Matte 3 Øving 7

1.8

3) $T: \mathbb{R}^4 \to \mathbb{R}^5$

$T(\vec{x}) = A\vec{x} \quad , \vec{x} \in \mathbb{R}^4$

Since $A\vec{x}$ only is defined when A has the same number of columns as $\vec{x}$ has rows, A has 4 columns. If A is $M \times 4$ it transforms a vector from $\mathbb{R}^4$ to $\mathbb{R}^M$. Thus M is 5 and A has 5 rows.

$\underline{A \text{ is } 5 \times 4}$

17) $\vec{u} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \quad , \quad \vec{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

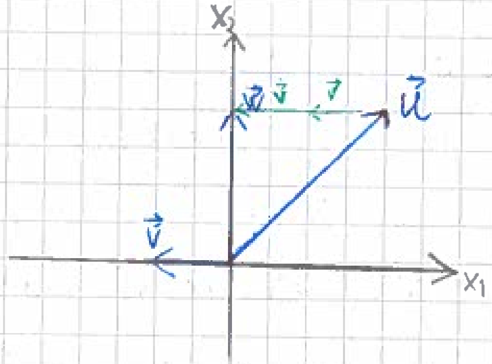
$T(\vec{u}) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad , \quad T(\vec{v}) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$T(3\vec{u}) = 3 \cdot T(\vec{u}) = 3 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \underline{\begin{pmatrix} 6 \\ 3 \end{pmatrix}}$

$T(2\vec{v}) = 2 \cdot T(\vec{v}) = 2 \cdot \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \underline{\begin{pmatrix} -2 \\ 6 \end{pmatrix}}$

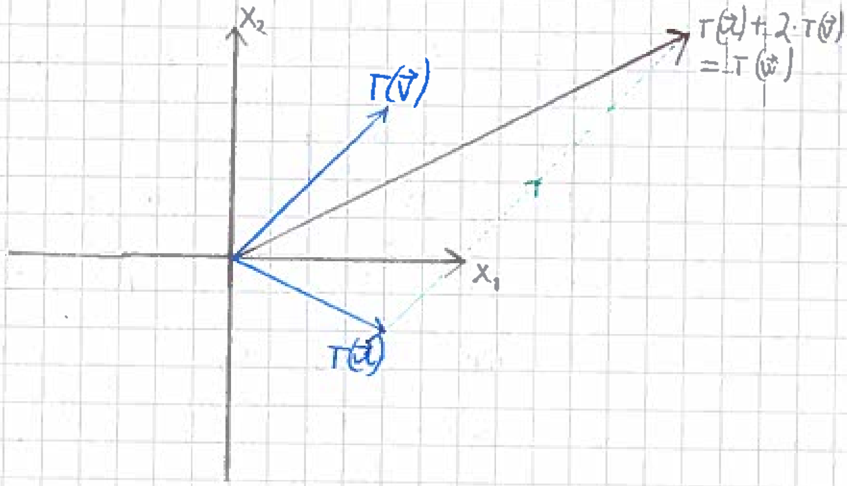
$$T(3\vec{u} + 2\vec{v}) = T(3\vec{u}) + T(2\vec{v})$$
$$= \begin{pmatrix} 6 \\ 3 \end{pmatrix} + \begin{pmatrix} -2 \\ 6 \end{pmatrix}$$
$$= \underline{\begin{pmatrix} 4 \\ 9 \end{pmatrix}}$$

18)



$\vec{w} = \vec{u} + 2\vec{v}$

$\Rightarrow T(\vec{w}) = T(\vec{u}) + 2 \cdot T(\vec{v})$



25) $\vec{v} \neq \vec{0}$, $\vec{p} \in \mathbb{R}^N$

$\vec{x} = \vec{p} + t \cdot \vec{v}$

A linear transform $T: \mathbb{R}^N \to \mathbb{R}^N$ maps $\vec{x}$ to $T(\vec{x})$.

Since $\vec{x} = \vec{p} + t \cdot \vec{v}$, $T(\vec{x}) = T(\vec{p} + t \cdot \vec{v})$.

By linearity, $T(\vec{p} + t \cdot \vec{v}) = T(\vec{p}) + t \cdot T(\vec{v})$.

- If $T(\vec{v}) \neq 0$, then $T(\vec{p}) + t\,T(\vec{v})$ is a line through $T(\vec{p})$ with direction $T(\vec{v})$.
- If $T(\vec{v}) = \vec{0}$, then $T(\vec{p}) + t\,T(\vec{v}) = T(\vec{p})$, which is a point in $\mathbb{R}^N$.

26) $\vec{u}, \vec{v} \in \mathbb{R}^3$

$T: \mathbb{R}^3 \to \mathbb{R}^3$ (linear)

Define $\vec{x} = s \cdot \vec{u} + t \cdot \vec{v}$, $s,t \in \mathbb{R}$ as the plane through $\vec{0}$, $\vec{u}$ and $\vec{v}$.

The transform of that plane is then $T(\vec{x}) = T(s \cdot \vec{u} + t \cdot \vec{v})$. By linearity this is equal to:

$T(\vec{x}) = T(s \cdot \vec{u} + t \cdot \vec{v}) = s \cdot T(\vec{u}) + t \cdot T(\vec{v})$

If $T(\vec{u}) = \vec{0}$, then $T(\vec{x}) = t \cdot T(\vec{v})$ which is a line.

If $T(\vec{v}) = \vec{0}$, then $T(\vec{x}) = s \cdot T(\vec{u})$ which is a line

If both are zero, then $T(\vec{x}) = \vec{0}$ which is a point.

If both $T(\vec{u})$ and $T(\vec{v})$ are non-zero, then the image of P under T will be a plane.

30) Define $T: \mathbb{R}^n \to \mathbb{R}^m$ as $T(\vec{x}) = A\vec{x} + \vec{B}$ with A $m \times n$ matrix and $\vec{B} \neq \vec{0}$.

Proof by counterexample/(contradiction)

Let $\vec{x} = 2\vec{u}$, $\vec{u} \in \mathbb{R}^n$

Then $T(2\vec{u}) = A(2\vec{u}) + \vec{B} = 2A\vec{u} + \vec{B}$

which is not equal to $2(T\vec{u}) = 2A\vec{u} + 2\vec{B}$

So T can't be linear. ▥

31) Let $T: \mathbb{R}^n \to \mathbb{R}^m$ and $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be a linearly dependent set in $\mathbb{R}^n$.
Since $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependent, we can write $\vec{v}_1 = a \cdot \vec{v}_2 + b \cdot \vec{v}_3$ for some $a, b \in \mathbb{R}$.
The transform of $\vec{v}_1$ can then be written as $T(\vec{v}_1) = T(a \cdot \vec{v}_2 + b \cdot \vec{v}_3)$ which by linearity is equal to $a \cdot T(\vec{v}_2) + b \cdot T(\vec{v}_3)$.
So $T(\vec{v}_1)$ is a linear combination of $T(\vec{v}_2)$ and $T(\vec{v}_3)$ and thus the set $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\}$ is a linearly dependent set. ▥

1.9

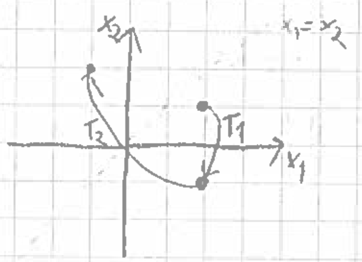
1) $T(\vec{x}) = \begin{pmatrix} T(e_1) & T(e_2) \end{pmatrix} \vec{x} = \begin{pmatrix} 3 & -5 \\ 1 & 2 \\ 3 & 0 \\ 1 & 0 \end{pmatrix} \vec{x}$

5) $T(e_1) = e_1 - 2e_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$T(e_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$T(\vec{x}) = \begin{pmatrix} T(e_1) & T(e_2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$.

8)



Reflection through x_1-axis: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} =: A$

Reflection through $x_1 = x_2$: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} =: B$

$$T(\vec{x}) = B(A\vec{x}) = (BA)\vec{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\vec{x}$$

$$T(\vec{x}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\vec{x}$$

12) A rotation about the origin can be written as the transform

$$T(\vec{x}) = \begin{pmatrix} \cos & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \vec{x}$$

Let $\varphi = \frac{\pi}{2}$ such that $T(\vec{x}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{x}$

which is precisely what we found in (8). The angle is then $\frac{\pi}{2}$ or $90°$.

15) $\begin{pmatrix} 3 & 0 & -2 \\ 4 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_1 - 2x_3 \\ 4x_1 \\ x_1 - x_2 + x_3 \end{pmatrix}$

29) $T: \mathbb{R}^3 \to \mathbb{R}^4$

$$\begin{pmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{pmatrix}$$

30) $T: \mathbb{R}^4 \to \mathbb{R}^3$

$$\begin{pmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \end{pmatrix}$$

32) "T maps $\mathbb{R}^n$ into $\mathbb{R}^m$ if and only if A has $\underline{m}$ pivot columns."

To span all of $\mathbb{R}$ A needs to consists of at least m linearly independent vectors. This will give at least m pivot columns.

31) "T is one-to-one if and only if A has $\underline{n}$ pivot columns."
The columns of A have to be linearly independent (by theorem 12). The size of A is $m \times n$, which is essentialy has n variables. In order to have no free variables we need n pivot columns.

35) $T: \mathbb{R}^n \to \mathbb{R}^m$
From ex. 32 $m \leq n$
If T is one-to-one, then $n = m$.

36) $S: \mathbb{R}^p \to \mathbb{R}^n$ and $T: \mathbb{R}^n \to \mathbb{R}^m$
Let S and T be linear transforms.
Since the codomain of S is the domain of T, we can compose T with S.

$$(T \circ S)(\vec{x}) = T(S(\vec{x})), \quad \vec{x} \in \mathbb{R}^p$$

Let $\vec{x} = c\vec{u} + d\vec{v}$, $\vec{u}, \vec{v} \in \mathbb{R}^p$, $c, d \in \mathbb{R}$
Want to show that $T \circ S$ is linear and that

$$(T \circ S)(c\vec{u} + d\vec{v}) = c(T \circ S)(\vec{u}) + d \cdot (T \circ S)(\vec{v}).$$

Since S is linear:

$$S(c \cdot \vec{u} + d\vec{v}) = c \cdot S(\vec{u}) + d \cdot S(\vec{v})$$

Since T is linear:

$$T(c\, S(\vec{u}) + d\, T(\vec{v})) = c \cdot T(S(\vec{u})) + d \cdot T(S(\vec{v}))$$

By the definition of composition,

$$c \cdot T(S(\vec{u})) + d \cdot T(S(\vec{v}))$$
$$= c \cdot (T \circ S)(\vec{u}) + d \cdot (T \circ S)(\vec{v})$$

which is what we wanted to show.
Thus we can conclude that the mapping $\vec{x} \longmapsto T(S(\vec{x}))$ is linear.

1.10 4a)

Nutrient	Non-fat milk	Soy flour	Whey	Soy protein
Protein	36	51	13	80
Carbohydrate	52	34	74	0
Fat	0	7	1.1	3.4
Calcium	1.26	0.19	0.8	0.18

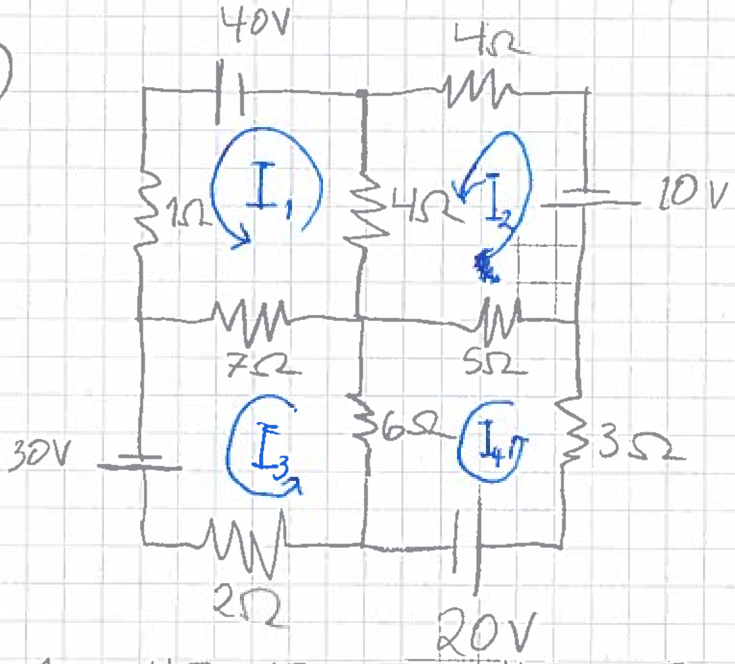
Nutrient	Cambridge Diet
Protein	33
Carbohydrate	45
Fat	3
Calcium	0.8

Let $A = \begin{pmatrix} 36 & 51 & 13 & 80 \\ 52 & 34 & 74 & 0 \\ 0 & 7 & 1.1 & 3.4 \\ 1.26 & 0.19 & 0.8 & 0.18 \end{pmatrix}$

and $\vec{b} = \begin{pmatrix} 33 \\ 45 \\ 3 \\ 0.8 \end{pmatrix}$ and $\vec{x} = \begin{pmatrix} \text{amount of milk} \\ \text{— '' — Soy flour} \\ \text{— '' — whey} \\ \text{— '' — soy protein} \end{pmatrix}$

Then $A\vec{x} = \vec{b}$ determines the amount of different ingredients.

7)



1: $4I_1 + 1I_1 + 7I_1 - 4I_2 - 7I_3 = 40$

$\Leftrightarrow \quad 12I_1 - 4I_2 - 7I_3 = 40$

2: $13I_2 - 4I_1 - 5I_4 = -10$

$\Leftrightarrow \quad -4I_1 + 13I_2 - 5I_4 = -10$

3: $15I_3 - 6I_4 - 7I_1 = 30$

$\Leftrightarrow \quad -7I_1 + 15I_3 - 6I_4 = 30$

4: $14I_4 - 5I_2 - 6I_3 = 20$

$\Leftrightarrow \quad -5I_2 - 6I_3 + 14I_4 = 20$

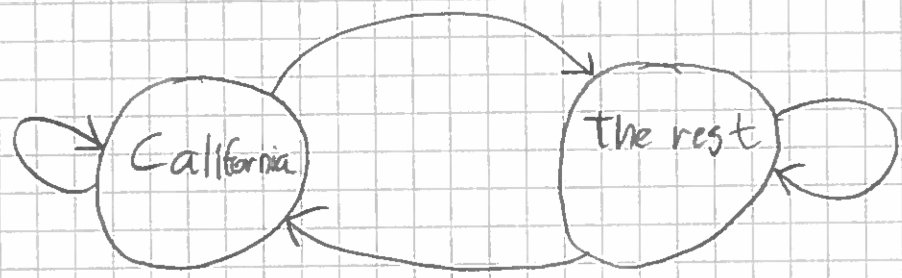
The matrix equation is then:

$$\begin{pmatrix} 12 & -4 & -7 & 0 \\ -4 & 13 & 0 & -5 \\ -7 & 0 & 15 & -5 \\ 0 & -5 & -6 & 14 \end{pmatrix} \cdot \begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix} = \begin{pmatrix} 40 \\ -10 \\ 30 \\ 20 \end{pmatrix}$$

11a) Population in California: 38 041 430 (2008)

Population outside California: 275 872 610 (2008)

$\Rightarrow$ Remain in city: 37 293 178
Move out of city: 748 252
Remain outside city: 275 378 969
Move into city: 493 641



Percentages:

City:

- remain: $\frac{\#\text{remain}}{\#\text{population}} = .98033$

- move: $\frac{\#\text{move}}{\#\text{population}} = .01967$

Rest of country:

- remain: $\frac{\#\text{remain}}{\#\text{population}} = .99821$

- move: $\frac{\#\text{move}}{\#\text{population}} = .00179$

The migration matrix is then:

From

$$\begin{array}{ccc} & \text{City} & \text{Rest} \\ \text{To} \quad \text{City:} & .98033 & .00179 \\ \text{Rest} & .01967 & .99821 \end{array} =: M$$

$$M = \begin{pmatrix} .98033 & .00179 \\ .01967 & .99821 \end{pmatrix}$$