# TTK4135 Optimization and Control
# Solution to Final Exam — Spring 2013

Department of Engineering Cybernetics

## 1 QP (20 %)

We have the QP problem

$$
\begin{aligned}
\min_{x} \quad & f(x) = \frac{1}{2}x_1^2 + x_2^2 - x_1x_2 - 2x_1 - 6x_2 \\
\text{s.t.} \quad & x_1 + x_2 \leq 2 \\
& -x_1 + 2x_2 \leq 2 \\
& x_1 \geq 0 \\
& x_2 \geq 0
\end{aligned}
$$

**a** (5 %) The problem can be written in the standard form (A.7) with

$$
\begin{aligned}
G &= \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} & c &= \begin{bmatrix} -2 \\ -6 \end{bmatrix} && \text{(1a)} \\
a_1 &= \begin{bmatrix} -1 & -1 \end{bmatrix} & b_1 &= -2 && \text{(1b)} \\
a_2 &= \begin{bmatrix} 1 & -2 \end{bmatrix} & b_2 &= -2 && \text{(1c)} \\
a_3 &= \begin{bmatrix} 1 & 0 \end{bmatrix} & b_3 &= 0 && \text{(1d)} \\
a_4 &= \begin{bmatrix} 0 & 1 \end{bmatrix} & b_4 &= 0 && \text{(1e)} \\
\mathcal{E} &= \emptyset & \mathcal{I} &= \{1, 2, 3, 4\} && \text{(1f)}
\end{aligned}
$$

**b** (3 %) The QP problem is (strictly) convex since $G$ is positive definite. This is easiest to verify by checking that the leading principle minors are positive; the first leading principle minor is $1 > 0$ while the second is $1 \times 2 - (-1) \times (-1) = 2 - 1 = 1 > 0$. One can also find the eigenvalues of $G$, which are $(3 \pm \sqrt{5})/2$ (both positive).

**c** (3 %) Before we derive the KKT conditions we define the Lagrangean (see (A.2))

$$
\begin{aligned}
\mathcal{L}(x, \lambda) &= f(x) - \sum_{i \in \mathcal{I} \cup \mathcal{E}} \lambda_i c_i(x) \\
&= \frac{1}{2}x^\top G x + x^\top c - \sum_{i \in \mathcal{I}} \lambda_i (a_i^\top x - b_i) \qquad (2)
\end{aligned}
$$

The KKT conditions (A.3) are then

$$
\begin{aligned}
Gx^* + c - \sum_{i \in \mathcal{I}} \lambda_i^* a_i &= 0 && && \text{(3a)} \\
a_i^\top x^* - b_i &\geq 0, && i \in \mathcal{I} && \text{(3b)} \\
\lambda_i^* &\geq 0, && i \in \mathcal{I} && \text{(3c)} \\
\lambda_i^*(a_i^\top x^* - b_i) &= 0, && i \in \mathcal{I} && \text{(3d)}
\end{aligned}
$$

with $G$, $c$, $a_i$, $b_i$, and $\mathcal{I}$ given in (1).

**d** (9 %) Since the first and second inequality constraints are active at the solution, we can find the solution by solving the equation set

$$
\begin{aligned}
x_1 + x_2 &= 2 && \text{(4a)} \\
-x_1 + 2x_2 &= 2 && \text{(4b)}
\end{aligned}
$$

Adding the two equations gives $3x_2 = 4$ and we then know that $x_2^* = 4/3$. This gives $x_1^* = 2/3$. That is, $x^* = \begin{bmatrix} 2/3 & 4/3 \end{bmatrix}^\top$.

Since the last two constraints are inactive (both variables are strictly positive at the solution) we know that $\lambda_3^* = \lambda_4^* = 0$; $\lambda_1^*$ and $\lambda_2^*$ must be found from (3a). We have that

$$
Gx^* + c - \sum_{i \in \mathcal{I}} \lambda_i^* a_i = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2/3 \\ 4/3 \end{bmatrix} + \begin{bmatrix} -2 \\ -6 \end{bmatrix} - \lambda_1^* \begin{bmatrix} -1 \\ -1 \end{bmatrix} - \lambda_2^* \begin{bmatrix} 1 \\ -2 \end{bmatrix} - 0 - 0 = 0
$$

which we can write as

$$
\begin{aligned}
\lambda_1^* - \lambda_2^* &= 8/3 && \text{(5a)} \\
\lambda_1^* + 2\lambda_2^* &= 4 && \text{(5b)}
\end{aligned}
$$

Subtracting the second equation from the first gives $-3\lambda_2^* = -4/3$ or $\lambda_2^* = 4/9$. We then have $\lambda_1^* = 28/9$. That is,

$$
\lambda^* = \begin{bmatrix} 28/9 & 4/9 & 0 & 0 \end{bmatrix}^\top \tag{6}
$$

## 2 Optimization problem formulation (20 %)

We first study optimization problem (A.1) in the Appendix.

**a** (2 %) The number of decision variables in (A.1) equals the number of elements in $x$. Since $x \in \mathbb{R}^n$, there are $n$ decision variables in (A.1)?

**b** (3 %) The statement "If $c_i(x)$ is a nonlinear function when $i \in \mathcal{E}$, then (A.1) is always a non-convex problem" is *true*. All equality constraint functions are linear in convex optimization problems.

**c** (3 %) The statement "If $c_i(x)$ is a nonlinear function when $i \in \mathcal{I}$, then (A.1) is always a non-convex problem" is *not true*. An optimization can still be convex with nonlinear inequality constraint functions, as long as the nonlinear inequality constraint functions are concave.

**d** (2 %) There are 2 equality constraints and 0 inequality constraints if $\mathcal{E} = \{1, 2\}$ and $\mathcal{I} = \emptyset$?

For each of the following five problems, classify the problem and suggest a suitable algorithm for solving problems of that class.

**e** (2 %)

$$\begin{aligned} \min_{x \in \mathbb{R}^2} \quad & x_1 + x_2 \\ \text{s.t.} \quad & 3x_1 - 2x_2 \leq 2 \\ & x_1 \geq 0 \end{aligned}$$

This is a linear programming (LP) problem since all the functions are linear. A suitable algorithm for LP problems is the simplex algorithm.

**f** (2 %)

$$\begin{aligned} \min_{x \in \mathbb{R}^2} \quad & 3x_1^2 - x_2 \\ \text{s.t.} \quad & x_1^2 + x_2^2 \leq 4 \end{aligned}$$

This is a nonlinear programming (NLP) problem since the inequality constraint function is nonlinear. A suitable algorithm for NLP problems is sequential quadratic programming (SQP).

**g** (2 %)

$$\begin{aligned} \min_{x \in \mathbb{R}^2} \quad & x_1 \sin(x_2) \\ \text{s.t.} \quad & x_1 = x_2^2 \end{aligned}$$

Here the objective function and the equality constraint function are both nonlinear, so this problem is also a NLP. A suitable algorithm for NLPs is SQP.

**h** (2 %)

$$
\begin{aligned}
\min_{x \in \mathbb{R}^2} \quad & x_1^2 + x_2^2 + x_1 \\
\text{s.t.} \quad & x_1 = x_2 - 1 \\
& x_2 \geq 3
\end{aligned}
$$

This problem has a quadratic objective function and linear constraint functions, making it a quadratic programming (QP) problem. An active set method is a suitable approach to solve QP problems.

**i** (2 %)

$$
\begin{aligned}
\min_{x \in \mathbb{R}^2} \quad & x_2^2 + x_1 + 3x_2 \\
\text{s.t.} \quad & x_1 - 3x_2 = 1
\end{aligned}
$$

This problem has a quadratic objective function and one linear equality constraint function; hence, this is also a (QP) problem. An active set method is a suitable approach to solve QP problems. Since this QP problem has no inequality constraints, we could also set up the KKT conditions on matrix form and solve the resulting equation set to obtain the solution.

# 3 Various topics (26 %)

**Linear Independence Constraint Qualification (LICQ)**

**a** (6 %) We consider the optimization problem

$$\begin{aligned} \min_{x \in \mathbb{R}^2} \quad & f(x) \\ \text{s.t.} \quad & (x_1 - 1)^2 + (x_2 - 1)^2 \leq 2 \\ & (x_1 - 1)^2 + (x_2 + 1)^2 \leq 2 \\ & x_1 \geq 0 \end{aligned}$$

where the solution is $x^* = \begin{bmatrix} 0 & 0 \end{bmatrix}^\top$. We need to find which constraints are active in order to check the LICQ condition. For the first constraint, we have

$$(x_1^* - 1)^2 + (x_2^* - 1)^2 - 2 = (0 - 1)^2 + (0 - 1)^2 - 2 = 1 + 1 - 2 = 0 \tag{7}$$

meaning the constraint is active. Similarly, for the second constraint,

$$(x_1^* - 1)^2 + (x_2^* + 1)^2 - 2 = (0 - 1)^2 + (0 + 1)^2 - 2 = 1 + 1 - 2 = 0 \tag{8}$$

meaning this constraint is active as well. The third constraint $x_1 \geq 0$ is clearly active at $x^* = (0, 0)^\top$. That is, all three constraints are active.

Since we have three active constraints in $\mathbb{R}^2$, the LICQ condition cannot hold, since three gradients in $\mathbb{R}^2$ cannot be linearly independent.

**Lagrange multipliers**

We here consider the two convex two-dimensional optimization problems in Figures 1 and 2 with the solution $x^*$ and the inequality constraints $c_1(x)$ and $c_2(x)$ indicated.

**b** (3 %) For the problem illustrated in Figure 1 (page 6), the first constraint is active and the second is inactive. Hence, $\lambda_1^*$ is strictly positive while $\lambda_2^*$ is zero.

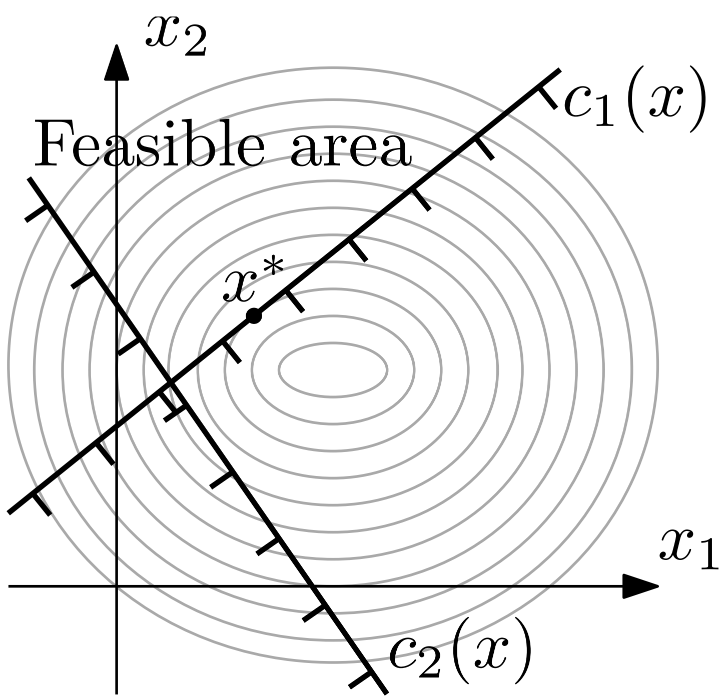
**c** (3 %) For the problem illustrated in Figure 2 (page 6), the first constraint is active and the second is inactive. However, we see that the unconstrained and constrained minimizers are the same. We say that the first constraint is *weakly active* in this situation, meaning that the solution does not "push against" the first constraint. Accordingly, both multipliers are zero at the solution. We also say that strict complementarity does not hold in this situation.

**Nonlinear programming and SQP**

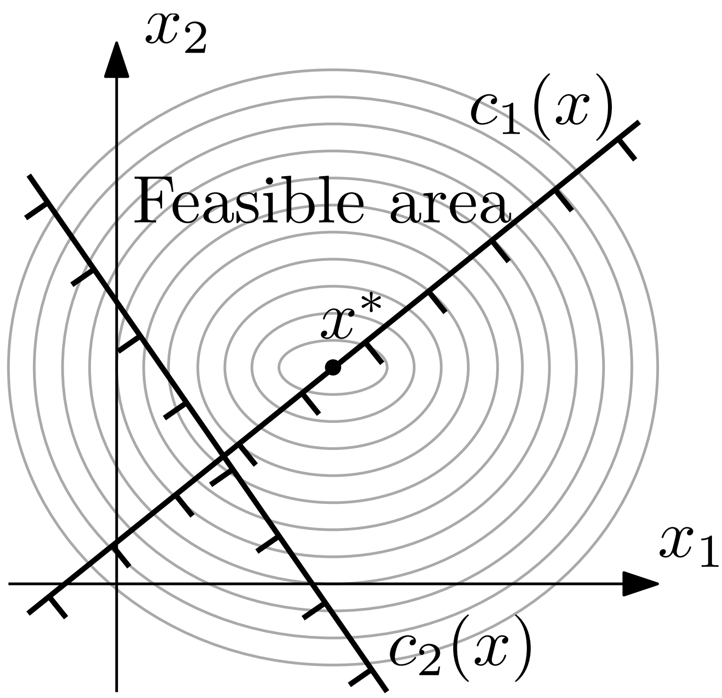
This part is about the SQP algorithm on page 13 in the Appendix (Algorithm 18.3 in Nocedal and Wright).

**d** (3 %) A suitable merit function for problem (A.1) when $\mathcal{E} = \{1\}$ and $\mathcal{I} = \emptyset$ is the $\ell_1$ merit function

$$\phi_1(x; \mu) = f(x) + \mu \|c(x)\|_1 = f(x) + \mu |c_1(x)| \tag{9}$$




Figur 1: Illustration for Problem 3 b.




Figur 2: Illustration for Problem 3 c.

**e** (3 %) The parameter $\mu$ is a part of the merit function, and it is calculated at each iteration of the SQP algorithm. Normally, $\mu$ increases from one iteration $k$ to the next iteration $k + 1$. A small positive $\mu$ allows a step length that is not too concerned about constraint satisfaction, but makes good progress toward the solution. Constraint satisfaction is more important as we get closer to the solution, and we tell the line search method this by increasing $\mu$.

**f** (4 %) The merit function $\phi_1$ is used in the line search part of the SQP algorithm. The merit function decreases from one iteration $k$ to the next; this is necessary for satisfying the termination criterion in the line search. However, the objective $f$ does not always decrease from one iteration point $k$ to the next. Some steps that reduce constraint violation do increase the objective function, this may be necessary to reach a (feasible) solution.

**g** (4 %) A merit function is exact it there is a positive scalar $\mu^*$ such that for any $\mu > \mu^*$, any local solution of the nonlinear programming problem (A.1) is a local minimizer of the merit function.

# 4 MPC and dynamic optimization (34 %)

Here we consider the dynamic optimization problem (A.9) in the Appendix.

**a** (3 %) The prooblem (A.9) is called an open loop optimization problem because no feedback is involved. That is, we calculate an optimal control sequence and an optimal future state trajectory based on the current state (estimate). Since future measurements of the state is not part of the model we use for optimization, we can think of the feedback loop as open (as opposed to a closed loop).

**b** (3 %) The inequality constraints (A.9f) can be included to ensure that the input is not asked to change faster than possible. One example is a valve that has a maximum rate of change.

**c** (2 %) The dimension of the vector (the number of elements) $z$ is $N \cdot (n_x + n_u)$. When $N = 15$, $n_x = 10$, and $n_u = 2$ we have that $n = 15 \times (10 + 2) = 180$. That is, $z \in \mathbb{R}^{180}$.

**d** (4 %) Reducing the number of decision variables by eliminating $x_1^T, \ldots, x_N^T$ from $z$ in (A.9l) using the equality constraints (A.9b) may or may not be a good idea. The number of decision variables will often be significantly reduced (it is common that $n_x \gg n_u$), which means that the reformulated optimization problem is much smaller. However, the full space formulation usually gives a very sparse KKT matrix, which is taken advantage by all efficient solvers. The sparsity property will usually be lost if the states are removed from the problem.

**e** (6 %) When not all the states in $x_t$ are measured, but rather a subset $y_t = Cx_t$, it is common to estimate the the states. A suitable algorithm for output feedback or observer feedback MPC is

**for** $t = 0, 1, 2, \ldots$ **do**
  Compute an estimate of the current state $\hat{x}_t$ based on the data up until time $t$.
  Solve a dynamic optimization problem on the prediction horizon from $t$ to $t + N$ with $\hat{x}_t$ as the initial condition.
  Apply the first control move $u_t$ from the solution above.
**end for**

**f** (8 %) In order to ensure feasibility at all times, it is possible to soften the state constraints

$$x^{\text{low}} \leq x_t \leq x^{\text{high}}, \qquad t = 1, \ldots, N \tag{A.9d}$$

through the use of slack variables. We reformulate (A.9d) to

$$x^{\text{low}} - \epsilon \leq x_t \leq x^{\text{high}} + \epsilon, \qquad t = 1, \ldots, N \tag{10}$$

$$0 \leq \epsilon \in \mathbb{R}^{n_x} \tag{11}$$

We then have to penalize a positive slack in the objective, and accomplish this through adding a term like

$$\rho^\top \epsilon \tag{12}$$

to the objective function, with $0 \leq \rho \in \mathbb{R}^{n_x}$. Note that $\rho$ has to be large. It is also possible to use a quadratic penalty formulation (which would not be an exact penalty). We could also have chosen to use a time varying slack variable $\epsilon_t$, but this would have increased the number of variables in the problem.

**g** (2 %) If we replace the linear model (A.9b) with a nonlinear discrete time model

$$x_{t+1} = g(x_t, u_t) \tag{13}$$

in an MPC controller, the open loop optimization problem is no longer a QP problem; it is now an NLP problem. SQP is a suitable algorithm for solving NLP problems.

**h** (6 %) Instead of using the nonlinear model (13) in the open loop optimization problem, we approximate the nonlinear model with a linear time varying (LTV) model

$$x_{t+1} = A_t x_t + B_t u_t \tag{14}$$

around a stationary point $\bar{x}_t, \bar{u}_t$. We can compute $A_t$ and $B_t$ from $g(x_t, u_t)$ with the formulas

$$A_t = \left.\frac{\partial g(x_t, u_t)}{\partial x_t}\right|_{\substack{x_t = \bar{x}_t \\ u_t = \bar{u}_t}}, \qquad B_t = \left.\frac{\partial g(x_t, u_t)}{\partial u_t}\right|_{\substack{x_t = \bar{x}_t \\ u_t = \bar{u}_t}} \tag{15}$$

under the condition that $g(x_t, u_t)$ is continuously differentiable (sufficiently smooth). This is derived using a first-order Taylor series expansion.

# Appendix

### Part 1 Optimization Problems and Optimality Conditions

A general formulation for constrained optimization problems is

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f(x) && && \text{(A.1a)} \\ \text{s.t.} \quad & c_i(x) = 0, && i \in \mathcal{E} && \text{(A.1b)} \\ & c_i(x) \geq 0, && i \in \mathcal{I} && \text{(A.1c)} \end{aligned}$$

where $f$ and the functions $c_i$ are all smooth, differentiable, real-valued functions on a subset of $\mathbb{R}^n$, and $\mathcal{E}$ and $\mathcal{I}$ are two finite sets of indices.

The Lagrangean function for the general problem (A.1) is

$$\mathcal{L}(x, \lambda) = f(x) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i c_i(x) \tag{A.2}$$

The KKT-conditions for (A.1) are given by:

$$\begin{aligned} \nabla_x \mathcal{L}(x^*, \lambda^*) &= 0 && && \text{(A.3a)} \\ c_i(x^*) &= 0, && i \in \mathcal{E} && \text{(A.3b)} \\ c_i(x^*) &\geq 0, && i \in \mathcal{I} && \text{(A.3c)} \\ \lambda_i^* &\geq 0, && i \in \mathcal{I} && \text{(A.3d)} \\ \lambda_i^* c_i(x^*) &= 0, && i \in \mathcal{E} \cup \mathcal{I} && \text{(A.3e)} \end{aligned}$$

2nd order (sufficient) conditions for (A.1) are given by:

$$w \in \mathcal{C}(x^*, \lambda^*) \Leftrightarrow \begin{cases} \nabla c_i(x^*)^\top w = 0 & \text{for all } i \in \mathcal{E} \\ \nabla c_i(x^*)^\top w = 0 & \text{for all } i \in \mathcal{A}(x^*) \cap \mathcal{I} \text{ with } \lambda_i^* > 0 \\ \nabla c_i(x^*)^\top w \geq 0 & \text{for all } i \in \mathcal{A}(x^*) \cap \mathcal{I} \text{ with } \lambda_i^* = 0 \end{cases} \tag{A.4}$$

**Theorem 1:** (Second-Order Sufficient Conditions) *Suppose that for some feasible point $x^* \in \mathbb{R}^n$ there is a Lagrange multiplier vector $\lambda^*$ such that the KKT conditions* (A.3) *are satisfied. Suppose also that*

$$w^\top \nabla_{xx}^2 \mathcal{L}(x^*, \lambda^*) w > 0, \qquad \text{for all } w \in \mathcal{C}(x^*, \lambda^*),\ w \neq 0. \tag{A.5}$$

*Then $x^*$ is a strict local solution for* (A.1).

LP problem in standard form:

$$\min_{x} \quad f(x) = c^\top x \tag{A.6a}$$
$$\text{s.t.} \quad Ax = b \tag{A.6b}$$
$$x \geq 0 \tag{A.6c}$$

where $A \in \mathbb{R}^{m\times n}$ and $\operatorname{rank} A = m$.

QP problem in standard form:

$$\min_{x} \quad f(x) = \frac{1}{2}x^\top Gx + x^\top c \tag{A.7a}$$
$$\text{s.t.} \quad a_i^\top x = b_i, \qquad i \in \mathcal{E} \tag{A.7b}$$
$$a_i^\top x \geq b_i, \qquad i \in \mathcal{I} \tag{A.7c}$$

where $G$ is a symmetric $n \times n$ matrix, $\mathcal{E}$ and $\mathcal{I}$ are finite sets of indices and $c$, $x$ and $\{a_i\}, i \in \mathcal{E} \cup \mathcal{I}$, are vectors in $\mathbb{R}^n$. Alternatively, the equalities can be written $Ax = b$, $A \in \mathbb{R}^{m\times n}$.

Iterative method:

$$x_{k+1} = x_k + \alpha_k p_k \tag{A.8a}$$
$$x_0 \text{ given} \tag{A.8b}$$
$$x_k, p_k \in \mathbb{R}^n, \ \alpha_k \in \mathbb{R} \tag{A.8c}$$

$p_k$ is the search direction and $\alpha_k$ is the line search parameter.

**Part 2 Optimal Control**

A typical open-loop optimal control problem on the time horizon 0 to $N$ is

$$\min_{z\in\mathbb{R}^n} f(z) = \sum_{t=0}^{N-1} \frac{1}{2}x_{t+1}^\top Q_{t+1}x_{t+1} + d_{xt+1}x_{t+1} + \frac{1}{2}u_t^\top R_t u_t + d_{ut}u_t \tag{A.9a}$$

subject to

$$\begin{aligned}
x_{t+1} &= A_t x_t + B_t u_t, && t = 0, \dots, N-1 && \text{(A.9b)}\\
x_0 &= \text{given} && && \text{(A.9c)}\\
x^{\text{low}} &\le x_t \le x^{\text{high}}, && t = 1, \dots, N && \text{(A.9d)}\\
u^{\text{low}} &\le u_t \le u^{\text{high}}, && t = 0, \dots, N-1 && \text{(A.9e)}\\
-\Delta u^{\text{high}} &\le \Delta u_t \le \Delta u^{\text{high}}, && t = 0, \dots, N-1 && \text{(A.9f)}\\
Q_t &\succeq 0 && t = 1, \dots, N && \text{(A.9g)}\\
R_t &\succeq 0 && t = 0, \dots, N-1 && \text{(A.9h)}
\end{aligned}$$

where

$$\begin{aligned}
u_t &\in \mathbb{R}^{n_u} && \text{(A.9i)}\\
x_t &\in \mathbb{R}^{n_x} && \text{(A.9j)}\\
\Delta u_t &= u_t - u_{t-1} && \text{(A.9k)}\\
z^\top &= (x_1^\top, \dots, x_N^\top, u_0^\top, \dots, u_{N-1}^\top) && \text{(A.9l)}
\end{aligned}$$

The subscript $t$ denotes discrete time sampling instants.

The optimization problem for linear quadratic control of discrete dynamic systems is given by

$$\min_{z\in\mathbb{R}^n} f(z) = \sum_{t=0}^{N-1} \frac{1}{2}x_{t+1}^\top Q_{t+1}x_{t+1} + \frac{1}{2}u_t^\top R_t u_t \tag{A.10a}$$

subject to

$$\begin{aligned}
x_{t+1} &= A_t x_t + B_t u_t && \text{(A.10b)}\\
x_0 &= \text{given} && \text{(A.10c)}
\end{aligned}$$

where

$$\begin{aligned}
u_t &\in \mathbb{R}^{n_u} && \text{(A.10d)}\\
x_t &\in \mathbb{R}^{n_x} && \text{(A.10e)}\\
z^\top &= (x_1^\top, \dots, x_N^\top, u_0^\top \dots, u_{N-1}^\top) && \text{(A.10f)}
\end{aligned}$$

**Theorem 2:** *The solution of (A.10) with $Q_t \succeq 0$ and $R_t \succ 0$ is given by*

$$u_t = -K_t x_t \tag{A.11a}$$

*where the feedback gain matrix is derived by*

$$K_t = R_t^{-1} B_t^\top P_{t+1} (I + B_t R_t^{-1} B_t^\top P_{t+1})^{-1} A_t, \qquad t = 0, \dots, N-1 \tag{A.11b}$$

$$P_t = Q_t + A_t^\top P_{t+1} (I + B_t R_t^{-1} B_t^\top P_{t+1})^{-1} A_t, \qquad t = 0, \dots, N-1 \tag{A.11c}$$

$$P_N = Q_N \tag{A.11d}$$

## Part 3 Sequential quadratic programming (SQP)

**Algorithm 18.3** (Line Search SQP Algorithm).

Choose parameters $\eta \in (0, 0.5)$, $\tau \in (0, 1)$, and an initial pair $(x_0, \lambda_0)$;
Evaluate $f_0$, $\nabla f_0$, $c_0$, $A_0$;
If a quasi-Newton approximation is used, choose an initial $n \times n$ symmetric positive definite Hessian approximation $B_0$, otherwise compute $\nabla^2_{xx} \mathcal{L}_0$;
**repeat** until a convergence test is satisfied
    Compute $p_k$ by solving (18.11); let $\hat{\lambda}$ be the corresponding multiplier;
    Set $p_\lambda \leftarrow \hat{\lambda} - \lambda_k$;
    Choose $\mu_k$ to satisfy (18.36) with $\sigma = 1$;
    Set $\alpha_k \leftarrow 1$;
    **while** $\phi_1(x_k + \alpha_k p_k; \mu_k) > \phi_1(x_k; \mu_k) + \eta \alpha_k D_1(\phi(x_k; \mu_k) p_k)$
        Reset $\alpha_k \leftarrow \tau_\alpha \alpha_k$ for some $\tau_\alpha \in (0, \tau]$;
    **end (while)**
    Set $x_{k+1} \leftarrow x_k + \alpha_k p_k$ and $\lambda_{k+1} \leftarrow \lambda_k + \alpha_k p_\lambda$;
    Evaluate $f_{k+1}$, $\nabla f_{k+1}$, $c_{k+1}$, $A_{k+1}$, (and possibly $\nabla^2_{xx} \mathcal{L}_{k+1}$);
    If a quasi-Newton approximation is used, set
        $s_k \leftarrow \alpha_k p_k$ and $y_k \leftarrow \nabla_x \mathcal{L}(x_{k+1}, \lambda_{k+1}) - \nabla_x \mathcal{L}(x_k, \lambda_{k+1})$,
    and obtain $B_{k+1}$ by updating $B_k$ using a quasi-Newton formula;
**end (repeat)**