Øving 5, Matte 4K

Rendell Cale, gruppe 2

Ønsker tilbakemelding :)

11.7:

ok

$$f(x) = \begin{cases} \frac{\pi}{2}\sin x, & 0 \leq x \leq \pi \\ 0, & x > \pi \end{cases}$$

Note that f is undefined for $x<0$, so we have to extend it to $x<0$.
We choose

$$g(x) = \begin{cases} f(x), & x \geq 0 \\ -f(-x), & x<0 \end{cases}$$

$$= \begin{cases} \frac{\pi}{2}\sin x, & -\pi \leq x \leq \pi \\ 0, & |x| > \pi \end{cases}$$

such that $g(x)$ is odd.
Since g is odd we get a Fourier sine integral.

$$g(x) = \int_0^\infty B(\omega)\sin(\omega x)\,d\omega$$

$$B(\omega) = \frac{2}{\pi}\int_0^\infty g(x)\sin(\omega x)\,dx$$

$$\Rightarrow \quad B(\omega) = \frac{2}{\pi}\int_0^{\pi} \frac{\pi}{2}\sin x \sin \omega x \, dx$$

$$= \int_0^{\pi} \sin(x)\sin(\omega x)\,dx$$

$$= \frac{1}{\omega}\int_0^{\pi} \cos x \cos \omega x \, dx$$

$$= -\left.\frac{\cos x \sin \omega x}{\omega^2}\right|_0^{\pi} + \frac{1}{\omega^2}\int_0^{\pi} \sin x \sin \omega x \, dx$$

$$= -\frac{\sin \pi \omega}{\omega^2} + \frac{B(\omega)}{\omega^2}$$

$$\Leftrightarrow \quad B(\omega)(\omega^2 - 1) = -\sin(\pi\omega)$$

$$\Leftrightarrow \quad B(\omega) = -\frac{\sin(\pi\omega)}{\omega^2 - 1}$$

$$= \frac{\sin(\pi\omega)}{1-\omega^2}$$

So $g(x) = \int_0^{\infty} \frac{\sin(\pi\omega)}{1-\omega^2} \sin \omega x \, d\omega$ R

$$= f(x), \; x \geq 0.$$

10)

$$f(x) = \begin{cases} \sin x & , \quad 0<x<\pi \\ 0 & , \quad x>\pi \end{cases}$$

$$f(x) = \int_0^\infty A(\omega)\cos(\omega x)\,d\omega$$

where $A(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos(\omega x)\,dx$

$$= \frac{2}{\pi}\int_0^\pi \sin x \cos \omega x \,dx$$

Let $I = \int_0^\pi \sin x \cos(\omega x)\,dx$

$$= \frac{\sin x \sin \omega x}{\omega}\Big|_0^\pi - \frac{1}{\omega}\int_0^\pi \cos x \sin(\omega x)\,dx$$

$$= +\frac{\cos x \cos(\omega x)}{\omega^2}\Big|_0^\pi + \frac{1}{\omega^2}\int_0^\pi \sin x \cos \omega x \,dx$$

$$= \frac{-\cos(\omega\pi)-1}{\omega^2} + \frac{I}{\omega^2}$$

$$\Leftrightarrow I(\omega^2-1) = -(\cos\omega\pi+1)$$

$$\Leftrightarrow I = \frac{\cos(\omega\pi)+1}{1-\omega^2}$$

So $A(\omega) = \frac{2}{\pi} I = \frac{2}{\pi}\cdot\frac{\cos(\omega\pi)+1}{1-\omega^2}$

So $f(x) = \int_0^\infty A(w) \cos wx \, dw$

$$= \int_0^\infty \frac{2}{\pi} \frac{\cos(w\pi)+1}{1-w^2} \cos wx \, dw$$

$$\Leftrightarrow f(x) = \frac{2}{\pi} \int_0^\infty \frac{\cos(w\pi)+1}{1-w^2} \cdot \cos(wx) \, dw$$

R

19) $f(x) = \begin{cases} e^x, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$

Using (11):

$$B(w) = \frac{2}{\pi} \int_0^1 e^x \sin(wx) \, dx$$

$$\Leftrightarrow \frac{\pi}{2} B(w) = -\frac{e^x \cos wx}{w}\Big|_0^1 + \frac{1}{w} \int_0^1 e^x \cos wx \, dx$$

$$= \frac{1 - e\cos w}{w} + \frac{e^x \sin wx}{w^2}\Big|_0^1 - \frac{\pi}{2} \frac{B(w)}{w^2}$$

$$\frac{\pi}{2} B(w)\left(1 + \frac{1}{w^2}\right) = \frac{1 - e\cos w}{w} + \frac{e \sin w}{w^2}$$

$$\frac{\pi}{2} B(w) = \frac{w^2}{w^2+1} \cdot \left(\frac{1 - e\cos w}{w} + \frac{e \sin w}{w^2}\right)$$

$$= \frac{w - we\cos w + e\sin w}{w^2+1}$$

$$\Leftrightarrow B(w) = \frac{2}{\pi} \cdot \frac{w + e(\sin w - w\cos w)}{w^2+1}$$

So $$f(x) = \frac{2}{\pi}\int_0^\infty \frac{w + e(\sin w - w\cos w)}{w^2+1}\sin wx \, dw$$

R

11.9:

5) $$f(x) = \begin{cases} e^{kx} & , -a < x < a \\ 0 & , \text{otherwise} \end{cases}$$

Du skulle gjort 11.9.4 :)

Want to find $\hat{f}$.

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{-iwx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{kx} e^{-iwx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{(k-iw)x} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{k-iw} e^{(k-iw)x}\Big|_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{k-iw}\left[e^{(k-iw)a} - e^{-(k-iw)a}\right]$$

$$= \underline{\underline{\frac{e^{(k-iw)a} - e^{-(k-iw)a}}{\sqrt{2\pi}\,(k-iw)}}}$$

4) $f(x) = \begin{cases} e^{kx} & , x<0 \; , \; k>0 \\ 0 & , x>0 \end{cases}$

$$\hat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{kx} e^{-i\omega x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{(k-i\omega)x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{k-i\omega} \cdot e^{(k-i\omega)x} \Big|_{-\infty}^{0}$$

$$= \underline{\underline{\frac{1}{\sqrt{2\pi}(k-i\omega)}}}$$

R nice

9) $f(x) = \begin{cases} |x| & , -1<x<1 \\ 0 & , \text{otherwise} \end{cases}$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} |x| e^{-i\omega x} dx$$

$$= \frac{|x| e^{-i\omega x}}{\sqrt{2\pi}(-i\omega)} \Big|_{-1}^{1} - \frac{i}{\sqrt{2\pi}\omega} \int_{-1}^{1} \frac{|x|}{x} e^{-i\omega x} \quad (*)$$

Since $\frac{d}{dx}(|x|) = \frac{|x|}{x}$

$$\int_{-1}^{1} \frac{|x|}{x} e^{-i\omega x}\,dx = \int_{-1}^{0} -e^{-i\omega x}\,dx + \int_{0}^{1} e^{-i\omega x}\,dx$$

$$= \frac{1}{i\omega} e^{-i\omega x}\Big|_{-1}^{0} + \frac{1}{-i\omega} e^{-i\omega x}\Big|_{0}^{1}$$

$$= \frac{1}{i\omega}\left[1 - e^{i\omega}\right] - \frac{1}{i\omega}\left[e^{-i\omega} - 1\right]$$

$$= \frac{1}{i\omega}\left(1 - e^{i\omega}\right) + \frac{1}{i\omega}\left(1 - e^{-i\omega}\right)$$

$$= \frac{1}{i\omega}\left(1 + 1 - (e^{i\omega} + e^{-i\omega})\right)$$

$$e^{i\omega} + e^{-i\omega} = \cos\omega + i\sin\omega + \cos\omega - i\sin\omega$$
$$= 2\cos\omega$$

$$= -\frac{i}{\omega}(2 - 2\cos\omega)$$

$$= \frac{2i}{\omega}(\cos\omega - 1)$$

Going back to (*) we get

$$\hat{f}(\omega) = \frac{e^{-i\omega} - e^{i\omega}}{\sqrt{2\pi}(-i\omega)} - \frac{i}{\sqrt{2\pi}\,\omega} \cdot \frac{2i}{\omega}(\cos\omega - 1)$$

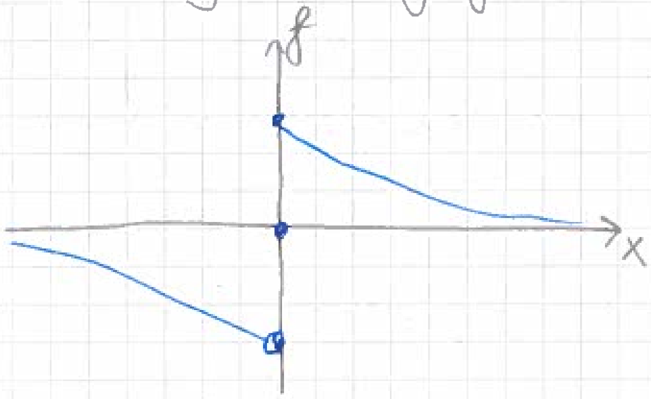
$$= \frac{-2i\sin\omega}{\sqrt{2\pi}(-i\omega)} + \frac{2(\cos\omega - 1)}{\sqrt{2\pi}\,\omega^2}$$

$$= \frac{\sqrt{2}\sin\omega}{\sqrt{\pi}\,\omega} + \frac{\sqrt{2}(\cos\omega - 1)}{\sqrt{\pi}\,\omega^2}$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}}\left[\frac{\sin\omega}{\omega} + \frac{\cos\omega - 1}{\omega^2}\right]$$ R

H) $f(x) = \begin{cases} e^{-x} & , x \geq 0 \\ -e^{x} & , x < 0 \end{cases}$

We start by sketching f



Note that f is odd so the Fourier transform will be a sine integral.

$$f(x) = \frac{2}{\pi}\int_0^\infty B(\omega)\sin(\omega x)\,d\omega$$

where $B(\omega) = \int_{\cancel{-\infty}\ 0}^{\infty} f(x)\sin(\omega x)\,dx$

$$= \underbrace{\cancel{\int_{-\infty}^{0} -e^{x}\sin(\omega x)\,dx}}_{I_1} + \underbrace{\int_0^\infty e^{-x}\sin(\omega x)\,dx}_{I_2}$$

$$\cancel{I_1 = \frac{+e^{x}\cos(\omega x)}{\omega}\Big|_{-\infty}^{0} - \frac{1}{\omega}\int_{-\infty}^{0} e^{x}\cos(\omega x)\,dx}$$

$$\cancel{= \frac{1}{\omega} - \underbrace{\frac{e^{x}\sin(\omega x)}{\omega^2}\Big|_{-\infty}^{0}}_{=0} - \frac{I_1}{\omega^2}}$$

$$\Leftrightarrow \cancel{I_1\left(1+\frac{1}{\omega^2}\right) = \frac{1}{\omega}}$$

$$\Leftrightarrow \cancel{I_1\frac{(\omega^2+1)}{\omega^2} = \frac{1}{\omega}}$$

$$\Leftrightarrow \cancel{I_1 = \frac{\omega}{\omega^2+1}}$$

$$I_2 = \frac{-e^{-x}\cos\omega x}{\omega}\Big|_0^\infty - \frac{1}{\omega}\int_0^\infty e^{-x}\cos(\omega x)\,dx$$

$$= \frac{1}{\omega} - \underbrace{\frac{e^{-x}\sin(\omega x)}{\omega^2}\Big|_0^\infty}_{=0} - \frac{1}{\omega^2}I_2$$

$$\Leftrightarrow I_2\left(1+\frac{1}{\omega^2}\right) = \frac{1}{\omega}$$

$$\Leftrightarrow I_2 = \frac{\omega}{\omega^2+1}$$

$$B(\omega) = \cancel{I_1} + I_2$$

$$= \cancel{\frac{\omega}{\omega^2+1}} + \frac{\omega}{\omega^2+1}$$

$$= \frac{\cancel{2}\omega}{\omega^2+1}$$

So $f(x) = \frac{2}{\pi}\int_0^\infty \frac{\omega\sin\omega x}{\omega^2+1}\,d\omega$

$$\Rightarrow \int_0^\infty \frac{\omega\sin\omega}{\omega^2+1}\,d\omega = \frac{\pi}{2}f(1) = \frac{\pi}{2e}$$

R

I) $f(t) = \cos(t)e^{-t^2}$

Note that f is even so we compute the Fourier cosine integral.

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\cos(\omega t)\,dt$$

$$= \frac{2}{\pi}\int_0^\infty \cos t \cos \omega t \, e^{-t^2}\,dt$$

Since $\cos t \cos \omega t = \frac{1}{2}\cos(t-\omega t) + \frac{1}{2}\cos(t+\omega t)$

we get

$$A(\omega) = \frac{1}{\pi}\int_0^\infty \underbrace{\cos((1-\omega)t)e^{-t^2}}_{I_1} + \underbrace{\cos((1+\omega)t)e^{-t^2}}_{I_2}\,dt$$

$$I_1 = \int_0^\infty \cos((1-\omega)t)e^{-t^2}\,dt$$

$$= \frac{1}{2}\int_{-\infty}^\infty \cos((1-\omega)t)e^{-t^2}\,dt$$

$$= \frac{e^{-t^2}\sin((1-\omega)t)}{2(1-\omega)}\Bigg|_{-\infty}^{\infty} + \frac{2}{2(1-\omega)}\int_{-\infty}^{\infty} te^{-t^2}\sin((1-\omega)t)\,dt$$

$$= \frac{-1}{1-\omega}\cdot\frac{te^{-t^2}\cos((1-\omega)t)}{1-\omega}\Bigg|_{-\infty}^{\infty}$$

I) $f(t) = \cos t \, e^{-t^2}$

Consider $g(t) = \cos t \, e^{-t^2} + i \sin t \, e^{-t^2}$

$= e^{it} e^{-t^2}$

If $\mathcal{F}\{e^{-t^2}\}(\omega) = \int_{-\infty}^{\infty} e^{-t^2} e^{-i\omega t} dt,$

then $\mathcal{F}\{e^{it} e^{-t^2}\}(\omega)$

$= \int_{-\infty}^{\infty} e^{it} e^{-t^2} e^{-i\omega t} dt$

$= \int_{-\infty}^{\infty} e^{-t^2} e^{it - i\omega t} dt$

$= \int_{-\infty}^{\infty} e^{-t^2} e^{-i(\omega-1)t} dt$

$= \underline{\mathcal{F}\{e^{-t^2}\}(\omega-1)}$

So $\hat{g}(\omega) = \mathcal{F}\{e^{-t^2}\}(\omega-1)$

$= \frac{1}{\sqrt{2}} e^{-(\omega-1)^2/4}$

$f(t) = g(t) - \underbrace{i \sin(t) e^{-t^2}}_{h(t)}$

So $\hat{f}(\omega) = \hat{g}(\omega) - \hat{h}(\omega)$ (1)

because of linearity.

$$h(t) = i\sin(t)e^{-t^2}$$

$$\hat{h}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} i\sin t\, e^{-t^2} e^{-i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overbrace{\underbrace{\sin t\cos\omega t}_{\text{odd}}\, \underbrace{e^{-t^2}}_{\text{even}}}^{0=} - i^2 \sin t\sin\omega t\; e^{-t^2}\, dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \sin t\sin\omega t\, e^{-t^2}\, dt$$

$$h(t) = i\sin(t)e^{-t^2}$$

$$\Leftrightarrow -h(t) + f(t) = \cos(t)e^{-t^2} - i\sin(t)e^{-t^2}$$

$$= e^{-it}e^{-t^2}$$

$$\Leftrightarrow \hat{f}(\omega) - \hat{h}(\omega) = \mathcal{F}\{e^{-t^2}\}(\omega+1)$$

$$\hat{f}(\omega) - \hat{h}(\omega) = \frac{1}{\sqrt{2}} e^{-\frac{(\omega+1)^2}{4}} \quad (2)$$

So we have:

$$(1)\quad \hat{f}(\omega) + \hat{h}(\omega) = \frac{1}{\sqrt{2}} e^{-\frac{(\omega-1)^2}{4}}$$

$$(2)\quad \hat{f}(\omega) - \hat{h}(\omega) = \frac{1}{\sqrt{2}} e^{-\frac{(\omega+1)^2}{4}}$$

$\frac{①+②}{2} \Longleftrightarrow$

$$\hat{f}(\omega) = \frac{1}{2\sqrt{2}}\left[e^{-\frac{(\omega-1)^2}{4}} + e^{-\frac{(\omega+1)^2}{4}}\right]$$ R

J) $h(x) = e^{-x^2} * e^{-x^2}$

$$\hat{h}(\omega) = \mathcal{F}\{e^{-x^2}\}(\omega) \cdot \mathcal{F}\{e^{-x^2}\}(\omega)\sqrt{2\pi}$$

$$= \frac{1}{\sqrt{2}} e^{-\frac{\omega^2}{4}} \cdot \frac{1}{\sqrt{2}} e^{-\frac{\omega^2}{4}} \sqrt{2\pi}$$

$$= \sqrt{\frac{\pi}{2}}\, e^{-\frac{\omega^2}{4} - \frac{\omega^2}{4}}$$

$$= \sqrt{\frac{\pi}{2}}\, e^{-\frac{\omega^2}{2}} \qquad (*)$$

If $\hat{h}(\omega) = \frac{b}{\sqrt{2a}} e^{-\frac{\omega^2}{4a}}$, then

$$h(x) = b \cdot e^{-ax^2} \qquad (a>0)$$

In $(*)$ we have

$$a = \frac{1}{2}, \quad b = \sqrt{\frac{\pi}{2}}$$

so

$$h(x) = \frac{\sqrt{\pi}}{\sqrt{2}} e^{-\frac{1}{2}x^2}.$$ R

K) $f(x) - \int_{-\infty}^{\infty} e^{-3|x-t|} f(t)\,dt = e^{-3|x|}$

$\Leftrightarrow f(x) - e^{-3|x|} * f(x) = e^{-3|x|}$ (*)

Taking the Fourier transform we get:

$$\hat{f}(\omega) - \mathcal{F}\{e^{-3|x|}\}\hat{f}(\omega)\cdot\sqrt{2\pi} = \mathcal{F}\{e^{-3|x|}\} \quad (*)$$

We compute $\mathcal{F}\{e^{-3|x|}\}(\omega) = \hat{g}(\omega)$.

$$\hat{g}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-3x} \cos \omega x \, dx$$

since $g(x) = e^{-3|x|}$ is even.

$$\hat{g}(\omega) = \cancel{\frac{\sqrt{2}\,e^{-3x}\sin \omega x}{\sqrt{\pi}\,\omega}\Big|_0^{\infty}} + \frac{3\sqrt{2}}{\omega\sqrt{\pi}} \int_0^{\infty} e^{-3x} \sin(\omega x)\,dx$$

$$= -\frac{3\sqrt{2}}{\omega\sqrt{\pi}} \cdot \frac{e^{-3x}\cos \omega x}{\omega}\Big|_0^{\infty} - \frac{3^2\sqrt{2}}{\omega^2\sqrt{\pi}}\,\hat{g}(\omega)$$

$$= \frac{3\sqrt{2}}{\omega^2\sqrt{\pi}} - \frac{9\sqrt{2}}{\omega^2\sqrt{\pi}}\,\hat{g}(\omega)$$

$$\Leftrightarrow \hat{g}(\omega)\left(\frac{\omega^2\sqrt{\pi} + 9\sqrt{2}}{\omega^2\sqrt{\pi}}\right) = \frac{3\sqrt{2}}{\omega^2\sqrt{\pi}}$$

$$\therefore \hat{g}(\omega) = \frac{3\sqrt{2}}{\omega^2\sqrt{\pi} + 9\sqrt{2}}$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \cdot \frac{3}{\omega^2 + 9\sqrt{2}/\sqrt{\pi}}$$

k) $$f(x) - e^{-3|x|} * f(x) = e^{-3|x|} \qquad (*)$$

$$\mathcal{F}\{e^{-3|x|}\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-3|x|} e^{-i\omega x}\, dx$$

$$\Leftrightarrow \sqrt{2\pi}\, \mathcal{F}\{e^{-3|x|}\}$$

$$= \int_{-\infty}^{0} e^{3x} e^{-i\omega x}\, dx + \int_{0}^{\infty} e^{-3x} e^{-i\omega x}\, dx$$

$$= \int_{-\infty}^{0} e^{(3-i\omega)x}\, dx + \int_{0}^{\infty} e^{(-3-i\omega)x}\, dx$$

$$= \left. \frac{e^{(3-i\omega)x}}{3-i\omega} \right|_{-\infty}^{0} + \left. \frac{e^{(-3-i\omega)x}}{-3-i\omega} \right|_{0}^{\infty}$$

$$= \frac{1}{3-i\omega} + \frac{1}{3+i\omega}$$

$$= \frac{3+\cancel{i\omega}+3-\cancel{i\omega}}{3^2+\omega^2}$$

$$= \frac{6}{9+\omega^2} \quad R$$

Taking the Fourier transform of (*) we get

$$\hat{f}(\omega) - \sqrt{2\pi}\cdot \mathcal{F}\{e^{-3|x|}\}\hat{f}(\omega) = \mathcal{F}\{e^{-3|x|}\}$$

$$\Leftrightarrow \hat{f}(\omega) - \frac{6}{9+\omega^2}\hat{f}(\omega) = \frac{6}{9+\omega^2}$$

$$\Leftrightarrow \hat{f}(\omega)\left(\frac{9+\omega^2-6}{\cancel{9+\omega^2}}\right) = \frac{6}{\cancel{9+\omega^2}}$$

$$\Leftrightarrow \hat{f}(\omega) = \frac{6}{\omega^2+3}$$

Since $\hat{f}(w) = \frac{6}{w^2+3}$, we get

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{6}{w^2+3} e^{iwx} \, dw$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{6}{w^2+3} \cos wx \, dw$$

$$f(x) = \frac{6\sqrt{2}}{\sqrt{\pi}} \int_{0}^{\infty} \frac{\cos wx}{w^2+3} \, dw$$

R

L) $f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & \text{otherwise} \end{cases}$

$g(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$

a) $$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} \, dx$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \int_{0}^{1} \cos wx \, dx$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \frac{\sin w}{w}$$

R

$$\hat{g}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-i\omega x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-x} e^{-i\omega x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{(-1-i\omega)x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{-1-i\omega} \left[e^{(-1-i\omega)x} \right]_{0}^{\infty}$$

$$= \frac{-1}{\sqrt{2\pi}} \cdot \frac{-1+i\omega}{1^2+\omega^2}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1-i\omega}{1+\omega^2} \quad R$$

b) $h(x) = (f * g)(x)$

$$\Rightarrow \hat{h}(\omega) = \sqrt{2\pi}\, \hat{f}(\omega)\, \hat{g}(\omega)$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \cdot \frac{\sin\omega}{\omega} \cdot \frac{1-i\omega}{1+\omega^2}$$

Vi vet at

$$h(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{h}(\omega) e^{i\omega x} d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\sqrt{2}}{\sqrt{\pi}} \frac{\sin\omega}{\omega} \cdot \frac{1-i\omega}{1+\omega^2} e^{i\omega x} d\omega$$

$$\Leftrightarrow \quad h(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(1-i\omega)\sin\omega}{\omega(1+\omega^2)} e^{i\omega x} d\omega$$

Want to calculate

$$\int_{-\infty}^{\infty} \frac{(1-i\omega)\sin\omega}{\omega(1+\omega^2)}\, d\omega$$

using $h(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(1-i\omega)\sin\omega}{\omega(1+\omega^2)} e^{i\omega x}\, d\omega$ R

Have to find x_0 such that *

$$(1-i\omega)\cdot e^{i\omega x_0} = 1$$

Then $h(x_0)$ is the value we want.

$$(1-i\omega)e^{i\omega x_0} = \pi$$

$$\Leftrightarrow \quad 1 - i\omega = \pi e^{-i\omega x_0}$$

$$\Rightarrow \quad 1 = \pi\cos\omega x_0 \qquad (1)$$

$$-\omega = -\pi\sin\omega x_0 \qquad (2)$$

$$(1) \Rightarrow \omega x_0 = \cos^{-1}\left(\frac{1}{\pi}\right)$$

$$(2) \Rightarrow -\omega = -\pi\sin\left(\cos^{-1}\left(\frac{1}{\pi}\right)\right)$$

$$\Rightarrow \omega = \pi\sqrt{1-\frac{1}{\pi^2}}$$

$$= \pi\frac{\sqrt{\pi^2-1}}{\pi}$$

$$= \sqrt{\pi^2-1}$$

$$(1) \Rightarrow x_0 = \frac{\cos^{-1}\left(\frac{1}{\pi}\right)}{\sqrt{\pi^2-1}}$$ g

* Du kan også sette $x = 0$ og si at $h(0)$ må være lik Realdelen av integralet.

So $\displaystyle\int_{-\infty}^{\infty} \frac{\sin w}{w(1+w^2)}\, dw = h\left(\frac{\cos^{-1}(1/\pi)}{\sqrt{\pi^2-1}}\right)$

$$= \underline{\underline{e^{-\frac{\cos^{-1}(1/\pi)}{\sqrt{\pi^2-1}}}}}$$

$$\approx 0{,}65$$

M) $f(x) = \begin{cases} 1, & |x|<1 \\ 0, & |x|\geq 1 \end{cases}$

This is the same as f in sup. L where we got

$$\hat{f}(w) = \frac{\sqrt{2}}{\sqrt{\pi}} \frac{\sin w}{w}$$

Since $\sin(w) = \dfrac{(\cos w + i\sin w) - (\cos w - i \sin w)}{2i}$

$$= \frac{e^{iw} - e^{-iw}}{2i}$$

$$= \frac{i}{2} \cdot \left(e^{-iw} - e^{iw}\right)$$

we get

$$\hat{f}(w) = \frac{\sqrt{2}}{\sqrt{\pi}} \frac{\frac{i}{2} \cdot \left(e^{-iw} - e^{iw}\right)}{w}$$

$$= \frac{i}{\sqrt{2\pi}} \cdot \frac{e^{-iw} - e^{iw}}{w}$$

Want to compute

$$\mathcal{F}\{f*f\}(\omega)$$

$$= \sqrt{2\pi}\,\hat{f}(\omega)\hat{f}(\omega)$$

$$= \cancel{\sqrt{2\pi}}\,\frac{i}{\cancel{\sqrt{2\pi}}}\,\frac{e^{-i\omega}-e^{i\omega}}{\omega}\cdot\frac{i}{\sqrt{2\pi}}\,\frac{e^{-i\omega}-e^{i\omega}}{\omega}$$

$$= \frac{-1}{\sqrt{2\pi}}\,\frac{(e^{-i\omega}-e^{i\omega})^2}{\omega^2}$$

$$= \frac{-e^{-2i\omega}+2e^{-i\omega}e^{i\omega}-e^{2i\omega}}{\sqrt{2\pi}\cdot\omega^2}$$

$$= \frac{-e^{-2i\omega}-e^{2i\omega}+2}{\sqrt{2\pi}\,\omega^2}$$

We know that (because we assume f is continous)

$$(f*f)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{-e^{-2i\omega}-e^{2i\omega}+2}{\sqrt{2\pi}\,\omega^2}\,e^{i\omega x}\,d\omega$$

$$I(x) = (-e^{-2i\omega}-e^{2i\omega}+2)e^{i\omega x}$$

$$= -e^{(-2+x)\omega i}-e^{(2+x)i\omega}+2e^{i\omega x}$$

$x=3$ gives

$$I(3) = -\cos\omega + i\sin\omega - \cos 5\omega - i\sin 5\omega + 2\cos 3\omega + i2\sin 3\omega$$

All the sine functions are even so when we integrate from $-\infty$ to ∞ we get that they make no contribution.

(This also applies when dividing by ω^2 (even).)

$$\text{So } (f*f)(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{I(x)}{\omega^2}\, d\omega$$

$$(f*f)(3) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-\cos\omega - \cos 5\omega + 2\cos 3\omega}{\omega^2}\, d\omega$$

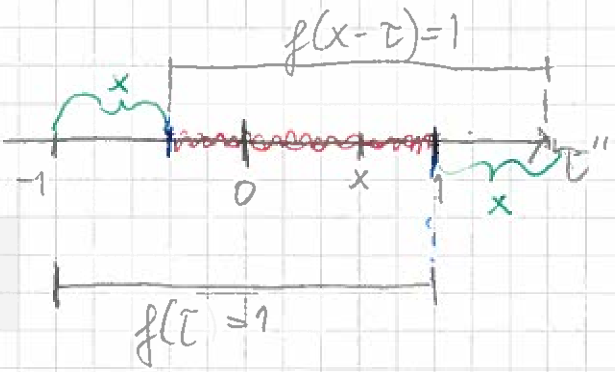
$$\Leftrightarrow \int_{-\infty}^{\infty} \frac{\cos 5\omega - 2\cos 3\omega + \cos \omega}{\omega^2}\, d\omega$$

$$= -2\pi \cdot (f*f)(3)$$

$$(f*f)(x) = \int_{-\infty}^{\infty} f(x-\tau) f(\tau)\, d\tau$$

$$f(x-\tau) = \begin{cases} 1, & |x-\tau| < 1 \\ 0, & |x-\tau| \geq 1 \end{cases}$$

if $x \geq 2$ then $f(x-\tau) = 0$ when $f(\tau) = 1$ and $f(\tau) = 0$ when $f(x-\tau) = 1$.
If $|x| < 2$, then the two functions will be 1 in some overlapping range.



For $|x| \geq 2$ $(f*f)(x) = 0$

For $|x| < 2$ we integrate from

$\max(-1, -1+x)$ to $\min(1, 1+x)$

with integrand 1.

$$(f*f)(x) = \int_{\max(-1,-1+x)}^{\min(+1,1+x)} dx$$

$$= \min(1, 1+x) - \max(-1, -1+x)$$

$$x \geq 0 \Rightarrow (f*f)(x) = 1 - (-1+x)$$
$$= 2 - x$$

$$x < 0 \Rightarrow (f*f)(x) = 1+x - (-1)$$
$$= 2+x$$

So in total

$$(f*f)(x) = \begin{cases} 2-x, & 0 \leq x < 2 \\ 2+x, & -2 < x < 0 \\ 0, & |x| \geq 2 \end{cases}$$

$$\Rightarrow (f*f)(3) = 0$$

So

$$\int_{-\infty}^{\infty} \frac{\cos 5w - 2\cos 3w + \cos(w)}{w^2}\, dw = 0$$

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