PUSH GAMING™

Basics of slot machine maths

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Please send any corrections to thomas.barnard@pushgaming.com

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1 Introduction

This is a basic overview of the maths behind slot machines. In it, I will expect the reader to know only basic maths. I will explain some maths along the way but this will be kept to a minimum as I will try to explain more why things are done instead of the theory behind every little detail.

2 Win Line Games

Whilst the hit rate and volatility of a win line game depend on the order of symbols on the reel strips, calculating the RTP doesn't require this information, only the number of each symbol on each reel. This leaves you free to play with the order of the symbols to change the feel of the game without the fear of altering the RTP.

2.1 Basic Example

First, let us think of a very basic slot machine with a three reels, a single win line, and the game only paying out for three of a kind (3 OAK). Let the following be the symbol distribution and pay table for the slot machine:

	REEL 1	**REEL 2**	**REEL 3**	**3 OAK PAY**
HIGH	5	4	6	10
MID	10	9	11	5
LOW	17	15	14	2
SUM	32	28	31	N/A

In this game, there are 32 symbols on the first reel, 28 on the second, and 31 on the third. This means there are $32 \times 28 \times 31 = 27,776$ different possible stopping positions for the reels. Let us first see how many of these ways pays out a high win. We need to have one of the five highs from the first reel turn up, one of the four highs from the second reel to turn up and one of the six highs from the third reel to turn up. Therefore we have $5 \times 4 \times 6 = 120$ possible ways for a high win to turn up. Similarly we have $10 \times 9 \times 11 = 990$ ways of getting a mid win, and $17 \times 15 \times 14 = 3,570$ ways of getting a low win. As we have now found all of the positions which pay, we can see that there are rather a lot of stopping positions which don't pay, specifically $27,776 - (120 + 990 + 3,570) = 23,096$. If each of the $27,776$ positions are just as likely to land as any other, and as we only have one win line, we can calculate the hit rate as: $(120 + 990 + 3,570)/27,776 = 0.1685 = 16.85\%$. This game is a special case in which I can calculate hit rate without the need of a simulation. This is because there is only one win line. In proper games, what turns up on one win line influences what will turn up on other win lines meaning that a simulation is needed to calculate the hit rate, which is why it was said earlier that the hit rate depends on reel structure.

We can now calculate the RTP for this game. We have 120 different ways of getting a high win and there are 27,776 different possible landing positions, therefore the probability of getting a high win is: $(120/27,776) \times 100\% = 0.432\%$. The pay from a high win is 10 times the bet and so the RTP from the high symbols is: $0.432\% \times 10 = 4.32\%$. We will now do the same with the mids and lows. The RTP from the mids is: $(990/27,776) \times 100\% \times 5 = 17.82\%$, and the RTP from the lows is: $(3,570/27,776) \times 100\% \times 2 = 25.71\%$. And so for this very simple slot machine, the total RTP is $4.32\% + 17.82\% + 25.71\% = 47.85\%$.

2.1.1 Introducing Wilds

The wild symbol complicates matters somewhat. At the very least wilds pay out as much as the top symbol and normally they pay out more than that. As only one win is ever payed out from any one win line, we must know what we will do in the event that we have more than one win to pick from. This can happen if we can pay out a 3 OAK wild or a 4 OAK low. In this situation, we will look at the pay table and determine which pays out more (normally the wild). This is easy to understand normally, but it makes us have to do more to calculate how many ways we can get a normal win from a non-wild symbol.

Going back to our last example, let us add some wilds in.

	REEL 1	REEL 2	REEL 3	3 OAK PAY
WILD	2	3	2	15
HIGH	5	4	6	10
MID	10	9	11	5
LOW	17	15	14	2
SUM	34	31	33	N/A

Now we will do the same thing as we did before. First, from the table below, we have that there are $2 \times 3 \times 2 = 12$ positions which give a full line of wilds. Now we go onto the normal symbols. On the first reel there are five high symbols and two wild ones. This means that there are seven positions which can be a high symbol. Similarly there are seven on the second reel and eight on the third reel. This means that there are $7 \times 7 \times 8 = 392$ different landing positions for the high symbol. We get also that there are $(10+2)\times(9+3)\times(11+2) = 1,872$ landing positions for the mid symbol and $(12+2)\times(15+3)\times(14+2) = 5,472$ positions for the low symbol.

But we have a problem here. In these calculations, we have counted all of the full lines of wilds as well. And so, of the 392 possible ways of spinning the reels to get a full line of highs, 12 of them are full lines of wilds. This means that we have to take off any positions which give a better win when counted in a different category. And so in this case, as the wilds pay more than any other symbol, we have 12 positions which give a 3 OAK wild win, $392 - 12 = 380$ ways for highs, $1,872 - 12 = 1,860$ ways for mids, and $5,472 - 12 = 5,460$ ways for lows. Now we have found all of the wins available and aren't double counting any wins like we were before. This means that we have a hit rate of $(12+380+1,860+5,460)/(34*31*33) = 7,712/34,782 = 0.2217 = 22.17\%$. This hit rate is over 5% higher than we had before we introduced wilds, which demonstrates how much difference just a few wilds can make to the hit rate.

Just as before, the RTP for wilds is: $(12/34,782) \times 100\% \times 15 = 0.52\%$. For the high symbol it is: $(380/34,782) \times 100\% \times 10 = 10.93\%$. For mids: $(1,860/34,782) \times 100\% \times 5 = 26.74\%$. And for lows: $(5,460/34,782) \times 100\% \times 2 = 31.40\%$. And so the total RTP for this game is: $0.52\% + 10.93\% + 26.74\% + 31.40\% = 69.59\%$, which is over 20% higher than it was before. Again, this demonstrates how much wilds change the game and how often they contribute to a win.

2.1.2 Discounting

When we are calculating how often a certain win comes up, we saw that sometimes you double count one reel position in multiple sections (almost always because of wilds). Taking this into account by taking these instances off from the sections which pay out less is called discounting. We have to do a lot more discounting when we have a normal slot machine. Consider a slot machine with five reels and win lengths of 3, 4, or 5. For the 5 OAK wilds, no discounting is required. For the 5 OAK normal symbols, we must discount the 5 OAK wild wins we included, and if the pay for 4 OAK or 3 OAK wilds is the same as or more than the pay for the symbol we are considering, we must also discount these as well. When we look at the 4 OAK wilds, we must be careful not to count any positions which might pay out more than 4 OAK wilds do (for example wilds in reels 1, 2, 3, and 4 and then a high 1 in the fifth reel). The 4 OAK normal symbols are more simple than the 5 OAK ones, as we only have to look at the 4 OAK wilds and 3 OAK wilds this time (but no 5 OAK wilds). When we get to the 3 OAK wilds, we have to consider all 4 OAK and 5 OAK normal symbols which can take a while to wrap your head around but the 3 OAK normal symbols only require you to look at the 3 OAK wilds, which is now rather simple.

2.1.3 Multiple Win Lines

It may seem like we have forgotten slot machines are normally more complex than a single win line, but in fact we don't need to do any more than we have already done to calculate the RTP. This is because the more win lines that are added, the more it will cost to play. For one win line, it will cost one coin. For ten win lines, it costs ten coins. Each win line uses the same calculations as we have used above and so for a ten win line game, there are ten times the number of places we can get a win from, but it costs ten times more to play. And so the RTP is the same as what was calculated above, regardless of the number of win lines used.

2.1.4 SCATTERS

Scatters don't care about win lines. This means that when we calculate how often a scatter comes up, we need to know the height of the window of the slot machine we are playing. Let us go back to the game we we just looking at before, this time with 5 scatters on each reel, and assume that the height of the window is three symbols.

	REEL 1	**REEL 2**	**REEL 3**
SCATTER	5	5	5
NON-SCATTER	34	31	33
SUM	39	36	38

Assume this game requires three scatters to turn up for a feature to be triggered. Also assume that we have made the reels such that we can only have one scatter appear on a single reel at once. As the window is three high, there are three stopping positions for the reel which will reveal the scatter. And so for the first reel, there are $5 \times 3 = 15$ different stopping positions for the reel which will give us a scatter. For the second and third reels, it is the same as there are five scatters on each reel. And so the probability of getting a scatter on the first reel is 15/39, for the second reel it is 15/36, and for the third it is 15/38. For the feature to be triggered, we need all three reels to produce a scatter and the probability of this happening is $(15/39) \times (15/36) \times (15/38) = 0.0633 = 6.33\%$. We also want to know how often this event will turn up now that we have the probability of it happening. If we have some event A and its probability $P(A)$, then we can calculate the frequency of the event by using the relationship:

$$FREQ(A) = \frac{1}{P(A)}$$

And so going back to our example the frequency is $1/6.33\% = 15.8$. This means that on average it will take 15.8 spins for this event to take place.

The game we have looked at is a very simple game and probably wouldn't be very exciting to play. Now that the basic ideas of a slot machine have been described, we will look at a more complex game and go into a few extra sections like anticipations and the RTP distribution over the different symbols.

2.2 A five reel example

Now we will make the maths a little more complicated by considering a game with five reels and wins for three, four, or five of a kind. There will also be scatters and wilds in this game (where wilds count as anything apart from scatters). When making a game, you will never be able to start off writing down the correct symbol distribution and pay table immediately. This means that you have to write down a rough estimate of what you want, see what that gives you, and then refine it to get the RTP closer to where you want it. We will start off with the following initial symbol distribution:

	REEL 1	REEL 2	REEL 3	REEL 4	REEL 5
WILD	8	12	15	12	10
HIGH 1	6	8	10	11	9
HIGH 2	10	8	7	10	9
HIGH 3	12	10	14	16	13
MID 1	15	16	13	18	19
MID 2	17	19	19	20	16
MID 3	20	23	25	21	23
LOW 1	27	29	30	27	24
LOW 2	30	29	34	31	32
LOW 3	35	37	34	38	36
LOW 4	40	45	42	43	47
SCATTER	5	10	10	20	5
SUM	225	246	253	267	243

And let us also use the following pay table:

	1 OAK	2 OAK	3 OAK	4 OAK	5 OAK
WILD	0	0	30	100	300
HIGH 1	0	0	25	75	250
HIGH 2	0	0	20	50	175
HIGH 3	0	0	20	40	150
MID 1	0	0	15	30	100
MID 2	0	0	10	20	75
MID 3	0	0	10	20	60
LOW 1	0	0	8	15	50
LOW 2	0	0	8	15	45
LOW 3	0	0	5	10	30
LOW 4	0	0	5	10	25

2.2.1 5 OAK Win Calculations

Before we start with any calculations, we need to know what to do in the event that we get two different possible wins on the same win line. We always pay out the maximum win out of the two, but to make sure we don't double count any wins or miss any, we also need to know which win we will count if two wins are the same. Here I will always choose to use the win from the best symbol if I am given the choice. For example, if I had three wilds and then a mid 1 and then a low 1 on a win line, I could either choose to pay out a 3 OAK wild win or a 4 OAK mid 1 win and the result would be the same. In this situation, I would choose wilds to be the symbol to take the win. If you are doing this yourself, you could make your own rules up for this, but this isn't necessary unless there is some feature which depends on the length of each win or if you don't like assigning a mixed win to only one symbol for the RTP breakdown of your game. In addition to this, you need to be consistent with how you do this in excel and in your simulation so that they both match up, otherwise you could be looking for a bug in one of the two documents for ages when there is no bug, just a difference in how you have implemented the game.

First, we shall look at the wilds. As the wilds always pay out the top amount, no discounting is required here. And so the number of hits we get for 5 OAK wilds is just $8 \times 12 \times 15 \times 12 \times 10 = 172,800$. This means that there are 172,800 different stopping positions for the reels which give a 5 OAK payout for wilds. There are $225 \times 246 \times 253 \times 267 \times 243 = 908,564,327,550$ different stopping positions for this game, and so the probability of stopping on 5 OAK wilds is $172,800/908,564,327,550 = 0.00000019$. As the pay for 5 OAK wilds is 300 coins, the RTP for 5 OAK wilds is $(172,800/908,564,327,550) \times 300 = 0.00005706 = 0.005706\%$.

Looking now at the 5 OAK hits for high 1, for each position we see that we can either have a wild or a high 1 in the place. This means that we have $(8+6) \times (12+8) \times (15+10) \times (12+11) \times (10+9) = 3,059,000$ hits for high 1. But we still need to discount the wild wins from this. As we have only got either wilds or high 1s in these hits, we can only have wild or high 1 wins. We are trying to count the high 1 hits, but the times when wilds would take priority have to be taken away to get there. Looking at the pay table, we see that only 5 OAK wild wins take outrank 5 OAK high 1 wins (and not 4 OAK wilds or 3 OAK wilds). We have just calculated this to be 172,800 hits, and so there are $3,059,000 - 172,800 = 2,886,200$ high 1 hits in total, from which the RTP is easy to calculate. We can then calculate the number of hits and the RTP for each of the symbols in the same manner as we have just done for high 1s. But it will change slightly. When we look at the mid 1 wins, the pay out for a 5 OAK mid 1 win is the same as a 4 OAK wild win and so we would have to take off any hits we counted which had a wild in the first four reels, not just the ones which were all wild like we have had up until now. And similarly, we will have to take off 3 OAK wild wins when we reach low 3 as well.

If we do all of these calculations, we get the following results:

	HITS	RTP
WILD	172,800	0.005706%
HIGH 1	2,886,200	0.079417%
HIGH 2	3,137,760	0.060437%
HIGH 3	8,044,640	0.132813%
MID 1	15,186,720	0.167151%
MID 2	21,476,920	0.177263%
MID 3	42,118,560	0.278144%
LOW 1	85,038,930	0.467985%
LOW 2	137,147,892	0.679276%
LOW 3	234,146,900	0.773133%
LOW 4	484,395,120	1.332859%
SUM	1,033,749,442	4.154181%

2.2.2 4 OAK and 3 OAK Win Calculations

When we calculate the hits and RTP distribution for 4 OAK and 3 OAK wins, we need to discount for 4 OAK wilds and possibly 3 OAK wilds for all of the normal symbols, like we did in the previous section. But the wilds are more complicated.

When we look at the 4 OAK wild wins, we need to pay attention to what comes after the four wilds. We know that we can't have a wild following the four wilds, as otherwise it would be counted as a 5 OAK wild win, but what other symbols can be on the fifth reel? Looking at the pay table, we can't have a high 1, high 2, or high 3. This is because the win would then go to the 5 OAK high symbol, instead of the 4 OAK wild. And so we have to discount for these symbols in our 4 OAK wild calculations. And finally, for 3 OAK wins, the normal symbols are now rather easy to do as all we have to pay attention to are the 3 OAK wild wins, but we have to do rather a lot for the 3 OAK wilds. Again, looking at the pay table we see that we can't have any highs following our three wilds as this would pay out more as a 4 OAK win. But then we also have to see if any 5 OAK wins need to be looked at as well. If we have the three wilds followed by a mid 1, then we can't have a mid 1 or a wild in the fifth position as this would pay out more as a 5 OAK mid 1. This is also true for the other two mids and the first two lows as well. Now, if we do all of these calculations, we get the following results for 4 OAK wins:

	HITS	RTP
WILD	3,490,560	0.038418%
HIGH 1	32,193,280	0.265748%
HIGH 2	35,159,040	0.193487%
HIGH 3	74,800,000	0.329311%
MID 1	106,520,640	0.351722%
MID 2	172,975,040	0.380766%
MID 3	261,676,800	0.576023%
LOW 1	514,613,385	0.849604%
LOW 2	647,377,986	1.068793%
LOW 3	1,002,759,550	1.103675%
LOW 4	1,580,657,760	1.739731%
SUM	4,432,224,041	6.897278%

And the following results for 3 OAK wins:

	HITS	RTP
WILD	70,587,360	0.233073%
HIGH 1	329,663,520	0.907100%
HIGH 2	385,786,800	0.849223%
HIGH 3	657,431,640	1.447188%
MID 1	955,549,872	1.577571%
MID 2	1,422,485,550	1.565641%
MID 3	2,147,109,120	2.363189%
LOW 1	3,497,931,540	3.079964%
LOW 2	4,077,065,664	3.589897%
LOW 3	5,368,173,993	2.954207%
LOW 4	7,959,840,192	4.380449%
SUM	26,871,625,251	22.947502%

2.2.3 Free Spins Calculations

As with our previous example, we will assume that this game requires three scatters to enter the free spins. We have a problem. There is more than one way for three scatters to appear. For example, they could turn up on the first, second, and third reels, or they could turn up on the second, fourth, and fifth reels. They could also turn up on more than three reels. We will start off by looking at the cases where three scatters are collected. There are ten different ways to do this (if you want to know why, look up combinatorics or Pascal's triangle. We are doing the calculation 5 choose 3 here). The very first case is where we have a scatter appear on the first three reels and the last two don't contain scatters. This will be denoted by $\{S, S, S, \#S, \#S\}$, where the hash means "not". The probability of this happening is:

$$P(\{S, S, S, \#S, \#S\}) = \frac{5 \times 3}{225} \times \frac{10 \times 3}{246} \times \frac{10 \times 3}{253} \times \frac{267 - (20 \times 3)}{267} \times \frac{243 - (5 \times 3)}{243} = 0.000701 = 0.0701\%$$

We can then go through all of the other different combinations of three scatters as well as the five states which give four scatters and the single state which gives five scatters to get the following table:

STATE	PROBABILITY	FREQUENCY
{S, S, S, #S, #S}	0.0701%	1,425.99
{S, S, #S, S, #S}	0.1511%	661.84
{S, S, #S, #S, S}	0.0343%	2,915.93
{S, #S, S, S, #S}	0.1464%	683.29
{S, #S, S, #S, S}	0.0332%	3,010.43
{S, #S, #S, S, S}	0.0716%	1,397.22
{#S, S, S, S, #S}	0.2846%	351.40
{#S, S, S, #S, S}	0.0646%	1,548.22
{#S, S, #S, S, S}	0.1392%	718.57
{#S, #S, S, S, S}	0.1348%	741.85
{S, S, S, S, #S}	0.0203%	4,919.67
{S, S, S, #S, S}	0.0046%	21,675.06
{S, S, #S, S, S}	0.0099%	10,059.95
{S, #S, S, S, S}	0.0096%	10,385.97
{#S, S, S, S, S}	0.0187%	5,341.35
{S, S, S, S, S}	0.0013%	74,778.96
SUM	1.1943%	83.73

2.2.4 Anticipations

We now look at something we haven't looked at before: anticipations. For an anticipation to happen, we need to be in the position where getting one more scatter would take us into the free spins. This means that if we got no scatter on the first, second, and third reels, we would not have an anticipation as it would be impossible for us to get into the free spins during the current spin, regardless of what happens on the final two reels.

When looking at anticipations, there are normally two things which give us a good indication of how the game is going to play. The first is the probability of getting an anticipation. The second is the probability of getting into the free spins given that we have had an anticipation. First we will look at the probability of getting an anticipation. I am ignoring the anticipations which happen after we have collected three or four scatters as, to get to this position, we would already have had an anticipation. This means that these situations don't come into the calculations for getting any anticipation. There are only six different states we need to consider to calculate the probability of getting any anticipation. The first I am going to consider is the probability of getting a scatter on the first and second reels. This is very easy to calculate. I shall denote this state by $\{S, S, A, A, A\}$ where A stands for anything.

$$P(\{S, S, A, A, A\}) = \frac{5 \times 3}{225} \times \frac{10 \times 3}{246} = 0.008130 = 0.8130\%$$

If I now look at all the different possible states, I get the following table.

STATE	PROBABILITY	FREQUENCY
{S, S, A, A, A}	0.8130%	123
{S, #S, S, A, A}	0.6941%	144.07
{S, #S, #S, S, A}	1.1594%	86.25
{#S, S, S, A, A}	1.3497%	74.09
{#S, S, #S, S, A}	2.2545%	44.36
{#S, #S, S, S, A}	2.1837%	45.79
SUM	8.4544%	11.83

And so the probability of getting an anticipation during a spin is 8.4544%.

Now we want to know how likely it is that we get into the free spins after an anticipation. This is an example of conditional probability. We could look at each state we just looked at and calculate the probability of each taking us into the free spins, but maths has a shortcut we can use. What we want is the probability of going into the free spins given that we have had an anticipation. If we denote the free spins by FS and an anticipation by A, then this can be written mathematically as $P(FS|A)$. The vertical bar in the brackets means "given". So we read it as the probability of free spins given an anticipation. There is a very helpful theorem we can use called Bayes' Theorem. It states that if we have two events A and B then:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}$$

Here the $\cap$ means intersection (which means that they both happen), and so $P(A \cap B)$ means the probability that we have both A and B taking place. Using this theorem, we get the formula:

$$P(FS|A) = \frac{P(A|FS)P(FS)}{P(A)} = \frac{P(FS)}{P(A)} = \frac{1.1943\%}{8.4544\%} = 14.1264\%$$

In the above formula, we used the fact that $P(A|FS) = 1$ as if we have had a free spins session, we must have had an anticipation. And so the frequency of the free spins given an anticipation is one in 7.08 (1/14.1264% = 7.08). This is a lot quicker to calculate than it would be to calculate the probability of the free spins being entered given each of the states we looked at earlier.

2.3 Balancing

Once we have an initial idea of how the game works and we have put some numbers into our excel sheet to see what the RTP distribution looks like, we can then make refinements. At the moment in our example, the RTP for the base game is $4.154181\% + 6.897278\% + 22.947502\% = 33.999361\%$. This is rather low and so should probably be boosted, unless there is some base game feature which will have a good share of the RTP. If we left it as it is, the free spins would take up a lot of the RTP and the player would probably feel as if they are losing money too quickly unless the free spins turned up a lot.

If it is decided that the RTP needs to be boosted, you can increase the pay table but if you have already balanced a feature to a point you like and the feature depends on the pay table, you will have to go back to it and redo it after you have altered the pay table. Another way to change the base game RTP is to change the number of each symbol on the reels. Adding more high symbols may increase the RTP but they may also lower the hit rate (which is determined using a simulation). One way to increase the RTP and the hit rate at the same time is to add more wilds to the reels. But this strategy is a rather brutish one as a small change in the number of wilds affects all the other symbols as well. One must also determine where to place wilds. Adding them to the first three reels will increase the hit rate as any win must have the correct symbols on the first three reels but the last two don't matter. Adding them to the fourth doesn't make any more wins appear, it just makes them pay more. The same thing happens when you add them to the fifth reel. If you add a lot of wilds to either the fourth or fifth reel, it starts taking wins away from the three of a kind or four of a kind wins respectively. This is because whatever win you have already, adding a wild onto the end of it will increase it. As a result, this will tend to increase the volatility of the game whereas adding wilds to the first three reels will reduce the volatility of the game.

Moving wilds around can alter the RTP a lot. Having wilds on the first two or three reels means that you have to pay out for wild wins, which are normally more than the highest paying normal symbol. But if you skip having wilds on the first or second (or third if you aren't paying out any 2 OAK wins) reels, then you don't have to pay these out.

If after all this, the RTP still isn't where you want it to be you can take even more drastic action and remove one of the symbols, for example the lowest symbol you have. But this isn't normally an option and is also only really done at the beginning of a game's development, as afterwards the game has to be completely re-balanced and is rather disruptive if done at a later stage.

Moving on to the free games, we have three different things we might want to alter before we enter the free games. The first is the probability of entering the free games, the second is getting an anticipation, and the third is entering the free games after getting an anticipation. Now although we have three different variables here, we can only alter two independently. Once two of them are set, the third will be determined already and there will be nothing we can do to change it. Normally, I will try to get the probability of getting into the free games nailed down and then try to balance the other two simultaneously.

Balancing these values is rather straight forward. All we have to change is the number of scatters in each reel. If we want more free spins to be given, we increase the number of scatters in the reels, and we reduce the number of scatters if we want fewer free games to be given. I normally consider the probability of getting into the free spins given an anticipation next. If we think we don't enter the free spins enough after getting an anticipation, we can move some of the scatters from the earlier reels to the later ones. This means we will get an anticipation less often but when we do, we are more likely to enter the free games. If we think we are getting into the free games too frequently after getting an anticipation, we can move scatters from the latter reels to the front. When I say move scatters, this is slightly misleading. We can't normally just "move" them across, as the probability of entering the free games will most likely be altered if we do this. And so we will most likely need to add in more scatters than we take away, or add in fewer scatters than we took away.

There is another thing that needs to be considered. Adding or taking scatters from reels will alter the RTP of the base game. This is because scatters normally act as blockers in the base game. And so adding in more scatters makes it harder for a win to take place, and "moving" scatters makes also changes how likely we are to get wins of different lengths.

2.4 Making the Reels

The structure of the reels is one of the factors which decides the hit rate and the volatility of a line game. For example, imagine you make a reel set with no stacks in and no symbol appearing close to another instance of the same symbol. This will mean that if you get a win for one symbol, there won't be any other wins of the same type appearing at the same time (not counting wilds). But if you only have stacks on the reels, then if you get a win for one type of symbol, you are rather likely to have wins for the same type of symbol appearing on other win lines. And so having stacks means that when we have a win, we are more likely to get other wins as well. This means that the average win the player receives when they get a win will be larger. Because the RTP is not altered when we change how the reels are structured, but the average win we receive when we get a win increases as we introduce more stacks, the hit rate must go down. And so the more stacks we have, the lower the hit rate is. When we introduce stacks into a game, it makes the average non-zero win higher, but it also makes the hit rate lower. These two facts combine to make the game more volatile.

The same is true for a ways game as well: the more stacks we put on the reels, the lower the hit rate and the higher the volatility. The reasoning behind this is pretty much the same as it is for win line games.

Finally, we need to consider what to do with scatters. When making the reels, it is normally necessary to ensure that scatters are spread out enough to only allow one scatter to turn up on one reel at any one time. If this doesn't happen, then the calculations we went through earlier have to be altered to allow multiple scatters to appear on a single reel at once. It will also become a lot easier to get the free spins as if you land two scatters on a single reel, you will only need one more reel to land a scatter and not the normal two.

3 Ways Games

Ways games can be rather confusing. You can think of them having all possible win lines but with the rule that if we have a win then we can't have another win made up entirely of symbols which make up the first win. To illustrate this point, consider the following slot window:

LOW 1	LOW 1	LOW 1	MID 1	HIGH 1
MID 1	MID 1	HIGH 3	LOW 1	LOW 1
MID 1	MID 1	MID 1	HIGH 3	LOW 2

Here we have a four of a kind LOW 1 win. And so we don't also have a three of a kind LOW 1 win using the win line which goes across the top of the window.

There is an easy way to figure out how much to pay for a specific symbol. First we look at how long the longest win is. Then we count how many of that symbol are on each of these reels and multiply them together. You pay out this number times the pay for the longest win. So for the above example, MID 1 has a length of four, there are two MID 1s on the first and second reels, and one MID 1s on the third and fourth reels. Therefore we pay $2 \times 2 \times 1 \times 1 = 4$ times the pay for 4 OAK MID 1s.

3.1 Names of Ways Games

Ways games normally have something telling the player how many ways the game has. This is number represents how many different win lines it is possible to draw on the game. So for example, for a game with a window height of three and with five reels, there are three possibilities for a win line on each reel and so there are $3 \times 3 \times 3 \times 3 \times 3 = 243$ ways. For a game with a window height of four with five reels there are $4 \times 4 \times 4 \times 4 \times 4 = 1,024$ ways. And for a game with five reels, with reel heights 2, 3, 4, 5, 6, the total number of ways is $2 \times 3 \times 4 \times 5 \times 6 = 720$. Sometimes this number is called something else, for example big time gaming calls them megaways.

3.2 Differences between Ways and Lines Games

Ways games typically have a much lower pay table as there is a lot more potential in a ways game. For example, on a ten line game with a height of three and five reels, the maximum win would be ten times the pay for five of a kind of the highest paying symbol (assuming it is possible for this to occur). Whereas for a ways game, it would be 243 times the pay for five of a kind of the highest paying symbol.

On a lines game, it is a rather normal thing to have a symbol appearing on the first three reels and not paying out, or getting a three of a kind win even though the symbol appears on all five reels. This never happens on a ways game as the order that the symbols which are displayed doesn't matter as long as they remain on the same reels. So for example, the following two windows would pay out exactly the same win for a ways game but a lines game would most likely have different wins.

HIGH 1	MID 2	MID 2	HIGH 1	LOW 3
MID 2	HIGH 1	MID 2	LOW 1	HIGH 1
MID 2	MID 2	HIGH 1	HIGH 1	LOW 4

MID 2	MID 2	MID 2	LOW 1	LOW 3
MID 2	MID 2	MID 2	HIGH 1	LOW 4
HIGH 1	HIGH 1	HIGH 1	HIGH 1	HIGH 1

On both of the above windows, you have eight ways for 3 OAK MID 2, and you also have two ways for 5 OAK HIGH 1. This means that there is a lot more potential in a ways game and so the pay table needs to be lower.

3.3 The Maths of Ways Games

The RTP calculations for ways games are harder to do mathematically than line games. This is because in a line game, the win lines are independent of one another. In a ways game they are not. This means that when calculating the RTP for a ways game, you must do so for the entire window all at once and not just one win line at a time. This is because you might not pay a win out for one win line as the win you are considering is the same as, or is a subset of, a win on another win line.

4 Volatility Calculations

4.1 Introduction and Definition

When we are making a game, we are normally aiming for the RTP to be between $96\% - 98\%$. This is the average win. But what about how spread out the wins are? We could have a game which always returns your bet amount, or we could have a game which returns $1,000,000$ times your bet amount one in million times you play and nothing the remaining $999,999$ times out of a million. Both of these games would have an RTP of 100% but would be very different games. The variance of a game is what is used to look at how spread out the wins are.

The formula for variance is written down below, it may look nasty but it isn't. What we are doing when we are calculating the variance is adding together all of the squares of how far from the expected value each result is, and then dividing this by the number of results we looked at. So if we have a game with an expected return of one, and we play the game ten times getting the results 0, 0, 2, 3, 0, 1, 0, 0, 0, 4, then the variance would be:

$$\frac{(0-1)^2+(0-1)^2+(2-1)^2+(3-1)^2+(0-1)^2+(1-1)^2+(0-1)^2+(0-1)^2+(0-1)^2+(4-1)^2}{10}$$

$$= \frac{1+1+1+4+1+0+1+1+1+9}{10} = \frac{20}{10} = 2$$

Another way which is used of showing this information is the standard deviation which is just the square root of the variance, and so the standard deviation of this game is $\sqrt{2}$.

The equation for variance is:

$$\begin{aligned} Var(X) = \sigma^2 &= \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 \\ &= \frac{1}{n}\sum_{i=1}^{n}(x_i^2 - 2\bar{x}x_i + \bar{x}^2) \\ &= \left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right) - 2\bar{x} + \bar{x} \\ &= \left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right) - \bar{x} \\ &= E\left(X^2\right) - E(X)^2 \end{aligned}$$

Where the operator E is just the expected value.

4.2 An Example: Roulette

Let us consider a European roulette wheel. There are 37 different places (or pockets) the ball can land on. 18 are red, 18 are black, and one is green (the number 0). If we bet on red (or black) then if the ball lands on our chosen colour, we get twice our bet back. The probability of landing on a winning pocket is 18 in 37. If the ball doesn't land on our chosen colour, we lose our bet (we aren't using the rule which gives you half of your bet back when the ball lands on zero). Another betting strategy is to bet on a single number. The odds of us winning when we do this are 1 in 37. If we win, we get paid 36 times our bet amount. For this example, these will be the only bets we will consider. The RTP for betting on red (or black) is the same as the RTP for betting on one number, 97.30%. But these two betting strategies have different levels of volatility.

First, we will calculate the standard deviation of the first betting strategy. Let us assume that our bet size is 1. There are 37 different outcomes, 18 of these pay out 2, and 19 pay out 0. The expected value is $36/37 = 0.9730$. Therefore the standard deviation is:

$$\sqrt{\frac{18 \times (2 - \frac{36}{37})^2 + 19 \times (0 - \frac{36}{37})^2}{37}} = \sqrt{0.99927} = 0.999635$$

Now for the second betting strategy, again with a bet size of 1. There is only one pocket which pays, at 36. All the other pockets give you nothing. Again, the expected value is 36/37. The standard deviation is:

$$\sqrt{\frac{1 \times (36 - \frac{36}{37})^2 + 36 \times (0 - \frac{36}{37})^2}{37}} = \sqrt{34.08035} = 5.837838$$

From these values, we can see that the second betting strategy is much more volatile. This is because the possible results in this strategy are much further away from the expected value in the second strategy than in the first.

4.3 Back to Slot Machines

The previous example is a good illustration of what we are trying to measure when we are calculating the standard deviation of a slot machine. If all of the wins are close to the expected value, then the game won't be very volatile and so the standard deviation will be quite small. However, if the wins are more spread out, then the game will be more volatile and so the standard deviation will be larger. When balancing a game, we need to strike a balance. If the game isn't volatile enough, the player will become bored quickly. If it is too volatile, the player may feel that they don't have any chance of winning.