NORGES TEKNISK- NATURVITENSKAPELIGE UNIVERSITET
INSTITUTT FOR TEKNISK KYBERNETIKK

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Eksamen - TTK 4115 Lineær systemteori
Exam - TTK 4115 Linear systems theory

16. desember 2015, 09:00 – 13:00

Hjelpemidler: D - Ingen trykte eller håndskrevne hjelpemidler tillatt. Bestemt, enkel kalkulator tillatt.
Supporting materials: D - No printed or handwritten material allowed. Specific, simple calculator allowed.

Merk at ingen deloppgave avhenger av at du har greid å løse noen av de andre deloppgavene. Oppgitt informasjon fra tidligere deloppgaver skal være tilstrekkelig for å komme videre.
Note that no parts of this problem assume that you have solved any of the previous parts. The given information from previous parts should be sufficient to move on.

Oppgave 1 (8 %)

Forklar følgende begreper:
Explain the following concepts:

a) (4%)
Eksakt diskretisering.
Exact discretization.

b) (4%)
Realisering.
Realization.

Oppgave 2 (21 %)

Gitt systemet:
Given the system

$$\begin{aligned}\dot{\mathbf{x}} &= \begin{bmatrix} 1 & 2 \\ -3 & -6 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ b \end{bmatrix} u, \qquad b \in \mathbb{R} \\ y &= \begin{bmatrix} 1 & 1 \end{bmatrix}\mathbf{x}\end{aligned}$$

a) (9%)
Transformer systemet til Jordanform:
Transform the system to Jordan form:

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= \mathbf{J}\hat{\mathbf{x}} + \hat{\mathbf{B}}u \\ y &= \hat{\mathbf{C}}\hat{\mathbf{x}}\end{aligned}$$

b) (4%)
Hvilke av disse egenskapene innehar systemet? Forklar dine valg.
Which of these properties does the system possess? Explain your choice.

A Marginalt stabilt, *Marginally stable*

B Ustabilt, *Unstable*

C Asymptotisk stabilt, *Asymptotically stable*

c) (2%)
Er systemet observerbart?
Is the system observable?

d) (6%)
Bestemme for hvilke verdier av $b \in \mathbb{R}$ systemet er: (Forklar dine valg)
Determine for which values of $b \in \mathbb{R}$ the system is: (Explain your choice)

1) styrbart, *controllable*

2) BIBO stabilt, *BIBO stable*

Oppgave 3 (14 %)

Lyapunovs ligning er:
The Lyapunov equation is:

$$\mathbf{A}^T\mathbf{M} + \mathbf{M}\mathbf{A} = -\mathbf{N}$$

a) (14%)
Bruk Lyapunovs ligning til å vise at systemet nedenfor er asymptotisk stabilt.
Use the Lyapunov equation to show that the system below is asymptotically stable.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \quad \mathbf{A} = \begin{bmatrix} -1 & -2 & 0 \\ 2 & -4 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Hint: Bruk en diagonal $\mathbf{N}$ (en mulighet er $\mathbf{N} = \mathbf{I}$)
Hint: Use a diagonal $\mathbf{N}$ (one possible choice is $\mathbf{N} = \mathbf{I}$)

Hint 2: Se etter en symmetrisk løsning $\mathbf{M}$ med samme blokkdiagonale struktur som $\mathbf{A}$
Hint 2: Look for a symmetric solution $\mathbf{M}$ with the same block-diagonal structure as $\mathbf{A}$

Oppgave 4 (18 %)

Gitt følgende system
Consider the system

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x} \end{aligned}$$

a) (2%)
Sjekk om systemet er styrbart og observerbart
Check controllability and observability

b) (16%)
Gitt observeren
Given the observer

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}(y - \mathbf{C}\hat{\mathbf{x}})$$

og tilbakekoblingen
and the feedback control

$$u = -\mathbf{K}\hat{\mathbf{x}},$$

bestem forsterkningsmatrisene $\mathbf{K}, \mathbf{L}$ slik at
determine the gains $\mathbf{K}, \mathbf{L}$ *such that*

A) egenverdiene til lukket-sløyfe matrisen $\mathbf{A} - \mathbf{BK}$ er $\lambda_1 = -1, \lambda_2 = -2$,
the eigenvalues of the closed-loop matrix $\mathbf{A} - \mathbf{BK}$ *are* $\lambda_1 = -1, \lambda_2 = -2$,

B) egenverdien til feiltilstandssystemmatrisen $\mathbf{A} - \mathbf{LC}$ er $\mu_1 = -3, \mu_2 = -5$.
the eigenvalues of the error system matrix $\mathbf{A} - \mathbf{LC}$ *are* $\mu_1 = -3, \mu_2 = -5$.

Oppgave 5 (17 %)

Spektraltetthetsfunksjonen til en stasjonær stokastisk prosess $x(t)$ er gitt ved:
The spectral density function of a stationary random process $x(t)$ *is given by:*

$$S_x(j\omega) = \frac{4}{\omega^4 + 5\omega^2 + 4}$$

a) (2%)
Vis at polene til $S_x(s)$ er plassert i ± 1 og ± 2.
Show that the poles of $S_x(s)$ *are placed in* ± 1 *and* ± 2.

b) (2%)
$x(t)$ kan representeres som filtrert hvitstøy hvor spektraltettheten til hvitstøyen på inngangen er $S_v(j\omega) = 1$.Finn transferfunksjonen $G(s)$ til dette filteret.
$x(t)$ can be represented as filtered white noise where the spectral density of the white input noise is $S_v(j\omega) = 1$. *Find the transfer function* $G(s)$ *of this filter.*

c) (13%)

Gitt transferfunksjonen:

Given the transfer function:

$$\frac{x(s)}{v(s)} = G(s) = \frac{2}{(s+1)(s+2)}$$

La $v(t)$ være en hvitstøyprosess med spektraltetthet $S_v(j\omega) = 1 = \tilde{Q}$. Finn differensiallikninga som beskriver utviklinga av variansen til $x(t)$ og finn stasjonærverdien til variansen for $x(t)$.

Let $v(t)$ be a white noise process with spectral density $S_v(j\omega) = 1 = \tilde{Q}$. Find the differential equation which describe the variance of $x(t)$ and find the stationary value of the variance of $x(t)$.

Oppgave 6 (22 %)

a) (2%)

Skriv opp likningene for systemet det diskrete kalmanfilteret er en optimalt estimator for, og forklar hvilke antagelser som er gjort for støyen. Husk at ligningene for det diskrete kalmanfilteret er oppgitt i vedlegget.

Write the system equations for which the discrete Kalman filter is an optimal estimator and state which assumptions are made on the noise. Remember that discrete Kalman Filter equations are given in Appendix.

b) (2%)

Skriv opp likningene for systemet det kontinuerlige kalmanfilteret er en optimalt estimator for, og forklar hvilke antagelser som er gjort for støyen. Husk at ligningene for kalmanfilteret er oppgitt i vedlegget.

Write the system equations for which the continuous Kalman filter is an optimal estimator and state which assumptions are made on the noise. Remember that Kalman Filter equations are given in Appendix.

I resten av oppgaven vil kovariansmatrisen til støyen være gitt av R (diskret system) og $\tilde{R}$ (tidskontinuerlig system).

For the remainder of this problem, the covariance matrix of the noise will be denoted by R (discrete-time case) and $\tilde{R}$ (continuous-time case).

c) (4%)

Gitt målingene

Given the measurements

$$z_k = x + w_k; \quad w_k \sim \mathcal{N}(0, R)$$

av den ukjent konstanten x så kan vi beregne et estimatet ved

of the unknown constant x. The estimate is then given by

$$\hat{x}_k = \frac{1}{k}\sum_{i=1}^{k} z_i$$

Utled en tilsvarende rekursiv form av denne likninga som den vi har for det diskrete kalmanfilteret (med likningene splittet i tidsoppdatering og måleoppdatering), dvs finn likningene for beregning av $\hat{x}_k$ og K_k. Vis at estimatet kan skrives som ligning (1).

Derive a comparable recursive equation to the discrete Kalman filter equations (time and measurement update equations), i.e derive the equations for calculating $\hat{x}_k$ and K_k. Show that the estimate can be expressed as Eq. (1).

$$\hat{x}_k = \hat{x}_{k-1} + \frac{1}{k}\left(z_k - \hat{x}_{k-1}\right) \tag{1}$$

d) (8%)

Gitt systemet

Given the system

$$\begin{aligned} x_{k+1} &= x_k; \quad x_0 \sim \mathcal{N}\left(\hat{x}_0, \hat{P}_0\right) \\ z_k &= x_k + w_k; \quad w_k \sim \mathcal{N}(0, R) \end{aligned}$$

Finn formlene for $\hat{P}_k$ og K_k fra likningene for det diskrete kalmanfilteret. Sammenlikn kalmanfilterforsterkninga funnet i (6.d) med forsterkninga i ligning (1), og kommenter.

Derive the formulas for $\hat{P}_k$ and K_k from the discrete Kalman filter equations. Compare the Kalman filter gain found in (6.d) and the one given in Eq. (1), and give a comment.

e) (6%)

Gitt systemet

Given the system

$$\begin{aligned} \dot{x} &= 0; \quad x(0) \sim \mathcal{N}\left(\hat{x}(0), \hat{P}(0)\right) \\ z(t) &= x(t) + w(t); \quad w(t) \sim \mathcal{N}\left(0, \tilde{R}\right) \end{aligned}$$

Skriv opp differentiallikninga for $\hat{P}(t)$ og løs den. Finn $K(t)$

Find the differential equation for $\hat{P}(t)$ *and solve it and calculate* $K(t)$.

Vedlegg til eksamen (noen nyttige formler og uttrykk):
Appendix to the exam (some useful formulas and expressions):

Solutions:

$$\begin{aligned}
\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau \\
\mathbf{x}[k] &= \mathbf{A}^k\mathbf{x}[0] + \sum_{m=0}^{k-1} \mathbf{A}^{k-1-m}\mathbf{B}\mathbf{u}[m]
\end{aligned}$$

Controllability/Observability:

$$\begin{aligned}
\mathcal{C} &= [\mathbf{B},\ \mathbf{AB},\ \mathbf{A}^2\mathbf{B},\ \cdots,\ \mathbf{A}^{n-1}\mathbf{B}] \\
\mathcal{O} &= \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}
\end{aligned}$$

Realization:

$$\begin{aligned}
\mathbf{G}(s) &= \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D} \\
\mathbf{G}(s) &= \mathbf{G}(\infty)+\mathbf{G}_{sp}(s) \\
d(s) &= s^r + \alpha_1 s^{r-1} + \cdots + \alpha_{r-1}s + \alpha_r \\
\mathbf{G}_{sp}(s) &= \frac{1}{d(s)}[\mathbf{N}_1 s^{r-1} + \mathbf{N}_2 s^{r-2} + \cdots + \mathbf{N}_{r-1}s + \mathbf{N}_r] \\
\dot{\mathbf{x}} &= \begin{bmatrix} -\alpha_1\mathbf{I}_p & -\alpha_2\mathbf{I}_p & \cdots & -\alpha_{r-1}\mathbf{I}_p & -\alpha_r\mathbf{I}_p \\ \mathbf{I}_p & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_p & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I}_p & \mathbf{0} \end{bmatrix}\mathbf{x} + \begin{bmatrix} \mathbf{I}_p \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}\mathbf{u} \\
\mathbf{y} &= [\ \mathbf{N}_1\ \ \mathbf{N}_2\ \ \cdots\ \ \mathbf{N}_{r-1}\ \ \mathbf{N}_r\ ]\mathbf{x} + \mathbf{G}(\infty)\mathbf{u}
\end{aligned}$$

LQR:

$$J_{\mathsf{LQR}} = \int_0^\infty \mathbf{x}^\mathsf{T}(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^\mathsf{T}(t)\mathbf{R}\mathbf{u}(t)\,dt$$

$$\mathbf{A}^\mathsf{T}\mathbf{P} + \mathbf{PA} + \mathbf{Q} - \mathbf{PBR}^{-1}\mathbf{B}^\mathsf{T}\mathbf{P} = \mathbf{0}$$

$$\mathbf{u}(t) = -\mathbf{R}^{-1}\mathbf{B}^\mathsf{T}\mathbf{P}\mathbf{x}(t)$$

Lyapunov equation:

$$\mathbf{A}^{\mathsf{T}}\mathbf{M} + \mathbf{MA} = -\mathbf{N}$$

Discrete-time Kalman filter:

$$\begin{aligned}
\underline{\bar{x}}_{k+1} &= \Phi \underline{\hat{x}}_k; \quad \underline{\hat{x}}_0 \text{ given;} \\
\bar{P}_{k+1} &= \Phi \hat{P}_k \Phi^T + \Gamma Q \Gamma^T; \quad \hat{P}_0 \text{ given} \\
\underline{\hat{x}}_k &= \underline{\bar{x}}_k + K_k \left(\underline{z}_k - H \underline{\bar{x}}_k \right) \\
\hat{P}_k &= \left(I - K_k H \right) \bar{P}_k \left(I - K_k H \right)^T + K_k R K_k^T = \left(I - K_k H \right) \bar{P}_k \\
K_k &= \bar{P}_k H^T \left(H \bar{P}_k H^T + R \right)^{-1}
\end{aligned}$$

Continuous-time Kalman filter:

$$\begin{aligned}
\dot{\underline{\hat{x}}}(t) &= F \underline{\hat{x}}(t) + K(t) \left(\underline{z}(t) - H \underline{\hat{x}}(t) \right); \quad \underline{\hat{x}}(t_0) \text{ given} \\
\dot{\hat{P}}(t) &= F \hat{P}(t) + \hat{P}(t) F^T + G \tilde{Q} G^T - \hat{P}(t) H^T \tilde{R}^{-1} H \hat{P}(t); \quad \hat{P}(t_0) \text{ given} \\
K(t) &= \hat{P}(t) H^T \tilde{R}^{-1}
\end{aligned}$$

Auto-correlation:

$$\begin{aligned}
R_X(\tau) &= E[X(t)X(t+\tau)] \text{ (Stationary process)} \\
R_X(t_1, t_2) &= E[X(t_1)X(t_2)] \text{ (Non-stationary process)} \\
Y(s) &= G(s)U(s) \Rightarrow \\
R_y(t_1, t_2) &= E[y(t_1)y(t_2)] = \int_0^{t_2} \int_0^{t_1} g(\xi) g(\eta) E\left[u(t_1 - \xi) u(t_2 - \eta) \right] d\xi d\eta
\end{aligned}$$

Laplace transform pairs:

$f(t)$	$\Longleftrightarrow$	$F(s)$
$\delta(t)$	$\Longleftrightarrow$	1
1	$\Longleftrightarrow$	$\frac{1}{s}$
e^{-at}	$\Longleftrightarrow$	$\frac{1}{s+a}$
t	$\Longleftrightarrow$	$\frac{1}{s^2}$
t^2	$\Longleftrightarrow$	$\frac{2}{s^3}$
te^{-at}	$\Longleftrightarrow$	$\frac{1}{(s+a)^2}$
$\sin\omega t$	$\Longleftrightarrow$	$\frac{\omega}{s^2+\omega^2}$
$\cos\omega t$	$\Longleftrightarrow$	$\frac{s}{s^2+\omega^2}$