Exercise 3

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Problem 1

$$\min_x c^T x \quad \text{s.t.} \quad Ax = b, \; x \geq 0$$

KKT conditions:

$$A^T\lambda + s = c \qquad (1)$$
$$Ax = b \qquad (2)$$
$$x \geq 0 \qquad (3)$$
$$s \geq 0 \qquad (4)$$
$$x_i s_i = 0 \qquad (5)$$

a) Calculation of Newton direction assumes that the Hessian of $c^T x$ is invertible, but

$$\nabla^2 c^T x = 0 \quad (\text{zero-matrix})$$

Since it is linear, and thus $(\nabla^2 c^T x)^{-1}$ doesn't exist.

b) The objective function is linear so it is clearly convex.

Proof: $f(\alpha x + (1-\alpha) y)$

$$= c^T(\alpha x + (1-\alpha) y)$$

$$= \alpha c^T x + (1-\alpha) c^T y$$

$$= \alpha f(x) + (1-\alpha) f(y)$$

$$\leq \alpha f(x) + (1-\alpha) f(y).$$

To show that the feasible set Ω is convex we'll consider two points $x, y \in \Omega$ and the line connecting them $\alpha x + (1-\alpha) y$, $\alpha \in [0,1]$. If this line is fully contained in Ω, then Ω is convex.

$$A(\alpha x + (1-\alpha) y) = \alpha A x + (1-\alpha) A y$$

$$= \alpha b + (1-\alpha) b$$

$$= b \quad \checkmark$$

Since $x \geq 0$, $y \geq 0$, $\alpha \geq 0$ and $(1-\alpha) \geq 0$ we can say

$$\alpha x + (1-\alpha) y \geq 0 \quad \checkmark$$

This shows that the line is feasible and thus that the set is convex.

c) $\max_{\lambda} b^T\lambda$ s.t. $A^T\lambda \leq c$

We first rephrase it as a minimization problem with equality constraints

$$\min_{\lambda} -b^T\lambda \quad \text{s.t.} \quad c - A^T\lambda - s = 0, \; s \geq 0$$

The Lagrangian is given by

$$\mathcal{L}(\lambda, s, x) = -b^T\lambda + x^T(c - A^T\lambda - s) + x^T s$$

Optimality requires $\nabla_\lambda \mathcal{L} = 0$, $x \geq 0$, $x^T s = 0$

$\nabla_\lambda \mathcal{L} = 0 \Rightarrow 0 = -b + Ax$

$\Leftrightarrow \quad Ax = b$

$x^T s = 0 \Leftrightarrow \quad x_i s_i = 0$ for $i = 1, \ldots, n$

Summarizing these requrements:

$$A^T\lambda + s = c$$

$$Ax = b$$

$$x \geq 0$$

$$s \geq 0$$

$$x_i s_i = 0$$

d) $$c^T x^* \overset{\text{(KKT-1)}}{=} (A^T \lambda^* + s^*)^T x^*$$
$$= (\lambda^{*T} A + s^{*T}) x^*$$
$$= \lambda^{*T} \underbrace{A x^*}_{} + s^{*T} x^*$$
$$= \lambda^{*T} b + 0$$
$$= (b^T \lambda^*)^T$$
$$= b^T \lambda^* \quad \text{(scalar)}$$

They are <u>equal</u>

e) A point x is feasible if there exists a a subset $\mathcal{B}$ of the indices, such that

- $|\mathcal{B}| = m$
- For $i \notin \mathcal{B}$ we have $x_i = 0$
- The matrix B defined by the i'th columns of A, where $i \in \mathcal{B}$ is non-singular.

f) If a_i is the i'th row of A, then we can write the equality constraints as

$$c_i(x) = a_i x - b_i = 0$$

$$\Rightarrow \nabla c_i = a_i^T$$

As long as $x_i \neq 0$ for all i, the the active set is given by the equality constraints. Then the linear independence of the rows of A guarantees that the LICQ holds.

Problem 2

a) $x_1 = x_A = \overbrace{\text{amount}}^{\#/1000\text{kg}}$ of A

$x_2 = x_B =$ —— " —— B

$f(x) =$ "profit"

$= \frac{3}{2} x_1 + x_2 = -c^T x, \quad c = \begin{pmatrix} -3/2 \\ -1 \end{pmatrix}$

Constraints:

$0 \leq R_I \leq 8$

$0 \leq R_{II} \leq 15$

where

$R_I = 2 \cdot x_1 + 1 \cdot x_2$

$R_{II} = 1 \cdot x_1 + 3 \cdot x_2$

By adding slack variables $x_3, x_4 \geq 0$ we can enforce the equality constraint.

This constraint can be stated as

$Ax = b, \quad A = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 8 \\ 15 \end{pmatrix}$

Also have $x \geq 0$ since we can't have a net loss of of product.

We can now state it as a maximization LP:

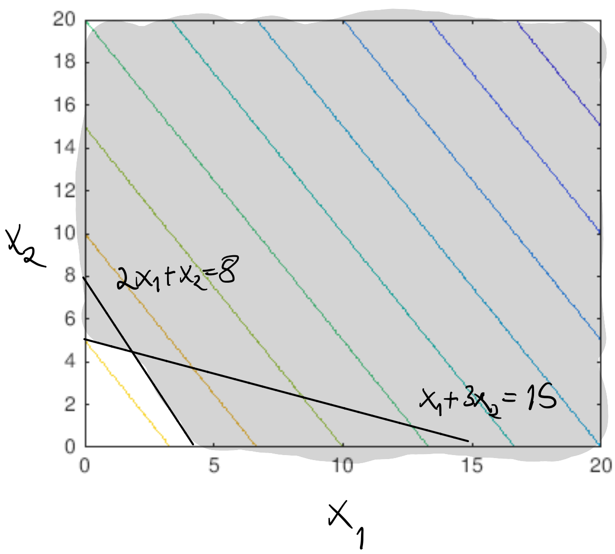
$$\max_{x} -c^T x \quad , Ax=b, \ x \geq 0$$

Flipping it a standard form LP we get

$$\min_{x} c^T x \quad \text{s.t.} \quad Ax=b, \quad x \geq 0$$

where $c = \begin{pmatrix} -3/2 \\ -1 \end{pmatrix}, \ A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, \ b = \begin{pmatrix} 8 \\ 15 \end{pmatrix}$

b)



c) Running the simplex algorithm gives solution

$A: x_1 = 1.8 \quad (\cdot 1000 \text{ kg})$

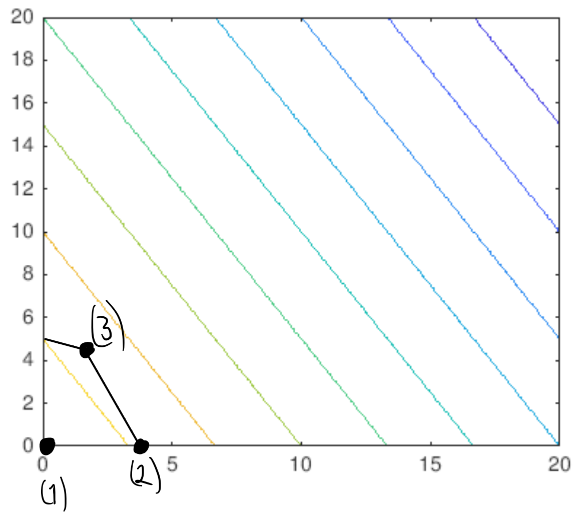
$B: x_2 = 4.4 \quad (\cdot 1000 \text{ kg})$

The constraint $x \geq 0$ is not active.

The solution is at a point where both the slack variables are zero, so it is at the intersection of the constraints

$$R_I = 8, \quad R_{II} = 15$$

d)



e)

Problem 3

$$\min_x \quad q(x) = \frac{1}{2}x^T G x + x^T c$$

$$\text{s.t.} \quad a_i^T x = b_i \qquad i \in \mathcal{E}$$

$$a_i^T x \geq b_i \qquad i \in \mathcal{I}$$

a)

$$A(x^*) = \mathcal{E} \cup \{ i \mid i \in \mathcal{I},\ a_i^T x^* = b_i \}$$

b) $\mathcal{L}(x, \lambda) = q(x) - \sum \lambda_i (a_i^T x - b_i)$

$$\Rightarrow \nabla_x \mathcal{L}(x, \lambda) = Gx + c - \sum \lambda_i a_i$$

For $i \notin A(x^*)$ we know that $\lambda_i = 0$, thus

$\lambda_i c_i(x^*) = \lambda_i (a_i^T x^* - b_i) = 0$ for all $i \in \mathcal{E} \cup \mathcal{I}$

This allows us to simplify the $\nabla_x \mathcal{L}$ at x^*

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = Gx^* + c - \sum_{i \in A(x^*)} \lambda_i^* a_i$$

The inactive inequality constraints must satisfy

$$a_i^T x^* > b_i$$

The active constraints must satisfy

$$a_i^T x^* = b_i$$

If the solution is optimal the Lagrangian multipliers must be non-negative for inequality constraints.

$$\lambda_i^* \geq 0 \quad , i \in \mathcal{I}$$

If the solution is optimal, then $\nabla_x \mathcal{L}(x^*, \lambda^*) = 0$, so

$$Gx^* + c = \sum_{i \in A(x^*)} \lambda_i^* a_i$$