DEPARTMENT OF ELECTRONICS AND TELECOMMUNICATIONS • NTNU

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WAVES

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Figure 1: http://www.crystalblueent.com/healthy-natural-frequencies.html

Contents

1 Introduction

> The sooner you fall behind, the more time you have to catch up
>
> *Ogden's Law*

What is a wave? It is exactly what you think it is! That may sound a bit disrespectful since there are many books written about waves. Nevertheless, we all know what it is. So why this chapter? Because you need to know how we describe them *mathematically* and you need to know about the underlying equation that describes their motion - the WAVE EQUATION.

To write this short chapter I have primarily been guided by the expositions in [1, 2]. Additionally, I recommend that you consult the following books if you need some additional information [3, 4, 5]. In [6] there are a few informative simulations of different types of waves. Finally, *googling* is both permitted and encouraged!

2 Wave Basics

Important: WAVES TRANSPORT ENERGY WITHOUT TRANSPORTING MATTER !

2.1 Harmonic Waves

The simplest wave is the harmonic or sinusoidal wave.[1]

Important: HARMONIC (SINUSOIDAL) WAVE

$$u(x,t) = A\cos(kx - \omega t + \varphi) \tag{1}$$

A is the amplitude and $\Phi(x,t) = (kx - \omega t + \varphi)$ is the time and space varying phase. φ is the phase constant and determines the value of $u(x,t)$ when x and t are both zero. For the wave in Fig. 2 we gather that for this specific wave $\varphi = 0$. k is the propagation constant, $k = \frac{2\pi}{\lambda}$ $[\frac{1}{m}]$. ω is the angular frequency, $\omega = 2\pi f$ $[\frac{rad}{sec}]$ where f is the frequency, $f = \frac{1}{T}$ $[\frac{1}{sec}]$. T is the time for a complete cycle to occur and f is the number of times a cycle (period)) occurs in one second. On the spatial side, λ is the distance for a full period, the distance between two consecutive crests or troughs. So, compared with the oscillations that you have encountered earlier, a wave, with the added kx in the phase, is then a collection of oscillations with

[1] In the case of oscillations and waves, sinusoidal and harmonic are synonymous.

the phase varying in both time and space. Since A is a constant in Eq. 1 we realize that if Φ is a constant too, and

$$x = vt \tag{2}$$

we must have

$$v = \frac{\omega}{k} \tag{3}$$

So, Φ is the *phase* of the wave and v is the *phase velocity*. In other words, the phase velocity is the speed with which each point on the upper sinus curve in Fig. 2 moves to the right as time goes on.

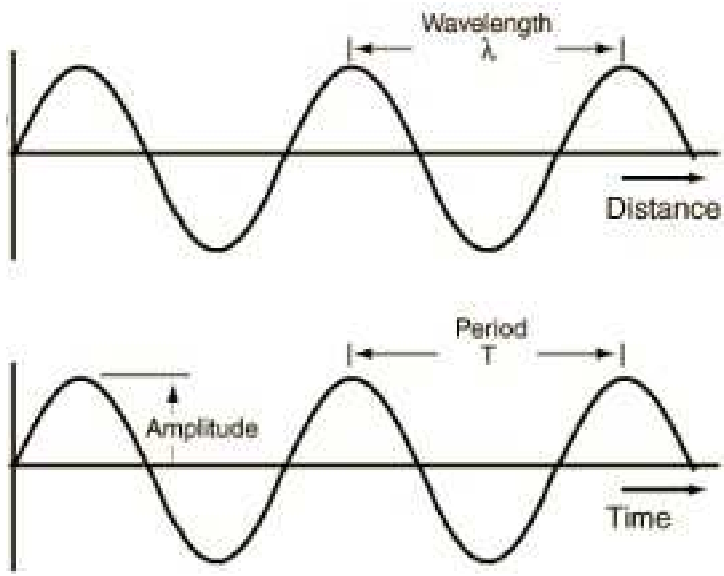

Figure 2: From http://www.physicsmynd.com/?page_id=1529

2.2 Phasor Formalism and Complex Notation

A phasor is a representation of a sinusoidal wave whose amplitude, A, phase constant, φ, and radial frequency, ω, are *independent of time* (time-invariant). The time-invariance allows us to to make the following definition:

$$A\sin(\omega t + \varphi) = \mathrm{Re}[\underbrace{Ae^{i\varphi}}_{\tilde{A}} e^{i\omega t}] \tag{4}$$

where $\tilde{A} = Ae^{i\varphi}$ is the phasor. The phasor spins around the complex plane as a function of time, Fig. 3. Phasors of the same frequency can be added.

With the help of de Moivre's formula,

$$e^{i\alpha} = \cos\alpha + i\sin\alpha \tag{5}$$

it is possible re-write our harmonic wave in complex notation,

$$A\cos(kx - \omega t + \varphi) = \mathrm{Re}[Ae^{i(kx-\omega t+\varphi)}] \tag{6}$$

Often, the Re is left out and we tacitly acknowledge that we only mean the real part. Remember that complex notation is only used to simplify the algebra. ALL PHYSICAL PROPERTIES ARE REAL, so after the mathematics is done we better make sure that we end up with something that is real.

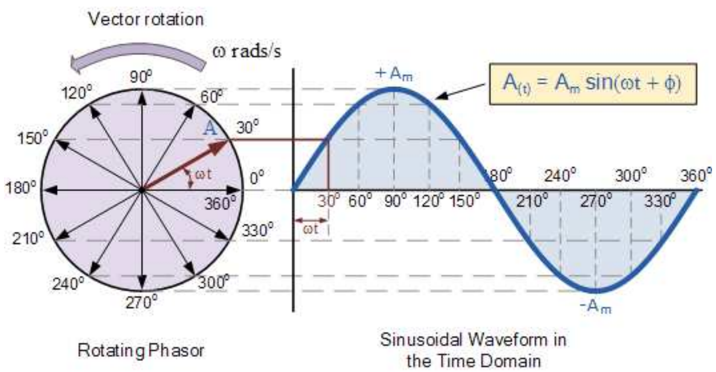

Figure 3: From http://www.electronics-tutorials.ws/accircuits/phasors.html

3 Wave Equation

In physics there is always an *equation of motion* for describing the dynamic properties of our system of interest. For example, in Newtonian mechanics we have,

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{v}\frac{\mathrm{d}m}{\mathrm{d}t} + m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = m\mathbf{a} \tag{7}$$

if we assume a system where the mass is a constant.

The equation of motion for wave systems is the wave equation and under certain conditions it takes the following form in 1D, Eq. 8,

Important: THE WAVE EQUATION

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u(x,t)}{\partial t^2} \tag{8}$$

3.1 String

There are many ways to derive the wave equation, using a common string as the physical medium is perhaps the most established method for deriving the wave equation in physics. You will watch Prof. Shankar do it in the video lectures on scalable learning. However, I also include a derivation in these notes so that you have easy access to the mathematics and I will also provide you with a little more background than what Shankar is capable of doing in his lectures. There are several different types of waves. In Fig. 4 a *longitudinal* and *transverse* wave are depicted, respectively. Acoustic (sound) waves are only longitudinal, the waves on a string are only transverse and electromagnetic waves can be both longitudinal and transverse. As the names implies, the longitudinal wave oscillates in the direction of the wave propagation while transverse waves oscillate perpendicular to the direction of propagation.

We are mainly following the outline given by Joel Feldman in [7]. For the derivation we need

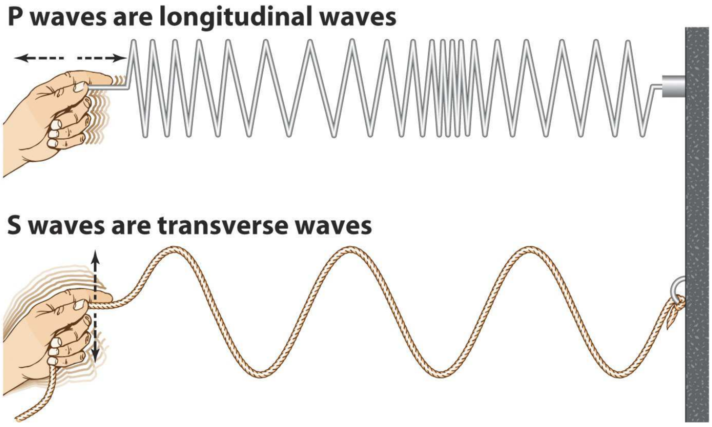

Figure 4: From http://lightcoalition.org/properties-waves-wave-cycles-scalar-transverse-energy/

Notation:

$u(x,t) =$ vertical displacement of the string at a given instant of time t and position x. Units = [m].

$\theta(x,t) =$ angle between the string and a horizontal line parallel with the x-axis at a given instant of time t and position x. Units = [rad].

$T(x,t) =$ tension in the string at a given instant of time t and position x. Units = [N/m]

$\varrho(x) =$ density of mass at position x. Units = [kg/m]

$F(x,t) =$ gravity. Force per unit length. If additional forces are present they can be added to F. Units = [N/m]

Assumptions:

- The string is perfectly flexible, i.e. there is no resistance to any external forces trying to bend the string.
- The tension, T, acts tangentially to the string at every point.
- At each point the tension is the same along both directions of the string, otherwise the string would change length.
- The tension is large compared to gravity and there are no other forces acting on the string.
- The vibration of the string occurs in the plane of the paper, i.e. the motion is perpendicular to the x-axis and the direction of propagation.
- Neglecting the width and height of the string, focussing solely on the length in the x-direction.
- The string is assumed to be infinitely long, otherwise end effects need to be considered. This we will deal with later with the help of boundary conditions.

We now focus on a short segment, Δx, of the string.

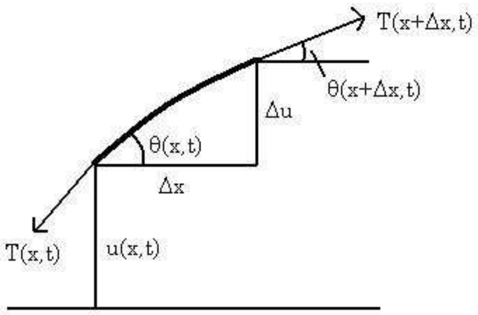

Figure 5: From http://gozips.uakron.edu/~crm23/wave/derivation/derivation.htm

Forces acting on Δx

1. At point x the tension is $T(x,t)$ and it points in the direction towards smaller values of x.
2. At point $x+\Delta x$ the tension is $T(x+\Delta x,t)$ and it points towards larger x values.
3. Gravity per unit length, $F(x,t)$ acts vertically and $F(x,t)\Delta x$ is the net magnitude acting on a segment Δx. Any other forces acting vertically on the string would also be lumped in with F.
4. The mass of the segment Δx is $m = \varrho(x))\sqrt{\Delta x^2 + \Delta u(x,t)^2}$

Whenever we deal with mechanical systems it is Newton's third law, $ma = F$ that is our equation of motion from which we start our derivations. As usual with mechanical systems we balance the forces both vertically and horizontally.

VERTICAL FORCES:

$$\underbrace{\varrho(x))\sqrt{\Delta x^2 + \Delta u(x,t)^2}}_{mass} \times \underbrace{\frac{\partial^2 u(x,t)}{\partial^2 t}}_{acceleration} = \underbrace{T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t) + F(x,t)}_{vertical\,forces} \tag{9}$$

Because of the $T(x+\Delta x) - T(x)$ combination in Eq. 9 we anticipate a derivate hiding in there so we divide both sides with Δx and take the limit as $\Delta x \to 0$. Hence,

$$\begin{aligned}\varrho(x)\sqrt{1+\left(\frac{\partial u(x,t)}{\partial x}\right)^2} \times \frac{\partial^2 u(x,t)}{\partial^2 t} &= \frac{\partial}{\partial x}[T(x,t)\sin\theta(x,t)] + F(x,t) \\ &= \frac{\partial T(x,t)}{\partial x}\sin\theta(x,t) + T(x,t)\cos\theta(x,t)\frac{\partial\theta(x,t)}{\partial x} + F(x,t)\end{aligned} \tag{10}$$

where you have to make sure that you keep track of the chain rule. From Fig. 5 we find

$$\tan\theta(x,t) = \frac{\Delta u}{\Delta x}$$

which continues to be true even when $\Delta x \to 0$, and for this reason

$$\tan\theta(x,t) = \lim_{\Delta x \to 0}\frac{\Delta u}{\Delta x} = \frac{\partial u(x,t)}{\partial x} \tag{11}$$

How to proceed? There are almost as many approaches to the next step as there are textbooks written about it, nevertheless they all have in common that the underlying idea that we take advantage of is that the amplitudes of the wave are small which results in the angle θ being small. From this we gather that

$$\cos\theta(x,t) \approx 1\,;\, \sin\theta(x,t) \approx \theta(x,t) \;\to\; \tan\theta(x,t) \approx \theta(x,t) = \frac{\partial u(x,t)}{\partial x} \;\to\; \frac{\partial\theta(x,t)}{\partial x} \approx \frac{\partial^2 u(x,t)}{\partial x^2} \tag{12}$$

and we also conclude that $\left(\frac{\partial u(x,t)}{\partial x}\right)^2 \ll 1$. Substituting these approximations into Eq. 10 we obtain

$$\varrho(x)\frac{\partial^2 u(x,t)}{\partial^2 t} = \frac{\partial T(x,t)}{\partial x}\frac{\partial u(x,t)}{\partial x} + T(x,t)\frac{\partial^2 u(x,t)}{\partial x^2} + F(x,t) \tag{13}$$

We are again at a bit of an impasse, we have one equation and two unknowns, $u(x,t)$ and $T(x,t)$. Luckily, we still have the equation for the horisontal forces to help us narrow things down even more.

HORIZONTAL FORCES:
We follow the outline we had for the vertical forces and the equivalent of Eq. 9 is, for the horizontal case,

$$T(x+\Delta x,t)\cos\theta(x+\Delta x,t) - T(x,t)\cos\theta(x,t) = 0 \tag{14}$$

again, dividing by Δx and letting $\Delta x \to 0$ we obtain

$$\frac{\partial}{\partial x}[T(x,t)\cos\theta(x,t)] = 0 \tag{15}$$

WAVE EQUATION:
Since, as before, $\cos\theta(x,t) \approx 1$ we get $\frac{\partial T(x,t)}{\partial x} \approx 0$. If we use this in Eq. 13 we find that

$$\varrho(x)\frac{\partial^2 u(x,t)}{\partial^2 t} = T(x,t)\frac{\partial^2 u(x,t)}{\partial x^2} + F(x,t) \tag{16}$$

at this point you may argue that $T(x,t)$ is still in there as an unknown. However, if we assume that the string density $\varrho(x)$ is not a function of x but a constant then the tension T is also a constant independent of t. Finally, if we only allow gravity to contribute to F and remember that $F \ll T$ we, at long last, end up with the wave equation

$$\boxed{\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u(x,t)}{\partial t^2}} \tag{17}$$

where

$$v = \sqrt{\frac{T}{\varrho}} \tag{18}$$

This velocity is the phase-velocity and it is always a function of parameters that determine the properties of the material in which the wave is propagating.

You may think that this equation is not of much value since we had to make so many assumptions to derive it. It turns out that most of these assumptions are realistic and the wave equation for the string we derived is of great value.

3.2 General Derivation

As a matter of fact, the wave equation is *ubiquitous*. It shows up in all branches of science and engineering. We derived it specifically for the simple case of a disturbance moving along a string but we could equally well have derived it for an acoustic wave propagating through air, an electromagnetic wave moving through space, a wave made by humans around a track and field arena, a vibrating drum head, a wave moving through the earth's crust generated by an earthquake, just to mention a few. You will learn how to derive the wave equation for electromagnetic and acoustic waves next semester in your wave propagation course. What I want to do in this section is to show you that one can derive the wave equation without referring to any specific physical system; only to some basic, underlying properties of the wave itself. We need to make two assumptions for this derivation to work.

Assumptions:

- No dissipation
- No dispersion

The first one means that the wave does not lose energy as it propagates and the second one that the velocity v is a constant, not a function of frequency (wavelength).

The upshot of this is that the wave retains its shape as it propagates, Fig. 6.

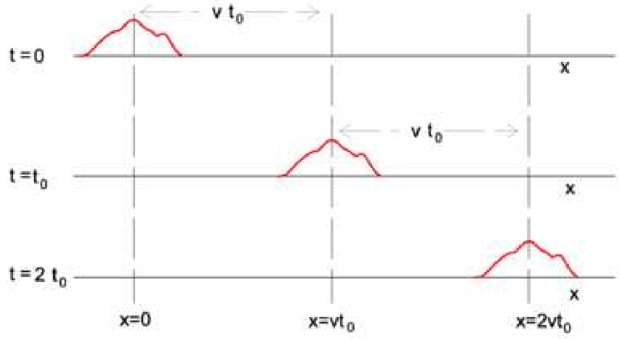

Figure 6: From http://en.citizendium.org/wiki/Wave_equation_(classical_physics)

At first, these assumptions may seem to exclude any realistic pulses but it turns out that even though the assumptions are, strictly speaking, never true they are true enough for many systems when the pulse does not travel a long distance. One last thing before the math, what about a pulse versus a wave. I have sneakily changed the vocabulary and Fig. 6 is distinctly different from Fig. 4. Although this will all be explained in sections 5 and 7, let me just give you a quick version. The pulse is a sum of many sinusoidal waves with slightly different amplitudes and phase constants. With the help of Fourier analysis we are able to create *ANY* pulse from a sum of sinusoidal waves. The effect of this is therefore that if we can describe *one* harmonic wave we can describe any combination of sinusoidal waves with the help of the superposition principle.

So, let's focus our attention on the pulse in Fig. 6. At $t = 0$ the pulse maximum is at $x = 0$ and its shape is given by the function $u(x, 0) = f(x)$. A time t later this *shape* has moved over a distance vt in the positive x-direction and we have $u(x, t) = f(x - vt)$. In words, the shape stays intact and simply moves like a unit with speed v.

$$\frac{\partial u(x,t)}{\partial x} = \frac{\partial f(x-vt)}{\partial x} \qquad \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 f(x-vt)}{\partial x^2}$$

$$\frac{\partial u(x,t)}{\partial t} = -v\frac{\partial f(x-vt)}{\partial t} \qquad \frac{\partial^2 u(x,t)}{\partial t^2} = v^2\frac{\partial^2 f(x-vt)}{\partial t^2}$$

We observe from the two right-most equations that, indeed

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u(x,t)}{\partial t^2} \tag{19}$$

as anticipated. Interestingly, if we look at the two left-most equations we find that

$$\frac{\partial u(x,t)}{\partial t} = -v\frac{\partial u(x,t)}{\partial x} \tag{20}$$

so why do we need to use the second-order equations? The reason is that Eq. 20 shows a pulse propagating, *solely*, to the right (positive x-direction) as time moves on. The second order derivatives in the wave equation, Eq. 19, on the other hand, describe both left and right moving pulses. As mentioned before the velocity v is a function of the medium used. We can perform the same derivation with the function $u(x, t) = g(x + vt)$ and we obtain as expected the wave equation, Eq. 19, and the equivalent of Eq. 20 is

$$\frac{\partial u(x,t)}{\partial t} = v\frac{\partial u(x,t)}{\partial x} \tag{21}$$

Since the wave equation is a linear differential equation we can always form a new solution by forming the sum of existing ones, e.g. $u(x, t) = f(x - vt) + g(x + vt)$. This is called superposition and is the reason why waves interfere with each other, as we will see in section 5.

4 Energy

Energy is a very important concept. And as we will see, over and over again in this course, it determines largely why certain atoms bond and others do not. It also drives the dynamic in devices and components. It is therefore reasonable that we investigate what is the energy of a wave. Let's use the string as our medium to find out what the energy density per unit length is by considering the region $[x, x + \Delta x]$. The total energy is the sum of kinetic and potential parts. From Fig. 5 and the earlier assumption that the motion comes solely from the transverse oscillations we obtain

$$K_{\mathrm{d}x} = \frac{1}{2}(\varrho \Delta x) v_y^2 = \frac{1}{2}(\varrho \Delta x)\left(\frac{\partial u}{\partial t}\right)^2 \tag{22}$$

Referring, again, to Fig. 5, we deduce that the length of the string for our segment $[x, x + \Delta x]$ is

$$\sqrt{(\Delta x)^2 + (\Delta u)^2} \underset{\Delta x \to 0}{\Longrightarrow} \mathrm{d}x\sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2} \approx \mathrm{d}x + \frac{1}{2}\mathrm{d}x\left(\frac{\partial u}{\partial x}\right)^2 \tag{23}$$

Our short segment is therefore *stretched* the distance $\mathrm{d}\ell \approx \frac{1}{2}\mathrm{d}x\left(\frac{\partial u}{\partial x}\right)^2$. This stretch is due to the tension forces T at the two ends of our string interval $[x, x + \Delta x]$. These tension forces do the work $T\mathrm{d}\ell$ and this work accumulates potential energy in the string segment. The potential energy can then be calculated as (remember, $pot.energy = force \times distance$)

$$U_{\mathrm{d}x} = \frac{1}{2}T\mathrm{d}x\left(\frac{\partial u}{\partial x}\right)^2 \tag{24}$$

Putting it all together we get for the total energy per unit length, ϵ

$$\epsilon(x,t) = \frac{K_{\mathrm{d}x} + U_{\mathrm{d}x}}{\mathrm{d}x} = \frac{\varrho}{2}\left(\frac{\partial u}{\partial t}\right)^2 + \frac{T}{2}\left(\frac{\partial u}{\partial x}\right)^2 \tag{25}$$

if we use $v = \sqrt{T/\varrho}$ in Eq. 25 we finally arrive at

$$\epsilon(x,t) = \frac{\varrho}{2}\left[\left(\frac{\partial u}{\partial t}\right)^2 + v^2\left(\frac{\partial u}{\partial x}\right)^2\right] \tag{26}$$

It is instructive to look at the traveling waves we analyzed in section 3.2, $u(x,t) = f(x \pm vt)$. The energy expression in Eq. 26 can be simplified further for these waves since they have to obey

$$\frac{\partial u}{\partial t} = \pm v\frac{\partial u}{\partial x} \tag{27}$$

We realize from Eq. 27 that the kinetic and potential energies are equal at a given point and time (you knew this from your first physics course, right?). Subsequently,

$$\epsilon(x,t) = \frac{Z}{v}\left(\frac{\partial u}{\partial t}\right)^2, \quad \text{or} \quad \epsilon(x,t) = Zv\left(\frac{\partial u}{\partial x}\right)^2 \tag{28}$$

where we have introduced the impedance $Z = \sqrt{Tv}$. A consequence of this is that the energy is maximum when the wave amplitude u is a minimum. The energy and the wave that transports this energy are out of phase!

In this section I followed the chapter on transverse waves by David Morin [8]. If you read his version you will find that there are a few subtleties that I have swept under the carpet but nothing that changes the end result.

5 Superposition

Many useful and illustrative simulations of waves are found in [9].

In the case of a single-frequency wave (monochromatic), can be expressed as (it really doesn't matter if I use sin or cos)

$$u(x,t) = a \cdot \cos(\omega t - kx) \tag{29}$$

there is only need for *one velocity*, the phase velocity v_p. And as we saw in section 2.1 it is

$$v_{ph} = \frac{\omega}{k} \tag{30}$$

where, as usual, ω is the angular frequency and k is the wave propagation constant. The phase of the harmonic wave is defined as

$$\Phi = (\omega t - kx) \tag{31}$$

which means that the phase velocity describes the speed with which constant phase is moving. Mathematically we can verify this by writing the expression for a small change in the phase

$$\Delta\Phi(x,t) = \frac{\partial\Phi}{\partial x}\Delta x + \frac{\partial\Phi}{\partial t}\Delta t \tag{32}$$

Figure 7 depicts our wave defined in Equation 29 as a function of x at two different time instances t_1 and

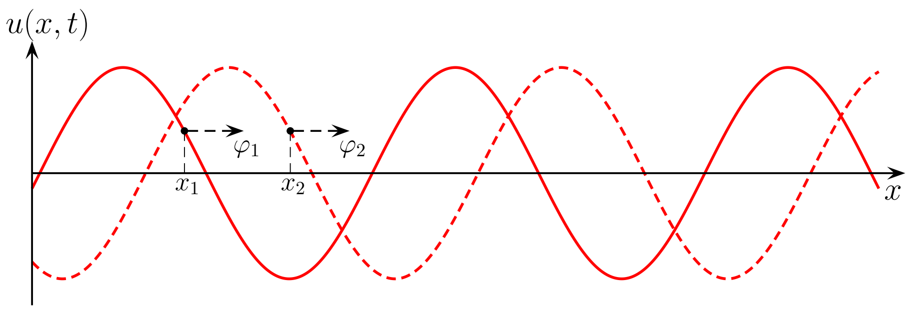

Figure 7: Phase velocity.

t_2, respectively. For these two times the phase function $\Delta\Phi$ is evaluated at points x_1 and x_2, respectively. We are interested in the speed of constant phase, i.e. we choose $\Phi_1(x_1,t_1) = \Phi_2(x_2,t_2)$ which results in

$$\Delta\Phi = \Phi_2(x_2,t_2) - \Phi_1(x_1,t_1) = 0 \tag{33}$$

From Equations 32 and 33 we find

$$v_{ph} = \frac{\Delta x}{\Delta t} = \frac{-\frac{\partial\Phi}{\partial t}}{\frac{\partial\Phi}{\partial x}} = \frac{-\omega}{-k} = \frac{\omega}{k} \tag{34}$$

If we have two monochromatic waves present simultaneously,

$$u(x,t) = u_1(x,t) + u_2(x,t) = a_1 \cdot \cos(\omega_1 t - k_1 x) + a_2 \cdot \cos(\omega_2 t - k_2 x) \tag{35}$$

the situation changes drastically and we have *three* velocities to account for; phase velocity (v_{ph} as before), group velocity (v_g) and energy velocity (v_E). For the single frequency wave all these velocities are the same and typically represented by the phase velocity, v_{ph} The group velocity concept naturally emerges when we rewrite Equation 35 as a product. To do this, with a minimum of mathematical inconvenience and still maintaining physical insight, we allow the two amplitudes, a_1 and a_2 to be equal, $a_1 = a_2 = a$. We then obtain, using basic trigonometric formulas,

$$u(x,t) = 2a\cos\left[\frac{(\omega_1 - \omega_2)t - (k_1 - k_2)x}{2}\right]\cos\left[\frac{(\omega_1 + \omega_2)t - (k_1 + k_2)x}{2}\right] \tag{36}$$

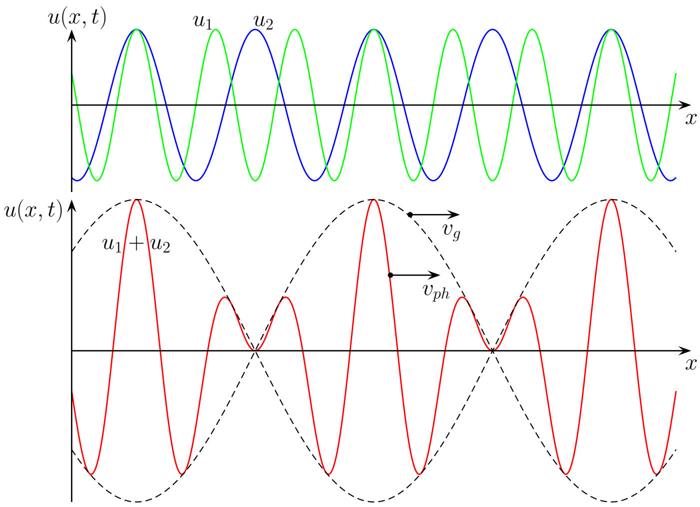

Figure 8: Group and phase velocities.

In Figure 8 we have plotted both u_1 and u_2 individually (top) as well as their sum (bottom, red line). In addition, we have outlined the *envelope* (black dashed line). Rewriting Equation 36 as

$$u(x,t) = 2a \cos\left[\frac{1}{2}(\Delta\omega\, t - \Delta k\, x)\right] \cos\left[\overline{\omega}t - \overline{k}x\right] \tag{37}$$

where $\Delta\omega = \omega_1 - \omega_2$, $\Delta k = k_1 - k_2$ and $\overline{\omega} = (\omega_1 + \omega_2)/2$, $\overline{k} = (k_1 + k_2)/2$ are the *average* frequency and propagation constant, respectively. Using the same technique as for deriving the phase velocity for the monochromatic wave in Equation 34 we find that the envelope moves with the velocity v_g, derived from

$$v_g = \frac{\Delta\omega/2}{\Delta k/2} = \frac{\Delta\omega}{\Delta k} \Longrightarrow \frac{\mathrm{d}\omega}{\mathrm{d}k} \tag{38}$$

and the phase velocity for the combined wave is

$$v_{ph} = \frac{\overline{\omega}}{\overline{k}} \tag{39}$$

It is clear from the simple arithmetic in this case that $v_{ph} > v_g$, which means that the red line moves in and out of the black dashed line at a higher velocity[2].

What about the energy velocity, v_E? That will have to wait until you have taken more courses in electromagnetics.

6 Standing Waves

The traveling waves that we have analyzed so far have all propagated in the same direction. What happens if two waves propagating in *opposite* directions interact?

Following the superposition principle in Eq. 35 we add the waves again but notice that the phase of one is $(kx - \omega t)$ while the wave propagating in the opposite direction has the phase $(kx + \omega t)$. Adding them results in,

$$\begin{aligned} u(x,t) &= a\cos(kx - \omega t) + a\cos(kx + \omega t) \\ &= 2a\cos\left[\frac{kx - \omega t}{2} + \frac{kx + \omega t}{2}\right] \cdot \cos\left[\frac{kx - \omega t}{2} - \frac{kx + \omega t}{2}\right] \\ &= 2a\cos(kx) \cdot \cos(\omega t) \end{aligned} \tag{40}$$

[2] As often is the case, history has imposed on us a technically incorrect terminology since the word velocity is associated with a vector that has both magnitude and direction while our velocities are speeds since they only have magnitude. Unfortunately, there is more than 100 years of precedence so we will continue to use velocity in these cases.

What you have now, is a wave with a spatially varying part times a temporal oscillation. The distance between two nodes (and anti-nodes) is $\lambda/2$ (check this!) and between a node and an anti-node it is $\lambda/4$. There is no transport of energy for a standing wave, which is why it is sometimes referred to as a stationary wave. If you had used the sin expression for the two waves traveling in the opposite directions instead of

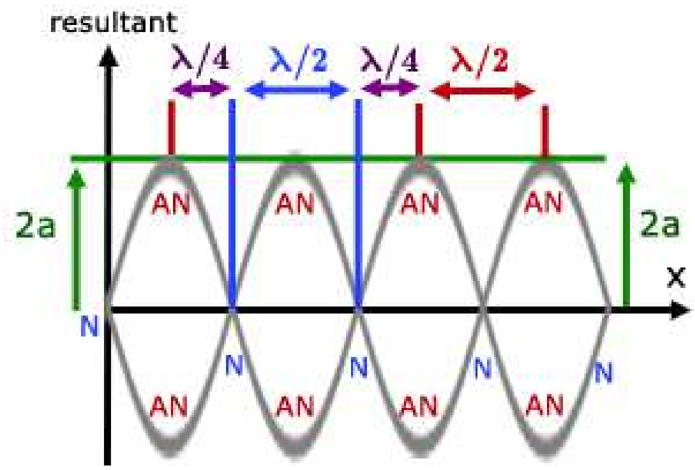

Figure 9: Basic portions of a standing wave. From http://www.a-levelphysicstutor.com/wav-stat-wavs.php.

the cos in Eq. 40 you would have obtained $u(x,t) = 2a\sin(kx)\cdot\cos(\omega t)$. Think about how this standing wave differs, or not, from the one in Eq. 40. We still get a standing wave even if the opposite moving wave has a smaller amplitude[3]. Actually, in reality it is much more common that the wave traveling in the opposite direction has a smaller amplitude since it usually originates from an imperfect reflection. See a typical example of a standing wave from a realistic scenario in an electrical antenna system, Fig. 10.

7 Fourier Analysis

To describe phase and group velocity for more realistic waves and pulses (aperiodic waves) we need a more powerful mathematical tool than just trigonometrically adding up the different frequencies. This tool is called the Fourier transform and is a natural extension of the Fourier series that you were introduced to in your mathematics course. The Fourier series for a periodic function $f(x) = f(x+2\pi)$ can be written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} a_n \sin(nx) \tag{41}$$

This Fourier series may not look identical to ones that you have been introduced to previously. The interval over which the trigonometric functions are orthogonal to each other is $[-\pi, \pi]$, and we have also both sine and cosine terms to make it completely general. The coefficients a_n and b_n are, as usual, obtained through the orthogonality relations for sine and cosine,

$$a_0 = \frac{1}{\pi}\int_{-}^{\pi} \pi f(x)\mathrm{d}x \tag{42}$$

$$a_n = \frac{1}{\pi}\int_{-}^{\pi} \pi f(x)\cos(nx)\mathrm{d}x \tag{43}$$

$$b_n = \frac{1}{\pi}\int_{-}^{\pi} \pi f(x)\sin(nx)\mathrm{d}x \tag{44}$$

[3] We let the amplitudes have the same magnitude in our derivation to make the mathematics simpler.

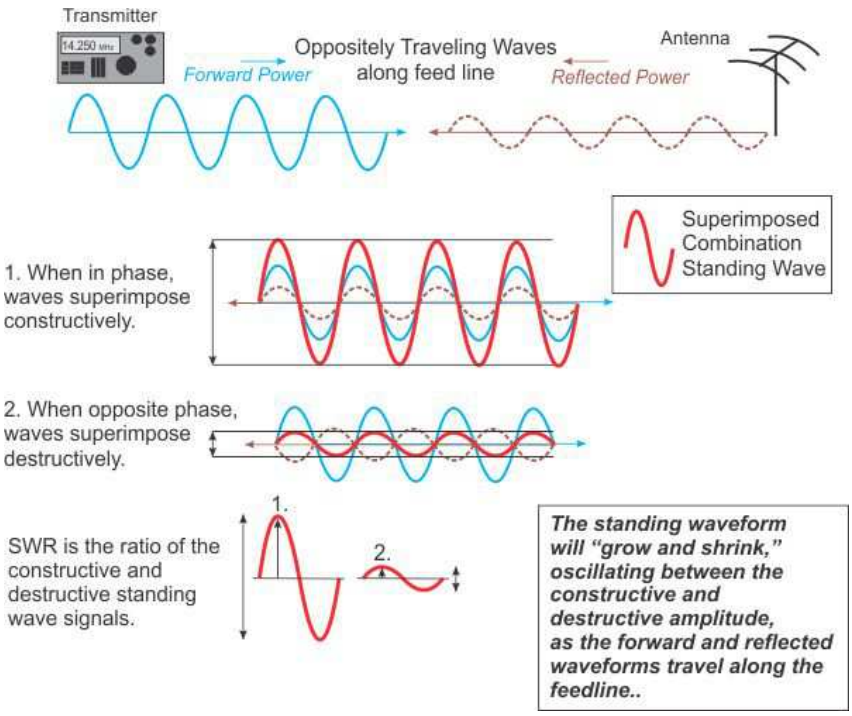

Figure 10: Illustration of a standing wave when the reflection is in- and out- of phase and has a smaller amplitude that the incoming wave. From http://www.hamradioschool.com/t7c04-swr/.

Next step we take is to convert from the spatial domain to the time domain, $x \implies \frac{2\pi}{T}t$. This changes the interval from $[-\pi, \pi]$ to $[-T/2, T/2]$. The Fourier series for $f(t)$ then yields

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t) \tag{45}$$

where we have substituted $\frac{2\pi}{T}$ with the radial frequency ω. Using Euler's formulas we may rewrite the sine/cosine series in terms of complex exponentials,

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t} \tag{46}$$

where

$$c_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) e^{-in\omega t}\,\mathrm{d}t \tag{47}$$

For a non-periodic function, such as a pulse, we let $T \to \infty$. We rewrite Equation 46 with the help of Equation 47 to obtain

$$f(t) = \sum_{n=-\infty}^{\infty} \left(\frac{2}{T} \int_{-T/2}^{T/2} f(t) e^{-in\omega t}\,\mathrm{d}t \right) e^{in\omega t} \tag{48}$$

using $T = 2\pi/\omega$, and a change of variables,

$$f(t) = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \omega \int_{-T/2}^{T/2} f(\xi) e^{in\omega(t-\xi)}\,\mathrm{d}\xi \tag{49}$$

which in the limit $T \to \infty$ brings

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t}\,\mathrm{d}\omega \int_{-\infty}^{\infty} f(\xi) e^{-i\omega\xi}\,\mathrm{d}\xi \tag{50}$$

Getting rid of the temporary "dummy variable" ξ we can then define the Fourier transform as

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,\mathrm{d}t \tag{51}$$

and the inverse Fourier transform as

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\,\mathrm{d}\omega \tag{52}$$

8 Pulses

With the additional tool of the Fourier transform we are now ready to look at waves with more than 2 frequencies, as a matter of fact there is no limit to the number of frequencies that we can mathematically handle. You saw in Figure 8 that the simple addition of just one more frequency produced a much more interesting and complicated wave. If we add many different harmonic frequency waves and all of them with different amplitudes and phase we can e.g. generate a pulse or, as it is also sometimes called, a wave packet, Figure 11. Mathematically, this wave can be described as

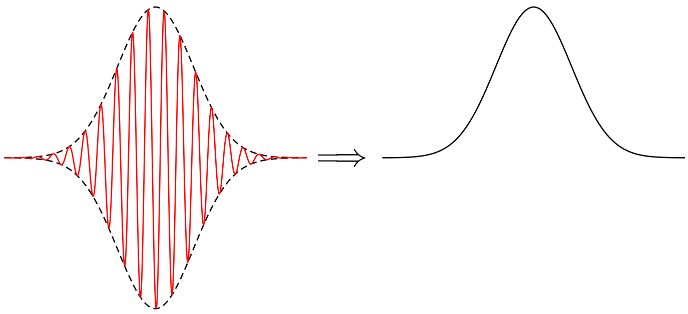

Figure 11: Gaussian pulse envelope with underlying carrier signal together with its usual depiction as only the envelope.

$$u(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(\omega)\,e^{i[\omega t - k(\omega)x]}\,\mathrm{d}\omega \tag{53}$$

The rationale behind this description of a pulse is that the integral adds all the different harmonic waves required to produce the pulse and then, as usual, we tack on the $k(\omega)x$ in the phase to make the whole thing move. Notice that the spatial part is not affected by, or contributing to, the Fourier integral. Additionally, we are allowing $k = k(\omega)$ to be a function of ω. Realistic systems are typically *dispersive*, meaning that harmonic waves with different frequencies travel with different velocities. Most pulses of interest, in all areas of engineering and sciences, can be described using the quasi monochromatic approximation $\omega_c >> \Delta\omega$, see Figure 12. Here, ω_c is the carrier frequency and it is usually much larger than the pulse/signal spectral width, $\Delta\omega$. The carrier frequency, ω_c, is usually defined as the average frequency $\overline{\omega}$,[4]. This weighted frequency is, when many frequencies are involved, most easily calculated using the standard statistical formulas for average,

$$\overline{\omega} = \omega_c = \frac{\int_{-\infty}^{\infty} \omega\cdot|U(\omega)|^2\,\mathrm{d}\omega}{\int_{-\infty}^{\infty} |U(\omega)|^2\,\mathrm{d}\omega} \tag{54}$$

[4]Just like in the case for two frequencies.

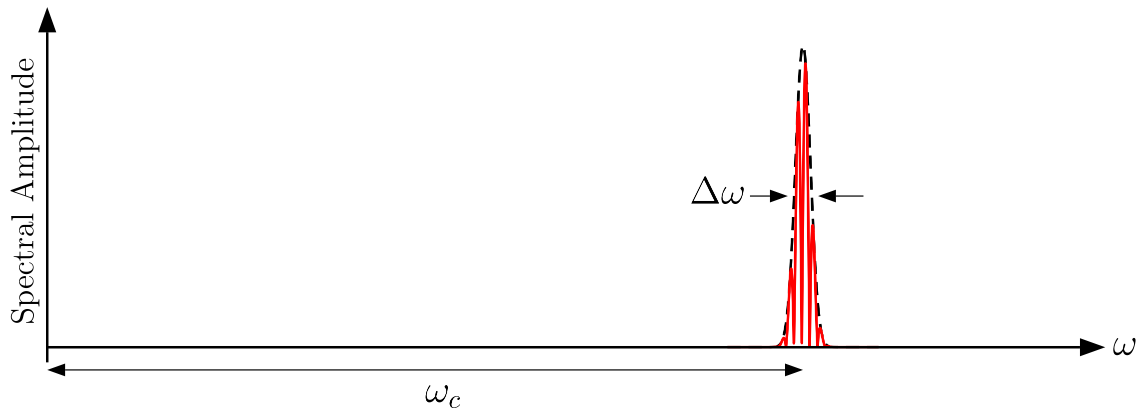

Figure 12: Common spectral scenario, the pulse carrier frequency is much larger than the spectral pulse width.

The frequency-dependence for $k(\omega)$ can be complicated, specifically around a system resonance, but over the narrow spectral region $\Delta\omega$ it is usually varying so little that a Taylor expansion around the carrier frequency ω_c is sufficient to accurately describe the pulse propagating through the medium described by $k(\omega)$. A Taylor expansion up to second-order is often sufficient,

$$k(\omega) = k(\omega_c) + \frac{\partial k}{\partial \omega}\Big|_{\omega_c}(\omega-\omega_c) + \frac{1}{2}\frac{\partial^2 k}{\partial \omega^2}\Big|_{\omega_c}(\omega-\omega_c)^2 + \ldots \tag{55}$$

Inserting Equation 55 into Equation 53 we obtain

$$u(t,x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} U(\omega)\, e^{i\left[\omega t - (k(\omega_c) + \frac{\partial k}{\partial \omega}\big|_{\omega_c}(\omega-\omega_c) + \frac{1}{2}\frac{\partial^2 k}{\partial \omega^2}\big|_{\omega_c}(\omega-\omega_c)^2)x\right]}\, \mathrm{d}\omega \tag{56}$$

This expression is easier to interpret if we parse it into units that can be explained physically

$$u(t,x) = A(t,x)\, e^{i[\omega_c t - k(\omega_c)x]} \tag{57}$$

where

$$A(t,x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} U(\omega)\, e^{i\left((\omega-\omega_c)\left[t - \frac{\partial k}{\partial \omega}\big|_{\omega_c} x\right]\right)} \cdot e^{i\left(\frac{1}{2}(\omega-\omega_c)^2\left[t - \frac{\partial^2 k}{\partial \omega^2}\big|_{\omega_c} x\right]\right)}\, \mathrm{d}\omega \tag{58}$$

Equation 57 says that a pulse[5] can be described as a carrier wave times an amplitude. The only difference from earlier is that the amplitude is now also a function of time and space. This amplitude is the same as the envelope for the case when we added only harmonic waves. The carrier wave, at frequency ω_c, travels with the phase velocity $v_{ph} = \frac{\omega_c}{k(\omega_c)}$ while the amplitude/envelope travels with the group velocity $v_g = \frac{\partial k}{\partial \omega}\Big|_{\omega_c}$. This only leaves the second term in the amplitude to be explained. It describes how the group velocity varies with frequency and it is referred to as group velocity dispersion, GVD.

We can illustrate the properties of the various terms in the Taylor expansion using a pulse with a Gaussian temporal profile,

$$u(x=0,t) = A\, e^{-\frac{1}{4}\left[\frac{t}{\Delta t_0}\right]^2} \tag{59}$$

where A and Δt are, in general, known quantities. The Δt_0 is the RMS width of the pulse defined as

$$\Delta t_0 = \sqrt{\overline{t^2} - \overline{t}^2} \tag{60}$$

[5]Which is a wave consisting of the sum of many different frequencies.

where

$$\bar{t} = \frac{\int_{-\infty}^{\infty} t \cdot |u(x,t)|^2 \, \mathrm{d}t}{\int_{-\infty}^{\infty} |u(x,t|^2 \, \mathrm{d}t} \tag{61}$$

and

$$\overline{t^2} = \frac{\int_{-\infty}^{\infty} t^2 \cdot |u(x,t)|^2 \, \mathrm{d}t}{\int_{-\infty}^{\infty} |u(x,t|^2 \, \mathrm{d}t} \tag{62}$$

The RMS width is the preferred option to describe pulse width since it is independent of shape. The choice of a Gaussian pulse is very common for describing many realistic situations. We use the phasor formalism out of mathematical convenience but it should be remembered that the actual field is a real function of time and space. To propagate this pulse in space and time is performed most conveniently in the frequency-domain. The whole process is outlined in Figure 13 and is a straightforward 4-step process which requires only a few lines of code in a computer program. Linear pulse propagation can be cast in the words of linear system theory where $e^{-ik(\omega)x}$ is the transfer function which propagates our pulse a distance x. The first step

$$\begin{tikzcd} u(x=0,t) \arrow[d, "\mathrm{FFT}"'] & u(x,t) \\ U(x=0,\omega) \arrow[r, "\times\, e^{-ik(\omega)x}"] & U(x,\omega) \arrow[u, "\mathrm{IFFT}"'] \end{tikzcd}$$

Figure 13: Linear propagation of a wave/pulse in a medium defined by the wave propagation constant $k(\omega)$.

is then to convert over to the frequency domain,

$$U(x=0,\omega) = \sqrt{4\pi}\Delta t_0 \, e^{-\Delta t_0^2 \omega^2} \tag{63}$$

The next step is to multiply with the transfer function $e^{-ik(\omega)x}$. The remarkable thing to notice here is that this transfer function does the whole propagation distance in one move! In other words, it doesn't matter if the distance is one nanometer or one lightyear, same procedure, just change the distance x. To finally obtain the temporal pulse at x, we perform the inverse Fourier transform,

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} U(x=0,\omega) \, e^{-ik(\omega)x} \, \mathrm{d}\omega \tag{64}$$

Naturally, to perform this integral we need to know the function $k(\omega)$. This function is generally not known so we have to rely on some type of physical model or other strategy. In the next section we will discuss some of the physical models available to us but for know we will use the Taylor expansion approach. You may argue that we don't know the three constants, $k_c = k(\omega_c)$, $\left.\frac{\partial k}{\partial \omega}\right|_{\omega_c}$, and $\left.\frac{\partial^2 k}{\partial \omega^2}\right|_{\omega_c}$, so what's the point? Well, the point is that these three constants can be measured quite accurately for most materials of interest.

9 Dispersion relations

An important part of analyzing dynamic systems is to find the dispersion relation for the particular wave equation. The dispersion relation is a function $\omega(k)$ or $k(\omega)$, where ω is the usual radial frequency and k is the propagation constant. The dispersion relation connects time and space, and thereby displays what types of waves are allowed in the particular wave system studied. It, also, shows if there are frequencies that cannot sustain a propagating wave, so called *cut-off*, and if there are any resonances in the system, in that case the dispersion relation "blows up". In other words, with some practice there is a lot of information that can be obtained from a dispersion relation.

9.1 Propagation constant k is a function of frequency

From the dispersion relation we obtain the phase and group velocities. For many different materials the propagation constant k is a function of frequency ω. This means that our wave - which is proportional to $e^{i(\omega t - kx)}$ - has a phase Φ which varies with frequency as the wave propagates through the material and as a consequence the phase and group velocities are different. This is shown more formally below,

$$v_{ph} = \frac{\omega}{k} \Rightarrow \frac{\partial v_{ph}}{\partial \omega} = \frac{\partial}{\partial \omega}\left(\frac{\omega}{k}\right) = \frac{1}{k} + \omega \frac{\partial}{\partial \omega}\left(\frac{1}{k}\right) \tag{65}$$

$$\frac{\partial v_{ph}}{\partial \omega} = \frac{1}{k} + \omega \frac{-\frac{\partial k}{\partial \omega}}{k^2} = \frac{1}{k}\left[1 - v_{ph} \cdot \frac{\partial k}{\partial \omega}\right] = \frac{1}{k}\left[1 - \frac{v_{ph}}{v_g}\right] \Rightarrow \frac{v_{ph}}{v_g} = 1 - k\frac{\partial v_{ph}}{\partial \omega}$$

$$v_g = \frac{v_p}{1 - k\frac{\partial v_{ph}}{\partial \omega}} \tag{66}$$

Except for electromagnetic waves in space (vacuum) the phase and group velocities are usually different. Note that a monochromatic wave always have the same phase and group velocity.

9.2 Finding the dispersion relation

The most common and perhaps simplest way of finding a dispersion relation, when the relevant wave/d-iffusion equation is known is to invoke the solution of a plane wave, $e^{i(\omega t - kx)}$. Specifically,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \tag{67}$$

Inserting $u = e^{i(\omega t - kx)}$, gives

$$(-ik)(-ik) \cdot u = \frac{(i\omega)(i\omega)}{v^2} \cdot u \tag{68}$$

which leads to the dispersion relation

$$\omega^2 = v^2 \cdot k^2 /, \Longrightarrow \omega = \pm v \cdot k \tag{69}$$

where the minus sign represents backwards traveling waves. We find therefore that the frequency ω is a linear function of the propagation constant k. For this system the phase and group velocities are the same, v. It is important to note that even this seemingly straightforward expression can become complicated in materials where the propagation constant k is a function of ω. In Figure 14 we show qualitatively a few important dispersion relations to illustrate their nonlinear behaviour. It should be clear from these curves that the group-velocity can be: positive, negative and smaller or larger than the phase velocity. These dispersion relations will all be derived in the following sections.

9.3 Complex propagation constant

A consequence of using the complex formalism for describing a wave is that most constants and variables can and will be complex too. That means that both the real and imaginary parts have to have physical interpretations. Since the propagation constant k and the frequency ω are of interest for our dispersion relations let's investigate what their real and imaginary parts mean. We continue to use the plane wave formalism $u = e^{i(\omega t - kx)}$, and a complex propagation constant

$$k = k_r - ik_i \tag{70}$$

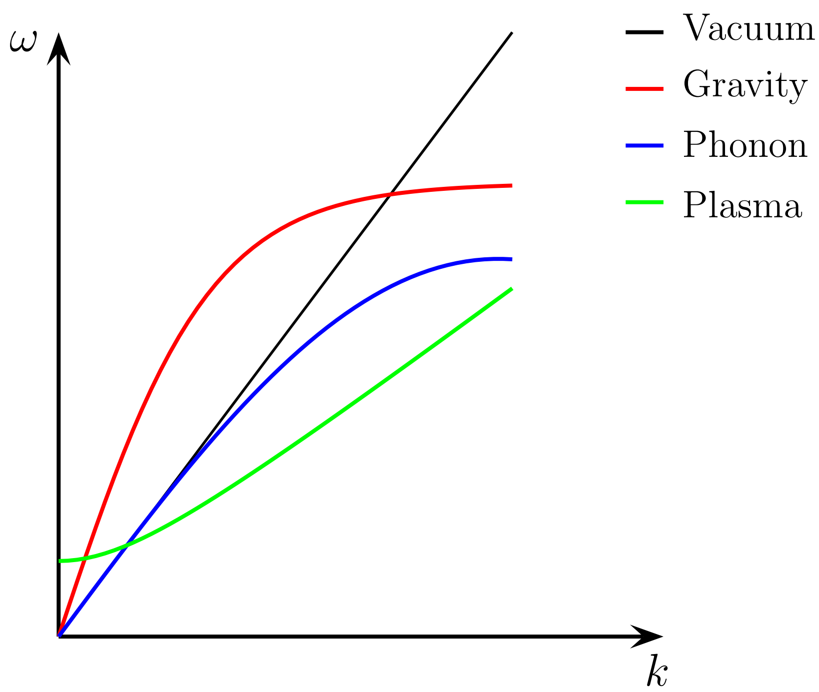

Figure 14: Depiction of four important dispersion relationships.

which turns into

$$u = e^{-ik_i x} e^{i(\omega t - k_r x)} \tag{71}$$

The meaning of the complex propagation constant is now clear, the real part, k_r, is the "usual" propagation constant, determining direction and velocity of the wave. The imaginary part, k_i, determines the exponential decay of the amplitude of the wave with distance x. k_i is proportional to the attenuation constant. You may argue that it was very convenient for us to define the propagation constant with a minus sign between the real and imaginary parts such that the amplitude decay came out with the correct sign. Turns out it really doesn't matter, if we had chosen to define $k = k_r + ik_i$ our interpretation of k_i would have been as the *negative* of the attenuation constant instead. You will see later in this chapter as we find $k(\omega)$ for real physical situations that the sign takes care of itself. The natural state of affairs is that a wave propagating through a medium will loose energy as part of the interaction. For a complex frequency

$$\omega = \omega_r + i\omega_i \tag{72}$$

and again using the plane wave formalism

$$u = e^{-i\omega_i t} e^{i(\omega_r t - kx)} \tag{73}$$

We find that the real part of the frequency acts as the "usual" frequency and the imaginary part, ω_i, controls how fast the wave dampens out in time.You will find that when a system is described by a second-order differential equation with a first-order time derivative there will be damping that can be expressed using a complex frequency.

9.4 Dispersion relations - Lorentz and Drude models

A famous dispersion relation is the Lorentz model for electromagnetic waves interacting with a dielectric. Each atom in the material can be treated as a harmonic oscillator with spring constant κ that describes how strongly the positive nucleus and negative electron are coupled and the damping coefficient Γ which indicates how long it takes for the induced oscillations to dampen after the external excitation has been removed.

$$m\mathbf{a} = \mathbf{F}_E + \mathbf{F}_{damping} + \mathbf{F}_{spring} \tag{74}$$

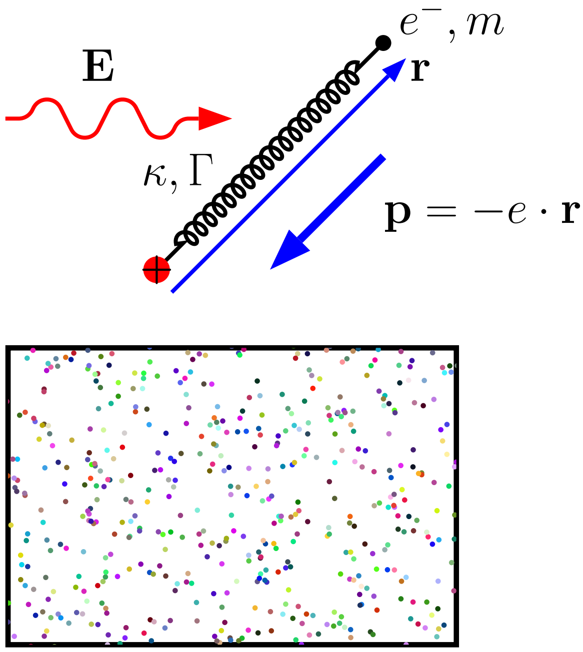

Figure 15: Lorentz model for dielectrics.

To find the dispersion relation we follow the suggestion above and assume that $\mathbf{F}_E = -e\mathbf{E_0}\exp(i\omega t)$ where $\mathbf{E} = \mathbf{E}_0 \exp(i\omega t)$ is the external electric field that induces the oscillation onto the electron-nuclei pair. Since the nuclei is approximately three orders of magnitude heavier than the electron we may assume that the nuclei is stationary and only the electron moves. The restoring spring force is $\mathbf{F}_{spring} = -\kappa\mathbf{r}$ and the damping is proportional to the rate of change of distance between the nuclei and electron, $\mathbf{F}_{damp} = -m\Gamma \mathrm{d}\mathbf{r}/\mathrm{d}t$. With these assumptions the ordinary differential equation governing the oscillations is

$$m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} + m\Gamma\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} + \kappa\mathbf{r} = -e\mathbf{E}e^{i\omega t} \tag{75}$$

Historically, there are two cases, $\kappa = 0$ (Drude) and $\kappa \neq 0$ (Lorentz). The latter case provides a, surprisingly, accurate description of an electromagnetic wave interacting with an individual atom. The $\kappa = 0$ case describes free electrons in a plasma or in a metal. This is the case we will pursue. Introducing the velocity $\mathbf{v} = \mathrm{d}\mathbf{r}/\mathrm{d}t$ into Equation 75 with $\kappa = 0$ we obtain,

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + m\Gamma\mathbf{v} = -e\mathbf{E} \tag{76}$$

Introducing the convection current density as $\mathbf{J} = -eN\mathbf{v}$ we can rewrite the previous equation as

$$\frac{\mathrm{d}\mathbf{J}}{\mathrm{d}t} + \Gamma\mathbf{J} = \frac{Ne^2}{m}\mathbf{E} \tag{77}$$

where N is the electron density (# of electrons per m^3). Assuming a time-harmonic varying current and electric field, $\mathbf{J} = \mathbf{J}_0 \exp(i\omega t)$ and $\mathbf{E} = \mathbf{E}_0 \exp(i\omega t)$, for this differential equation we obtain,

$$\mathbf{J}(i\omega + \Gamma) = \frac{Ne^2}{m}\mathbf{E} \implies \mathbf{J} = \sigma\mathbf{E} \tag{78}$$

where

$$\sigma = \sigma_{ac} = \frac{\frac{Ne^2}{m}}{\Gamma + i\omega} \tag{79}$$

$\sigma_{dc} = Ne^2/m\Gamma$ describes how easily the electrons move when a dc electric field is applied across a medium and the σ_{ac} describes the electron movements for an alternating field at frequency ω. The physical interactions contributing to the damping coefficient Γ for the plasma current is primarily from collisions between electrons and to some degree from recapture by the ions in the plasma. For electrons in a metal it is collisions between the electrons themselves and with the atoms in the metal that contribute to the value of the damping coefficient.

Next step on the way to find the dispersion relation for electromagnetic waves traveling in plasmas or metals is to obtain the relevant wave equation. It is based on Equation **??**, you will show this in Problem 8.x,

$$\nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} + \frac{1}{\epsilon_0 c^2}\frac{\partial \mathbf{J}}{\partial t} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} + \frac{\sigma}{\epsilon_0 c^2}\frac{\partial \mathbf{E}}{\partial t} \tag{80}$$

For the sake of simplicity and without sacrificing the underlying physics we limit ourselves to the one-dimensional wave equation,

$$\frac{\partial^2 E_x}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2} + \frac{\sigma_{ac}}{\epsilon_0 c^2}\frac{\partial E_x}{\partial t} \tag{81}$$

Using the time-harmonic, plain wave approach, $E_x = E_0 e^{i(\omega t - kx)}$ in Equation 81, we obtain the dispersion relation

$$-k^2 = -\frac{\omega^2}{c^2} + \frac{\sigma}{\varepsilon_0 c^2} i\omega \tag{82}$$

which can be simplified to

$$k^2 = \frac{\omega^2}{c^2}\left(1 - i\frac{\sigma}{\varepsilon_0 \omega}\right) \tag{83}$$

To gain some insight into this dispersion relation let's look at the real and imaginary parts. Unfortunately, the complete expressions for the real and imaginary part of the propagation constant k are very complicated. What is usually done is to make use of the relative dielectric constant, ε_r,

$$k^2 = \frac{\omega^2}{c^2}\varepsilon_r \tag{84}$$

Defining $\varepsilon_r = \varepsilon_{rr} + i\varepsilon_{ri}$, you will show in Problem 8.x, that they are

$$\varepsilon_{rr} = 1 - \frac{\omega^2 \omega_p^2}{(\omega^2)^2 + (\omega\Gamma)^2} \tag{85}$$

$$\varepsilon_{ri} = -\frac{\omega\Gamma\omega_p^2}{(\omega^2)^2 + (\omega\Gamma)^2} \tag{86}$$

We conclude from these equations that attenuation can occur for a propagating wave in a Drude medium if there is no damping, i.e. $\Gamma = 0$, or if $\omega < \omega_p$, where $\omega_p^2 = e^2 N/m\varepsilon_0$ is the plasma frequency of the electron oscillations excited by the electromagnetic wave $E_x = E_0 e^{i(\omega t - kx)}$.

9.5 Dispersion relations - Phonons

Phonons are particles associated with the movements of atoms in the lattice. As with every particle there is an associated wave[6] and consequently this wave is described through the propagation constant k. In most practical situations these movements are small and can be described mathematically as harmonic vibrations. If we limit ourselves to a one-dimensional lattice we can use the formalism in section 2.4.3 for the mass-spring chain to describe the motion of an atom numbered n

$$M\frac{\mathrm{d}^2 u_n}{\mathrm{d}t^2} = \sum_p K_p[u_{n+p} - u_n] \tag{87}$$

This is the same equation as Eq. 2.29 with the only difference that we are using the displacement u from equilibrium as the dependent variable rather than the absolute position characterized by x. As in section 2.4.3 K_n represents the spring constant for atom n. The sum indicates that *all* atoms in the lattice interacts

[6]This is the famous de Broglie hypothesis

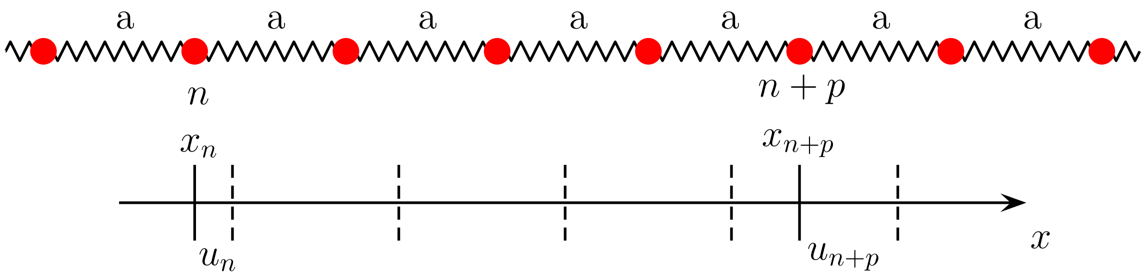

Figure 16: One-dimensional linear lattice modeled as a mass-spring chain.

with a given atom n. We know from experience (chapter 4!) that the solution to Equation 87 can be written as

$$u_n = A\, e^{i(\omega t - kx_n)} \tag{88}$$

where, for this particular example with a perfectly periodic lattice, $x_n = n \cdot a$, and k is the wavenumber. This solution represents a traveling wave. Due to the inherent periodicity from the lattice we also have

$$u_{n+p} = A\, e^{i(\omega t - k(n+p)a)} \tag{89}$$

Inserting our two wave solutions into our equation of motion, Eq.87 we obtain,

$$MA(i\omega)^2\, e^{i(\omega t - kna)} = \sum_p A\, K_p[e^{-ik(n+p)a} - e^{-ik(n)a}]e^{i\omega t} \tag{90}$$

This equation can be simplified by recognizing identical factors on both sides of the equal sign

$$M\omega^2 = \sum_p K_p[e^{-ikpa} - 1] \tag{91}$$

Furthermore, symmetry requires that $K_p = K_{-p}$. Specifically, this requirement comes from the assumption that all the atoms and forces between them (represented by the springs) are identical (perfect lattice) so whether we observe the interactions to the left or right of our atom n it is the same. This leads to

$$M\omega^2 = \sum_{p>0} K_p[e^{-ikpa} + e^{ikpa} - 2] \tag{92}$$

Using Euler's formula this can be rewritten as

$$\omega^2 = \frac{2}{M}\sum_{p>0} K_p[1 - \cos kpa] = \frac{4}{M}\sum_{p>0} K_p \sin^2 \frac{kpa}{2} \tag{93}$$

Finally, if we only allow interactions between nearest neighbours, $p = 1$ we end up with the following dispersion relation

$$\omega^2 = \frac{4}{M} K_1 \sin^2 \frac{ka}{2} \tag{94}$$

Compare with homogeneous string highest frequency and periodicity.

9.6 Dispersion relationships - summary

Dispersion - the fact that different frequencies in a wave *may* propagate with different speeds - has primarily two different causes; firstly, the medium in which the wave is propagating has different properties for different wavelengths/frequencies. This is because all matter is quantized into different energy levels and subsequently different energy quanta ($E = hf = hc/\lambda$) resonate differently with the medium. Energy at a frequency that gets absorbed by the medium is *slowed* down relative to a frequency that isn't absorbed. Secondly, the specific geometry that the wave may be constrained to will also contribute to the wave vectors frequency dependence since different wavelengths will relate/compare differently to the *size* of the structure that the wave is propagating through. From our examples in this chapter we have seen both cases illustrated. The spring-mass model for electromagnetic radiation interacting with an infinite medium consisting of N identical atoms resulted in a $k(\omega)$ that only depended on the resonant frequency ω_0 of the medium. The surface gravity waves, on the contrary, are not affected by its medium properties but rather by the limited depth H, a geometrical constraint. For the waves propagating along the coaxial cable described through the Telegrapher's equation we are, again, finding that it is the material properties ($R, G, C,$ and$L$) that influence the dispersion relation. Then for the final example it is again geometrical considerations, distance a between atoms in relationship to the wavelength, that determine the dispersion of the wave.

10 Appendices

10.1 Waves - General Stuff

In chapers 3 and 4 we introduced one-dimensional systems described by the diffusion and wave equations. With the help of chapters 5-7 we are now in a position to describe three-dimensional waves. It is natural to assume that we began with one-dimensional wave systems for pedagogical reasons - to keep the math as simple as possible - and even though this is partially true it is also correct to say that a one-dimensional description of realistic structures is often more than adequate for sufficiently describing/simulating a system. So, why then devote a whole chapter to three-dimensional waves? The two main reasons are:
(1) every now and then a 3D description is absolutely necessary
and, perhaps more importantly,
(2) to be able to understand how we can approximate a three-dimensional system with a one-dimensional system.

The reason waves, in general, are so important to understand is that they transport, well... pretty much everything: mass, charge, energy, momentum, pressure, pollutants, etc. Waves can be divided into *mechanical* and *electromagnetic*. Mechanical waves require a medium to propagate in, electromagnetic (E&M) waves do not. Although they are, of course, perfectly able to propagate in any medium. How are waves related to

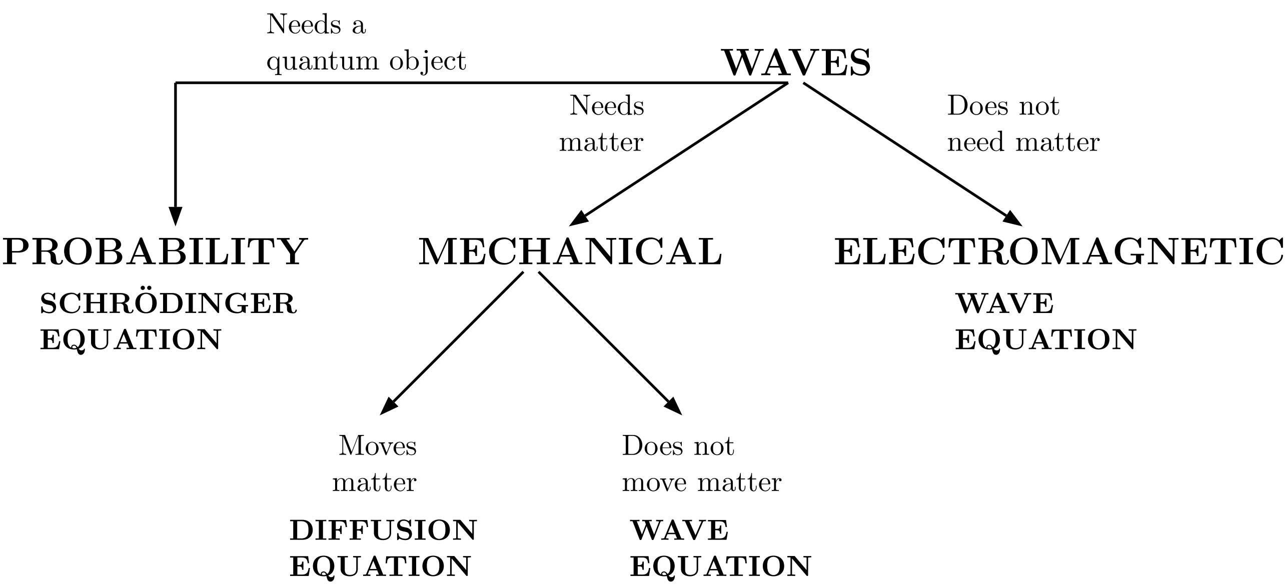

Figure 17: Overview of waves.

our vector fields?
In short, a wave is a time-varying vector field. The wave is a manifestation of changes moving across a

field. A closer look also shows that a wave governed by the wave equation

$$\nabla^2 \mathbf{F} + \frac{1}{v^2}\frac{\partial^2 \mathbf{F}}{\partial^2 t} = 0$$

in contrast to the diffusion equation

$$\nabla^2 \mathbf{F} + \frac{1}{D}\frac{\partial \mathbf{F}}{\partial t} = 0$$

always have an accompanying field. For example, an electromagnetic wave is a combination of the electric (**E**) and the magnetic (**H**) fields. Similarly, a sound wave is a combination of a velocity field (**v**) and a pressure field (p). **F** in the equations above can be substituted by any of the fields **E**, **H**, **v**, and p. A distinguishing feature is then that the wave is moving forward by "oscillating" between the two accompanying fields, one being more "inductive" and the other more "capacitive". As a consequence the wave equation is typically derived from a set of equations that link the accompanying fields through a set of temporal and spatial derivatives. A wave always originates from equations linking time and space. The above mentioned waves are derived from the following set of equations

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \mathbf{H} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

and similarly for the sound wave

$$\nabla \cdot \mathbf{v} = -\frac{1}{P_0 \gamma}\frac{\partial p}{\partial t}$$

$$\nabla p = \varrho_0 \frac{\partial \mathbf{v}}{\partial t}$$

The sound wave is an example of a mechanical wave and it is longitudinal, recognized by the divergence in its equation. E&M waves in vacuum are characterized by their curl and are transversal. The complete set of electromagnetic equations (Maxwell) have, additionally, two equations for the divergence and as a consequence electromagnetic waves can be both transversal and longitudinal.

Additionally, mechanical waves differ from electromagnetic waves in that they can transport matter (particles). This is described via the diffusion equation (second order in space but only first order in time). Both the diffusion and the wave equation have mathematically similar solutions but their interpretations are quite different. Specifically, the wave equation waves are associated with a velocity (energy) that ultimately cannot be larger than the speed of light in vacuum while diffusive waves do not have such speed limitations.

10.2 Uniform Plane Waves

A solution to the one-dimensional wave equation, as we saw in chapter 4, is the time and space harmonic function,

$$u(z,t) = A\cos(\omega t - kz)$$

In free space the equivalent solution is

$$\mathbf{E}(\mathbf{r},t) = \mathbf{A}\cos(\omega t - \mathbf{k}\cdot\mathbf{r}) \tag{95}$$

This solution is referred to as a monochromatic, uniform, plane wave and as in previous chapters we use the *phasor notation* to write it as

$$\mathbf{E}(\mathbf{r},t) = \mathbf{A}\, e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})} = \mathbf{A}\, e^{i\omega t}\cdot e^{-i\mathbf{k}\cdot\mathbf{r}} \tag{96}$$

The monochromatic part comes from the fact that there is only *one* ω - or one frequency - in this wave. This is not a limitation since many engineering systems are linear we can use Fourier analysis as in previous chapters to add up all the frequencies that may be part of a more realistic, complicated wave. Therefore, assuming a time-harmonic solution ($E \propto e^{i\omega t}$) we can focus on the Helmholtz' equation and the spatial properties of the wave.

The spatial part of the wave can now be written as

$$\mathbf{E}(\mathbf{r}) = \mathbf{A}e^{-i\mathbf{k}\cdot\mathbf{r}} \tag{97}$$

where

$$\mathbf{k} = \hat{\mathbf{k}}k = \hat{\mathbf{k}}\frac{2\pi}{\lambda} = \hat{\mathbf{x}}k_x + \hat{\mathbf{y}}k_y + \hat{\mathbf{z}}k_z$$

and $\hat{\mathbf{k}}$ is the unit vector pointing in the direction of propagation, $\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$ and, as usual, the position vector $\mathbf{r}$ is

$$\mathbf{r} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z$$

From Figure 18 we gather, from geometrical considerations, that there exist, for the $e^{-i\mathbf{k}\cdot\mathbf{r}}$ wave, a plane for

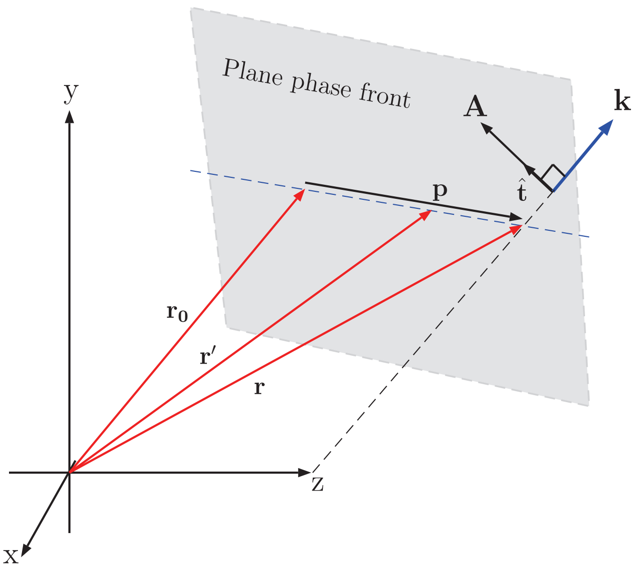

Figure 18: Geometry for a uniform plane wave.

which

$$\hat{\mathbf{k}}\cdot\mathbf{r} = constant \tag{98}$$

We call this plane the phase front. It is uniform because the amplitude $\mathbf{A}$ is a constant, not a function of (x, y, z). For a vector wave, such as the electromagnetic field $\mathbf{E}$ the direction of its amplitude $\mathbf{A}$ represents the *polarization* of the wave. Fortunately, without any loss of generality, we can set $A = A_x = A_y = A_z$ and let the direction of $\mathbf{A}$ be in a general direction, e.g. $\hat{t}$, see Figure 18. Our Equation 97 is then a *plane wave* since the planes perpendicular to the direction of propagation have the same magnitude $|\mathbf{E}|$. More specifically, using the *normal form* to describe the plane

$$\mathbf{k}\cdot\mathbf{p} = 0 \tag{99}$$

where $\mathbf{p}$ is a vector in the plane, $\mathbf{p} = \mathbf{r} - \mathbf{r}_0$ and $\mathbf{k}$ is by definition perpendicular to the phase plane. As a result, $0 = \mathbf{k}\cdot\mathbf{p} = \mathbf{k}\cdot[\mathbf{r} - \mathbf{r}_0] \implies \hat{\mathbf{k}}\cdot\mathbf{r} = \hat{\mathbf{k}}\cdot\mathbf{r}_0$. The constant in Equation 98 is then equal to $|\mathbf{k}||\mathbf{r}_0|$ and its magnitude is a function of the distance from the origin represented via the position vector $\mathbf{r}_0$.

A plane wave is, naturally, a 1D wave since the second-order partial derivatives in the directions perpendicular to the direction of propagation are zero[7]. This is most easily verified by letting the direction of propagation be along one of the coordinate axes, e.g. the z-direction and perform the partial derivations for the x- and y-directions.

Associated with deriving the wave equation for a transversal wave was the rotation of the field and similarly for the wave equation for the longitudinal field was the divergence of the field. Let us see what we obtain when we take the rotation and the divergence of a uniform plane wave,

[7] Remember that second-order derivatives informs us how much curvature there is and since we are dealing with a plane these are, of course, zero.

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Ae^{-i\mathbf{k}\cdot\mathbf{r}} & Ae^{-i\mathbf{k}\cdot\mathbf{r}} & Ae^{-i\mathbf{k}\cdot\mathbf{r}} \end{vmatrix} \tag{100}$$

If we solve this determinant we obtain

$$Ae^{-i\mathbf{k}\cdot\mathbf{r}} \cdot \left[\hat{\mathbf{x}}(ik_y - ik_z) + \hat{\mathbf{y}}(ik_z - ik_x) + \hat{\mathbf{z}}(ik_x - ik_y)\right]$$

This can be cleaned up to read $i\mathbf{k} \times \mathbf{E}$ which leads to

$$\nabla \times \mathbf{E} = i\mathbf{k} \times \mathbf{E} \tag{101}$$

This equation tells us that the **E**-field is directed perpendicular to the direction of propagation **k**. An equivalent exercise for the divergence will reveal that

$$\nabla \cdot \mathbf{E} = i\mathbf{k} \cdot \mathbf{E} \tag{102}$$

In the case of electromagnetic waves in vacuum we assumed $\nabla \cdot \mathbf{E} = 0$ and consequently we have no longitudinal components for the electric field. This is contrary to the acoustic field where

$$\nabla \cdot \mathbf{v} = -\frac{1}{P_0\gamma}\frac{\partial p}{\partial t} \Longrightarrow i\mathbf{k} \cdot \mathbf{v} = -\frac{1}{P_0\gamma}\frac{\partial p}{\partial t}$$

This shows that the temporal changes in the pressure fluctuations drive the longitudinal velocity component of the acoustic wave. It is a good exercise to confirm that $\nabla \times \mathbf{v} = 0$ proving that there is no transversal component of the acoustic wave.

Can there ever be a longitudinal electromagnetic wave or a transversal acoustic wave? The answer is yes in both cases but to show that we need more complicated geometries and/or materials so that will have to wait for later courses. Assuming a plane, monochromatic, uniform wave we can infer a few more properties about our fields. First thing we do is to take the spatial and temporal derivatives for our two sets of equations for the transversal and longitudinal waves resulting in

$$-i\mathbf{k} \times \mathbf{E} = -i\mu_0\omega\mathbf{H} \tag{103}$$
$$-i\mathbf{k} \times \mathbf{H} = i\varepsilon_0\omega\mathbf{E} \tag{104}$$

and

$$-i\mathbf{k}p = -i\varrho_0\omega\mathbf{v} \tag{105}$$
$$-i\mathbf{k} \cdot \mathbf{v} = -\frac{i\omega}{P_0\gamma}p \tag{106}$$

The first set of equations show that both the electric and magnetic fields are perpendicular to each other, Figure 19. Additionally, using Equation 102 we find

$$\mathbf{k} \cdot \mathbf{E} = 0 \tag{107}$$

which proves that both the electric and magnetic fields are perpendicular to the direction of propagation.

Similarly, the second set shows that both the velocity of the acoustic disturbance and the pressure wave itself are both oscillating in the direction of propagation.

Returning to the E&M waves and rewriting Equation 103 we find that

$$\mathbf{H} = \frac{1}{\eta}\hat{\mathbf{k}} \times \mathbf{E} \tag{108}$$

where $\eta = \sqrt{\frac{\mu_o}{\varepsilon_0}}$ is the intrinsic impedance of space and it is $\approx 377\Omega$. This intrinsic impedance is a consequence of the fact that an electromagnetic wave (light !) cannot propagate faster than $c \approx 3 \cdot 10^8\ m/s$. The relationship of the magnitudes in Equation 108 is a manifestation of Ohm's law in a point $|E| = \eta|H|$ where E is associated with the voltage and H with the current.

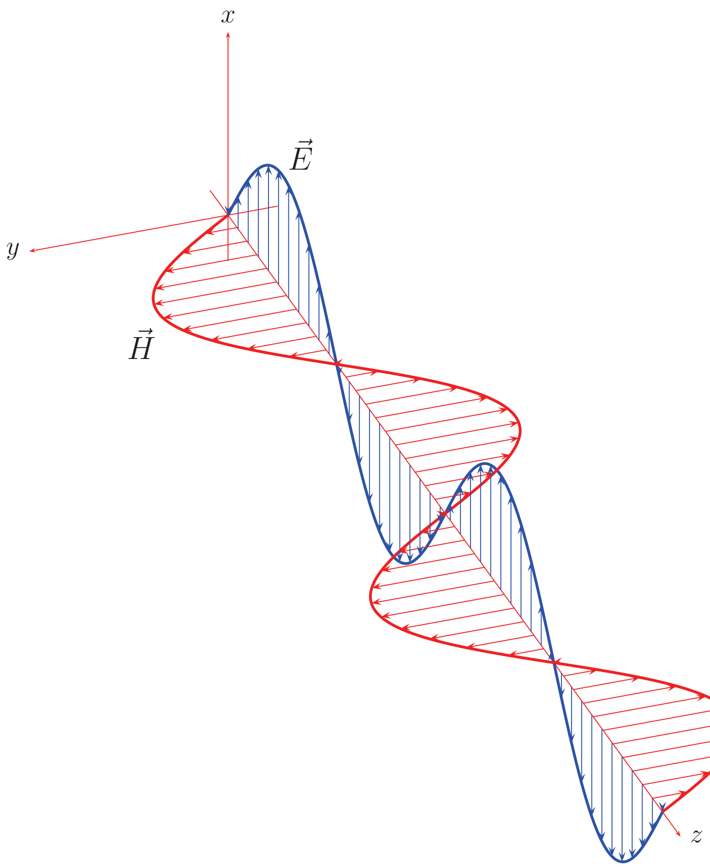

Figure 19: Your generic TEM electromagnetic wave in vacuum.

Likewise we can define an acoustic impedance Z. To do so we write the two planar acoustic waves as

$$\mathbf{v} = \hat{\mathbf{k}}\tilde{v}e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})}$$

and

$$p = \tilde{p}e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})}$$

The acoustic impedance is defined as

$$Z = \frac{\tilde{p}}{\tilde{v}} \tag{109}$$

which, with the help of Equations 105 becomes

$$Z = v\varrho_0 = \sqrt{\gamma P_0 \varrho_0} = \sqrt{\mathrm{E}\varrho_0} \tag{110}$$

where E is the elasticity module for the medium in which the acoustic wave is propagating[8] .

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[8] We apologize that E is used for different physical variables. This is a problem trying to write a cross-disciplinary text.