

Exercise 9

TTK4130 Modeling and Simulation

Problem 1 (Sliding stick (Exam 2010))

Consider a stick of length ℓ with uniformly distributed mass m . It has center of mass/gravity C , about which it has a moment of inertia I_z . The stick is in contact with a frictionless horizontal surface, and moves due to the influence of gravity. See Figure 1.

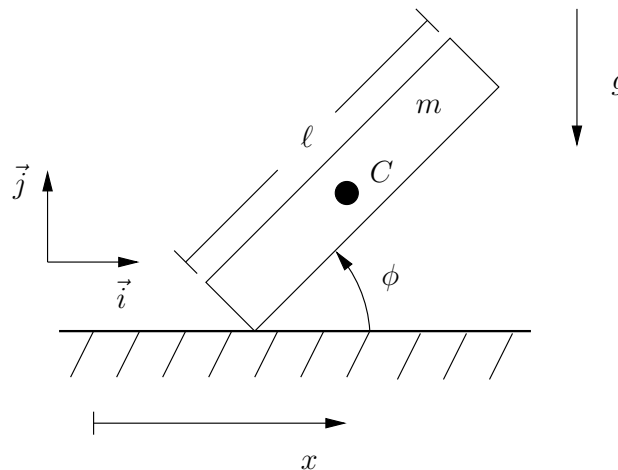


Figure 1: Stick sliding on frictionless surface

- (a) Choose appropriate generalized coordinates (the figure should give you some hints). What are the corresponding generalized (actuator) forces?

Solution: A natural choice for generalized coordinates are the horizontal position of the center of mass/gravity (denoted x), and ϕ , the angle between the stick and the surface. An alternative to x could be the contact point between the stick and the surface.

There are no (generalized) actuator forces corresponding to these coordinates. (A candidate answering 'gravity' might get full score if he/she uses it correctly in the rest of the Problem.)

- (b) What are the position, velocity, and angular velocity of the center of mass, as function of your chosen generalized coordinates (and/or their derivatives)?

Solution:

$$\begin{aligned}\vec{r}_c &= x\vec{i} + \frac{\ell}{2} \sin \phi \vec{j} \\ \vec{v}_c &= \dot{x}\vec{i} + \frac{\ell}{2} \dot{\phi} \cos \phi \vec{j} \\ \vec{\omega}_{ib} &= \dot{\phi} \vec{k}\end{aligned}$$

(Coordinate vectors also accepted for the position and velocity, scalar accepted for angular velocity.)

- (c) Write up the kinetic and potential energy of the stick, as function of your chosen generalized coordinates (and/or their derivatives).

Solution: The kinetic energy for the rigid body is

$$\begin{aligned} T &= \frac{1}{2} m \vec{v}_c \cdot \vec{v}_c + \frac{1}{2} \vec{\omega}_{ib} \cdot \vec{M}_{b/c} \cdot \vec{\omega}_{ib} \\ &= \frac{1}{2} m \left(\dot{x}^2 + \frac{\ell^2}{4} \dot{\phi}^2 \cos^2 \phi \right) + \frac{1}{2} I_z \dot{\phi}^2. \end{aligned}$$

The potential energy due to gravity is

$$U = mg \frac{\ell}{2} \sin \phi.$$

(d) Derive the equations of motion for the stick.

Solution: It is probably easiest to use Lagrange's equation of motion. Define the Lagrangian $L = T - U$. Then the first equation of motion is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0,$$

which reduces to

$$\ddot{x} = 0.$$

The second equation of motion is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\phi}} \right) - \frac{\partial L}{\partial \phi} = 0$$

which gives

$$\left(\frac{m\ell^2}{4} \cos^2 \phi + I_z \right) \ddot{\phi} - \frac{m\ell^2}{4} \dot{\phi}^2 \cos \phi \sin \phi + mg \frac{\ell}{2} \cos \phi = 0.$$

Problem 2 (Double inverted pendulum)

The double inverted pendulum on a cart (DIPC) poses a challenging control problem. In a DIPC system, two rods are connected together on a moving cart as shown in Figure 2. The rod is located above the centre of mass of the cart. The length of the first rod is denoted by l_1 and the length of the second rod by l_2 . The mass of the cart is denoted by m_0 , its length by l_0 and its width by b_0 . The height of the cart is denoted by h_0 . Both rods have a mass, which are denoted by m_1 and m_2 . All masses are assumed to be concentrated into the centre of mass. The moments of inertia are denoted by I_i . Furthermore, the force τ is acting on the cart.

(a) Find the position of cart and the two rods.

Solution: The position of the cart is given as

$$r_0 = \begin{pmatrix} \theta_0 \\ 0 \end{pmatrix}, \quad (1)$$

the position of the centre of mass of the first rod is given by

$$r_1 = \begin{pmatrix} \theta_0 + \frac{1}{2} l_1 \sin \theta_1 \\ h_0 + \frac{1}{2} l_1 \cos \theta_1 \end{pmatrix} \quad (2)$$

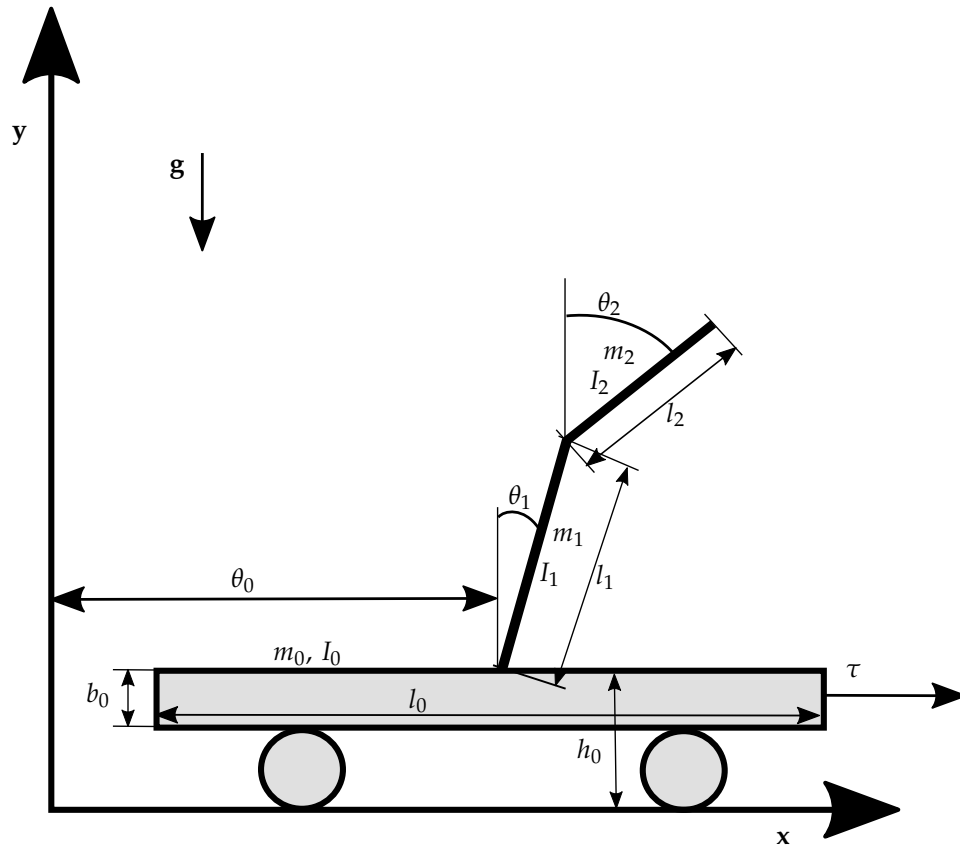


Figure 2: Double inverted pendulum on a cart

and the position of the second rod is given by

$$r_2 = \begin{pmatrix} \theta_0 + l_1 \sin \theta_1 + \frac{1}{2} \sin \theta_2 \\ h_0 + l_1 \cos \theta_1 + \frac{1}{2} l_2 \cos \theta_2 \end{pmatrix}. \quad (3)$$

If the origin of the coordinate system is chosen differently the terms can look slightly different. However, if the coordinate system is chosen differently then in Fig. 2, it has to be mentioned.

- (b) Find the kinetic energy of the system. (*Hint: $\cos(x - y) = \cos x \cos y + \sin x \sin y$*)

Solution:

- The velocities can be calculated by derivation of the positions

$$v_0 = \begin{pmatrix} \dot{\theta}_0 \\ 0 \end{pmatrix}, \quad (4a)$$

$$v_1 = \begin{pmatrix} \dot{\theta}_0 + \frac{1}{2} l_1 \dot{\theta}_1 \cos \theta_1 \\ -\frac{1}{2} l_1 \dot{\theta}_1 \sin \theta_1 \end{pmatrix}, \quad (4b)$$

$$v_2 = \begin{pmatrix} \dot{\theta}_0 + l_1 \dot{\theta}_1 \cos \theta_1 + \frac{1}{2} l_2 \dot{\theta}_2 \cos \theta_2 \\ -l_1 \dot{\theta}_1 \sin \theta_1 - \frac{1}{2} l_2 \dot{\theta}_2 \sin \theta_2 \end{pmatrix}. \quad (4c)$$

- The kinetic energy is given by

$$T = \sum_i T_i, \quad (5)$$

with

$$T_i = \frac{1}{2} m_i v_i^T v_i + \frac{1}{2} I_i \omega_i^T \omega_i. \quad (6)$$

Consequently, the kinetic energy of each body part is given by

$$T_0 = \frac{1}{2} m_0 \dot{\theta}_0^2, \quad (7a)$$

$$T_1 = \frac{1}{2} m_1 \left(\dot{\theta}_0^2 + l_1 \dot{\theta}_0 \dot{\theta}_1 \cos \theta_1 + \frac{1}{4} l_1^2 \dot{\theta}_1^2 \right) + \frac{1}{2} I_1 \dot{\theta}_1^2, \quad (7b)$$

$$T_2 = \frac{1}{2} m_2 \left(\dot{\theta}_0^2 + l_1^2 \dot{\theta}_1^2 + \frac{1}{4} l_2^2 \dot{\theta}_2^2 + 2 l_1 \dot{\theta}_0 \dot{\theta}_1 \cos \theta_1 + l_2 \dot{\theta}_0 \dot{\theta}_2 \cos \theta_2 + l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos (\theta_1 - \theta_2) \right) + \frac{1}{2} I_2 \dot{\theta}_2^2, \quad (7c)$$

with

$$\cos (\theta_1 - \theta_2) = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2. \quad (8)$$

- (c) Find the potential energy of the system.

Solution: The potential energy is given by

$$U = \sum_i U_i \quad (9)$$

$$U_0 = m_0 g (h_0 - \frac{1}{2} b_0), \quad (10a)$$

$$U_1 = m_1 g (h_0 + \frac{1}{2} l_1 \cos \theta_1), \quad (10b)$$

$$U_2 = m_2 g (h_0 + l_1 \cos \theta_1 + \frac{1}{2} l_2 \cos \theta_2). \quad (10c)$$

If the origin of the coordinate system is chosen differently the terms can look slightly different. However, if the coordinate system is chosen differently then in Fig. 2, it has to be mentioned.

- (d) Find the equation of motion of the system.

Solution: The Lagrange equation of motion

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \tau_i \quad (11)$$

is used. L is defined as

$$L(\mathbf{q}, \dot{\mathbf{q}}, t) = T(\mathbf{q}, \dot{\mathbf{q}}, t) - U(\mathbf{q}), \quad (12)$$

where T is the kinetic energy and U the potential energy of the system.

- The resulting Lagrangian is

$$L = \frac{1}{2} \left[(m_0 + m_1 + m_2) \dot{\theta}_0^2 + \left(\frac{1}{2} m_1 l_1^2 + m_2 l_1^2 + I_1 \right) \dot{\theta}_1^2 + \left(\frac{1}{2} m_2 l_2^2 + I_2 \right) \dot{\theta}_2^2 + (m_1 l_1 + 2m_2 l_1) \dot{\theta}_0 \dot{\theta}_1 \cos \theta_1 + m_2 l_2 \dot{\theta}_0 \dot{\theta}_2 \cos \theta_2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right] - \left(\frac{1}{2} m_1 + m_2 \right) g l_1 \cos \theta_1 - \frac{1}{2} m_2 g l_2 \cos \theta_2 - (m_0 + m_1 + m_2) h_0 g. \quad (13)$$

- The derivation of the Lagrangian with respect to $\dot{\mathbf{q}}$ is

$$\frac{\partial}{\partial \dot{\mathbf{q}}} L(\mathbf{q}, \dot{\mathbf{q}}, t) = \begin{pmatrix} (m_0 + m_1 + m_2) \dot{\theta}_0 + \frac{1}{2} ((m_1 + 2m_2) l_1 \dot{\theta}_1 \cos \theta_1 + m_2 l_2 \dot{\theta}_2 \cos \theta_2) \\ \left(\frac{1}{2} m_1 l_1^2 + m_2 l_1^2 + I_1 \right) \dot{\theta}_1 + \frac{1}{2} ((m_1 + 2m_2) l_1 \dot{\theta}_0 \cos \theta_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)) \\ \left(\frac{1}{2} m_2 l_2^2 + I_2 \right) \dot{\theta}_2 + \frac{1}{2} m_2 l_2 (\dot{\theta}_0 \cos \theta_2 + l_1 \dot{\theta}_1 \cos(\theta_1 - \theta_2)) \end{pmatrix} \quad (14)$$

- The derivation of Eq. ?? with respect to time is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \right) = \begin{pmatrix} (\sum_i m_i) \ddot{\theta}_0 + \frac{1}{2} (l_1 (m_1 + 2m_2) (\ddot{\theta}_1 \cos \theta_1 - \dot{\theta}_1^2 \sin \theta_1) + m_2 l_2 (\ddot{\theta}_2 \cos \theta_2 - \dot{\theta}_2^2 \sin \theta_2)) \\ \left(\frac{1}{2} m_1 l_1^2 + m_2 l_1^2 + I_1 \right) \ddot{\theta}_1 + \frac{1}{2} (l_1 (m_1 + 2m_2) (\ddot{\theta}_0 \cos \theta_1 - \dot{\theta}_0 \dot{\theta}_1 \sin \theta_1) + \dots \\ m_2 l_1 l_2 [\ddot{\theta}_2 \cos(\theta_1 - \theta_2) - \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)]) \\ \left(\frac{1}{2} m_2 l_2^2 + I_2 \right) \ddot{\theta}_2 + \frac{1}{2} (m_2 l_2 (\ddot{\theta}_0 \cos \theta_2 - \dot{\theta}_0 \dot{\theta}_2 \sin \theta_2) + m_2 l_1 l_2 [\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dots \\ \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2)]) \end{pmatrix} \quad (15)$$

- The derivation of the Lagrangian with respect to \mathbf{q} is

$$\frac{\partial}{\partial \mathbf{q}} L(\mathbf{q}, \dot{\mathbf{q}}, t) = \frac{1}{2} \begin{pmatrix} 0 \\ -(m_1 + 2m_2) l_1 \dot{\theta}_0 \dot{\theta}_1 \sin \theta_1 - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + \dots \\ (m_1 + 2m_2) g l_1 \sin \theta_1 \\ -m_2 l_2 \dot{\theta}_0 \dot{\theta}_2 \sin \theta_2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin \theta_2 \end{pmatrix}. \quad (16)$$

- The results (??) and (??) are put in Eq. ??, which gives

$$\tau = (\sum m_i) \ddot{\theta}_0 + \frac{1}{2} (l_1 (m_1 + 2m_2 l_1) (\ddot{\theta}_1 \cos \theta_1 - \dot{\theta}_1^2 \sin \theta_1) + m_2 l_2 (\ddot{\theta}_2 \cos \theta_2 - \dot{\theta}_2^2 \sin \theta_2)), \quad (17a)$$

$$0 = \left(\frac{1}{2} m_1 l_1^2 + m_2 l_1^2 + I_1 \right) \ddot{\theta}_1 + \frac{1}{2} (l_1 (m_1 + 2m_2) \ddot{\theta}_0 \cos \theta_1 + m_2 l_1 l_2 [\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)] - (m_1 + 2m_2) g l_1 \sin \theta_1), \quad (17b)$$

$$0 = \left(\frac{1}{2} m_2 l_2^2 + I_2 \right) \ddot{\theta}_2 + \frac{1}{2} (m_2 l_2 \ddot{\theta}_0 \cos \theta_2 + m_2 l_1 l_2 [\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1^2 \sin(\theta_1 - \theta_2)] - m_2 g l_2 \sin \theta_2), \quad (17c)$$

which are the equations of motion for the DIPC system.

(e) The moment of inertia is defined as

$$I = \int_Q r^2 dm \quad (18)$$

where r is the distance from each point to the axis of rotation and Q is the entire mass. Derive the moment of inertia for the rectangular plate with the length l and height h for the centre of mass of the plate (Fig. 3). The axis of rotation is the z -axis (perpendicular to the plate). Assume that the plate has a constant width b and a homogeneous density ρ .

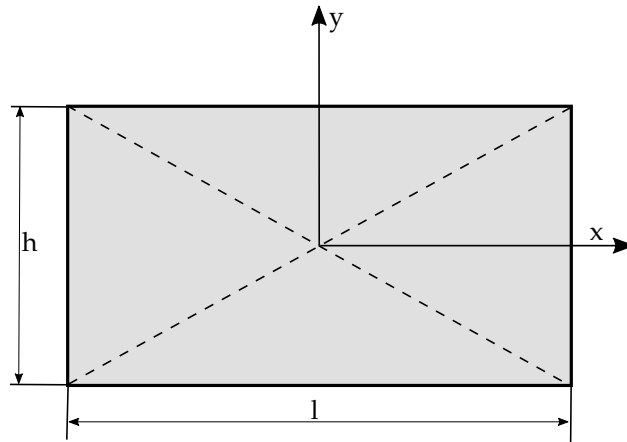


Figure 3: Rectangular plate

Solution: The distance r of each point on the plate to the z -axis is $y^2 + x^2$. Consequently, the moment of inertia is

$$I = \int (x^2 + y^2) dm. \quad (19)$$

In a next step the dm has to be replaced. We know that the infinitesimal mass is

$$dm = \rho dV = \rho b \cdot dA = \rho b \cdot dx dy, \quad (20)$$

where ρ is the density and dV is a infinitesimal volume, dA the infinitesimal area and dx and dy the infinitesimal lengths (Fig. ??).

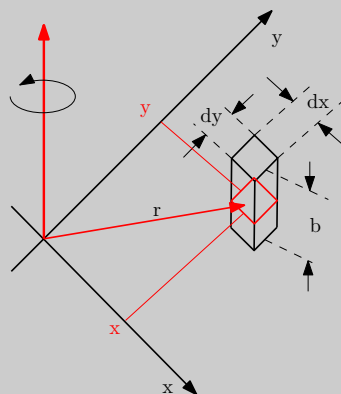


Figure 4: Infinitesimal volume of rectangular plate

Eq. ?? and ?? together with the borders of the integral, which are given by the dimensions of the plate, are

$$I = \rho b \int_{-h/2}^{h/2} \int_{-l/2}^{l/2} (x^2 + y^2) dx dy. \quad (21)$$

In the next step the integral has to be solved over the dimensions of the plate

$$\begin{aligned} I &= \rho b \int_{-h/2}^{h/2} \int_{-l/2}^{l/2} (x^2 + y^2) dx dy \\ &= \rho b \int_{-h/2}^{h/2} \left(\frac{1}{3} x^3 + y^2 x \right) \Big|_{-l/2}^{l/2} dy \\ &= \rho b \int_{-h/2}^{h/2} \left(\frac{1}{12} l^3 + y^2 l \right) dy \\ &= \rho b \left(\frac{1}{12} l^3 y + \frac{1}{3} y^3 l \right) \Big|_{-h/2}^{h/2} \\ &= \frac{1}{12} \rho b (l^3 h + h^3 l) \\ I &= \frac{1}{12} m (l^2 + h^2). \end{aligned} \quad (22)$$

I is the moment of inertia through the centre of mass.

- (f) Use the parallel axes theorem to change the point of rotation from the centre of mass to point **A** given in Fig. 5.

(Help: If you were not able to solve the previous task, use as moment of inertia $I = \frac{1}{4} m (l^2 + h^2)$ Please state explicitly, if you use this for further calculations.)

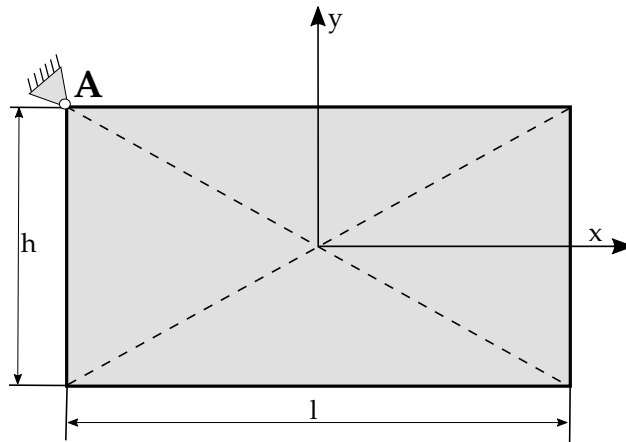


Figure 5: Rectangular plate with attachment on one corner

Solution: The parallel axes theorem states that the inertia I_A through some other axes is equal to the inertia through the centre of mass plus the mass multiplied with the distance of the new axes to the centre of mass squared

$$I_A = I + m d^2, \quad (23)$$

where d is the distance of the new axes to the centre of mass. It can be easily seen that the distance of **A** to the centre of mass is

$$d = \sqrt{x^2 + y^2} = \sqrt{(-l/2)^2 + (h/2)^2}. \quad (24)$$

Consequently, the moment of inertia I_A is

$$\begin{aligned} I_A &= I + m((-l/2)^2 + (h/2)^2) \\ &= \frac{1}{12}m(l^2 + h^2) + \frac{1}{4}m(l^2 + h^2) \\ I_A &= \frac{1}{3}m(l^2 + h^2) \end{aligned} \quad (25)$$

If the **hint** was use the result is

$$I_A = \frac{1}{2}m(l^2 + h^2). \quad (26)$$

- (g) Make an appropriate simplification for the moment of inertia of the rods used in the DIPC system. Justify your decision!

Solution: In case of the rod is the dimension of one axes much larger than the other dimension. It holds $l \gg h$. Therefore, the moment of inertia can be simplified to

$$I_A = \frac{1}{3}ml^2 \quad \text{or} \quad I = \frac{1}{12}ml^2. \quad (27)$$