{\pi^2 n^2} \cos(\pi n x) \right|_0^1$$

$$= -\frac{2}{\pi^2 n^2} \left((-1)^n - 1 \right)$$

$$\Rightarrow f(x) = \frac{1}{2} - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos(\pi n x)$$

R

25)



Cosine series:

Same as 11.2.17 except a scaling factor of π and "stretch" of π.

This gives

$$f_{even}(x) = \frac{\pi}{2} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos(nx)$$ R

Sine series:

Same as 11.1.21:

$$a_0 = a_n = 0, \quad b_n = \frac{2}{n}$$

$$f_{odd}(x) = \sum_{n=1}^{\infty} \frac{2}{n} \sin(nx)$$ R

13.1:

$z_1 = -2+5i \quad , \quad z_2 = 3-i$

a) $\mathrm{Re}(z_1^2) = \mathrm{Re}(-5^2 - 4 \cdot 5i + 4)$
$= \mathrm{Re}(-21 - 20i)$
$= \underline{-21}$ R

$\mathrm{Re}(z_1)^2 = (-2)^2 = \underline{4}$ R

12) $\frac{z_1}{z_2} = \frac{-2+5i}{3-i} \cdot \frac{3+i}{3+i}$

$= \frac{-6-2i+15i-5}{3^2+1}$

$= \underline{\underline{-\frac{11}{10} + \frac{13}{10}i}}$ R

$\frac{z_2}{z_1} = \frac{3-i}{-2+5i} \cdot \frac{-2-5i}{-2-5i}$

$= \frac{-6-15i+2i-5}{2^2+5^2}$

$= -\frac{11}{29} - \frac{13i}{29}$ R

1a) $z = x + yi$

$$\frac{z}{\bar{z}} = \frac{x+yi}{x-yi} \cdot \frac{x+yi}{x+yi}$$

$$= \frac{x^2 + 2xyi - y^2}{x^2+y^2}$$

$$= \frac{x^2-y^2}{x^2+y^2} + \frac{2xy}{x^2+y^2} i$$

$$\mathrm{Re}(z/\bar{z}) = \frac{x^2-y^2}{x^2+y^2}$$ R

$$\mathrm{Im}(z/\bar{z}) = \frac{2xy}{x^2+y^2}$$ R

1b) $(1+i)^{16} = \left(\sqrt{2}\, e^{i\frac{\pi}{4}}\right)^{16}$

$$= 256\, e^{i4\pi}$$

$$= 256\,(1 + 0i)$$

$$z^2 = x^2 + 2xyi - y^2$$

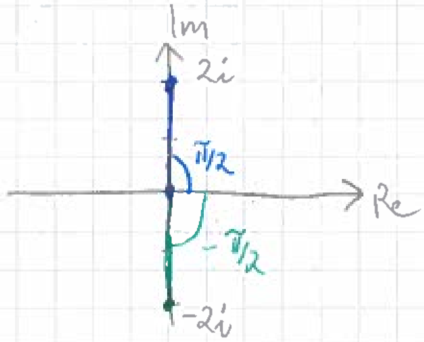
So $\mathrm{Re}\left((1+i)^{16} z^2\right)$

$$= \underline{\underline{256(x^2-y^2)}}$$ R

13.2:

3) $2i = 2e^{i\pi/2}$ R

$-2i = 2e^{i3\pi/2}$



8) $\dfrac{7+4i}{3-2i} = \dfrac{21+14i+12i-8}{3^2+2^2}$

$= \dfrac{13+26i}{13}$

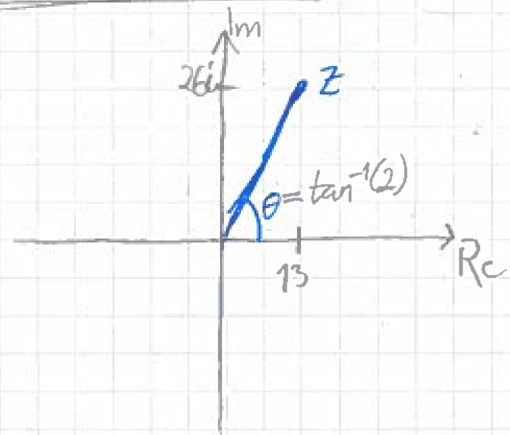
$= \cancel{13}(1+2i) = z$

$\arg z = \tan^{-1}\left(\frac{2}{1}\right)$

$|z| = 13\cdot\sqrt{1^2+2^2}$

$= 13\sqrt{5}$

$\Rightarrow z = \cancel{13}\sqrt{5}\,e^{i\cdot\tan^{-1}(2)}$ r



21) $z = \sqrt[3]{1-i}$

$1-i = \sqrt{2}e^{-\frac{\pi}{4}i}$

$$\Rightarrow \arg z = \frac{-\frac{\pi}{4}+2\pi n}{3}$$

$$= -\frac{\pi}{12} + \frac{2\pi}{3}n$$

$$= -\frac{3\pi}{4},\ -\frac{\pi}{12},\ \frac{7\pi}{12}$$

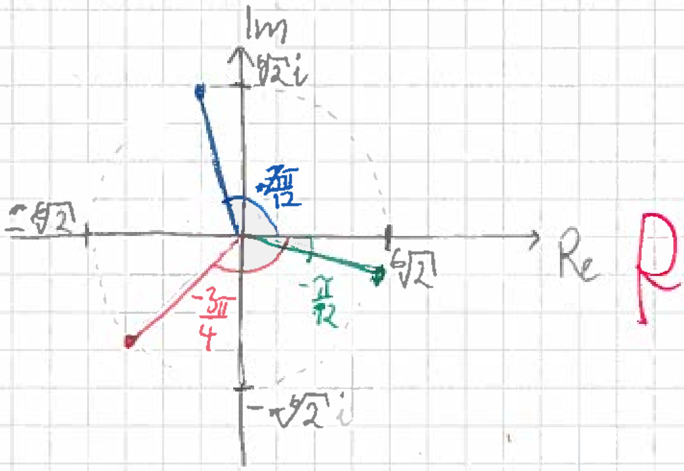
$$|z| = \sqrt[3]{\sqrt{2}}$$

$$= \sqrt[6]{2} = 2^{1/6}$$

$$\Rightarrow z_1 = \sqrt[6]{2}\,e^{-\frac{3\pi}{4}i}$$

$$z_2 = \sqrt[6]{2}\,e^{-\frac{\pi}{12}i}$$

$$z_3 = \sqrt[6]{2}\,e^{\frac{7\pi}{12}i}$$



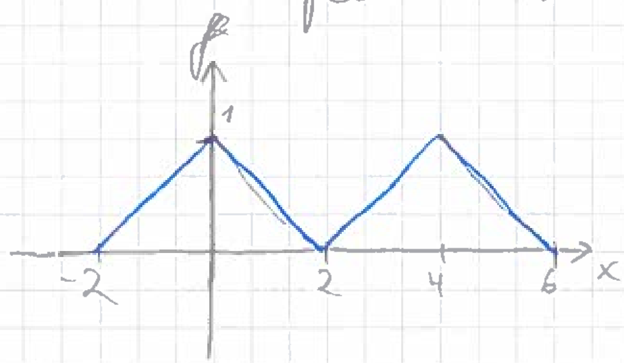
Sup. D:

$$f: \quad f(x) = f(-x)$$
$$f(x) = f(x+4)$$
$$f(x) = 1 - x, \quad 0 < x < 2$$

Since $f(x) = f(-x)$ we get

$$f(x) = 1 + x, \quad -2 < x < 0$$

or we can write $f(x) = 1 - |x|, -2 < x < 2$



We found the Fourier series for this in 11.2.17 except with period 2. Using this result but doubling the period gives us

$$f(x) = \frac{1}{2} - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos\left(\frac{\pi}{2} n x\right)$$