Øving 8

Ønsker tilbakemelding :)

2.1

1) $A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{pmatrix}$

$C = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 3 & 5 \\ -1 & 4 \end{pmatrix}$

a) $-2A = \underline{\underline{\begin{pmatrix} -4 & 0 & 2 \\ -8 & 10 & -4 \end{pmatrix}}}$

b) $B - 2A = \begin{pmatrix} 7-4 & -5+0 & 1+2 \\ 1-8 & -4+10 & -3-4 \end{pmatrix}$

$= \underline{\underline{\begin{pmatrix} 3 & -5 & 3 \\ -7 & 6 & -7 \end{pmatrix}}}$

c) $A \cdot C$ doesn't make sense because we need #columns of A to be equal to #rows of C.

d) $CD = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 3 & 5 \\ -1 & 4 \end{pmatrix}$

$= \left(\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 4 \end{pmatrix} \right)$

$= \begin{pmatrix} 3-2 & 5+4 \\ -6-1 & -10+4 \end{pmatrix}$

$= \underline{\underline{\begin{pmatrix} 1 & 9 \\ -7 & -6 \end{pmatrix}}}$

10) $A = \begin{pmatrix} 2 & -3 \\ -4 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 8 & 4 \\ 5 & 5 \end{pmatrix}$, $C = \begin{pmatrix} 5 & -2 \\ 3 & 1 \end{pmatrix}$

AB:

$$\begin{array}{cc} & \begin{pmatrix} 8 & 4 \\ 5 & 5 \end{pmatrix} \\ \begin{pmatrix} 2 & -3 \\ -4 & 6 \end{pmatrix} & \begin{matrix} 2\cdot 8-3\cdot 5 & 2\cdot 4-3\cdot 5 \\ -4\cdot 8+6\cdot 5 & -4\cdot 4+6\cdot 5 \end{matrix} \end{array}$$

$$\Rightarrow AB = \begin{pmatrix} 1 & -7 \\ -2 & 14 \end{pmatrix}$$

AC:

$$\begin{array}{cc} & \begin{pmatrix} 5 & -2 \\ 3 & 1 \end{pmatrix} \\ \begin{pmatrix} 2 & -3 \\ -4 & 6 \end{pmatrix} & \begin{matrix} 2\cdot 5-3\cdot 3 & -2\cdot 2-3\cdot 1 \\ -4\cdot 5+6\cdot 3 & +4\cdot 2+6\cdot 1 \end{matrix} \end{array}$$

$$\Rightarrow AC = \begin{pmatrix} 1 & -7 \\ -2 & 14 \end{pmatrix}$$

So $AB = AC = \begin{pmatrix} 1 & -7 \\ -2 & 14 \end{pmatrix}$ ✓

11)

$$A = \begin{pmatrix}1&1&1\\1&2&3\\1&4&5\end{pmatrix}, \quad D = \begin{pmatrix}2&0&0\\0&3&0\\0&0&5\end{pmatrix}$$

AD:

$$\begin{array}{cc} & \begin{pmatrix}2&0&0\\0&3&0\\0&0&5\end{pmatrix} \\ \begin{pmatrix}1&1&1\\1&2&3\\1&4&5\end{pmatrix} & \begin{matrix}2&3&5\\2&6&15\\2&12&25\end{matrix}\end{array}$$

$$\Rightarrow AD = \begin{pmatrix}2&3&5\\2&6&15\\2&12&25\end{pmatrix}$$

DA:

$$\begin{array}{cc} & \begin{pmatrix}1&1&1\\1&2&3\\1&4&5\end{pmatrix} \\ \begin{pmatrix}2&0&0\\0&3&0\\0&0&5\end{pmatrix} & \begin{matrix}2&2&2\\3&6&9\\5&20&25\end{matrix}\end{array}$$

$$\Rightarrow DA = \begin{pmatrix}2&2&2\\3&6&9\\5&20&25\end{pmatrix}$$

When D is multiplied with A on the left ($D \cdot A$), this amounts to doing three row operations on A

(i) $R_1 \leftarrow 2 \cdot R_1$

(ii) $R_2 \leftarrow 3 \cdot R_2$

(iii) $R_3 \leftarrow 5 \cdot R_3$

When D is multiplied on the right $(A \cdot D)$, D essentially does three column operations on A

(i) $C_1 \leftarrow 2 \cdot C_1$
(ii) $C_2 \leftarrow 3 \cdot C_2$
(iii) $C_3 \leftarrow 5 \cdot C_3$

Let $B = A^{-1}$, then $A \cdot B = B \cdot A = I$

Want to find A^{-1}

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 5 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_2 \leftarrow R_2 - R_1 \\ R_3 \leftarrow R_3 - R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 3 & 4 & -1 & 0 & 1 \end{array}\right) \quad R_3 \leftarrow R_3 - 3R_2$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & -2 & 2 & -3 & 1 \end{array}\right) \quad \begin{array}{l} R_2 \leftarrow R_2 + R_3 \\ R_1 \leftarrow R_1 + \frac{1}{2} R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 1 & -2 & 1 \\ 0 & 0 & -2 & 2 & -3 & 1 \end{array}\right) \quad \begin{array}{l} R_1 \leftarrow R_1 - R_2 \\ R_3 \leftarrow -\frac{1}{2} R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 1 & -2 & 1 \\ -4 & 6 & -2 \end{matrix}}_{A^{-1}} \\ 0 & 0 & 1 & \end{array}\right)$$

So if $B = A^{-1} = \begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 1 & -2 & 1 \\ -4 & 6 & -2 \end{pmatrix}$

then $A \cdot B = B \cdot A$

29) Let A be an $m \times n$ matrix and assume B and C are such that AB, AC and $A(B+C)$ are defined. We want to show that (b) $A(B+C) = AB + AC$ and (c) $(B+C)A = BA + CA$.

b) Consider the (i,j)-entry in AB. It is by definition equal to

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$$

The (i,j)-entry in $AB + AC$ is then

$$(AB + AC)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} + \sum_{k=1}^{n} a_{ik} c_{kj}$$

$$= \sum_{k=1}^{n} (a_{ik} b_{kj} + a_{ik} c_{kj})$$

$$= \sum_{k=1}^{n} a_{ik} (b_{kj} + c_{kj})$$

But this is exactly equal to the (i,j)-entry of $A(B+C)$ so $A(B+C)$ must be equal to $AB + AC$. ■

c) Right distrubitivity can be proven almost the same way.

The (i,j)-entry in $(B+C)A$ is

$$
\begin{aligned}
((B+C)A)_{ij} &= (b+c)_{i1}a_{1j} + \dots + (b+c)_{im}a_{mj} \\
&= \sum_{k=1}^{m} (b+c)_{ik}a_{kj} \\
&= \sum_{k=1}^{m} (b_{ik}+c_{ik})a_{kj} \\
&= \sum_{k=1}^{m} b_{ik}a_{kj} + c_{ik}a_{kj} \\
&= \sum_{k=1}^{m} b_{ik}a_{kj} + \sum_{k=1}^{m} c_{ik}a_{kj} \\
&= (BA)_{ij} + (CA)_{ij}
\end{aligned}
$$

So we conclude that

$$(B+C)A = BA + CA \qquad \blacksquare$$

33)

2.2

25) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

The equation $A\vec{x} = \vec{0}$ can be written as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Which reveals a system of equations.

(i) $ax_1 + bx_2 = 0$

(ii) $cx_1 + dx_2 = 0$

Solving for x_1 and x_2 gives

(i) $x_1 = -\frac{b}{a}x_2$

(ii) $c(-\frac{b}{a})x_2 + dx_2 = 0 \quad | \cdot a$

$\Leftrightarrow \quad adx_2 = bcx_2$

$\Leftrightarrow \quad ad = bc$

$\Leftrightarrow \quad ad - bc = 0$

If $ad-bc$ is zero, then there will always be some x_1, x_2 such that $A\vec{x} = \vec{0}$ (non-trivial), and if $ad-bc$ is not zero, then the system has no solutions except the trivial one, $\vec{x} = \vec{0}$.

For square matrices, having a determinant equal to zero is equivalent to not being invertible.

31)

$$\left(\begin{array}{ccc|ccc} \multicolumn{3}{c|}{\overbrace{\hphantom{1\quad 0\quad -2}}^{=A}} & & & \\ 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_2 \Leftarrow R_2 + 3R_1 \\ R_3 \leftarrow R_3 - 2R_1 \end{array}$$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 10 & 3 & 1 & 0 \\ 0 & -3 & 0 & -2 & 0 & 1 \end{array}\right) \quad R_3 \leftarrow R_3 + 3R_2$$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 10 & 3 & 1 & 0 \\ 0 & 0 & 30 & 7 & 3 & 1 \end{array}\right) \quad \begin{array}{l} R_2 \leftarrow R_2 - \frac{1}{3} R_3 \\ R_1 \leftarrow R_1 - \frac{1}{15} R_3 \end{array}$$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 8/15 & -1/5 & -1/15 \\ 0 & 1 & 0 & 2/3 & 0 & -1/3 \\ 0 & 0 & 30 & 7 & 3 & 1 \end{array}\right) \quad R_3 \leftarrow \frac{1}{30} R_3$$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 8/15 & -1/5 & -1/15 \\ 0 & 1 & 0 & 2/3 & 0 & -1/3 \\ 0 & 0 & 1 & 7/30 & 1/10 & 1/30 \\ & & & \multicolumn{3}{c}{\underbrace{\hphantom{7/30\quad 1/10\quad 1/30}}_{A^{-1}}} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 8/15 & -1/5 & -1/15 \\ 2/3 & 0 & -1/3 \\ 7/30 & 1/10 & 1/30 \end{pmatrix}$$

3)

$$\left(\overbrace{\begin{matrix}1 & 0 & -2\\ -3 & 1 & 4\\ 2 & -3 & 4\end{matrix}}^{A}\middle|\begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}\right)$$

$R_2 \leftarrow R_2 + 3R_1$
$R_3 \leftarrow R_3 - 2R_1$

$$\Rightarrow \left(\begin{array}{ccc|ccc}1 & 0 & -2 & 1 & 0 & 0\\ 0 & 1 & -2 & 3 & 1 & 0\\ 0 & -3 & 8 & -2 & 0 & 1\end{array}\right)$$

$R_3 \leftarrow R_3 + 3R_2$

$$\Rightarrow \left(\begin{array}{ccc|ccc}1 & 0 & -2 & 1 & 0 & 0\\ 0 & 1 & -2 & 3 & 1 & 0\\ 0 & 0 & -2 & 7 & 3 & 1\end{array}\right)$$

$R_1 \leftarrow R_1 - R_3$
$R_2 \leftarrow R_2 - R_3$

$$\Rightarrow \left(\begin{array}{ccc|ccc}1 & 0 & 0 & -6 & -3 & -1\\ 0 & 1 & 0 & -4 & -2 & -1\\ 0 & 0 & -2 & 7 & 3 & 1\end{array}\right)$$

$R_3 \leftarrow -\frac{1}{2}R_3$

$$\Rightarrow \left(\begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}\middle|\underbrace{\begin{matrix}-6 & -3 & -1\\ -4 & -2 & -1\\ -\frac{7}{2} & -\frac{3}{2} & -\frac{1}{2}\end{matrix}}_{A^{-1}}\right)$$

The inverse of the matrix (A) is

$$\begin{pmatrix}-6 & -3 & -1\\ -4 & -2 & -1\\ -\frac{7}{2} & -\frac{3}{2} & -\frac{1}{2}\end{pmatrix}$$

24) Suppose $A\vec{x}=\vec{b}$ has a solution for all $\vec{b}\in\mathbb{R}^n$, (A is $n{\times}n$). Then A has n pivot positions, so one in every row. This means that reduced echelon form of A is I_n, the $n{\times}n$ identity matrix. Because A can be reduced to I_n, A must be invertible, because the operations that transform A to I_n, also transform I_n to A^{-1}. (theorem 7)

2.3 2) $\det\begin{pmatrix}-4 & 6\\ 6 & -9\end{pmatrix} = 4\cdot 9 - 6\cdot 6 = 0$

It is not invertible

7) $A = \begin{pmatrix}-1 & -3 & 0 & 1\\ 3 & 5 & 8 & -3\\ -2 & -6 & 3 & 2\\ 0 & -1 & 2 & 1\end{pmatrix}$ Want to make A triangular. $R_2 \leftarrow R_2 + 3R_1$, $R_3 \leftarrow R_3 - 2R_1$

$\sim \begin{pmatrix}-1 & -3 & 0 & 1\\ 0 & -4 & 8 & 0\\ 0 & 0 & 3 & 0\\ 0 & -1 & 2 & 1\end{pmatrix}$ $R_4 \leftarrow R_4 - \frac{1}{4}R_2$

$\sim \begin{pmatrix}-1 & -3 & 0 & 1\\ 0 & -4 & 8 & 0\\ 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$

$\det A = (-1)\cdot(-4)\cdot(3)\cdot(1)$
$= 12$

$\det A \neq 0$ so A is invertible

11) a. True
b. True
c. False
d. True
e. True

13) A square triangular matrix is invertible when the determinant is non-zero. The determinant is the product of the main diagonal because the matrix is triangular. If no entries in the diagonal are zero, the the product will also be non-zero. This is of course the same as having n pivot positions, so a square triangular matrix is invertible if all entries in the main diagonal are non-zero.

2.5

1) Row reduction:

$$\left(\begin{array}{ccc|c} 3 & -7 & -2 & -7 \\ -3 & 5 & 1 & 5 \\ 6 & -4 & 0 & 2 \end{array}\right) \quad \begin{array}{l} R_2 \leftarrow R_2 + R_1 \\ R_3 \leftarrow R_3 - 2R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 3 & -7 & -2 & -7 \\ 0 & -2 & -1 & -2 \\ 0 & 10 & 4 & 16 \end{array}\right) \quad R_3 \leftarrow R_3 + 5R_2$$

$$\sim \left(\begin{array}{ccc|c} 3 & -7 & -2 & -7 \\ 0 & -2 & -1 & -2 \\ 0 & 0 & -1 & 6 \end{array}\right) \quad \begin{array}{l} R_1 \leftarrow R_1 - 2R_3 \\ R_2 \leftarrow R_2 - R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 3 & -7 & 0 & -19 \\ 0 & -2 & 0 & -8 \\ 0 & 0 & -1 & 6 \end{array}\right) \quad \begin{array}{l} R_1 \leftarrow \frac{1}{3} R_1 \\ R_2 \leftarrow -\frac{1}{2} R_2 \\ R_3 \leftarrow -R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & -\frac{7}{3} & 0 & -\frac{19}{3} \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -6 \end{array}\right) \quad R_1 \leftarrow R_1 + \frac{7}{3} R_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -6 \end{array}\right)$$

$$\Rightarrow \vec{x} = \begin{pmatrix} 3 \\ 4 \\ -6 \end{pmatrix}$$

LU-factorization:

$$A = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{pmatrix}}_{L} \cdot \underbrace{\begin{pmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{pmatrix}}_{U}$$

Want solve $LU\vec{x} = \begin{pmatrix} -7 \\ 5 \\ 2 \end{pmatrix}$

Step 1: $L\vec{y} = \begin{pmatrix} -7 \\ 5 \\ 2 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -7 \\ -1 & 1 & 0 & 5 \\ 2 & -5 & 1 & 2 \end{array}\right) \quad \begin{array}{l} R_2 \leftarrow R_2 + R_1 \\ R_3 \leftarrow R_3 - 2R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -2 \\ 0 & -5 & 1 & 16 \end{array}\right) \quad R_3 \leftarrow R_3 + 5R_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 6 \end{array}\right) \Rightarrow \vec{y} = \begin{pmatrix} -7 \\ -2 \\ 6 \end{pmatrix}$$

Step 2: $U\vec{x} = \vec{y}$

$$\left(\begin{array}{ccc|c} 3 & -7 & -2 & -7 \\ 0 & -2 & -1 & -2 \\ 0 & 0 & -1 & 6 \end{array}\right) \quad \begin{array}{l} R_1 \leftarrow R_1 - 2R_3 \\ R_2 \leftarrow R_2 - R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 3 & -7 & 0 & -19 \\ 0 & -2 & 0 & -8 \\ 0 & 0 & -1 & 6 \end{array}\right) \quad \begin{array}{l} R_1 \leftarrow R_1 - \frac{7}{2} R_2 \\ R_3 \leftarrow -R_3 \end{array}$$

$$\left(\begin{array}{ccc|c} 3 & 0 & 0 & 9 \\ 0 & 2 & 0 & -8 \\ 0 & 0 & 1 & -6 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -6 \end{array}\right)$$

$$\vec{x} = \begin{pmatrix} 3 \\ 4 \\ -6 \end{pmatrix}$$

4) $L = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -5 & 1 \end{pmatrix}$ $\quad U = \begin{pmatrix} 2 & -2 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{pmatrix}$

$\vec{b} = \begin{pmatrix} 0 \\ 5 \\ -7 \end{pmatrix}$

Step 1: $L\vec{y} = \vec{b}$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 5 \\ 3/2 & -5 & 1 & -7 \end{array}\right) \quad \begin{array}{l} R_2 \leftarrow R_2 - \frac{1}{2} R_1 \\ R_3 \leftarrow R_3 - \frac{3}{2} R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & -5 & 1 & -7 \end{array}\right) \quad R_3 \leftarrow R_3 + 5 R_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 18 \end{array}\right) \Rightarrow \vec{y} = \begin{pmatrix} 0 \\ 5 \\ 18 \end{pmatrix}$$

Step 2: Solve $U\vec{x} = \vec{y}$

$$\left(\begin{array}{ccc|c} 2 & -2 & 4 & 0 \\ 0 & -2 & -1 & 5 \\ 0 & 0 & -6 & 18 \end{array}\right) \quad \begin{array}{l} R_3 \leftarrow \frac{1}{6} R_3 \\ R_1 \leftarrow \frac{1}{2} R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & -2 & -1 & 5 \\ 0 & 0 & 1 & -3 \end{array}\right) \quad \begin{array}{l} R_2 \leftarrow R_2 + R_3 \\ R_1 \leftarrow R_1 - 2R_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & 6 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 1 & -3 \end{array}\right) \quad \begin{array}{l} R_2 \leftarrow -\frac{1}{2} R_2 \\ \text{then } R_1 \leftarrow R_1 + R_2 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \end{array}\right)$$

$$\underline{\underline{\vec{x} = \begin{pmatrix} 7 \\ 1 \\ -3 \end{pmatrix}}}$$

9) Let $A = \begin{pmatrix} 3 & -1 & 2 \\ -3 & -2 & 10 \\ 9 & -5 & 6 \end{pmatrix} \quad \begin{array}{l} R_2 \leftarrow R_2 - (-1) R_1 \\ R_3 \leftarrow R_3 - 3R_1 \end{array}$

$$\sim \begin{pmatrix} 3 & -1 & 2 \\ 0 & -3 & 12 \\ 0 & -2 & 0 \end{pmatrix} \quad R_3 \leftarrow R_3 - \frac{2}{3} R_2$$

$$\sim \begin{pmatrix} 3 & -1 & 2 \\ 0 & -3 & 12 \\ 0 & 0 & -8 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & \frac{2}{3} & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 3 & -1 & 2 \\ 0 & -3 & 12 \\ 0 & 0 & -8 \end{pmatrix}$$

24) Let $A = QR$, where R is square and upper triangular, ~~and~~ and Q is such that $Q \cdot Q^T = I$ (orthogonal)

The equation $A\vec{x} = \vec{b}$ can then be written as

$$(QR)\vec{x} = \vec{b} \quad \Big| \cdot Q^T$$

$$\iff Q^TQ(R\vec{x}) = Q^T\vec{b}$$

$$\iff R\vec{x} = Q^T\vec{b}$$

We can view $Q^T\vec{b}$ as a linear transform from $\mathbb{R}^n$ to $\mathbb{R}^n$ and thus write $\vec{c} = Q^T\vec{b}$, $\vec{c} \in \mathbb{R}^n$.

This gives $R\vec{x} = \vec{c}$.

By the definition of R, R is invertible so it must have a pivot in every row. Since R then has n pivots the columns of R must span all of $\mathbb{R}^n$, and thus the system also has a unique solution, $\vec{x}$, for all $\vec{c}$ in $\mathbb{R}^n$. $\blacksquare$

The system $R\vec{x} = Q^T\vec{b}$ is a triangular system that can be solved via, for instance backward substitution.

3.1

2) Along first row:

$$\det\begin{pmatrix} 0 & 4 & 1 \\ 5 & -3 & 0 \\ 2 & 3 & 1 \end{pmatrix} = -4\cdot\begin{vmatrix} 5 & 0 \\ 2 & 1 \end{vmatrix} + 1\cdot\begin{vmatrix} 5 & -3 \\ 2 & 3 \end{vmatrix}$$
$$= -4\cdot 5 + 5\cdot 3 + 3\cdot 2$$
$$= -20 + 15 + 6$$
$$= \underline{\underline{1}}$$

Along second column:

$$\begin{vmatrix} 0 & 4 & 1 \\ 5 & -3 & 0 \\ 2 & 3 & 1 \end{vmatrix} = -4\begin{vmatrix} 5 & 0 \\ 2 & 1 \end{vmatrix} + (-3)\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} - 3\cdot\begin{vmatrix} 0 & 1 \\ 5 & 0 \end{vmatrix}$$
$$= -4\cdot 5 - 3\cdot(-2) - 3\cdot(-5)$$
$$= -20 + 6 + 15$$
$$= \underline{\underline{1}}$$

9) Along 3rd row

$$\begin{vmatrix} 4 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 3 & 0 & 0 & 0 \\ 8 & 3 & 1 & 7 \end{vmatrix} = 3\cdot\begin{vmatrix} 0 & 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 7 \end{vmatrix} - 0 + 0 - 0$$
$$= 3\cdot 5\cdot\begin{vmatrix} 7 & 2 \\ 3 & 1 \end{vmatrix}$$
$$= 15\cdot(7 - 3\cdot 2)$$
$$= \underline{\underline{15}}$$

12) Along 1st row

$$\begin{vmatrix} 3 & 0 & 0 & 0 \\ 7 & -2 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ 3 & -8 & 4 & -3 \end{vmatrix} = 3\cdot\begin{vmatrix} -2 & 0 & 0 \\ 6 & 3 & 0 \\ -8 & 4 & -3 \end{vmatrix}$$
$$= 3\cdot(-2)\cdot\begin{vmatrix} 3 & 0 \\ 4 & -3 \end{vmatrix}$$
$$= -6\cdot 3\cdot(-3)$$
$$= \underline{\underline{54}}$$

37) $A = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$, $\begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} = 3\cdot2 - 4\cdot1 = 2$

$5A = \begin{pmatrix} 5\cdot3 & 5\cdot1 \\ 5\cdot4 & 5\cdot2 \end{pmatrix} = \begin{pmatrix} 15 & 5 \\ 20 & 10 \end{pmatrix}$

$\det(5A) = \begin{vmatrix} 15 & 5 \\ 20 & 10 \end{vmatrix} = 15\cdot10 - 20\cdot5 = 50$

Clearly $\det 5A \neq 5 \det A$.
In this case $\det 5A = 5^2 \det A$

38) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix and K a constant scaar.

$KA = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$

$\det A = ad - bc$

$\det KA = kakd - kckb = k^2(ad - cd)$

$= k^2 \det A$

So for 2x2 matrices $\det KA = K^2 \det A$

3.2

10) Let $A = \begin{pmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -2 & -6 \\ -2 & -6 & 2 & 3 & 10 \\ 1 & 5 & -6 & 2 & -3 \\ 0 & 2 & -4 & 5 & 9 \end{pmatrix}$

Want reduce to echelon form

$R_3 \leftarrow R_3 + 2R_1$, $R_4 \leftarrow R_4 - R_1$

$$\sim \begin{pmatrix} 1 & 3 & -1 & 0 & 2 \\ 0 & 2 & -4 & -2 & -6 \\ 0 & 0 & 0 & 3 & -6 \\ 0 & 2 & -5 & 2 & -1 \\ 0 & 2 & -4 & 5 & 9 \end{pmatrix}$$

$R_3 \longleftrightarrow R_5$
then: $R_3 \leftarrow R_3 - R_5$
$R_4 \leftarrow R_4 - R_5$

$$\sim \begin{pmatrix} 1 & 3 & -1 & 0 & 2 \\ 0 & 2 & -4 & -2 & -6 \\ 0 & 0 & 0 & 9 & 15 \\ 0 & 0 & -1 & 4 & 5 \\ 0 & 0 & 0 & 3 & 6 \end{pmatrix}$$

$R_3 \leftrightarrow R_4$
then: $R_5 \leftarrow R_5 - \frac{1}{3} R_4$

$$A_E = \sim \begin{pmatrix} 1 & 3 & -1 & 0 & 2 \\ 0 & 2 & -4 & -2 & -6 \\ 0 & 0 & -1 & 4 & 5 \\ 0 & 0 & 0 & 9 & 15 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We did 2 row interchanges so
$\det A_E = -(-)\det A$
$= \det A$

$\det A_E = 1 \cdot 2 \cdot (-1) \cdot 9 \cdot 1$
$= -18$

The determinant is -18

40) Let A and B be 4×4 matrices such that $\det A = -3$ and $\det B = -1$.

a) $\det AB = \det A \det B$

$= (-3)\cdot(-1)$

$= 3$

b) $\det B^5 = (\det B)^5 = (-1)^5 = -1$

c) $\det 2A = 2^4 \cdot \det A = 16\cdot(-3)$

$= -48$

d) $\det A^T BA = \det A^T \det B \det A$

$= \det A \det B \det A$

$= (-3)(-1)(-3)$

$= -9$

e) $A^{-1}\cdot A = I$

$\Rightarrow \det A^{-1} \cdot \det A = \det I$

$\det A^{-1} = \frac{1}{\det A}$

$\det(B^{-1}AB) = \frac{1}{\cancel{\det B}} \cdot \det A \cdot \cancel{\det B}$

$= \det A$

$= -3$

33) Let A and B be square matrices.
Then $\det AB = \det A \cdot \det B$
$\det A$ and $\det B$ are just "some" numbers,
so we can use commutativity to see
that $\det A \det B = \det B \det A$
But $\det B \det A$ is just $\det BA$, so
we can conclude $\det AB = \det BA$
for all square matrices.