DEPARTMENT OF ELECTRONIC SYSTEMS • NTNU

TFE 4130

BOUNDARY CONDITIONS

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Contents

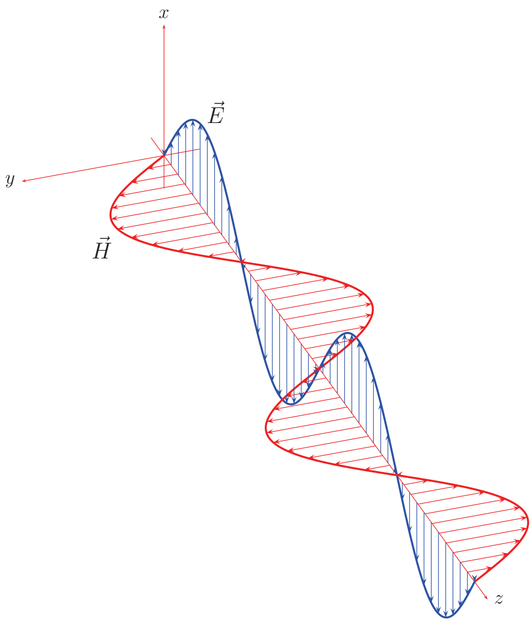

Figure 1: Your generic TEM electromagnetic wave in vacuum.

1 Introduction and background

In this handout we will expand on the discussion in the book (pp. 364-367) regarding polarization of an electromagnetic wave. Please note that, *polarization is a property of the electromagnetic field* and all we are trying to provide you with is a mathematical descripton of this vector property, we're not trying to explain why nature has electromagnetic fields with this particular attribute ! This material is based on section 2.5 in S.J. Orfanidis book on Electromagnetic Waves and Antennas.

We have earlier derived a time-harmonic field with both a forward and backward traveling wave;

$$\mathbf{E}(z) = \mathbf{a}_x \left\{ E_0^+ e^{-jk_0 z} + E_0^- e^{+jk_0 z} \right\} \tag{1}$$

For the purposes of introducing the concept of polarization we will only be concerned with the forward traveling wave. On the other hand we will have to add another vector component of the field,

$$\mathbf{E}(z) = (\mathbf{a}_x E_{10} + \mathbf{a}_y E_{20})\, e^{-jk_0 z} \tag{2}$$

You may ask, what about the polarization of the **H**-field ? Well, as you see from Figure 1 as long as we determine all the properties of the **E**-field we will have all the information about the **H**-field from the equation

$$\mathbf{H} = \frac{1}{\eta}\mathbf{a}_n \times \mathbf{E} \tag{3}$$

So, pursuing the **E**-field our next step will be to bring back the time variation $e^{j\omega t}$, i.e. the instantaneous field is

$$\mathbf{E}(z,t) = (\mathbf{a}_x E_{10} + \mathbf{a}_y E_{20})\, e^{j(\omega t - k_0 z)} \tag{4}$$

The reason for this is that if we don't have the time variation in Equation 4 the wave isn't moving and subsequently it couldn't oscillate having a polarization, making this whole exercise moot !

Introducing the new variables A, B, φ_x, and φ_y we can write

$$E_{10} = Ae^{j\varphi_x} \tag{5}$$

$$E_{20} = Be^{j\varphi_y} \tag{6}$$

Subsequently,

$$\mathbf{E}(z,t) = \mathbf{a}_x E_x(z,t) + \mathbf{a}_y E_y(z,t) \tag{7}$$

where (going back to the trigonometric description)

$$E_x(z,t) = A\cos(\omega t - k_0 z + \varphi_x) \tag{8}$$

$$E_y(z,t) = B\cos(\omega t - k_0 z + \varphi_y) \tag{9}$$

Let's analyze the polarization for a given z (e.g. $z = 0$, why not, any z is eligible so let's pick an easy one !)

$$\begin{aligned} E_x(0,t) &= A\cos(\omega t + \varphi_x) \\ E_y(0,t) &= B\cos(\omega t + \varphi_y) \end{aligned} \tag{10}$$

Leading us to the "new" field

$$\mathbf{E}(t) = \mathbf{a}_x E_x(t) + \mathbf{a}_y E_y(t) \tag{11}$$

which rotates in the xy-plane with an angular frequency ω. To move on we first need to expand the cosine terms in Equations 10,

$$E_x(t) = A[\cos\omega t\cos\varphi_x - \sin\omega t\sin\varphi_x] \tag{12}$$

$$E_y(t) = B[\cos\omega t\cos\varphi_y - \sin\omega t\sin\varphi_y] \tag{13}$$

Introducing $\varphi = \varphi_x - \varphi_y$ and solving for $\cos\omega t$ and $\sin\omega t$ in terms of E_x, E_y, A, B, and φ, we obtain

$$\cos\omega t\sin\varphi = \frac{E_y(t)}{B}\sin\varphi_x - \frac{E_x(t)}{A}\sin\varphi_y \tag{14}$$

$$\sin\omega t\sin\varphi = \frac{E_y(t)}{B}\cos\varphi_x - \frac{E_x(t)}{A}\cos\varphi_y \tag{15}$$

$$\tag{16}$$

Next thing to do is to square both sides and add the two equations which results in

$$\sin^2\varphi = \left(\frac{E_y(t)}{B}\sin\varphi_x - \frac{E_x(t)}{A}\sin\varphi_y\right)^2 + \left(\frac{E_y(t)}{B}\cos\varphi_x - \frac{E_x(t)}{A}\cos\varphi_y\right)^2 \tag{17}$$

This can be simplified by completing the squares to finally yield the equation for the **POLARIZATION ELLIPSE**

$$\boxed{\frac{E_x^2}{A^2} + \frac{E_y^2}{B^2} - 2\cos\varphi\frac{E_x E_y}{AB} = \sin^2\varphi} \tag{18a}$$

Depending on the the values of A, B, and φ, this "ellipse" may be an ellipse, circle or a straight line.

2 Linear Polarization

We will give two examples of how linear polarization can be obtained. For both cases we have $A \neq B$ and $\varphi = 0$ and $\varphi = \pi$, respectively. It is important to note that it doesnt matter what φ_x and φ_y are individually, it is the difference $\varphi = \varphi_x - \varphi_y$ that determines in what direction the linearly polarized light will oscillate. E.g., for the first case we are describing here we could have $\varphi_x = 0$ and $\varphi_y = 0$ resulting in $\varphi = 0$, however, $\varphi_x = \pi/2$ and $\varphi_y = \pi/2$ would result in the same linear polarization. The only period of time when you could possibly spot the difference between these two scenarios is in the very beginning when the light is "starting up" its oscillations ! So, the possible phasor amplitudes associated with Equation 11 are $\mathbf{E} = \mathbf{a}_x A \pm \mathbf{a}_y B$ for $\varphi = 0$ and $\varphi = \pi$, respectively. Substituting φ with 0 and π in the equation for the polarization ellipse above results in,

$$\frac{E_x^2}{A^2} + \frac{E_y^2}{B^2} \mp \frac{E_x E_y}{AB} = 0 \quad \Longrightarrow \quad \left(\frac{E_x}{A} \mp \frac{E_y}{B}\right)^2 = 0 \tag{19}$$

which represents the two straight lines

$$E_y = \pm \frac{B}{A} E_x \tag{20}$$

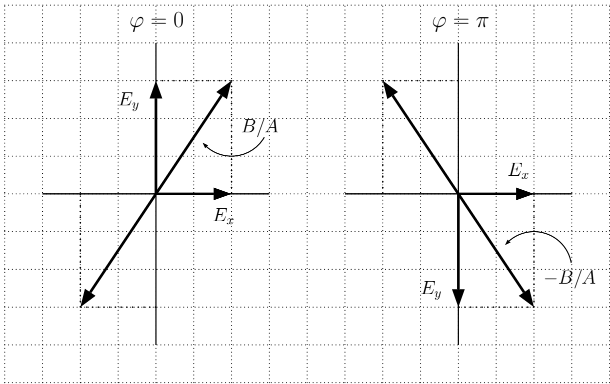

Figure 2: Two possible cases resulting in a linearly polarized field

For these two specific cases, resulting in linear polarized fields we obtain the instantaneous fields

$$E_x(t) = A\cos\omega t \qquad\qquad E_x(t) = A\cos\omega t \tag{21}$$

$$E_y(t) = B\cos\omega t \qquad\qquad E_y(t) = B\cos(\omega t - \pi) = -B\cos\omega t \tag{22}$$

for $\varphi = 0$ and $\varphi = \pi$, respectively.

3 Circular Polarization

To obtain circular polarization is much more stringent than for any of the other polarizations since $A \equiv B$ and the phase difference φ has to be either $+\pi/2$ or $-\pi/2$. If we substitute $A = B$ and $\varphi = \pm\pi/2$ into the equation for the polarization ellipse we obtain,

$$\frac{E_x^2}{A^2} + \frac{E_y^2}{A^2} = 1 \quad \Longrightarrow \quad E_x^2 + E_y^2 = A^2 \tag{23}$$

i.e. the equation of a circle with radius A in the xy-plane. The instantaneous fields are,

$$E_x(t) = A\cos\omega t \tag{24}$$

$$E_y(t) = A\cos\left(\omega t \mp \frac{\pi}{2}\right) = \pm A\sin\omega t \tag{25}$$

for $\varphi = \pm\pi/2$, respectively. To determine if the field is *right-* or *left-* circularly polarized we need a convention and we choose the one put forward by the IEEE organization. It states, To determine the parity of a circularly polarized electromagnetic field, put both of your thumbs in the direction of the propagation of the field. If the total field vector (phasor) follows the direction of the fingers on your right hand it is said to be right-circularly polarized, otherwise it is left-circularly polarized.

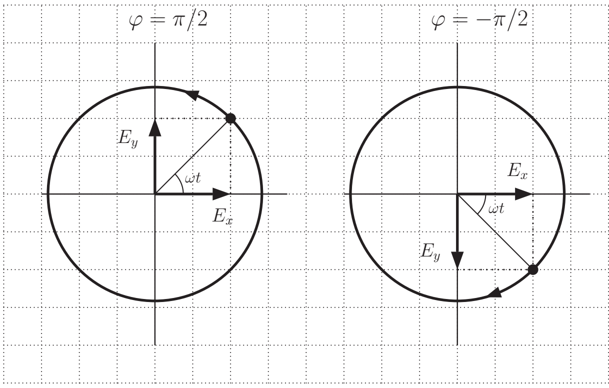

Figure 3: The two cases resulting in a circularly polarized field

4 Elliptical Polarization

Last but not least, elliptical polarization. To describe elliptical polarization we recognize that $A \neq B$, always, and that there are two cases for the phase, either $\varphi = \pm\pi/2$ or it can be completely arbitrary.

If we substitute $A \neq B$ and $\varphi = \pm\pi/2$ into our equation for the polarization ellipse we obtain

$$\frac{E_x^2}{A^2} + \frac{E_y^2}{B^2} = 1 \tag{26}$$

which corresponds to the ellipse in Figure 4a. It's an ellipse with its major and minor axes aligned with the x- and y-axis. For the arbitrary phase we again get an ellipse

$$\frac{{E'}_x^2}{{A'}^2} + \frac{{E'}_y^2}{{B'}^2} = 1 \tag{27}$$

but in a rotated coordinate system for which we have

$$E'_x = E_x \cos\theta + E_y \sin\theta \tag{28}$$
$$E'_y = -E_x \sin\theta + E_y \cos\theta \tag{29}$$

and its new major and minor axes,

$$A' = \sqrt{\frac{1}{2}(A^2 + B^2) + \frac{s}{2}\sqrt{(A^2 - B^2)^2 + 4A^2B^2\cos^2\varphi}} \tag{30}$$
$$B' = \sqrt{\frac{1}{2}(A^2 + B^2) - \frac{s}{2}\sqrt{(A^2 - B^2)^2 + 4A^2B^2\cos^2\varphi}} \tag{31}$$

where $s = \mathit{signum}(A - B)$. The angle θ, see Figure 4b, is given by,

$$\tan 2\theta = \frac{2AB}{A^2 - B^2}\cos\varphi \tag{32}$$

For those of you who are curious how you derive Equation 32 please note that the two ellipses in Figure 4 have to be related via the following *similarity transformation*

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \frac{1}{A^2} & -\frac{\cos\varphi}{AB} \\ -\frac{\cos\varphi}{AB} & \frac{1}{B^2} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \sin^2\varphi \begin{bmatrix} \frac{1}{A'^2} & 0 \\ 0 & \frac{1}{B'^2} \end{bmatrix} \tag{33}$$

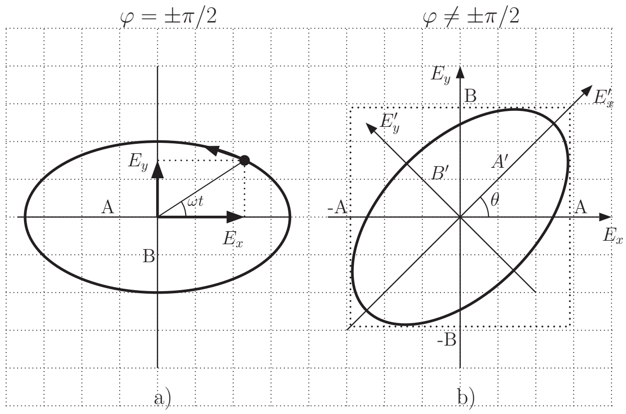

Figure 4: The two cases resulting in an elliptically polarized field

where the matrix

$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{34}$$

defines the rotation between the un-primed and primed coordinate systems.

Finally, what about the case when $A = B$ and the phase φ is anything, EXCEPT FOR, $0, \pm\frac{\pi}{2}, or \pm\pi$. In that case we need to go back to our *polarization ellipse* equation together with the coordinate transformations in Equation 28. After some more algebra we find that the polarization is again *elliptical*, the ellipse is rotated 45 degrees, i.e. $\theta = 45°$, regardless what the phase difference φ is, and the semi-axis are $A' = \sqrt{2}A\cos\frac{\varphi}{2}$ and $B' = \sqrt{2}A\sin\frac{\varphi}{2}$.

5 Example

So this is all very well, but how do you actually use all this math !?

A common problem (at least as long as you take classes at the university !) can be that you are given an instantaneous field, e.g.

$$\mathbf{E}(z) = \left(\mathbf{a}_x 4 + \mathbf{a}_y 3e^{-j\pi/4}\right) e^{j(\omega t - k_0 z)} \tag{35}$$

and you want to know everything there is about the polarization properties of this field. The first thing to do is to re-write the field for $z = 0$ as we did in our derivation of the polarization ellipse.

$$E_x(t) = 4\cos\omega t \tag{36}$$
$$E_y(t) = 3\cos(\omega t - \frac{\pi}{4}) \tag{37}$$

Since $A = 4$ is not equal to $B = 3$ and the phase difference $\varphi = \varphi_x - \varphi_y = 0 - (-\pi/4) = \pi/4 \neq \pm\pi/2$ we immediately realize that this is an elliptically polarized field with its major and minor axes at some angle θ to the original x-axis. The perhaps easiest way to proceed is to write down all the variables we have defined and see what we get:

A	B	φ	$\sin\varphi$	A'	B'	θ
4	3	45°	0.5	4.66	1.82	33.8°

This means that we have an elliptically polarized field with its major axis at an angle 33.8° to the x-axis and it is right hand polarized since $\sin\varphi > 0$. By the way, why is that ?