Soft Gluon Resummation in QCD

Applications using Wilson Lines

by

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THESIS

for the degree of

MASTER OF SCIENCE

Faculty of Mathematics and Natural Sciences
University of Oslo

Autumn 2021

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http://www.duo.uio.no/

Printed: Reprosentralen, University of Oslo

Abstract

Near the kinematical production threshold scattering cross sections gain large logarithmic corrections. This is due to the fact that close to the threshold the radiation of gauge bosons is restricted to be soft. This leads to an imbalance in the cancelling between real and virtual contributions at higher order, leading to large logarithmic contributions. In order to make reliable predictions such contributions must be resummed. In this thesis we make use of an object called a Wilson line to describe the soft radiation. Wilson lines are path-ordered exponentials of the gauge fields. They contain all the kinematical and dynamical information from the gauge sector and are central in taking a geometrical viewpoint of quantum field theory, and in particular quantum chromodynamics. We mainly consider semi-infinite Wilson lines on linear paths, which naturally describes radiation from highly energetic particles. By constructing a special class of Wilson lines, namely Wilson lines on closed paths called Wilson loops, we show how the soft radiation can be fully characterized by a so-called eikonal cross section. From an explicit one-loop calculation of a Wilson loop expectation value we find the universal cusp anomalous dimension. This cusp anomalous dimension is shown to be a central ingredient in evolution equations for Wilson lines, perturbative distributions and subsequently for eikonal cross sections. With the aid of the cusp anomalous dimension we find an exponentiated form of the Drell-Yan cross section up to leading logarithmic order.

Acknowledgments

First of all I would like to thank my supervisor, Are Raklev, for giving me the trust and encouragement to explore what I found most fascinating on the subject of this thesis. Your enthusiasm for physics and dedication to detail has been of enormous value. I am grateful that you always found time to answer my questions and put me back on track. I also wish to thank Anders Kvellestad, Jeriek Vda and Eli Rye for all the help and encouragement.

Thanks to the whole theory group for making these last years so enjoyable, and a special thanks to Cecilie Glittum, a good friend and vital physics-partner the last five years. I also would like to thank Kristian Wold, Lars Dean and Jonathan Waters for all the fruitful knowledge exchanges and especially for the warm friendships. Many thanks to my family, for always supporting and believing in me.

Finally, huge thanks to my girlfriend Elin. Words cannot describe how grateful I am for all her love, support and patience. This thesis would not have been possible without her.

Contents

Introduction

The best current understanding of the fundamental interaction between elementary particles are combined in the extraordinary successful theory called the Standard Model (SM). This model describes the electromagnetic, weak and strong interactions between particles. After the discovery of the Higgs boson in 2012, the last piece in this theory was in place. The only force that is not incorporated yet is gravitation, and there are several reasons to believe that there must be new physics beyond the Standard Model. In order to test the SM and search for new physics the Large Hadron Collider (LHC) was built. Because of the high energies the LHC is able to reach, one are able to study the SM at even higher precision. Unfortunately, so far there has been no signs of new physics. However, to fully appreciate high precision measurements, we need high precision predictions. To calculate observables in quantum field theory, we make a perturbative expansion in the coupling of the theory, and these mathematical expressions are often given graphical representation in the form of Feynman diagrams. Calculating these diagrams at leading order is usually trivial, but higher-order calculations are notoriously difficult, as they are ridden with divergences. The program for dealing with these divergences is what we call regularization and renormalization, which is well known. But even after renormalization has been performed there are sources for large corrections. Near the kinematical production threshold scattering cross sections gain large logarithmic corrections. This is due to the fact that close to the threshold the radiation of gauge bosons is restricted to be soft. This leads to an imbalance in the cancelling between real and virtual contributions at higher order, and in order to make reliable predictions such contributions must be resummed.

In this thesis we will focus on the theory of strong interactions, which is formulated in a field theory called Quantum Chromodynamics (QCD). This is a non-abelian gauge theory based on the symmetry group $SU(3)_c$. The formulation of this theory is fairly straightforward, but the physical implications are very complex. Perhaps the clearest confirmation of the complexity is that the fundamental constituents in QCD, the quarks and gluons, cannot be detected as free particles. They exist in colour-neutral states we call hadrons. This is known as confinement, and is still not theoretically understood. Another complication is that the force carriers, i.e. the gluons, are self-interacting and as a result QCD calculations tend to be very complicated.

Another important aspect of QCD is *asymptotic freedom*. It states that at very large energies the coupling between coloured particles becomes small, entering the realm of perturbation theory. However, for low energy interactions the coupling blows up and perturbation theory does not apply. This is a problem, because in real life experiments there will always be low energy radiation coming from high energy particles. In order to give reliable predictions, this part of the process must be taken account of and here enters the use of *Wilson lines*. With Wilson lines the perturbation series can be re-exponentiated, such that the large coupling does not invalidate the perturbation series anymore.

Wilson lines are fascinating objects, first introduced by Kenneth Wilson in attempting to explain confinement in QCD by considering a Wilson loop expectation value on the lattice [1]. They are path-ordered exponentials of the gauge fields and their definition follows directly from a parallel transport equation in gauge theory. They can be used to render bi-local operators gauge invariant and also to construct all terms that appear in a renormalizable and gauge invariant Lagrangian. They contain all the kinematical and dynamical information from the gauge sector and are central in taking a geometrical viewpoint on quantum field theory and in particular QCD.

However, this is not the only usage of Wilson lines. As they are path-ordered exponentials and the coupling appears in the exponent, they can be expanded in the coupling and used in perturbation theory. Such an expansion will naturally describe gauge boson radiation. Here, we will mainly consider semi-infinite Wilson lines on linear paths, which can be used to describe radiation from highly energetic particles. Also, by constructing a special class of Wilson lines, namely Wilson lines on closed paths called Wilson loops, one can fully characterize the soft radiation. Further, by using factorization theorems in QCD one can define perturbative parton distributions

$f_{i/i}(x)$. These distributions can in the $x \to 1$ limit be constructed in terms of a Wilson loop expectation value. Hence, by studying Wilson lines and Wilson loops in detail will give all we need in order to find an exponentiated cross section.

Outline of the Thesis

This thesis is divided into four chapters. Chapter 1 is meant to be a basic introduction to quantum field theory, where the chapter is loosely divided into three parts. The first part is focused on the construction of Green's functions and the quantization procedure. In the second part the link between Green's functions and scattering amplitudes are made, and the last part focuses on explaining renormalization of quantum field theories.

In Chapter 2 we take a closer look at the geometry of gauge theories. This chapter has two main parts, where the first is meant to give a review of the mathematical concepts in geometrizing gauge theories. In the second part we make use of several concepts from the geometrical formalism and construct the Yang-Mills Lagrangian from purely geometrical arguments. This is a natural introduction to Wilson lines and Wilson loops, as they are geometrical objects that governs the dynamics of gauge fields. After Wilson lines have been introduced, we go on and derive some of their properties that we will have use for in later chapters.

In Chapter 3 we go into more detail about the properties of the strong force and its field theoretical description in QCD. This chapter can also be divided into two parts. In the first part important concepts such as factorization, parton distribution functions and running coupling are introduced and discussed. In introducing these concepts we use the most common experimental setup, namely deep inelastic scattering. From there we go on and define gauge invariant parton distribution functions by using Wilson lines, before we use Wilson lines to introduce perturbative parton-in-parton distributions. In the last part we take a closer look at another important process, namely the Drell-Yan process, which we follow in the remainder of the thesis. Here we make an explicit next-to-leading order (NLO) calculation by using dimensional regularization. Even after the process has been dealt with by dimensional regularization we have a collinear divergence. By using factorization theorems in QCD we show how to fully renormalize the cross section. We also show that even after renormalization and regularization the cross section has large logarithmic contributions in certain parts of phase space. Because of these large contributions the need for resummation is made clear.

Lastly, in Chapter 4 we make use of everything that we have introduced and discussed in the previous chapters. The chapter is meant to investigate the procedure of threshold resummation by the use of factorization theorems, Wilson lines and in particular calculations of Wilson loop expectation values. We begin with looking at how scattering amplitudes exponentiate, and how they naturally factorizes into hard and soft regimes. Then we go on and factorize the hadronic Drell-Yan cross section by using Mellin space techniques. After the cross section is fully factorized we construct Wilson loop expectation values and calculate their perturbative behaviour, which we show results in a exponentiated and resummed cross section.

Chapter 1

Introduction to Quantum Field Theory

In this chapter we aim to review some basic knowledge in quantum field theory. The idea is that before we tackle more advanced calculations, this chapter will serve as a smooth transition to later chapters. We assume that the reader is already acquainted with classical and quantum field theory to a basic level. Therefore, we will try to avoid too much detail and instead sketch the main ideas.

We begin with a very brief discussion on why we need quantum field theory and the basic playing ground. From there we look at the canonical quantization procedure. This part will be very brief as we aim to use the path integral formulation of quantum field theory. We derive the path integral by looking at path integrals in quantum mechanics, where the transition to quantum field theory is made by replacing quantum mechanical operators with fields. After we have introduced path integrals, the path integral quantization procedure for free theories is explored. However, the main focus in this chapter is to explore Green's functions and how to relate them to scattering amplitudes and Feynman diagrams. Hence, we will explicitly show how one goes about formulating interacting theories using path integrals and perturbative expansions. For simplicity, we will mainly focus on scalar ϕ^4-theory as it serves as an easy example to introduce the main concepts.

Lastly, we consider the consequences of quantum fluctuations and introduce the concepts of regularization and renormalization. The Callan-Symanzik equation will be derived and the notion of running coupling is discussed. We will try not to focus on all the details of calculations and rather state common results and discuss some important implications. For a more complete treatment on quantum field theory and its construction, see e.g. [2–4].

1.1 Basic Construction and Lagrangians

The first thing we want to do when constructing a theory is to define what space it lives in. In general, the space is a smooth Riemannian (or pseudo-Riemannian) manifold M with general metric g. In particle physics, we have that M is the four-dimensional Minkowski spacetime and g is the Minkowski metric. There are a wide variety of interesting choices to study, but a general feature is that we always keep the metric fixed, as we would need quantum gravity to describe quantum fluctuations of the metric.

The next ingredient is to define what kind of fundamental objects that lives in this space. These objects are what we call *fields*, and the simplest choice is a scalar field. A scalar field ϕ is just a function on M, which we can see as a map

$$\phi : M \to \mathbb{R} \,(\text{or}\, \mathbb{C})\,. \tag{1.1}$$

The main goal of quantum field theory is to construct a framework where special relativity (SR) and quantum mechanics (QM) are compatible. To that end, we have to look at a theory built on a Hilbert space $\mathcal{H}$ with states represented by unit rays and observables by self-adjoint operators, respecting the symmetry group of special relativity, the *Poincare* group. This is the group of Minkowski spacetime isometries, consisting of a semidirect product of the group of translations and the Lorentz group

$$\mathcal{P} = \mathbb{R}^4 \rtimes SL(2, \mathbb{C})\,, \tag{1.2}$$

where $SL(2,\mathbb{C})$ is the universal cover of $SO^+(1,3)$, the group of proper orthocronous Lorentz transformations. It appears in quantum field theory since we need the representations to describe particles with spin. From *Wigner's symmetry representation theorem* we view particles as irreducible representations of $\mathcal{P}$ on $\mathcal{H}$, and from *Schur's lemma* we have that any irreducible representation can be classified using *Casimir-invariants*. This leads to the remarkable result that particles can be classified only in terms of their mass and spin.

To keep the next discussion as brief and simple as possible, we mainly consider real scalar fields, i.e. fields with spin zero. Let $\mathcal{C}$ denote the the space of field configurations on M, meaning every point $\phi \in \mathcal{C}$ corresponds to a configuration of the field. Typically $\mathcal{C}$ is an infinite-dimensional function space, and trying to understand the geometry and topology of this infinite space of fields, and then trying to do something useful with it is what makes quantum field theory so difficult, but at the same time so interesting.

The first attempts of unifying SR with QM, physicists encountered problems like conservation of particles, negative energy states, violation of causality and Born's probability interpretation of QM. These problems could not be fixed in a consistent way using the known theories at the time. Thus, to reconcile these problems one needed a multi-particle theory, i.e a field theory. To model this we must replace $\mathcal{H}$ with the direct sum of n-particle Hilbert spaces, also called the *Fock space*

$$\mathcal{H} \equiv \bigoplus_{n=0}^{\infty} S(\mathcal{H}^{\otimes n}) \,, \tag{1.3}$$

symmetrized (S) as we are here describing bosonic fields. This construction is only formally valid for non-interacting fields, but let us not worry about such technicalities here[1].

The next fundamental piece we need to determine is the *action* for the theory. The action is a map from the configuration space to the real space

$$S[\phi] : \mathcal{C} \to \mathbb{R} \,, \tag{1.4}$$

meaning it is a real function on the space of fields, or in other words, given a field configuration, the action yields a real number. The action is what we call a *functional*, just to remind ourselves that the configuration space is infinite-dimensional. From Hamilton's principle, we define the critical set

$$\mathrm{Crit}_{\mathcal{C}}(S) = \{\phi \in \mathcal{C} \,|\, \delta S[\phi] = 0\} \,, \tag{1.5}$$

which is a fancy way of saying that the fields in our theory obey the Euler-Lagrange equations.

With a Minkowskian metric, we define the action as

$$S[\phi] = \int d^4x \, \mathcal{L}(\phi, \partial_\mu \phi) \,, \tag{1.6}$$

where $\mathcal{L}$ is our Lagrangian[2] of the theory. So by applying Hamilton's principle $\delta S = 0$, one finds the Euler-Lagrange equation

$$\partial_\mu \Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Big) - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \,, \tag{1.7}$$

and if the Lagrangian contains more than one field, there is one such equation for each.

With this brief introduction, we can now introduce a few important Lagrangians that we will use in this thesis.

Lagrangians

Without reviewing classical field theory we will simply state the Lagrangians of interest and explore a couple of details that we will cover more on later.

First of all, in any Lagrangian, we can group the different terms into three categories. These are kinetic terms, mass terms and interaction terms. The first two terms are quadratic in the same field, while the latter is either a combination of the same field or a combination of different fields.

[1] For more details on the axiomatic formulation of quantum field theory and *Haag's theorem*, see e.g. [5, 6].
[2] More correctly, it is the Lagrangian density, but this is standard terminology.

The kinetic term contains derivatives of the same field and describes the dynamics of that field, while the mass term describes the statics of this field. Upon quantizing a theory these terms will form what is called a *propagator*, which we will cover the details on later. There are of course exceptions to the requirement that quadratic terms must involve the same field, but we will not cover this scenario here in this thesis.

Interaction terms are combinations of fields that are higher than quadratic in the fields. These terms are either a combination of the same field or a combination of different fields. Interactions are characterized by a constant of proportionality, which we call a *coupling constant*. Eventually we want to have gauge invariant and renormalizable theories, which constrains the possible interaction terms. The restriction on renormalizable terms follow as the action is a dimensionless quantity[3]. Hence, from Eq. (1.6) the Lagrangian must have mass dimension four, and it can be shown that the coupling constant in renormalizable theories must be dimensionless. This has the implication that only field combinations with mass dimension four are 'allowed'.

Real Scalar Field

Let us begin with the simplest case of a Lagrangian describing a real scalar field[4]. A theory without interaction is what we call a *free* theory, and for a real scalar field the Lagrangian can be written as

$$\mathcal{L}_S = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2\,, \tag{1.8}$$

where the minus sign in front of the 'mass' term is crucial to keep the corresponding Hamiltonian bounded from below. We have to be careful with interpreting this as the physical mass of a field excitation. This is a crucial observation when we eventually encounter divergences in perturbation theory, but more on that later. The common factor of $1/2$ is just a convention that is chosen such that the propagator of the theory does not contain any factors in front.

By applying the Euler-Lagrange equation on this Lagrangian, we find the *Klein-Gordon* equation

$$(\partial^2 + m^2)\phi(x) = 0\,, \tag{1.9}$$

where $\partial^2 = \partial_\mu\partial^\mu$. We will study free theories and quantize them in the next sections, but they are not very interesting. That is not to say they are not useful to study, but ultimately they just describe a theory without interactions.

Let us look at a possible interaction term that we will explore in some detail in this chapter. From dimensional analysis we observe that $[\phi] = m$, implying that for a renormalizable scalar theory, we can only have interaction terms up to ϕ^4. So if we add such a term to the free Lagrangian, we have the interacting theory called scalar ϕ^4-theory

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\,, \tag{1.10}$$

where λ is dimensionless and in Sec. 1.5 we will show how to formalise this as a renormalized theory.

By applying the Euler-Lagrange equation we find a inhomogenous differential equation

$$(\partial^2 + m^2)\phi(x) = -\frac{1}{3!}\lambda\phi^3\,. \tag{1.11}$$

The implication is that while the free theories can be expanded using Fourier theory, interacting theories does not have this property. Hence, interactions must be evaluated by interpreting them as small deviations from the free theory, which is done by a power expansion in a small coupling. We will discuss how this works in Sec. 1.4.1.

Dirac Fields

The next Lagrangian we have use for is the Dirac Lagrangian. This Lagrangian describes fermionic fields, also called *matter fields*. Because of the Pauli exclusion principle, these fields have the property of being anti-commutative, i.e. $\psi(x)\psi(y) = -\psi(y)\psi(x)$. Hence, the square of a fermionic field is zero. From classical quantum mechanics we know that fermionic fields obey the Dirac equation

$$(i\partial\!\!\!/ - m)\psi = 0\,, \tag{1.12}$$

[3] Lorentz invariance is always required.

[4] We also have complex scalar fields, but we keep this discussion as simple and brief as possible.

where ψ is a complex valued spinor field. The contraction is defined as $\not{\partial} = \partial_\mu \gamma^\mu$, where γ^μ are the Dirac gamma matrices, see Appendix B.1. From Lorentz symmetry one can deduce that the correct combination of spinor fields that reproduces the Dirac equation is $\bar{\psi}\psi$, where $\bar{\psi} \equiv \psi^\dagger \gamma^0$. Hence, the free Dirac Lagrangian is given by

$$\mathcal{L}_D = \bar{\psi}(i\not{\partial} - m)\psi \,, \tag{1.13}$$

where it follows from dimensional analysis that $[\psi] = [\bar{\psi}] = m^{3/2}$. An example of a interaction term is to add a scalar field that interacts with the fermions, known as Yukawa theory

$$\mathcal{L}_Y = \mathcal{L}_D + \mathcal{L}_S - g'\bar{\psi}\psi\phi \,, \tag{1.14}$$

where g' is the coupling between fermions and scalars. The most interesting interaction term for fermions are those to vector (gauge) bosons, but we will see an example of such a interaction in the next section.

Vector Fields

The last example that is relevant for us, is the Lagrangian for vector fields. Vector fields are also bosonic by nature, and the most general free Lagrangian can be written as

$$\mathcal{L}_V = \frac{1}{2}\partial_\mu A_\nu - \frac{1}{2}\partial_\nu A_\mu - m^2 A_\mu A_\nu \,, \tag{1.15}$$

where the opposite sign in the kinetic term is to be in accordance with classical electrodynamics. Hence, one can instead write this in terms of the electromagnetic field tensor $F_{\mu\nu}$

$$\mathcal{L}_V = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - m^2 A_\mu A_\nu \,, \tag{1.16}$$

where the minus sign in front of the kinetic term is a convention, and the field tensor is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,. \tag{1.17}$$

There are several possible interactions when introducing vector fields, but we are only interested in interactions with matter fields[5]. However, let us discuss the mass term for vector fields. In a realistic quantum field theory, the Lagrangian must be invariant under what we call a local phase rotation (or gauge transformation). The mass term is not invariant under such a transformation, and this term must be dropped.

Let us take the example of a photon coupled to fermions. Instead of writing this down in a heuristic fashion as we did above for Yukawa theory, we can find this term in a more natural way. The Dirac Lagrangian in Eq. (1.13) is not invariant under the transformation

$$\psi(x) \to e^{i\alpha(x)}\psi \,. \tag{1.18}$$

This is a local phase rotation through an 'angle' $\alpha(x)$ that varies arbitrarily from point to point. The difficulty arises from the partial derivative in the Dirac Lagrangian, which brings down $\alpha(x)$ and spoils the invariance. However, by replacing $\partial_\mu \to D_\mu$, we can form a gauge invariant Lagrangian

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi \,, \tag{1.19}$$

where we define the gauge-covariant derivative

$$D_\mu \equiv \partial_\mu + ieA_\mu \,. \tag{1.20}$$

With this construction one can add the kinetic term for the photons, giving the well known Lagrangian for Quantum Electrodynamics

$$\mathcal{L}_{QED} = \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - e\bar{\psi}\not{A}\psi \,, \tag{1.21}$$

where e is the electromagnetic coupling.

This example is slightly unmotivated and brief, but this was just meant to introduce the concept of gauge invariance. We will spend a major part of Chapter 2 discussing gauge theories and the physical implications that follows. There we will go into more detail why we have to make the replacement $\partial_\mu \to D_\mu$ from geometric arguments and not just demand it in a ad-hoc fashion.

Having introduced the different Lagrangians we will encounter in this thesis, we will now move on and discuss quantization of free theories.

[5] Self-interacting terms are also relevant, but more on such theories in Chapter 2.

1.2 Canonical Quantization

In this section we will briefly discuss the main ideas behind the canonical quantization procedure. Instead of going through all derivations, we will discuss the most important steps and ideas[6].

Let us continue the discussion of a scalar field theory, and in particular a real free scalar field theory. The Lagrangian for a free theory was given in Eq. (1.8), and by applying the Euler-Lagrange equation we obtained the Klein-Gordon equation

$$(\partial^2 + m^2)\phi(x) = 0\,. \tag{1.22}$$

As this is a partial differential equation involving continuous functions, the solution must be interpreted in the sense of distributions[7]. For a real scalar field a general Fourier expansion can be written as

$$\phi(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}\left(a_\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}} + a^*_\mathbf{k}\, e^{-i\mathbf{k}\cdot\mathbf{x}}\right), \tag{1.23}$$

where $a_\mathbf{k}$ and $a^*_\mathbf{k}$ are Fourier coefficients. To quantize this theory we simply promote the Fourier coefficients to operators $a_\mathbf{k}$ and $a^\dagger_\mathbf{k}$ in Hilbert space. Then $\phi(\mathbf{x})$ becomes a time-independent operator valued distribution, i.e. we are working in the Schrödinger picture. This quantization procedure introduces the equal-time bosonic commutation relations for the operators $a_\mathbf{k}, a^\dagger_\mathbf{k} \in \mathcal{H}$

$$[a_\mathbf{k}, a^\dagger_{\mathbf{k}'}] = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\,, \tag{1.24}$$

$$[a_\mathbf{k}, a_{\mathbf{k}'}] = [a^\dagger_\mathbf{k}, a^\dagger_{\mathbf{k}'}] = 0\,. \tag{1.25}$$

We can now replace the Fourier coefficients with operators and write down the operator valued fields

$$\phi(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}\left(a_\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}} + a^\dagger_\mathbf{k}\, e^{-i\mathbf{k}\cdot\mathbf{x}}\right) \tag{1.26}$$

$$\pi(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3}(-i)\sqrt{\frac{E_\mathbf{k}}{2}}\left(a_\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}} - a^\dagger_\mathbf{k}\, e^{-i\mathbf{k}\cdot\mathbf{x}}\right), \tag{1.27}$$

where $\pi(\mathbf{x})$ is known as the *conjugated field*.

By performing a Legendre transformation on the Lagrangian the Hamiltonian of the system is given by

$$\begin{aligned} H &= \int d^3x\left(\pi(\mathbf{x})\dot{\phi}(\mathbf{x}) - \mathcal{L}\right) \\ &= \int d^3x\left(\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right), \end{aligned} \tag{1.28}$$

giving the *normal ordered* Hamiltonian

$$H = \int \frac{d^3k}{(2\pi)^3} E_\mathbf{k}\, a^\dagger_\mathbf{k} a_\mathbf{k}\,, \tag{1.29}$$

which follows by using the commutator relations in Eq. (1.24)[8]. The normal ordered Hamiltonian in Eq. (1.29) is an infinite sum of harmonic oscillators with infinitely many degrees of freedom. Hence, we interpret $a_\mathbf{k}$ and $a^\dagger_\mathbf{k}$ as annihilation and creation operators, which introduces the existence of a vacuum state $|0\rangle$, satisfying the condition $a_\mathbf{k}|0\rangle = 0$.

However, we want our theory to be manifestly Lorentz invariant, so it is useful to transform to the Heisenberg picture

$$\phi(x) = e^{iHt}\phi(\mathbf{x})e^{-iHt} = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}\left(a_\mathbf{k}\, e^{-ik\cdot x} + a^\dagger_\mathbf{k}\, e^{ik\cdot x}\right), \tag{1.30}$$

[6] For a more elaborate treatment on the canonical quantization procedure, we refer the reader to e.g. [2, 7].
[7] The mathematics of distributions are necessary when constructing consistent quantum field theories at a formal level.
[8] For a detailed derivation of this expression, see [2].

where $k^0 = E_k$, and $k^\mu x_\mu = k \cdot x$.

Another quantity of interest is to look at the commutator of fields at different spacetime points $[\phi(x), \phi(y)]$. By evaluating this commutator in the following way

$$\langle 0|[\phi(x), \phi(y)]|0\rangle \,, \tag{1.31}$$

one finds after choosing a Feynman pole prescription that the propagator in a scalar field theory can be written as

$$D_F(x-y) \equiv \int \frac{d^4k}{(2\pi)^2} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik\cdot(x-y)} \,, \tag{1.32}$$

where the poles have been shifted as $k^0 = \pm(E_k - i\epsilon)$, such that they are displaced properly above and below the real axis. This is called the Feynman propagator and can also be written as the time-ordered product

$$D_F(x-y) \equiv \langle 0|\mathcal{T}(\phi(x)\phi(y))|0\rangle \,. \tag{1.33}$$

Another way of deriving the Feynman propagator is by using that it is the Green's function of the Klein-Gordon operator, i.e. it must satisfy the following Green's function equation

$$(\partial^2 + m^2) D_F(x-y) = -i\delta^4(x-y) \,, \tag{1.34}$$

where Eq. (1.32) follows by Fourier expanding D_F. In all essence the Feynman propagator can be interpreted as the probability amplitude for a 'particle' to travel between two spacetime points. For practical application one often just use the momentum space representation, which is just found by reading of the integrand in Eq. (1.32)

$$D_F(p) = \frac{i}{p^2 - m^2 + i\epsilon} \,. \tag{1.35}$$

From this discussion we have that the field $\phi(x)$ is the fundamental quantity that permeates all of space, while the particles we observe in nature are excitations of this field.

The same kind of analysis can be made for fermionic fields $\psi(x)$, but with the crucial difference that we have to impose anti-commutation relations. This is not surprising as fermions obey Fermi-Dirac statistics, while scalars obey Bose-Einstein statistics. The Dirac Lagrangian was given in Eq. (1.13), and by applying the Euler-Lagrange equation the Dirac equation follows

$$(i\partial\!\!\!/ - m)\psi(x) = 0 \,, \tag{1.36}$$

and if we multiply with $(-i\partial\!\!\!/ - m)$ from the left, it can be shown that the Dirac equation implies the Klein-Gordon equation. Hence, a free particle solution can be written as a linear combination of plane waves

$$\psi(x) = \sum_s u^s(p) e^{-ip\cdot x} + v^s(p) e^{ip\cdot x} \,, \qquad p^2 = m^2 \,, \tag{1.37}$$

where $u(p)$ and $v(p)$ are interesting objects known as a spinors, while the sum is over spin states. The Fourier expansion for $\psi(x)$ and $\bar{\psi}(x)$ can then be written as

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \left(a^s_{\mathbf{p}} u^s(p) e^{-ip\cdot x} + b^{s\,\dagger}_{\mathbf{p}} v^s(p) e^{ip\cdot x} \right) , \tag{1.38}$$

$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \left(a^{s\,\dagger}_{\mathbf{p}} \bar{u}^s(p) e^{ip\cdot x} + b^s_{\mathbf{p}} \bar{v}^s(p) e^{-ip\cdot x} \right) , \tag{1.39}$$

where the creation and annihilation operators are not hermitian conjugates of each other, as they were for real scalar fields[9]. We interpret this expression as $a^\dagger$ creates fermions, while $b^\dagger$ creates anti-fermions[10].

[9] This is because a fermionic field is not real valued and describes two different particles, i.e. a particle and its anti-particle.
[10] The operators a and b will therefore annihilate fermions and anti-fermions respectively.

The next object to define is the Feynman propagator for fermions, which is found by using that it is the Green's function of the Dirac operator

$$(i\partial\!\!\!/ - m)S_F(x-y) = i\delta^{(4)}(x-y)\,, \tag{1.40}$$

which by a Fourier transform give the momentum space solution

$$S_F(p) = \frac{i(p\!\!\!/ + m)}{p^2 - m^2 + i\epsilon}\,, \tag{1.41}$$

and is often written as the time-ordered product of spinor fields

$$S_F(x-y) \equiv \langle 0|\mathcal{T}(\psi(x)\bar{\psi}(x))|0\rangle\ . \tag{1.42}$$

When we later perform perturbative calculations with so-called Feynman diagrams, we will associate $S_F(p)$ with each internal fermion line.

To end the discussion we will just review some properties of Dirac fields and spinor theory that we will have use for when calculating scattering amplitudes. If a Dirac field acts on a particle momentum state $|p,s\rangle$, we get

$$\psi(x)|p,s\rangle_{\text{in}} = e^{-ip\cdot x}u^s(p)|0\rangle\,, \tag{1.43}$$

$$_{\text{out}}\langle p,s|\bar{\psi}(x) = \langle 0|\bar{u}^s(p)e^{ip\cdot x}\,, \tag{1.44}$$

and an antiparticle momentum state

$$\bar{\psi}(x)|p,s\rangle_{\text{in}} = e^{-ip\cdot x}\bar{v}^s(p)|0\rangle\,, \tag{1.45}$$

$$_{\text{out}}\langle p,s|\bar{\psi}(x) = \langle 0|v^s(p)e^{ip\cdot x}\,, \tag{1.46}$$

where all other combinations are zero. When we eventually write down amplitudes from Feynman diagrams, each external fermion leg is represented by a spinor. The exponential factors combine into a delta function ensuring momentum conservation in each vertex. These spinors satisfy the very useful spin sum rules

$$\sum_s u^s(p)\bar{u}^s(p) = p\!\!\!/ + m\,, \tag{1.47}$$

$$\sum_s \bar{v}^s(p)v^s(p) = p\!\!\!/ - m\,, \tag{1.48}$$

which is a combination that always appear when calculating scattering amplitudes and simplifies the calculation significantly.

For completeness the most natural thing to do would be to quantize a vector field in this formalism. However, there is another formalism that is much more elegant for treating these types of fields. This formalism is known as the Feynman *Path Integral Formalism*. While the canonical quantization procedure have many natural extensions from regular quantum mechanics, we will eventually generalize to interacting theories and for that treatment the path integral formalism is more suited. Hence, the next sections are devoted to exploring this formalism in more detail.

1.3 The Path Integral Formalism

In this section we will consider the path integral formulation of quantum field theory. When we eventually use Feynman diagrams to describe physical processes, we will have the mind-set that the system will take on every configuration possible as it evolves from the intial state to the final state. E.g. photons will split into electrons and positrons which recombines to different photons, leptons (or quarks) annihilate one another and the resulting energy is used to create particles of a different flavour. The point is; anything that can happen will happen. Each of these distinct histories can be thought of as a path through the space of all configurations that describe the state of the system. As previously mentioned the configuration space for a quantum field theory is a Fock space, see Eq. (1.3).

But most importantly, each path that the system takes comes with a probabilistic amplitude. The sum over amplitudes associated with each path connecting the initial and final states in the Fock space is interpreted

as the probability that some initial state will end up in some final state. So a perturbative expansion using Feynman diagrams is a description of all the possible ways the system can behave. However, the derivation of the path integral in field theory is quite extensive.

But ultimately quantum field theory is grounded in ordinary quantum mechanics; with the essential difference of the number of degrees of freedom [11]. Hence, we will instead derive the path integral in quantum mechanics and make the necessary replacements of fields instead of operators. For a more complete treatment and derivation of path integrals in quantum field theory, see e.g. [2, 3].

1.3.1 Path Integral in Quantum Mechanics

In order to derive the basic idea of the path integral, let us consider the simplest case of one-dimensional quantum mechanics. The time evolution for a state $|\psi(t)\rangle$ can be written as

$$|\psi(t)\rangle = e^{-i(t-t_0)H}|\psi(t_0)\rangle\,, \tag{1.49}$$

with the time-independent Hamiltonian

$$H = \frac{\hat{p}^2}{2m} + V(\hat{x})\,, \tag{1.50}$$

where $\hat{p}$ and $\hat{x}$ are momentum and position operators acting on states in Hilbert space in the following way

$$\hat{x}|x\rangle = x|x\rangle\,, \tag{1.51}$$
$$\hat{p}|p\rangle = p|p\rangle\,. \tag{1.52}$$

To derive the path integral we will look at an alternative way of describing the time-evolution, using propagators. From the path integral expression of these propagators, we can derive the most important quantity in the path integral formulation, namely Green's functions. To find an expression for the propagator we will consider Eq. (1.49) and multiply with a position eigenket from the left, giving

$$\psi(x,t) = \langle x|\psi(t)\rangle = \langle x|e^{-i(t-t_0)H}|\psi(t_0)\rangle\,, \tag{1.53}$$

and if we insert an identity operator, we can write this as

$$\begin{aligned}\psi(x,t) &= \int dx_0 \langle x|e^{-i(t-t_0)H}|x_0\rangle\,\langle x_0|\psi(t_0)\rangle \\ &\equiv \int dx_0\, U(x,t;x_0,t_0)\,\psi(x_0,t_0)\,,\end{aligned} \tag{1.54}$$

where we have defined

$$U(x,t;x_0,t_0) \equiv \langle x|e^{-i(t-t_0)H}|x_0\rangle = \langle x,t|x_0,t_0\rangle\ , \tag{1.55}$$

which is the definiton of the propagator in quantum mechanics. The interpretation of this object goes as follows: we have that $e^{-i(t-t_0)H}|x_0\rangle$ is the state at time t, supposed that it was in state $|x_0\rangle$ at time t_0. Hence, $\langle x|e^{-i(t-t_0)H}|x_0\rangle$ is the probability amplitude that the state has evolved to a state $|x\rangle$ at time t, given that it was in the state $|x_0\rangle$ at t_0. In other words, say we measure a particle's position at t_0 and find it to be x_0, then the propagator tells us the probability amplitude for finding the particle at x if we make a new measurent at time t.

Let us then see what happens if we split the time interval in two smaller parts, $(t_1 - t_0)$ and $(t - t_1)$ and insert the identity

$$1 = \int dx_1\,|x_1\rangle\langle x_1|\,, \tag{1.56}$$

[11] With the additional requirement that the system has to be invariant under Lorentz transformations, but this is manifest in the path integral formalism.

giving

$$\begin{aligned}\langle x,t|x_0,t_0\rangle &= \langle x|e^{-i(t-t_1)H}e^{-i(t_1-t_0)H}|x_0\rangle \\ &= \int dx_1\, \langle x|e^{-i(t-t_1)H}|x_1\rangle\langle x_1|e^{-i(t_1-t_0)H}|x_0\rangle \\ &= \int dx_1\, \langle x,t|x_1,t_1\rangle\, \langle x_1,t_1|x_0,t_0\rangle\ . \end{aligned} \tag{1.57}$$

This expression is identical to the original, only that we have separated it into smaller intermediate pieces. The logic goes as this: we are evolving the system from x_0 at time t_0 to x_1 at t_1, then from x_1 at t_1 to x at t, where all positions x_1 are integrated over. This must be the same as evolving from x_0 to x by passing any arbitrary point x_1.

We can then break the problem into n small time intervals δt and consider the amplitude for each of those infinitesimal steps. We label the intermediate times $t_j = j\delta t$ where $j = 0, 1, \ldots, n$. With this choice we have that $t_0 = 0$ and $t_n = t$. If we also insert the identity Eq. (1.56) for each intermediate step, we get the following expression

$$\begin{aligned}\langle x|e^{-it\hat{H}}|x_0\rangle &= \int \prod_{k=1}^{n-1} dx_k\, \langle x|e^{-i\delta t H}|x_{n-1}\rangle\langle x_{n-1}|\ldots|x_2\rangle\langle x_2|e^{-i\delta t H}|x_1\rangle\langle x_1|e^{-i\delta t H}|x_0\rangle \\ &= \int \prod_{k=1}^{n-1} dx_k\, \langle x_n,t_n|x_{n-1},t_{n-1}\rangle\, \langle x_{n-1},t_{n-1}|x_{n-2},t_{n-2}\rangle \ldots \langle x_1,t_1|x_0,t_0\rangle\ . \end{aligned} \tag{1.58}$$

where we used that $t_n = t$ and $x_n \equiv x$. Each of these individual propagators can be evaluated by inserting a complete set of momentum eigenstates

$$\begin{aligned}\langle x_{j+1}|e^{-iH\delta t}|x_j\rangle &= \int \frac{dp}{2\pi}\, \langle x_{j+1}|p\rangle\, \langle p|e^{-i\left(\frac{\hat{p}^2}{2m}+V(\hat{x}_j)\right)\delta t}|x_j\rangle \\ &= e^{-iV(x_j)\delta t}\int \frac{dp}{2\pi} e^{-i\frac{p^2}{2m}\delta t} e^{ip(x_{j+1}-x_j)}\,, \end{aligned} \tag{1.59}$$

where we used that $\langle p|x\rangle = e^{-ipx}$. We observe that this is now a Gaussian integral, which has the general solution

$$\int dp\, e^{-ap^2+bp} = \sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}}\,. \tag{1.60}$$

Using this solution we can write each individual propagator as

$$\begin{aligned}\langle x_{j+1}|e^{-iH\delta t}|x_j\rangle &= \sqrt{\frac{m}{2\pi i\delta t}}\, \exp\left(i\big[\frac{m}{2}\frac{(x_{j+1}-x_j)^2}{(\delta t)^2} - V(x_j)\big]\delta t\right) \\ &= \mathcal{N}\exp\left(iL(x_j,\dot{x}_j)\delta t\right), \end{aligned} \tag{1.61}$$

where we defined the integration constant $\mathcal{N}$, which is independent of x and t. We also used that the Lagrangian is given by

$$L(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 - V(x)\,, \tag{1.62}$$

and by inserting Eq. (1.61) for each step into Eq. (1.58), we find that

$$\langle x|e^{-it\hat{H}}|x_0\rangle = \mathcal{N}^n \int \prod_{k=1}^{n-1} dx_k\, \exp\left(i\sum_{j=0}^{n-1}\big(\frac{m}{2}\frac{(x_{j+1}-x_j)^2}{(\delta t)^2} - V(x_j)\big)\delta t\right). \tag{1.63}$$

If we take the limit $n\to\infty$, and $|t-t_0|$ fixed and finite, the interval $\delta t\to 0$ and the exponentials will turn into an integral over dt. This will lead to the following expression

$$\langle x|e^{-it\hat{H}}|x_0\rangle = \mathcal{N}\int \mathcal{D}x\, e^{i\int dt L(x,\dot{x})}\,, \tag{1.64}$$

which is the expression for the path integral in quantum mechanics. We can also write it on the following form

$$\langle x,t|x_0,t_0\rangle = \mathcal{N}\int \mathcal{D}x\, e^{iS[x]}\,. \tag{1.65}$$

The normalization constant $\mathcal{N}$ has been redefined, and is an infinite constant that will cancel in physical quantities. The measure is defined as

$$\int \mathcal{D}x = \int_{x_0(t_0)=x_0}^{x(t)=x} \mathcal{D}x(t') \equiv \lim_{n\to\infty}\int \prod_{k=1}^{n-1} dx_k\,, \tag{1.66}$$

where $\mathcal{D}x$ means the sum over all paths $x(t')$ with the given boundary conditions[12].

One of the most important features of path integrals is that it allows for an easy computation of matrix elements of operators. These matrix elements are what is called correlators or Green's functions as we call them, and are probably the most important object for any quantum field theory. For example, we saw in Eq. (1.33), that the two-point Green's function is the Feynman propagator, which describes the probability for a particle to travel from a spacetime point to another.

To see how we can construct these correlators in the path integral formalism, we insert a function $x(t_1)$ and $x(t_2)$ into the path integral. This amount to inserting an operator $\hat{x}(t_1)$ and $\hat{x}(t_2)$, between the initial and final states. We write this as

$$\langle x,t|\mathcal{T}(\hat{x}(t_1)\hat{x}(t_2))|x_0,t_0\rangle = \mathcal{N}\int \mathcal{D}x\, x(t_1)x(t_2)\, e^{iS[x]}\,. \tag{1.67}$$

In order to show that this the correct form, we start from the right hand side and insert functional delta functions

$$\int \mathcal{D}x\, x(t_1)x(t_2)\, e^{iS[x]} = \int \mathcal{D}x\mathcal{D}x_1\mathcal{D}x_2\, \delta(x_1 - x(t_1))\delta(x_2 - x(t_2))x_1x_2\, e^{iS[x]}\,. \tag{1.68}$$

For $t_2 > t_1$, we can write this as

$$\begin{aligned}\int \mathcal{D}x\, x(t_1)x(t_2)\, e^{iS[x]} &= \int \mathcal{D}x_1\mathcal{D}x_2 \int_{x(t_1)=x_1}^{x(t_2)=x_2} \mathcal{D}x\, x_1x_2\, e^{iS[x]} \\ &= \mathcal{N}^{-1}\int \mathcal{D}x_1\mathcal{D}x_2 \langle x|e^{-i(t-t_2)H}x_2|x_2\rangle\langle x_2|e^{-i(t_2-t_0)H}x_1|x_1\rangle\langle x_1|e^{-i(t_1-t_0)}|x_0\rangle \\ &= \mathcal{N}^{-1}\langle x|e^{-i(t-t_2)H}\hat{x}e^{-i(t_2-t_1)H}\hat{x}e^{-i(t_1-t_0)H}|x_0\rangle \end{aligned} \tag{1.69}$$

where we in the last line used the functional spectral decomposition

$$\hat{x} = \int \mathcal{D}x\, x|x\rangle\langle x|\,. \tag{1.70}$$

Now, the relation between the time-dependent operators and time-independent operators are

$$\hat{x}(t) = e^{-itH}\hat{x}e^{itH}\,, \tag{1.71}$$

which mean that

$$\begin{aligned}\mathcal{N}\int \mathcal{D}x\, x(t_1)x(t_2)\, e^{iS[x]} &= \langle x|e^{-i(t-t_2)H}\hat{x}e^{-i(t_2-t_1)H}\hat{x}e^{-i(t_1-t_0)H}|x_0\rangle \\ &= \langle x|e^{-iHt}\hat{x}(t_2)\hat{x}(t_1)e^{iHt_0}|x_0\rangle \\ &= \langle x,t|\hat{x}(t_2)\hat{x}(t_1)|x_0,t_0\rangle \end{aligned} \tag{1.72}$$

The exact same result can be found for $t_1 > t_2$, giving that in general we write this as in Eq. (1.67). This is because in the path-integral formalism time ordering is automatic. This can also be generalized to n products of operators $\hat{x}$. Furthermore, it can be shown that

$$\lim_{t_0,t\to\mp\infty}\langle x,t|\mathcal{T}(\hat{x}(t_1)\dots\hat{x}(t_n))|x_0,t_0\rangle \propto \langle 0|T\big(\hat{x}(t_1)\dots\hat{x}(t_n)\big)|0\rangle\,. \tag{1.73}$$

[12]This is formally infinite, but we will later see that it does not appear in physical quantities so we will not bother with any formal technicalities.

This is not a trivial result do derive, but it says that at asymptotic times, the initial and final states are the vacuum states of the system. With these consideration we can write what we call n-point Green's functions[13] as

$$\langle 0|\mathcal{T}(\hat{x}(t_1)\dots\hat{x}(t_n))|0\rangle = \mathcal{N}\int \mathcal{D}x\, x(t_1)\dots x(t_n)\, e^{iS[x]}\,. \tag{1.74}$$

Note that the left hand side involve position operators $\hat{x}(t)$, while on the right hand side we have functions $x(t)$. This observation is important to remember when we start talking about fields, and constitutes an important distinction between the canonical and the path integral approach.

In all practicality, the time ordering operator is reduntant in the path integral formulation, but it is common to keep it there. So, when working in the path integral formalism one often use the left hand side of Eq. (1.74) for notational simplicity.

1.3.2 Path Integrals in Quantum Field Theory

In quantum mechanics we have that position and momentum are promoted to operators, and formulating this in the path integral formalism we have integrals over trajectories in coordinate space. This is often referred to as *first quantization*. In quantum field theory we promote the fields to operators and invoke the canonical commutation relations in Eq. (1.24). This procedure is often referred to as *second quantization*.

Formulated as a path integral we no longer have an integration over trajectories in coordinate space, but an integration over field configurations. With the work done in the last section, we can move on to the field theoretical description. In Eq. (1.33) we defined the Feynman propagator as the two-point Green's function of the Klein-Gordon operator. In general, we can write the n-point Green's function as the field correlator

$$\mathcal{G}^n(x_1,\dots,x_n) = \langle 0|\mathcal{T}(\phi(x_1)\dots\phi(x_n))|0\rangle\,. \tag{1.75}$$

By replacing the operators in Eq. (1.74) that the path-integral in quantum field theory can be written in terms of the n-point Green's function as

$$\mathcal{G}^{(n)}(x_1,\dots,x_n) = \mathcal{N}\int \mathcal{D}\phi\,\phi(x_1)\dots\phi(x_n)\, e^{iS[\phi]}\,. \tag{1.76}$$

It is important to point out that the fields in the path integral are the classical fields, and not the quantum operators. In the canonical quantization formalism, quantization itself lies in the canonical commutation relations by promoting fields to operators. In the path integral formalism we skip the commutation relations and will directly identify the propagator as the Green's function of the differential operator of the theory. This is the main difference of the two approaches, but we will see below that they give the same result.

There is a way to proceed that makes path integral calculations much simpler, and that is to use a *generating gunctional*. The generating functional is defined as the vacuum amplitude in the presence of a source $J(x)$

$$\mathcal{Z}[J] \equiv \langle 0|0\rangle_J = \mathcal{N}\int \mathcal{D}\phi\, \exp\left(iS[\phi] + i\int d^4x\, J(x)\phi(x)\right), \tag{1.77}$$

where we require the vacuum to be normalized, i.e. $\langle 0|0\rangle = 1$, giving

$$\mathcal{N}^{-1} = \int \mathcal{D}\phi\, \exp\left(iS[\phi]\right). \tag{1.78}$$

To go from the generating functional to the Green's function is straigthforward by applying the *functional derivative*[14] on $\mathcal{Z}[J]$. The effect of the functional derivative is made clear by observing that for

$$J(y) = \int d^4x \delta^4(x-y)J(x)\,, \tag{1.79}$$

we can define

$$\frac{\partial J(x)}{\partial J(y)} = \delta^4(x-y)\,, \tag{1.80}$$

[13] Keep in mind that we will sometimes call these correlators, but it stands for the same object.
[14] We will write this as a partial derivative, but it is formally a *variational partial deriative*.

giving

$$\frac{\partial}{\partial J(x_1)} \int d^4x J(x)\phi(x) = \phi(x_1)\,, \tag{1.81}$$

implying that

$$-i\frac{\partial \mathcal{Z}[J]}{\partial J(x_1)} = \mathcal{N} \int \mathcal{D}\phi\, \phi(x_1) \exp\left(iS[\phi] + i\int d^4x\, J(x)\phi(x)\right). \tag{1.82}$$

The generalization to n functional derivatives evaluated at $J = 0$, is given by

$$(-i)^n \frac{\partial^n \mathcal{Z}[J]}{\partial J(x_1)\cdots\partial J(x_n)}\Big|_{J=0} = \frac{\int \mathcal{D}\phi\, \phi(x_1)\ldots\phi(x_n)\, e^{iS[\phi]}}{\int \mathcal{D}\phi\, e^{iS[\phi]}}\,. \tag{1.83}$$

Comparing with Eq. (1.76), the n-point Green's function can be written as

$$\mathcal{G}^{(n)}(x_1,\ldots,x_n) = (-i)^n \frac{\partial^n \mathcal{Z}[J]}{\partial J(x_1)\cdots\partial J(x_n)}\Big|_{J=0}\,. \tag{1.84}$$

In practical calculations this generating functional corresponds to all possible Feynman diagrams, also those that are not describing scattering, i.e. parts of the so-called S-matrix that is trivial[15]. Therefore, it is convenient to define a generating functional that corresponds to *connected* Feynman diagrams.

This generating functional for connected diagrams is defined as

$$\mathcal{W}[J] \equiv -i\ln \mathcal{Z}[J]\,, \tag{1.85}$$

giving the connected Green's function

$$\mathcal{G}_c^{(n)}(x_1,\ldots,x_n) = (-i)^{n+1} \frac{\partial^n \mathcal{W}[J]}{\partial J(x_1)\cdots\partial J(x_n)}\Big|_{J=0}\,. \tag{1.86}$$

This is a simple[16] way of calculating time ordered products. We simply find the generating functional of the theory and perform derivatives to get the time ordered products.

The generating functional is the quantum field theory analog of the partition function in statistical mechanics, where all the information about the theory is encoded in it. That is, if you have an exact closed-form expression for $\mathcal{Z}[J]$, for any particular theory, you have solved it completely. Unfortunately, this is only possible for free theories. For interacting theories we have to use perturbation theory, but in Sec. 1.4 we will see a elegant procedure using the Feynman diagram expansion and its connections to the so-called LSZ-reduction formula.

Before we talk about interacting theories, we would like to show how to solve a quantum field theory exactly. In order to do this, we will first discuss the most important path integral of them all, namely Gaussian path integrals.

Gaussian Path Integrals

As we have seen Green's functions are given as a path integral weighted with an exponential of the action. The action will always have kinetic terms that are quadratic by nature, i.e. they are Gaussian path integrals. A general Gaussian path integral can be written as

$$\int \mathcal{D}\phi\, e^{-\phi_x K_{xy}\phi_y}\,, \tag{1.87}$$

with the abbreviation

$$\phi_x K_{xy}\phi_y = \int d^4x\, d^4y\, \phi(x)K(x,y)\phi(y)\,, \tag{1.88}$$

[15] The S-matrix will be defined in Sec. 1.4.2.

[16] That is not to say it is simple in practice, but in principle.

where $K(x,y)$ is a real symmetric operator. If we expand the fields ϕ and K in the same orthonormal basis

$$\phi(x) = \sum_i^\infty \xi_i \, \phi_i(x) \tag{1.89}$$

$$K(x,y) = \sum_{i,j}^\infty \phi_i(x) K_{ij} \phi_j(y) \,, \tag{1.90}$$

we can pick out K_{ij} by projecting on the basis functions

$$K_{ij} = \int d^4x \, d^4y \, \phi_i(x) K(x,y) \phi_j(y) \,, \tag{1.91}$$

giving that we can write Eq. (1.88) as

$$\phi_x K_{xy} \phi_y = \sum_{i,j} \xi_i K_{ij} \xi_j \,. \tag{1.92}$$

and with this rewriting the Gaussian integral takes the form

$$\lim_{n\to\infty} \Big(\prod_i^n \int d\xi_i \Big) \exp\left[- \sum_{i,j} \xi_i K_{ij} \xi_j \right] . \tag{1.93}$$

To evaluate this integral we use that K is symmetric, and write

$$K_{ij} = \sum_{k,l} O_{ik} A_{kl} O_{lj} \,, \tag{1.94}$$

where O is an orthogonal matrix that diagonalizes K. The matrix A is diagonal, i.e. $A_{kl} = a_k \delta_{kl}$, where a_k are the eigenvalues of K. By redefining $x_i = O_{ij}\xi_j$, we get that

$$\begin{aligned} \Big(\prod_i^n \int dx_i \Big) \exp\left[- \sum_i a_i x_i^2 \right] &= \prod_i^n \sqrt{\frac{\pi}{a_i}} \\ &= \sqrt{\pi^n} [\det K]^{-1/2} \,, \end{aligned} \tag{1.95}$$

and the general Gaussian path integral is therefore a product of n Gaussian integrals

$$\int \mathcal{D}\phi \, e^{-\phi_x K_{xy} \phi_y} = N_G [\det K]^{-1/2} \,, \tag{1.96}$$

where the normalization constant N_G is an infinite constant that will be divided out when calculating Green's functions and $\det K$ is referred to as a *functional determinant*. Functional determinants are interesting and has a wide range of applicability in quantum field theory, but we will not cover this here.

When we want to calculate Green's functions in practice, we will also have terms in the exponential that are linear in the fields. We will then have the following path integral

$$\int \mathcal{D}\phi \, e^{-\phi_x K_{xy} \phi_y + J_x \phi_x} \,. \tag{1.97}$$

The general approach to compute these kinds of integrals is to complete the square. The easiest way to do this is to make the following shift of the fields

$$\phi'(x) = \phi(x) - \frac{1}{2} \int d^4z \, K^{-1}(x,z) J(z) \,, \tag{1.98}$$

with K^{-1} satisfying[17]

$$\int d^4z \, K(x,z) K^{-1}(z,y) = \delta^4(x-y) \,. \tag{1.99}$$

[17] That is, assuming the inverse exist. In general this is not true as we will see when we want to quantize gauge theories.

We observe that if K is a differential operator, this technique resembles the Green's function method of solving partial differential equations[18]. The path integral is translational invariant, so a shift will not change the integration measure. Inserting this shift into Eq. (1.97), the exponent takes the form

$$\int d^4x\, d^4y\, \phi'(x)K(x,y)\phi'(y) - \frac{1}{4}J(x)K^{-1}(x,y)J(y)\,. \tag{1.100}$$

The second term is independent on ϕ', so this will result in a Gaussian path integral times an exponential function. Since all fields in the path integral formulation are functions, they commute and we can just pull the extra term with source fields J outside the path integral.

The result after completing the square is then given by

$$\int \mathcal{D}\phi\, e^{-\phi_x K_{xy}\phi_y + J_x\phi_x} = e^{\frac{1}{4}J_x K^{-1}_{xy}J_y}\int \mathcal{D}\phi\, e^{-\phi_x K_{xy}\phi_y}\,. \tag{1.101}$$

This is now on a form where it is easier to see the action of functional derivaties on the path integral. From Eq. (1.81) we know that by acting with functional derivatives on a path integral, we 'pull' down a field from the exponent, i.e.

$$\frac{\partial^2}{\partial J(x_1)\partial J(x_2)}\int \mathcal{D}\phi e^{-\phi_x K_{xy}\phi_y + J_x\phi_x}\Big|_{J=0} = \int \mathcal{D}\phi\, \phi(x_1)\phi(x_2)e^{-\phi_x K_{xy}\phi_y}\,, \tag{1.102}$$

but since we have completed the square in Eq. (1.101), all the J dependence is in the factor in front of the path integral, so we can also write

$$\int \mathcal{D}\phi\, \phi(x_1)\phi(x_2)e^{-\phi_x K_{xy}\phi_y} = \Big[\frac{\partial^2}{\partial J(x_1)\partial J(x_2)}e^{\frac{1}{4}J_x K^{-1}_{xy}J_y}\Big]_{J=0}\int \mathcal{D}\phi e^{-\phi_x K_{xy}\phi_y}\,. \tag{1.103}$$

Calculating the bracket yields

$$\frac{\partial^2}{\partial J(x_1)\partial J(x_2)}e^{\frac{1}{4}J_x K^{-1}_{xy}J_y}\Big|_{J=0} = \frac{1}{2}K^{-1}(x_1,x_2)\,, \tag{1.104}$$

giving the two-point Gaussian path integral

$$\frac{1}{2}K^{-1}(x_1,x_2) = \frac{\int \mathcal{D}\phi\, \phi(x_1)\phi(x_2)e^{-\phi_x K_{xy}\phi_y}}{\int \mathcal{D}\phi e^{-\phi_x K_{xy}\phi_y}}\,, \tag{1.105}$$

and generalizing this to n-point Gaussian path integrals is straightforward

$$\Big[\frac{\partial^n}{\partial J(x_1)\cdots\partial J(x_n)}e^{\frac{1}{4}J_x K^{-1}_{xy}J_y}\Big]_{J=0} = \frac{\int \mathcal{D}\phi\, \phi(x_1)\cdots\phi(x_n)e^{-\phi_x K_{xy}\phi_y}}{\int \mathcal{D}\phi e^{-\phi_x K_{xy}\phi_y}}\,. \tag{1.106}$$

This looks very much like the expression we found for the generating functional Eq. (1.84), and will be helpful after the action has been specified.

1.3.3 Quantization of Scalar Field Theory

Let us then move on and finally try to quantize a theory in this formalism. The scalar theory is easiest so we show the main steps, which we will use when we come to quantization of fermionic fields and vector fields.

The action for a free real scalar theory follows from Eq. (1.8)

$$S_0[\phi] = \int d^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2\right), \tag{1.107}$$

and the generating functional can be written as

$$\mathcal{Z}_0[J] = \mathcal{N}_0\int \mathcal{D}\phi\, \exp\left(iS_0[\phi] + i\int d^4x\, J(x)\phi(x)\right). \tag{1.108}$$

[18] Not surprisingly we will later relate K to the differential operator of the theory.

We would like to rewrite the action on a form such that we can use the results we found for Gaussian path integrals. To do this, we use partial integration and rewrite the action as

$$S_0[\phi] = -\frac{1}{2}\int d^4x\, d^4y\, \phi(x)\delta^{(4)}(x-y)(\partial^2 + m^2)\phi(y)\,. \tag{1.109}$$

If we insert this expression into Eq. (1.108) and compare with the Gaussian path integral Eq. (1.97), we observe that

$$K(x,y) = \frac{i}{2}\delta^{(4)}(x-y)(\partial^2 + m^2)\,, \tag{1.110}$$

and from Eq. (1.99) it follows that the inverse must satisfy

$$\frac{1}{2}(\partial^2 + m^2)K^{-1}(x,y) = -i\delta^{(4)}(x-y)\,. \tag{1.111}$$

If this step is obscure, just insert Eq. (1.110) into Eq. (1.99), giving the above equation. We observe that Eq. (1.111) is a Green's function equation for the Klein-Gordon operator. The recipe of quantization in the path integral formalism is then by defining the propagator as

$$D_F(x,y) \equiv \frac{1}{2}K^{-1}(x,y)\,, \tag{1.112}$$

which reproduces Klein-Gordon propagator equation Eq. (1.34) in canonical quantization

$$(\partial^2 + m^2)D_F(x,y) = -i\delta^{(4)}(x-y)\,, \tag{1.113}$$

which after a Fourier transform results in the coordinate space propagator given in Eq. (1.32). This identification shows that the path integral formalism and the canonical quantization procedure give the same result. This might seem backwards and a lot of work for nothing new, but it shows that the Feynman propagator follows from a general treatment of the path integral.

With the identification Eq. (1.112) the generating functional for a free scalar theory follows by completing the square of Eq. (1.108), giving

$$\mathcal{Z}_0[J] = \exp\left(-\frac{1}{2}\int d^4x\, d^4y\, J(x)D_F(x,y)J(y)\right). \tag{1.114}$$

This is an exact result, and from it we can learn all there is about the theory.

Then we can also see that by performing the following functional derivatives on Eq. (1.114), will give

$$\mathcal{G}^{(2)}(x,y) = -\frac{\partial^2\mathcal{Z}_0[J]}{\partial J(x)\partial J(y)}\Big|_{J=0} = D_F(x,y)\,, \tag{1.115}$$

and as a path integral it follows from Eq. (1.74) that

$$D_F(x,y) = \frac{\int \mathcal{D}\phi\, \phi(x)\phi(y)e^{iS_0[\phi]}}{\int \mathcal{D}\phi\, e^{iS_0[\phi]}}\,. \tag{1.116}$$

We already know that the two-point function is the Feynman propagator, but we wanted to show that there are many different ways of representing Green's functions in the path integral formalism.

The most important point here is that the quantization procedure did not involve promoting the fields to operators or invoking commutation relations, but instead by imposing the identity in Eq. (1.112).

1.3.4 Quantization of Fermionic Fields

To quantize fermionic theories the general features of the quantization procedure is the same as for scalar theory. The main difference is that one have to use Grassmann calculus as fermionic fields are anti-commuting. It would take to much space to fully cover Grassmann numbers and Gaussian path integrals over Grassmannian fields, so we will instead cover the basic results for Green's functions in fermionic theory. For more detail on Grassmanian path integrals we refer the reader to [2].

The generating functional can be written in a similar way as for scalar theory, only that we need two Grassmann valued sources $\bar{\eta}$ and η. In general, one has that the Gaussian path integral over Grassmannian fields is given by

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{-\bar{\psi}_x K_{xy}\psi_y} = \det K\,, \tag{1.117}$$

and the result after completing the square

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{-\bar{\psi}_x K_{xy}\psi_y+\bar{\eta}_x\psi_x+\bar{\psi}_x\eta_x} = \det K\, e^{\bar{\eta}_x K^{-1}_{xy}\eta}\,. \tag{1.118}$$

The free Dirac generating functional can then be written as

$$\mathcal{Z}_0[\bar{\eta},\eta] = \mathcal{N}\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left(iS_0[\bar{\psi},\psi] + i\int d^4x\, \bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x)\right), \tag{1.119}$$

with the free Dirac action

$$S_0[\bar{\psi},\psi] = \int d^4x\, \bar{\psi}(i\partial\!\!\!/ - m)\psi\,, \tag{1.120}$$

and if we do the same as for the scalar theory in Eq. (1.109), it follows that

$$K(x,y) = -i\delta^{(4)}(x-y)(i\partial\!\!\!/ - m)\,, \tag{1.121}$$

satisfying

$$\int d^4z\, K(x,z)K^{-1}(y,z) = \delta^{(4)}(x-y)\,, \tag{1.122}$$

which implies that the inverse must satisfy the Green's function equation for the Dirac operator, i.e.

$$(i\partial\!\!\!/ - m)K^{-1}(x,y) = i\delta^{(4)}(x-y)\,. \tag{1.123}$$

Everything is similar to the scalar theory, so we identity the inverse as the Green's function of the Dirac operator

$$S_F(x,y) \equiv K^{-1}(x,y)\,, \tag{1.124}$$

where again the solution is obtained by a Fourier transform of the Green's function equation, giving the momentum space solution

$$S_F(x,y) = \frac{i}{\not{p} - m + i\epsilon}\,, \tag{1.125}$$

reproducing the same propagator given in Eq. (1.41) defined in canonical quantization.

For completeness, let us complete the square by shifting the fields in Eq. (1.119), giving[19]

$$\mathcal{Z}_0[\bar{\eta},\eta] = \exp\left(-\int d^4x\, d^4y\, \bar{\eta}(x)S_F(x,y)\eta(y)\right). \tag{1.126}$$

Applying functional derivatives to this we can find n-point Green's functions, which as path integrarls can be written as

$$\begin{aligned}\mathcal{G}^{(n)}(x_1,\dots x_n;y_1,\dots y_n) &= \langle 0|\mathcal{T}(\psi(x_1)\dots\psi(x_n)\bar{\psi}(y_1)\dots\bar{\psi}(y_n))|0\rangle \\ &= \frac{\int \mathcal{D}\bar{\psi}\mathcal{D}\,\psi\,\psi(x_1)\dots\psi(x_n)\bar{\psi}(y_1)\dots\bar{\psi}(y_n)\, e^{iS_0[\bar{\psi},\psi]}}{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{iS_0[\bar{\psi},\psi]}}\,.\end{aligned} \tag{1.127}$$

With these constructions one have all information needed to specify the whole theory, but this is not very interesting as the fields are non-interacting. After we have connected Green's functions to the S-matrix and subsequently to scattering amplitudes, we have all we need to compute physical processes using Feynman diagrams.

[19] See Eq. (1.98) how to shift the fields.

1.3.5 Quantization of Abelian Gauge Fields

Lastly, in this subsection we will investigate how to quantize an Abelian gauge theory, or Maxwell theory, as we are ultimately interested in describing the dynamics of the electromagnetic field. The quantization of non-Abelian gauge theories will be made after we have derived the Yang-Mills Lagrangian from a geometric viewpoint in Chapter 2. But the basic set up is the same for both Abelian and non-Abelian theories, so it is useful to do it in detail for the easiest case and point out the differences in the non-Abelian case.

The action for a Maxwell theory is given by[20]

$$S[A] = -\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu}\,, \tag{1.128}$$

and as usual we want to find the two-point Green's function for this theory. The relevant path integral for this theory is

$$\int \mathcal{D}A\, e^{iS[A]}\,, \tag{1.129}$$

where

$$\int \mathcal{D}A = \int \mathcal{D}A^0 \int \mathcal{D}A^1 \int \mathcal{D}A^2 \int \mathcal{D}A^3\,, \tag{1.130}$$

since Abelian fields commute we have one scalar path integral per component.

Let us use the prescription we have used for both the scalar field and Dirac field quantization, i.e. use the Gaussian path integral to find K^{-1} and relate it to the propagator. To identify the differential operator we integrate by parts and use a delta function to write

$$S[A] = \frac{1}{2}\int d^4x\, d^4y\, A_\mu(x)\delta^{(4)}(x-y)(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu)A_\nu(y)\,, \tag{1.131}$$

from which we identity

$$K^{\mu\nu}(x,y) = -\frac{i}{2}\delta^{(4)}(x-y)(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu)\,. \tag{1.132}$$

Let us continue in the naive way and assume the inverse exist, giving the Green's function equation

$$\frac{1}{2}(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu)K^{-1}_{\nu\rho}(x,y) = i\delta^\mu_\rho\delta^{(4)}(x-y)\,, \tag{1.133}$$

or in terms of the Fourier transform

$$\frac{1}{2}(-k^2 g^{\mu\nu} + k^\mu k^\nu)K^{-1}_{\nu\rho}(k) = i\delta^\mu_\rho\,. \tag{1.134}$$

However, this is not a well defined equation since $(-k^2 g^{\mu\nu} + k^\mu k^\nu)$ is singular, i.e. it has an eigenvector k_ν with eigenvalue zero, resulting in that K^{-1} has no solution.

The difficulty is due to the gauge freedom of the Maxwell action, i.e. the Maxwell action is invariant under the transformation

$$A_\mu(x) \to A_\mu(x) + \partial_\mu\alpha(x)\,, \tag{1.135}$$

where the troublesome modes are those for which $A_\mu(x) = \partial_\mu\alpha(x)$, i.e. those that are gauge-equivalent to $A_\mu(x) = 0$. As a result, the path integral is badly divergent, as we are redundantly integrating over an infinite set of equivalent fields.

To fix the problem there is a trick due to Fadeev and Popov [8], where we can isolate the interesting part of the functional integral. To this end, we define a functional $G[A]$ that we wish to set equal to zero as a gauge-fixing condition. There are a wide range of possibilities, but the ones we will focus on here is

$$\text{Lorentz gauge:} \qquad G[A] = \partial_\mu A^\mu\,, \tag{1.136}$$

$$\text{Axial gauge:} \qquad G[A] = n_\mu A^\mu\,. \tag{1.137}$$

[20] This follows from the vector field Lagrangian in Eq. (1.16).

For the Abelian case we will only consider Lorentz gauge, but for the non-Abelian case we will take a closer look at axial gauges.

We can constrain the path integral to cover only the configurations with $G[A] = 0$ by inserting a functional delta function. However, we can not blindly put this in as it would change the path integral. Instead, we can insert an identity by using the functional identity

$$\int \mathcal{D}\alpha\, \delta(G[A^\alpha]) = \det\Big(\frac{\partial G[A^\alpha]}{\partial \alpha}\Big)^{-1}, \tag{1.138}$$

or

$$1 = \det\left(\frac{\partial G[A^\alpha]}{\partial \alpha}\right) \int \mathcal{D}\alpha\, \delta(G[A^\alpha])\,, \tag{1.139}$$

where A^α is the gauge-transformed field. Since the gauge transformation is just a shift, we have that $S[A] = S[A^\alpha]$ and $\mathcal{D}A = \mathcal{D}A^\alpha$. Hence, we can write the path integral as

$$\int \mathcal{D}\alpha \int \mathcal{D}A^\alpha e^{iS[A^\alpha]}\, \delta(G[A^\alpha]) \det\left(\frac{\partial G[A^\alpha]}{\partial \alpha}\right). \tag{1.140}$$

Now, A^α is just a dummy integration variable, so we can just rename it back to A. Also, for Abelian fields we have that the functional determinant is independent of A. This is easily seen by choosing Lorentz gauge $G[A^\alpha] = \partial^\mu A_\mu + \partial^2 \alpha$, giving

$$\det\left(\frac{\partial G[A^\alpha]}{\partial \alpha}\right) = \det\big(\partial^2\big)\,. \tag{1.141}$$

Hence, we can pull it out of the path integral and obtain

$$\int \mathcal{D}A\, e^{iS[A]} = \det\left(\frac{\partial G[A^\alpha]}{\partial \alpha}\right) \int \mathcal{D}\alpha \int \mathcal{D}A\, e^{iS[A]}\, \delta(G[A])\,, \tag{1.142}$$

where we kept the α-dependence in the functional determinant as it is gauge-invariant and is unimportant as it will cancel for Green's functions. The integration over α is an infinite constant, but this will also cancel so we will not concern ourselves with it. The effect of rewriting the path integral in this fashion is that we have due to the delta function restricted ourselves to physical inequivalent field configurations.

Let us then specify the gauge fixing functional to be the class of Lorentz gauges

$$G[A] = \partial^\mu A_\mu - \omega(x)\,, \tag{1.143}$$

where $\omega(x)$ is a scalar function. The functional determinant does not change, so we get

$$\begin{aligned}\int \mathcal{D}A\, e^{iS[A]} &= \Big(\det\big(\partial^2\big) \int \mathcal{D}\alpha\Big) \int \mathcal{D}A\, e^{iS[A]}\, \delta(\partial^\mu A_\mu - \omega(x)) \\ &= N(\alpha) \int \mathcal{D}A\, e^{iS[A]}\, \delta(\partial^\mu A_\mu - \omega(x))\,.\end{aligned} \tag{1.144}$$

Since ω is unspecified, this equation remains valid if we replace the right hand side with any linear combination of ω. The final trick is then to add such a linear combination by adding 1 again. To do this we consider the following Gaussian integral

$$\int \mathcal{D}\omega\, \exp\left(-i \int d^4x\, \frac{\omega^2}{2\xi}\right) = \lim_{n\to\infty} \sqrt{(2i\xi\pi)^n} \equiv N^{-1}(\xi)\,, \tag{1.145}$$

which mean that can then safely add the following factor to the path integral

$$N(\xi) \int \mathcal{D}\omega\, \exp\left(-i \int d^4x\, \frac{\omega^2}{2\xi}\right) = 1\,, \tag{1.146}$$

giving

$$\int \mathcal{D}A\, e^{iS[A]} = \tilde{N} \int \mathcal{D}\omega \int \mathcal{D}A\, \delta(\partial^\mu A_\mu - \omega(x)) \exp\Big(iS[A] - i \int d^4x\, \frac{\omega^2}{2\xi} \Big) , \tag{1.147}$$

where $\tilde{N} = N(\xi)N(\alpha)$ is still just an infinite constant that is unimportant. The above insertion corresponds to a Gaussian weighting function that is normalized to one, centered around $\omega = 0$, and integrated over all possible ω. This is the same as the continous limit of making all possible linear combinations of ω, just as we wanted. Also, we can choose ξ to be any finite constant and is referred to as a gauge fixing parameter.

Let us then use the delta function to perform the integration over ω, giving

$$\int \mathcal{D}A\, e^{iS[A]} = \tilde{N} \int \mathcal{D}A \exp\Big(iS[A] - i \int d^4x\, \frac{1}{2\xi} (\partial^\mu A_\mu)^2 \Big) . \tag{1.148}$$

Effectively, we have added a term to the action that ensures that the previously divergent path integral has a nice behaviour.

We can then write the gauge-fixed action as

$$S_{GF}[A] = \frac{1}{2} \int d^4x\, d^4y\, A_\mu(x) \delta^{(4)}(x-y) \big(\partial^2 g^{\mu\nu} - \Big(1 - \frac{1}{\xi}\Big) \partial^\mu \partial^\nu \big) A_\nu(y) , \tag{1.149}$$

and it follows that the Green's function equation is given by

$$\frac{1}{2} \big(\partial^2 g^{\mu\nu} - \Big(1 - \frac{1}{\xi}\Big) \partial^\mu \partial^\nu \big) K^{-1}_{\nu\rho}(x,y) = i \delta^\mu_\rho \delta^{(4)}(x-y) . \tag{1.150}$$

which is non-singular, i.e. the propagator exist. Thus, we define the Abelian gauge field (photon) propagator as

$$D_F^{\mu\nu}(x,y) \equiv \frac{1}{2} K^{-1}(x,y) . \tag{1.151}$$

To find the explicit solution to the propagator equation we use that the most general second rank symmetric tensor can be expanded as (in momentum space)

$$D_F^{\mu\nu}(k) = A g^{\mu\nu} + B k^\mu k^\nu , \tag{1.152}$$

which inserted into Eq. (1.150) will give $A = -i/k^2$ and $B = -i(1-\xi)/k^4$, and the the momentum space propagator takes the form

$$D_F^{\mu\nu}(k) = \frac{-i}{k^2 + i\epsilon} \Big(g^{\mu\nu} - (1-\xi) \frac{k^\mu k^\nu}{k^2} \Big) . \tag{1.153}$$

To actually use this expression in calculations we have to specify a gauge choice for ξ, and the most common is the Feynman gauge $\xi = 1$. Other choices are possible as well, e.g. the Landau gauge $\xi = 0$ or the Yennie gauge $\xi = 3$.

Finally, we can show that the infinite constants $\mathcal{N}(\xi)\mathcal{N}(\alpha)$ we have in the path integral vanishes for physical quantities, i.e. we have that the n-point Green's function takes the form

$$\langle 0 | \mathcal{T}\, F[A] | 0 \rangle = \frac{\int \mathcal{D}A\, F[A] \exp\Big(iS[A] - i \int d^4x\, \frac{1}{2\xi} (\partial^\mu A_\mu)^2 \Big)}{\int \mathcal{D}A \exp\Big(iS[A] - i \int d^4x\, \frac{1}{2\xi} (\partial^\mu A_\mu)^2 \Big)} , \tag{1.154}$$

where $F[A]$ is a collection of fields.

This leads us to the end of the discussion regarding free fields. We have now explored all the relevant concepts we will use in the transition to interacting theories using the the path integral formalism.

1.4 Perturbation Theory in Quantum Field Theory

In this section we would like to develop a method for calculating the physical consequences of a small interaction in a nearly free quantum field theory. The fundamental assumption as that we express quantities as a power series in the coupling strength. These powers series will have many unpleasant mathematical properties to be discussed later. In this chapter, we shall ignore such problems and show how the power series can be calculated in principle.

1.4.1 Perturbation Theory from Path Integrals

In order to develop perturbation theory form the path integral formalism, we split the action into a free part and an interacting part. In general, the exactly solvable parts of a theory, i.e. the parts that are quadratic in the fields are placed in the free part. Those terms that are higher order than quadratic are interaction terms and are placed in the interaction part. We will talk more about the structure of this splitting when we want to relate Green's function to the S-matrix, but for now we just use that this as the procedure.

We write the full action as the sum over a free part and a interacting part, i.e.

$$S = S_0 + S_{\text{int}} \,. \tag{1.155}$$

The interaction term causes problems if we want to complete the square by shifting the fields, as we did for the free theory. However, there is a trick that makes it possible to formulate the interacting theory in terms of the free theory.

Let us start by looking at the generating functional for a scalar theory

$$\mathcal{Z}[J] \equiv \langle \Omega | \Omega \rangle_J = \mathcal{N} \int \mathcal{D}\phi \exp\Big(i S_0 + i S_{\text{int}} + i \int d^4x \, J(x)\phi(x) \Big) \,. \tag{1.156}$$

where $|\Omega\rangle$ is the interacting vacuum, with normalization $\langle \Omega | \Omega \rangle = 1$. To rewrite this we can use that the functional derivative acts as[21]

$$\Big(-i \frac{\partial}{\partial J(y)} \Big) e^{i \int d^4x \, J(x)\phi(x)} = \phi(y) e^{i \int d^4x \, J(x)\phi(x)} \,, \tag{1.157}$$

so we can make the replacement

$$\mathcal{S}_{\text{int}}[\phi] \to \mathcal{S}_{\text{int}}\Big[-i \frac{\partial}{\partial J(x)} \Big] \,, \tag{1.158}$$

which is independent of ϕ and can be moved outside the path integral. This enable us to write the generating functional as

$$\begin{aligned} \mathcal{Z}[J] &= \mathcal{N} \exp\Big(i \mathcal{S}_{\text{int}}\Big[-i \frac{\partial}{\partial J(x)} \Big] \Big) \int \mathcal{D}\phi \exp\Big(i S_0 + i \int d^4x \, J(x)\phi(x) \Big) \\ &= \mathcal{N} \exp\Big(i \mathcal{S}_{\text{int}}\Big[-i \frac{\partial}{\partial J(x)} \Big] \Big) \mathcal{N}_0^{-1} \mathcal{Z}_0[J] \,, \end{aligned} \tag{1.159}$$

where $\mathcal{Z}_0$ and $\mathcal{N}_0$ are the generating functional and normalization factor for the free theory, see Eq. (1.108). Now if we set $J = 0$ we have $\mathcal{Z}[0] = 1$, giving the normalization factor

$$\mathcal{N}^{-1} = \mathcal{N}_0^{-1} \exp\Big(i \mathcal{S}_{\text{int}}\Big[-i \frac{\partial}{\partial J(x)} \Big] \Big) \mathcal{Z}_0[J] \Big|_{J=0} \,, \tag{1.160}$$

and the generating functional in the interacting theory can be written as

$$\mathcal{Z}[J] = \frac{\exp\Big(i \mathcal{S}_{\text{int}}\Big[-i \frac{\partial}{\partial J(x)} \Big] \Big) \mathcal{Z}_0[J]}{\exp\Big(i \mathcal{S}_{\text{int}}\Big[-i \frac{\partial}{\partial J(x)} \Big] \Big) \mathcal{Z}_0[J] \Big|_{J=0}} \,. \tag{1.161}$$

The general approach from here is then to use the small coupling assumption and expand the exponential, giving that the numerator can be calculated as

$$\mathcal{Z}_{\text{num}}[J] = \Big[1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int d^4x_1 \ldots d^4x_n \, \mathcal{L}_{\text{int}}\Big(-i \frac{\partial}{\partial J(x_1)} \Big) \ldots \mathcal{L}_{\text{int}}\Big(-i \frac{\partial}{\partial J(x_n)} \Big) \Big] \mathcal{Z}_0[J] \,. \tag{1.162}$$

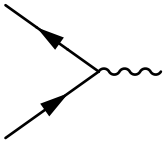
We observe that in order to obtain Green's functions we have to perform an awful lot of functional derivatives. First we have to calculate the functional derivatives in Eq. (1.162), then we have to perform even more functional

[21] This follows from Eq. (1.81).

derivatives by using Eq. (1.83). This looks kind of horrible, but it can be worked out order by order in perturbation theory. This procedure will result in a sum over products of propagators D_F and each term can be associated to a Feynman diagram. In the end one finds that all the Feynman rules necessary for practical calculations can be read off from the exponential of the generating functional of the theory. For example, for a Dirac theory coupled to an electromagnetic field A_μ we have the Lagrangian[22]

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - e\bar{\psi}A\!\!\!/\psi\,, \tag{1.163}$$

where the last term describes the following vertex rule



$$= -ie\gamma^\mu\,, \tag{1.164}$$

which follows from the fact that the Lagrangian appear in the action, which subsequently appear in the exponent of the generating functional times i. Interaction terms involving derivatives are a bit more complicated, but since each fields can be Fourier expanded the derivative will pull down a momentum.

In order to show that vertex rules are as simple as this we can take a look at the example of ϕ^4-theory, with the interaction term

$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{4!}\phi^4\,, \tag{1.165}$$

where we as always assume small coupling λ in order to do perturbation theory. In general, we have to calculate

$$\mathcal{Z}[J] = \frac{\exp\Big(-i\frac{\lambda}{4!}\int d^4z\Big(-i\frac{\partial}{\partial J(z)}\Big)^4\Big)\mathcal{Z}_0[J]}{\exp\Big(-i\frac{\lambda}{4!}\int d^4z\Big(-i\frac{\partial}{\partial J(z)}\Big)^4\Big)\mathcal{Z}_0[J]\Big|_{J=0}}\,, \tag{1.166}$$

up to which order we are interested in. Say we wanted to calculate the four-point Greens function, i.e. find the Feynman rule for the four point vertex. Following the discussion above we expect it to be $-i\lambda$[23], and we will show that this is indeed the case.

In order to do the calculation we only need to calculate the numerator as the denonminator follows by just setting $J = 0$. We expand to $\mathcal{O}(\lambda)$, giving

$$\mathcal{Z}_{\text{num}}[J] = \Big(1 - i\frac{\lambda}{4!}\int d^4z\Big(-i\frac{\partial}{\partial J(z)}\Big)^4\Big)\mathcal{Z}_0[J] + \mathcal{O}(\lambda^2)\,, \tag{1.167}$$

and if we use the abbreviation

$$\mathcal{Z}_0[J] = \exp\Big(-\frac{1}{2}\int d^4x\, d^4y\, J(x)D_F(x,y)J(y)\Big) = e^{-\frac{1}{2}J_xD_{xy}J_y}\,, \tag{1.168}$$

we find after a page or two of calculation that

$$\mathcal{Z}_{\text{num}}[J] = \Big(1 - i\frac{\lambda}{4!}\int d^4z\big[3D_{zz}^2 - 6D_{zz}J_xD_{xz}J_yD_{yz} + J_xD_{xz}J_yD_{yz}J_xD_{xz}J_yD_{yz}\big]\Big)\mathcal{Z}_0[J] \tag{1.169}$$

where it immediately follows that the denominator is

$$\mathcal{Z}_{\text{den}}[J=0] = 1 - i\frac{\lambda}{4!}\int d^4z\, 3D_{zz}^2\,. \tag{1.170}$$

[22] We will see how this term comes about when we discuss gauge theories, but for now we take it as a given. The dynamics of the gauge fields are also neglected.

[23] We have to symmetrize bosons, so for four bosons we get a factor of 4! canceling the denominator. We will see that this falls out of this procedure.

If we were to perform further functional derivatives on $\mathcal{Z}[J]$ at this point, it would lead to a lot of work because of the denominator. Instead, we can use that the connected Green's functions is found by taking the logarithm of the generating functional, i.e.

$$i\mathcal{W}[J] = \ln \mathcal{Z}[J] = \ln \mathcal{Z}_{\text{num}}[J] - \ln \mathcal{Z}_{\text{den}}[J]\,, \tag{1.171}$$

and since we always assume small coupling, we can use the approximation $\ln(1+\lambda) = \lambda$. Then it follows that the generating functional for connected Green's functions to $\mathcal{O}(\lambda)$ is given by

$$i\mathcal{W}[J] = -\frac{1}{2}J_x D_{xy} J_y - i\frac{\lambda}{4!}\int d^4z \big(J_x D_{xz} J_y D_{yz} J_x D_{xz} J_y D_{yz} - 6 D_{zz} J_x D_{xz} J_y D_{yz}\big)\,, \tag{1.172}$$

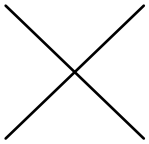
and to find the four point function we have to calculate

$$\mathcal{G}_c^{(4)}(x_1,x_2,x_3,x_4) = \frac{\partial^4 i\mathcal{W}[J]}{\partial J(x_1)\partial J(x_2)\partial J(x_3)\partial J(x_4)}\Big|_{J=0}\,. \tag{1.173}$$

However, instead of actually performing the differentiation we can see from Eq. (1.172) that the only term surviving all the functional derivatives are the one with four sources. This results in 4! terms as the first defivative has *four* J's to act on, the second has *three* and so on. The final result is

$$\mathcal{G}_c^{(4)}(x_1,x_2,x_3,x_4) = -i\lambda\int d^4z\, D_F(x_1-z)D_F(x_2-z)D_F(x_3-z)D_F(x_4-z)\,, \tag{1.174}$$

showing that the four point vertex rule is



$$= -i\lambda\,. \tag{1.175}$$

This confirms that the Feynman rules for vertices can in general be read of the generating functional of the theory, or the Lagrangian if one remembers to multiply with an i coming from the exponential of the action.

Greens functions are fundamental and important objects in any quantum field theory, but they are difficult to work with in practical calculations. Therefore, in the next section we will derive an expression that relates Green's functions with simpler quantities that are easier to work with when we want to calculate physical processes.

1.4.2 LSZ-Reduction Formula and Cross Section

The derivations so far have only concerned n-point field correlators or Green's functions as we have called them. However, by themselves, they are not observable quantities and in some instances not even gauge invariant. Therefore, we need to relate them to observable quantities, and the most common approach is to connect them to the S-matrix via the LSZ-reduction formula.

To find the formal definition of the S-matrix, let us consider the *interaction picture* formalism. To define it we will go to the Hamiltonian formulation and split the Hamiltonian up in a free part and an interacting part $H = H_0 + H_{\text{int}}$. We did the same for the action in Eq. (1.155), but we did not specify what we meant by the splitting. The interaction picture is defined by requiring that the operators carry the time dependence defined by the free Hamiltonian, and the states carry the time dependence defined by the interacting Hamiltonian. This is a hybrid of the relation between the Heisenberg and Schödinger picture, and one can deduce that a state $|\psi,t\rangle_I$ in this picture obeys the following equation

$$i\frac{d}{dt}|\psi,t\rangle_I = (H_{\text{int}})_I|\psi,t\rangle_I\,, \tag{1.176}$$

and the operators $\mathcal{O}$ obey

$$\mathcal{O}_I(t) = \exp(i(H_0)_S t)\mathcal{O}_S\exp(-i(H_0)_S t)\,, \tag{1.177}$$

where the subscripts S and I denotes the Schrödinger and interaction picture respectively. We can write the time-evolution in the interaction picture in terms of a new time-evolution operator[24] U, where

$$|\psi, t\rangle = U(t, t_0)|\psi, t_0\rangle \,. \tag{1.178}$$

where the time evolution operator satisfies the following equation

$$i\frac{d}{dt}U(t, t_0) = H_{\text{int}}(t)U(t, t_0) \,, \tag{1.179}$$

which is a matrix equation, and can be solved by the use of *Chen iterated integrals*, see [9]. The solution is known as the *Dyson formula*

$$U(t, t_0) = \mathcal{T} \exp\left(-i \int_{t_0}^{t} dt' \, H_{\text{int}}(t')\right) , \tag{1.180}$$

where $\mathcal{T}$ is the time-ordering operator.

To calculate amplitudes, we assume that at asymptotic times, i.e. $t_i \to -\infty$ and $t_f \to +\infty$, the initial and final state are eigenstates of the free theory. Further, we assume that all the interactions happen in some finite time, which is true in real scattering experiments. There we have control of the initial state and place detectors far away from the interaction point such that the final state contains no interaction[25]. Without interactions at asymptotic times, we can define these states as on-shell one-particle states of given momenta, known as *asymptotic states*.

Let us then assume we prepare a state $|i\rangle$ at $t_i \to -\infty$, let it evolve and particles interact briefly, before they depart and go into a final state $|f\rangle$ at $t_f \to +\infty$. The transition amplitude for this process is defined as

$$\lim_{t_{i,f} \to \mp\infty} \langle f|U(t_f, t_i)|i\rangle \equiv \langle f|S|i\rangle \,, \tag{1.181}$$

where the S-matrix is simply the time-evolution operator in the interaction picture between these two extreme times, i.e.

$$S = U(\infty, -\infty) = \mathcal{T} \exp\left(-i \int_{-\infty}^{\infty} dt H_{\text{int}}(t)\right) , \tag{1.182}$$

or in terms of the Hamiltonian density[26]

$$S = \mathcal{T} \exp\left(-i \int d^4x \, \mathcal{H}_{\text{int}}(x)\right) . \tag{1.183}$$

The S-matrix has the following structure: if the particles in question do not interact at all, S is simply the identity. There is also a probability that even for interactions, the particles just fly by one another. Thus, to isolate the interesting part of the S-matrix, we define the transition matrix as the shift from the identity

$$S = 1 + iT \,. \tag{1.184}$$

Let us consider an $m \to n$ process, where the states are created at $t = \pm\infty$ by using creation operators defined at asymptotic times. We can write this construction as

$$|i\rangle = \prod_{i=1}^{m} \sqrt{2E_{p_i}} \, a^\dagger_{p_i}(-\infty)|\Omega\rangle \,, \tag{1.185}$$

$$|f\rangle = \prod_{j=1}^{n} \sqrt{2E_{k_i}} \, a^\dagger_{k_j}(\infty)|\Omega\rangle \,, \tag{1.186}$$

[24] We will drop the interaction subscript I from now.

[25] Well, in field theory a particle is never 'alone' without interaction. For example, an electron is surrounded by a cloud of virtual photons, but this will be taken into account with renormalization.

[26] We generally use the Lagrangian in field theory, but the Lagrangian is just a Legendre transformation of the Hamiltonian and since the interaction contains no derivative we simply have $\mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}}$.

where $\sqrt{2E_{p_i}}$ is a conventional normalization factor and the creation and annihilation operators are time-independent at $t = \pm\infty$. We are only interested in the case where scattering actually happens, so we assume $|i\rangle \neq |f\rangle$, in which case we have that

$$\langle f|iT|i\rangle = \langle\Omega| \prod_{j=1}^{n} \sqrt{2E_{k_j}}\, a_{k_j}(\infty) \prod_{i=1}^{m} \sqrt{2E_{p_i}}\, a^\dagger_{p_i}(-\infty)|\Omega\rangle\,, \tag{1.187}$$

where the 1 cancel since the in and out states are different. This expression is already time-ordered, so we can instead write

$$\langle f|iT|i\rangle = N\,\langle\Omega|\mathcal{T}\big(a_{k_1}(\infty)\dots a_{k_n}(\infty)a^\dagger_{p_1}(-\infty)\dots a^\dagger_{p_m}(-\infty)\big)|\Omega\rangle\,, \tag{1.188}$$

where N is the collection of all normalization factors.

We want to relate this S-matrix element to Green's functions, so we need to find a way of rewriting the time-ordered product of creation and annihilation operators. To do this, we remember that a free real scalar field can be expanded as

$$\phi_0(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p(t)\, e^{-ip\cdot x} + a^\dagger_p(t)\, e^{ip\cdot x}\right), \tag{1.189}$$

where we have used Heisenberg picture creation and annihilation operators, which are equal to the time-independent Schrödinger operators for a fixed time t. We can invert this expression to yield the following relations[27]

$$i \int d^3x\, e^{ip\cdot x} \overset{\leftrightarrow}{\partial_t}\, \phi_0(x) = \sqrt{2E_p}\, a_p(t) \tag{1.190}$$

$$-i \int d^3x\, e^{-ip\cdot x} \overset{\leftrightarrow}{\partial_t}\, \phi_0(x) = \sqrt{2E_p}\, a^\dagger_p(t)\,. \tag{1.191}$$

which can be shown to hold by inserting for ϕ_0.

We have used the notation where ϕ_0 are free fields, however, these relations does not hold for interacting fields as they can not be simply expanded in terms of creation and annihilation operators. Nevertheless, we expect that at $t \to \pm\infty$ the theory reduces to a free theory, that is, since all incoming particles are infinitely far apart and, if the interaction decreases sufficiently fast with distance, there will be no difference between a free and an interacting theory[28]. We formalise this with the following hypothesis for the incoming part

$$\lim_{t\to-\infty} \phi(x) \to \sqrt{Z}\phi_{\text{in}}(x)\,, \tag{1.192}$$

where $\phi_{\text{in}}(x)$ is a free field and Z is known as the field renormalization. We will discuss the meaning of Z when we discuss renormalization in Sec. 1.5, but for now we ignore its wider interpretation. Similarly, for the outgoing part we have

$$\lim_{t\to\infty} \phi(x) \to \sqrt{Z}\phi_{\text{out}}(x)\,, \tag{1.193}$$

with $\phi_{\text{out}}(x)$ again a free field, and the same constant Z. These are not operator relations, but understood to hold inside matrix elements. We can then use Eq. (1.191) to define

$$\lim_{t\to-\infty} -iZ^{-1/2} \int d^3x\, e^{-ip\cdot x} \overset{\leftrightarrow}{\partial_t}\, \phi(x) = \sqrt{2E_p}\, a^\dagger_p(-\infty)\,, \tag{1.194}$$

$$\lim_{t\to\infty} -iZ^{-1/2} \int d^3x\, e^{-ip\cdot x} \overset{\leftrightarrow}{\partial_t}\, \phi(x) = \sqrt{2E_p}\, a^\dagger_p(+\infty)\,. \tag{1.195}$$

We want this on covariant form, so we can use that for any integrable function $f(x) = f(t, \mathbf{x})$, we have the identity

$$\Big(\lim_{t\to\infty} - \lim_{t\to-\infty}\Big) \int d^3x\, f(t,\mathbf{x}) = \int_{-\infty}^{\infty} dt\, \partial_t \int d^3x\, f(t,\mathbf{x})\,, \tag{1.196}$$

[27] The double arrow notation just means that it works on functions to the left and right.

[28] It is important to note that this is only possible for particles that are not bound.

giving that

$$\sqrt{2E_p}\big(a_p^\dagger(+\infty)-a_p^\dagger(-\infty)\big) = -iZ^{-1/2}\int d^4x\,\partial_t(e^{-ip\cdot x}\overset{\leftrightarrow}{\partial_t}\phi(x)) \\ = -iZ^{-1/2}\int d^4x\big(e^{-ip\cdot x}\partial_t^2\phi(x)-\phi(x)(\nabla^2-m^2)e^{-ip\cdot x}\big)\,, \tag{1.197}$$

where we have used that $p^2=m^2$, since we are considering on-shell particles. It is understood here that the asymptotic states are states with definite momenta, i.e. they are plane waves that form wave packets, so at each given time they are localized in space. We can then use partial integration on ∇^2, resulting in the algebraic relation

$$-iZ^{-1/2}\int d^4x\,e^{-ip\cdot x}(\partial^2+m^2)\phi(x) = \sqrt{2E_p}\big(a_p^\dagger(\infty)-a_p^\dagger(-\infty)\big)\,, \tag{1.198}$$

and its hermitian conjugate

$$iZ^{-1/2}\int d^4x\,e^{ip\cdot x}(\partial^2+m^2)\phi(x) = \sqrt{2E_p}\big(a_p(\infty)-a_p(-\infty)\big)\,. \tag{1.199}$$

To use these relations we exploit that the time ordering operator in Eq. (1.188) shuffles the operators inside the product regardless if they commute or not. Hence, we might as well just write

$$\langle f|S|i\rangle = N\,\langle\Omega|\mathcal{T}\big([a_{k_1}(\infty)-a_{k_1}(-\infty)]\dots[a_{k_n}(\infty)-a_{k_n}(-\infty)] \\ \times[a_{p_1}^\dagger(\infty)-a_{p_1}^\dagger(-\infty)]\dots[a_{p_m}^\dagger(\infty)-a_{p_m}^\dagger(-\infty)]\big)|\Omega\rangle\,, \tag{1.200}$$

where all the unwanted operators $a_{p,k}^\dagger(\infty)$ and $a_{p,k}(-\infty)$ migrates inside the time ordering and annihilates, so in effect we have just added zeros to Eq. (1.188).

Then, we can use Eq. (1.198) and Eq. (1.199) to write

$$\langle k_1\dots k_n|iT|p_1\dots p_m\rangle = (iZ^{-1/2})^{n+m}\Big[\int\prod_{i=1}^m d^4x_i\prod_{j=1}^n d^4y_j\,\exp\big(i\sum_{j=1}^n k_j\cdot y_j - i\sum_{i=1}^m p_i\cdot x_i\big)\Big] \\ \times(\partial_{x_1}^2+m^2)\dots(\partial_{y_n}^2+m^2)\langle\Omega|T\big(\phi(x_1)\dots\phi(y_n)\big)|\Omega\rangle\,. \tag{1.201}$$

where we have pulled the factors ∂^2 outside the time-ordering product. By pulling this factor out, there are additional additional terms appearing that in general has to be taken into account, but these can be shown not to contribute to the S-matrix [10].

We can rewrite this expression further by using that the n-point Green's $\mathcal{G}$ function can be written in terms of its Fourier transform $\tilde{\}$ as

$$\langle\Omega|\mathcal{T}\big(\phi(x_1)\dots\phi(x_n)\big)|\Omega\rangle = \mathcal{G}^{(n)}(x_1,\dots,x_n) = \int\prod_i^n\frac{d^4p_i}{(2\pi)^4}\,\exp\left(-i\sum_i p_i\cdot x_i\right)\tilde{\mathcal{G}}(p_1\dots,p_n)\,, \tag{1.202}$$

and if we act with the differential, we get

$$(\partial_{x_j}^2+m^2)\mathcal{G}^{(n)}(x_1,\dots,x_n) = -\int\prod_i^n\frac{d^4p_i}{(2\pi)^4}(p_j^2-m^2)\,\exp\left(-i\sum_i p_i\cdot x_i\right)\tilde{\mathcal{G}}(p_1\dots,p_n)\,. \tag{1.203}$$

Then, we can write Eq. (1.201) as

$$\Big(\prod_{i=1}^m\frac{i\sqrt{Z}}{p_i^2-m^2+i\epsilon}\Big)\Big(\prod_{j=1}^n\frac{i\sqrt{Z}}{k_j^2-m^2+i\epsilon}\Big)\langle k_1\dots k_n|iT|p_1\dots p_m\rangle \\ = \int\prod_{i=1}^m d^4x_i\,e^{-i\sum_{i=1}^m p_i\cdot x_i}\prod_{j=1}^n\int d^4y_j\,e^{i\sum_{i=1}^n k_j\cdot y_j}\,\mathcal{G}^{(m+n)}(x_1,\dots,x_m;y_1\dots,y_n)\,, \tag{1.204}$$

which is known as the LSZ reduction formula, named after Lehmann, Symanzik and Zimmermann [11]. The equality in this equation is understood in the sense that the right hand side and the left hand side has an equal pole structure, that will eventually cancel each other.

In words, the LSZ formula says that an S-matrix element can be calculated by computing the appropriate Fourier transformed Green's function. We know that such a calculation will give propagators that are of the form $i/(p^2 - m^2 + i\epsilon)$[29], which develop poles as the particles go on-shell. These poles will then effectively cancel with the factors on the left-hand side of Eq. (1.204), and we remain with a finite result.

Four-point function and Feynman rules in ϕ^4

Let us work through an example to get a feeling for how this works. We have already calculated the connected four-point Green's function for ϕ^4-theory in Sec. 1.4.1. We found that the $\mathcal{O}(\lambda)$ expansion was given by

$$\mathcal{G}_c^{(4)}(x_1, x_2; y_1, y_2) = -i\lambda \int d^4x\, D_F(x_1 - x) D_F(x_2 - x) D_F(y_1 - x) D_F(y_2 - x)\,. \tag{1.205}$$

By inserting this into the right-hand side of the LSZ formula, we find

$$\begin{aligned} &\int d^4x\, d^4x_1\, d^4x_2\, d^4y_1\, d^4y_2\, e^{i(p_1\cdot x_1 + p_2\cdot x_2 - k_1\cdot y_1 - k_2\cdot y_2)} \\ &\quad \times (-i\lambda)\, D_F(x_1 - x) D_F(x_2 - x) D_F(y_1 - x) D_F(y_2 - x)\,. \end{aligned} \tag{1.206}$$

If we change variable $z_i = x_i - x$ and $z_i' = y_i - x$, we get

$$\begin{aligned} &\int d^4x\, e^{i(p_1 + p_2 - k_1 - k_2)\cdot x} \Big[\int d^4z_1\, d^4z_2\, d^4z_1'\, d^4z_2'\, e^{i(p_1\cdot z_1 + p_2\cdot z_2 - k_1\cdot z_1' - k_2\cdot z_2')} \\ &\times (-i\lambda)\, D_F(z_1) D_F(z_2) D_F(z_1') D_F(z_2') \Big]\,. \\ &= (-i\lambda)(2\pi)^4 \delta^{(4)}(p_1 + p_2 - k_1 - k_2)\, D_F(p_1) D_F(p_2) D_F(k_1) D_F(k_2) \\ &= (-i\lambda)(2\pi)^4 \delta^{(4)}(p_1 + p_2 - k_1 - k_2) \Big(\prod_{i=1}^{2} \frac{i}{p_i^2 - m^2 + i\epsilon} \Big) \Big(\prod_{j=1}^{2} \frac{i}{k_j^2 - m^2 + i\epsilon} \Big)\,. \end{aligned} \tag{1.207}$$

To leading order in λ we do not need any information from renormalization, i.e. we can set $Z = 1$. Hence, we get from the LSZ formula that the propagators cancel, and we are left with the finite S-matrix element

$$\langle k_1 k_2 | iT | p_1 p_2 \rangle = (-i\lambda)(2\pi)^4 \delta^{(4)}(p_1 + p_2 - k_1 - k_2)\,. \tag{1.208}$$

The delta function reflects momentum conservation and the factor $(-i\lambda)$ is the transition amplitude.

This is a general structure in scattering amplitudes, which can be written as

$$\langle f | iT | i \rangle = (2\pi)^4 \delta^{(4)}\Big(\sum p_i - \sum p_f \Big)\, i\mathcal{M}\,, \tag{1.209}$$

where the amplitude $\mathcal{M}$ is a shorthand for $\langle f | \mathcal{M} | i \rangle$. From this definition of the matrix element one can construct an observable known as the *cross section*. The cross section is a measure of the statistical importance of a certain interaction, which we often just refer to as the probability of that interaction. Probabilities in quantum field theory are found by taking the matrix elements squared, so one has to evaluate $|\langle f | iT | i \rangle|^2$. A detailed derivation of the cross section formula can be found in, e.g. [2, 3, 10], so we will just state the final result.

For a $2 \to n$ process it takes the form

$$d\sigma = \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}\, d\mathcal{P}^{(n)}\, |\mathcal{M}|^2\,, \tag{1.210}$$

where the first factor is known as the Lorentz invariant *flux factor* and is dependent on the incoming particles four momenta. The measure $d\mathcal{P}^{(n)}$ is the Lorentz invariant n-body phase space, and represents the on-shell condition for the final state particles

$$d\mathcal{P}^{(n)} = \Big(\prod_i^n \frac{d^3k_i}{(2\pi)^3} \frac{1}{2E_{k_i}} \Big) (2\pi)^4 \delta^{(4)}(p_1 + p_2 - \sum_i^n k_i)\,, \tag{1.211}$$

[29] For scalar propagators.

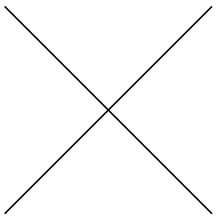


Figure 1.1: Four point function in ϕ^4-theory.

where the on-shell condition is reflected through the identity

$$\int \frac{d^3k}{(2\pi)^3}\frac{1}{2E_k} = \int \frac{d^4k}{(2\pi)^4}(2\pi)\delta^+(k^2-m^2)\,, \tag{1.212}$$

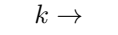
where

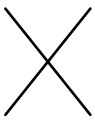
$$\delta^+(k^2-m^2) \equiv \delta(k^2-m^2)\,\theta(k^0)\,, \tag{1.213}$$

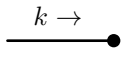
and $\theta(k^0)$ is the Heaviside function.

For practical calculations of scattering amplitudes it is easiest to calculate the amplitude $\mathcal{M}$ via Feynman diagrams. In fact, $\mathcal{M}$ is the sum of all Feynman diagrams, so in practice we will often draw the diagram and apply the relevant Feynman rules instead of going through the machinery of calculating S-matrix elements via Green's functions. For example, the Feynman diagram for the amplitude in Eq. (1.208) is given in Fig. 1.1. We can view this diagram as the leading order scattering $\phi(p_1)\phi(p_2) \to \phi(p_3)\phi(p_4)$, where p_1 and p_2 are the incoming momenta and p_3 and p_4 are the outgoing momenta.

Let us summarize the procedure how to go about calculating amplitudes by using Feynman rules. For ϕ^4-theory we have the following momentum space Feynman rules:

For each propagator:  $= \dfrac{i}{k^2-m^2+i\epsilon}\,,$ (1.214)

For each vertex:  $= -i\lambda\,,$ (1.215)

For each external line:  $= 1\,,$ (1.216)

Integrate over each undetermined loop momenta: $\displaystyle\int \frac{d^4k}{(2\pi)^4}\,,$ (1.217)

Impose momentum conservation at each vertex and divide by symmetry factor . (1.218)

The Feynman rules for more complex theories are of course different, but the procedure is still the same. For example, in a theory like QED the vertex rule is given in Eq. (1.164), the fermion propagator in Eq. (1.41), the photon propagator in Eq. (1.153) and the fermion external states are given by the spinor relations in Eq. (1.43) and Eq. (1.45)[30].

[30]Without the exponentials that combine into a momentum conserving delta function.

In this section we have managed to relate Green's functions with scattering amplitudes, but we have only considered the simplest case of a leading order expansion. However, the idea of perturbation theory is to look at higher and higher-order in the expansion series, which causes several problems we have to tackle.

1.5 Regularization and Renormalization

In this section we will go through the basic ideas of regularization and renormalization. The point is not to go through all the technical details of calculations, but to recognize where the divergences appear and possible ways of managing them. The main focus is based on the region of phase space where the momentum of particles appearing in so-called loops go to infinity. This is known as the *ultraviolett* (UV) region. We also have regions where massless particles have a momentum that goes to zero, which creates additional divergences, known as the *infrared* (IR) region. We will spend a large amount of time tackling these IR-divergences in Chapter 3 and Chapter 4, so we will briefly discuss them here, and focus on UV-divergences.

In Sec. 1.3, we formulated our quantum theory as a path integral. This path integral is an integral in configuration space weighted by the classical action. Hence, it might not be clear at first what separates a quantum field from a classical field, but because of quantum fluctuations, a particle in quantum field theory is never 'alone'. For example, an electron is constantly surrounded by a cloud of virtual photons and virtual electron-positron pairs. When considering higher-order perturbative expansion in the coupling, these fluctuations are visualized as closed loops that have to be integrated over. Loop integration requires appropriate regularization, because it generates unphysical divergences. These regularized divergences have to be treated, and this procedure is what is called renormalization. This has a deep impact on what we actually perceive as physical parameters. There are two ways of dealing with renormalization. The first one is known as Wilsonian renormalization, while the second is known as perturbative renormalization. We will not cover the Wilsonian renormalization procedure, but instead focus on perturbative renormalization and the continous renormalization group equation. To keep the discussion as simple as possible, we will stick to scalar ϕ^4-theory.

Loops and Divergences

Before we can talk about the general ideas and systematics of renormalization, we have to encounter an example of a divergence. We continue with the simple model of ϕ^4-theory and look at the second-order perturbative expansion of the four-point function, i.e. to $\mathcal{O}(\lambda^2)$. Using the Feynman rules for ϕ^4-theory, we have that the one-loop amplitude for the process $\phi(p_1)\phi(p_2) \to \phi(p_3)\phi(p_4)$, can be written as[31]

$$i\mathcal{M} = \frac{(-i\lambda_0)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m_0^2 + i\epsilon} \frac{i}{(p_1 + p_2 - k)^2 - m_0^2 + i\epsilon}\,, \tag{1.219}$$

where the factor of $1/2$ is the symmetry factor due to bosonic nature of ϕ, and the integration over k is from the undetermined loop momenta. We observe that this integral will diverge for large values of k, and illustrates the UV-divergence. In order to tackle such divergences, we need to regularize the integral. There are several different options, and the most popular is called *dimensional regularization*[32], but for now we will only use a momentum cut-off to show the basic idea. To proceed with the analysis, we will set $p_1 + p_2 = 0$ for the incoming momenta, which is just a statement of k dominating in the UV-region. A more detailed treatment of this integral would be to use what are called *Feynman parameters*, enabling one to rewrite the denominator in a more convenient form. We will cover Feynman parameters when we talk about dimensional regularization in Sec. 1.5.1, so we proceed with our simplification.

The amplitude can then be written as

$$i\mathcal{M} = \frac{\lambda_0^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m_0^2 + i\epsilon)^2}\,. \tag{1.220}$$

Because of the Minkowski signature we use the $i\epsilon$ prescription to shift the poles in propagators. The integral above can be evaluated using Cauchy's contour integral formula, but an easier way is to use what is called *Wick*

[31] This is only the s-channel contribution, but we are only interested in encountering a divergence so we will not bother with the other channels for now. Also note that we have for later convenience chosen a subscript on the parameters, which we will comment on later.

[32] We will discuss this method in more detail below.

rotation. The $i\epsilon$ prescription tells us that in the k^0-plane, the pole at $k^0 > 0$ is shifted below the real axis and the pole at $k^0 < 0$ is shifted above the real axis. Therefore we can change the integration path in the complex k^0-plane, by rotating counterclockwise from the real axis to the imaginary axis[33]. The effect of this rotation is that we change from an Minkowskian signature to an Euclidean signature, i.e. we can define the Euclidean four momentum variable $k_E = (ik_E^0, \mathbf{k}_E)$, giving $k_E^2 = -k^2$.

With these manipulations we obtain the amplitude

$$i\mathcal{M} = i\frac{\lambda_0^2}{2}\int \frac{d^4k_E}{(2\pi)^4}\frac{1}{(k_E^2+m_0^2)^2}\,. \tag{1.221}$$

For our purposes the exact computation of this integral is not necessary. The point is that the integral diverges in the UV region. We impose a momentum cut-off Λ and write

$$\int \frac{d^4k_E}{(2\pi)^4}\frac{1}{(k_E^2+m_0^2)^2} = \frac{1}{(2\pi)^4}\int^{\Lambda}\frac{d^4k_E}{k_E^4} + \text{finite} \tag{1.222}$$

$$= \frac{1}{8\pi^2}\ln\Lambda + \text{finite}\,. \tag{1.223}$$

At the end of each regulated calculation we would like to remove the regulator. In this case we used a momentum cut-off, so to make it independent of the cut-off we would like to send Λ to infinity. In this large limit we neglect the constant term in Eq. (1.222) and write the amplitude up to one-loop order as

$$i\mathcal{M}_{2\to 2} = -i\lambda_0 + i\frac{\lambda_0^2}{16\pi^2}\ln\Lambda + \text{finite}\,, \tag{1.224}$$

where it is clear that this amplitude is logarithmically divergent for $\Lambda \to \infty$. This is in contrast to what we should expect for higher-order perturbative expansions. We should expect corrections to become smaller and smaller at higher-order, so there is something fundamentally wrong with our starting point.

But before we proceed with a more systematic treatment, we will introduce dimensional regularization as a more useful way of regularizing the integral.

1.5.1 Dimensional Regularization and Feynman Paramerization

While the idea of integrating out momenta up to a cut-off Λ is very intuitive, in more complicated scenarios it becomes very cumbersome to perform. More seriously, in a gauge theory, simply imposing a cut-off does not preserve gauge invariance. That is not to say that momentum cut-off is entirely useless, as it gives a clear indication to the origin of the divergence[34]. We should, however, use a method that preserves all the symmetries of the theory. Probably the most used and arguably most useful is that of dimensional regularization. There are other choices that are valid for certain theories; for example, Pauli-Villars regularization works for Abelian gauge theories, but not for non-Abelian gauge theories. Dimensional regularization preserves all the symmetries, so we will mostly stick with this method. However, the disadvantage is that it is not as transparent in which momentum region the divergence originate.

Whether an integral diverges or not is largely determined by power counting, e.g. we have that the integral

$$\int^{\Lambda}\frac{d^4k}{k^4} \sim \ln\Lambda\,, \tag{1.225}$$

is logarithmically divergent. We observe that if the physical dimension were less than four, this integral would no longer diverge in the UV-region. That suggests we can regulate an integral by computing in some generic number of dimensions d[35]. Since the dimension is generic, the integral is not divergent until we specify the physical dimension. In order to 'approach' the physical dimension, we have to analytically continue the result of our d-dimensional theory through a non-integer value of d. Hence, we introduce a regulator ϵ and define that $d = 4 - 2\epsilon$[36], where ϵ is in general complex. In momentum cut-off, we send the regulator to infinity at the end of a calculation, while in dimensional regularization the divergence will manifest itself for $\epsilon \to 0$.

[33] We have to rotate counterclockwise to avoid crossing the poles.
[34] Meaning if the divergence is in the UV or IR region of momentum space.
[35] This argument is kind of heuristic, but using the Wilsonian renormalization procedure one can deduce that this is indeed true.
[36] We should not confuse this ϵ with the Feynman prescription $i\epsilon$ of shifting the poles in propagators.

A we have 'changed' the space-time dimensions, we have to re-evaluate the mass dimension of fields and couplings in the Lagrangian. In order to keep the action dimensionless, we must have that the Lagrangian has the mass dimension

$$\Big[\int d^d x \mathcal{L}\Big] = m^0 \implies [\mathcal{L}] = m^d \,, \tag{1.226}$$

i.e. the mass dimension of fields changes as well. E.g. for a scalar field we must have

$$[(\partial_\mu \phi)^2] = m^d \implies [\phi] = m^{1-\epsilon/2} \,, \tag{1.227}$$

leading to a dimensionful coupling

$$[\lambda] = m^{(4-d)/2} \,. \tag{1.228}$$

In order to keep the coupling dimensionless, we can replace

$$\lambda \to \mu^{(4-d)/2} \lambda(\mu) \,, \tag{1.229}$$

where $\lambda(\mu)$ is dimensionless, and μ is some arbitrary energy scale. It is important to stress that μ is not a cut-off; it is merely a scale we introduce, allowing us to use dimensionless couplings. This scale is most often set at the typical scale of the experiment we are interested in explaining and is referred to as the *renormalization scale*. In practice, this means that instead of calculating with the coupling that appears in the Lagrangian, we will replace it with $\lambda\mu^\epsilon$. Just to be clear, observables should not depend on this scale, which is a useful property we will exploit later on.

To explain the method of dimensional regularization, let us consider the simplest Feynman integral (after Wick rotation)

$$\int \frac{d^d k_E}{(2\pi)^d} \frac{1}{k_E^2 + m^2} = \frac{1}{(2\pi)^d} \int d\Omega_d \int_0^\infty dk_E \frac{k_E^{d-1}}{k_E^2 + m^2} \,. \tag{1.230}$$

This integral is UV-divergent for $d \geq 2$ and IR-divergent for $d \leq 2$. In this way we can treat them both by considering the different cases separately. For example, one would first treat the UV-divergence by considering $\epsilon > 0$ and then the IR-divergence by setting $\epsilon < 0$. We will encounter this scenario later, but for now we will only consider UV-divergences.

To compute the area integral of the unit sphere in d-dimensions we can use the following trick

$$\begin{aligned}
(\sqrt{\pi})^d &= \int d^d x \, \exp\left(-\sum_{i=1}^d x_i^2\right) \\
&= \int d\Omega_d \int_0^\infty dx \, x^{d-1} e^{-x^2} \\
&= \frac{\Gamma(d/2)}{2} \int d\Omega_d \,,
\end{aligned} \tag{1.231}$$

where we used the integral representation of the gamma function

$$\Gamma(z) = \int_0^\infty dx \, x^{z-1} e^{-x} \,, \tag{1.232}$$

giving

$$\Omega_d = \int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \,. \tag{1.233}$$

The remaining momentum integral can be evaluated as

$$\begin{aligned}
\int_0^\infty dk_E \frac{k_E^{d-1}}{k_E^2 + m^2} &= \frac{1}{2} (m^2)^{d/2-1} \int_0^\infty dx \, x^{d/2-1} (1-x)^{-d/2} \\
&= \frac{1}{2} (m^2)^{d/2-1} \, \Gamma(d/2) \Gamma(1 - d/2) \,,
\end{aligned} \tag{1.234}$$

where we made the substitution $x = k_E^2/m^2$ and used the integral representation of the Euler-Beta function

$$B(\alpha,\beta) = \int_0^\infty dx\, x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,. \tag{1.235}$$

The final result for the d-dimensional integral is

$$\int \frac{d^dk_E}{(2\pi)^d}\frac{1}{k_E^2+m^2} = \frac{(m^2)^{d/2-1}}{(4\pi)^{d/2}}\Gamma(1-d/2)\,, \tag{1.236}$$

where the divergence is manifest in $d = 4$ inside the gamma function. The generalized result takes the form

$$\int \frac{d^dk_E}{(2\pi)^d}\frac{1}{(k_E^2+m^2)^n} = \frac{(m^2)^{d/2-n}}{(4\pi)^{d/2}}\frac{\Gamma(n-d/2)}{\Gamma(n)}\,, \tag{1.237}$$

Another useful integral is

$$\int \frac{d^dk_E}{(2\pi)^d}\frac{k_E^2}{(k_E^2+m^2)^n} = \frac{(m^2)^{d/2-n+1}}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma(n-d/2-1)}{\Gamma(n)}\,. \tag{1.238}$$

When dealing with massless external particles, we will often meet integrals that are scaleless. An example is the integral $\int \frac{d^4k}{k^4}$, which is both IR and UV-divergent. However, since there are no available non-zero mass dimension, the integral $\int \frac{d^4k}{k^4}$ must vanish in d-dimensions. We can exploit this to extract the UV and IR divergences, i.e.

$$0 = \int \frac{d^dk}{(2\pi)^d}\frac{1}{k^4} = i\frac{\Omega_d}{(2\pi)^d}\Big(\frac{1}{\epsilon_{\rm UV}} - \frac{1}{\epsilon_{\rm IR}}\Big) = \frac{i}{8\pi^2}\Big(\frac{1}{\epsilon_{\rm UV}} - \frac{1}{\epsilon_{\rm IR}}\Big)\,, \tag{1.239}$$

which we can write as

$$\Big[\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^4}\Big]_{\rm UV} = \frac{i}{8\pi^2}\frac{1}{\epsilon_{\rm UV}}\,, \tag{1.240}$$

$$\Big[\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^4}\Big]_{\rm IR} = -\frac{i}{8\pi^2}\frac{1}{\epsilon_{\rm IR}}\,. \tag{1.241}$$

It is important to point out that $\epsilon_{\rm UV}$ and $\epsilon_{\rm IR}$ are not different, we just give them different names to be able to extract different parts. This is a very important feature which can be used to extract IR-divergences from UV-divergences. This can be useful in very complicated processes where the IR parts of a process is cumbersome to evaluate.

Feynaman parameters

In subsequent chapters we will mostly make calculations involving fermions and vector bosons, which leads to a more complex structure in the integrand. The relevant part for us here is that the denominator will consist of products of different propagators. By using *Feynman's parametric integral formula*, these can also be brought onto the form of Gamma functions. For two different propagators A^a and B^b we have the identity

$$\begin{aligned}\frac{1}{A^aB^b} &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^1 dx\frac{x^{a-1}(1-x)^{b-1}}{(Ax+(1-x)B)^{a+b}}\\ &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^1 dxdy\,\delta(x+y-1)\frac{x^{a-1}y^{b-1}}{(Ax+yB)^{a+b}}\,,\end{aligned} \tag{1.242}$$

where x and y are known as the Feynman parameters. The generalization to arbitrary many propagators is straightforward, and the proof can be made by induction. We will not do that proof here, so we just state the result

$$\frac{1}{A_1^{a_1}\dots A_n^{a_n}} = \frac{\Gamma(a_1+\dots+a_n)}{\Gamma(a_1)\cdots\Gamma(a_1)}\int_0^1 dx_1\dots dx_n\,\frac{\delta(1-x_1+\dots+x_n)\,x_1^{a_1-1}\cdots x_n^{a_n-1}}{(x_1A_1+\dots+x_nA_n)^{a_1+\dots+a_n}}\,, \tag{1.243}$$

which is valid for integer vaules of a_i, but with analytic continuation also holds for complex values of a_i.

To see how this parametrization works in practice we will consider an example we will have use for when we consider gauge boson radiation in Chapter 3. The diagram producing these propagators is given in the bottom diagram of Fig. 3.5. We have that k is the loop momentum, p and p' is the momentum of the incoming particles. We assume massless particles, so the propagator factors can be written as

$$\frac{1}{[k^2+i\epsilon][(p'+k)^2+i\epsilon][(k-p)^2+i\epsilon]} = \int_0^1 dx\, dy\, dz\, \delta(x+y+z-1)\frac{2}{\mathcal{D}^3}\,, \tag{1.244}$$

where the factor of 2 follows from $\Gamma(3) = 2$. The denominator is given by

$$\begin{aligned}\mathcal{D} &= zk^2 + x(p'+k)^2 + y(k-p)^2 + (x+y+z)i\epsilon \\ &= k^2 + 2k\cdot(xp' - yp) + i\epsilon\,,\end{aligned} \tag{1.245}$$

where we have used that $x+y+z = 1$ and that the particles are massless, i.e. $p^2 = p'^2 = 0$. Let us complete the square by shifting k in the following way

$$l = k + xp' - yp\,, \tag{1.246}$$

such that we get

$$\mathcal{D} = l^2 + xyQ^2 + i\epsilon\,, \tag{1.247}$$

where $Q^2 = 2p\cdot p'$, and let us define $\Delta = -xyQ^2$. We observe that the integrand is independent of z so we can use the delta function, giving

$$\frac{1}{[k^2+i\epsilon][(p'+k)^2+i\epsilon][(k-p)^2+i\epsilon]} = \int_0^1 dx \int_0^{1-x} dy \frac{2}{(l^2-\Delta+i\epsilon)^3}\,. \tag{1.248}$$

This is now on a form where we can perform Wick rotation and evaluate the momentum integrals, with the downside of later having to perform integrals over Feynman parameters.

Note that we only considered factors of propagators, but when we calculate with fermions and gauge bosons, we will have a complex tensor structure in the numerator. The shift to complete the square must also be performed in the numerator.

1.5.2 Renormalized Perturbation Theory

At this point we have seen the apperance of UV-divergences and how to regularize them using both momentum cutoff and dimensional regularization. In this section we will make the procedure of removing the divergence in a more systematic way.

The first important point to make is that the amplitude in Eq. (1.224) depends on the parameters m_0 and λ_0 appearing in the parametrization of the Lagrangian. We have interpreted these as the mass and coupling of the theory. These certainly describe 'mass' and 'coupling' terms, but there is no reason to believe that the physical mass and coupling is equal to the ones we write down in a theoretical construction. In real life experiments, observables are all order processes, meaning that there is no perturbative expansion. However, we have parametrized our theory at the lowest order and want it to cover an all order process, which does not sound like the right thing to do. We call these parameters without corrections 'bare' parameters.

The best way to see our failing intuition is to look at the concept of electric charge. We interpret the electric charge e as the strength of a charged particles interaction with the electromagnetic field. In quantum field theory, we know that charged particles interact electromagnetically by exchanging a photon. Hence, we can interpret the electric charge as a measure of how 'easily' a photon propagates. Because of quantum fluctuations, a propagating photon can spontaneously transform into an electron-positron pair, which subsequently annihilates and form a photon again. This process repeats itself all the time, which certainly affect the ease with which the photon propagates. The consequence is that the charge of a particle depends on how 'closely' we probe the strength of electromagnetism, i.e. the more energy we use more and more virtual particles appear and the 'harder' it is for the photon to propagate. Thus the quantity e we write down in a Lagrangian to describe the strength of the electromagnetic interaction, can not describe the physical charge we observe in experiments.

So, instead of using finite bare parameters, we allow these bare parameters to be infinite, such that they become finite after including all order corrections. This is the key concept of renormalization. A natural question that arises is the validity of defining an infinite bare charge. From a physical standpoint, this is perfectly fine as the bare charge describes the charge of an electron without the quantum fluctuations, which is not physically possible for a real-life electron constantly dressed with virtual particles.

The next question that arises is how many infinite constants we need in order to render a theory as renormalizable. In other words, is it sufficient to use lower order loops to define these divergences or do we get 'new' divergences at each new order. For a renormalizable theory, there is a finite number of these infinite constants, while a theory that is non-renormalizable has an infinite number of them. That does not mean a non-renormalizable theory is meaningless to study, just that it can be challenging to make precise predictions with them.

The general rule for a theory to be renormalizable follows from dimensional analysis. We know that because the action is dimensionless, the Lagrangian in a four-dimensional theory must have a mass dimension $[\mathcal{L}] = m^4$. One can show that for a theory to be renormalizable the coupling must either be dimensionless or have positive mass dimension, and theories with negative mass dimension are non-renormalizable.

Field Renormalization and Counterterms

Let us write down the bare Lagrangian in ϕ^4-theory, where the subscript on the parameters refer to the bare quantities

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_0)^2 - \frac{1}{2}m_0^2 - \frac{\lambda_0}{4!}\phi_0^4 \,. \tag{1.249}$$

This theory has a dimensionless coupling, so the theory is renormalizable, and we need to specify three infinite constants that will be absorbed by three unobservable parameters: the bare mass m_0, the bare coupling λ_0 and the field renormalization constant Z. First, we should explain where Z comes from and why we included it in the LSZ formula.

We know that if we act with a field $\phi_0(x)$ on a free vacuum state $|0\rangle$, we create a superposition of one-particle eigenstates of the free Hamiltonian. Let us then look at the following expectation value $\langle 0|\phi(x)|p\rangle$,

$$\begin{aligned} \langle 0|\phi_0(x)|p\rangle &= \langle 0|e^{iP\cdot x}\phi(0)e^{-iP\cdot x}|p\rangle \\ &= \langle 0|\phi_0(0)|p\rangle e^{-ip\cdot x}|_{p^0=E_p} \\ &= e^{-ip\cdot x}|_{p^0=E_p} \,, \end{aligned} \tag{1.250}$$

where we have used the translation operator Eq. (3.49) and we have that $\langle 0|\phi(0)|p\rangle = 1$. This means that the probability of creating or annihilating a one-particle state from the free vacuum is one, i.e. $|\langle p|\phi(0)|0\rangle|^2 = |\langle 0|\phi(0)|p\rangle|^2 = 1$.

We want to see how this property transfer to the interacting theory. Because of interactions, we have that multiple particles can be created. Hence, we have to take into account for the possibility of multiparticle states. Let us denote $|n_p\rangle$ as a multiparticle state, where p is the momentum, and let us do the same calculation as we did for the free theory using the interacting vacuum. We have that the expecation value is

$$\langle \Omega|\phi_0(x)|n_p\rangle = \langle \Omega|\phi(0)|n_p\rangle e^{-ip\cdot x}|_{p^0=E_p} \,, \tag{1.251}$$

where it is not necessarily true that $\langle \Omega|\phi(0)|n_p\rangle$ is equal to one. To rewrite this a bit, we can use that under a scalar field transform under a Lorentz transformation as $U(\Lambda)\phi(0)U^{-1}(\Lambda) = \phi(0)$ and a momentum state as $U(\Lambda)|n_p\rangle = |n_0\rangle$[37]. Then we can write

$$\langle \Omega|\phi_0(x)|n_p\rangle = \langle \Omega|\phi_0(0)|n_0\rangle e^{-ip\cdot x}|_{p^0=E_p} \,. \tag{1.252}$$

The field renormalization constant is then defined as

$$Z = |\langle n_0|\phi_0(0)|\Omega\rangle|^2 \,, \tag{1.253}$$

[37] n_0 means that we have boosted from a state with momentum p to a frame where p is zero.

i.e. the probability for $\phi_0(0)$ to create a given state from the interacting vacuum. This explains the appearance of Z when we derived the LSZ-reduction formula, see Eq. (1.204). However, we have said that this constant is infinite, but again it is not an observable quantity so we will not bother with ambiguities in the interpretation.

As we want to describe scattering observables, we can eliminate Z from the LSZ formula Eq. (1.204) by rescaling the field

$$\phi_0(x) = Z^{1/2}\phi(x)\,, \tag{1.254}$$

which alters the form of the Lagrangian, giving

$$\mathcal{L} = \frac{1}{2}Z(\partial_\mu\phi)^2 - \frac{1}{2}m_0^2 Z\phi^2 - \frac{\lambda_0}{4!}Z^2\phi^4\,. \tag{1.255}$$

The bare mass and coupling still appear in the Lagrangian, but they can be eliminated by making the following definitions

$$Z = 1+\delta_Z\,, \qquad m_0^2 Z = m^2+\delta_m\,, \qquad \lambda_0 Z^2 = \lambda+\delta_\lambda \tag{1.256}$$

where m and λ are the physical mass and coupling measured in experiments, also called renormalized mass and coupling. Then the Lagrangian becomes

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \\ &+ \frac{1}{2}\delta_Z(\partial_\mu\phi)^2 - \frac{1}{2}\delta_m\phi^2 - \frac{\delta_\lambda}{4!}\phi^4\,,\end{aligned} \tag{1.257}$$

where we observe that the first line is identical to the original bare Lagrangian, with the difference that the parameters are the physical ones. The second line consists of what we call *counterterms*, i.e. they can be chosen in such a way that they cancel the divergences coming from loop calculations. These counterterms have their own Feynman rules, where $i(p^2\delta_Z - \delta_m)$ is the counterterm for propagators and $(-i\delta_\lambda)$ is the counterterm for the vertex (or four point function)[38]. We can find δ_m and δ_z by calculating corrections to the propagator and we can find δ_λ by calculating the one-loop vertex correction[39].

We have already argued that we exclude disconnected and bubble[40] diagrams as they can not describe propagation, i.e. we are looking at corrections to the connected Green's function

$$\mathcal{G}_c^{(2)} = \langle\Omega|T(\phi(x)\phi(y))|\Omega\rangle\,. \tag{1.258}$$

It turns out that we only need to compute the *1-particle irreducible* or 1PI diagrams. These are defined as diagrams that are still connected after any one internal line is cut, and we denote them as $i\Pi(p)$.

The reason we only need to consider 1PI diagrams is because we can write the full propagator as a geometric series

$$\begin{aligned}D(p) &= \frac{i}{p^2-m^2+i\epsilon} + \frac{i}{p^2-m^2+i\epsilon}\big(i\Pi(p)\big)\frac{i}{p^2-m^2+i\epsilon} \\ &\quad + \frac{i}{p^2-m^2+i\epsilon}\big(i\Pi(p)\big)\frac{i}{p^2-m^2+i\epsilon}\big(i\Pi(p)\big)\frac{i}{p^2-m^2+i\epsilon} + \cdots \\ &= \frac{i}{p^2-m^2+i\epsilon}\sum_{n=0}^{\infty}\Big[\frac{-\Pi(p)}{p^2-m^2+i\epsilon}\Big]^n \\ &= \frac{i}{p^2-m^2+\Pi(p)+i\epsilon}\,,\end{aligned} \tag{1.259}$$

meaning that all non-1PI contributions are part of the geometric series and need not be computed independently.

In order to find the counterterms we also have to specify renormalization conditions. From the above definition of the full propagator we define the physical mass as the pole of the propagator and it's residue to be one, i.e.

$$\Pi(p)|_{m^2=p^2} = 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}p^2}\Pi(p)|_{p^2=m^2} = 0\,. \tag{1.260}$$

[38] These are directly read of the Lagrangian, as we have seen is identical to computing Green's functions by functional derivatives.
[39] This is the same calculation as we made for $\phi(p_1)\phi(p_2) \to \phi(p_3)\phi(p_4)$ by setting the external momenta to zero.
[40] These are diagrams that close in on themselves without any external legs.

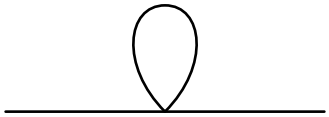


Figure 1.2: Tadpole diagram in scalar ϕ^4-theory.

The first condition specifies m^2 as the location of the pole and the second that the residue is equal to one. This last statement is equivalent to expanding the full propagator around $p^2 = m^2$ and read of the coefficient in front of the $(p^2 - m^2)^{-1}$ term, just like in a regular Laurent series.

The one-loop correction to the propagator (or two-point function) is given by the tadpole diagram Fig. 1.2, which has a single propagator in a loop that closes in on itself. Explicitly, by including the counterterm for propagators we have to one-loop order

$$\begin{aligned} i\Pi(p) &= \frac{i}{2}\lambda\mu^{(4-d)/2}\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2+i\epsilon} + i(p^2\delta_z - \delta_m) \\ &= \frac{i\lambda\, m^2}{2(4\pi)^{d/2}}\Big(\frac{\mu}{m}\Big)^{(4-d)/2}\Gamma(1-d/2) + i(p^2\delta_z - \delta_m) \end{aligned} \tag{1.261}$$

where we Wick rotated and used Eq. (1.237) to write the integral in terms of a gamma function. We can expand around $\epsilon = 0$, giving

$$i\Pi(p) = -i\frac{\lambda\, m^2}{2(4\pi)^2}\Big(\frac{1}{\epsilon} - \gamma_E + \ln 4\pi + \ln\Big(\frac{\mu^2}{m^2}\Big)\Big) + i(p^2\delta_z - \delta_m) + \mathcal{O}(\epsilon) \tag{1.262}$$

where γ_E is the Euler-Mascheroni constant. We observe that the first term does not depend on any momenta p^2, so we have that $\delta_z = 0$. There are different choices for choosing δ_m, but the most common and the one we will stick to in this thesis is the $\overline{MS}$-scheme

$$\delta_m = -\frac{\lambda\, m^2}{2(4\pi)^2}\Big(\frac{1}{\epsilon} - \gamma_E + \ln 4\pi\Big)\,, \tag{1.263}$$

meaning that together with the divergence $1/\epsilon$, we always subtract the finite numerical factor $-\gamma_E + \ln 4\pi$. It is easy to check that the results obtained for these counterterms obey the renormalization conditions in Eq. (1.260). This does not mean that $\delta_z = 0$ to all order, but we would have to calculate the $\mathcal{O}(\lambda^2)$ correction to the propagator to find the first non-zero contribution.

The δ_λ counterterm is found by considering the one-loop amplitude in Eq. (1.220), but using dimensional regularization. We write the full amplitude as

$$i\mathcal{M} = -i\lambda - i\lambda^2\,\Gamma(p) - i\delta_\lambda\,, \tag{1.264}$$

where p denotes the external particle momentum, i.e. in the general expression Eq. (1.219) we have the initial particle momentum $p = p_1 + p_2$, which we set to zero. This is a simplification and not the exact result, but it is sufficient for our purposes. By setting the external momenta to zero, we find the correction

$$\begin{aligned} \Gamma(0) &= \frac{3}{2}\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-m^2+i\epsilon)^2} \\ &= \frac{3}{2(4\pi)^2}\Big(\frac{\mu^2}{m^2}\Big)^{(4-d)/2}\Gamma(2-d/2)\,. \end{aligned} \tag{1.265}$$

The factor of 3 originates from that to this order we have three channels, so summing over all will give this factor. Again we can expand around $\epsilon = 0$, giving

$$\Gamma(0) = \frac{3}{32\pi^2}\Big(\frac{1}{\epsilon} - \gamma_E + \ln 4\pi + \ln\Big(\frac{\mu^2}{m^2}\Big)\Big) + \mathcal{O}(\epsilon)\,. \tag{1.266}$$

The counterterm in the $\overline{MS}$-scheme is then given by

$$\delta_\lambda = -\frac{3\lambda^2}{32\pi^2}\big(\frac{1}{\epsilon} - \gamma_E + \ln 4\pi\big)\,, \tag{1.267}$$

giving that to one-loop order we have the amplitude

$$i\mathcal{M} = -i\lambda - i\frac{3\lambda^2}{32\pi^2}\ln\Big(\frac{\mu^2}{m^2}\Big) + \mathcal{O}(\lambda^3)\,. \tag{1.268}$$

In general, after divergences have been subtracted using counterterms, the correction term will almost always have a logarithmic dependency. However, since μ is not a cut-off, this is not a problem as we can always choose it such that the logarithm has a nice behaviour.

1.5.3 The Callan-Symanzik Equation

We have looked at how to handle divergences by introducing a regulator, e.g. ϵ in dimensional regularization. These regulators are then removed by renormalizing the theory using counterterms, but in this procedure, an energy scale μ is introduced to compensate for the shift in dimension. After renormalizing the fields, all parameters can potentially depend on this scale. The renormalized n-point Green's function will then also depend on this scale through the parameters, but possibly also directly

$$\mathcal{G}^{(n)}(p_i, \lambda(\mu), m(\mu), \mu) = Z^{-n/2}(\mu)\mathcal{G}_0^{(n)}(p_i, \lambda_0, m_0)\,. \tag{1.269}$$

The choice of μ is arbitrary, so we might as well choose another scale μ' with renormalized parameters $\lambda'(\mu')$ and $m'(\mu')$. How the renormalized Green's function change under a continuous change of scale is then governed by a differential equation. To find this differential equation we can exploit that the bare Green's function is independent of this continous change of scale, i.e. we have that

$$\mu\frac{d}{d\mu}\mathcal{G}_0^{(n)} = 0\,, \tag{1.270}$$

which leads to

$$\begin{aligned} 0 &= \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\big(Z^{n/2}\mathcal{G}^{(n)}\big) \\ &= Z^{n/2}\Big(\mu\frac{\partial}{\partial\mu} + \mu\frac{\partial\lambda}{\partial\mu}\frac{\partial}{\partial\lambda} + \mu\frac{\partial m}{\partial\mu}\frac{\partial}{\partial m} + \frac{n}{2}\frac{\mu}{Z}\frac{\partial Z}{\partial\mu}\Big)\mathcal{G}^{(n)}\,. \end{aligned} \tag{1.271}$$

Then let us define the following functions

$$\gamma_\phi = \frac{\mu}{2Z}\frac{\partial Z}{\partial\mu} \tag{1.272}$$

$$\gamma_m = \frac{\mu}{m}\frac{\partial m}{\partial\mu} \tag{1.273}$$

$$\beta(\lambda) = \mu\frac{\partial\lambda}{\partial\mu}\,, \tag{1.274}$$

where γ_ϕ and γ_m are known as anomalous dimensions and $\beta(\lambda)$ is known as the beta function. Using these definitions we find what is called a *Callan-Symanzik* equation

$$\Big(\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} + m\gamma_m\frac{\partial}{\partial m} + n\gamma_\phi\Big)\mathcal{G}^{(n)} = 0\,. \tag{1.275}$$

This equation states that there exists universal functions $\beta(\lambda)$, γ_m and γ_ϕ, related to the change in coupling, mass and field, that compensates for the change of scale μ. The generalization to theories with several fields, e.g. QED with fermions and photons or QCD with quarks and gluons is straightforward. The difference—apart from different masses and couplings—is that we have several fields, giving several different anomalous dimensions.

To find the beta function for ϕ^4-theory we can use that the amplitude in Eq. (1.268) should not depend on the choice of scale μ, giving to $\mathcal{O}(\lambda^2)$

$$\mu\frac{\mathrm{d}\lambda}{\mathrm{d}\mu} = \frac{3\lambda^2}{16\pi^2}\,, \tag{1.276}$$

which is the definition of the beta function, i.e. we find

$$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3)\,. \tag{1.277}$$

We observe that the sign of the beta function is positive, which mean that the coupling increases as the scale increases. This is most easily seen by solving Eq. (1.276),

$$\int_{\lambda(\mu_0)}^{\lambda(\mu)} \frac{d\lambda}{\lambda^2} = \frac{3}{16\pi^2}\int_{\mu_0}^{\mu}\frac{d\mu}{\mu}\,, \tag{1.278}$$

giving

$$\lambda(\mu) = \frac{\lambda(\mu_0)}{1 - \frac{3\lambda(\mu_0)}{16\pi^2}\ln\left(\frac{\mu}{\mu_0}\right)}\,, \tag{1.279}$$

where μ_0 is a reference scale. This gives us a new scale in the theory which is physical and is defined through the coupling.

One important observation of the solution in Eq. (1.279) is that if we expand the solution, we find that

$$\lambda(\mu) = \lambda(\mu_0) + \frac{3\lambda^2(\mu_0)}{16\pi^2}\ln\left(\frac{\mu}{\mu_0}\right) + \mathcal{O}(\lambda^3)\,, \tag{1.280}$$

meaning that by using the renormalization group equation, the logarithmic dependence is effectively resummed. Resummation by the use of renormalization group equations is a feature we will use when we want to handle corrections that have large logarithms. Resummation of these is a way of organizing them such that they don't ruin the perturbative expansion. We will come back to resummation after we have considered corrections and divergences for cross sections in Chapter 3.

If the coupling varies with energy, it should raise a question about at which point is the coupling too large to be used in a power series expansion. For example, we can define the value of $\mu = \Lambda_L$ where the coupling is 1, i.e. the scale where perturbation theory breaks down. This scale is known as the Landau pole, hence the subscript L. For theories like QED, this Landau pole occurs at such a high energy that it is really not a problem. However, in a theory like QCD this pole occur at low enegies and invalidates the perturbative expansion. We will talk more about the coupling constant for QCD in Chapter 3.

Chapter 2

Geometry of Gauge Theories and Wilson Lines

The notion of gauge invariance was first presented by Hermann Weyl in 1918 when he tried to unify electromagnetism with gravity from his own purely infinitesimal geometry. This unification was met with scepticism by other physicists as it lead to unphysical results, and Weyl first abandoned the idea. The problem in Weyl's gauge theory was that he looked at transformations of the metric, but as Weyl himself and others pointed out a decade later, the transformation had to be performed on the fields. Based on this formalism Wolfgang Pauli presented the first widely recognized gauge theory in 1941 [12]. Pauli also tried to generalize this to higher dimensional internal spaces, but could not find a way of giving mass to the gauge fields, so he figured this was a dead end and did not publish any of his results. An almost complete generalization of these concepts came about in 1954 by C.N.Yang and R.L.Mills, called Yang-Mills theories [13]. Yang and Mills encountered the same problems as Pauli related to the mass of the gauge fields, but they figured their idea was so important they published it either way[41]. The problem of gauge boson masses were solved after the introduction of the *Higgs mechanism*. Hence, Yang-Mills theories is the basis of one of the most successfull theories in all of physics, namely the *Standard Model* of particle physics.

In this chapter we will not take the same approach as Yang and Mills, where they generalized $U(1)$ invariance to $SU(N)$. We will instead derive Yang-Mills theories from completely geometrical concepts. To this end, we will introduce the mathematical language known as *fibre bundle theory*. The theory of fibre bundles is purely mathematical and is intriguing in its own sense, but it also plays a prominent role in modern theoretical physics. Fundamental theories, like General Relativity and the Standard Model, are gauge field theories and the theory of fibre bundles provides a natural mathematical framework for these theories. The framework provides a clear separation between the kinematics and the dynamics of the theory. The kinematics is provided by the structure of the base manifold—in physics this represents spacetime—and the dynamics by the identification of a Lagrangian. In the case of the Standard Model, the internal symmetries of the Lagrangian, is made by a local construction of a fibre bundle with the fibre being the symmetry group G. From this construction the notion of sections, connection and curvature can be defined on the bundle, which represents physical fields in spacetime. After we have introduced the basic concepts and the structure that follows from fibre bundle theory, we will make use of them by constructing Wilson lines and Wilson loops. By using Wilson lines and Wilson loops we can construct the Yang-Mills Lagrangian.

There is of course much more to be said than we cover here, so we refer the reader to [14–17] for a more elaborate treatment on differential geometry, fibre bundle theory and Wilson lines.

2.1 Mathematical Concepts in Gauge Theories

Before introducing fibre bundle theory and its relation with gauge theories, it is instructive to review some basic concepts from differential geometry and group theory as manifolds, tangent spaces, connections, curvature, Lie groups and Lie algebras. The objective here is not to dive into all the fundamental details, but a brief

[41] According to Yang, Pauli was furious when Yang presented his and Mills ideas and could not explain why the gauge bosons were massless.

introduction to the most important concepts. We should mention that in order to follow the definition of the Wilson line and Wilson loop in Sec. 2.2, the mathematical details of fibre bundle theory and differential geometry is strictly not needed. So the reader can jump straight to Sec. 2.2 and start from there. We will refer to the equations that we explicitly use, which can be sought out if necessary.

2.1.1 Basics in Differential Geometry

In this section we will briefly cover the basic concepts of a manifold, tangent space, contangent space, differential forms, Lie groups and Lie algebras, curvature, covariant derivative and parallel transport. As previously mentioned, the details are not at the level of mathematical rigour. We aim to introduce the simplest explanations and nomenclature that we will use throughout the chapter.

Manifold

The concept of Manifolds is central in almost all theories of modern physics, from general relativity to quantum field theories, so we will have to define what a manifold is:

Definition 2.1.1: Manifold

Suppose M is a topological space, then M is a topological manifold if it has the following properties:

- M is a Hausdorff space: For every pair of points $x, y \in M$, there are disjoint open subsets U,V $\in M$ such that $x \in U$ and $y \in V$
- M is second countable: There exists a countable basis for the topology of M
- M is locally Euclidean of dimension n: Every point has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$

In general the manifold might have a complicated global structure, but the defining property is to be homeomorphic to $\mathbb{R}^n$. The homeomorphism ϕ_i from M to an open subset U_i of $\mathbb{R}^n$ is called a chart

$$\phi_i : M \to U_i \in \mathbb{R}^n \,, \tag{2.1}$$

which permits us to assign co-ordinates to the manifold using those from U_i. Because a manifold in general globally differs from $\mathbb{R}^n$, we need to provide sets of charts (U_i, ϕ), called an open covering, such that all of M is covered. Transitions between charts is described by smooth transition functions defined by

$$\phi_i \circ \phi_j^{-1} : U_i \to U_j \,, \tag{2.2}$$

which may be denoted as ϕ_{ij}. Finally one needs to define some properties of the transition functions in case there is a overlap of the charts, so

$$\phi_{ii} = \mathrm{id}_{U_i} \tag{2.3}$$
$$\phi_{ij} = \phi_{ji}^{-1} \tag{2.4}$$
$$\phi_{ik} = \phi_{ij} \circ \phi_{jk} \,. \tag{2.5}$$

The above description may seem a little abstract, so what we basically mean and what is most relevant for our purposes is: a manifold M is a set that can be continously parametrized. The number of independent parameters needed to specify uniquely any point of M is its dimension n, and these parameters $x = \{x^1, \ldots, x^n\}$ are called co-ordinates. Manifolds are then a generalization of the familiar space $\mathbb{R}^n$, in the sense that they can be viewed as smooth surfaces which locally look like $\mathbb{R}^n$, but in general has a completely different global structure. We demand the manifold to be smooth, i.e the transition from one set of co-ordinates to another $x^i = f(\tilde{x}^i, \ldots, \tilde{x}^n)$, is C^∞[42]. A simple example of a manifold is the surface S^2 of a sphere in $\mathbb{R}^3$. Even if the sphere is in $\mathbb{R}^3$, the surface is a two dimensional manifold, because it locally looks like $\mathbb{R}^2$.

[42] This means infinitely differentiable.

Tangent Spaces and Differential Forms

A common type of field is what is called the tangent vector field, which assigns to each point $x \in M$ a vector $v(x)$ tangent to M. These vectors can be used to describe what is meant by vectors 'moving' on the manifold. The space of all vectors tangent to $x \in M$ is a vector space, which is denoted as T_xM. An easy and visualizable example is again the surface of a sphere, S^2, where at every point on the surface there is a plane T_xS^2 that is tangent to S^2 at every point x.

For a coordinate x^μ defined on some neighborhood on the manifold, there is a tangent vector $e_\mu \in T_xM$ pointing in the direction of x^μ on the manifold and denotes 'travel' at unit speed. These vectors can be denoted by $e_\mu \equiv \partial_\mu$, from the correspondence with the differential operator. Thus any vector in the neighborhood can be expanded in this basis, so $v = v^\mu \partial_\mu$, and these vector fields are called co-ordinate basis fields for the neighborhood.

Furthermore, to every tangent space T_xM there is the notion of a dual space, called cotangent space T_x^*M. As it is dual to T_xM it consists of linear maps $dx : T_xM \to \mathbb{R}$, such that

$$dx^\mu(\partial_\nu) \equiv \delta^\mu_\nu \,, \tag{2.6}$$

and elements of the cotangent space T_x^*M are called covectors or one-forms.

In the same sense that a vector field is a function that at each point in the manifold assigns a vector $v(x)$ in the tangent space T_xM, a one-form is a map w which takes a point x on the manifold to an element $w(x)$ of the corresponding cotangent space T_x^*M. In order to define n-forms we must first define a wedge product. For two elements of a vector space $v, u \in V$, their wedge product is defined as

$$v \wedge u \equiv \frac{1}{2}(vu - uv) \,, \tag{2.7}$$

which can be generalized to a wedge product for n-vectors, called the nth exterior product of V and is a vector space denoted $\Lambda(V)$. A n-form is a map $w : M \to \Lambda^n(T^*M)$, which takes a point x on the manifold M to an element of $\Lambda^n(T^*M)$. We also have that a one-form can be expanded in the dual basis dx^μ of the cotangent space T_x^*M in the following way

$$w = w_\mu dx^\mu \,, \tag{2.8}$$

where w_μ is the local co-ordinate representation of a one-form. Similarly a two-form may be expanded as

$$\Omega = \Omega_{\mu\nu} dx^\mu \wedge dx^\nu \,, \tag{2.9}$$

and so on for higher forms. The set of all n-forms on a manifold M is often denoted by $\Omega^n(M)$.

Lie Groups and Lie Algebras

In this section a brief introduction of Lie groups an Lie algebras will be given. As all gauge theories involve invariance under some symmetry operation, it has a natural formulation in terms of groups. The groups of gauge theories have the structure of a manifold, and so the concept of Lie groups and Lie algebras are fundamental in gauge theories.

Let us begin by defining a Lie group:

Definition 2.1.2: Lie Group

A Lie group G is a group that is a finite-dimensional differentiable manifold, with the properties that the group operations are smooth. For group elements $g, g' \in G$ we specifically have that the product

$$\pi : G \times G \to G\,, \tag{2.10}$$
$$(g, g') \mapsto g \cdot g'\,, \tag{2.11}$$

and the inverse

$$\rho : G \to G\,, \tag{2.12}$$
$$g \mapsto g^{-1}\,, \tag{2.13}$$

are smooth maps.

Particularly important Lie groups in theoretical physics are the general linear groups $GL(n, V)$ of invertible $n \times n$ matrices. Some examples are the subgroups of $GL(n, V)$: the orthogonal group $O(n)$, the special orthogonal group $SO(n)$, the unitary group $U(n)$ and the special unitary group $SU(n)$. The special stands for the condition that the determinant of the matrices are one.

The concept of homomorphism is important in study of Lie groups, and in group theory in general. A homomorphism of Lie groups is given by a map $\rho : G \to H$ between elements of the groups, such that $\forall\ g,\ g' \in G$

$$\rho(g \cdot g') = \rho(g) \cdot \rho(g')\,. \tag{2.14}$$

As physicists we are interested in groups where the elements of the groups act on physical states, say a quantum field or a wavefunction. Hence, we want to 'represent' Lie group elements by linear transformations on some vector space V. Such a mapping is called a Lie group representation, and is a group representation where we have a homomorphism of Lie groups $\rho : G \to GL(V)$. To be more specific we are interested in the map

$$\rho : G \times V \to V\,, \tag{2.15}$$
$$(g, v) \mapsto \rho(g)v\,, \qquad g \in G, v \in V\,. \tag{2.16}$$

The physical states are members of a vector field, and we want our group members to act on them. With the definition of a representation this makes the transformation properties of the group to be written in terms of matrices.

Lie groups are complicated geometric objects and can be difficult to study directly, but the Lie algebra corresponding to a Lie group becomes important as it is closely related to the Lie group but easier to study.

Definition 2.1.3: Lie Algebra

A Lie algebra is a vector space with a skew-symmetric bilinear map $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the Lie bracket, satisfying the Jacobi identity for $X, Y, Z \in \mathfrak{g}$

$$[X, [Y, Z] + [Y, [Z, X]] + [Z, [X, Y]] = 0\,. \tag{2.17}$$

It is because of this underlying vector space structure that the Lie algebra is easier to study than the Lie group itself. The Lie algebra encodes most of the group structure of the entire Lie group and many of the topological properties. If G is a Lie group, then the Lie algebra $\mathfrak{g}$ of G is the tangent space of the identity element of G, denoted T_eG.

Since Lie groups have the structure of a manifold, we can consider vector fields on G. The vector fields of interest in connection with Lie algebras are left and right-invariant vector fields[43]. A vector field V is said to be left-invariant if $L_g^* V = V$, $\forall$ g $\in G$. The set of all left-invariant vector fields on a Lie group G is denoted $L(G)$, and for any two left-invariant vector fields their Lie bracket is also a left-invariant vector field. This means that $L(G)$ is isomorphic to the Lie algebra $\mathfrak{g}$, hence $L(G)$ can be considered to be the Lie algebra of G. So, for two

[43] We only cover left-invariant vector fields here.

left-invariant vector fields V, W, and $v, w \in T_eG$ we have that $V(e) = v$ and $W(e) = w$, therefore we can define the Lie bracket $[u, w] \in T_eG$ as the unique element in T_eG, such that

$$[v, w] \equiv [V, W](e)\,, \tag{2.18}$$

which turns T_eG into an algebra. It is then possible to define a homomorphism of Lie algebras as a linear map $\rho : \mathfrak{g} \to \mathfrak{h}$, such that

$$\rho([u, w]) = [\rho(u), \rho(w)] \qquad \forall\, u, v \in \mathfrak{g}\,. \tag{2.19}$$

One important property of Lie algebras is that if we have a basis set $\{M_1, M_2 \dots M_n\}$ for $L(G) \cong T_eG$, then the commutator of these fields must be equal to a linear combination of the fields, i.e we may write

$$[M_\alpha, M_\beta] = C^\gamma_{\alpha\beta} M_\gamma\,, \tag{2.20}$$

where $C^\gamma_{\alpha\beta}$ are real numbers. These numbers are called the structure constants of the Lie group, i.e. they characterize the structure of the group.

There is one important way of characterizing the Lie algebra $\mathfrak{g}$, via the exponential map. As we saw above the Lie algebra is the tangent space of the Lie group at the identity, thus one say that the Lie algebra gives a linearization of the Lie group near the identity. The exponential map can then be viewed as a delinearization, i.e. it take us back to the group. Thus we define:

Definition 2.1.4: Exponential map

The exponential map from the Lie algebra $\mathfrak{gl}(\mathfrak{n})$ to the general Lie group $GL(n)$ is defined by

$$\exp : \mathfrak{gl} \to GL\,, \tag{2.21}$$

where

$$\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!}\,. \tag{2.22}$$

We observe that this is just the definition of an exponential of a matrix X. For any subgroup of GL, the Lie algebra of that group can be mapped into the group from the exponential map, meaning that any group that can be written in term of matrices can be constructed from the algebra in this precise manner. As physicists we often want our group elements to act on complex vector spaces, thus in order to preserve the inner product on a Hilbert space these transformations must be unitary. It can be shown that for compact Lie groups it is always possible to choose the unitary representation. If the transformation is unitary it means that one multiplies the argument of the exponential with a factor i, and this forces the matrices X to hermitian.

Connection and Covariant Derivative

In physics one typically apply differential geometry by a setup where the underlying manifold represents all of spacetime, where fields on that manifold is used to describe physical quantities. In order to describe evolution of these physical quantities one needs a precise mathematical description of calculus on manifolds. In order to see why this is needed, let us look at the problems that arise when defining the derivative of a vector field. The naive definition is

$$\partial_\mu v^\nu = \lim_{dx^\mu \to 0} \frac{v^\nu(x + dx) - v^\nu(x)}{dx^\mu}\,, \tag{2.23}$$

which is not correct as the term $v^\nu(x+dx)$ and $v^\nu(x)$ live in different tangent spaces, meaning that the subtraction can not be made in a meaningful way. In order to properly define a derivative operator on a manifold, one needs to be able to compare tensors (fields) at different points. For this we need the concept of a linear connection.

Definition 2.1.5: Connection

A linear connection ∇ is defined as a map which sends a pair of smooth vector fields V, U to a new smooth vector field:

$$\nabla : V,U \mapsto \nabla_V U \,. \tag{2.24}$$

Satisfying the following requirements:

$$\nabla_V(U+W) = \nabla_V U + \nabla_V(W)\,, \tag{2.25}$$
$$\nabla_{fV+U} W = f\nabla_V(W) + \nabla_U(W)\,, \tag{2.26}$$
$$\nabla_V(f) = V(f)\,, \tag{2.27}$$
$$\nabla_V(fU) = f\nabla_V(U) + V(f)U\,, \tag{2.28}$$

where f is a function.

The object $\nabla_V U$ is named the covariant derivative of U with respect to V. From the last requirement, which is the Leibnitz rule, ∇ is not a tensor as it is not linear in U. However, as a map $\nabla U : V \mapsto \nabla_V U$, which is a linear map $T_xM \to T_xM$, ∇U is a $(1,1)$ tensor known as the covariant derivative of U.

In general we want to decompose vectors into components, so we choose a basis $\{e_\mu\}$, which is a basis in the tangent space. The conventional approach is to choose basis vectors that are tangential vectors along the coordinate lines x^μ in M, so

$$e_\mu = \partial_\mu \,. \tag{2.29}$$

As defined above the object ∇_{∂_μ} is a map taking ∂_μ to some vector field, so we define

$$\nabla_{\partial_\mu}\partial_\nu \equiv \nabla_\mu \partial_\nu \,. \tag{2.30}$$

This is now a new vector field, which can be expanded as a linear combination of the basis vectors

$$\nabla_\mu \partial_\nu = \Gamma^\sigma_{\nu\mu}\partial_\sigma \,, \tag{2.31}$$

where $\Gamma^\sigma_{\nu\mu}$ are connection coefficients, also known as Christoffel symbols[44]. It can be shown that they do not transform as a tensor, so therefore the indices does not describe the components of a tensor.

We can now use our definition of the covariant derivative to see how it acts on vector fields described in terms of their components. Thus we write two vector fields in component form as $v = v^\mu\partial_\mu$ and $u = u^\mu\partial_\mu$, and then define what the covariant derivative on components are

$$\nabla_v u = (\nabla_v u)^\mu \partial_\mu \equiv (u^\mu{}_{;\nu} v^\nu)\partial_\mu \,. \tag{2.32}$$

From the definition of the covariant derivative we can calculate the left hand side of Eq. (2.32),

$$\begin{aligned}(\nabla_v u)^\mu \partial_\mu &= \nabla_v(u^\mu\partial_\mu)\\ &= v(u^\mu\partial_\mu) + u^\mu(\nabla_v\partial_\mu)\\ &= v^\nu\partial_\nu u^\mu\partial_\mu + u^\mu(\nabla_{(v^\nu\partial_\nu)}\partial_\mu)\\ &= \partial_\nu u^\mu v^\nu\partial_\mu + u^\mu(v^\nu\nabla_\nu\partial_\mu)\\ &= \partial_\nu u^\mu v^\nu\partial_\mu + u^\mu(v^\nu\Gamma^\sigma_{\mu\nu}\partial_\sigma)\,,\end{aligned} \tag{2.33}$$

which mean we can write

$$(u^\mu{}_{;\nu}v^\nu)\partial_\mu = \partial_\nu u^\mu v^\nu\partial_\mu + u^\sigma(v^\nu\Gamma^\mu_{\sigma\nu}\partial_\mu)\,. \tag{2.34}$$

This is to hold for all v^ν and all ∂_μ. Thus, the covariant derivative on a vector field in a co-ordinate induced basis is given by

$$\nabla_\nu u^\mu \equiv u^\mu{}_{;\nu} = \partial_\nu u^\mu + \Gamma^\mu_{\nu\sigma}u^\sigma \,, \tag{2.35}$$

[44] Also known as components of the Levi-Civita connection.

i.e. if ∇ is to obey the Leibnitz rule, it can be written as the partial derivative plus some linear transformation, where this linear transformation describes the correction in order to make the derivative covariant. Hence, for each direction μ, the covariant ∇_μ will be given by the partial derivative ∂_μ plus a correction specified by a set of n matrices $(\Gamma_\mu)^\sigma{}_\nu$, where n is the dimension of the manifold.

Parallel Transport and Curvature

Parallel transport is the curved space generalization of keeping a vector (tensor) constant as we move it along a path[45]. The crucial difference between flat and curved spaces is that, in a curved space, the result of parallel transporting a vector from one point to another will depend on the path taken between the two points. In flat space, the requirement that a vector is constant as we move it along a curve $x^\mu(\lambda)$, is that the components are constant, and is expressed as

$$\frac{d}{d\lambda}v^\sigma = \frac{dx^\mu}{d\lambda}\partial_\mu v^\sigma = 0\,. \tag{2.36}$$

As we have shown, the partial derivative of a vector is not tensorial, and therefore the generalization is to use the covariant derivative and define the directional covariant derivative

$$\frac{D}{d\lambda} \equiv \frac{dx^\mu}{d\lambda}\nabla_\mu = n^\mu \nabla_\mu\,, \tag{2.37}$$

where n^μ follows from the parametrization $x^\mu = \lambda n^\mu$. This is now a map from (k,l) tensors to (k,l) tensors, defined only along the path. For a general tensor T, we define the parallel transport of T along $x^\mu(\lambda)$ to be the requirement that the directional covariant derivative of T is zero

$$n^\mu \nabla_\mu T^{\mu_1\cdots\mu_k}{}_{\nu_1\ldots\nu_k} = 0\,, \tag{2.38}$$

which in particular, for a vector field, gives that

$$\frac{dv^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}v^\beta = 0\,. \tag{2.39}$$

Hence, the requirement that the covariant derivative of a tensor in a direction which it is parallel transported is zero, gives that the covariant derivative of a tensor measures how much the tensor changes as it is parallel transported.

Given what we now know of covariant derivatives and parallel transportation, we can investigate what curvature is. Given the paths λ_1 and λ_2 with the same endpoint p, then parallel transporting along these two paths are in general not the same. Thus, a vector being parallel transported around a loop will be transformed, but the resulting transformation will depend on the total curvature around the loop. Therefore it would be more convenient to have a local description of the curvature at each point, and this is what the Riemann curvature tensor provides.

We can then consider parallel transportation around an infinitesimal loop, and as the manifold looks flat in sufficiently small regions, the loop will be specified by two infinitesimal vectors a^μ and b^ν. First we parallel transport a vector v^μ in the direction of a^μ, then along b^ν, before moving backwards along a^μ and b^ν, returning to the starting point. Since parallel transportation is independent of co-ordinates, there should be some tensor describing how much the vector changes when it comes back to its starting point. Instead of actually performing the calculation for the change of the vector as it is parallel transported, it is easier to look at the related operation of covariant derivatives. The commutator of two covariant derivatives compares the difference between parallel transporting a tensor along a^μ, then along y^ν, versus first along b^ν , then along a^μ. In other words, the curvature is the measure of the failure of covariant derivatives to commute. The result of this operation on a vector is given by

$$[\nabla_\mu, \nabla_\nu]v^\sigma = \big(\partial_\mu\Gamma^\sigma_{\nu\rho} - \partial_\nu\Gamma^\sigma_{\mu\rho} + \Gamma^\sigma_{\mu\alpha}\Gamma^\alpha_{\nu\rho} - \Gamma^\sigma_{\nu\alpha}\Gamma^\alpha_{\mu\rho}\big)v^\rho - 2\Gamma^\alpha_{[\mu\nu]}\nabla_\alpha v^\sigma\,, \tag{2.40}$$

where $\Gamma^\alpha_{[\mu\nu]} = \Gamma_{\mu\nu} - \Gamma_{\nu\mu} = 0$, as the connection coefficients are symmetric in the interchange of lower indices. The term inside the bracket of Eq. (2.40) is known as the Riemann curvature tensor,

$$R^\sigma_{\rho\mu\nu} = \partial_\mu\Gamma^\sigma_{\nu\rho} - \partial_\nu\Gamma^\sigma_{\mu\rho} + \Gamma^\sigma_{\mu\alpha}\Gamma^\alpha_{\nu\rho} - \Gamma^\sigma_{\nu\alpha}\Gamma^\alpha_{\mu\rho}\,. \tag{2.41}$$

[45] Or, as we shall see for fibre bundles a path in internal space.

In general relativity the Riemann curvature tensor is important as the curvature describes how objects move. If we look at the $_{\mu\nu}$ components of the Riemann tensor, it has a striking resemblance with the field strength tensor $F_{\mu\nu}$ in Yang-Mills theories. This is not an accident, but it took physicists a long time to understand that Yang-Mills theories and General Relativity can both be formulated in the same mathematical language of fibre bundles[46].

Up to this point we have restricted our discussion to vector and tensor fields, but in general we are interested in other types of fields as well, like for example spinor and scalar fields. Hence, we have to extend the notion of a connection such that we can do calculus with all types of fields, and in order to do this properly we use fibre bundle theory. Naturally this leads us down a path where several mathematical concepts must be introduced, but it will prove useful to see the strength of fibre bundle theory to naturally describe several concepts we use in quantum field theory. It is important to emphasize that the material we will cover on fibre bundles is by no means a full treatment, but we will try to cover what is most important for physics. Several of the definitions and statements made might seem obscure and unmotivated, which they sometimes are, but it would take too much space to write all there is about fibre bundles.

2.1.2 Gauge Fields as Connections on Principal Fibre Bundles

The defining property of a manifold M is to be locally homeomorphic to $\mathbb{R}^n$, but in general it differs from $\mathbb{R}^n$ globally. Hence, we needed sets of homeomorphisms, which we called charts, to locally map M to a open subset of $\mathbb{R}^n$. This gave a Euclidean structure to the manifold, which allowed us to use conventional multivariable calculus. A fiber bundle has the property of being locally disseomorphic to a direct product of topological spaces, thus we need disseomorphisms to define a local map. Let us jump straight into the definition of a fibre bundle, and then try to clarify some of the structure.

Definition 2.1.6: Fibre Bundle

A fibre bundle is a structure (E, π, M, F, G), often denoted $E \xrightarrow{\pi} M$, which consists of the following elements:

- A smooth manifold E called the **total space**
- A smooth manifold M called the **base space**, and in physics this is spacetime.
- A smooth manifold F called the **fibre**
- A surjective map $\pi : E \to M$ called the **projection**. The subset of elements $\{q\} \in E$ which are projected to a point $p \in M$ is called the fibre at p, given by the inverse image $\pi^{-1}(p) \equiv F_p \cong F$.
- A Lie group G, which acts on F from the left called the **structure group**
- A set of open covering $\{U_i\}$ of M with a diffeomorphism $\phi_i : U_i \times F \to \pi^{-1}(U_i)$, such that $\pi \circ \phi_i(p, f) = p$. As the inverse ϕ_i^{-1} maps $\pi^{-1}(U_i)$ to the direct product $U_i \times F$, ϕ_i is called a **local trivialization**
- A way to smoothly paste the direct products $\{U_i \times F\}$, such that we cover all of the total space E. As $\phi_i(p, f) : F \to F_p$ is a disseomorphism we introduce **transition functions** $t_{ij}(p) \equiv \phi_{i,p}^{-1} \circ \phi_{j,p} : F \to F$, which we require to be elements of G. Then ϕ_i and ϕ_j is related by a smooth map $t_{ij} : U_i \cup U_j \to G$ as:

$$\phi_j(p, f) = \phi_i(p, t_{ij}(p)f) \tag{2.42}$$

The requirement of a local trivialization comes from the fact that fibre bundles are in general extremely complex structures, so in order to use them in practical applications we need to restrict ourselves with simpler ones. The simplest case is what is called a trivial bundle, where E is isomorphic to the product $M \times F$. As it turns out we

[46] Have to emphasize that this is at the classical level. Both of these theories have to be quantized and to this day only Yang-Mills theories have given reliable physical results after quantization.

need to define a local isomorphism where $U \subset M$ such that $E|_U$ is locally isomorphic to $U \times F$, and the bundles of interest in gauge theories has this property.

Another way phrasing the last requiremtnt is: if we have a overlap between charts $U_i \cup U_j \neq \emptyset$ in the base space M, we have two maps ϕ_i and ϕ_j on the overlap. Given a point q such that $\pi(q) = p \in U_i \cup U_j$, we can assign two elements of F, one by $\phi_i^{-1}(q) = (p, f_i)$ and the other by $\phi_j^{-1}(u) = (p, f_j)$. Then there exists a map $t_{ij} : U_i \cup U_j \to G$ which relates f_i and f_j as $f_i = t_{ij} f_j$, and in order to glue the local pieces of the fibre bundle together consistently we need the transition functions to obey the following requirements

$$t_{ii} = \mathrm{id}\,, \tag{2.43}$$
$$t_{ij} = g_{ji}^{-1}\,, \tag{2.44}$$
$$t_{ik} = g_{ij} \circ g_{jk}\,. \tag{2.45}$$

For a given fibre bundle $E \xrightarrow{\pi} M$, an important feature is that the transition functions are not unique. So let $\{U_i\}$ be a covering of M and let $\{\phi_i\}$ and $\{\tilde{\phi}_i\}$ be two sets of local trivializations giving rise to the *same* fibre bundle. Then the transition functions are given by

$$t_{ij} = \phi_{i,p}^{-1} \circ \phi_{j,p}\,, \tag{2.46}$$
$$\tilde{t}_{ij} = \tilde{\phi}_{i,p}^{-1} \circ \tilde{\phi}_{j,p}\,, \tag{2.47}$$

and we have a homeomorphism $g_i : F \to F$ at each point $p \in M$ that belongs to G, we define it to be

$$g_i(p) \equiv \phi_{i,p}^{-1} \tilde{\phi}_{i,p}\,, \tag{2.48}$$

which must be the case if the local trivializations is to describe the same fibre bundle, but then we see that the relation between the two transition functions are

$$\tilde{t}_{ij}(p) = g_i(p)^{-1} \circ t_{ij}(p) \circ g_j(p)\,. \tag{2.49}$$

All of this might seem very abstract and without meaning, but let us clarify some points: t_{ij} can be viewed as gauge transformations for gluing patches together, and g_i can be viewed as gauge transformation within a certain patch. In physics, we will often meet the case when $U_i = U_j$, i.e. is the same set U, and we are just comparing two different ways of associating the fibre at $\pi^{-1}(U)$ to $U \times G$. In this case the local trivializations can be viewed as choices of gauges, and the transitions functions as gauge transformations. For example, when the base space is flat spacetime $M \cong \mathbb{R}^4$, and the symmetry group is $G \cong U(1)$. Then the transition functions at a spacetime point x is expressed as $e^{i\alpha(x)} \in U(1)$, where $\alpha(x)$ is a spacetime dependent group parameter. Another example is when we have curved spacetime and the group is $GL(n, \mathbb{R})$, which is the case for general relativity. A choice of local trivialization is then a choice of co-ordinate system over an open patch U, and the transition function is just general co-ordinate transformations.

Sections and the Pull-back

In order to define fields on a fiber bundle we need the concepts of *sections*:

Definition 2.1.7: Section

A section of E is a smooth map

$$s : M \to E\,,$$

that satisfies

$$\pi \circ s = \mathrm{id}_M\,,$$

where π is the projection in E.

The set of all sections on M is denoted $\Gamma(M, F)$. As usual we want the local description, so for $U \subset M$ we have a local section defined only on U, where the set off all local sections is naturally denoted as $\Gamma(U, F)$.

To define gauge fields in terms of sections we will need the *pull-back*:

Definition 2.1.8: Pull-back

Let $\phi : M \to N$ be a smooth map of manifolds and let $\phi(q) = p$. Then we let

$$\phi_* : T_pM \to T_qN\,, \tag{2.50}$$

be the differential of ϕ. The pull-back ϕ^* is the linear transformation taking covectors at q to covectors at p, $\phi^* : N^*(q) \to M^*(p)$, defined by

$$\phi^*(\beta)(v) \equiv \beta(\phi_*(v))\,, \tag{2.51}$$

for all covectors β at q and vectors v at p.

Vector Bundles

The fields in physics are objects in a vector space, so we need the concept of vector bundle to describe these fields in this formalism. A vector bundle $E \xrightarrow{\pi} M$ is a fibre bundle whose fibre F is a vector space. More loosely spoken, this means that we attach a vector space at each point p on the base manifold M. If $F = \mathbb{R}^n$, the transition functions belong to $GL(n, \mathbb{R})$, and if $F = \mathbb{C}^n$, they belong to $GL(n, \mathbb{C})$.

A prime example of a vector bundle is the tangent bundle TM, where the fiber is $\mathbb{R}^n$. Let $\{U_i\}$ be an open covering of M, and let q be a point in TM such that the projection satisfies $\pi(q) = p \in U_i \cup U_j$. If we define x^μ to be the local co-ordinate system of U_i and y^μ to be the local co-ordinate system of U_j, then the vector V corresponding to q can be expanded in two different ways

$$V = V^\mu \frac{\partial}{\partial x^\mu} = \tilde{V}^\mu \frac{\partial}{\partial y^\mu}\,, \tag{2.52}$$

where the local trivializations become

$$\phi_i^{-1} = (p, \{V^\mu\})\,, \tag{2.53}$$
$$\phi_j^{-1} = (p, \{\tilde{V}^\mu\})\,, \tag{2.54}$$

and the fibre coordinates are related by a general linear transformation

$$V^\mu = G^\mu{}_\nu \tilde{V}^\nu\,, \tag{2.55}$$

where $G^\mu{}_\nu$ is the transition function, found by performing a change of frame

$$V^\mu = \frac{\partial x^\mu}{\partial y^\nu}\tilde{V}^\mu = G^\mu_\nu(p)\tilde{V}^\nu\,, \tag{2.56}$$

where $G^\mu{}_\nu$ is an element of the structure group $GL(n, \mathbb{R})$. Hence, the tangent bundle can be identified by the structure $(TM, \pi, M, \mathbb{R}^n, GL(n, \mathbb{R}^n))$.

Sections on vector bundles pointwisely obey the usual vector multiplication and addition with scalars,

$$(s + s')(p) = s(p) + s'(p)\,, \tag{2.57}$$
$$(fs)(p) = f(p)s(p.)\,, \tag{2.58}$$

where $p \in M$ and $f \in F$. Any vector bundle admits a global section which is called the null section $s_0 \in \Gamma(M, E)$, that satisfies the property $\phi_i^{-1}(s_0(p)) = (p, 0)$ in any local trivialization. What we want in physical applications is to use that any field or wavefunction can be represented as a section of a vector bundle. For example, a $U(1)$ gauge group, a complex scalar field $\Psi(x)$ defined on $U \subset M$ is represented by a local section of a complex line bundle.

Principal Bundles and Gauge Transformations

In order to describe gauge transformations on fields, we need the concept of principal bundles. A principal bundle $P(M, G, \pi)$ is a fibre bundle where the fibre F is identical to the structure group G, also called a G-bundle over M. The action of the group G on F becomes simple left multiplication within G. It is also possible to construct a right multiplication of G on P as: let $\phi_i : U_i \times G \to \pi^{-1}(U_i)$ be a local trivialization given by

$$\phi_i^{-1}(q) = (p, g_i)\,. \tag{2.59}$$

Then the right multiplication is defined as

$$qa = \phi_i(p, g_i a)\,, \tag{2.60}$$

for any $a \in G$ and $q \in \pi^{-1}(p)$, with property $\pi(qa) = \pi(q) = p$. Because of the associativity of the group, this is true for any local trivialization. Let $p \in U_i \cup U_j$, then

$$qa = \phi_j(p, g_j a) = \phi_j(p, t_{ji}(p) g_i a) = \phi_i(p, g_i a)\,, \tag{2.61}$$

which mean one can just write the action as $P \times G \to P : (q, a) \mapsto qa$, without reference to any local choices. In other words, if q is a point in the fibre over p then acting with a group element a gives another point qa in the fibre over p, with the property $\pi(qa) = p$ so that both q and qa lie in the same copy of the fibre. This means that the group action enables us to move within each copy of the fibre, or equivalently for principal bundles each copy of G, but does not move you around in the base space M. Since the fibre is equal to the structure group, also called symmetry group in physics, principal bundles is central to the description of gauge theories, and the right action on P can be identified with gauge transformations.

Since a section on a principal bundle is a map $s : M \to P$, then the value of a section at a point p corresponds to an element of the structure group G through a local trivialization

$$s(p) = \phi_i(p, g_i)\,. \tag{2.62}$$

Unless the principal bundle is a direct product $M \times F$ we need local sections. Given a local section s_i on U_i and a $q \in \pi^{-1}(U_i)$, we can always find a $g_i \in G$ such that $q = s_i(p) g_i$. Then the section itself may be represented as a canonical local trivialization

$$s_i(p) = \phi_i(p, e)\,, \tag{2.63}$$

where e is the identity element of G. All other local sections can then be expressed in terms of these by the right action as

$$\tilde{s}_i(p) = \phi_i(p, g_i(p)) = \phi_i(p, e) g_i(p) = s_i(p) g_i(p)\,. \tag{2.64}$$

The different $\tilde{s}_i$ is viewed as different gauges, while $g_i(p)$ is the corresponding gauge transformations between them.

Associated bundles and Field Transformations

We have seen how sections on vector bundles can be used to describe fields, and how sections on principal bundles describe gauge transformations. We are now ready to see how we may associate these two concepts such that the gauge transformations act on the fields.

Given a principal bundle $P(M, G)$ and a faithful representaiton $\rho : G \to GL(n, V)$ which acts on a vector space V from the left. The group action on elements (q, v) in the product space $P \times_\rho V$ can then be defined as

$$(q, v) \to (qg, \rho(g)^{-1} v)\,, \tag{2.65}$$

where $q \in P$, $g \in G$ and $v \in V$. The associated vector bundle $E_\rho = E \times_\rho V$ is then defined by identifying the points related by such a group action, so

$$(q, v) \sim (qg, \rho(g)^{-1} v)\,, \tag{2.66}$$

which also implies that $(qg, v) = (q, \rho(g)v)$. This is the same as saying that we change the fiber from G to V and use transition functions $\rho(t_{ij})$ instead of t_{ij}. As every element of P over a point $p \in M$ can be found from (p, e) by an element of G, the equivalence relation replaces the fibre over p with V, and thus replacing a principal bundle with a vector bundle. We can introduce the projection $\pi_{E_\rho} : E_\rho \to M$, defined by acting on elements (q, v) as

$$\pi_{E_\rho}((q, v)) = \pi(q)\,. \tag{2.67}$$

Then E_ρ is a fibre bundle with the same structure group G as its associated principal bundle $P(M, G)$.

Let us consider the $U(1)$ group and a complex scalar matter field $\phi : M \to \mathbb{C}$. The relevant fibre bundles are the principal $U(1)$-bundle $P(M, U(1))$ and its associated vector bundle $E_\rho = P \times_\rho \mathbb{C}$. A local section on a principal bundle may be represented as a canonical local trivialization

$$s_i(p) = \phi_i(p, e)\,, \tag{2.68}$$

with e the identity element of $U(1)$. Within a trivializing neighborhood on a principal bundle we can define a local identity section $\sigma_i(p) \equiv \phi^{-1}(e)$, which mean that if we act with ϕ^{-1} on the left on the local trivialization we obtain

$$\phi_i^{-1}(s_i(p)) = (p, e)\,. \tag{2.69}$$

All other local sections may then be constructed by the right action of the group $\tilde{s}_i(p) = s_i(p)g_i(p)$, where $g_i \in U(1)$. To make $\tilde{s}_i$ act on $\phi(p)$ we use the representation $\rho : g_i(p) \to e^{i\alpha(p)}$ and define a base section on the associated vector bundle

$$\mathbf{e} = [(s_i(p), 1)] \qquad \in E_\rho\,, \tag{2.70}$$

where 1 is a basis vector in the complex line bundle. Naturally any other section may then be expressed in terms of these base sections, so we define our field as a section in the following way

$$\phi(p)\sigma_e = [(s_i(p), \phi(p))]\,, \tag{2.71}$$

where $\phi \in \mathbb{C}$, and a local gauge transformation corresponds to

$$\begin{aligned}
\phi^{'}(p)\sigma_e &= [(\tilde{s}_i(p), \phi(p))] \\
&= [(s_i(p)g_i(p), \phi(p))] \\
&= [(s_i(p), \rho(g_i)\phi(p))] \\
&= [(s_i(p), e^{i\alpha(p)}\phi(p))] \\
&= e^{i\alpha(p)}[(s_i(p), \phi(p)] \\
&= e^{i\alpha(p)}\phi(p)\sigma_e\,.
\end{aligned} \tag{2.72}$$

In physics we always choose local co-ordinates, so for x^μ we simply write this transformation as

$$\phi^{'}(x) = e^{i\alpha(x)}\phi(x)\,, \tag{2.73}$$

which is the well known form of the local $U(1)$ gauge transformation in quantum electrodynamics.

We have now seen how sections on associated bundles can be formulated as matter fields and we have shown how these are transformed under a local $U(1)$ gauge transformation. The next objective is then to investigate how one can construct gauge fields and their corresponding behaviour under gauge transformations in this language.

Connection and Gauge Fields on Principal Bundles

There are several ways of defining a connection on a principal bundle, but the approach we use here is to decompose tangent spaces into vertical and horizontal ones. In Eq. (2.35) we defined a linear connection to make the derivative of vector fields covariant, there in terms of the Levi-Civita connection. Here we will take another approach, where we instead define the connection in terms of sections, from which the gauge fields and the field strength tensor follows. This is approach is very abstract, but it is necessary in order to show that the

gauge fields has a geometrical basis. In Sec. 2.2.1 we will use the results derived here and take a more physical approach at the level of Lagrangians, by the use of Wilson lines.

First, we want to investigate how we can decompose the tangent space of a principal bundle, but to do that we must describe how we may construct a tangent vector in $P(M,G)$. A fundamental vector field $\mathbf{v}$ may be generated through an element A of a Lie algebra $\mathfrak{g}$ of the G-bundle $P(M,G)$, in the following way

$$\mathbf{v}f(q) = \frac{d}{dt} f(qe^{tA})|_{t=0}\,. \tag{2.74}$$

Now, since $e^{tA} \in G$ we have that the projection $\pi(qe^{tA}) = \pi(q) = p$. This means that qe^{tA} defines a curve that lies within the fibre at p, and thus $\mathbf{v}$ is tangent to the fibre at p at every point $q \in P$.

The tangent space T_qP at $q \in P$, can be decomposed into a horizontal and vertical subspaces in the following way

$$T_qP = V_qP \otimes H_qP\,, \tag{2.75}$$

such that every vector $\mathbf{x}$ in the tangent space may also be decomposed into vertical and horizontal components $\mathbf{x} = \mathbf{x}^V + \mathbf{x}^H$. With this decomposition we are ready to define what a connection in this language is:

Definition 2.1.9: Connection one-form

A connection on a Principal G-bundle $P(M,G)$ is a Lie algebra $\mathfrak{g}$ valued one-form $w \in \mathfrak{g} \otimes \Omega^1(P)$ that projects elements in T_qP onto $V_qP \cong \mathfrak{g}$, satisfying the following requirements

- $w(\mathbf{v}) = A \qquad A \in \mathfrak{g}$
- $R_g w = g^{-1} w g \qquad g \in G$

where R_g describes the right action of the group.

Given $U \subset M$ and local sections $s_i : U_i \to \pi^{-1}(U_i)$, a local connection $A_i \in \mathfrak{g} \otimes \Omega^1(U_i)$ is defined by the pull-back of the global connection one-form w,

$$A_i \equiv s_i^* w\,. \tag{2.76}$$

Local connections is what we in physics call a gauge potential. Instead of this abstract definition, in practice we use that since A_i is a Lie algebra valued one-form it can be expanded in a dual basis dx^μ and in terms of Lie algebra generators $t^a : U \to G$, in the following way

$$A_i = (A_i)_\mu dx^\mu = (A_i)^a_\mu t^a dx^\mu\,. \tag{2.77}$$

Given two local sections s_i and s_j over patches U_i and U_j[47], $X \in T_qM$ and $q \in U_i \cap U_j$, it can be shown that

$$s_{j\,*}X = R_{t_{ij}}(s_{i\,*}X) + \left(t_{ij}^{-1}\mathbf{d}t_{ij}\right), \tag{2.78}$$

where $t_{ij} \in G$ are the transition functions and $\mathbf{d}$ is the de-Rham-differential. If we use the connection w on this equation, with the relation $w(s_{j\,*}) = s_j^* w$ from Eq. (2.51), we have that the local connection transform as

$$A_j = t_{ij}^{-1} A_i t_{ij} + t_{ij}^{-1}\mathbf{d}t_{ij}\,, \tag{2.79}$$

which in the more familiar component form is written as

$$A'_\mu = g^{-1} A_\mu g + g^{-1}\partial_\mu g\,, \tag{2.80}$$

which is the transformation for the gauge fields in quantum field theory.

[47] With the requiremnt that $U_i \cap U_j \neq \emptyset$.

2.1.3 Curvature and Field Strength

In order to talk about curvature (or field strength) in this language, we have to define an exterior covariant derivative:

Definition 2.1.10: Exterior Covariant Derivative

The exterior covariant derivative of a general vector valued n-form $\Phi(x) \in \Omega^n \otimes V$, $x_1, \ldots, x_{n+1} \in T_qP$, is defined as:

$$\mathbf{d}_w \Phi(x_1, \ldots, x_{p+1}) \equiv \mathbf{d}\Phi(x_1^H, \ldots, x_{p+1}^H)\,, \tag{2.81}$$

where $\mathbf{d}\Phi = \mathbf{d}\Phi^\alpha \otimes e_\alpha$, and $x^H \in H_qP$.

The curvature is a Lie algebra valued two-form $\Omega \in \Omega^2 \otimes \mathfrak{g}$, and is defined as the exterior covariant derivative of the one-form connection $w \in \Omega^1 \otimes \mathfrak{g}$

$$\Omega \equiv \mathbf{d}_w w\,, \tag{2.82}$$

which for $x, y \in T_qP$ satisfies Cartan's structure equation

$$\Omega(x, y) = \mathbf{d}w(x, y) + [w(x), w(y)]\,, \tag{2.83}$$

where the bracket is a tensor product of the Lie bracket and the wedge product, i.e. the curvature can be written as

$$\Omega = \mathbf{d}w + w \wedge w\,. \tag{2.84}$$

Just as the local connection the local curvature is given by the pull-back of a local section

$$F_i \equiv s_i^* \Omega\,, \tag{2.85}$$

and by using Cartan's structure equation the local curvature may be written in terms of the local connection as

$$\begin{aligned} F_i &= s_i^*(\mathbf{d}w + w \wedge w)\,, \\ &= \mathbf{d}(s_i^* w) + s_i^* w \wedge s_i^* w\,, \\ &= \mathbf{d}A_i + A_i \wedge A_i\,. \end{aligned} \tag{2.86}$$

where it follows from the transformation of A_i that the transformation of F_i is given by

$$F_j = t_{ij}^{-1} F_i\, t_{ij}\,. \tag{2.87}$$

Let us then choose local co-ordinates x^μ on a patch $U \subset M$, and as we always consider local objects we neglect the subscript i. The connection one-form and curvature two-form can then be expanded as

$$A = A_\mu dx^\mu\,, \tag{2.88}$$

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu\,, \tag{2.89}$$

and the de-Rham differential takes the form

$$\mathbf{d}A = (\partial_\mu A_\nu - \partial_\nu A_\mu) dx^\mu \wedge dx^\nu\,, \tag{2.90}$$

giving the field strength in terms of components

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\,, \tag{2.91}$$

where the bracket is the usual Lie commutator. Since both the gauge field and the field strength are Lie algebra valued, they can be expanded in the the group generators t^a as well. Using the commutator relation between the generators $[t^a, t^b] = f^{abc} t^c$ the field strength takes the form

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c\,, \tag{2.92}$$

which is the well known field strength tensor for a Yang-Mills theory[48], and the transformation of the field strength in local co-ordinates is given by

$$F'_{\mu\nu} = g^{-1} F_{\mu\nu} g \,, \tag{2.93}$$

which we also recognize from gauge theories in quantum field theory.

Horizontal Lift and Parallel Transport Equation

In the previous section we found that the tangent space TP of the principal G-bundle $P(M,G)$ can be split into a vertical and horizontal part. This splitting allows us to define a *horizontal lift* of a curve in the base manifold M:

Definition 2.1.11: Horizontal lift

Let $P(M,G,\pi)$ be a principal G-bundle and

$$\gamma : [0,1] \to M \,,$$

a curve in M. Then a curve

$$\tilde{\gamma} : [0,1] \to P \,,$$

is said to be a horizontal lift of γ from the the base space up in the fibre if the tangent vector $\tilde{\gamma} \in H_{\tilde{\gamma}(t)}P$.

From this is follows that for a curve γ in M and a point p in the fibre, i.e. $p \in \pi^{-1}(\gamma(0))$, there exist a unique horizontal lift $\tilde{\gamma}(t)$ in P such that $\tilde{\gamma}(0) = p$. Further, it follows that another horizontal lift $\tilde{\gamma}'$ of γ can be found by applying the right group action, $\tilde{\gamma}'(0) = \tilde{\gamma}(0)g$. Then, we also have that

$$\tilde{\gamma}'(t) = \tilde{\gamma}(t)g \,, \qquad \forall t \in [0,1] \,, \tag{2.94}$$

which is saying that the global right action does not change the connection on the principal fibre.

Then let X be the tangent vector at $\gamma(0)$, and by using the horizontal lift we have that

$$\tilde{X} = \tilde{\gamma}_* X \,, \tag{2.95}$$

is tangent a tangent vector at $\tilde{\gamma}(0) = p$. By construction, this vector is horizontal and if we act with the connection w, we get

$$w(\tilde{X}) = 0 \,, \tag{2.96}$$

which is saying that acting with the connection one-form on a horizontal lifted tangent vector yields zero.

Now, since the transition functions are elements of the group, we can use Eq. (2.78) to write

$$\tilde{X} = g_i^{-1}(t)\big(s_{i*}X\big)g_i(t) + \big(g_i^{-1}\mathbf{d}g_i\big) \,, \tag{2.97}$$

and if we apply the connection w on this equation, we get

$$0 = g_i^{-1}(t) w(s_{i*}X) g(t) + g_i^{-1}(t)\frac{dg_i(t)}{dt} \,, \tag{2.98}$$

and if we use that $w(s_{i*}X) = s_i^* w(X) = A_i(X)$[49], we find the local *parallel transport equation* for a gauge theory

$$\frac{dg_i(t)}{dt} = -A_i(X) g_i(t) \,. \tag{2.99}$$

[48] In physics we also need the coupling g, but we will come to that later.

[49] This follows from the definition of the pullback in Eq. (2.51).

This is a matrix equation and can be solved by the use of *Chen iterated integrals* [9]. Given the initial condition $g_i(0) = e$[50], a local solution takes the form of a functional of an arbitrary path $\gamma(t)$

$$g_i[\gamma(t)] = \mathcal{P} \exp\Big(- \int_{\gamma(0)}^{\gamma(t)} dx^\mu A_{i\,\mu}(x) \Big) , \tag{2.100}$$

where in general $A_{i\,\mu} = A^a_{i\,\mu} t^a$[51], and $\mathcal{P}$ is a path-ordering operator which orders the gauge fields along the path in the manifold[52]. It works in a similar fashion as the time-ordering operator we have for the time-evolution operator in QM.

Parallel Transport and Holonomy

The last important concept of this section is that of *holonomy*. Given a principal fibre bundle $P(M, G, \pi)$, we let γ_1 and γ_2 be two curves in M, such that

$$\gamma_1(0) = \gamma_2(0) = p_0 , \tag{2.101}$$
$$\gamma_1(1) = \gamma_2(1) = p_1 . \tag{2.102}$$

Let us consider the horizontal lift of these two curves

$$\tilde{\gamma}_1(0) = \tilde{\gamma}_2(0) = q_0 , \tag{2.103}$$

then it is not in general true that

$$\tilde{\gamma}_1(1) = \tilde{\gamma}_2(1) = q_1 . \tag{2.104}$$

Then if we consider a loop γ in M

$$\gamma(0) = \gamma(1) , \tag{2.105}$$

we have that in general

$$\tilde{\gamma}(0) \neq \tilde{\gamma}(1) . \tag{2.106}$$

The loop then induces a map on the fibre at p

$$\tau_\gamma : \pi^{-1}(p) \to \pi^{-1}(p) . \tag{2.107}$$

We can then consider loops with fixed base-point in the manifold M, denoted $L_p M$. Now, τ_γ can only reach certain elements of the group G, but combining them with a gauge transformation we can reach all elements of G. Hence, the set of all elements that can be reached starting from the point (p, q) in the principal fibre bundle form a subgroup of G, called the *holonomy group* at q, where the projection of q is the point p, i.e. $\pi(q) = p$. We write this as

$$\Phi_q = \big\{ g \in G | \tau_\gamma(q) = qg,\, \gamma \in L_p M \big\} . \tag{2.108}$$

Holonomy elements are generated by considering parallel transport around a closed loop and it follows from Eq. (2.100) that it can be written as

$$g_\gamma = \mathcal{P} \exp\big(- \oint_\gamma dz^\mu A_\mu(z) \big) . \tag{2.109}$$

Parallel transport around closed loops are special objects with many interesting features, and some of them will be discussed in more detail in Sec. 2.2. The reason why these are important in physics follows from the following theorem:

[50] Here e is the unit element in the gauge group G.
[51] In physics we use $A_{i\,\mu} = igA^a_{i\,\mu} t^a$, where g is the coupling, but more on that later.
[52] In general the matrices does not commute, so the path ordering is needed.

Theorem 2.1.1: Ambrose-Singer

Let $P(M,G)$ be a principal fibre bundle with connection w, and curvature Ω. Let Φ_q be the holonomy group with reference point $q \in P(M,G)$, and $P(q)$ the holonomy bundle of w through q. Then the Lie algebra of $\Phi(q)$ is equal to the Lie sub-algebra $\mathfrak{g}$, generated by all elements of the form $\Omega_p(v_1, v_2)$ for $p \in P(q)$ and v_1, v_2 horizontal vectors at p, where $\mathfrak{g}$ is the Lie algebra of G.

This is quite technical, but instead of tackling all the mathematical language we will instead explain what it means for physics. The theorem states that all the information contained in the curvature Ω at a point in the principal fibre bundle P with connection w, can also be found in the holonomy group $\Phi(q)$ at that point. The implication is that—at least in principal—all physical observables can be expressed as functions of the holonomies, instead as functions of the gauge potentials. The holonomy group is not gauge invariant, so real observables need to be expressed in terms of gauge invariant functions of these holonomies. Such an object is what we in physics call a *Wilson loop*. Hence, we have that the Wilson loop contains all the information about the curvature two-form we need to describe the dynamics of gauge fields.

2.2 Yang-Mills Theories from Wilson Lines

In this section we will construct the Yang-Mills Lagrangian from a purely geometric setting by using the concept of connection, covariant derivative and parallel transportation. Our basic starting point is the Dirac Lagrangian, which we require to be invariant under a local gauge transformation. To do this, we will use what we have learned from the previous sections of connections on principal fibre bundles, and in particular use the solution of the parallel transport equation to generate gauge invariant objects. The requirement of local gauge invariance introduces interactions between matter fields and gauge fields, and if these gauge fields are to describe physical fields we need to find a gauge invariant term for them in the Lagrangian. This is where the Ambrose-Singer theorem can be applied by using the gauge invariant elements of the holonomy group, i.e. the Wilson loop.

2.2.1 Covariant Derivative and Wilson Lines

We have seen how the matter fields transform under local gauge transformation and how gauge fields naturally appear from connection one-forms. However, the main objective regarding gauge invariance in physics is that the Lagrangian (or action) remain invariant under local gauge transformations, i.e we need to define connections such that derivative terms are gauge invariant. To construct the covariant derivative for gauge theories we will make use of the important object called a *Wilson line*. This object is central in almost all modern theories of physics, as it can be used as a building block for gauge invariance. Before we define the Wilson line we will introduce its use from a physical perspective.

Let us start with the Dirac Lagrangian

$$\mathcal{L}_D = \bar{\psi}(i\not{\partial} - m)\psi\,, \tag{2.110}$$

which because of the partial derivative cannot describe a gauge invariant theory. The problem with the partial derivative when working with manifolds was discussed from Eq. (2.23), where we looked at the directional derivative of a vector field. The directional partial derivative of the fields ψ can in the naive way be expressed in a similar fashion

$$n^\mu \partial_\mu \psi(x) \equiv \lim_{\epsilon \to 0} \frac{\psi(x + \epsilon n) - \psi(x)}{\epsilon}\,, \tag{2.111}$$

where the fields $\psi(x + \epsilon n)$ and $\psi(x)$ have completely different transformation properties under local gauge transformations and can therefore not be subtracted in a meaningful way. With a curved background this was solved by parallel transporting the fields using the Levi-Civita connection Eq. (2.32). We are now in a position to do the same thing for a gauge theory, where the gauge fields themselves act as connections[53]. The parallel transporter for a gauge theory is given by Eq. (2.100), which we use as the definition of a Wilson line:

[53] This was the main motivation for introducing connections on principal fibre bundles.

Definition 2.2.1: Wilson line

Given a path γ in the spacetime manifold M^4, a Wilson line is defined as the solution to the parallel transport equation:

$$\mathcal{U}_\gamma[y,x] = \mathcal{P}\exp\left(ig\int_x^y dz^\mu A_\mu^a(z)t^a\right), \tag{2.112}$$

where g is the coupling, γ is the path along which one parallel transports between spacetime points x and y and $\mathcal{P}$ denotes the path-ordering operator.

The matter fields transform through the unitary representation of G[54]

$$\psi(x) \to U(x)\psi(x)\,, \tag{2.113}$$

where

$$U(x) = \exp\left(ig\alpha^a(x)t^a\right)\ . \tag{2.114}$$

The Wilson line transforms under gauge transformation as[55]

$$\mathcal{U}_\gamma[y,x] \to U(y)\mathcal{U}_\gamma[y,x]U^\dagger(x)\,, \tag{2.115}$$

such that we can parallel transport $\psi(x)$ in the following way

$$\psi(y) = \mathcal{U}_\gamma[y,x]\psi(x)\,, \tag{2.116}$$

which transform as

$$\begin{aligned}\psi(y) &\to U(y)\mathcal{U}_\gamma[y,x]U^\dagger(x)U(x)\psi(x) &(2.117)\\ &= U(y)\mathcal{U}_\gamma[y,x]\psi(x) &(2.118)\\ &= U(y)\psi(y)\,. &(2.119)\end{aligned}$$

This is the result we need to compare the matter fields at different points. Hence, we define the directional covariant derivative[56]:

Definition 2.2.2: Directional Covariant Derivative

A directional covariant derivative is defined through the Wilson line in the following way

$$n^\mu D_\mu\psi(x) = \lim_{\epsilon\to 0}\frac{\psi(x+\epsilon n) - \mathcal{U}[x+\epsilon n, x]\psi(x)}{\epsilon}\,.$$

We want to expand the Wilson line, so we can use the *gradient theorem*:

Theorem 2.2.1: Gradient Theorem

Given an analytic function f on the continous path $\gamma \in [x,y]$, we have

$$\int_x^y dz^\mu \partial_\mu f(z) = f(y) - f(x)\,. \tag{2.120}$$

Expanding the Wilson line using the gradient theorem and expanding the field $\psi(x+\epsilon n)$, we get

$$\begin{aligned}\mathcal{U}[x+\epsilon n, x] &= 1 + ig\epsilon n^\mu A_\mu(x) + \mathcal{O}(\epsilon^2) &(2.121)\\ \psi(x+\epsilon n) &= \psi(x) + \epsilon n^\mu\partial_\mu\psi(x) + \mathcal{O}(\epsilon^2)\,, &(2.122)\end{aligned}$$

[54] We are specifically talking about the gauge (Lie) group $SU(N)$, so we have a local $\mathfrak{su}(n)$ phase rotation.
[55] This is not trivial to show as one has to discretize the integral in the exponent.
[56] We usually drop the subscript γ in applications. In this case γ is a infinitesimal straight line along n^μ.

inserting these expansions into the definition of the covariant derivaticve, we find that

$$D_\mu\psi(x) = \partial_\mu\psi(x) - igA_\mu(x)\psi(x)\,, \tag{2.123}$$

which is the well known covariant derivative acting on fields in the fundamental representation of the gauge group.

The transformation property of the gauge fields follow directly from the Wilson line transformation. To write this out we use the well known identity of exponentiated matrices

$$e^{YXY^{-1}} = YXY^{-1}\,, \tag{2.124}$$

which applied to Eq. (2.115), will give

$$U(y)\mathcal{U}_\gamma[y,x]U^\dagger(x) = \mathcal{P}\exp\Big[U(y)\big(ig\int_x^y dz^\mu A^a_\mu(z)t^a\big)U^\dagger(x)\Big]\,, \tag{2.125}$$

which after partial integration and application of the gradient theorem takes the form

$$\mathcal{P}\exp\Big[ig\int_x^y dz^\mu U(z)\Big(A^a_\mu(z)t^a + \frac{i}{g}\partial_\mu\Big)U^\dagger(z)\Big]\,, \tag{2.126}$$

meaning that the gauge fields transform as

$$A^a_\mu(x)t^a \to U(x)A^a_\mu t^a U^\dagger(x) + \frac{i}{g}U(x)\partial_\mu U^\dagger(x)\,, \tag{2.127}$$

which is the same we found in Eq. (2.80), with the difference of the factor i/g we use in physics.

With these results we have that the gauge invariant Dirac Lagrangian can be written as

$$\mathcal{L}_D = \bar\psi(i\not{D} - m)\psi\,. \tag{2.128}$$

The missing piece of the Yang-Mills Lagrangian is the kinetic term[57] for the gauge fields. To find the field strength, we can do the same as we did for the Riemann curvature and consider the commutator of the covariant derivatives, see Eq. (2.40). We find[58]

$$[D_\mu, D_\nu] = -igF^a_{\mu\nu}t^a\,, \tag{2.129}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu\,, \tag{2.130}$$

which apart from the coupling g is identical to the curvature two-form we defined in Eq. (2.92). From the transformation property in Eq. (2.93), the field strength is not gauge invariant. Meaning we have to use a certain combination of them that are gauge invariant to use it as a term in the Lagrangian. We could follow in a heuristic manner and postulate what this combination is, but we can show that the term appear in a natural way from geometric arguments.

We will encounter the use for Wilson lines on numerous occasions throughout this thesis, for example when we want to ensure gauge invariance of bi-local operators or when we want to describe soft gauge boson radiation in scattering amplitudes.

2.2.2 Field Strength Tensor and Wilson Loops

In this section we will take a closer look at the definition of a Wilson line on a closed path and from it construct the kinetic term in the Lagrangian for gauge bosons.

If the solution of the parallel transport equation is defined on a path γ that is a closed loop, we have the Wilson line operator

$$\Gamma_\gamma = \mathcal{P}\exp\Big(ig\oint_\gamma dz^\mu A^a_\mu(z)t^a\Big)\,. \tag{2.131}$$

[57] No mass terms as it would not be gauge invariant.

[58] The commutator is not a differential operator, but merely a matrix that is understood to act on ψ.

If one expands this exponential, this becomes an infinite series that converges absolutely to an element $g \in G$, see [18]. It is important to note that this object is not gauge invariant. However, a Wilson line operator acts as an operator on Hilbert space, and because of the convergent behaviour, we can just as well consider its trace[59]:

Definition 2.2.3: Wilson loop

Given a Wilson line Γ_γ defined along a closed loop γ, a Wilson loop is then defined as its trace

$$W[\gamma] = \operatorname{tr} \mathcal{P} \exp\left(ig \oint_\gamma dz^\mu A^a_\mu(z) t^a \right).$$

Because of the trace this is gauge invariant, which can be shown by calculating

$$\begin{aligned} \operatorname{tr}[\Gamma_\gamma] &\to \operatorname{tr}[U(x)\Gamma_\gamma U^{-1}(x)] \\ &= \operatorname{tr}[\Gamma_\gamma U^{-1}(x) U(x)] \\ &= \operatorname{tr}[\Gamma_\gamma], \end{aligned} \tag{2.132}$$

where $U(x)$ is the usual gauge transformation.

To show that the Wilson loop contains all the dynamical information we have to expand the Wilson loop. We can first use Stokes' theorem to write the loop integral into a surface integral

$$\oint_\gamma dz \cdot A = \int_\Sigma d\sigma \cdot \partial \wedge A. \tag{2.133}$$

We can always parametrize a path in spacetime in terms of a parameter λ in the following way

$$\gamma : z^\mu(\lambda), \tag{2.134}$$

$$dz^\mu = d\lambda \frac{\partial z^\mu}{\partial \lambda}. \tag{2.135}$$

Hence, we can always parametrize a surface in spacetime in terms of two parameteres λ, λ'

$$\Sigma : z^\mu(\lambda, \lambda'), \tag{2.136}$$

$$\begin{aligned} d\sigma^{\mu\nu} &= dz^\mu \wedge dz^\nu \\ &= d\lambda d\lambda' \left(\frac{\partial z^\mu}{\partial \lambda} \frac{\partial z^\nu}{\partial \lambda'} - \frac{\partial z^\nu}{\partial \lambda} \frac{\partial z^\mu}{\partial \lambda'} \right), \end{aligned} \tag{2.137}$$

and we can write the surface integral as

$$\int_\Sigma d\sigma \cdot \partial \wedge A = \int_\Sigma d\lambda d\lambda' \frac{\partial z^\mu}{\partial \lambda} \frac{\partial z^\nu}{\partial \lambda'} \left(\partial_\mu A_\nu - \partial_\nu A_\mu \right), \tag{2.138}$$

where we used that $\partial \wedge A = (\partial_\mu A_\nu - \partial_\nu A_\mu)/2$. We want to consider an infinitesimal loop, and to do this we discretize spacetime and define the theory on a lattice with grid spacing ϵ[60]. However, Minkowski space can not be discretized in a well defined manner so we have to transform to Euclidean space. This is done by performing a Wick rotation. We can then write the Euclidean Wilson loop

$$W_E = \operatorname{tr} \mathcal{P} \exp\left(ig \int_\Sigma d\lambda d\lambda' \frac{\partial z^\mu_E}{\partial \lambda} \frac{\partial z^\nu_E}{\partial \lambda'} \left(\partial_\mu A^a_{E\,\nu} - \partial_\nu A^a_{E\,\mu} \right) t^a \right). \tag{2.139}$$

The expansion is tedious and takes quite some work to bring on a nice form, so we will just state the relevant terms. The first order term vanishes identically since $\operatorname{tr}[t^a] = 0$. Ignoring constants the expansion to $\mathcal{O}(g^2)$ is given by

$$-g^2 \frac{\epsilon^4}{4} \left(\partial_\mu A^a_{E\,\nu} - \partial_\nu A^a_{E\,\mu} \right)^2, \tag{2.140}$$

[59] The trace over an operator converges absolutely and is independent of the basis $\{\psi_n\}$ in Hilbert space.

[60] A square with corners x, $x + \epsilon n$, $x + \epsilon n + \epsilon n'$ and $x + \epsilon n'$.

and the contributions from the third and fourth order expansion take the form

$$-g^3\epsilon^4 f^{abc} A^a_{E\,\mu} A^b_{E\,\mu} \partial^\mu A^{\nu\,c}_E - g^4 \frac{\epsilon^4}{4} f^{abe} f^{ecd} (A^a_{E\,\mu} A^b_{E\,\nu})(A^{\mu\,c}_E A^{\nu\,d}_E)\,. \tag{2.141}$$

Collecting terms will give

$$\begin{aligned} W_E \approx & -\frac{g^2\epsilon^4}{4}\left(\partial_\mu A^a_{E\,\nu} - \partial_\nu A^a_{E\,\mu}\right)^2 - g^3\epsilon^4 f^{abc} A^a_{E\,\mu} A^b_{E\,\mu} \partial^\mu A^{\nu\,a}_E \\ & - g^4 \frac{\epsilon^4}{4} f^{abe} f^{ecd} (A^a_{E\,\mu} A^b_{E\,\nu})(A^{\mu\,c}_E A^{\nu\,d}_E) + \mathcal{O}(\epsilon^5)\,, \end{aligned} \tag{2.142}$$

valid up to a constant term that is unimportant for the present discussion. If we compare this expression with Eq. (2.130), this looks very much the square of the field strength. The difference lies in the powers of g and of course the small increment ϵ. However, eventually we would like to take the continuum limit by sending $\epsilon \to 0$. This is a highly non-trivial task, and to reproduce the correct continuum limit one has to take into account for rescaling and renormalization of all quantities in the theory. Also, the continuum limit corresponds to summing over all lattice points, therefore we have to divide by the lattice spacing to the fourth ϵ^4 before letting $\epsilon \to 0$[61].

But first we use Eq. (2.130) to write Eq. (2.142) as

$$W_E \approx -g^2\epsilon^4 \frac{1}{4} F^a_{E\,\mu\nu} F^{\mu\nu\,a}_E + \mathcal{O}(\epsilon^5)\,, \tag{2.143}$$

which after rescaling of the coupling and a Wick rotation back to Minkowski space will in the continuum limit take the form [62]

$$W[A] = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu\,a}\,, \tag{2.144}$$

which is the well known gauge invariant kinetic term for the gauge fields. The gauge invariance naturally follows as the original Wilson loop is gauge invariant.

To construct a Yang-Mills theory of gauge fields interacting with fermions, we simply add Eq. (2.144) to the gauge invariant Dirac Lagrangian. The result takes the form

$$\mathcal{L}_{YM} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu\,a} + \bar{\psi}(i\not{D} - m)\psi \tag{2.145}$$

which is known as the Yang-Mills Lagrangian and is what we call a *non-Abelian* gauge theory.

From the above expansion of the Wilson loop we observe that there are quartic and cubic terms in A^a_μ, meaning that this is a nontrivial, interacting field theory[63]. It is important to note that this discussion is at the classical level, so we have to quantize the Yang-Mills theory.

2.2.3 Quantization of Yang-Mills Theories

In this section we will quantize the Yang-Mills theory. We will use several of the concepts we used in the quantization of Abelian gauge theories, so for more detail see Sec. 1.3.5.

When we quantized the Abelian gauge field in Sec. 1.3.5, we saw that because of gauge invariance we are integrating over an infinite number of identical field configurations leading to a divergent path integral. To fix the problem we introduced a gauge fixing condition that ensured integration over those configurations that were different. Due to the non-commutivity between non-Abelian gauge fields this is more complicated.

To begin with, we use that an infinitesimal transformation of non-Abelian gauge fields can be written as

$$(A^\alpha)^a_\mu = A^a_\mu + \frac{1}{g} D_\mu \alpha^a\,, \tag{2.146}$$

where the covariant derivative acting on fields in the adjoint representation is given by

$$D^{ab}_\mu = \delta^{ab}\partial_\mu - g f^{abc} A^c_\mu\,. \tag{2.147}$$

[61] For a rigorous treatment of the continuum limit we refer to [19].
[62] An example for $SU(2)$ can be found in [2].
[63] As opposed to Abelian gauge theories where there are no such interactions between the gauge bosons.

We can do as in the Abelian case and insert the identity Eq. (1.138) and write the path integral as[64]

$$\int \mathcal{D}A e^{iS[A]} = \Big(\int \mathcal{D}\alpha\Big) \int \mathcal{D}A\, e^{iS[A]}\, \delta(G[A])\, \det\left(\frac{\partial G[A]}{\partial \alpha}\right). \tag{2.148}$$

In contrast to the Abelian case we can not move the functional determinant out of the path integral and combine it with the infinite constant. To see this we calculate the argument inside the determinant and find

$$\frac{\partial G[A]}{\partial \alpha} = \frac{1}{g}\partial^\mu D_\mu\,, \tag{2.149}$$

which is not independent of A. To circumvent this problem we use that functional determinants can be written as a path integral, a representation introduced by Fadeev and Popov [8]. The trick of it is that we need to write the determinant as a path integral of anticommuting Grassmann fields that transform in the adjoint representation, i.e.

$$\det(\partial^\mu D_\mu) = \int \mathcal{D}c\mathcal{D}\bar{c} \exp\left(-i\int d^4x\, \bar{c}(\partial^\mu D_\mu)c\right). \tag{2.150}$$

These fields can be shown to transform under Lorentz transformation as scalars, meaning that due to their Grassmannian nature do not obey the correct spin-statistics relation. Therefore, they can not be dynamical fields that we can observe. However, they can be shown to serve as negative degrees of freedom in the sense that given their Feynman rules they can cancel the unphysical timelike and longitudinal polarization states of the gauge bosons. With this representation of the determinant one can use Eq. (1.146) and integrate using the delta function, leading to the Fadeev-Popov Lagrangian

$$\mathcal{L}_{FP} = -\frac{1}{4}(F^a_{\mu\nu})^2 + \bar{\psi}(i\not{D} - m)\psi - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 - \bar{c}^a(\partial^\mu D^{ac}_\mu)c^c\,. \tag{2.151}$$

As mentioned the practical use of ghosts is to cancel unphysical polarizations appearing in gauge boson diagrams. For example, when calculating a gauge boson self energy diagram, one also have to add a ghost diagram.

In some cases it can be cumbersome and difficult to calculate these extra ghost diagrams. There is another quantization procedure that circumvent the introduction of ghosts. That is, one can choose a different class of gauges, namely the axial gauges, i.e.

$$G[A] = n^\mu A_\mu - \omega\,, \tag{2.152}$$

for some arbitrary directional vector n_μ. With this choice the functional determinant can be written as

$$\det\left(\frac{\partial G[A]}{\partial \alpha}\right) = \det(n^\mu D_\mu) = \det(n^\mu\partial_\mu - g n^\mu A_\mu) = \det(n^\mu\partial_\mu - g\omega)\,, \tag{2.153}$$

which is independent of A, and can be pulled out of the path integral. Not being able to pull the functional determinant out of the path integral was the problem that led to the introduction of ghosts. The general result is that any gauge choice involving derivatives, will give rise to ghosts. The downside with axial gauges is that the propagator will become more complicated.

As for the Abelian gauge fixing we can now make an integration over ω using a Gaussian weight, see Eq. (1.146). The path integral can then be written as

$$\begin{aligned}\int \mathcal{D}A\, e^{iS[A]} &= N(\xi)\det(n^\mu\partial_\mu)\Big(\int \mathcal{D}\alpha\Big)\int \mathcal{D}\omega \int \mathcal{D}A\, e^{iS[A]}\,\delta(n^\mu A_\mu - \omega) e^{-i\int d^4x \frac{\omega^2}{2\xi}} \\ &= N(\xi)N(\alpha)\int \mathcal{D}A\, \exp\left(iS[A] - i\int d^4x \frac{1}{2\xi}(n^\mu A_\mu)^2\right)\end{aligned} \tag{2.154}$$

This is completely analogous to Eq. (1.149), with the difference of n^μ instead of ∂^μ in the gauge fixing term. The gauge-fixed action is then given by

$$S_{GF}[A] = \frac{1}{2}\int d^4x\, d^4y\, A_\mu(x)\delta^{(4)}(x-y)\big(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu + \frac{1}{\xi}n^\mu n^\nu\big)A_\nu(y)\,, \tag{2.155}$$

[64] Due to notational simplicity we will not always write out the group indicies, but for non-Abelian field it is always understood to be there.

giving the propagator equation

$$\left(\partial^2 g_{\mu\nu} - \partial_\mu\partial_\nu - \frac{1}{\xi}n_\mu n_\nu\right)D_F^{\nu\rho}(x,y) = i\delta_\mu^\rho\delta^{(4)}(x-y)\,. \tag{2.156}$$

As in Eq. (1.152) this can be solved exactly by exploiting that the gauge boson propagator is a second rank symmetric tensor. Using this property we have that the momentum space propagator in axial gauge is given by

$$\text{Axial Gauge}: \qquad D^{ab}_{\mu\nu}(k) = \frac{-i\delta^{ab}}{k^2+i\epsilon}\left(g_{\mu\nu} - \frac{k_\mu n_\nu + k_\nu n_\mu}{k\cdot n} + (n^2+\xi k^2)\frac{k_\mu k_\nu}{(k\cdot n)^2}\right). \tag{2.157}$$

This expression looks like a huge price to pay for avoiding introducing ghosts, but we can use an even smaller set of axial gauges. The one we will mostly use is the so-called *light-cone gauges*, which amounts to choosing n^μ as a lightlike vector, i.e. $n^2 = 0$. Also, like for Lorentz gauges we have an additional gauge choice for ξ. We can choose $\xi = 0$, which is called the *homogenous light-cone gauge*, such that the term proportional to k^2 vanishes entirely. We are then left with the light-cone propagator

$$\text{LC Gauge}: \qquad D^{ab}_{\mu\nu}(k) = \frac{-i\delta^{ab}}{k^2+i\epsilon}\left(g_{\mu\nu} - \frac{k_\mu n_\nu + k_\nu n_\mu}{k\cdot n}\right). \tag{2.158}$$

For completeness, the gauge boson propagator in covariant gauge is given by

$$\text{Lorentz Gauge}: \qquad D^{ab}_{\mu\nu}(k) = \frac{-i\delta^{ab}}{k^2+i\epsilon}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right). \tag{2.159}$$

Most of the time, calculations are easier in Lorentz gauges, but for some QCD calculations it is easier to use the light-cone gauge.

2.3 Wilson Line Properties

In Sec. 2.2 we found that Wilson lines emerged naturally in gauge theories from a geometrical viewpoint. Because of its bi-local transformation property, it is used as a parallel transporter to render non-local terms gauge invariant. It could also be used to derive a gauge invariant Lagrangian for a non-Abelian gauge theory. However, the application of Wilson lines are much broader than this and in this section we will look at some their properties in more detail.

The Wilson line we defined in Sec. 2.2 is valid for any gauge theory, but as our main focus is on QCD we mostly use that the gauge fields are non-Abelian in nature. The physical consequence of involving Wilson lines in a theory becomes apparent if we expand Eq. (2.112),

$$\begin{aligned}\mathcal{U}_\gamma &= \mathcal{P}\exp\left(ig\int_\gamma dz^\mu A_\mu(z)\right)\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}(ig)^n\mathcal{P}\int_\gamma dz_n^{\mu_n}\ldots dz_1^{\mu_1}A_{\mu_n}(z_n)\ldots A_{\mu_1}(z_1)\,,\end{aligned} \tag{2.160}$$

i.e. the n-th order expansion represents radiation of n gauge bosons. The point z_i at which the gauge field is radiated is integrated over, meaning that all possible configurations are taken into account. The path-ordering makes sure that the radiated field are ordered such that $A_{\mu_i}(z_i)$ is radiated between $A_{\mu_{i-1}}(z_{i-1})$ and $A_{\mu_{i+1}}(z_{i+1})$. Thus, the full exponential is a resummation of gauge boson radiation from the path. One important consequence of this is that we can 'dress' a particle, say a quark, with a Wilson line and this would then correspond to resummation of gluon radiation from a quark line. The same can be done for an electron with an Abelian Wilson line, with the simplification that the gauge fields commute and the path-ordering is redundant.

Before we discuss how we can dress a fermion with a Wilson line, we summarize some important properties of Wilson lines.

Wilson Line Properties

A Wilson line $\mathcal{U}_\gamma$ defined on a path γ with endpoints x and y have the following properties:

Hermiticity: The hermitian conjugate of a Wilson line is given by

$$\mathcal{U}_\gamma^\dagger[y,x] = \mathcal{U}_{-\gamma}[x,y]\,, \tag{2.161}$$

i.e. it gives the same line in the opposite direction.

Causality: Because of path-ordering the Wilson line is path-transitive, i.e. we can continously glue several paths together to form one. For a path $\gamma = \gamma_1 + \gamma_2$ going from x to z, then from z to y, is the same as going directly from x to y,

$$\mathcal{U}_\gamma[y,x] = \mathcal{U}_{\gamma_1}[z,x]\,\mathcal{U}_{\gamma_2}[y,z]\,. \tag{2.162}$$

Unitarity: If we have a Wilson line from x to y and back from y to x in the opposite direction, this composition is equal to one, i.e.

$$\mathcal{U}_\gamma[y,x]\mathcal{U}_\gamma^\dagger[y,x] = 1\,. \tag{2.163}$$

Bi-Locality: A Wilson line transform in function of it's endpoints only

$$\mathcal{U}_\gamma[y,x] \to e^{ig\alpha^a(y)t^a}\mathcal{U}_\gamma[x,y]e^{-ig\alpha^a(x)t^a}\,. \tag{2.164}$$

The ultimate goal going forward is to use Wilson lines in perturbation theory, so we have to evaluate the line integrals in Eq. (2.160). To that end, it is convenient to parametrize the path γ as we did in Eq. (2.134), i.e.

$$\gamma:\ z^\mu(\lambda)\,, \qquad \lambda = a\ldots b\,, \tag{2.165}$$

where $z(a)$ and $z(b)$ are the endpoints of the path. Then we can make the change of variable

$$dz^\mu \to d\lambda \frac{dz^\mu}{d\lambda}\,, \tag{2.166}$$

Such that the expansion in Eq. (2.160) can be written as

$$\begin{aligned}&\mathcal{P}\int_\gamma d\lambda_n\ldots d\lambda_1 \frac{\mathrm{d}z^{\mu_n}(\lambda_n)}{\mathrm{d}\lambda_n}\ldots\frac{\mathrm{d}z^{\mu_1}(\lambda_1)}{\mathrm{d}\lambda_1}A_{\mu_n}(z_n)\ldots A_{\mu_1}(z_1)\\ &\quad = n!\int_a^b d\lambda_n\int_a^{\lambda_n} d\lambda_{n-1}\cdots\int_a^{\lambda_2} d\lambda_1\frac{\mathrm{d}z^{\mu_n}(\lambda_n)}{\mathrm{d}\lambda_n}\ldots\frac{\mathrm{d}z^{\mu_1}(\lambda_1)}{\mathrm{d}\lambda_1}A_{\mu_n}(z_n)\ldots A_{\mu_1}(z_1)\,,\end{aligned} \tag{2.167}$$

which tells us that the i-th gauge boson with parameter λ_i is radiated between the $i-1$-th and $i+1$-th. Thus, the field with highest value of λ is written leftmost in the integral, meaning that this is the field that will be written rightmost in a Feynman diagram. This implies that we read a Wilson line from right to left, like we do for Dirac lines.

We want to use these Wilson lines in scattering amplitudes, so there is a specific Wilson line that is relevant for us, and that is Wilson lines on linear paths. In Feynman diagrams we draw particle lines as linear paths, so these are the most relevant for our purposes. We want to integrate over the path dependence, so it is convenient to disentagle the gauge field content from the path content of the line integral. This is most easily done by using the Fourier transform

$$A_\mu(z) = \int\frac{d^dk}{(2\pi)^d}\,A_\mu(k)\,e^{-ik\cdot z}\,, \tag{2.168}$$

such that the n-th order term in the expanded Wilson line Eq. (2.170) can be written as

$$\mathcal{U}_\gamma^{(n)} = \int\frac{d^dk_n}{(2\pi)^d}\cdots\frac{d^dk_1}{(2\pi)^d}\,A_{\mu_n}(k_n)\ldots A_{\mu_1}(k_1)\,\mathcal{I}^{(n)}\,, \tag{2.169}$$

where we have collected all the path content in the following integral

$$\mathcal{I}_A^{(n)} = (ig)^n \int_a^b d\lambda_n \cdots \int_a^{\lambda_2} d\lambda_1 \frac{\mathrm{d}z^{\mu_n}(\lambda_n)}{\mathrm{d}\lambda_n} \cdots \frac{\mathrm{d}z^{\mu_1}(\lambda_1)}{\mathrm{d}\lambda_1} \exp\left(-i \sum_{i=1}^{n} k_i \cdot z_i\right). \tag{2.170}$$

We want to consider paths that are bounded from below and bounded from above. The result in Eq. (2.170) is convenient to use when we have a line that is bounded from above[65]. For a path that is bounded from below, we change the integration chaining in Eq. (2.167) and flip the order of integration[66]. We summarize the two different cases in the following integrals

$$\mathcal{I}_A^{(n)} = (ig)^n \int_a^b d\lambda_n \cdots \int_a^{\lambda_2} d\lambda_1 \frac{\mathrm{d}z^{\mu_n}(\lambda_n)}{\mathrm{d}\lambda_n} \cdots \frac{\mathrm{d}z^{\mu_1}(\lambda_1)}{\mathrm{d}\lambda_1} \exp\left(-i \sum_{i=1}^{n} k_i \cdot z_i\right), \tag{2.171}$$

$$\mathcal{I}_B^{(n)} = (ig)^n \int_a^b d\lambda_1 \cdots \int_{\lambda_{n-1}}^{b} d\lambda_n \frac{\mathrm{d}z^{\mu_n}(\lambda_n)}{\mathrm{d}\lambda_n} \cdots \frac{\mathrm{d}z^{\mu_1}(\lambda_1)}{\mathrm{d}\lambda_1} \exp\left(-i \sum_{i=1}^{n} k_i \cdot z_i\right). \tag{2.172}$$

Semi-Infinite Wilson Lines

The most interesting and physical relevant paths are those that contain so-called *cusps*, i.e. there are points in the path that are not smooth. They are continous, but the derivative is not. Fortunately, since Wilson lines are path-transitive we can split the path at the cusp and continue with a product of two Wilson lines.

To explain why we are interested in paths with cusps, we take the example of annihilation of two fermions. Highly energetic particles that radiates gauge bosons can be described by a Wilson line[67]. The point of interaction is then what we call a cusp, e.g. when we draw Feynman diagrams we have two particles lines that meet and there is a crack at the interaction. Such cusps will lead to divergences and one has to renormalize the Wilson line. This will lead to the so-called *cusp anamalous dimensions* containing all the interesting dynamics. We will come back these issues in Chapter 4.

The idea of Wilson lines coming in from infinity and meets at a point or created at a point and goes out to infinity, naturally leads to the notion of what we call semi-infinite Wilson lines. Let us start with a Wilson line that is bounded from above, which we parametrize as

$$z_i^\mu = a^\mu + n^\mu \lambda_i \,, \qquad \lambda = -\infty \ldots 0 \,, \tag{2.173}$$

which is a linear path going from $-\infty$ up to a point a^μ along a direction n^μ. If we insert this parametrization into Eq. (2.171), we get

$$\begin{aligned}\mathcal{I}_A^{(m)} =& (ig)^m \, n^{\mu_1} \cdots n^{\mu_m} \exp\left(-i \sum_{i=1}^{m} a \cdot k_i\right) \\ & \int_{-\infty}^{0} d\lambda_m \int_{-\infty}^{\lambda_m} d\lambda_{m-1} \cdots \int_{-\infty}^{\lambda_2} d\lambda_1 \exp\left(-i \sum_{i=1}^{m} (n \cdot k_i + i\epsilon)\lambda_i\right),\end{aligned} \tag{2.174}$$

where we have used that the integrals has the form of a Fourier transformed Heaviside θ-function, which is not convergent unless we use a $i\epsilon$ prescription. The solution of a Fourier transformed Heaviside function is well known, so the innermost integral is given by

$$\int_{-\infty}^{\lambda_2} d\lambda_1 e^{-i(n\cdot k_1 + i\epsilon)\lambda_1} = \frac{i}{n \cdot k_1 + i\epsilon} e^{-i(n\cdot k_1 + i\epsilon)\lambda_2} \,, \tag{2.175}$$

and to reveal the pattern we can also solve the next integral

$$\int_{-\infty}^{\lambda_3} d\lambda_2 e^{-i(n\cdot k_1 + n\cdot k_2 + i\epsilon)\lambda_2} = \frac{i}{n \cdot k_1 + n \cdot k_2 + i\epsilon} e^{-i(n\cdot k_1 + n\cdot k_2 + i\epsilon)\lambda_3} \,. \tag{2.176}$$

[65] Hence, the subscript.
[66] This is done to make sure that the radiation happens in the correct order.
[67] We will show this later.

By collecting all factors that are brought down by the integration, the n-th integral takes the form

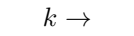
$$\mathcal{I}^{(m)} = (ig)^m\, n^{\mu_1}\cdots n^{\mu_m}\, \exp\left(-i\sum_{i=1}^{m} a\cdot k_i\right)\prod_{i=1}^{m}\frac{i}{n\cdot\sum_{j=1}^{i}k_j + i\epsilon}\,, \tag{2.177}$$

giving

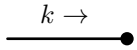
$$\mathcal{U}[0,-\infty] = \sum_{m=0}^{\infty}(ig)^m\int\frac{d^dk}{(2\pi)^d}\, n\cdot A(k_m)\cdots n\cdot A(k_1)\, e^{-ia\cdot\sum_{i=1}^{m}k_i}\prod_{i=1}^{m}\frac{i}{n\cdot\sum_{j=1}^{i}k_j + i\epsilon}\,. \tag{2.178}$$

Just to clarify the notation used here and later: when we have a Wilson line defined from x to y, we use the bracket $\mathcal{U}[y,x]$. But when we have a composition of Wilson lines meeting at a point x coming from $-\infty$, we typically write this as $\mathcal{U}(x) = \mathcal{U}_{\gamma_1}[x,-\infty]\,\mathcal{U}_{\gamma_2}[x,-\infty]$.

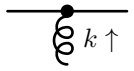
The expansion in Eq. (2.178) gives rise to the following Feynman rules:



$$= \frac{i}{n\cdot k + i\epsilon}\,, \qquad \text{Wilson line propagator} \tag{2.179}$$



$$= e^{-ia\cdot k}\,, \qquad \text{External point} \tag{2.180}$$



$$= ig\, n^\mu\, t^a\,. \qquad \text{Wilson vertex} \tag{2.181}$$

The next option to study is a line starting at a point a^μ and moving out to $+\infty$ along n^μ. The parametrization reads

$$z_i^\mu = a^\mu + n^\mu\lambda_i\,, \qquad \lambda = 0\ldots\infty\,. \tag{2.182}$$

If we insert this parametrization into Eq. (2.172), we get

$$\begin{aligned}\mathcal{I}_B^{(m)} =& (ig)^m\, n^{\mu_1}\cdots n^{\mu_m}\, \exp\left(-i\sum_{i=1}^{m} a\cdot k_i\right)\\ &\int_0^\infty d\lambda_1\int_{\lambda_1}^\infty d\lambda_2\cdots\int_{\lambda_{n-1}}^\infty d\lambda_n\, \exp\left(-i\sum_{i=1}^{m}(n\cdot k_i - i\epsilon)\lambda_i\right),\end{aligned} \tag{2.183}$$

where the innermost integral is

$$\int_{\lambda_{n-1}}^\infty d\lambda_n e^{-i(n\cdot k_1 - i\epsilon)\lambda_n} = \frac{-i}{n\cdot k_1 - i\epsilon}e^{-i(n\cdot k_1 - i\epsilon)\lambda_{n-1}}\,, \tag{2.184}$$

giving the Wilson line

$$\mathcal{U}[\infty,0] = \sum_{m=0}^{\infty}(ig)^m\int\frac{d^dk}{(2\pi)^d}\, n\cdot A(k_m)\cdots n\cdot A(k_1)\, e^{-ia\cdot\sum_{i=1}^{m}k_i}\prod_{i=1}^{m}\frac{-i}{n\cdot\sum_{j=1}^{i}k_j - i\epsilon}\,. \tag{2.185}$$

We observe that the Feynman rules defined above apply if we just make the change $k\to -k$ in the propagator. There is much more to be said of different paths and structures, but we will only use Wilson lines on linear paths so we restrict ourselves to the ones discussed here.

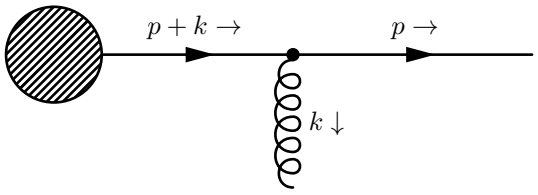


Figure 2.1: Radiation of a gauge boson from a outgoing fermion in a non-Abelian theory.

Eikonal Particles

We will now take a closer look at an amplitude and see the appearance of a Wilson line from it. A highly energetic fermion will always radiate soft gauge bosons. When the momentum carried by the gauge boson is much smaller than the momentum of the fermion this will lead to infrared divergences. As previously mentioned, we will investigate and treat IR divergences in more detail in Chapter 3. But even after these have been treated we will have logarithmic contributions that can become large in certain regions of phase space. This problem can be solved by using Wilson lines, where we can treat diagrams perturbatively and re-exponentiate to an exact expression.

To see an example, let us investigate the process where a massless fermion radiates a gauge boson, see Fig. 2.1[68]. The amplitude for this process is given by[69]

$$\mathcal{M}_1 = \bar{u}(p)(-igt^a \not{\varepsilon}(k))\frac{i(\not{p}+\not{k})}{(p+k)^2}\,\mathcal{B}\,, \tag{2.186}$$

where $\mathcal{B}$ is the blob containing all the information that is independent of the radiation. As we consider massless fermions, we have that $(p+k)^2 = 2\,p\cdot k$. If the radiated gauge boson is soft, we can make the approximation $\not{p}+\not{k} \approx \not{p}$. This is known as the *eikonal approximation*. We can also use that the fermion is massless and obeys the massless Dirac equation, i.e. we can use that $\bar{u}(p)\not{p} = 0$. This allows us to substitute $\not{\varepsilon}\not{p}$ with $\{\not{\varepsilon}, \not{p}\} = 2\varepsilon_\mu(k)p_\nu g^{\mu\nu}$, i.e. we have just added zero to the amplitude. With these adjustments, we can write Eq. (2.186) as

$$\mathcal{M}^{(1)} = gt^a\bar{u}(p)\not{\varepsilon}(k)\frac{\not{p}}{2p\cdot k}\,\mathcal{B} = gt^a\bar{u}(p)\frac{p\cdot\varepsilon(k)}{p\cdot k - i\epsilon}\,\mathcal{B}\,. \tag{2.187}$$

where we inserted the pole prescription as it is understood that the gauge boson momenta is to be integrated over in observables. We observe that the 'new' fermion propagator looks very much like the Wilson propagator in Eq. (2.179). To see that this will indeed give a description of radiation from a Wilson line, we can consider the process with two soft gauge bosons. If we use the same approximations as above, we find the amplitude

$$\begin{aligned}\mathcal{M}^{(2)} &= \bar{u}(p)(-igt^b\not{\varepsilon}(k_2))\frac{i(\not{p}+\not{k}_2)}{(p+k_2)^2}(-igt^a\not{\varepsilon}(k_1))\frac{i(\not{p}+\not{k}_1+\not{k}_2)}{(p+k_1+k_2)^2}\,\mathcal{B}\\ &= g^a t^b t^a \bar{u}(p)\frac{p\cdot\varepsilon(k_2)}{p\cdot k_2 - i\epsilon}\frac{p\cdot\varepsilon(k_1)}{p\cdot k_1 + p\cdot k_2 - i\epsilon}\,\mathcal{B}\,.\end{aligned} \tag{2.188}$$

We can now use that the polarization vector is used when Fourier expanding gauge fields, so we can make the substitution $\varepsilon^a_\mu(k) \to A^a_\mu(k)$. Further, we observe that Eq. (2.186) and Eq. (2.188) is invariant under the rescaling

[68] These are gluons, but we restrain from casually mention them before we have introduced the framework of QCD.
[69] Remember that we read the diagram against the particle flow. We also use the notation $\not{\varepsilon} = \gamma^\mu\varepsilon^a_\mu$.

$p^\mu = p\, n^\mu$. Lastly, we integrate over all external momenta, giving the amplitude to $\mathcal{O}(g^2)$

$$\begin{aligned}\mathcal{M} &= \mathcal{M}^{(0)} + \mathcal{M}^{(1)} + \mathcal{M}^{(2)} + \mathcal{O}(g^3) \\ &= \bar{u}(p)\Big(1 + ig\int \frac{d^dk_1}{(2\pi)^d}\, n\cdot A(k_1)\frac{-i}{n\cdot k_1 - i\epsilon} \\ &\quad + (ig)^2\int \frac{d^dk_2}{(2\pi)^d}\frac{d^dk_1}{(2\pi)^d} n\cdot A(k_2)\, n\cdot A(k_1)\frac{-i}{n\cdot k_1 - i\epsilon}\frac{-i}{n\cdot k_1 + n\cdot k_2 - i\epsilon} \\ &\quad + \mathcal{O}(g^3)\Big)\,\mathcal{B}\,, \end{aligned} \tag{2.189}$$

where we observe that the term inside the bracket is the $\mathcal{O}(g^2)$ expansion of the Wilson line given in Eq. (2.185). Hence, the definition of a dressed fermion, also called an *eikonal fermion*, is given by[70]

$$\overline{\Psi}(x) = \bar{\psi}(x)\mathcal{U}[\infty, 0]\,, \quad \text{and} \quad \Psi(x) = \mathcal{U}^\dagger[\infty,0]\psi(x)\,. \tag{2.190}$$

These are now resummed fermion lines, as all the radiation has been exponentiated. The crucial step for this to happen was to add zero, by using that $\bar{u}(p)\not{p} = 0$. Hence, Wilson lines as a resummation of gauge boson radiation can only appear next to on-shell fermions.

What this implies is that by taking the soft limit of a scattering process, the structure of that amplitude is fully described by a Wilson line. This feature is one reason that Wilson lines are such useful and important objects to use in scattering calculations. As a teaser to why this is important: in Chapter 4 we will use factorization theorems to separate a cross section into hard and soft parts. This soft part is what we will use Wilson lines to construct. This is an important concept to keep in mind when we delve into factorization of cross sections.

[70] These definitions are not operator valued, but ment to hold inside matrix elements.

Chapter 3

Perturbative Quantum Chromodynamics

In this chapter we take a closer look at QCD as the qunatum field theory of the strong interaction. We begin by introducing the Lagrangian and its constituents leading up to a discussion on the notion of asymptotic freedom and the running of the strong coupling. Then the process of DIS is explored, where important concepts as factorization and parton distribution functions are introduced. An operator valued expression for the parton distributions is derived and we show how Wilson lines can be used to render these gauge invariant. We also introduce perturbative parton-in-parton distributions by dressing them with Wilson lines. Thereafter, we investigate the Drell-Yan process and calculate a NLO quark-antiquark annihilation process. We show that even after renormalization and regularization of this cross section that there are certain regions of phase space where it gain large logarithmic corrections.

3.1 Field Theoretical Description

In Chapter 2 we introduced a geometric formulation behind non-Abelian gauge theories. We are now ready to build on this formalism and introduce the non-Abelian gauge theory of the strong interaction, namely Quantum Chromodynamics. It is formulated in terms of quark and gluon fields, where quarks are spin 1/2-fermions and gluons are spin-1 gauge bosons. The symmetry group is $SU(3)_c$, meaning that the fields carry a quantum number which we call colour.

The quark fields live in the fundamental representation of the gauge group, and is represented as a triplet in colour space

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \end{pmatrix} , \tag{3.1}$$

which we know transform under local gauge transformations as

$$\psi(x) \to U(x)\psi(x) , \tag{3.2}$$

where

$$U(x) = e^{i\alpha^a(x)t^a} . \tag{3.3}$$

The generators t^a are 3×3 hermitian matrices, where the hermiticity follows by insisting that the group matrices U are unitary, while keeping the group parameters $\alpha^a(x)$ real. The generators of $SU(3)$ are traceless, and the group index runs over eight values $a = 1, \ldots, 3^2 - 1 = 8$. This has the implication that there are three different coloured quarks and there are eight gluons. However, experimentally we know that there are three families of quarks, which are represented as doublets

$$\begin{pmatrix} u \\ d \end{pmatrix} , \begin{pmatrix} c \\ s \end{pmatrix} , \begin{pmatrix} t \\ b \end{pmatrix} , \tag{3.4}$$

giving that there are in total 36 different quarks. In a scattering calculations we will not explicitly specify whether we are dealing with an up-quark or a down-quark, just regard them as fermions with fractional electric charge and remember to sum over all of them. We will mostly work in the high energy limit where we use that the quarks are massless, and then we tend to ignore the heaviest generation of the top and bottom quarks.

From Eq. (2.20) we know that the generators satisfy the commutation relation

$$[t^a, t^b] = i f^{abc} t^c \,, \tag{3.5}$$

in which the structure constants are real and antisymmetric in the indices a, b and c. The normalization of the structure constants fixes the trace and Casimir invariants for any representation R, i.e.

$$\mathrm{tr}(t^a t^b) = C(R)\delta^{ab} \,, \qquad \sum_a t^a t^a = C_2(R) \,. \tag{3.6}$$

For the fundamental representation of the quarks we have that

$$C(\mathrm{fund}) = T_F = \frac{1}{2} \tag{3.7}$$

$$C_2(\mathrm{fund}) = C_F = \frac{4}{3} \,, \tag{3.8}$$

and for the adjoint representation of the gluons we have that

$$C(\mathrm{adj}) = C_2(\mathrm{adj}) = C_A = 3 \,. \tag{3.9}$$

The classical QCD Lagrangian follows directly from the Yang-Mills Lagrangian Eq. (2.145), and the quantized from the Fadeev-Popov Lagrangian Eq. (2.151). Using axial gauge we have that the gauge fixed QCD-Lagrangian is given by

$$\mathcal{L}_{QCD} = -\frac{1}{4}(F^a_{\mu\nu})^2 + \bar{\psi}(i\not{D} - m)\psi - \frac{1}{2\xi}(n^\mu A^a_\mu)^2 \,. \tag{3.10}$$

where it is understood that the quark fields carry both flavour index and colour index, i.e. $\psi = (\psi_{\alpha\, i})$ where $\alpha = u, d, s, c, b, t$ and $i = 1, 2, 3$. Expanding this Lagrangian will give all Feynman rules needed for calculating Feynman diagrams. The expansion is straightforward using Eq. (2.130), but tedious. We can group the different parts in the following way[71]

$$\mathcal{L}_{QCD} = \mathcal{L}_{\mathrm{GF}} + \mathcal{L}_{\mathrm{Dirac}} + \mathcal{L}_{\mathrm{int}} \,, \tag{3.11}$$

where

$$\mathcal{L}_{\mathrm{GF}} = -\frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2 - \frac{1}{2\xi}(n^\mu A^a_\mu)^2 \,, \tag{3.12}$$

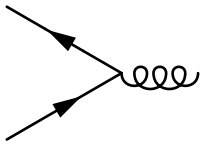
$$\mathcal{L}_{\mathrm{Dirac}} = \bar{\psi}(i\not{\partial} - m_0)\psi \,, \tag{3.13}$$

$$\begin{aligned}\mathcal{L}_{\mathrm{int}} =& g_0 \bar{\psi} \not{A}^a t^a \psi - g_0 f^{abc}(\partial_\mu A^a_\nu) A^{\mu\, b} A^{\nu\, c} \\ & - \frac{1}{2} g_0^2 f^{abe} f^{ecd} A^a_\mu A^b_\nu A^{\mu\, c} A^{\nu\, d} \,.\end{aligned} \tag{3.14}$$

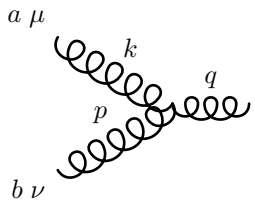
The first term of Eq. (3.14) describes the interaction between quarks and gluons, the second term the interaction between three gluons and the fourth term between four gluons. These terms give rise to the following vertex rules[72]

[71] These are the bare fields, but since there are too many subscripts and superscripts already we will not write this out explicitly.

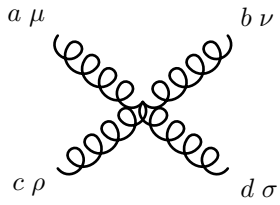
[72] In the three gluon vertex all momenta are pointing towards the vertex.



$$= ig\gamma^\mu t^a\,, \tag{3.15}$$



$$= gf^{abc}\big(g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu\big)\,, \tag{3.16}$$



$$\begin{aligned} &= -ig^2\big[f^{abc}f^{cde}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\ &\quad + f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})\big]\,. \end{aligned} \tag{3.17}$$

The propagators for fermions and gauge bosons can be found in e.g. Eq. (1.41) and Eq. (2.159).

The missing piece to this Lagrangian is to renormalize it by rescaling the fields and define counterterms that remove all UV-divergences. We have already seen in Sec. 1.5.2 how this works for a scalar theory. We will use that the exact same procedure applies for QCD, with the obvious difference that there are several fields and a more intricate relation between the parameters and the renormalization constants. We rescale the bare fields in the usual way

$$A^\mu \to \mathcal{Z}_3^{1/2} A^\mu\,, \tag{3.18}$$

$$\psi \to \mathcal{Z}_2^{1/2}\psi\,, \tag{3.19}$$

where we need different renormalization constants for different fields. We have neglected all subscript separating bare and renormalized fields, but keep in mind that with this rescaling it is the renormalized fields that appear in the 'new' Lagrangian. We can now define the counterterms

$$\delta_1 = Z_1 - 1\,, \qquad \delta_2 = \mathcal{Z}_2 - 1\,, \qquad \delta_3 = \mathcal{Z}_3 - 1\,, \tag{3.20}$$

$$\delta_m = \mathcal{Z}_2 m_0 - m\,, \qquad \delta_{A^3} = \mathcal{Z}_{A^3} - 1\,, \qquad \delta_{A^4} = \mathcal{Z}_{A^4} - 1\,, \tag{3.21}$$

where we defined

$$\mathcal{Z}_1 g = \mathcal{Z}_2\mathcal{Z}_3^{1/2} g_0\,, \qquad \mathcal{Z}_{A^3} g = \mathcal{Z}_3^{3/2} g_0\,, \qquad \mathcal{Z}_{A^4} g^2 = \mathcal{Z}_3^2 g_0^2\,, \tag{3.22}$$

such that the Lagrangian takes the form

$$\mathcal{L} = \mathcal{L}_\mathrm{R} + \mathcal{L}_\mathrm{CT}\,, \tag{3.23}$$

where the first part is identical to Eq. (3.11), but with the renormalized fields and parameters. The counterterm part is given by

$$\mathcal{L}_\mathrm{CT} = -\frac{1}{4}\delta_3\big(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\big)^2 - \frac{1}{2\xi}\delta_3(n^\mu A_\mu^a)^2 + \delta_2\bar\psi(i\partial\!\!\!/ - \delta_m)\psi\,, \tag{3.24}$$

$$\begin{aligned} &+ g\delta_1\bar\psi A\!\!\!/^a t^a\psi - g\delta_{A^3} f^{abc}(\partial_\mu A_\nu^a)A^{\mu\,b}A^{\nu\,c} \\ &- \frac{1}{2}g^2\delta_{A^4} f^{abe}f^{ecd}A_\mu^a A_\nu^b A^{\mu\,c}A^{\nu\,d}\,. \end{aligned} \tag{3.25}$$

The definition of the counterterms only makes sense if one gives a precise definition for the physical mass and coupling, i.e. one has to define renormalization conditions to put constraints on the counterterms. For example, the physical mass of the quarks are defined as the pole of the quark propagator at all orders, just like we did for ϕ^4-theory.

The notion of running coupling in non-Abelian gauge theories is different than in Abelian gauge theories or scalar theories. The concept of changing with scale is not different, but how it changes with scale. In Eq. (1.279) we found that the coupling in ϕ^4-theory increases with the change of scale. This is also true for an Abelian gauge theory like QED, but for a non-Abelian gauge theory like QCD this is no longer true. In QCD the coupling is large in the low energy regime, and perturbation theory breaks down, while it becomes smaller at higher energies. This phenomena is known as *asymptotic freedom* and is only present in non-Abelian gauge theories. To see how this comes about, we have to look at the beta function of the theory. The beta function in QCD is found by calculating the quark self-energy (giving $\mathcal{Z}_2$), corrections to the gluon propagator (giving $\mathcal{Z}_3$) and corrections to the vertex function (giving $\mathcal{Z}_1$). This is a very long and tedious calculation, so we will not perform it explicitly here, but instead state the results in order to understand the behaviour of the strong coupling.

In Eq. (3.22) the relation between the bare coupling and renormalized coupling is given by

$$g_0 = \frac{\mathcal{Z}_1}{\mathcal{Z}_2 \mathcal{Z}_3^{1/2}} g\mu^\epsilon \,, \tag{3.26}$$

where we have inserted μ^ϵ to make the coupling dimensionless in dimensional regularization. We can now use that the bare coupling is independent on μ, giving

$$0 = \mu \frac{\mathrm{d}g_0}{\mathrm{d}\mu} = \mu \frac{\mathrm{d}}{\mathrm{d}\mu}\Big[\frac{\mathcal{Z}_1}{\mathcal{Z}_2 \mathcal{Z}_3^{1/2}} g\mu^\epsilon\Big] \,, \tag{3.27}$$

and performing the differentiation will give the differential equation

$$\mu \frac{\mathrm{d}g}{\mathrm{d}\mu} = -\epsilon g - g\Big[\frac{\mu}{\mathcal{Z}_1}\frac{\mathrm{d}\mathcal{Z}_1}{\mathrm{d}\mu} - \frac{\mu}{\mathcal{Z}_2}\frac{\mathrm{d}\mathcal{Z}_2}{\mathrm{d}\mu} - \frac{1}{2}\frac{\mu}{\mathcal{Z}_3}\frac{\mathrm{d}\mathcal{Z}_3}{\mathrm{d}\mu}\Big] \,. \tag{3.28}$$

The μ dependence on $\mathcal{Z}_i$ is only through the coupling g, so we can write

$$\frac{\mu}{\mathcal{Z}_i}\frac{\mathrm{d}\mathcal{Z}_i}{\mathrm{d}\mu} = \frac{1}{\mathcal{Z}_i}\frac{\partial \mathcal{Z}_i}{\partial \mu}\mu\frac{\partial g}{\partial \mu} = \frac{1}{\mathcal{Z}_i}\frac{\partial \mathcal{Z}_i}{\partial \mu}\beta \,, \tag{3.29}$$

and use that $\mathcal{Z}_i = 1 + \delta_i$, where the relevant counterterms can be found in [3][73],

$$\delta_1 = -\frac{1}{\epsilon}\frac{g^2}{(4\pi)^2}(C_F + C_A) \tag{3.30}$$

$$\delta_2 = -\frac{1}{\epsilon}\frac{g^2}{(4\pi)^2}C_F \tag{3.31}$$

$$\delta_3 = \frac{1}{\epsilon}\frac{g^2}{(4\pi)^2}\big(\frac{5}{3}C_A - \frac{4}{3}n_f T_F\big) \,, \tag{3.32}$$

where n_f is the number of flavours. We observe that the counterterms are defined at $\mathcal{O}(g^2)$. We are only interested in the one-loop correction and since all one-loop diagrams have $\mathcal{O}(g^3)$, we can write

$$\frac{1}{\mathcal{Z}_i}\frac{\partial \mathcal{Z}_i}{\partial \mu} = \frac{\partial \delta_i}{\partial g} + \cdots \,, \tag{3.33}$$

and use that

$$\beta = -\epsilon g + \mathcal{O}(g^2) \,, \tag{3.34}$$

leading to the following differential equation

$$\mu\frac{\mathrm{d}g}{\mathrm{d}\mu} = -\epsilon g + \epsilon g^2 \frac{\partial}{\partial g}\big(\delta_1 - \delta_2 - \frac{1}{2}\delta_3\big) = -\epsilon g - \frac{g^3}{(4\pi)^2}\big(\frac{11}{3}C_A - \frac{4}{3}n_f T_F\big) \,. \tag{3.35}$$

[73] They use a different convention in dimensional regularization, with $d = 4 - \epsilon$. The difference is just a factor of two, so we adjust the answer to suit our convention.

We can safely send $\epsilon \to 0$, and define

$$\beta_0 = \frac{11}{3} C_A - \frac{4}{3} n_f T_F \,. \tag{3.36}$$

The solution to Eq. (3.35) can be found by using separation of variables, giving

$$g^2(\mu) = \frac{g^2(\mu_0)}{1 + \frac{g^2(\mu_0)}{(4\pi)^2}\beta_0 \ln\left(\frac{\mu^2}{\mu_0^2}\right)} \,. \tag{3.37}$$

In scattering observables we will mostly use that $\alpha_s = g^2/4\pi$, so we can write Eq. (3.37) as

$$\alpha_s(\mu) = \frac{\alpha_s(\mu_0)}{1 + \frac{\alpha_s(\mu_0)}{4\pi}\beta_0 \ln\left(\frac{\mu^2}{\mu_0^2}\right)} \,, \tag{3.38}$$

which is the well known one-loop coupling of QCD. The crucial point here is that: as long as $n_f < 17$, $\beta_0 > 0$, and the coupling decreases as a function of increasing μ. We know that there are six flavours of quarks, so QCD is an asymptotically free theory. This discovery was made independetly by D.Pulitzer, F.Wilczek and D.Gross, for which they received the 2004 Nobel Prize in physics.

The notion of asymptotic freedom is very counter-intuitive, as it implies that particles at an infinitesimally small separation do not attract each other (colourwise). There is a metaphor for this kind of behaviour, which is also useful to explain what we call *colour confinement*. Imagine we have two particles connected by a rubber band. If we move the particles closer and closer each other, the rubber band will relax and nothing happens. The moment we try to move the particles apart, the tension in the band will increase, and the more we pull trying to separate the particles, the stronger the tension. Hence, it can work as a metaphor for why coloured particles has never been seen to exist as free states, i.e. they are confined to exist as colour neutral composite particles. We can actually extend the analogy even further. Imagine that we stretch the rubber band with such force that it snaps, i.e. we have succeeded in separating them. However, the amount of energy needed to snap the gluon binding energy is so large that a particle-antiparticle pair is created and binds with the particles we try to separate, giving new colour neutral states. However, this is only an analogy as confinement is still not theoretically understood.

The closest attempt to understand and explain confinement is due to Kenneth Wilson by the use of Wilson loops on the lattice. It is well known that the attempt to define a expectation value of a QCD current describing the potential between two colour charges fails. This procedure works in QED, and can be shown to be equivalent to calculating the beta function, i.e. it gives a measure of the running of the coupling. The problem with this approach in QCD is that it is not gauge invariant. This led Kenneth Wilson to postulate that this potential could be described by instead defining the expectation value of a Wilson loop on the lattice. Wilson's idea was that if the expectation value of a Wilson loop could be shown to be proportional to the area of the Wilson loop, it would indicate that the potential between colour charges grew with distance. Wilson was able to show analytically that on the lattice the expectation value of a Wilson loop scales as the area of the loop, which indicates that confinement is present in QCD [1]. The problem is that Wilson's arguments is valid for any gauge theory, but QED does most certainly not have this behaviour. There is also a problem with preserving confinement in the continuum limit, i.e. when the lattice spacing is removed. However, there has been a lot of progress in this field the last decades, and that is due to the development of generalized loops and loop calculus, see e.g. [18, 20, 21]. We will not pursue confinement and the low energy regime any further, but it is worth mentioning that Wilson loops/lines are at the centre of both perturbative and non-perturbative QCD.

Apart from small coupling at large energies, the other consequence of asymptotic freedom is that at a given energy scale, the coupling becomes large and invalidates perturbation theory. In theories without asymptotic freedom, like QED, this scale is so large that it is not relevant to consider, but for QCD we have that for $\Lambda_{\text{QCD}} \approx 250\text{MeV}$ the coupling becomes larger than one. The consequence of this is that we can not use perturbation theory to understand the low energy behaviour of bound states. This separates QCD into two regimes; a part where $\mu > \Lambda_{\text{QCD}}$ where perturbation theory is valid and a region where we can not calculate. This latter part needs to be extracted from experiment, and this is the basis for the *factorization* framework we will investigate in more detail in this chapter.

3.2 Deep Inelastic Scattering

We have given a somewhat formal introduction of Quantum Chromodynamics as the theory of strong interactions. However, as this description only applies to the fundamental constituents, namely the quarks and gluons, we need to connect this formalism to scattering amplitudes involving particles that are composited of quarks and gluons. These composited particles are what particle physicists call *hadrons*. Hadrons are categorized into two families: baryons, made of an odd number of quarks – usually three quarks – and mesons, made of an even number of quarks—usually one quark and one antiquark. Protons and neutrons are examples of baryons and pions are an example of a meson. For more detail on the classification of particles, see [22].

As a non-abelian gauge theory, QCD has the peculiar behaviour of asymptotic freedom, and as described above, this is due to the anti-screening of gluons. As the coupling is 'small' at high energies, the intuitive approach is to use perturbation theory. However, before that can be accomplished in a meaningful way, we have to define exactly what we are to expand.

Asymptotic freedom effectively separates the strong interaction into two regimes: A perturbative regime – α_s is smaller than unity – where we can expand using standard field theory methods, and a non-perturbative regime where we are unable to calculate. Unfortunately, we can not ignore the non-perturbative regime in real-life scattering processes. Thus, we need a method where we can unite both regimes into useful observables. To achieve this we need the concept of *factorization*, which allows for separation of the low and high energy regime. This separation is made at an arbitrary energy scale, called the *factorization scale*, and just like the renormalization scale observables can not depend on it. This naturally leads to evolution equations that describe the behaviour of the non-perturbative part as a function of the energy scale.

The original, and still one of the most powerful, test of perturbative QCD is the Bjorken scaling in *deep inelastic scattering* (DIS). Here a hadron is probed by a highly relativistic lepton, breaking up the hadron, creating additional hadrons in the final state. Only the lepton needs to be measured in the final state, meaning the final state hadrons can be integrated out. At sufficiently high energies the DIS experiments indicated the lepton scattered of point-like particles. This led Richard Feynman to postulate that hadrons were composite objects made up of fundamental constituents, which he called partons. This was the birth of what is called the *parton model* [23].

James Bjorken was the first to formalise the parton model in DIS [24], where he was able to predict the behaviour of the hadronic system without any fundamental Lagrangian or knowledge of the hadron structure. He parametrized the process using so-called *structure functions*. In the high energy limit, he showed that these structure functions only depended on a dimensionless scaling variable.

In Sec. 3.2.1 we will take a closer look at this parametrization and via a calculation verify Bjorken scaling in the parton model before we in Sec. 3.2.2 include QCD effects and move on to the more precise concept of factorization.

3.2.1 DIS and the Parton Model

Protons are in the parton model envisioned as extended objects, made up of partons and glued together by their mutual interactions. Of course, the partons are the quarks and gluons of quantum chromodynamics, but we will proceed without referring to this fact just yet.

The essential assumption of the parton model is that at high energies and momentum transfer, the electron scatters of 'free' point-like particles. This seems like a strange assumption, as the strong interaction between the constituents is what make protons bound objects in the first place. To justify this assumption, we can use basic principles from special relativity. We consider the inclusive process of electron–proton scattering at high energy, where the interaction goes through an exchange of a virtual photon with momentum q. In the centre-of-mass frame, two key concepts happen to the proton: since it is ultra-relativistic the proton is Lorentz contracted in the direction of travel, and the internal interactions are time-dilated.

Therefore, if the time the electron uses to traverse the proton is shorter than the dilated time of interaction, the electron effectively scatters of non-interacting partons. This is known as the impulse approximation. Also, when the momentum transfer is very high, the virtual photon becomes short-lived. Then, if the parton density is sufficiently low, the electron can only interact with one single parton. Since the partons do not interact between themselves during this period, each carries a definite fraction ξ of the proton's momentum P. With this picture, it is possible to calculate the electron–parton interaction using perturbation theory without considering the proton as a whole.

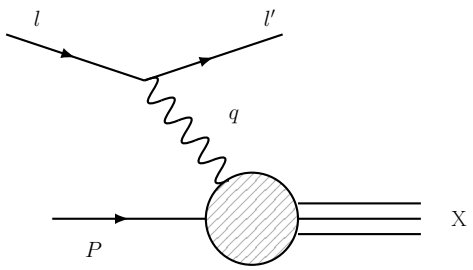


Figure 3.1: Kinematics of deep inelastic electron–proton scattering

The high energy inelastic process is therefore separated into a short-distance scattering part—the interaction between the photon and one of the partons—called the *hard* part, and a long-distance—everything inside the proton—*soft* part. Because of the different time scales of these two effects, one assumes that there is no quantum mechanical interference between them. The hadronic cross-section may thus be calculated by combining probabilities, rather than amplitudes. We define parton distribution functions $f_{i/h}(\xi)$ to describe the probability that a parton of type i has a fraction ξ of the hadrons momentum.

Kinematics

We consider the scattering of a high energy electron off a proton target, via the exchange of a virtual photon (see Fig. 3.1). The full treatment would involve weak interactions, but for our purposes we only consider the electromagnetic interaction. We lable the incoming and outgoing electron momenta with l^μ and l'^μ respectively, the momentum of the proton by P^μ and the momentum transfer by the photon $q^\mu = l^\mu - l'^\mu$. We will assume that the mass of the electron and the proton constituents are negligible compared to the scale of the process. The centre-of-mass energy squared is then

$$s = (P + l)^2 = m_p^2 + 2P \cdot l \,, \tag{3.39}$$

where m_p is the mass of the proton. It follows that the momentum transfer squared is given by

$$q^2 = 2E_l E_{l'}(\cos\theta_{ll'} - 1) \leq 0 \,. \tag{3.40}$$

Therefore, it is useful to instead define $Q^2 \equiv -q^2 \geq 0$. The invariant mass of the final state X is given by

$$m_X^2 = (P + q)^2 = m_p^2 + 2P \cdot q - Q^2 \,. \tag{3.41}$$

In order to have *deep* and *inelastic* scattering, we must have the requirement that $Q^2 \gg m_p^2$—the momentum transfer is so large that the proton target is very much excited—and inelastically $m_X^2 \gg m_p^2$.

The two independent Lorentz invariants for the hadron system are Q^2 and $P \cdot q$, but it is convenient to define additional invariant variables

$$x_B = \frac{Q^2}{2P \cdot q} \,, \tag{3.42}$$

$$y = \frac{P \cdot q}{P \cdot l} = \frac{Q^2}{x_B(s - m_p^2)} \,. \tag{3.43}$$

Here x_B is called *Bjorken-x*, which we from now will only denote by x. Kinematically x is restricted to the range $Q^2/s + Q^2 \leq x \leq 1$, neglecting terms of $\mathcal{O}(m_p^2/Q^2)$. In the parton model we will find that x gives an estimate for the hadron's momentum fraction that is carried by the struck parton. In the rest frame of the proton y is the fractional energy loss of the electron, but it is not an independent variable since it is given by Q^2 and x.

Factorization and Bjorken Scaling

From the above picture, the parton model describes DIS without the strong interaction participating, as all strong effects have been absorbed into the proton. Consequently, the proton is like a black box, where we have no idea of its structure, except that we can extract a parton from it.

Before we write down the amplitude and differential cross-section for this process, we must define some relations for current matrix elements. A fermion current is $j^\mu(z) = \bar{\psi}(z)\gamma^\mu\psi(z)$, with $\psi(z)$ the fermion field. A generic current matrix element for a fermion is then given by

$$\langle k'|j^\mu(z)|k\rangle = \bar{u}(k')\gamma^\mu u(k)\, e^{i(k'-k)\cdot z}\,, \tag{3.44}$$

where we used z as space-time variable in order to avoid confusion with the Bjorken-x. The relation in Eq. (3.44) can be derived using the expansion of the fermion fields in terms of creation and annihilation operators. Thus, a shorthand for the spinor product $\bar{u}(k')\gamma^\mu u(k)$ coming out of matrix elements is just the current matrix element at space-time point $z = 0$, i.e.

$$\langle k'|j^\mu(0)|k\rangle = \bar{u}(k')\gamma^\mu u(k)\,. \tag{3.45}$$

Then we can write the amplitude for DIS as

$$\mathcal{M} = \langle l'|\, j_L^\mu\, |l\rangle\langle X|\, j_H^\nu\, |P\rangle\, D_{\mu\nu}(q)\,, \tag{3.46}$$

where $D_{\mu\nu}(q)$ is just the regular photon propagator, see Eq. (1.153). The differential cross section then takes the form

$$d\sigma = \frac{1}{4\,P\cdot l}\frac{d^3l'}{(2\pi)^3\,2E_{l'}}\sum_X\int\frac{d^3p_X}{(2\pi)^3\,2E_X}(2\pi)^4\delta^{(4)}(P+l-p_X-l)|\mathcal{M}|^2\,,$$

giving

$$E_{l'}\frac{d\sigma}{d^3l'} = \frac{2}{s-m_p^2}\frac{\alpha^2}{Q^4}L_{\mu\nu}W^{\mu\nu}\,, \tag{3.47}$$

where α is the fine structure constant. As we only consider photon exchange, the lepton tensor is completely determined by QED

$$L_{\mu\nu} = \frac{1}{2}\mathrm{tr}[\not{l}\gamma_\mu\not{l}'\gamma_\nu] = 2\big(l_\mu l'_\nu + l_\nu l'_\mu - g_{\mu\nu}\, l\cdot l'\big)\,. \tag{3.48}$$

In contrast, the hadronic tensor contains all the information about the interaction between the electromagnetic current j_H^μ and the proton P

$$\begin{aligned} W^{\mu\nu} &= 4\pi^3\sum_X\int\frac{d^3p_X}{(2\pi)^3\,2E_X}\delta^{(4)}(P+q-p_X)\,\langle P|\, j^{\dagger\,\mu}(0)\,|X\rangle\langle X|\, j^\nu(0)\,|P\rangle \\ &= \frac{1}{4\pi}\int d^4z\sum_X\int\frac{d^3p_X}{(2\pi)^3\,2E_X}\,e^{iz\cdot(P+q-p_X)}\,\langle P|\, j^{\dagger\,\mu}(0)\,|X\rangle\langle X|\, j^\nu(0)\,|P\rangle\,, \end{aligned}$$

where we used the integral representation of the four dimensional delta function. Further, we can use the translation operator

$$\begin{aligned} \langle P|\, j^{\dagger\,\mu}(0)\,|X\rangle e^{iz\cdot(P-p_X)} &= \langle P|e^{iz\cdot\hat{P}}\, j^{\dagger\,\mu}(0)\, e^{-iz\cdot\hat{P}}|X\rangle \\ &= \langle P|\, j^{\dagger\,\mu}(z)\,|X\rangle\,, \end{aligned} \tag{3.49}$$

and integrate out a complete set of states by the use of the completeness relation:

$$\sum_X\int\frac{d^3p_X}{(2\pi)^3\,2E_X}|X\rangle\langle X| = \mathbf{1}\,. \tag{3.50}$$

The hadronic tensor can then be written as

$$W^{\mu\nu} = \frac{1}{4\pi}\int d^4z\, e^{iq\cdot z}\langle P|\, j^{\dagger\,\mu}(z)j^\nu(0)|P\rangle\,. \tag{3.51}$$

This tensor can now be decomposed in terms of tensors that governs the kinematics times scalar functions. To do this, we use that the electromagnetic current is conserved, $\partial_\mu j^\mu = 0$, so that the hadronic tensor satisfies the Ward identity $q_\mu W^{\mu\nu} = 0$. Further, using that the strong interaction is parity invariant and $W^{\mu\nu}$ is hermitian, the most general form of the hadronic tensor for unpolarized protons can be written as

$$W^{\mu\nu} = \Big(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\Big) F_1(x,Q^2) + \Big(P^\mu + \frac{1}{2x}q^\mu\Big)\Big(P^\nu + \frac{1}{2x}q^\nu\Big)\frac{1}{P\cdot q}F_2(x,Q^2)\,. \tag{3.52}$$

The scalar functions F_1 and F_2 are the structure functions we alluded to earlier. They contain the information of the hadron structure as 'seen' by the virtual photon. Combining this with the leptonic tensor we find that

$$L_{\mu\nu}W^{\mu\nu} = \frac{2Q^2}{xy^2}\Big[\Big(1-y+\frac{y^2}{2}\Big)2xF_1(x,Q^2) + (1-y)\big(F_2(x,Q^2) - 2xF_1(x,Q^2)\big)\Big]\,. \tag{3.53}$$

Plugging this into Eq. (3.47), and neglecting terms of $\mathcal{O}(m_p^2/Q^2)$ gives the unpolarized electron–proton DIS cross section

$$\frac{d^2\sigma}{dxdy} = \frac{4\pi\alpha^2 s}{Q^4}\Big[\Big(1-y+\frac{y^2}{2}\Big)2xF_1(x,Q^2) + (1-y)\big(F_2(x,Q^2) - 2xF_1(x,Q^2)\big)\Big]\,. \tag{3.54}$$

To find a parton model prediction for the behaviour of the structure functions, we calculate the partonic equivalent of *Eq.* (3.54). Thus, we are only interested in electron–quark scattering, $e^-(l)q(p) \to e^-(l')q(p')$. By using the Mandelstam variables

$$\hat{s} = (l+p)^2\,, \qquad \hat{t} = (l-l')\,, \qquad \hat{u} = (p-l')\,, \tag{3.55}$$

it is straightforward to show that the spin/colour averaged amplitude takes the form

$$\langle\,|\mathcal{M}|^2\rangle = 2Q_q^2 e^4\,\frac{\hat{s}^2+\hat{u}^2}{\hat{t}^2}\,, \tag{3.56}$$

where Q_q is the fractional charge of the quark. Using the standard result for the differential cross section for massless $2\to 2$ scattering:

$$\frac{d\hat{\sigma}}{d\hat{t}} = \frac{1}{16\pi\hat{s}^2}\langle\,|\mathcal{M}|^2\rangle\,, \tag{3.57}$$

which after some rewriting will give the partonic differential cross section

$$\frac{d\hat{\sigma}}{dy} = Q_q^2\frac{4\pi\alpha^2\hat{s}}{Q^4}\Big(1-y+\frac{y^2}{2}\Big)\,. \tag{3.58}$$

In order to relate the hard cross section with the full cross section, we define the quark momentum as a fraction of the proton momentum,

$$p^\mu = \xi P^\mu\,, \qquad 0<\xi<1\,, \tag{3.59}$$

such that $\hat{s} = \xi s$. If we approximate with an on-shell constraint for the outgoing quark, we find that

$$p'^2 = (p+q)^2 = 2\xi P\cdot q - Q^2 = 0\,, \tag{3.60}$$

implying that $\xi = x$. The on-shell constraint fixes the momentum fraction to equal the Bjorken variable, but this is of course not a general result. The Bjorken-x is a kinematical constraint defining the process, while ξ is just a momentum fraction that is independent of the process. To obtain a double differential cross-section as in Eq. (3.54), we simply use that

$$\int_0^1 dx\,\delta(x-\xi) = 1\,, \tag{3.61}$$

and write

$$\frac{d^3\hat{\sigma}}{dxdyd\xi} = Q_q^2\frac{4\pi\alpha^2 s}{Q^4}\Big(1-y+\frac{y^2}{2}\Big)\,\xi\,\delta(x-\xi)\,. \tag{3.62}$$

We can now use the parton model interpretation of $f_q(\xi)$ as a probability and convolute it with the partonic part. To find the electron–proton differential cross section we simply integrate over all possible fraction ξ and sum over quark flavour

$$\begin{aligned}\frac{d^2\sigma}{dxdy} &= \sum_q \int_0^1 d\xi\, f_q(\xi)\, \frac{d^3\hat{\sigma}}{dxdyd\xi} \\ &= \frac{4\pi\alpha^2 s}{Q^4}\Big(1-y+\frac{y^2}{2}\Big)\sum_q Q_q^2\, x\, f_q(x)\,. \end{aligned} \tag{3.63}$$

Then if we compare Eq. (3.63) with Eq. (3.54) we see that the proton structure functions in this simple model is given by

$$F_2(x) = 2xF_1(x) = \sum_q Q_q^2\, x\, f_q(x)\,. \tag{3.64}$$

This result shows that in the regime where Q^2 is very large, we have that the structure functions only depend on the Bjorken-x. This is what is called *Bjorken scaling*, and the result $F_2 = 2xF_1$ is known as the *Callan-Gross relation*. The Callan-Gross relation follows from the spin-1/2 nature of quarks and was later confirmed by structure function measurements. These predictions made the parton model an intriguing model to further explore the structure of hadrons.

In Eq. (3.63) we convoluted the partonic cross section with the parton distributions resulting in a hadronic cross section. Therefore, we can also define quark structure functions that we can convolute with the parton distribution, giving the proton structure functions. We see that if we define

$$\hat{F}_2 = Q_q^2\, x\, \delta(1-x)\,, \tag{3.65}$$

$$\hat{F}_1 = \frac{1}{2}Q_q^2\, \delta(1-x)\,, \tag{3.66}$$

we can write the parton model factorization formulas for the proton structure functions as,

$$F_2(x) = \sum_q \int_x^1 d\xi\, f_q(\xi)\hat{F}_2\Big(\frac{x}{\xi}\Big) = \sum_q Q_q^2\, x\, f_q(x)\,, \tag{3.67}$$

$$F_1(x) = \sum_q \int_x^1 \frac{d\xi}{\xi} f_q(\xi)\hat{F}_1\Big(\frac{x}{\xi}\Big) = \frac{1}{2}\sum_q Q_q^2 f_q(x)\,, \tag{3.68}$$

giving the same result as above. The integration bound follows from $x/\xi \leq 1$. In the same manner the differential cross section can be written in the following way:

$$d\sigma(x,Q^2) = \sum_q \int_x^1 d\xi\, f_q(\xi)\, d\hat{\sigma}\Big(\frac{x}{\xi}, Q^2\Big)\,, \tag{3.69}$$

$$\tag{3.70}$$

which is known as *collinear factorization* of DIS in the parton model. An important point to make is that the above factorized integrals are convolutions defined in *Mellin Space*, see Appendix D.1. Mellin transformations and their properties are very useful in studying QCD, which we will lay out in more detail in Chapter 4.

The collinear terminology used here refers to the fact that we have only considered the case where the quark momentum is in the same direction as the proton. We should, therefore, point out that $f_q(\xi)$ is formally defined as

$$f_q(\xi) = \int d^2k_\perp f_q(\xi, k_\perp)\,, \tag{3.71}$$

where the transverse momentum dependence has been integrated out. It should also be pointed out that this is only valid as we only measure the final state lepton. If we wanted to take into account the final state hadrons we would have to use *transverse momentum distributions* (TMDs), which is much more complicated. In this chapter we will only consider processes, like DIS and Drell-Yan, where the final state hadrons are integrated out.

We have seen that the parton model result can give predictions that have been confirmed by experiments, but it is a phenomenological model and is not a formal treatment of the strong interaction. However, the concept of factorization is exactly what makes one take a field theoretical approach to hadronic scattering processes. Including QCD into the parton model results in interactions between the partons, and the assumption of free partons does not hold anymore. Therefore, the above factorization formulas are invalid and we must make a more precise definition of factorization in QCD. To this end, we will consider the case of QCD corrections to the DIS process in the next section.

To end this section we will make a brief remark about the difference between structure functions and parton distribution functions. The structure functions appeared when we parametrized the hadronic tensor, which is process dependent. That is, if we considered DIS neutrino scattering, the structure functions would change as we, in that case, considered W or Z boson exchange. The main idea behind factorization is that, inside the structure functions, we can factorize out the proton content from the process dependent part. The factorization ansatz in Eq. (3.63) is required to be valid for any process, meaning that the PDFs are universal. Thus, the PDFs can be extracted from DIS electron–proton scattering experiments and re-used in another experiment like DIS neutrino–proton scattering and proton–proton collisions at the LHC.

3.2.2 Collinear Factorization in QCD

In Sec. 3.2.1 we saw that the observables in the parton model could be written on a factorized form. The crucial element of that result is the assumption that the quark momentum is a fraction of the proton's momentum, $p^\mu = \xi P^\mu$. This is in general not true, as the quark will also have transverse components. To highlight the validity of the parton model assumption it is useful to use *light-cone coordinates*, see Appendix A.1.

By considering the case where the proton has no transverse momentum components, the general form of the proton and parton momenta can be parametrized as

$$P^\mu = \left(P^+, \frac{m_p^2}{2P^+}, 0_\perp\right), \qquad p^\mu = (p^+, p^-, p_\perp). \tag{3.72}$$

It is safe to assume that in the protons rest frame, the distribution of partons is isotropic, and that the components of the parton momentum is of the order of the proton mass. We are free to choose frame, so by choosing the frame where $P^+ \to \infty$, the only remaining component of the proton momentum is its plus-component. It is also hard to imagine that the partons will not follow the proton along this direction—at least if we assume no gluon radiation—so we find that

$$P^\mu = (P^+, 0^-, 0_\perp), \qquad p^\mu \approx (p^+, 0^-, 0_\perp). \tag{3.73}$$

This is what is called the *infinite momentum frame* (IMF), and the partons transverse components has been neglected compared to the plus-component, $p^+ \gg p_\perp \sim m_p$. We can now assume that the struck quark has a fraction ξ of the proton's momentum, giving that in the infinite momentum frame the parton momentum is fully collinear to the proton momentum

$$p^\mu = \xi P^\mu, \tag{3.74}$$

which is the result we used for the parton model in Sec. 3.2.1.

However, if the struck parton is a quark that has just emitted a gluon this is no longer necessarily true, and we will need a more general parametrization of the momenta. We can use $p^+ = \xi P^+$ to parametrize the parton momentum in terms of the large plus-component P^+

$$P^\mu = \left(P^+, \frac{m_p^2}{2P^+}, 0_\perp\right), \qquad p^\mu = \left(\xi P^+, \frac{p^2 + p_\perp^2}{2\xi P^+}, p_\perp\right), \tag{3.75}$$

which reproduces the infinite momentum frame limit for $p^2, p_\perp^2, m_p^2 \ll P^+$. In turn the photon momentum can be parametrized as

$$q^\mu = \left(0, \frac{Q^2}{2xP^+}, q_\perp\right), \tag{3.76}$$

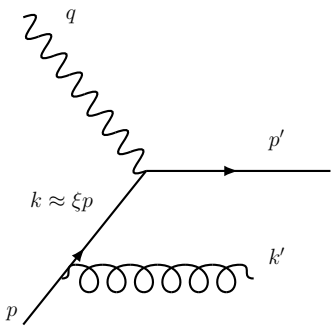


Figure 3.2: Gluon emission from initial quark.

where $-q^2 = q_\perp^2 = Q^2$ and $P \cdot q = Q^2/2x$, meaning that this choice reproduces the known kinematics in DIS, given in Eq. (3.42). Not only does light-cone coordinates formalise the parton model assumption of fully collinear partons in the IMF, but as we will later see it is in this formalism that the parton model has it's closest relation to field theory. In the following calculations we will parametrize all momenta as in Eq. (3.75).

In the parton model calculation we found that the structure functions scale, i.e. $F(x, Q^2) \to F(x)$ in the Bjorken limit $Q^2 \to \infty$. By including QCD into this picture, $F(x, Q^2)$ will have terms that are proportional to $\ln Q^2$, i.e. we have logarithmic breaking of Bjorken scaling. The key point is that the original quark can emit a gluon before being struck by the photon and will acquire a transverse component that can not be neglected. As we shall see below, when integrating over the intermediate quark momentum, the integral over this transverse component extends up to the kinematic limit Q^2, leading to this logarithmic dependancy of Q^2.

Factorization in QCD is therefore the determination of at which point the emitted gluon belongs to the soft part or the hard part of the scattering. The correct way to determine this — at least in the collinear case — is to define a separation in terms of an energy scale μ_F (factorization scale). Thus, if the gluon is emitted at a scale lower than μ_F it is part of the PDF (soft part) and if it is emitted at a scale larger than μ_F it is part of the partonic process (hard part). The all order proof of factorization in QCD is beyond the scope of this thesis, so here we will investigate the $\mathcal{O}(\alpha_s)$ correction to DIS and see how that changes the parton model factorization.

One gluon emission

At NLO there are two real gluon diagrams and three virtual gluon diagrams in DIS. The real diagrams are emission from initial and final quark line, while the virtual diagrams are self energy diagrams of the initial and final state together with exchange of a gluon between the initial and final quark lines. However, we will only consider gluon emission from the initial quark line, given in Fig. 3.2. The reason for this is that we will choose to work with gluons in light-cone gauge, and in this gauge this is the only diagram that gives a logarithmic divergence. The other diagrams will give finite results that are unimportant for the current discussion and will be omitted for brevity.

The way to proceed is to investigate corrections to the hadronic tensor and extract the structure functions from it. The proton strucure functions were in the parton model written as a convolution between the PDF and the quark structure functions, see Eq. (3.67) and Eq. (3.68). The hadronic tensor can also be written on this convoluted form

$$W^{\mu\nu}(x, Q^2) = \sum_q \int_x^1 \frac{d\xi}{\xi} f_q(\xi) \hat{W}^{\mu\nu}\left(\frac{x}{\xi}, Q^2\right), \tag{3.77}$$

where $\hat{W}^{\mu\nu}$ refers to the partonic part that are calculable in perturbation theory. Thus, we will calculate $\hat{W}^{\mu\nu}$ up to $\mathcal{O}(\alpha_s)$ and extract $\hat{F}_2$ from it, enabling us to compare with Eq. (3.67) to see the effect it has on the PDF.

In Eq. (3.52) we parametrized the hadronic tensor as

$$W^{\mu\nu} = \Big(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \Big) F_1(x, Q^2) + \Big(P^\mu + \frac{1}{2x} q^\mu \Big) \Big(P^\nu + \frac{1}{2x} q^\nu \Big) \frac{1}{P \cdot q} F_2(x, Q^2) \,. \tag{3.78}$$

However, in higher order calculations there is an alternative parametrization that comes in handy when we want to extract the structure functions. Hence, we define normalized basis vectors (factors and terms $\mathcal{O}(m_p^2/Q^2)$ are as usual neglected)

$$\hat{q}^\mu \equiv \frac{q^\mu}{Q}\,, \qquad \hat{t}^\mu \equiv \frac{1}{Q}(q^\mu + 2xP^\mu)\,. \tag{3.79}$$

These basis vector satisfy: $\hat{q}\cdot\hat{t}=0$, $\hat{q}^2=-1$ and $\hat{t}^2=1$, meaning that $\hat{q}$ is spacelike and $\hat{t}$ is timelike. With respect to these basis vectors one can also define a transverse tensor

$$g_\perp^{\mu\nu} \equiv g^{\mu\nu} + \hat{q}^\mu\hat{q}^\nu - \hat{t}^\mu\hat{t}^\nu\,, \tag{3.80}$$

from which the following relations follow

$$\hat{t}_\mu g_\perp^{\mu\nu} = 0 \tag{3.81}$$
$$\hat{q}_\mu g_\perp^{\mu\nu} = 0 \tag{3.82}$$
$$g_\perp^{\mu\nu} g_{\perp\,\mu\nu} = 2\,, \tag{3.83}$$

making it consistent with Eq. (A.14). With these definitions it is straightforward to rewrite the hadronic tensor as

$$W^{\mu\nu}(x,Q^2) = -g_\perp^{\mu\nu}F_1(x,Q^2) + \frac{\hat{t}^\mu\hat{t}^\nu}{2x}\big(F_2(x,Q^2) - 2xF_1(x,Q^2)\big)\,, \tag{3.84}$$

from which the structure functions can be extracted by applying appropriate tensors on the hadronic tensor

$$F_1 = -\frac{1}{2}g_\perp^{\mu\nu}W_{\mu\nu}\,, \tag{3.85}$$
$$F_2 = x(2\hat{t}^\mu\hat{t}^\nu - g_\perp^{\mu\nu})W_{\mu\nu}\,. \tag{3.86}$$

For the F_2 projection is is useful to note that $(2\hat{t}^\mu\hat{t}^\nu - g_\perp^{\mu\nu})g_{\mu\nu}=0$, which will cancel all terms involving $g^{\mu\nu}$ in the Dirac traces.

The amplitude for the gluon emission in Fig. 3.2, is given by

$$\mathcal{M}^\mu = -igQ_q t^a \varepsilon^*_\beta(k')\,\bar{u}(p')\,\gamma^\mu\frac{\not{k}}{k^2}\gamma^\beta\,u(p)\,. \tag{3.87}$$

We average over incoming spin and colour, sum over final spin, colour and gluon polarization,

$$\begin{aligned}\langle|\mathcal{M}|^2\rangle^{\mu\nu} &= \frac{1}{N_s N_c}\sum_{a,b}\sum_{\text{spin}}\sum_{\text{pol}}\big(|\mathcal{M}|^2\big)^{\mu\nu}\\ &= \frac{C_F}{N_s}Q_q^2 g^2\frac{1}{k^4}\sum_{\text{pol}}\varepsilon_\alpha(k')\varepsilon^*_\beta(k')\,\text{tr}[\gamma^\nu\not{p}'\gamma^\mu\not{k}\gamma^\alpha\not{p}\gamma^\beta\not{k}]\,.\end{aligned} \tag{3.88}$$

We can then use the gluon polarization sum in the light-cone gauge (see Eq. (A.16))

$$\sum_{\text{pol}}\varepsilon_\alpha(k')\varepsilon^*_\beta(k') = -g_{\alpha\beta} + \frac{k'_\alpha\,n_{-\,\beta}}{k'^+} + \frac{k'_\beta\,n_{-\,\alpha}}{k'^+}\,, \tag{3.89}$$

which will give the following expression for the Dirac trace

$$\begin{aligned}\sum_{\text{pol}}\varepsilon_\alpha(k')\varepsilon^*_\beta(k')\,\text{tr}[\gamma^\nu\not{p}'\gamma^\mu\not{k}\gamma^\alpha\not{p}\gamma^\beta\not{k}] &= -\,\text{tr}[\gamma^\nu\not{p}'\gamma^\mu\not{k}\gamma^\alpha\not{p}\gamma_\alpha\not{k}] + \frac{1}{k'^+}\text{tr}[\gamma^\nu\not{p}'\gamma^\mu\not{k}\not{k}'\not{p}\gamma^+\not{k}]\\ &\quad + \frac{1}{k'^+}\text{tr}[\gamma^\nu\not{p}'\gamma^\mu\not{k}\gamma^+\not{p}\not{k}'\not{k}]\,.\end{aligned}$$

These traces can be calculated using the relations given in Appendix B.1, but it is tedious so we will just write out the main steps. The first trace can be simplified to

$$\text{tr}[\gamma^\nu\not{p}'\gamma^\mu\not{k}\gamma^\alpha\not{p}\gamma_\alpha\not{k}] = -4p\cdot k\,\text{tr}[\gamma^\nu\not{p}'\gamma^\mu\not{k}] + 2k^2\,\text{tr}[\gamma^\nu\not{p}'\gamma^\mu\not{p}]\,, \tag{3.90}$$

and the last two traces can first be simplified by using that $k' = p - k$, giving

$$\mathrm{tr}[\gamma^\nu \not{p}' \gamma^\mu \not{k} \not{k}' \not{p} \gamma^+ \not{k}] = -k^2 \,\mathrm{tr}[\gamma^\nu \not{p}' \gamma^\mu \not{p} \gamma^+ \not{k}]\,, \tag{3.91}$$

$$\mathrm{tr}[\gamma^\nu \not{p}' \gamma^\mu \not{k} \gamma^+ \not{p} \not{k}' \not{k}] = -k^2 \,\mathrm{tr}[\gamma^\nu \not{p}' \gamma^\mu \not{k} \gamma^+ \not{p}]\,, \tag{3.92}$$

where the sum of these two traces can be rewritten as

$$\begin{aligned} -k^2 \mathrm{tr}[\gamma^\nu \not{p}' \gamma^\mu (\not{k} + \not{p}) \gamma^+ (\not{p} + \not{k})] = &-2(k^+ + p^+) k^2 \,\mathrm{tr}[\gamma^\nu \not{p}' \gamma^\mu \not{k}] \\ &- 2(k^+ + p^+) k^2 \,\mathrm{tr}[\gamma^\nu \not{p}' \gamma^\mu \not{p}] \\ &+ k^2 (k^2 + 2p \cdot k) \mathrm{tr}[\gamma^\nu \not{p}' \gamma^\mu \gamma^+]\,. \end{aligned}$$

Collecting all terms and using the cyclic property of the trace, we find that the sum of all traces can be written as

$$\begin{aligned} \sum \mathrm{tr}(...) = &\Big(4p \cdot k - 2k^2 \frac{k^+ + p^+}{k^+ - p^+}\Big) \mathrm{tr}[\not{k} \gamma^\mu \not{p}' \gamma^\nu] \\ &- 2k^2 \Big(1 + \frac{k^+ + p^+}{k^+ - p^+}\Big) \mathrm{tr}[\not{p} \gamma^\mu \not{p}' \gamma^\nu] \\ &+ \frac{k^2}{k^+ - p^+} \big(k^2 + 2p \cdot k\big) \mathrm{tr}[\not{n}_- \gamma^\mu \not{p}' \gamma^\nu]\,, \end{aligned}$$

where we used that $\gamma^+ = \not{n}_-$, such that all the traces take the same form. For generic vectors, a^μ and b^μ, these traces evaluate to[74]

$$\mathrm{tr}[\not{a} \gamma^\mu \not{b} \gamma^\nu] = 4(a^\mu b^\nu + a^\nu b^\mu - g^{\mu\nu} a \cdot b)\,. \tag{3.93}$$

Hence, we write the averaged amplitude as

$$\langle\, |\mathcal{M}|^2 \rangle^{\mu\nu} = \frac{C_F}{N_s} Q_q^2 g^2 \frac{1}{k^4} \sum \mathrm{tr}(...)\,. \tag{3.94}$$

To calculate $\hat{W}^{\mu\nu}$, we must integrate over the final state particles, giving

$$\hat{W}^{\mu\nu} = \frac{1}{4\pi} \int d\mathcal{P}_2 \,\langle\, |\mathcal{M}|^2 \rangle\,, \tag{3.95}$$

where 4π comes from the normalization in Eq. (3.51). The n-body phase space for on-shell massless particles is given by Eq. (1.211), so the two-body phase space takes the form

$$\begin{aligned} d\mathcal{P}_2 &= \int \frac{d^4k'}{(2\pi)^3} \frac{d^4p'}{(2\pi)^3} \,\delta^+(k'^2) \delta^+(p'^2) (2\pi)^4 \delta^{(4)}(p + q - k' - p') \\ &= \frac{1}{4\pi^2} \int d^4k \,\delta^+\big((p-k)^2\big) \delta^+\big((k+q)^2\big)\,, \end{aligned} \tag{3.96}$$

where we use that $\delta^+(k'^2) \equiv \delta(k'^2)\theta(k'^+)$[75]. Note that we work explicitly in four dimensions, which mean we will not use dimensional regularization in this calculation. For the Drell-Yan process (see Sec. 3.3), we will perform a full calculation using dimensional regularization, but for physical intuition it is more useful to use momentum cutoff in DIS.

In the partonic system we assume that the original quark has no transverse components and move in the plus direction, the virtual[76] quark on the other hand may have large transverse components due to the radiation

[74] See Appendix B.1 for more detail on traces.

[75] In light-cone coordinates the usual Heaviside function $\theta(k^0)$ is replaced by $\theta(k^+)$.

[76] All quarks are kind of virtual, but this is common terminology in scattering processes to specify that it is intermediate, i.e. propagating and off-shell.

of a gluon from the original quark. Thus, the relevant momenta in the partonic system is given by:

$$p^\mu = (p^+, 0^-, 0_\perp)\,, \tag{3.97}$$

$$k^\mu = \Big(\xi p^+, \frac{k_\perp^2 - |k^2|}{2\xi p^+}, k_\perp\Big)\,, \tag{3.98}$$

$$q^\mu = \Big(0, \frac{Q^2}{2xp^+}, q_\perp\Big)\,, \tag{3.99}$$

where we used that $k^2 = -|k^2|$ because the intermediate quark is virtual. The arguments of the delta functions in Eq. (3.96) is given by

$$(p-k)^2 = -2p\cdot k - |k^2| = -\frac{1}{\xi}\big(k_\perp^2 - (1-\xi)|k^2|\big)\,, \tag{3.100}$$

$$(k+q)^2 = 2k\cdot q - |k^2| - Q^2 = 2P\cdot q\Big(\xi - x - \frac{|k^2| + 2k_\perp\cdot q_\perp}{2P\cdot q}\Big)\,, \tag{3.101}$$

and the differential is given by

$$d^4k = dk^+dk^-d^2k_\perp = \frac{1}{4\xi}d\xi dk^2 dk_\perp^2 d\theta\,, \tag{3.102}$$

with $0 < \theta < \pi$. Inserting these expression into Eq. (3.96), we get

$$d\mathcal{P}_2 = \frac{1}{(4\pi)^2 P\cdot q}\int d\xi dk^2 dk_\perp^2 d\theta\;\; \delta\big(k_\perp^2 - (1-\xi)|k^2|\big)\,\delta\Big(\xi - x - \frac{|k^2| + 2k_\perp\cdot q_\perp}{2P\cdot q}\Big)\,. \tag{3.103}$$

Now, there is no a priori reason why the transverse momentum (or equivalently $|k^2|$) should be small, but in the second delta function the effect of these are damped by $P\cdot q \sim Q^2$ so we proceed by neglecting this term. This again fixes $\xi = x$ in the final integration. We are actually jumping ahead here, but with a more thorough analysis it can be shown that this is in fact the case [25]. The first delta function fixes $k_\perp^2 = (1-\xi)|k^2|$, so we can just use it directly to rewrite the averaged amplitude in Eq. (3.94), giving

$$\begin{aligned}\langle\,|\mathcal{M}|^2\rangle^{\mu\nu} =& 4\pi C_F\,Q_q^2\,\alpha_s\frac{1}{|k^2|}\Big[\frac{1}{\xi}\frac{1+\xi^2}{1-\xi}\big(8k^{(\mu}p'^{\nu)} - 4g^{\mu\nu}k\cdot p'\big)\\ &+ \frac{1}{1-\xi}\big(8p^{(\mu}p'^{\nu)} - 4g^{\mu\nu}p\cdot p'\big) + \frac{1}{1-\xi}\frac{|k^2|}{p^+}\big(8n_-^{(\mu}p'^{\nu)} - 4g^{\mu\nu}p'^+\big)\Big]\,.\end{aligned} \tag{3.104}$$

where we have evaluated the traces using Eq. (3.93) and for notational simplicity defined $a^{(\mu}b^{\nu)} = (a^\mu b^\nu + a^\nu b^\mu)/2$.

Let us then finally use Eq. (3.86) to project out $\hat{F}_2$,

$$\hat{F}_2 = \int d\mathcal{P}_2\; x(\hat{t}^\mu\hat{t}^\nu - g_\perp^{\mu\nu})\,\langle\,|\mathcal{M}|^2\rangle_{\mu\nu}\,. \tag{3.105}$$

In the following we will only list the divergent term as the finite ones are unimportant for this discussion. For the contraction with the matrix element we use that $p' = k + q$ and the basis vector $\hat{t}$ is found from Eq. (3.79). Putting everything together is messy, but after the effect of the delta functions the divergent part takes the form[77]

$$\hat{F}_2\,\big|_{\text{div}} = Q_q^2\frac{\alpha_s}{2\pi}\,xP_{q/q}(x)\int_{Q_0^2}^{Q^2}\frac{d|k^2|}{|k^2|}\,, \tag{3.106}$$

where the integral is regulated over $|k^2|$ in terms of a IR cut-off Q_0^2 and a UV cut-off Q^2, that follows from the kinematics. We have also defined a function $P_{q/q}(x)$, which is known as the *quark–quark splitting function*

$$P_{q/q}(x) = C_F\frac{1+x^2}{1-x}\,. \tag{3.107}$$

[77] For a more detailed derivation of this term, see [25].

Its form is specific to the quark-quark-gluon vertex of QCD, and it represents the probability for a quark to split into another quark with momentum fraction x and a gluon with momentum fraction $1-x$. Integrating Eq. (3.106), we find

$$\hat{F}_2\big|_{\text{div}} = Q_q^2 \frac{\alpha_s}{2\pi} x P_{q/q}(x) \ln \frac{Q^2}{Q_0^2} \,. \tag{3.108}$$

If we include the leading order contribution given in Eq. (3.67) and all finite terms, denoted $C_q(x)$, we have the quark structure function at $\mathcal{O}(\alpha_s)$

$$\hat{F}_2 = Q_q^2 x \Big(\delta(1-x) + \frac{\alpha_s}{2\pi} \big(P_{q/q}(x) \ln \frac{Q^2}{Q_0^2} + C_q(x) \big) \Big) \,. \tag{3.109}$$

The natural question that arises is how to interpret the divergence of the structure function. We observe that the singularity arises when the gluon is emitted fully collinear to the quark, i.e. $k_\perp = 0$, and is therefore referred to as a *collinear singularity*. To understand what is happening we need to realize that physically $k_\perp^2 \to 0$ corresponds to a long-range part of the strong interaction which is not calculable in perturbation theory. Hence, if we use the PDFs to describe physics at long-range, we can convolute the structure function $\hat{F}_2$ with 'bare' distributions f_q^0 to absorb the collinear singularity. This is a similar, but still different approach to the UV-renormalization procedure we discussed in Sec. 1.5.

To get a clear understanding of how this is done we investigate the cut-off we made in Eq. (3.106). By imposing the IR cut-off we effectively integrate $k_\perp^2$ from Q_0^2 up to Q^2. Now, the kinematics of the process justifies the upper limit of Q^2 as $k_\perp$ is always smaller than or equal to Q[78]. However, in the IR-region there is no kinematical restriction on $k_\perp$. The lower cut-off ensures that the gluons with transverse momentum $k_\perp \leq Q_0$ is neglected from the hard part of the scattering. We can not drop these gluons entirely, so we absorb this part of the process into the PDF. We can then renormalize the PDF up to the arbitrary energy scale μ_F.

To obtain the proton structure function we convolute the quark structure function $\hat{F}_2$ of Eq. (3.109) with the bare PDF f_q^0 as we did for the parton model in Eq. (3.67),

$$F_2(x,Q^2) = \sum_q Q_q^2 x \Big(f_q^0(x) + \frac{\alpha_s}{2\pi} \int_x^1 \frac{d\xi}{\xi} f_q^0(\xi) \big[P_{q/q}\big(\frac{x}{\xi}\big) \ln \frac{Q^2}{Q_0^2} + C_q\big(\frac{x}{\xi}\big) \big] \Big) \,. \tag{3.110}$$

Then we can absorb the collinear singularities into the bare distribution at the factorization scale μ_F, or in other words, we define a renormalized distribution

$$f_q(x,\mu_F^2) = f_q^0(x) + \frac{\alpha_s}{2\pi} \int_x^1 \frac{d\xi}{\xi} f_q^0(\xi) \big[P_{q/q}\big(\frac{x}{\xi}\big) \ln \frac{\mu_F^2}{Q_0^2} + C_q'\big(\frac{x}{\xi}\big) \big] \,, \tag{3.111}$$

which inserted into the expression for F_2 gives the finite result

$$F_2(x,Q^2) = \sum_q Q_q^2 x \int_x^1 \frac{d\xi}{\xi} f_q(\xi,\mu_F^2) \big[\delta\big(1-\frac{x}{\xi}\big) + \frac{\alpha_s}{2\pi} \Big(P_{q/q}\big(\frac{x}{\xi}\big) \ln \frac{Q^2}{\mu_F^2} + D_q\big(\frac{x}{\xi}\big) \Big) \big] \,. \tag{3.112}$$

This result says that we can choose an arbitrary energy scale μ_F to separate the process into two parts: a hard part where $k_\perp$ is larger than this scale, and a soft part where $k_\perp$ is smaller than this scale. Thus, we 'hide' the divergence inside a part that were non-perturbative in the first place. We have alluded to this separation repeatedly and made this statement in the discussion of the parton model, but now we have extended that result to the correct framework of QCD. We also note that the only difference in QCD factorization and the parton model factorization is that the PDF and the hard part acquires a dependancy on the factorization scale. Although we have only demonstrated factorization to $\mathcal{O}(\alpha_S)$ in DIS, it has been proven to all orders in perturbation theory [26]. Another important feature of extending to QCD is that Bjorken-scaling is violated as F_2 is explicitly dependent on Q^2.

One important aspect of the renormalization procedure we have made here is that even though the factorization specifies a prescription of dealing with the logarithmic singularities, there is still an arbitrariness in the choice of

[78] This follows from the fact that the transverse momentum of the gluon can not be larger than the scale of the process.

the finite terms. In Eq. (3.112) we have that $D(x) = C(x) - C'(x)$, where $C'(x)$ is absorbed by the PDF and $D(x)$ is the terms that remains. However, we could have made another choice and absorbed all finite terms into the PDF, and the exact choice made is referred to as a *factorization scheme*. The most common scheme is the $\overline{\text{MS}}$ scheme we mentioned in Sec. 1.5, where the absorbed term is $C' = \ln 4\pi - \gamma_E$.

Due to the non-perturbative nature of long distance physics in QCD, the distributions $f_q(x, \mu_F^2)$ are not directly calculable from first principles, and must therefore be extracted from experiments or, more recently in lattice QCD. However, what can be calculated perturbatively is the dependence on the scale μ_F^2. The proton structure function F_2 is an observable, meaning that it cannot depend on the choice of scale. This leads to the following requirement

$$\frac{\partial F_2}{\partial \ln \mu_F^2} = 0\,, \tag{3.113}$$

which leads to the following differential equation for the PDFs

$$\frac{\partial}{\partial \ln \mu_F^2} f_q(x, \mu_F^2) = \frac{\alpha_s(\mu_F^2)}{2\pi} \int_x^1 \frac{d\xi}{\xi} P_{q/q}\big(\frac{x}{\xi}\big)\, f_q(\xi, \mu_F^2)\,. \tag{3.114}$$

This evolution equation is similar to the β function equation describing the evolution of the coupling $\alpha_s(\mu^2)$, and is known as the Dokshitzer-Gribov-Altarelli-Parisi (DGLAP) equation. As it stands, this equation is only valid up to $\mathcal{O}(\alpha_s)$. To get an all order evolution equation we can expand the splitting function in the coupling,

$$P_{q/q}(x, \alpha_s) = \sum_{n=0}^{\infty} \Big(\frac{\alpha_s}{2\pi}\Big)^{n+1} P_{q/q}^{(n)}(x)\,, \tag{3.115}$$

and write the evolution equation as

$$\frac{\partial}{\partial \ln \mu_F^2} f_q(x, \mu_F^2) = \int_x^1 \frac{d\xi}{\xi} P_{q/q}\big(\frac{x}{\xi}, \alpha_s(\mu_F^2)\big)\, f_q(\xi, \mu_F^2)\,, \tag{3.116}$$

where the higher order information is contained inside the splitting function. We can then identify the splitting function in Eq. (3.107) and Eq. (3.114) as the leading order splitting function $P_{q/q}^{(0)}$. For a more rigorous derivation of this generalization see [27].

In order to obtain a complete discussion of deep inelastic scattering in terms of parton distribution functions, there is one more ingredient that we have not considered. That is, we also need to consider the initial scattering of gluons. The $\mathcal{O}(\alpha_s)$ contribution for initial gluon scattering is through a t-channel boson-gluon fusion process. Following a similar line of argument as for the quark initial state this will lead to the gluon structure function

$$\hat{F}_2^g(x, Q^2) = \sum_q Q_q^2\, x\, \frac{\alpha_s}{2\pi} \Big(P_{q/g}(x) \ln \frac{Q^2}{Q_0^2} + C_g(x) \Big)\,, \tag{3.117}$$

which is very similar to the quark structure function Eq. (3.109), apart from the fact that at zeroth order in α_s there is no gluon radiation. The splitting function for this process is specific for the gluon-quark-quark vertex, and is given by

$$P_{q/g}(x) = \frac{1}{2}(x^2 + (1-x)^2)\,, \tag{3.118}$$

and is interpreted as the probability for a gluon to split into a quark with momentum fraction x and another quark with momentum fraction $1 - x$.

As with the quark structure function we have to convolute the gluon structure function with a bare gluon distribution f_g^0, define a renormalized gluon distribution and add it to Eq. (3.112). This is equivalent to redefining our renormalized quark parton distribution to also absorb the singular gluon term. Thus, we define

$$\begin{aligned} f_q(x, \mu_F^2) =& f_q^0(x) + \frac{\alpha_s}{2\pi} \int_x^1 \frac{d\xi}{\xi} f_q^0(\xi) \big[P_{q/q}\big(\frac{x}{\xi}\big) \ln \frac{\mu_F^2}{Q_0^2} + C_q'\big(\frac{x}{\xi}\big) \big] \\ &+ \frac{\alpha_s}{2\pi} \int_x^1 \frac{d\xi}{\xi} f_g^0(\xi) \big[P_{q/g}\big(\frac{x}{\xi}\big) \ln \frac{\mu_F^2}{Q_0^2} + C_g'\big(\frac{x}{\xi}\big) \big]\,. \end{aligned} \tag{3.119}$$

Instead of writing the whole final expression out, let us be even more general and define hard functions with perturbative expansions

$$H_q(z) = \sum_{n=0}^{\infty} \Big(\frac{\alpha_s}{2\pi}\Big)^n H_q^{(n)}(z)\,, \tag{3.120}$$

$$H_g(z) = \sum_{n=1}^{\infty} \Big(\frac{\alpha_s}{2\pi}\Big)^n H_g^{(n)}(z)\,. \tag{3.121}$$

where the $\mathcal{O}(\alpha_s)$ expansion is given by

$$H_q(z) = H_q^{(0)}(z) + \frac{\alpha_s}{2\pi} H_q^{(1)}(z) = \delta(1-z) + \frac{\alpha_s}{2\pi}\Big(P_{q/q}^{(0)}(z)\ln\frac{Q^2}{\mu_F^2} + D_q(z)\Big)\,, \tag{3.122}$$

$$H_g(z) = \frac{\alpha_s}{2\pi} H_g^{(1)}(z) = \frac{\alpha_s}{2\pi}\Big(P_{q/g}^{(0)}(z)\ln\frac{Q^2}{\mu_F^2} + D_g(z)\Big)\,, \tag{3.123}$$

The collinear factorization formula for F_2 then takes the form

$$\begin{aligned} F_2(x,Q^2) = &\sum_q Q_q^2\, x\int_x^1 \frac{d\xi}{\xi} f_q(\xi,\mu_F^2) H_q\big(\frac{x}{\xi}, \frac{Q^2}{\mu_F^2}, \alpha_s(\mu_F^2)\big) \\ &+ \sum_q Q_q^2\, x\int_x^1 \frac{d\xi}{\xi} f_g(\xi,\mu_F^2) H_g\big(\frac{x}{\xi}, \frac{Q^2}{\mu_F^2}, \alpha_s(\mu_F^2)\big)\,. \end{aligned} \tag{3.124}$$

To actually have a full description of collinear factorization in QCD we would need to find the factorization of F_1 as well. To find F_1 we would have to project it out as in Eq. (3.85), but we will not consider the specific calculation as it follows in the same manner as for F_2.

If one considered even higher order corrections we would encounter the additional splitting functions $P_{g/q}(x)$ and $P_{g/g}(x)$. $P_{g/q}(x)$ decribes the splitting of a quark into a gluon with momentum fraction x and a quark with momentum fraction $1-x$, and similarly $P_{g/g}(z)$ describes the splitting of a gluon into a gluon with momentum fraction x and another gluon with momentum fraction $1-x$. For completeness we list all splitting functions at leading order

$$P_{q/q}^{(0)}(x) = C_F\frac{1+x^2}{1-x}\,, \tag{3.125}$$

$$P_{q/g}^{(0)}(x) = \frac{1}{2}(x^2+(1-x)^2)\,, \tag{3.126}$$

$$P_{g/q}^{(0)}(x) = C_F\frac{1+(1-x)^2}{x}\,, \tag{3.127}$$

$$P_{g/g}^{(0)}(x) = 2C_A\Big(\frac{x}{1-x} + \frac{1-x}{x} + x(1-x)\Big)\,. \tag{3.128}$$

We observe that $P_{q/q}(x)$ and $P_{g/g}(x)$ both have singular behaviour for $x\to 1$, so we will need to regularize these as well. This is done by using so-called *plus distributions*, see Appendix C.1. We will encounter plus distributions in Sec. 3.3 as well, so it is useful to write down the regulated splitting functions. The regularized splitting functions are at leading order given by [27],

$$P_{q/q}^{(0)}(x) = C_F\Big(\Big[\frac{1+x^2}{1-x}\Big]_+ + \frac{3}{2}\delta(1-x)\Big)\,, \tag{3.129}$$

$$P_{g/g}^{(0)}(x) = 2C_A\Big(\Big[\frac{1+x^2}{1-x}\Big]_+ + \frac{1-x}{x} + x(1-x)\Big) + \delta(1-x)\frac{(11C_A-2n_f)}{6}\,, \tag{3.130}$$

where n_f is the number of quarks.

This leads us to the end of our discussion of factorization in DIS. In Sec. 3.3 we will investigate one other process where factorization has been proven, namely the Drell-Yan process. But before we move on to the Drell-Yan cross section, there are some important details about parton distribution functions we have to discuss.

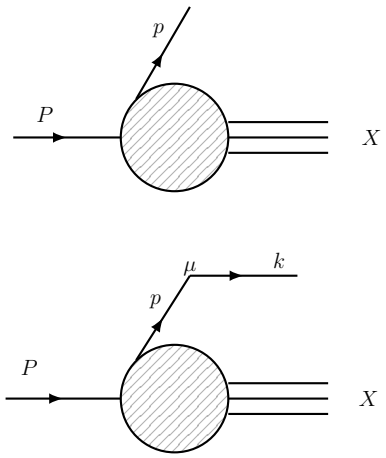


Figure 3.3: Diagramatic construction of hadronic tensor

That is, we can actually write PDFs as operator valued matrix elements, and use Wilson lines to render these gauge invariant. Not only that, but we can use the expansion of Wilson lines to describe gluon radiation from incoming partons. That is not to say that the universal distribtuins $f_{i/h}$ can be calculated in perturbation theory, but we can define parton-in-parton distributions that are calculable in perturbation theory. This is the topic of the next few chapters.

3.2.3 Operator Definition for PDFs

The interpretation of a parton distribution function as the probability of finding a parton inside a proton with momentum fraction ξ was crucial in finding factorized formulas for QCD scattering observables. In quantum field theory, we have that probabilities are matrix elements squared, so we would like to define the parton distribution functions as operator-valued matrix elements. In terms of operator definitions one can use them to prove the factorization properties of QCD, and one can also use them to include soft and collinear singularities. This last part is essential for what is called *re-factorization* in QCD resummation, which we will come back to in Chapter 4.

In Sec. 3.2.1 we found an expression for the hadronic tensor in terms of the electromagnetic currents in Eq. (3.51). Here we will construct it diagrammatically such that we can extract the so-called *quark correlator*, which are the building block for the parton distribution functions. The diagrammatic construction is illustrated in Fig. 3.3. The upper diagram pulls out a quark from the proton, which subsequently fragments into the final state X. Mathematically this is given by the matrix element

$$\mathcal{A}_i = \langle X|\psi_i(0)|P\rangle\,. \tag{3.131}$$

Assuming a gauge interaction in μ, the bottom diagram will give a quark in the final state. Then we will get the matrix element

$$\mathcal{A}^\nu = Q_q \bar{u}^s_j(k)\big(\gamma^\nu\big)^{ji}\langle X|\psi_i(0)|P\rangle\,. \tag{3.132}$$

The squared matrix element for this part of the process can then be written as:

$$\big(|\mathcal{A}|^2\big)^{\mu\nu} = \sum_q Q_q^2\big[\gamma^\mu(\not{k}+m)\gamma^\nu\big]^{ji}\langle P|\overline{\psi}_j(0)|X\rangle\langle X|\psi_i(0)|P\rangle\,, \tag{3.133}$$

where we have used the spin sum rule

$$\sum_s u^s(k)\bar{u}^s(k) = \not{k}+m\,. \tag{3.134}$$

To write down the hadronic tensor we integrate over the final states X and k and impose momentum conservation through a delta function. As usual, we use the exponential representation of the delta function, and the integral over k will be made by using the on-shell condition, see Eq. (1.211)

$$\int \frac{d^3k}{(2\pi)^3}\frac{1}{2k^0} = \int \frac{d^4k}{(2\pi)^4} 2\pi\delta(k^2-m^2)\theta(k^0)\,, \tag{3.135}$$

Taking the quark to be massless, we find the hadronic tensor to take the form

$$\begin{aligned}
W^{\mu\nu} &= 4\pi^3 \sum_q Q_q^2 \sum_X \int \frac{d^3p_X}{(2\pi)^3 2E_X} \int \frac{d^3k}{(2\pi)^3 2k^0} \delta^{(4)}(P+q-k-p_X) \\
&\quad \times \left(\gamma^\mu \not{k} \gamma^\nu\right)^{ji} \langle P|\overline{\psi}_j(0)|X\rangle\langle X|\psi_i(0)|P\rangle \\
&= 4\pi^3 \sum_q Q_q^2 \sum_X \int \frac{d^3p_X}{(2\pi)^3 2E_X} \int \frac{d^4k}{(2\pi)^4} 2\pi\delta(k^2)\theta(k^0) \int \frac{d^4z}{(2\pi)^4}\, e^{iz\cdot(P-q-k-p_X)} \\
&\quad \times \left(\gamma^\mu \not{k} \gamma^\nu\right)^{ji} \langle P|\overline{\psi}_j(0)|X\rangle\langle X|\psi_i(0)|P\rangle \\
&= \frac{1}{2} \sum_q Q_q^2 \sum_X \int \frac{d^3p_X}{(2\pi)^3 2E_X} \int d^4p\, \delta((p+q)^2)\theta(p^0+q^0) \\
&\quad \int \frac{d^4z}{(2\pi)^4}\, e^{iz\cdot(P-q-k-p_X)} \left(\gamma^\mu(\not{p}+\not{q})\gamma^\nu\right)^{ji} \langle P|\overline{\psi}_j(0)|X\rangle\langle X|\psi_i(0)|P\rangle\,,
\end{aligned} \tag{3.136}$$

where we in the last step used that $k = p+q$, where q is the photon momentum. By using the translation operator Eq. (3.49) and the completeness relation Eq. (3.50), we find

$$\begin{aligned}
W^{\mu\nu} &= \frac{1}{2} \sum_q Q_q^2 \int d^4p\, \delta((p+q)^2)\theta(p^0+q^0)\, \Phi_{ij}(p) \left(\gamma^\mu(\not{p}+\not{q})\gamma^\nu\right)^{ji} \\
&= \frac{1}{2} \sum_q Q_q^2 \int d^4p\, \delta((p+q)^2)\theta(p^0+q^0)\, \mathrm{tr}\Big(\Phi(p)\gamma^\mu(\not{p}+\not{q})\gamma^\nu\Big)\,,
\end{aligned} \tag{3.137}$$

where the quark correlator in momentum space is defined as

$$\Phi_{ij}(p) = \int \frac{d^4z}{(2\pi)^4}\, e^{-ip\cdot z} \langle P|\overline{\psi}_j(z)\psi_i(0)|P\rangle\,. \tag{3.138}$$

We want to simplify the hadronic tensor further, which is easiest by using light-cone coordinates, see Appendix A.1. We can then parametrize the proton, quark and photon momenta as in Eq. (3.75) and Eq. (3.76). With this parametrization, we can expand the delta function in Eq. (3.137) in the following way

$$\begin{aligned}
\delta((p+q)^2) &= \delta(p^2 + 2p\cdot q + q^2) \\
&= \delta(2p^+q^- + 2p^-q^+ - 2p_\perp\cdot q_\perp - Q^2) \\
&\approx \delta(2p^+q^- - Q^2) \\
&= \delta(2P\cdot q\,\xi - 2P\cdot q\, x) \\
&= \frac{1}{2P\cdot q}\delta(\xi - x)\,,
\end{aligned} \tag{3.139}$$

where we in the first step used that the quarks are massless, and in the third line we used that $p_\perp \ll Q^2$. To rewrite the argument we also used the Bjorken variable $x = Q^2/2P\cdot q$. We observe that this delta function fixes the momentum fraction to be equal the Bjorken-x, just as in the parton model calculation.

The second simplification we can make is to use the fact that in the infinite momentum frame, the incoming photon hits the quark head-on. Then the outgoing quark will move in the k^- direction, giving

$$\not{k} = \not{p} + \not{q} = \gamma^+ q^- + \mathcal{O}\Big(\frac{1}{P^+}\Big)\,, \tag{3.140}$$

and with these simplifications the hadronic tensor takes the form

$$\begin{aligned} W^{\mu\nu} &= \frac{1}{2}\sum_q Q_q^2 \int d^4p\, \delta^+((p+q)^2)\, \mathrm{tr}\Big[\Phi(p)\gamma^\mu(\not{p}+\not{q})\gamma^\nu\Big] \\ &= \frac{1}{4}\sum_q Q_q^2 \int dp^+dp^-d^2p_\perp \frac{1}{P\cdot q}\, \mathrm{tr}\big[\Phi(p)\gamma^\mu\gamma^+q^-\gamma^\nu\big]\delta(\xi-x) \\ &= \frac{1}{4}\sum_q Q_q^2 \int d\xi dp^-d^2p_\perp \frac{P^+q^-}{P\cdot q}\, \mathrm{tr}\big[\Phi(p)\gamma^\mu\gamma^+\gamma^\nu\big]\delta(\xi-x) \\ &= \frac{1}{4}\sum_q Q_q^2\, \mathrm{tr}\big[\Phi(x)\gamma^\mu\gamma^+\gamma^\nu\big]\,, \end{aligned} \tag{3.141}$$

where we have defined the fourier transformed integrated quark correlator

$$\begin{aligned} \Phi_{ij}(x) &= \int dp^-d^2p_\perp \Phi_{ij}(x,p^-,p_\perp) \\ &= \int \frac{dz^-}{2\pi}\, e^{-ixP^+z^-}\langle P|\overline{\psi}_j(0^+,z^-,0_\perp)\psi_i(0)|P\rangle\,. \end{aligned} \tag{3.142}$$

To find the quark parton distribution we use the light-cone contraction $\gamma^\mu\gamma^+\gamma_\mu = -2\gamma^+$, such that the trace in Eq. (3.141) can be written as

$$\begin{aligned} \mathrm{tr}[\Phi(x)\gamma^\mu\gamma^+\gamma^\nu] &= g^{\mu\nu}\mathrm{tr}[\Phi(x)\gamma^\mu\gamma^+\gamma_\mu] \\ &= -2g^{\mu\nu}\mathrm{tr}[\Phi(x)\gamma^+]\,. \end{aligned} \tag{3.143}$$

Further, we use that $g^{\mu\nu} = g_\perp^{\mu\nu} - \big(n_+^\mu n_-^\nu + n_+^\nu n_-^\mu\big)$, see Eq. (A.14), and insert it into Eq. (3.141). This will give the following expression for the hadronic tensor

$$W^{\mu\nu} = -\frac{1}{2}g_\perp^{\mu\nu}\sum_q Q_q^2\,\mathrm{tr}[\Phi(x)\gamma^+] + \big(n_+^\mu n_-^\nu + n_+^\nu n_-^\mu\big)\sum_q Q_q^2\,\mathrm{tr}[\Phi(x)\gamma^+]\,, \tag{3.144}$$

and if we compare this expression to Eq. (3.84), we find that we must have

$$F_1(x) = \frac{1}{2}\sum_q Q_q^2\,\mathrm{tr}[\Phi(x)\gamma^+]\,. \tag{3.145}$$

However, we already know from Eq. (3.68) that this structure function is given by

$$F_1(x) = \frac{1}{2}\sum_q Q_q^2 f_q(x)\,, \tag{3.146}$$

from which it follows that the operator definition for the integrated quark PDF is given by

$$f_{q/P}(x) = \int \frac{dz^-}{4\pi}\, e^{-ixP^+z^-}\langle P|\overline{\psi}(0^+,z^-,0_\perp)\gamma^+\psi(0)|P\rangle\,, \tag{3.147}$$

where the subscript $f_{i/P}$ is the common notation for integrated PDFs of a parton with flavour i inside the proton. From now we will not make this statement as it is implicit in the notation.

Having found an expression for the parton distribution function in terms of matrix elements, let us take a closer look at how we can make these gauge invariant and subsequently used as a motivation for deriving perturbative distributions.

3.2.4 Gauge Invariant Parton Distributions

The quark correlator in Eq. (3.138) is not gauge invariant, which can be made clear if we look at the gauge transformation of the fields. In general, fermionic fields transform under local gauge transformations as

$$\psi'(x) = e^{ig\alpha^a(x)t^a}\psi(x) \tag{3.148}$$

$$\overline{\psi}'(x) = \overline{\psi}(x)e^{-ig\alpha^a(x)t^a}\,, \tag{3.149}$$

and as a consequence the quark correlator in momentum space transform as

$$\Phi'(p) = \int \frac{d^4z}{(2\pi)^4}\, e^{-ip\cdot z}\langle P|\overline{\psi}(z)e^{-i\alpha^a(z)t^a}e^{ig\alpha^a(0)t^a}\psi(0)|P\rangle\,, \tag{3.150}$$

where the fields are defined at different space-time points and the exponentials inside the matrix element do not cancel.

A similar problem appeared when trying to define a directional derivative of the fermionic fields in Sec. 2.2. We used a Wilson line to parallel transport the fields such that they could be subtracted in a meaningful way, giving rise to a directional covariant derivative that ensured gauge invariance. The same procedure does not apply here, but in Sec. 2.2 we had that the Wilson line transformed under gauge transformation as

$$\mathcal{U}'[y,x] = e^{ig\alpha^a(y)t^a}\mathcal{U}[y,x])e^{-ig\alpha^a(x)t^a}\,, \tag{3.151}$$

which is exactly the transformation we need to cancel the exponentials. So, let us define the following Wilson line

$$\mathcal{U}[z,0] = \mathcal{P}\exp\Big(-ig\int_0^z dy^\mu A_\mu(y)\Big)\,, \tag{3.152}$$

where it is understood that $A_\mu = A_\mu^a t^a$. We can then use the transformation property of the Wilson line to write down the gauge invariant quark correlator as

$$\Phi(p) = \int \frac{d^4z}{(2\pi)^4}\, e^{-ip\cdot z}\langle P|\overline{\psi}(z)\mathcal{U}[z,0]\psi(0)|P\rangle\,. \tag{3.153}$$

The integrated quark correlator in Eq. (3.142) has fields that are fixed along the z^- direction, which mean that the quark correlator takes the form

$$\Phi(x) = \int \frac{dz^-}{2\pi}\, e^{-ixP^+z^-}\langle P|\overline{\psi}(0^+,z^-,0_\perp)\mathcal{U}[z^-,0]\psi(0)|P\rangle\,, \tag{3.154}$$

$$\mathcal{U}[z^-,0] = \mathcal{P}\exp\Big(-ig\int_0^{z^-} dy^- A^+(0^+,y^-,0_\perp)\Big)\,, \tag{3.155}$$

and it follows that the gauge invariant formulation of the quark parton distribution is

$$f_{q/P}(x) = \int \frac{dz^-}{2\pi}\, e^{-ixP^+z^-}\langle P|\overline{\psi}(0^+,z^-,0_\perp)\gamma^+\mathcal{U}(z^-;0)\psi(0)|P\rangle\,. \tag{3.156}$$

We observe that if we choose the light-cone gauge, $A^+ = 0$, the Wilson line is $\mathcal{U}[z^-,0] = 1$. Hence, we reduce the parton distribution to the one defined in Eq. (3.147), and as long as one stays in light-cone gauge the Wilson line can be neglected.

For a complete treatment we will also give the operator definition of the gluon distribution. The calculation is very similar to the one we made for the quark distribution, but this is a higher order effect through boson-gluon fusion via a quark, so we will not make the explicit derivation here. If one starts in the light-cone gauge the Wilson line can be ignored, and the integrated gluon PDF can be found to be [28, 29],

$$f_{g/P}(x) = \int \frac{dz^-}{2\pi} xP^+ e^{-ixP^+z^-}\langle P|A_i^a(z^-)A_i^a(0)|P\rangle\,, \tag{3.157}$$

where $i = -, 1, 2$ as we have set $A^+ = 0$.

We know from Sec. 2.2 that the gauge fields transform under gauge transformation as

$$A_\mu^a t^a \to \frac{i}{g}e^{ig\alpha^a t^a}D_\mu e^{-ig\alpha^a t^a}\,, \tag{3.158}$$

meaning that in the current form the gluon distribution is not gauge invariant. Because of the derivative it would not help if we naively insert a Wilson line as we did for the quark distribution. Let us instead investigate the field strength tensor $F_{\mu\nu}$, with the transformation

$$F_{\mu\nu} \to e^{ig\alpha^a t^a}F_{\mu\nu}e^{-ig\alpha^a t^a}\,, \tag{3.159}$$

which we observe transform in a similar fashion as the Wilson line in Eq. (3.151). However, it is not valid to just replace the gauge fields with the field strength. We observe that if we did, and inserted the same Wilson line as for the quark distribution, it would still not be gauge invariant. This is not surprising as the quark fields are defined in the fundamental representation, and the gluon fields in the adjoint representation of $SU(3)$. Therefore, we need to use a Wilson line in the adjoint representation, which we define as

$$\mathcal{U}^A_{ab}[z,0] = \mathcal{P}\exp\left\{-ig\int_0^z dy^\mu A^c_\mu(y)(t^c)_{ab}\right\}, \tag{3.160}$$

where $(t^c)_{ab} = -if^{abc}$ are the generators in the adjoint representation. If we insert this Wilson line between two field strength tensors, we would have a gauge invariant operator. We can then use the field strength and relate it to the gauge fields in the following standard way

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu\,, \tag{3.161}$$

and use that $A^+ = 0$, which gives

$$F^a_{+i} = \partial_+ A^a_i\,. \tag{3.162}$$

Inserting for A_i into Eq. (3.157) and integrating by parts yields the gauge invariant gluon PDF

$$f_{g/P}(x) = \int \frac{dz^-}{2\pi}\frac{1}{xP^+}e^{-ixP^+z^-}\langle P|F^{+i}_a(z^-)\mathcal{U}^A_{ab}[z^-,0]F^{+i}_b(0)|P\rangle\,, \tag{3.163}$$

where the factor of xP^+ is common for even spin particles, i.e for bosons. There are several additional subtleties when dealing with gluon PDFs that we have not covered, so for more details see [29].

3.2.5 Parton-in-Parton Distributions

In this section we will define one last object that we will have use for in Chapter 4, namely *parton-in-parton* distributions. As we shall see, these will be useful when we want to renormalize our hadron–hadron cross-section and when we want to refactorize the cross-section in the so-called *threshold region*, making it eligible for resummation.

To define the parton-in-parton distribution we start from the quark PDF Eq. (3.147) and insert a complete set of final states in the following way

$$\begin{aligned} f_{q/P}(x) &= \int \frac{dz^-}{4\pi}\, e^{-ixP^+z^-}\langle P|\overline{\psi}(0^+,z^-,0_\perp)\gamma^+\psi(0)|P\rangle \\ &= \int \frac{dz^-}{4\pi}\, e^{-iz^-(P_n^+ - P^+ + xP^+)}\sum_n \langle P|\overline{\psi}(0)|n\rangle\gamma^+\langle n|\psi(0)|P\rangle \\ &= \frac{1}{2}\sum_n \langle P|\overline{\psi}(0)|n\rangle\gamma^+\langle n|\psi(0)|P\rangle\,\delta(P_n^+ - (1-x)P^+)\,, \end{aligned} \tag{3.164}$$

where we used the translation operator to pick out the momentum $P_n^+ - P^+$ in the exponential, see Eq. (3.49). It is understood that the matrix element is an average and sum over spin, as $f_{q/P}$ is by construction unpolarized. Let us then consider the case where $P = q$, i.e. a quark. In that case we must also include an average and sum over colour. Then we can naively write down the quark-in-quark distribution as

$$f_{q/q}(x) = \frac{1}{4N_c}\sum_{colour}\sum_{spin}\sum_n \langle q|\overline{\psi}(0)|n\rangle\gamma^+\langle n|\psi(0)|q\rangle\,\delta(p_n^+ - (1-x)p^+)\,, \tag{3.165}$$

where we have explicitly written out the average and sum over colour and spin. To evaluate the matrix elements we use that if a quark operator ψ act on a quark state $|q\rangle$, we get

$$\psi(z)|q\rangle = e^{-ip\cdot z}u(p)|0\rangle\,, \tag{3.166}$$

which mean that the matrix elements in Eq. (3.165) only have a contribution if $n = 0$. This scenario corresponds to a quark that just 'travels' along without changing. Thus, we conclude that the expression we have written

down is the leading order expansion of $f_{q/q}$, where there is no gluon radiation from the quark line. It is therefore important to point out that the only way a quark can 'change' is by radiating a gluon, and in this scenario there is no gluon to make that change. Hence, by using Eq. (3.166), we find the leading order result for the quark-in-quark distribution,

$$\begin{aligned} f^{(0)}_{q/q}(x) &= \frac{1}{4p^+}\sum_s u^s_i(p)\bar{u}^s_j(p)\gamma^+_{ji}\,\delta(1-x) \\ &= \frac{1}{4p^+}\mathrm{tr}[\not{p}\gamma^+]\delta(1-x) \\ &= \delta(1-x)\,, \end{aligned} \tag{3.167}$$

which states that in the absence of interaction the quark remains itself. For general partons, we can therefore write

$$f^{(0)}_{i/j}(x) = \delta_{ij}\delta(1-x)\,, \tag{3.168}$$

where δ_{ij} is inserted to make sure that the parton does not change without a gauge interaction. If we assume that $f_{i/j}$ can be calculated in perturbation theory, the expansion take the form

$$f_{i/j}(x,\mu^2) = \delta_{ij}\delta(1-x) + \sum_{n=1}^{\infty}\left(\frac{\alpha_s}{2\pi}\right)^n f^{(n)}_{i/j}(x,\mu^2)\,. \tag{3.169}$$

Before we proceed to the general treatment of radiation from quark lines, we can actually 'guess' the first order correction. In Sec. 3.2.2 we calculated a diagram where a gluon was emitted from an incoming quark line, see Fig. 3.2. The divergent part of the diagram was found to be

$$\hat{F}_2\big|_{\mathrm{div}} = Q_q^2\,\frac{\alpha_s}{2\pi}x\,P^{(0)}_{q/q}(x)\ln\frac{Q^2}{Q_0^2}\,, \tag{3.170}$$

where the quark–quark splitting function naturally appeared as the process involved a quark emitting a gluon and continued on as another quark. Thus, if $f^{(1)}_{q/q}$ describes a quark emitting a gluon and continuing on as another quark, the natural guess would be

$$f^{(1)}_{q/q}(x) \propto P^{(0)}_{q/q}(x)\,, \tag{3.171}$$

i.e. it has to be proportional to the splitting functions as it describes exactly what the splitting function describes.

For the general treatment, we need to implement gauge interactions systematically. But we already know from Sec. 2.2 that this can be done by dressing the quark line with a Wilson line. Therefore, we can use our expression for the gauge invariant parton distributions, see Eq. (3.156), and expand the Wilson line. To do this, we can first use the Wilson line relation

$$\mathcal{U}[z,0] = \mathcal{U}^\dagger[+\infty,z]\mathcal{U}[+\infty,0]\,, \tag{3.172}$$

and use Eq. (2.190) to write

$$\Psi(z) = \mathcal{U}[\infty,0]\psi(z)\,, \tag{3.173}$$

$$\overline{\Psi}(z) = \bar{\psi}(z)\mathcal{U}^\dagger[\infty,z]\,. \tag{3.174}$$

With these definitions the quark-in-quark distribution can be defined as

$$f_{q/q}(x) = \int\frac{dz^-}{4\pi}e^{-ixp^+z^-}\langle q|\overline{\Psi}(z^-)\gamma^+\Psi(0)|q\rangle\,, \tag{3.175}$$

also commonly rewritten by inserting a complete set of states, giving

$$f_{q/q}(x) = \int\frac{dz^-}{4\pi}e^{-ixp^+z^-}\sum_n\langle q|\overline{\Psi}(z^-)|n\rangle\gamma^+\langle n|\Psi(0)|q\rangle\,. \tag{3.176}$$

The expansion of a semi-infinite Wilson line is given in Eq. (2.171) and Eq. (2.172), from which Eq. (3.176) can be calculated in perturbation theory. We will not perform this calculation explicitly, but we can find its pole structure by comparing with the amplitude in Eq. (2.189). When squaring this amplitude, one finds that the $\mathcal{O}(g^2)$ term describes a scaleless integral. Scaleless integrals has the pole structure given in Eq. (1.239). The UV-divergence can be removed by counterterms, leaving the IR-divergence in $1/\epsilon$. An explicit calculation to $\mathcal{O}(\alpha_s)$ in dimensional regularization gives [26],

$$f_{q/q}(x) = \delta(1-x) - \frac{\alpha_s}{2\pi}\left(\frac{4\pi\mu^2}{\mu_F^2}\right)^{\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\frac{1}{\epsilon}P_{q/q}^{(0)}(x) + \mathcal{O}(\alpha_s^2)\,, \tag{3.177}$$

where the $1/\epsilon$ factor appear as a consequence of a collinear singularity. From these considerations it follows that the first order correction $f_{q/q}^{(1)}$ has the structure given in Eq. (3.171). We could also have gluons or quarks radiating from a gluon in the initial state, and the same arguments would apply with the difference in which splitting function that would appear in the expression. We should emphasize that there are several ways of defining these parton-in-parton distributions. The choice made in Eq. (3.176) are lightcone distributions with a fixed momentum fraction. In Sec. 4.2 we will define distributions at fixed energy instead, which make them more suitable for resummation. The most important point here is that one can use these parton-in-parton distributions to absorb collinear singularities that appear when gluons radiate from quark lines in scattering processes. This is a very important feature when doing resummation that will be used on several occasions.

3.3 Drell-Yan Cross Section in QCD

In this section we will make another calculation that historically has been important in the study of QCD, namely the Drell-Yan process.

In 1970, the first observation of a $\mu^+\mu^-$ in hadron-hadron collision was observed [30]. By applying the parton model Drell and Yan were the first to give a theoretical prediction of this process [31]. In the modern framework of QCD, the impulse approximation of the parton model is as in DIS replaced by the more precise concept of factorization. Then it can be proven that the hadronic Drell-Yan cross section can be written as a convolution of a perturbative calculable partonic cross section and universal process independent parton distribution functions [26]. Without specifying the kinematics, the hadronic cross section can be written as

$$\sigma_{h_1\,h_2}(P_1,P_2) = \sum_{i,j}\int dx_1\,dx_2\,f_{i/h_1}(x_1,\mu_F)f_{j/h_1}(x_1,\mu_F)\,\hat{\sigma}_{ij}(x_1P_1,x_2P_2,\mu_F)\,, \tag{3.178}$$

which is the form we expected from the parton model, and as in DIS, the parton distribution functions and the partonic cross section has acquired an dependancy on the factorization scale.

In the simplest case, a Drell-Yan process is the annihilation of a quark-antiquark pair into a virtual photon, which subsequently produces a pair of leptons with invariant mass Q^2. As leptons are blind to the strong interaction, there will not be any final state gluon radiation. The consequence is that all radiation comes from the initial state quark-antiquark pair, and therefore we have a clear probe of the behaviour of coloured particles. Consequently, the Drell-Yan process is an effective way of studying the internal structure of hadrons.

The main objective of this section is to investigate the divergences appearing in higher-order calculations, and how to deal with them using renormalization techniques. The calculation will proceed through the use of dimensional regularization, where the poles will manifest themselves in $1/\epsilon$. Further, close to the threshold, the finite result has contributions that become large, and these must eventually be handled using resummation techniques. To this end, we will focus on the next-to-leading order (NLO) calculation, where the initial state quarks emit a gluon.

3.3.1 LO Drell-Yan Cross Section

For the sake of completeness we will first sketch the leading order result. At this order we write the partonic process as $q(p) + \bar{q}(p') \to l^-(k) + l^+(k')$, where the corresponding Feynman diagram is given in Fig. 3.4. Here p and p' denote the momenta of the incoming quark and antiquark, and likewise k and k' denote the momentum

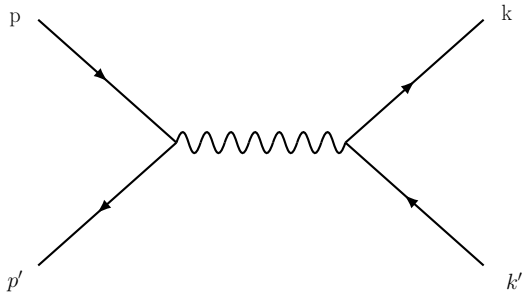


Figure 3.4: Drell-Yan process at leading order.

of the outgoing leptons. We can directly write down the amplitude from the basic Feynman rules[79]

$$i\mathcal{M} = i\delta_{ij}\frac{Q_q e^2}{q^2}\left[\bar{v}(p')\gamma^\mu u(p)\right]\left[\bar{u}(k)\gamma_\mu v(k')\right], \tag{3.179}$$

where δ_{ij} is a colour conserving factor for the quark vertex, and Q_q is the fractional charge of the quarks. As usual we are only interested in the spin and colour averaged amplitude,

$$\langle|\mathcal{M}|^2\rangle = \frac{1}{N_c N_s^2}\frac{Q_q^2 e^4}{\hat{s}^2}H^{\mu\nu}L_{\mu\nu}, \tag{3.180}$$

where N_c and N_s denote the number of colours and spin, and we have defined hadronic and leptonic tensors

$$H^{\mu\nu} = \mathrm{Tr}\left[\not{p}'\gamma^\mu\not{p}\gamma^\nu\right], \tag{3.181}$$

$$L_{\mu\nu} = \mathrm{Tr}\left[\not{k}\gamma_\mu\not{k}'\gamma_\nu\right], \tag{3.182}$$

and the contraction yields

$$H^{\mu\nu}L_{\mu\nu} = \hat{s}^2(1+\cos\theta^2), \tag{3.183}$$

where θ is the centre of mass scattering angle, and $\hat{s}$ is the partonic centre of mass energy, $\hat{s} = (p+p')^2$. The differential cross section is given by

$$d\hat{\sigma}^{(0)}_{q\bar{q}} = \frac{1}{2\hat{s}}\langle|\mathcal{M}|^2\rangle d\mathcal{P}^{(2)}, \tag{3.184}$$

where $d\mathcal{P}^{(2)}$ is the two-body phase space of the final state leptons, see Eq. (1.211). The integrated cross section is given by

$$\hat{\sigma}^{(0)} = \frac{4\pi Q_q^2\alpha^2}{3N_c\hat{s}}. \tag{3.185}$$

We can write this as a differential cross section by using that

$$1 = \int dQ^2\delta(\hat{s}-Q^2), \tag{3.186}$$

which is merely a statement that the invariant mass of the $q\bar{q}$ pair that annihilates into the photon, matches the invariant mass of the photon. This identity can also be written on the form

$$1 = \frac{1}{\hat{s}}\int dQ^2\delta(1-\frac{Q^2}{\hat{s}}), \tag{3.187}$$

[79] To see where all these term comes from, see Eq. (1.164) for the quark-photon vertex, Eq. (1.153) for the photon propagator and Sec. 1.2 for the appearance of spinors.

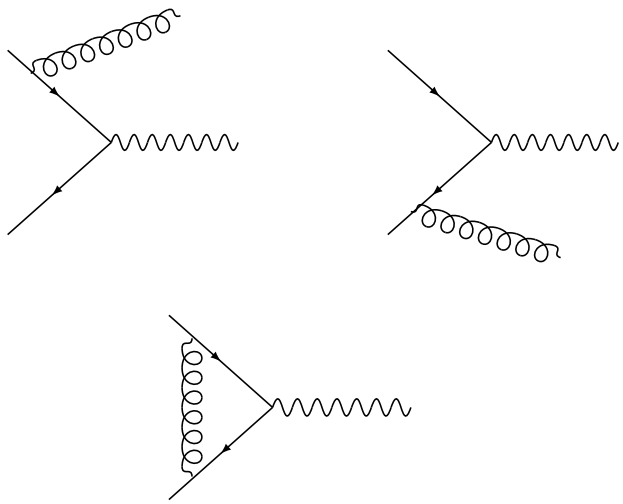


Figure 3.5: NLO diagrams contributing to the Drell-Yan process. Top: Real gluon emission. Bottom: Virtual correction.

giving the differential cross section in Q^2

$$\frac{d\hat{\sigma}^{(0)}_{q\bar{q}}}{dQ^2} = \frac{4\pi Q_q^2 \alpha^2}{3N_c \hat{s}\, Q^2}\delta(1-z) = \hat{\sigma}_0\, \delta(1-z)\,, \tag{3.188}$$

where we defined the partonic Born cross-section $\hat{\sigma}_0 = (4\pi Q_q^2\alpha^2)/(3N_c\hat{s}Q^2)$, and the partonic threshold variable $z = Q^2/\hat{s}$. For later purposes, we want to make some remarks about the structure of the leading order diagram. The photon and leptons do not feel the strong interaction, which means that even at higher orders, the final state part of the amplitude decouples from the initial part. Further, the electromagnetic corrections are negligible compared to the strong corrections so any photon radiation is neglected. To find the higher-order corrections, we only have to consider the corrections for the production of a virtual photon. The leptonic part of the hard process contributes with a factor $\alpha/3Q^2$, where Q^2 is the invariant mass of the final state leptons.

3.3.2 NLO Drell-Yan Cross Section

Let us now go beyond the leading order approximation and calculate the $\mathcal{O}(\alpha_s)$ correction to the Drell-Yan process. The contributing diagrams is illustrated in Fig. 3.5, where the upper diagrams corresponds to real gluon emission, and the lower diagram is the virtual correction. As we will see, the difficulty due to divergences in the integral is substantial. To treat these divergences we will use dimensional regularization, see Sec. 1.5.1 for more detail. One important feature to keep in mind is that the UV-divergences is treated by evaluating $\epsilon > 0$ and the IR-divergences for $\epsilon < 0$.

Now, by counting order in g_s in Fig. 3.5, we see that the virtual correction has one order higher than the real emission and can not be squared in order to contribute to NLO. Therefore, one must multiply the virtual correction amplitude with the leading order amplitude found from Fig. 3.4, which contributes to wanted order in α_s. The real contribution corresponds to the emission of gluons that are on mass-shell, i.e. there are no undetermined loop momenta that has to be integrated over. However, the phase space integration contains two kinds of singularities. The first is when the gluon is emitted collinearly to the emitting quark. The second singularity appears if the gluon momentum is soft, $k \to 0$. Both of these divergences are what is called infrared-singularities. Virtual corrections emerge when the initial quarks exchange a gluon. This gluon is virtual, and the integral is over undetermined loop momenta, leading to ultraviolet-divergences. To handle these divergences, we will use dimensional regularization, and work in $d = 4 - 2\epsilon$ dimensions.

Differential cross section

By applying the QCD Feynman rules, the amplitude for real gluon emission can be written as

$$i\mathcal{M} = Q_q\, e\, g_s\, \mu^{(4-d)/2}\, (t^a_{ij})\varepsilon^*_\alpha(k)\varepsilon^*_\mu(q)\left[\bar{v}(p')A^{\mu\alpha}u(p)\right], \tag{3.189}$$

where we have made the usual substitution $g_s \to \mu^\epsilon g_s$. We have also used the abbreviation

$$A^{\mu\alpha} = \gamma^\mu \frac{i(\not{p} - \not{k})}{(p-k)^2}\gamma^\alpha + \gamma^\alpha \frac{i(\not{p} - \not{q})}{(p-q)^2}\gamma^\mu \,, \tag{3.190}$$

where k is the gluon momenta and q is the massive photon momenta. The spin and colour averaged amplitude is given by

$$\begin{aligned}
\langle |\mathcal{M}|^2 \rangle &= \mathcal{C}\frac{Q_q^2 e^2 g_s^2 \mu^{(4-d)}}{N_s^2}\mathrm{Tr}[\not{p}' A^{\mu\alpha}\not{p} A_{\alpha\mu}] \\
&= C_F \frac{Q_q^2 e^2 g_s^2 \mu^{(4-d)}}{N_c\, N_s^2}\, 2(d-2)\Big(2\hat{s}\frac{Q^2}{\hat{t}\hat{u}} + 2(d-4) + (d-2)\Big[\frac{\hat{t}}{\hat{u}} + \frac{\hat{u}}{\hat{t}}\Big]\Big) \\
&= C_F \frac{Q_q^2 e^2 g_s^2 \mu^{2\epsilon}}{N_c\, N_s^2}\, 8(1-\epsilon)\Big(2\hat{s}\frac{Q^2}{\hat{t}\hat{u}} - 2\epsilon + (1-\epsilon)\Big[\frac{\hat{t}}{\hat{u}} + \frac{\hat{u}}{\hat{t}}\Big]\Big) \,,
\end{aligned} \tag{3.191}$$

where the terms with $d = 4 - 2\epsilon$ appears because of the modified Dirac algebra in d-dimensions Appendix B.1. The color factor for this process follows from the QCD group structure[80]

$$\mathcal{C} = \frac{1}{N_c}\mathrm{Tr}[t^a t^a] = \frac{C_F}{N_C} \,, \tag{3.192}$$

and the partonic Mandelstam variables is defined as

$$\hat{s} = (p+p')^2 = (q+k)^2 \,, \tag{3.193}$$
$$\hat{t} = (p-q)2 = (p'-k)^2 \,, \tag{3.194}$$
$$\hat{u} = (p-k)^2 = (p'-q)^2 \,. \tag{3.195}$$

The partonic differential cross section is then given by

$$\frac{d\hat{\sigma}^r_{q\bar{q}}}{dQ^2} = \frac{1}{2}\Big(\frac{\alpha}{3Q^2}\Big)\int d\mathcal{P}^{(2)}\, \langle |\mathcal{M}|^2 \rangle \,, \tag{3.196}$$

where the bracket represents lepton part appearing at all order. The differential phase space is over the massless gluon momenta k and the virtual photon momenta q. In d-dimensions, the differential phase space can be written as

$$d\mathcal{P} = \frac{1}{8\pi}\Big(\frac{4\pi}{Q^2}\Big)^\epsilon \frac{(1-z)^{1-2\epsilon} z^\epsilon}{\Gamma(1-\epsilon)}\big(y(1-y)\big)^{-\epsilon}\, dy \,, \tag{3.197}$$

where $y = \frac{1}{2}(1+\cos\theta)$, Γ is the Euler-Gamma function and z is the threshold variable $z = Q^2/\hat{s}$. Notice that the integral is to be taken over the dimensionless quantity y, so we should find a way of rewriting the averaged amplitude in terms y as well. We already have the partonic centre of mass-energy, $\hat{s}$ in terms of z, so by writing out $\hat{t}$ and $\hat{u}$ we find

$$\hat{t} = -\frac{Q^2}{z}(1-z)(1-y) \,, \tag{3.198}$$
$$\hat{u} = -\frac{Q^2}{z}(1-z)y \,. \tag{3.199}$$

With these definitions, the averaged amplitude in Eq. (3.191) takes the form

$$\begin{aligned}
\langle |\mathcal{M}|^2 \rangle = C_F \frac{Q_q^2 e^2 g_s^2 \mu^{2\epsilon}}{N_c\, N_s^2}\, 8(1-\epsilon)\Big(&\frac{2z}{(1-z)^2(1-y)y} \\
&+ (1-\epsilon)\Big[\frac{1-y}{y} + \frac{y}{1-y}\Big] - 2\epsilon\Big) \,.
\end{aligned} \tag{3.200}$$

[80] See [2] for a more elaborate treatment of the colour sums.

At this point it is worth commenting the different terms in this expression: we have that $(1-z)$ is the momentum fraction of the emitted gluon, and if the gluon is soft, $z=1$, we have a singularity. Further, if the gluon is emitted collinearly to the emitting quarks, we have that $y=1$, resulting in another singularity. Hence, we have both a collinear and soft divergence.

However, if we keep ϵ finite, the singularities can be extracted by using the Euler-Beta integral. The singularities will then manifest themselves in ϵ singularities. Writing out all terms, the differential cross-section for real emission takes the form

$$\begin{aligned}\frac{d\hat{\sigma}^r_{q\bar{q}}}{dQ^2} &= \frac{1}{2\hat{s}}\Big(\frac{\alpha}{3\pi Q^2}\Big)\Big[C_F\frac{Q_q^2e^2g_s^2\mu^{2\epsilon}}{N_c\,N_s^2}\Big]\frac{1}{8\pi}\Big(\frac{4\pi}{Q^2}\Big)^{\epsilon}\frac{(1-z)^{1-2\epsilon}\,z^{\epsilon}}{\Gamma(1-\epsilon)}8(1-\epsilon)\\ &\quad\times\int_0^1 dy\,\big(y(1-y)\big)^{-\epsilon}\Big(\frac{2z}{(1-z)^2(1-y)y}+(1-\epsilon)\Big[\frac{1-y}{y}+\frac{y}{1-y}\Big]-2\epsilon\Big)\\ &= C_F\hat{\sigma}_0\frac{\alpha_s}{2\pi}\Big(\frac{4\pi\mu^2}{Q^2}\Big)^{\epsilon}\frac{(1-\epsilon)}{\Gamma(1-\epsilon)}(1-z)^{1-2\epsilon}z^{\epsilon}\\ &\quad\times\Big[\frac{2z}{(1-z)^2}B(-\epsilon,-\epsilon)+2(1-\epsilon)B(-\epsilon,2-\epsilon)-2\epsilon B(1-\epsilon,1-\epsilon)\Big]\\ &= C_F\hat{\sigma}_0\frac{\alpha_s}{\pi}A(\epsilon)\frac{z^{\epsilon}}{\epsilon}\Big(-2z(1-z)^{-1-2\epsilon}-(1-z)^{1-2\epsilon}\Big)\,,\end{aligned}\tag{3.201}$$

where we for notational simplicity have neglected several factors that are finite for $\epsilon\to 0$. The integral over y was performed by the Euler-Beta integral

$$B(a,b)=\int_0^1 dt\,t^{a-1}(1-t)^{b-1}=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\,,\tag{3.202}$$

and we defined the prefactor $A(\epsilon)$ as

$$A(\epsilon)=\Big(\frac{4\pi\mu^2}{Q^2}\Big)^{\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\,.\tag{3.203}$$

We still have a singularity as $z\to 1$. To make all singularities manifest in ϵ, we can expand the terms inside the bracket by the use of plus distributions, see Appendix C.1. We write this as

$$(1-z)^{-1-2\epsilon}=-\frac{1}{2\epsilon}\delta(1-z)+\Big[\frac{1}{1-z}\Big]_+-2\epsilon\Big[\frac{\ln(1-z)}{1-z}\Big]_+ +\mathcal{O}(\epsilon^2)\,,\tag{3.204}$$

$$z^{\epsilon}=1+\epsilon\ln z+\mathcal{O}(\epsilon^2)\,.\tag{3.205}$$

With these considerations, the final result for real gluon emission takes the form

$$\begin{aligned}\frac{d\hat{\sigma}^r_{q\bar{q}}}{dQ^2} = C_F\hat{\sigma}_0\frac{\alpha_s}{\pi}A(\epsilon)\Big(&\frac{1}{\epsilon^2}\delta(1-z)-\frac{1}{\epsilon}\Big[\frac{1+z^2}{1-z}\Big]_+,\\ &+2(1+z^2)\Big[\frac{\ln(1-z)}{(1-z)}\Big]_+-\frac{1+z^2}{1-z}\ln z\Big)\,.\end{aligned}\tag{3.206}$$

We have now managed to bring all singularities on ϵ form. The term $1/\epsilon^2$ is an infrared singularity due to simultaneously soft and collinear emission, and the term $1/\epsilon$ is due to collinear emission.

To any given order in perturbation theory, we must consider all possible diagrams at that specific order and add them at the end. Therefore, we must find the virtual contribution and see if some of the singular terms cancel. The only virtual contribution comes from the lower diagram in Fig. 3.5. There are two additional diagrams at this order that could contribute, the quark self-energy diagrams, but it can be shown that these exactly cancel each other. The virtual contribution contains both infrared and ultraviolet-singularities. To separate these singularities, we do as in Eq. (1.239) and name them ϵ_{UV} and ϵ_{IR}.

The amplitude for the virtual correction is given by

$$\begin{aligned}\mathcal{M}^{\mu} =& t^a_{ik}t^a_{kj}\mu^{(4-d)}\int\frac{d^dk}{(2\pi)^2}\bar{v}(p')(ig_s\gamma^{\alpha})\frac{i(\not{p}'+\not{k})(ie\gamma^{\mu})i(\not{k}-\not{p}))}{(p'+k)^2(p-k)^2}(ig_s\gamma^{\beta})u(p)\,D_{\alpha\beta}(k)\\ &= eg_s^2C_F\mu^{(d-4)}\int\frac{d^dk}{(2\pi)^2}\bar{v}(p')\gamma^{\alpha}\frac{(\not{p}'+\not{k})\gamma^{\mu}(\not{k}-\not{p})}{k^2(p'+k)^2(p-k)^2}\gamma^{\beta}u(p)\\ &= \bar{v}(p')(\Gamma^{\mu})u(p)\,,\end{aligned}\tag{3.207}$$

where we have defined the vertex correction Γ^μ. The integrand can be simplified by the Dirac algebra in d-dimensions and the massless Dirac equation. Further, by applying the Feynman parameter method introduced in Sec. 1.5.1, the integral can be turned into a Euler-Beta integral.

The propagator factors in Eq. (3.207), are the same as the ones we derived in Eq. (1.248). The numerator is straightforward to manipulate, but tedious. The resulting vertex correction is given by

$$\begin{aligned}\Gamma^\mu &= \gamma^\mu C_F \frac{\alpha_s}{4\pi}\Big(\frac{4\pi\mu^2}{Q^2}\Big)^\epsilon (-1)^\epsilon \Gamma(1+\epsilon)\Gamma(1-\epsilon)\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \\ &\quad \Big(\frac{1}{\epsilon_{UV}} - \frac{2}{\epsilon_{IR}^2} - \frac{4}{\epsilon_{IR}} - 8 + \mathcal{O}(\epsilon)\Big) \\ &= \gamma^\mu C_F \frac{\alpha_s}{4\pi} A(\epsilon)\Big(\frac{1}{\epsilon_{UV}} - \frac{2}{\epsilon_{IR}^2} - \frac{4}{\epsilon_{IR}} - 8 + \frac{2\pi^2}{3} + \mathcal{O}(\epsilon)\Big)\,, \end{aligned} \tag{3.208}$$

where we expanded the factors

$$(-1)^\epsilon \Gamma(1+\epsilon)\Gamma(1-\epsilon) = 1 - \frac{\pi^2}{3}\epsilon^2 + \mathcal{O}(\epsilon^3)\,. \tag{3.209}$$

To regulate the UV-singularities we have to add a counterterm. We are not considering electroweak corrections, so for our vertex correction the only reasonable counterterm to add is the quark self-energy. In dimensional regularization this contribution yields a scaleless integral, which we encountered in Eq. (1.239). To remove the UV-divergence we add[81]

$$\delta_\Gamma \gamma^\mu = -\gamma^\mu C_F \frac{\alpha_s}{4\pi} A(\epsilon)\left(\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}}\right). \tag{3.210}$$

This results in the UV-finite vertex correction

$$\begin{aligned}\Gamma_R^\mu &= \gamma^\mu C_F \frac{\alpha_s}{4\pi} A(\epsilon)\Big(-\frac{2}{\epsilon_{IR}^2} - \frac{3}{\epsilon_{IR}} - 8 + \frac{2\pi^2}{3} + \mathcal{O}(\epsilon)\Big) \\ &= \gamma^\mu \mathcal{F}\,. \end{aligned} \tag{3.211}$$

The differential cross section for the virtual correction is obtained by considering the interference between the leading order Fig. 3.4 and the vertex correction Fig. 3.5. This corresponds to making the substitution

$$e\gamma^\mu \to \Gamma_R^\mu = e\gamma^\mu \mathcal{F} \tag{3.212}$$

which leads to the differential cross section

$$\frac{d\hat{\sigma}^v_{q\bar{q}}}{dQ^2} = \frac{d\hat{\sigma}^{(0)}_{q\bar{q}}}{dQ^2} 2\mathrm{Re}(\mathcal{F}) \tag{3.213}$$

$$= C_F \hat{\sigma}_0 \frac{\alpha_s}{\pi} A(\epsilon)\Big(-\frac{1}{\epsilon^2} - \frac{3}{2\epsilon} - 4 + \frac{\pi^2}{3}\Big)\delta(1-z)\,. \tag{3.214}$$

The real and virtual contributions are now on the same form, such that we can easily add them, giving the NLO result

$$\begin{aligned}\Big(\frac{d\hat{\sigma}_{q\bar{q}}}{dQ^2}\Big)_{\mathrm{NLO}} &= \frac{d\hat{\sigma}^r_{q\bar{q}}}{dQ^2} + \frac{d\hat{\sigma}^v_{q\bar{q}}}{dQ^2} \\ &= C_F \hat{\sigma}_0 \frac{\alpha_s}{\pi} A(\epsilon)\Big[-\frac{1}{\epsilon}\Big(\Big[\frac{1+z^2}{1-z}\Big]_+ + \frac{3}{2}\delta(1-z)\Big) + \Big(\frac{\pi^2}{3} - 4\Big)\delta(1-z) \\ &\quad + 2(1+z^2)\Big[\frac{\ln(1-z)}{1-z}\Big]_+ - \frac{1+z^2}{(1-z)}\ln z\Big]\,. \end{aligned} \tag{3.215}$$

We observe that in the sum over real and virtual contributions, some of the divergences have canceled. This is a manifestation of the Kinoshita-Lee-Nauenberg (KLN) theorem [32, 33]. The KLN-theorem states that in the

[81] As discussed in Sec. 1.5.2, this is always allowed for a renormalizable theory.

sum of all diagrams, the result is to be free of IR-divergences. But we still have a collinear divergence, which we soon will show how to deal with.

In general, we can write the partonic cross section as the perturbative expansion

$$\begin{aligned}\frac{d\hat{\sigma}}{dQ^2} &= \sum_{n=0}^{\infty}\Big(\frac{\alpha_s}{\pi}\Big)^n \frac{d\hat{\sigma}^{(n)}}{dQ^2}\,,\\ &= \frac{d\hat{\sigma}^{(0)}}{dQ^2} + \frac{\alpha_s}{\pi}\frac{d\hat{\sigma}^{(1)}}{dQ^2} + \cdots\\ &= \hat{\sigma}_0\big(\omega^{(0)} + \frac{\alpha_s}{\pi}\omega^{(1)} + \cdots\big)\,,\end{aligned} \tag{3.216}$$

where we separated out the Born cross section and defined a hard function $\omega^{(n)}$. With this notation, we have that the hard functions up to $\mathcal{O}(\alpha_s)$ are given by

$$\begin{aligned}\omega^{(0)} &= \delta(1-z)\,,\\ \omega^{(1)} &= -A(\epsilon)\frac{1}{\epsilon}P^{(0)}_{q/q} + A(\epsilon)\Big[\Big(\frac{\pi^2}{3}-4\Big)\delta(1-z)\\ &\qquad + 2(1+z^2)\Big[\frac{\ln(1-z)}{1-z}\Big]_+ - \frac{1+z^2}{1-z}\ln z\Big]\,,\end{aligned} \tag{3.217}$$

where we have inserted the splitting function $P^{(0)}_{q/q}$, see Eq. (3.129).

As mentioned above some of the IR-divergences has canceled, but there still remain a collinear singularity in $1/\epsilon$. The reason this collinear singularity is still present and did not cancel according to the KLN theorem is that we are considering massless particles. To treat the remaining singularity we can use the factorization property of QCD. In DIS we defined 'bare' parton distributions to cancel the divergence. However, we have already seen that we can define parton-in-parton distributions that contains collinear divergences, see Eq. (3.176). Hence, we can use these parton-in-parton distributions instead, which we will show how to do in the next section.

3.3.3 Renormalization of NLO cross section

According to the QCD factorization theorem, the hadronic Drell-Yan cross section can be written as[82]

$$\frac{d\sigma_{h_1h_2}}{dQ^2} = \sigma_0 W(\tau, Q, \mu, \alpha_s)\,, \tag{3.218}$$

where[83]

$$W(\tau, Q, \mu, \alpha_s) = \sum_{i,j} Q_q^2 \int \frac{dx_1}{x_1}\frac{dx_2}{x_2} f_{i/h_1}(x_1,\mu) f_{j/h_2}(x_2,\mu) w_{ij}\Big(z, Q, \mu, \alpha_s(\mu)\Big)\,, \tag{3.219}$$

where we set the factorization and renormalization scale to be the same, $\mu_F = \mu_R = \mu$. We have also factored out the hadronic Born cross section by using that $x_1x_2\bar{\sigma}_0 = Q_q^2\sigma_0$, where

$$\sigma_0 = \frac{4\pi\alpha^2}{3N_c s Q^2} \tag{3.220}$$

and defined the hadronic and partonic threshold variables as

$$\tau = \frac{Q^2}{s}\,, \qquad z = \frac{\tau}{x_1x_2}\,. \tag{3.221}$$

The singular part of the fixed order calculation needs to be separated out of the hard scattering part, such that we can absorb it into the parton distribution functions. To this end, we define the partonic analogue to

[82] This is proven to hold up to corrections of $\mathcal{O}(1/Q^2)$, see [26]. In Sec. 3.3.2 we only considered the $q\bar{q}$ process, but to be general we will use generic subscripts.

[83] This is really a Mellin convolution, see Appendix D.1.

Eq. (3.219)

$$h_{ij}(z,Q,\mu,\alpha_s(\mu),\epsilon) = \sum_{k,l}\int_z^1 \frac{dy_1}{y_1}\frac{dy_2}{y_2}\, f_{k/i}(y_1,\mu,\epsilon)\, f_{l/j}(y_2,\mu,\epsilon) \times \omega_{ij}\Big(\frac{z}{y_1},\frac{z}{y_2},Q,\mu,\alpha_s(\mu^2)\Big) \tag{3.222}$$

which is a refactorization of the hard scattering function using the parton-in-parton densities we discussed in Sec. 3.2.5. We can extract the singular part by making the following perturbative expansion

$$h_{ij} = h_{ij}^{(0)} + \frac{\alpha_s}{\pi}h_{ij}^{(1)} + \mathcal{O}(\alpha_s^2) \tag{3.223}$$

$$\omega_{ij} = \omega_{ij}^{(0)} + \frac{\alpha_s}{\pi}w_{ij}^{(1)} + \mathcal{O}(\alpha_s^2) \tag{3.224}$$

$$f_{i/j} = f_{i/j}^{(0)} + \frac{\alpha_s}{\pi}f_{i/j}^{(1)} + \mathcal{O}(\alpha_s^2)\,, \tag{3.225}$$

where $h_{ij}^{(1)}$ is the singular hard function we calculated at NLO in Sec. 3.3.2, i.e.

$$h_{ij}^{(1)} = -A(\epsilon)\frac{1}{\epsilon}P_{q/q}^{(0)} + A(\epsilon)\Big(\frac{\pi^2}{3}-4\Big)\delta(1-z) + 2(1+z^2)\Big[\frac{\ln(1-z)}{1-z}\Big]_+ - \frac{1+z^2}{1-z}\ln z\,, \tag{3.226}$$

and the $\mathcal{O}(\alpha_s)$ parton-in-parton density is given in Eq. (3.177). Let us expand $A(\epsilon)$, giving

$$A(\epsilon) = \Big(\frac{4\pi\mu^2}{Q^2}\Big)^{\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} = 1 - \epsilon(\gamma_E - \ln 4\pi) + \epsilon\ln\Big(\frac{\mu^2}{Q^2}\Big) + \mathcal{O}(\epsilon^2)\,. \tag{3.227}$$

By using the $\overline{\text{MS}}$ scheme we remove γ_E and $\ln(4\pi)$, giving

$$h_{ij}^{(1)}(z,\epsilon) = C_F\Big[\Big(\frac{\pi^2}{3}-4\Big)\delta(1-z) + 2(1+z^2)\Big[\frac{\ln(1-z)}{1-z}\Big]_+ - \frac{1+z^2}{1-z}\ln z\Big] - \frac{1}{\epsilon}\Big(1+\epsilon\ln\frac{\mu^2}{Q^2}\Big)P_{q/q}^{(0)}(z)\,, \tag{3.228}$$

$$f_{i/j}^{(1)}(z,\epsilon) = -\frac{1}{2\epsilon}\Big[1+\epsilon\ln\big(\frac{\mu^2}{\mu_F^2}\big)\Big]P_{q/q}^{(0)}(z)\,. \tag{3.229}$$

If we insert the perturbative expansions in Eq. (3.223) into Eq. (3.222), we find that

$$h_{ij}^{(0)}(z) = \omega_{ij}^{(0)}(z) = \delta_{ij}\delta(1-z)\,, \tag{3.230}$$

and the first order correction is given by

$$\begin{aligned}\omega_{ij}^{(1)}(z) &= h_{ij}^{(1)}(z,\epsilon) + \frac{1}{2\epsilon}\Big[1+\epsilon\ln\Big(\frac{\mu^2}{\mu_F^2}\Big)\Big]\sum_k\int_z^1\frac{dy_1}{y_1}P_{k/i}^{(0)}(y_1)h_{kj}^{(0)}(\frac{z}{y_1}) \\ &\quad + \frac{1}{2\epsilon}\Big[1+\epsilon\ln\Big(\frac{\mu^2}{\mu_F^2}\Big)\Big]\sum_l\int_z^1\frac{dy_2}{y_1}P_{l/j}^{(0)}(y_2)h_{il}^{(0)}(\frac{z}{y_2}) \\ &= h_{ij}^{(1)}(z,\epsilon) + \frac{1}{\epsilon}P_{ij}^{(0)}(z) + \ln\Big(\frac{\mu^2}{\mu_F^2}\Big)P_{ij}^{(0)}(z)\,,\end{aligned}$$

where we used the delta function to evaluate the integrals. We observe that the $1/\epsilon$ singularity will cancel in this sum, leading to the infrared safe hard function in the $\overline{MS}$-scheme

$$\omega_{q\bar{q}}^{(1)}(z) = P_{qq}^{(0)}\ln\frac{Q^2}{\mu_F^2} + C_F\Big[\Big(\frac{\pi^2}{3}-4\Big)\delta(1-z) + 2(1+z^2)\Big[\frac{\ln(1-z)}{1-z}\Big]_+ - \frac{1+z^2}{1-z}\ln z\Big]\,. \tag{3.231}$$

Hence, regularizing soft divergences gives rise to logarithmic plus distributions and regularizing collinear divergences gives rise to a logarithm of the factorization scale.

We can remove this last part by choosing $\mu_F = Q$, which also removes the splitting function that contains further plus distributions. This is the most common choice in the literature, which we also adopt. With this choice the partonic scattering function up to $\mathcal{O}(\alpha_s)$ takes the form

$$\omega_{q\bar{q}}(z, \alpha_s(\mu)) = \delta(1-z) + \frac{\alpha_s}{\pi} C_F \Big[\Big(\frac{\pi^2}{3} - 4 \Big) \delta(1-z) + 2(1+z^2) \Big[\frac{\ln(1-z)}{1-z} \Big]_+ - \frac{1+z^2}{1-z} \ln z \Big] . \tag{3.232}$$

The result in Eq. (3.232) contains no singularities, but the plus distributions is a source of potential large corrections. When we are close to threshold, there is little phase space left for the emission of gluons. In this case, most of the energy is used to produce the final state leptons, leading to a suppression of real soft gluon emission[84]. These plus distributions appeared when we summed real and virtual contributions. Thus, if real contributions are suppressed there will be an imbalance in the cancellation. These corrections appear to all order in perturbation theory, so to n-th order they take the form

$$\alpha_s^n \Big[\frac{\ln^{2n-1}(1-z)}{1-z} \Big]_+ . \tag{3.233}$$

As they are large in the threshold regime, they will spoil the convergent behaviour of the perturbative expansion. Therefore, they must be treated to all orders in order to make reliable predictions. The technique to do this is called *threshold resummation*, which we will lay out in more detail in the next chapter.

3.3.4 Numerical Evaluation of the Hadronic Drell-Yan Cross Section

In this section we will compare the LO and the full NLO differential cross section for the Drell-Yan process to experimental results from the CMS experiment [34]. Our basic result is shown in Fig. 3.6 where the LO result is given by the dashed line, the full NLO result in the solid line and the data from CMS is given by the red dots with errorbars [34]. The numerical evaluation is based on the analytical $\mathcal{O}(\alpha_s)$ result in Eq. (3.232) and performed by evaluating Eq. (4.26) for both the LO and LO+NLO cross section. The analytical cross section has been obtained at a centre of mass energy $\sqrt{s} = 13$ TeV and we consider production of leptons with invariant mass between 50 and 300 GeV. At LO there is only the $q\bar{q}$ channel that contributes, but at NLO we also have a contribution from a gluon initiated process. To be in accordance with our analytical calculation we have chosen to neglect this contribution in the numerical evaluation as well and only focused on the colour singlet process. Also, the plus distribution in Eq. (3.232) have been omitted as they are to be resummed in Chapter 4. We use the CT10 NLO [35] PDF set, and we choose the factorization scale and renormalization scale to be equal the invariant mass of the final state leptons, $\mu = Q$. We have also included the Z-boson resonance by using an effective coupling as discussed in [36].

The main theoretical uncertainty in our analysis comes from the dependence of the calculated cross section on the renormalization and factorization scale. This uncertainty is found by varying the scale between $Q/2$ and $2Q$. By calculating the uncertainty in this way the error-band should give a estimate of how higher order effects should affect the cross section. There are also theoretical uncertainties coming from the PDFs, but we have chosen not to take these into account.

We can see from Fig. 3.6 that the full NLO calculation we have performed is much closer to the true cross section found by the CMS experiment than the LO result. This is of course not surprising as the true cross section is an all order process and taking into account higher and higher order in the perturbative expansion should give better and better predictions. However, considering the simplifications we have made by neglecting gluon initiated processes and omitting the plus distribution, the result is surprisingly good. Another point to make is that the dependancy on the renormalization scale is shown to decrease when including QCD effects, implying that even higher order corrections are needed in order to reduce this dependancy even further. Also, according to our uncertainty estimation the full NLO result should lie inside the error-band coming from the LO calculation. But we observe that this is far off, especially at the tails. We suspect that this is due to the fact that at LO we have a pure electroweak process, and QCD effects are only manifest at higher orders.

[84] These are soft because near threshold most of the energy is used to produce the final state leptons.

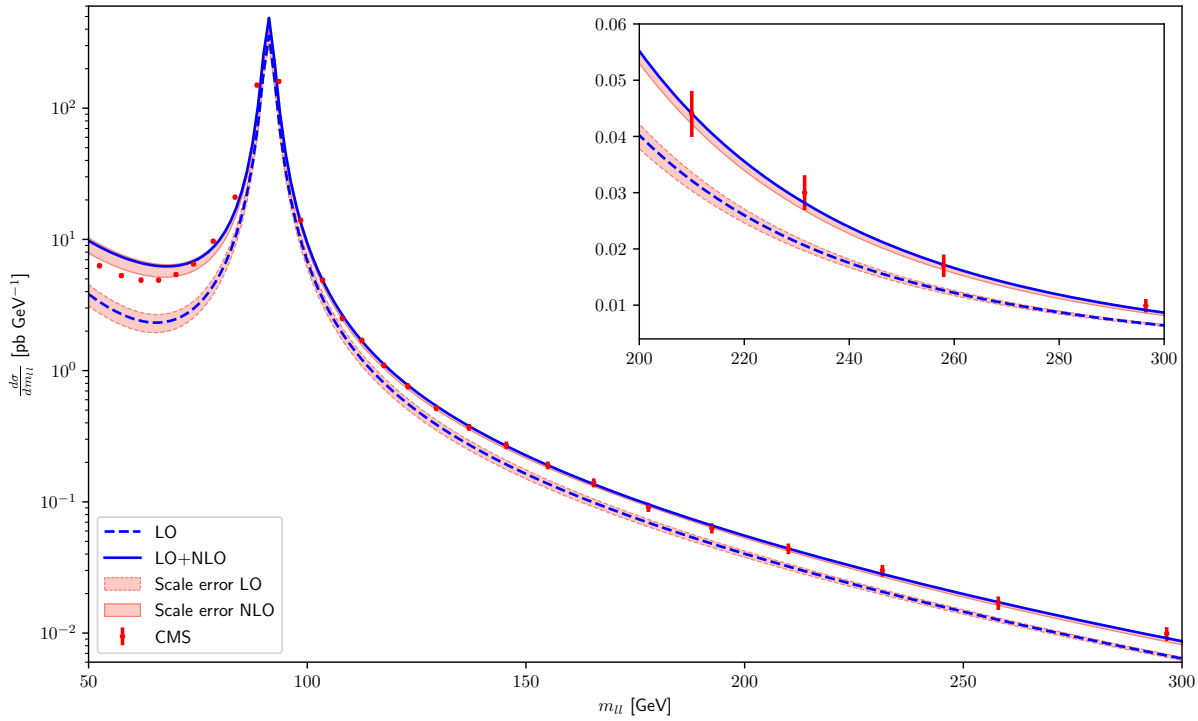


Figure 3.6: Differential cross section of neutral Drell-Yan production as function of final-state invariant mass. Shown are LO (dashed) and LO+NLO (solid) results with corresponding scale errors (pink bands). The figure also shows the differential cross section (red dots) measured by the CMS experiment [34].

Chapter 4

Resummation using Wilson Lines

In this chapter we will take a closer look at how we can deal with the large logarithms in the threshold region. First we will take a classical approach to eikonal exponentiation in an Abelian theory, following a similar line as Steven Weinberg [37]. The generalization to non-Abelian theories is not straightforward. This is of course due to the non-commutivity between non-Abelian gauge fields. The proof exist in the literature, and the result is known as the *non-Abelian eikonal exponentiation theorem* [38–40]. We will not cover this generalization as it would take us to far from our main purpose, but the above references can be sought out for more detail.

After we have seen how amplitudes exponentiate, we will move on to discuss several factorization properties of cross sections in QCD. First, we will introduce the Mellin space formalism where we take a Mellin transform of the hadronic cross section and show how it factorizes. Then we go on and use factorization theorems to fully factorize the cross section in the threshold regime by using parton-in-parton distributions. After the cross section has been fully factorized into hard, soft and collinear parts we will use Wilson lines to construct an eikonal cross section that governs the soft radiation. Then we will briefly discuss the renormalization properties of Wilson lines and make an explicit calculation of the cusp anomalous dimension to one-loop order.

With the introduction of the cusp anomalous dimension we discuss the renormalization group equation for parton-in-parton distributions and show that it is given in terms of the cusp anomalous dimension. The non-Abelian exponentiation theorem will then be used to find an exponentiated eikonal cross section. After we have found the exponentiated cross section we will take the discussion to the level of the hadronic cross section again. We will discuss possible ways of evaluating the inverse Mellin transform in order to obtain a cross section in real space.

4.1 Exponentiation

In Sec. 3.3, we dealt with a fixed order calculation for the Drell-Yan process. Due to soft gluon radiation from the hard quark lines, we identified large logarithmic contributions to the cross-section in the threshold region, i.e. $z \to 1$. As these large effects appear at all orders, they spoil the convergent behaviour of the series expansion in the strong coupling. To perform an all-order calculation in the full theory of QCD is an impossible task, but with certain approximations, these large contributions can be resummed.

Resummation of large logarithmic contributions was first demonstrated in [41], where the radiation of photons in QED was considered. The resummation was done by using the eikonal approximation, i.e. the photons are restricted to be soft. The crucial feature of the eikonal approximation is that scattering amplitudes exponentiate, with the consequence that one can compute the logarithm of the amplitude via a set of simplified rules. Therefore, we can access all order information from low-order perturbative calculations, which simplifies calculations substantially. Using this approximation, a scattering amplitude $\mathcal{M}$ describing multiple soft photon radiation can be written on the form

$$\mathcal{M} = \mathcal{M}_0 \exp\Big(\sum W_c\Big)\,, \tag{4.1}$$

where $\mathcal{M}_0$ is the amplitude without soft photon radiation, and the exponential has a sum over all connected Feynman diagrams W_c, involving soft photon emission only. Since QED is an Abelian gauge theory, the emitted photons will not interact and the amplitude factorizes, leading to an exponentiation of the amplitude.

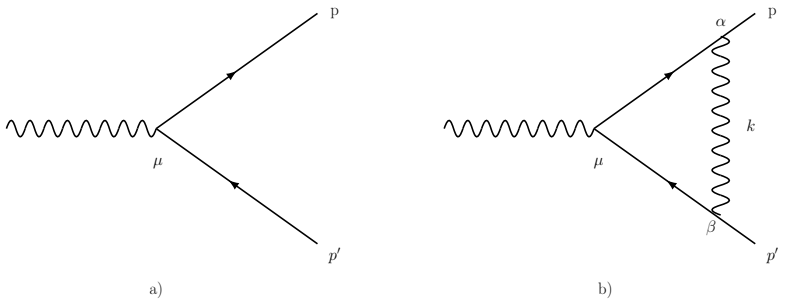


Figure 4.1: Leading order and $\mathcal{O}(\alpha)$ correction to virtual photon decay.

For a non-Abelian gauge theory, such as QCD, this simple factorization no longer applies as the gluons interact with each other. Nevertheless, in [42], it was observed that one could achieve exponentiation by using the eikonal approximation also in the case of QCD. The formal proof of this observation was later provided in [38, 39], where the structure of the exponent is much more complicated due to the non-commutativity of the colour matrices. The non-abelian analogue of Eq. (4.1) can then be written on schematic form as

$$\mathcal{M} = \mathcal{M}_0 \exp\Big(\sum \bar{c}_W W\Big)\,, \tag{4.2}$$

where the soft gluons are emitted from two hard partons connected by a colour singlet vertex, which is exactly the case considered in the Drell-Yan process. The sum in the exponent involves Feynman diagrams W, called *webs* in the literature, and $\bar{c}_W$ are modified colour factors which in general differ from the usual colour factors c_W using standard Feynman rules. We will encounter these webs later on, but we will not cover the details of their properties[85].

We will now show how the structure in Eq. (4.1) appears by considering the all order process of photon emission from a final state fermion-antifermion creation process. We will not do this by explicitly using Wilson lines, but we will again see that this it is equivalent.

Eikonal Exponentiation in QED

In this section we will look at the exponentiation of soft photon emission. Let us start by considering the amplitude of an off-shell photon decaying to a massless fermion-antifermion pair, see Fig. 4.1. The leading order amplitude is given by

$$\mathcal{M}_0 = \bar{u}(p)\gamma^\mu v(p')\,, \tag{4.3}$$

where coupling factors have been neglected for simplicity. If we now consider the correction, where a photon is emitted from the final state, we get the amplitude

$$\mathcal{M}_1 = \int \frac{d^dk}{(2\pi)^d}\bar{u}(p)\gamma^\alpha\frac{(\not{p}-\not{k})}{(p-k)^2}\gamma^\mu\frac{(\not{k}-\not{p}')}{(k-p')^2}\gamma^\beta v(p')D_{\alpha\beta}(k)\,, \tag{4.4}$$

where $D_{\alpha\beta}(k)$ is the photon propagator.

As we showed in Sec. 2.3 the eikonal approximation corresponds to taking the soft limit $k \to 0$, such that we can neglect $\not{k}$ in the numerator and k^2 in the denominator of the fermion propagators. If we also assume that the fermions are massless, $p^2 = 0$, we get the much simpler eikonal amplitude

$$\begin{aligned}\mathcal{M}_1 &= \int \frac{d^dk}{(2\pi)^d}\bar{u}(p)\gamma^\alpha\Big(-\frac{\not{p}}{2p\cdot k}\Big)\gamma^\mu\Big(\frac{\not{p}'}{2p\cdot k}\Big)\gamma^\beta v(p')D_{\alpha\beta}(k)\\ &= \int \frac{d^dk}{(2\pi)^d}\big[\bar{u}(p)\gamma^\mu v(p')\big]\Big(-\frac{p^\alpha}{p\cdot k}\Big)\Big(\frac{p'^\beta}{p\cdot k}\Big)D_{\alpha\beta}(k)\\ &= \mathcal{M}_0\int \frac{d^dk}{(2\pi)^d}\Big(-\frac{p^\alpha}{p\cdot k}\Big)\Big(\frac{p'^\beta}{p\cdot k}\Big)D_{\alpha\beta}(k)\,,\end{aligned} \tag{4.5}$$

[85] This is an extensive topic, so for more detail about webs, see e.g. [40, 43–45].

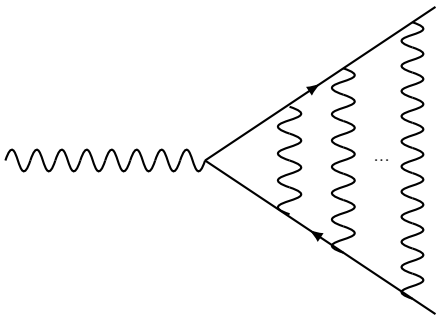
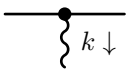


Figure 4.2: Ladder diagram of n soft photon emission.

where we in the second step used the Dirac algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ and the massless Dirac equation, $\bar{u}(p)\not{p} = 0$ and $\not{p}'v(p') = 0$. We observe that the tree-level amplitude has been factored out, and contains no divergences. This result is actually an example of factorization of soft physics from hard physics, where the soft part is described by the integral and the hard part is the leading order amplitude $\mathcal{M}_0$. The physical reason for this factorization is that the momentum of the soft photon is to low to resolve the inner structure of the hard process.

Another important observation is that we can define the terms inside the brackets as an eikonal Feynman rule[86]



$$= \frac{p^\mu}{p\cdot k}\,. \qquad \text{Eikonal vertex} \tag{4.6}$$

This actually means that we can think of the factors multiplying the tree-level amplitude in Eq. (4.5) as a new type of Feynman diagrams, which are *subdiagrams* of the full amplitude.

The next step is to generalize to the emission of n soft photons, so let us start with the diagram Fig. 4.2, where none of the photon lines cross each other, called a *ladder diagram*. This amplitude is given by

$$\mathcal{M}^{(n)} = \Big(\prod_{i=1}^{n}\int\frac{d^dk_i}{(2\pi)^d}\Big)\bar{u}(p)\mathcal{E}^{\alpha_1\ldots\alpha_n}(p,k_i)\gamma^\mu\mathcal{E}^{\beta_1\ldots\beta_n}(p',k_i)v(p')D_{\alpha_1\beta_1}(k_1)\ldots D_{\alpha_n\beta_n}(k_n)\,, \tag{4.7}$$

where we have collected the product of fermion propagators and the gamma matrices from the corresponding vertices into $\mathcal{E}$ in the following way

$$\mathcal{E}^{\alpha_1\ldots\alpha_n}(p,k_i) = \frac{\gamma^{\alpha_1}(\not{p}-\not{k}_1)\cdots\gamma^{\alpha_n}(\not{p}-\not{k}_1-\cdots-\not{k}_n)}{(p-k_1)^2\cdots(p-k_1-\ldots k_n)^2}\,, \tag{4.8}$$

and by taking the eikonal approximation this can be simplified to

$$\mathcal{E}^{\alpha_1\ldots\alpha_n}(p,k_i) = \frac{\bar{u}(p)\gamma^{\alpha_1}\not{p}\gamma^{\alpha_2}\not{p}\cdots\gamma^{\alpha_n}\not{p}}{(-2p\cdot k_1)\cdots(-2p\cdot(k_1+\cdots+k_n)}\,, \tag{4.9}$$

which can be further simplified by permuting the gamma matrices using the Dirac algebra and use that $\bar{u}(p)\not{p} = 0$, giving

$$\bar{u}(p)\mathcal{E}^{\alpha_1\ldots\alpha_n} = \frac{(-1)^n\bar{u}(p)p^{\alpha_1}\cdots p^{\alpha_n}}{p\cdot k_1\cdots p\cdot(k_1+\cdots+k_n)}\,, \tag{4.10}$$

$$\mathcal{E}^{\beta_1\ldots\beta_n}v(p') = \frac{p'^{\beta_1}\cdots p'^{\beta_n}\,v(p')}{p'\cdot k_1\cdots p'\cdot(k_1+\cdots+k_n)}\,, \tag{4.11}$$

where the factors of 2 cancels from those in the Dirac algebra, see Eq. (B.1). Combining these two expressions into Eq. (4.7), we get

$$\mathcal{M}^{(n)} = \mathcal{M}_0\Big(\prod_{i=1}^{n}\int\frac{d^dk_i}{(2\pi)^d}\Big)\Big[\frac{(-1)^np^{\alpha_1}\cdots p^{\alpha_n}\,p'^{\beta_1}\cdots p'^{\beta_n}\,D_{\alpha_1\beta_1}(k_1)\cdots D_{\alpha_n\beta_n}(k_n)}{p\cdot k_1\cdots p\cdot(k_1+\cdots+k_n)\,p'\cdot k_1\cdots p'\cdot(k_1+\cdots+k_n)}\Big]\,. \tag{4.12}$$

[86] This is of course closely related to the Wilson line rules we derived in Sec. 2.3, with the distinction that this is an Abelian theory.

This reveals an intricate dependency on the photon momenta k_i, as they are coupled along both the fermion and anti-fermion line. However, we have only considered the ladder diagram where the photons are emitted in the order $k_1 \dots k_n$. Therefore, we must sum over all diagrams to this order, meaning we must sum over all permutations of the emitted photon momenta. We start by fixing the order of photon emission on the anti-fermion line, and sum over permutations on the fermion line. If we let π denote a permutation of $(1, 2, \dots, n)$, which maps to $(\pi_1, \pi_2, \dots \pi_n)$, the sum over diagrams correspond to making the following substitution

$$\frac{1}{p \cdot k_1 \cdots p \cdot (k_1 + \cdots + k_n)} \to \sum_\pi \frac{1}{p \cdot k_{\pi_1} \cdots p \cdot (k_{\pi_1} + \cdots + k_{\pi_n})} \,.$$

Then we can make use of the so-called *eikonal identity*[87]

$$\sum_\pi \frac{1}{p \cdot k_{\pi_1} \cdots p \cdot (k_{\pi_1} + \cdots + k_{\pi_n})} = \prod_{i=1}^n \frac{1}{p \cdot k_i} \,. \tag{4.13}$$

If this seems mysterious, let us show a simple example that highlights how this works. For the simple case of $n = 2$, we have that

$$\frac{1}{p \cdot k_1 \, p \cdot (k_1 + k_2)} + \frac{1}{p \cdot k_2 \, p \cdot (k_1 + k_2)} = \frac{1}{p \cdot k_1 \, p \cdot k_2} \,, \tag{4.14}$$

which shows the structure.

So by using the eikonal identity we substantially simplify the dependence on the photon momenta for the fermion line, as they have decoupled from each other. We could hope to do the same on the anti-fermion line, but that is not possible as they have already been fixed. However, we can exploit that k_i are dummy variables inside the integral. The integrand has a symmetric Lorentz structure under the permutation of any two momenta, which means that we can make the replacement

$$\begin{aligned}
&\Big(\prod_{i=1}^n \int \frac{d^d k_i}{(2\pi)^d}\Big) \frac{1}{p' \cdot k_1 \cdots p' \cdot (k_1 + \cdots + k_n)} \\
&= \frac{1}{n!} \Big(\prod_{i=1}^n \int \frac{d^d k_i}{(2\pi)^d}\Big) \sum_\pi \frac{1}{p' \cdot k_{\pi_1} \cdots p' \cdot (k_{\pi_1} + \cdots + k_{\pi_n})} \\
&= \frac{1}{n!} \prod_{i=1}^n \int \frac{d^d k_i}{(2\pi)^d} \frac{1}{p' \cdot k_i} \,,
\end{aligned} \tag{4.15}$$

where we have used that there are $n!$ such permutations.

Substituting these simplifications into Eq. (4.12), we get

$$\begin{aligned}
\mathcal{M}^{(n)} &= \mathcal{M}_0 \frac{1}{n!} \prod_{i=1}^n \int \frac{d^d k_i}{(2\pi)^d} \Big(\frac{p^{\alpha_i}}{p \cdot k_i}\Big)\Big(\frac{-p'^{\beta_i}}{p' \cdot k_i}\Big) D_{\alpha_i \beta_i}(k_i) \\
&= \mathcal{M}_0 \frac{1}{n!} \Big[\int \frac{d^d k}{(2\pi)^d} \Big(\frac{p^\alpha}{p \cdot k}\Big)\Big(\frac{-p'^\beta}{p' \cdot k}\Big) D_{\alpha\beta}(k) \Big]^n \,.
\end{aligned} \tag{4.16}$$

This looks very much like the nth term in a Taylor expansion of an exponential. By using the eikonal approximation we have found the remarkable result that the sum over all n photon graphs is given by the one-loop graph to the nth power! If we take the sum over all diagrams for any number of single soft photon emission, we find the all order amplitude

$$\mathcal{M} = \sum_{n=1}^\infty \mathcal{M}^{(n)} = \mathcal{M}_0 \exp\Big(\int \frac{d^d k}{(2\pi)^d} \Big(\frac{p^\alpha}{p \cdot k}\Big)\Big(\frac{-p'^\beta}{p' \cdot k}\Big) D_{\alpha\beta}(k) \Big) \,, \tag{4.17}$$

[87] This identity can be proven by using the Wilson lines we considered in Sec. 2.3, but for a standard derivation we refer the reader to [2].

which demonstrates that the hard part factorizes from the soft part also in the all order case, with the addition that the soft part exponentiates. From the structure of Eq. (4.17), we can write the amplitude on the factorized form

$$\mathcal{M} = \mathcal{H}\mathcal{S}\,, \tag{4.18}$$

where $\mathcal{H}$ is a hard scattering function and $\mathcal{S}$ is a soft scattering function. The hard function is finite, while the soft function contains all the soft and collinear divergences. It is important to note that we have only considered single photon emissions, but to be general one should in fact consider the case where the emitted photons could connect off the external lines, via fermion loops. This would extend the result in Eq. (4.17) to include the sum over all connected diagrams including loops in the exponent, see [41].

As already mentioned the complexity increases substantially for a non-abelian gauge theory, and while the result has existed in literature for a long time [38, 39], the proof is restricted to involve only two coloured external particles. This proof covers Drell-Yan like processes, but if the final state contains coloured particles as well this proof does not hold. This scenario has been studied in great detail and recent work has been done to improve it to include several coloured particles [40, 43]. The proof is too advanced to go into detail on here, but the main idea is to use a path integral approach with Wilson lines as a source for creating particles from the vacuum. From there one uses the so-called replica method from statistical mechanics to show that the theory can be written as N replicas leading to the structure of webs.

4.2 Mellin Space Factorization

In the NLO calculation Sec. 3.3.3, we found that the final hard scattering function was infrared safe. However, in the cancellation of infrared divergences, we found the emergence of logarithmic distributions. These logarithmic distributions appear at every order and ruin the convergence of the perturbative expansion in the threshold regime. To handle these large corrections, we have to use threshold resummation techniques. Threshold resummation was first derived for the Drell-Yan process [46, 47]. These papers laid the groundwork for resummation of large contributions in many hard QCD processes.

4.2.1 Phase Space Factorization

A fundamental ingredient for resummation in QCD, is that the eikonal approximation leads to exponentiation. Near the threshold, most of the available energy goes to producing the final state particles. Therefore, the emitted gluons are soft, and the eikonal approximation corresponds to taking the partonic threshold limit. However, the derivation of eikonal exponentiation in Sec. 4.1 was at the level of amplitudes. Eventually, we want the full cross-section to exponentiate, including the underlying hard process. The problem is that the hard part contains phase space integrals, which may lead to an intricate dependency between the final state gluon momenta. If the cross-section is to exponentiate the gluon momenta must disentangle from each other. To make this problem explicit, let us consider the Drell-Yan process where the differential phase space for the emission of n soft gluons can be written as

$$\begin{aligned} d\mathcal{P}^{(n)} &= \frac{d^4q}{(2\pi)^4}\prod_{i=1}^{n}\frac{d^3k_i}{(2\pi)^3}\frac{1}{2k_i^0}\,2\pi\delta^+(q^2-Q^2)(2\pi)^4\delta^{(4)}\Big(p_1+p_2-q-\sum_i k_i\Big) \\ &= \prod_{i=1}^{n}\frac{d^3k_i}{(2\pi)^2}\frac{1}{2k_i^0}\,\delta\Big(\Big[p_1+p_2-\sum_i k_i\Big]^2-Q^2\Big)\,, \end{aligned} \tag{4.19}$$

where we began with the general expression for clarity and applied the delta function. Here k_i is the gluon momenta, q is the photon momenta, p_1 and p_2 is the incoming quarks and Q^2 is the final state invariant mass. In the soft gluon region we can make the approximation

$$\begin{aligned} d\mathcal{P}^{(n)} &\approx \prod_{i=1}^{n}\frac{d^3k_i}{(2\pi)^2}\frac{1}{2k_i^0}\delta\Big(\hat{s}-2\sum_i k_i^0\sqrt{\hat{s}}-Q^2\Big) \\ &= \prod_{i=1}^{n}\frac{d^3k_i}{(2\pi)^2}\frac{1}{2k_i^0}\frac{1}{\hat{s}}\delta\Big(1-z-\sum_i \omega_{k_i}\Big)\,, \end{aligned} \tag{4.20}$$

where we have defined the fractional energy of the ith gluon as $\omega_{k_i} = 2k_i^0/\sqrt{\hat{s}}$, and z is the usual partonic threshold variable, $z = Q^2/\hat{s}$. We observe that the delta function forces the emitted gluon momenta to depend on each other, spoiling the factorization of the phase space.

There is a neat solution to this problem via the Laplace transform. In general, the Laplace transform of a delta function is given by

$$\mathcal{L}[\delta(x-y)](N) = e^{-Ny}\,. \tag{4.21}$$

where $N \in \mathbb{C}$. The goal now is to try to disentagle the delta function, so if we take the inverse Laplace of the delta function in Eq. (4.20) we obtain the expression

$$\delta(1-z-\sum_i \omega_{k_i}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dN\, e^{N(1-z-\sum_i \omega_{k_i})}\,. \tag{4.22}$$

However, we are only interested in the threshold regime $z \to 1$, so let us consider the following Taylor series

$$\ln z = \sum_{n=1}^{\infty}(-1)^n \frac{(1-z)^n}{n}\,, \tag{4.23}$$

leading to the approximate identity, $e^{N(1-z)} \approx e^{-N\ln z} = z^{-N}$. With this approximation Eq. (4.22) takes the form

$$\delta(1-z-\sum_i \omega_{k_i}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dN\, z^{-N}\prod_{i=1}^{n} e^{-N\omega_{k_i}}\,, \tag{4.24}$$

which is an inverse Mellin transform, see Appendix D.1. The consequence of this result is made clear if we take the Mellin transform of the differential phase space

$$\int_0^1 dz\, z^{N-1}\, d\mathcal{P}_n = \prod_{i=1}^{n}\frac{d^3k_i}{(2\pi)^2}\frac{1}{2k_i^0}\frac{1}{\hat{s}}e^{-N\omega_{k_i}}\,, \tag{4.25}$$

giving that the phase space has taken a factorized form, enabling it to exponentiate.

4.2.2 Hadronic Cross Section in Mellin Space

Factorization of the phase space is not the only advantage of using the Mellin transform. In Sec. 3.3, we wrote the factorized Drell-Yan cross section as

$$\frac{d\sigma_{h_1h_2}}{dQ^2} = \sigma_0\sum_{i,j}\int\frac{dx_1}{x_1}\frac{dx_2}{x_2}f_{i/h_1}(x_1,\mu)f_{j/h_2}(x_2,\mu)\,\omega_{ij}\Big(z,Q,\mu,\alpha_s(\mu^2)\Big)\,. \tag{4.26}$$

Instead of working with the convoluted integrals, we consider the Mellin transform in the hadronic threshold variable $\tau = x_1x_2z$. In this expression it is only the hard function that is calculable in perturbation theory and is the main function of interest. Hence, let us divide out the Born cross section and define the Mellin space hadronic cross section[88]

$$\begin{aligned}\tilde{\sigma}_{h_1h_2}(N) &= \int_0^1 d\tau\tau^{N-1}\,\frac{1}{\sigma_0}\frac{d\sigma_{h_1h_2}}{dQ^2}\\ &= \sum_{i,j}\tilde{f}_{i/h_1}(N,\mu)\tilde{f}_{j/h_2}(N,\mu)\,\tilde{\omega}_{ij}\Big(N,Q,\mu,\alpha_s(\mu^2)\Big)\,,\end{aligned} \tag{4.27}$$

where

$$\tilde{\omega}_{ij}(N) = \int_0^1 dz\, z^{N-1}\,\omega_{ij}(z)\,, \tag{4.28}$$

$$\tilde{f}_{i/h_1}(N) = \int_0^1 dx\, x^{N-1}\, f_{i/h_1}(x)\,. \tag{4.29}$$

[88] We will recover the Born cross section when we return to the resummed hadronic cross section. We have also neglected the fractional charge of the quarks for simplicity.

Here we observe that one of the advantages of working in Mellin space is that the convolution has turned into simple products. If this transformation seems obscure, see Eq. (D.6).

To see how the threshold logarithms manifest themselves in Mellin space, we take the limit $z \to 1$ of Eq. (3.232), giving

$$\omega_{q\bar{q}} = \delta(1-z) + \frac{\alpha_s}{\pi} C_F \Big(4 \Big[\frac{\ln(1-z)}{1-z} \Big]_+ + \Big(\frac{\pi^2}{3} - 4 \Big) \delta(1-z) \Big) + \mathcal{O}(\alpha_s^2) \,. \tag{4.30}$$

By using the Mellin transforms given in Appendix D.1, we find the Mellin space expression

$$\tilde{\omega}_{q\bar{q}} = 1 + \frac{\alpha_s}{\pi} C_F 2 \ln^2 \bar{N} + \mathcal{O}(\alpha_s^2) \,, \tag{4.31}$$

where $\bar{N} = N e^{\gamma_E}$, and we omitted constant terms that has no relevance compared to the logarithm. We observe that the threshold limit $z \to 1$, corresponds to $N \to \infty$. Hence, the constant factors are unimportant for large N.

To n-th power of the logarithm, we have that the plus distributions give terms of the form

$$\int_0^1 dz z^{N-1} \Big[\frac{\ln^n(1-z)}{(1-z)} \Big]_+ = \frac{(-1)^{n+1}}{n+1} \ln^{n+1}(\bar{N}) + \mathcal{O}(\ln^{n-1}(\bar{N})) \,, \tag{4.32}$$

after Mellin transformation. These are the logarithms we now want to resum, and try to reproduce after the resummation procedure has been performed. In order to make progress, we will use the factorization property of QCD to refactorize the cross section by using parton-in-parton distributions. We also used this property in the fixed order Drell-Yan calculation, where we had that the hard function ω_{ij} were collinear divergent. By refactorizing the hard functions by using parton-in-parton distributions we showed that the hard function could be rendered infrared safe as the parton-in-parton distributions were responsible for these. This is a very important feature that we will exploit heavily in the sections to come.

Kinematics

Let us briefly look at some of the kinematics that is used. We choose the quarks to be fully collinear to the incoming protons, i.e.

$$p_1 = x_1 P_1 \,, \tag{4.33}$$
$$p_2 = x_2 P_2 \,. \tag{4.34}$$

We work in the centre of mass frame of P_1 and P_2, with $P_1^0 = P_2^0 = E = Q/2$. The hadrons have the following momenta

$$P_1 = \frac{Q}{2}(1,1,0) \tag{4.35}$$
$$P_2 = \frac{Q}{2}(1,-1,0) \,, \tag{4.36}$$

such that

$$s = (P_1 + P_2)^2 = Q^2 \,. \tag{4.37}$$

In the threshold limit $\tau \to 1$, we have that τ coincides with z when $x_1 \to 1$ and $x_2 \to 1$. Hence, we have that the momenta of the quarks are $p_1 = P_1$ and $p_2 = P_2$. Also, if a gluon radiates from an incoming quark, it follows that the energy of that gluon is given by

$$Q^2 = (p_1 + p_2 - k)^2 = s - 2\sqrt{s} k^0 \tag{4.38}$$
$$\to k^0 = (1-\tau) Q/2 \,. \tag{4.39}$$

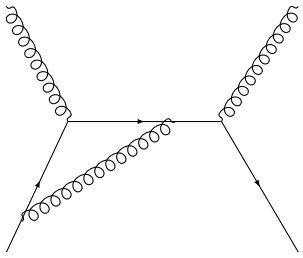


Figure 4.3: Diagram that has no IR-divergences.

4.3 Threshold Factorization

In this section we will use factorization theorems to find a fully factorized form of the hard function $\tilde{w}_{ij}(N)$.

First we want to extract the universal collinear singularities we encountered in the fixed order NLO calculation. We do the same as in Sec. 3.3.3 and define a partonic analogue to the hadronic cross section Eq. (4.27),

$$\tilde{\sigma}_{ij}(N) = \tilde{f}_{i/i}(N,\mu,\epsilon)\tilde{f}_{j/j}(N,\mu,\epsilon)\,\tilde{\omega}_{ij}(N,Q,\mu,\alpha_s)\,, \tag{4.40}$$

where $f_{i/i}$ are the light-cone distributions responsible for the collinear singularities. Thus, the partonic function $\tilde{\omega}_{ij}$ is with this refactorization infrared safe. We can go even further with the factorization theorem.

As we saw in Sec. 3.3.2, the cancellation of soft divergences are responsible for the large logarithms. Hence, we want to factorize out these soft parts and use the non-Abelian eikonal exponentiation theorem to resum them. We begin with defining a threshold analogue to Eq. (4.27) [46]

$$\tilde{\sigma}_{ij}(N) = H_{ij}(Q,\alpha_s(Q))\tilde{J}_{i/i}(N,Q,\epsilon)\tilde{J}_{j/j}(N,Q,\epsilon)\tilde{S}_{ij}(N,Q,\mu,\alpha_s(\mu))\,, \tag{4.41}$$

where $H_{ij}(Q)$ is a hard function that is free of singular distributions. We have also defined jet functions $J_{i/i}(N)$. We specifically defined these as functions of Q and not μ, which we will comment on later. These contain additional collinear divergences, that will cancel the collinear singularities contained in the light-cone distributions $f_{i/i}(x,\mu,\epsilon)$. Lastly, we have a soft function $S_{ij}(N)$ that is responsible for all wide angle soft radiation. The derivation of this formula is quite technical, but we will try to give the main arguments behind it[89].

The first step is to decouple H from S. Diagrams such as Fig. 4.3 does not generate IR-divergences. This is true to all order, i.e. a soft line can only generate an IR divergence if it is connected to an on-shell external line [7]. Thus, gluons that connect the soft and the hard part do not contribute to the IR-divergences, i.e. the soft and hard part cannot be connected directly. The physical interpretation of such a decoupling is that soft gluons correspond to large length scales, while the hard part takes place at small length scales. Therefore the soft gluons are unable to resolve the internal structure of the hard process. This fact also follows from the eikonal approximation we made in Sec. 2.3, where we showed that the eikonal approximation led to the concept of particles dressed with Wilson lines. This was only possible if the soft emission was connected to an on-shell external line.

The second step of the argument is to decouple the jets from each other. By definition all jets move in different directions, so lines in different jets are proportional to different momenta. We have two jets that meet at the hard interaction, but before that they cannot combine. Thus their collinear divergences will not mix with each other [48]. Again, based on length scales the small momentum of the gluons in the J_i cannot resolve the inner structure of the hard process. The decoupling of the jets from the soft part is more subtle, but since jets have large total momentum their substructure cannot be resolved by the soft gluons in S. However, close to the threshold there are energy restrictions on the gluons, leading to the large logarithmic corrections. Hence, the soft function does not completely decouple and we have to take it into account. This soft function can be constructed by taking the eikonal approximation of the partonic process, i.e. we can build it out of Wilson lines. We will come back to this procedure in the next section.

Close to the threshold we have that Eq. (4.40) and Eq. (4.41) must be equal, giving the fully factorized hard function

$$\tilde{\omega}_{ij}(N,Q,\mu,\alpha_s(\mu)) = \frac{\tilde{J}_{i/i}(N,Q,\epsilon)\tilde{J}_{j/j}(N,Q,\epsilon)}{\tilde{f}_{i/i}(N,\mu,\epsilon)\tilde{f}_{j/j}(N,\mu,\epsilon)}H_{ij}(Q,\alpha_s(\mu))\tilde{S}_{ij}(N,Q,\mu,\alpha_s(\mu))\,. \tag{4.42}$$

[89] For a more comprehensive explanation, see [26].

Since the partonic function is defined to be infrared safe, the ratio of the jet and parton distributions must cancel the collinear divergences. The ratio of distributions might seem strange and it may not be obvious at this point how to evaluate them. But we will show later how this is done using renormalization group equations.

The next step going forward is to make use of what we know of Wilson lines in order to construct the soft function. We will do this by constructing an eikonal cross section, i.e. a cross section where the radiation is restricted to be soft. Then we will show that the soft function can be replaced from Eq. (4.42) by the eikonal cross section. The motivation behind this is to use the non-Abelian eikonal exponentiation theorem to calculate the eikonal cross section.

4.4 Factorization of Soft Gluons

In this section we will take a closer look at how we can organize the soft contributions coming from $\tilde{S}$ by constructing the eikonal cross section.

Near threshold all radiation is restricted to be soft compared with the hard scattering function. This naturally leads to an eikonal approximation for the cross section, i.e. we can construct this cross section by using Wilson lines. To this end, we consider the following Wilson lines[90]

$$\mathcal{U}_p[x,\infty] = \mathcal{P}\exp\left\{ig\int_\infty^0 ds\, p\cdot A(x+ps)\right\}, \tag{4.43}$$

$$\mathcal{U}_{-p}[\infty,x] = \mathcal{P}\exp\left\{-ig\int_0^\infty ds\, p\cdot A(x-ps)\right\}, \tag{4.44}$$

where we have parametrized the path as $z^\mu = x^\mu + p^\mu s$, where p is the light-like momentum of the incoming massless quark (or anti-quark) and s is the proper time. From this we can construct the Drell-Yan Wilson line[91]

$$\mathcal{U}_{DY}(0) = \mathcal{U}_{-p_2}[\infty,0]\,\mathcal{U}_{p_1}[0,\infty], \tag{4.45}$$

where $\mathcal{U}_{p_1}[\infty,0]$ and $\mathcal{U}_{-p_2}[\infty,0]$ are the Wilson lines evaluated along the classical trajectories of the incoming massless quark and anti-quark. The classical trajectory means that the particles are so energetic that they will not recoil as the soft gluons are emitted, such that they move along a straight line. This is necessary as we want to use Wilson lines on linear paths. From this we can construct the expectation value[92]

$$\mathcal{W}_{DY}(0) = \langle 0|\bar{\mathcal{T}}\mathcal{U}^\dagger_{DY}(0)\mathcal{T}\mathcal{U}_{DY}(0)|0\rangle, \tag{4.46}$$

where $\mathcal{T}$ and $\bar{\mathcal{T}}$ are the time and anti-time ordering operators.

The main idea from here is to construct a Wilson loop expectation value. To this end, we rewrite Eq. (4.46) by inserting a complete set of final states

$$\mathcal{W}_{DY}(0) = \sum_n \langle 0|\bar{\mathcal{T}}\mathcal{U}^\dagger_{DY}(0)|n\rangle\langle n|\mathcal{T}\mathcal{U}_{DY}(0)|0\rangle, \tag{4.47}$$

and the eikonal cross section is then constructed by inserting a energy conserving delta function, i.e.

$$\sigma^{(\text{eik})}_{ij}(\tau,Q) = \sum_n \delta((1-\tau)Q/2 - E_n)\langle 0|\bar{\mathcal{T}}\mathcal{U}^\dagger_{DY}(0)|n\rangle\langle n|\mathcal{T}\mathcal{U}_{DY}(0)|0\rangle, \tag{4.48}$$

where $E_n = \sum_n k^0_n$ is the total energy of the emitted gluons, which is restricted to be $(1-\tau)Q/2$.

By using the Fourier representation of the delta function, we find

$$\begin{aligned}\sigma^{(\text{eik})}_{ij}(\tau,Q,\mu,\alpha_s,\epsilon) &= \frac{Q}{2}\sum_n\int_{-\infty}^\infty \frac{dy^0}{2\pi}\, e^{iy^0((1-\tau)Q/2-\sum_n k^0_n)}\langle 0|\bar{\mathcal{T}}\mathcal{U}^\dagger_{DY}(0)|n\rangle\langle n|\mathcal{T}\mathcal{U}_{DY}(0)|0\rangle \\ &= \frac{Q}{2}\int_{-\infty}^\infty \frac{dy^0}{2\pi}\, e^{iy^0(1-\tau)Q/2}\,\mathcal{W}_{DY}(y),\end{aligned} \tag{4.49}$$

[90] This definition is slightly different in appearance to the ones we derived in Sec. 2.3, but all the rules are equivalent.

[91] If the notation $\mathcal{U}(0)$ is confusing it just mean that the composition are connected such that the two Wilson lines meet at space-time point 0.

[92] There is an implicit average and sum over colour in this expectation value.

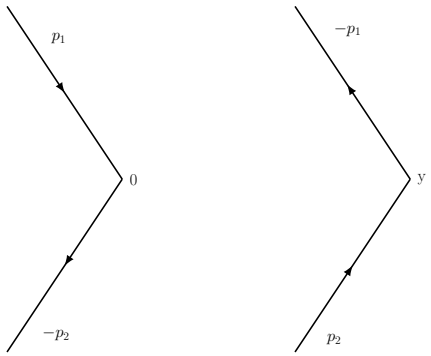


Figure 4.4: Integration contour for the eikonal approximation in the Drell-Yan process, and $y = (y^0, \vec{0})$.

where we have explicitly included the dimensional regulator ϵ as an argument in the eikonal cross section, as it contains divergences. We also used the translation property $\mathcal{U}_{DY}(y) = e^{iP\cdot y}\mathcal{U}_{DY}(0)e^{-iP\cdot y}$, where $y^\mu = (y^0, \vec{0})$, to define

$$\mathcal{W}_{DY}(y) = \langle 0|\bar{\mathcal{T}}\mathcal{U}^\dagger_{DY}(y)\mathcal{T}\mathcal{U}_{DY}(0)|0\rangle \equiv \langle 0|\mathcal{P}\exp\left(ig\oint_{\gamma_{DY}} dx^\mu A_\mu(x)\right)|0\rangle \tag{4.50}$$

which is an expectation value of a gauge invariant Wilson loop integrated over the path γ_{DY}, see Fig. 4.4[93].

We can now do the same as we did in Sec. 4.3 and use factorization properties of cross sections to define eikonal distributions responsible for the collinear divergences in $\sigma^{(\text{eik})}_{ij}$. So the next object to consider is the eikonal analogue to the light-cone distributions $f_{i/i}$. In the limit $x \to 1$, parton-in-parton distributions can be shown to take the form [49][94]

$$\begin{aligned} f^{(\text{eik})}_{i/i}(x,\mu,\epsilon) &= \frac{Q}{2}\int_{-\infty}^{\infty}\frac{dy^-}{2\pi}\, e^{iy^-(1-x)Q/2}\langle 0|\bar{\mathcal{T}}\{\mathcal{U}_{-p_1}[y,\infty]\}\mathcal{T}\{\mathcal{U}_{p_1}[0,\infty]\}|0\rangle \\ &= \frac{Q}{2}\int_{-\infty}^{\infty}\frac{dy^-}{2\pi}\, e^{iy^-(1-x)Q/2}\mathcal{W}_{\gamma_{p_1}}(y)\,, \end{aligned} \tag{4.51}$$

where the path γ_{p_1} is the p_1 part of γ_{DY}.

By using these eikonal distributions the eikonal cross section can be written as

$$\begin{aligned} \sigma^{(\text{eik})}_{ij}(w,Q,\mu,\alpha_s,\epsilon) = \int dw_1 dw_2 dw'\, f^{(\text{eik})}_{i/i}(w_1,\mu,\epsilon)\, f^{(\text{eik})}_{i/i}(w_2,\mu,\epsilon)\, \omega^{(\text{eik})}_{ij}(w',Q,\mu,\alpha_s) \\ \delta(w-w_1-w_2-w')\,. \end{aligned} \tag{4.52}$$

where we defined the energy fractions $w = 1-\tau$, $w_1 = 1-x_1$, $w_2 = 1-x_2$ and $w' = 1-z$.

In Sec. 4.3 we were working in Mellin space, so by taking the Mellin transform of Eq. (4.52) we obtain

$$\tilde{\sigma}^{(\text{eik})}(N,Q,\mu,\alpha_s,\epsilon) = \tilde{f}^{(\text{eik})}_{i/i}(N,\mu,\epsilon)\,\tilde{f}^{(\text{eik})}_{j/j}(N,\mu,\epsilon)\,\tilde{\omega}^{(\text{eik})}_{ij}(N,Q,\mu,\alpha_s)\,, \tag{4.53}$$

where we have used that the Mellin transform of the delta function is

$$\int_0^1 d\tau\tau^{N-1}\,\delta(1-\tau-(1-x_1)-(1-x_2)-(1-z)) = e^{-N(1-x_1+1-x_2+1-z)}\,, \tag{4.54}$$

where $e^{-N(1-x)} = x^{N-1}$ in the large N limit. The result in Eq. (4.53) is the eikonal approximation of Eq. (4.40).

We can also make an eikonal approximation of the near threshold cross section Eq. (4.41), which can be constructed in a similar fashion as we have done for Eq. (4.53), given by

$$\tilde{\sigma}^{(\text{eik})}(N,Q,\mu,\alpha_s,\epsilon) = \tilde{J}^{(\text{eik})}_{i/i}(N,\mu,\epsilon)\,\tilde{J}^{(\text{eik})}_{j/j}(N,\mu,\epsilon)\,\tilde{S}_{ij}(N,Q,\mu,\alpha_s)\,, \tag{4.55}$$

[93] This might not seem as a Wilson loop, but the Wilson lines go out to infinity where they combine.

[94] Again, there is an implicit average and sum over colour in the expectation value.

where we have used that the soft function by definition contains the soft contributions, i.e. $S_{ij} = S_{ij}^{(\text{eik})}$. Then we can use that Eq. (4.53) and Eq. (4.55) must be equal near threshold, giving

$$\tilde{\omega}_{ij}^{(\text{eik})}(N,Q,\mu,\alpha_s) = \frac{\tilde{J}_{i/i}^{(\text{eik})}(N,\mu,\epsilon)\,\tilde{J}_{j/j}^{(\text{eik})}(N,\mu,\epsilon)}{\tilde{f}_{i/i}^{(\text{eik})}(N,\mu,\epsilon)\,\tilde{f}_{j/j}^{(\text{eik})}(N,\mu,\epsilon)}\tilde{S}_{ij}(N,Q,\mu,\alpha_s)\,. \tag{4.56}$$

Apart from contributions from hard virtual gluons, Eq. (4.56) is the eikonal approximation of Eq. (4.42). This means that we have an expression for the soft function $\tilde{S}$, so if we solve for the soft function in Eq. (4.56) and insert it into Eq. (4.42), we find that the hard partonic function can be written as

$$\begin{aligned}\tilde{\omega}_{ij}(N,Q,\mu,\alpha_s(\mu)) =& \Big[\frac{\tilde{J}_{i/i}(N,Q,\epsilon)\tilde{J}_{j/j}(N,Q,\epsilon)}{\tilde{f}_{i/i}(N,\mu,\epsilon)\tilde{f}_{j/j}(N,\mu,\epsilon)}\Big]\Big[\frac{\tilde{f}_{i/i}^{(\text{eik})}(N,\mu,\epsilon)\,\tilde{f}_{j/j}^{(\text{eik})}(N,\mu,\epsilon)}{\tilde{J}_{i/i}^{(\text{eik})}(N,\mu,\epsilon)\,\tilde{J}_{j/j}^{(\text{eik})}(N,Q,\epsilon)}\Big] \\ & H_{ij}(Q,\alpha_s(\mu))\,\tilde{\omega}_{ij}^{(\text{eik})}(N,Q,\mu,\alpha_s)\,,\end{aligned} \tag{4.57}$$

where both of these ratios are defined such that they cancel each others collinear divergences. This expression looks daunting, but as previously mentioned the ratios can be simplified by using the renormalization group properties of the distributions, which we will do in Sec. 4.6. The eikonal function $\omega_{ij}^{(\text{eik})}$ is the main function of interest, as it contains the parts where soft contributions cancel to give the large logarithms.

We have managed to bring the hard partonic function $\tilde{w}_{ij}$ on a fully factorized form in terms of the eikonal function $\tilde{\omega}_{ij}^{(\text{eik})}$. It might not have been obvious what the point of this whole refactorization is, but the idea is that we now have an expression where the IR-divergences are grouped into different terms responsible for different regions of phase space. Before we go into details of how to find the eikonal function $\tilde{w}_{ij}^{(\text{eik})}$ we will take a closer look at the renormalization properties of Wilson lines and the renormalization group equations for the distributions in Eq. (4.57). The reason for taking this slight detour is to introduce the cusp anomalous dimension, which we will have use for later.

Overview of IR-divergences

We have tried to point out as we went along where the divergences in the different expressions above are, but let us try to make that more clear.

First of all, the hadronic cross section $\sigma_{h_1h_2}$ is of course finite. On the other hand, we observed in a fixed order calculation at NLO that the partonic function ω_{ij} has collinear divergences. To single this contribution out we defined a partonic analog to the hadronic cross section Eq. (4.40) in Mellin space, in such a way that the parton-in-parton distributions $f_{i/i}$ were responsible for these collinear divergences, rendering ω_{ij} IR-finite. The consequence of this definition is that the parton-in-hadron distributions $f_{i/h}$ do not contain any singularities, which is important as one wants to take these from experimental measurements.

From there we went on to write down a near threshold form of the partonic cross section Eq. (4.41), where all collinear singularities are included in the jet subprocesses $J_{i/i}$. The hard subprocess H includes only lines that are off-shell and does not contain any large logarithms. The soft subprocess S_{ij} is defined to include all wide angle soft radiation.

Finally we made an eikonal cross section $\sigma_{ij}^{(\text{eik})}$ in Eq. (4.49), which contains collinear singularities due to the light-like momenta p_1 and p_2 of the incoming partons[95]. We factorized this cross section in Eq. (4.53) and Eq. (4.55) such that $f_{i/i}^{(\text{eik})}$ and $J_{i/i}^{(\text{eik})}$ are responsible for these collinear divergences. The eikonal cross section $\sigma_{ij}^{(\text{eik})}$ also contains soft divergences, but according to the KLN-theorem these cancel in the sum over all final states, leading to large logarithms. Hence, the soft function S_{ij} and the eikonal function $\omega_{ij}^{(eik)}$ are free of IR-divergences.

4.5 Renormalization of Wilson Lines

In this section we will look at the renormalization properties of Wilson lines, and in particular find the cusp anomalous dimension. Later we will see that the cusp anomalous dimension appear in evolution equations for

[95] Or rather due to the light-like directional vectors n_1 and n_2 along the momenta p_1 and p_2.

parton distributions and in the exponent of the exponentiated eikonal cross section. Hence, it is a fundamental ingredient in resummation with Wilson lines.

There are two kinds of cusp anomalous dimensions, one is for Wilson lines on light-cone and one for Wilson lines off light-cone. We are considering the case of massless quarks, so the Wilson lines on light-cone is those of main focus, i.e. we need the on light-cone cusp anomalous dimension. We will show one way of calculating it in the next section, but first we will show how it appears.

In order to find the behaviour of Wilson lines at different scales, we can use the basic principle that the bare definition must be independent on the renormalization scale, i.e. the bare Wilson line satifies

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \mathcal{U}_\gamma^0 = 0 \,. \tag{4.58}$$

The bare Wilson line is given in terms of the bare coupling g_0 and the bare gauge field. Let us then rescale the field as in Eq. (3.18), and use Eq. (3.20) to define the relation

$$\mathcal{Z}_3^{1/2} g_0 = \mathcal{Z}_g g \,, \tag{4.59}$$

giving the renormalized Wilson line[96]

$$\mathcal{U}_\gamma(g,\mu) = \mathcal{P} \exp\left\{ ig\mathcal{Z}_g \int_\gamma dz^\mu A_\mu(z) \right\} . \tag{4.60}$$

A smooth Wilson with no cusps is completely renormalized as long as the coupling and the field is renormalized [50, 51]. Hence, by applying Eq. (4.58) we would find a regular Callan-Symanzik equation.

However, since we are studying a quark–antiquark pair that meets at a point and annihilates we are in interested in paths with cusps. These cusps will contribute with additional UV-divergences, so-called cusp divergences. A cusp in a Wilson line is characterized by two directional vectors n_1^μ and n_2^μ and the cusp divergence is a function of the angle χ between these two vectors. In Minkowski space this angle is defined as

$$\cosh \chi = \frac{n_1 \cdot n_2}{\sqrt{n_1^2 n_2^2}} \,, \tag{4.61}$$

and the Wilson line will acquire a dependency on the regulator ϵ in dimensional regularization, i.e. $\mathcal{U}_\gamma(g,\mu,\epsilon)$. If both vectors are off light-cone, i.e. $n_1^2 \neq 0$ and $n_2^2 \neq 0$, the cusp divergences can be treated multiplicatively by introducing a multiplicative factor $\mathcal{Z}_{\text{cusp}}$ [51, 52]

$$\widetilde{\mathcal{U}}_\gamma(g,\mu) = \mathcal{Z}_{\text{cusp}}(\chi,g,\mu,\epsilon)\mathcal{U}_\gamma(g,\mu,\epsilon) \tag{4.62}$$

giving the Callan-Symanzik equation

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}\right) \ln \widetilde{\mathcal{U}}_\gamma(g,\mu) = \Gamma_{\text{cusp}}(\chi,g) \,, \tag{4.63}$$

where the cusp anomalous dimension is given by

$$\Gamma_{\text{cusp}}(\chi,g) = \lim_{\epsilon\to 0} \frac{\mu}{\mathcal{Z}_{\text{cusp}}} \frac{\mathrm{d}}{\mathrm{d}\mu} \mathcal{Z}_{\text{cusp}}(\chi,g,\epsilon) = \lim_{\epsilon\to 0} \frac{\mathrm{d}}{\mathrm{d}\ln\mu} \ln \mathcal{Z}_{\text{cusp}}(\chi,g,\epsilon) \,. \tag{4.64}$$

In regular UV-renormalization we have that the renormalization factor removes the ϵ dependence via counterterms, see Sec. 1.5. Hence, we can calculate a Wilson line with cusps in perturbation theory and use $\mathcal{Z}_{\text{cusp}}$ to pull out the divergent part.

If one or both vectors are on the light-cone, we can no longer use the multiplicative renormalization technique. This follows from the fact that for light-like vectors, Eq. (4.61) blows up, and creates additional divergences. In dimensional regularization these additional divergences are double poles, i.e. of the form $1/\epsilon^2$. In Sec. 3.3.2 we found that by adding the real and virtual gluon emission the double pole vanished and the cross section acquired large logarithmic dependancy. Thus, treating the $1/\epsilon^2$ divergence would give a way of managing these large contributions by the renormalization properties of Wilson lines.

[96] The scale factor μ is as usual hidden in g, i.e. we always make the substitution $g(\mu) \to \mu^{d-4} g$.

Now, there is a relation between the on light-cone and off light-cone cusp anomalous dimension that we can use. In [53] it was found that the relation between the two in the limit of large χ, is given by

$$\lim_{\chi\to\infty}\Gamma_{\text{cusp}}(\chi,g)=\chi\Gamma_{\text{cusp}}(g)+\mathcal{O}(\chi^0)\,. \tag{4.65}$$

In this large limit, it follows from Eq. (4.61) that

$$\chi=\ln\left(\frac{2n_1\cdot n_2}{\sqrt{n_1^2n_2^2}}\right). \tag{4.66}$$

We observe that if we differentiate Eq. (4.65) with respect to $\ln n_1\cdot n_2$ we remove the troublesome denominator that blows up for light-like vectors. Therefore, we can write the on light-cone cusp anomalous dimension as[97]

$$\Gamma_{\text{cusp}}(g)=\lim_{\epsilon\to0}\frac{\mathrm{d}}{\mathrm{d}\ln n_1\cdot n_2}\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\ln\mathcal{Z}_{\text{cusp}}(\chi,g,\epsilon)\,, \tag{4.67}$$

which is the expression we will use after we have calculated $\Gamma(\chi,g)$ in the next section.

Wilson lines with endpoints will also have their own renormalization factors, and a corresponding endpoint anomalous dimension [21]. But we will only consider semi-infinite Wilson lines with endpoint at infinity. These contains IR-divergnces, which we will treat with an exponential regulator that suppress such contributions.

4.5.1 One-Loop Cusp Anomalous Dimension

To calculate the one-loop cusp anomalous dimension, we consider the case of two semi-infinite Wilson lines bounded from below, see Eq. (2.185). We denote these as

$$\mathcal{U}_{\gamma_1}[\infty,0]=\mathcal{P}\exp\left(ig\int_0^\infty d\lambda_1\,n_1\cdot A(\lambda_1n_1)\right), \tag{4.68}$$

$$\mathcal{U}_{\gamma_2}[\infty,0]=\mathcal{P}\exp\left(ig\int_0^\infty d\lambda_2\,n_2\cdot A(\lambda_2n_2)\right). \tag{4.69}$$

To construct the geometry of the diagrams in Fig. 4.5, we use that Wilson lines are path-transitive and can be written as the composition[98]

$$\mathcal{U}_\wedge(0)=\mathcal{U}_{\gamma_1}[\infty,0]\mathcal{U}_{\gamma_2}[\infty,0]\,. \tag{4.70}$$

Expanding Eq. (4.70) to $\mathcal{O}(g^2)$, we find

$$\begin{aligned}\mathcal{U}_\wedge(0)=1&+igt^an_1^\mu\int_0^\infty d\lambda_1\,A_\mu^a(\lambda_1n_1)+igt^bn_2^\mu\int_0^\infty d\lambda_2\,A_\mu^b(\lambda_2n_2)\\&-g^2t^at^bn_1^\mu n_2^\nu\int_0^\infty d\lambda_1\int_0^\infty d\lambda_2\,A_\mu^a(\lambda_1n_1)A_\nu^b(\lambda_2n_2)\,,\end{aligned} \tag{4.71}$$

which follows from the expansions we discussed in Sec. 2.3. But in Sec. 2.3 we integrated over λ directly by Fourier transforming the fields, giving Eq. (2.185). In momentum space, we have IR-divergences when $n\cdot k\to0$ and $k^2\to0$. These originate from the Wilson line propagator and after the gauge fields have been Wick contracted to give the gauge field propagator. However, it is easier to work in coordinate space for this calculation. The IR-divergence in coordinate space originates from $\lambda\to\infty$, so to treat it we insert an exponential regulator in the exponent of the Wilson lines

$$\mathcal{U}^\delta[\infty,0]=\mathcal{P}\exp\left(ig\int_0^\infty d\lambda\,n\cdot A(\lambda n)e^{-\delta\lambda\sqrt{-n^2}}\right), \tag{4.72}$$

which was proposed for Wilson line calculations in [45]. The idea here is that $\delta\sqrt{-n^2}>0$, so that the exponential factor smoothly cuts off the $\lambda\to\infty$ contribution. This is guaranteed to yield an IR-finite result for the integral in Eq. (4.72), and all the remaining poles are of UV origin, i.e. $\lambda\to0$.

[97] If the $\epsilon\to0$ limit seem sketchy it is ment to be happen after the differentiation has been performed.

[98] We could have used $\mathcal{W}_{DY}$ to calculate the cusp anomalous dimension, but that calculation is more complicated.

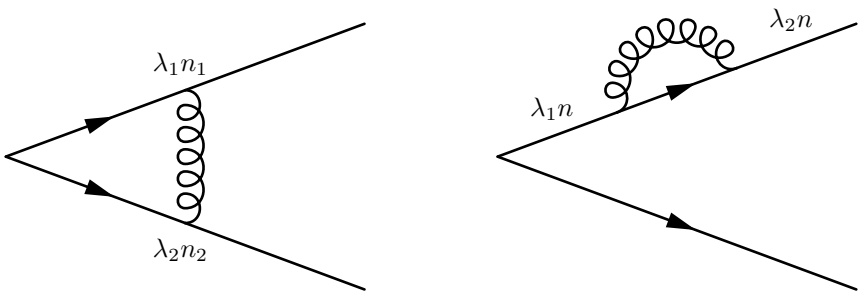


Figure 4.5: Wilson line diagrams contributing to the one-loop cusp anomalous dimension $\Gamma(\chi, g)$.

One-Loop Calculation

To calculate the full cusp anomalous dimension $\Gamma_{\text{cusp}}(\chi, g)$, we would have to calculate both diagrams in Fig. 4.5. But as we can see, the diagram on the right-hand does not depend on the cusp angle as the radiation is from the same line, so to find $\Gamma_{\text{cusp}}(g)$ we only focus on the left-hand diagram.

To calculate this contribution we consider the expectation value

$$\mathcal{W}_\wedge = \langle 0|\mathcal{T}\mathcal{U}_\wedge(0)|0\rangle \tag{4.73}$$

where the $\mathcal{T}$ is the time-ordering operator. Expanding Eq. (4.73) to $\mathcal{O}(g^2)$ using the expansion in Eq. (4.71), will give

$$\begin{aligned}\mathcal{W}_\wedge &= 1 + \mathcal{W}_\wedge^{(1)} \\ &= 1 - g^2 t^a t^b n_1^\mu n_2^\mu \int_0^\infty d\lambda_1 \int_0^\infty d\lambda_2 \, D_{\mu\nu}^{ab}(\lambda_1 n_1 - \lambda_2 n_2)\,,\end{aligned} \tag{4.74}$$

where we Wick contracted the emitted gluons to give the propagator. The propagator in coordinate space is given by [45],

$$D_{\mu\nu}^{ab}(x-y) = -\mathcal{N}\frac{g_{\mu\nu}\delta^{ab}}{(-(x-y)^2 + i\epsilon)^{d/2-1}}\,, \tag{4.75}$$

where

$$\mathcal{N} = \frac{\Gamma(d/2-1)}{4\pi^{d/2}}\,. \tag{4.76}$$

Here we should keep in mind that the Feynman prescription $i\epsilon$ and the regulator ϵ are not the same when we expand in $d = 4 - 2\epsilon$.

Let us then insert the IR-regulator given in Eq. (4.72), giving the expression

$$\begin{aligned}\mathcal{W}_\wedge^{(1)} &= -g^2 t^a t^b n_1^\mu n_2^\mu \int_0^\infty d\lambda_1 \int_0^\infty d\lambda_2 \, D_{\mu\nu}^{ab}(\lambda_1 n_1 - \lambda_2 n_2)\, e^{-\delta(\lambda_1\sqrt{-n_1^2}+\lambda_2\sqrt{-n_2^2})} \\ &= g^2 C_F \mathcal{N}(\epsilon) n_1 \cdot n_2 \int_0^\infty d\lambda_1 \int_0^\infty d\lambda_2 \, \frac{e^{-\delta(\lambda_1\sqrt{-n_1^2}+\lambda_2\sqrt{-n^2})}}{(-(\lambda_1 n_1 - \lambda_2 n_2)^2)^{1-\epsilon}}\,.\end{aligned} \tag{4.77}$$

To evaluate the integrals we can make the change of variables

$$\lambda_1 = \frac{\alpha x}{\sqrt{-n_1^2}}\,, \tag{4.78}$$

$$\lambda_2 = \frac{\alpha(1-x)}{\sqrt{-n_2^2}}\,, \tag{4.79}$$

where $x \in [0,1]$ and $\alpha \in [0,\infty)$, giving the Jacobian

$$\mathcal{J} = \frac{\alpha}{\sqrt{n_1^2 n_2^2}}\,. \tag{4.80}$$

With these changes we get the following integral

$$\begin{aligned} I &= \int_0^\infty d\lambda_1 \int_0^\infty d\lambda_2 \, \frac{e^{-\delta(\lambda_1\sqrt{-n_1^2}+\lambda_2\sqrt{-n^2})}}{(-(\lambda_1 n_1 - \lambda_2 n_2)^2)^{1-\epsilon}} \\ &= \frac{1}{\sqrt{n_1^2 n_2^2}} \int_0^1 dx \frac{1}{(x^2+(1-x)^2+2x(1-x)\cosh\gamma)^{1-\epsilon}} \int_0^\infty d\alpha \, e^{-\delta\alpha}\alpha^{-1+2\epsilon} \,, \end{aligned} \tag{4.81}$$

where we defined

$$\cosh\gamma = -\frac{n_1 \cdot n_2}{\sqrt{n_1^2 n_2^2}} \,, \tag{4.82}$$

and by inserting this back into Eq. (4.77), we get

$$\mathcal{W}_\wedge^{(1)} = -g^2 C_F \mathcal{N}(\epsilon) \int_0^1 dx \frac{\cosh\gamma}{(x^2+(1-x)^2+2x(1-x)\cosh\gamma)^{1-\epsilon}} \int_0^\infty d\alpha \, e^{-\delta\alpha}\alpha^{-1+2\epsilon} \,. \tag{4.83}$$

Let us evaluate the α integral by another change of variable $y = \alpha\delta$, giving

$$\int_0^\infty d\alpha \, e^{-\delta\alpha}\alpha^{-1+2\epsilon} = \delta^{-2\epsilon} \int_0^\infty dy \, e^{-y} y^{-1+2\epsilon} = \delta^{-2\epsilon}\,\Gamma(2\epsilon) \,, \tag{4.84}$$

where we used the integral representation of the Gamma function. This gamma function has the expansion as $\epsilon \to 0$

$$\Gamma(2\epsilon) = \frac{1}{2\epsilon} + \mathcal{O}(\epsilon^0) \,. \tag{4.85}$$

The x integral can be rewritten in terms of the hypergeometric function ${}_2F_1$. However, we want the expansion in the limit $\epsilon \to 0$, so it is inconvenient to use this representation. Let us instead set $\epsilon = 0$ in this integral, giving

$$\int_0^1 dx \frac{\cosh\gamma}{(x^2+(1-x)^2+2x(1-x)\cosh\gamma)} = \gamma\coth\gamma \,. \tag{4.86}$$

This integral would not be convergent without the definition of γ in Eq. (4.82). But we want our result in terms of χ, and from Eq. (4.61) these are related in the following way

$$\cosh\gamma = -\cosh\chi = \cosh(\chi + i\pi) \,, \tag{4.87}$$

giving

$$\gamma = \chi + i\pi \tag{4.88}$$

and

$$\coth(\chi + i\pi) = \coth\chi \,. \tag{4.89}$$

Also, we can safely neglect the terms that are non singular for $\epsilon \to 0$, i.e. $\delta^{-2\epsilon} \to 1$ and $\mathcal{N}(\epsilon) \to 1/4\pi^2$. This removes the IR regulator δ from the expression in a smooth way. Using all these relations, we find that the $\mathcal{O}(g^2)$ expansion of the Wilson loop expectation value takes the form

$$\begin{aligned} \mathcal{W}_\wedge^{(1)} &= -g^2\, C_F\, \mathcal{N}(\epsilon)\, \delta^{-2\epsilon}\, \Gamma(2\epsilon)\, \gamma \coth\gamma \\ &= -g^2\, C_F\, \frac{1}{4\pi^2}\, \frac{1}{2\epsilon}\, (\chi + i\pi)\coth\chi \,. \end{aligned} \tag{4.90}$$

As mentioned above, this is only one contribution to the cusp anomalous dimension $\Gamma_{cusp}(\chi, g)$. But the other contribution does not depend on the cusp angle, so when we perform the differentiation with respect to $\ln n_1 \cdot n_2$ it will not contribute to $\Gamma_{cusp}(g)$ that we are interested in.

Cusp Anomalous Dimension $\Gamma_{\text{cusp}}(g)$

In Eq. (4.67) we had that the cusp anomalous dimension for Wilson lines on light-cone could be written as

$$\Gamma_{\text{cusp}}(g) = \lim_{\epsilon\to 0} \frac{\mathrm{d}}{\mathrm{d}\ln n_1\cdot n_2}\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\ln \mathcal{Z}_{\text{cusp}}(\chi, g, \epsilon)\,. \tag{4.91}$$

To find $\Gamma_{\text{cusp}}(g)$ we can now use that $\mathcal{Z}_{\text{cusp}}$ is used to cancel the ϵ divergence from the Wilson line in Eq. (4.90). We can also introduce the dependence on the scale μ in the usual way $g^2 \to \mu^{2\epsilon} g^2$, giving the cusp factor

$$\mathcal{Z}_{\text{cusp}} = 1 + g^2\mu^{2\epsilon} C_F \frac{1}{4\pi^2}\frac{1}{2\epsilon}(\chi + i\pi)\coth\chi\,. \tag{4.92}$$

Performing the differentiation and keeping only terms to $\mathcal{O}(g^2)$, we find

$$\begin{aligned}\Gamma_{\text{cusp}}(g) &= \lim_{\epsilon\to 0}\frac{\mathrm{d}}{\mathrm{d}\ln n_1\cdot n_2}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\ln\left(1 + g^2\mu^{2\epsilon}C_F\frac{1}{4\pi^2}\frac{1}{2\epsilon}(\chi+i\pi)\coth\chi\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}\ln n_1\cdot n_2}\left(g^2 C_F\frac{1}{4\pi^2}(\chi+i\pi)\coth\chi\right)\\ &= \frac{g^2}{4\pi^2}C_F\,,\end{aligned} \tag{4.93}$$

where we in the last differentiation used that χ is given by Eq. (4.66) in the large limit, and that $\coth\chi = 1$ in this limit. At first sight the this might seem a little fishy as the derivative of the logarithm gives the argument in the denominator. But if we expand this denominator it will give a $\mathcal{O}(g^4)$ term and we are only considering the $\mathcal{O}(g^2)$ correction. As usual we use that $\alpha_s = g^2/4\pi$, giving

$$\Gamma_{\text{cusp}}(\alpha_s) = \frac{\alpha_s}{\pi}C_F\,, \tag{4.94}$$

which is the well known one-loop cusp anomalous dimension for a Wilson line in the fundamental representation [52].

4.6 Exponentiation of Parton-In-Parton Distributions

In Sec. 3.2.2, we discussed the renormalization group equation for the parton distribution functions $f_{i/P}(x,\mu)$, i.e. the DGLAP equation. Now we would like to discuss the renormalization group equations for the parton-in-parton disttribution functions $f_{i/i}$. In Sec. 3.2.5 we derived the parton-in-parton distributions, given by

$$f_{i/i}(x) = \int\frac{dy^-}{4\pi}e^{-ixp^+y^-}\langle q|\overline{\Psi}(y^-)\gamma^+\Psi(0)|q\rangle\,, \tag{4.95}$$

where the product in the matrix element are eikonal fermions[99]. In [49] it was found that in the limit $x\to 1$, the parton-in-parton distributions obeys the evolution equation

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}f_{i/i}(x,\mu) = \int_x^1\frac{dz}{z}P_{i/i}\big(\frac{x}{z},\alpha_s\big)f_{i/i}(z,\mu) + \mathcal{O}((1-x)^0)\,, \tag{4.96}$$

where the splitting functions has the asymptotic behaviour

$$P_{i/i}(z,\alpha_s) = 2\Gamma^{(i)}_{\text{cusp}}(\alpha_s)\Big[\frac{1}{1-z}\Big]_+ + 2C^{(i)}(\alpha_s)\delta(1-z) + \mathcal{O}((1-z)^0)\,, \tag{4.97}$$

which is true to all order [49], and the one-loop $\Gamma^{(q,\bar q)}_{\text{cusp}}$ is the one we found in Eq. (4.94). We can verify this behaviour by taking the limit $z\to 1$ of the splitting functions we found in Eq. (3.129) and Eq. (3.130), giving

$$P_{q/q}(z) = 2C_F\frac{\alpha_s}{\pi}\left(\Big[\frac{1}{1-z}\Big]_+ + \frac{3}{4}\delta(1-z)\right) + \mathcal{O}(\alpha_s^2)\,, \tag{4.98}$$

$$P_{g/g}(z) = 2\frac{\alpha_s}{\pi}\left(C_A\Big[\frac{1}{1-z}\Big]_+ + \frac{\beta_0}{4}\delta(1-z)\right) + \mathcal{O}(\alpha_s^2)\,, \tag{4.99}$$

[99] Or particles dressed with Wilson lines.

where β_0 is the one-loop beta coefficient, see Eq. (3.35), and we observe that the cusp anomalous dimension for $q, \bar{q}$ corresponds to the one we found in Eq. (4.94). We also observe that the cusp anomalous dimension for $i = g$ is given by

$$\Gamma^{(g)}_{\text{cusp}}(\alpha_s) = \frac{\alpha_s}{\pi} C_A + \mathcal{O}(\alpha_s^2)\,, \tag{4.100}$$

which is just a matter of making the calculation we did in Eq. (4.94) by using Wilson lines in the adjoint representation giving the Casimir invariant C_A. We can also read of the one-loop expression for $C^{(i)}$, given by

$$C^{(q,\bar{q})}(\alpha_s) = \frac{\alpha_s}{\pi} \frac{3}{4} C_F + \mathcal{O}(\alpha_s^2)\,, \tag{4.101}$$

$$C^{(g)}(\alpha_s) = \frac{\alpha_s}{\pi} \frac{\beta_0}{4} + \mathcal{O}(\alpha_s^2)\,, \tag{4.102}$$

In order to solve Eq. (4.96) we take the Mellin transform, giving the large N equation[100]

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} f_{i/i}(N,\mu) = P_{i/i}(N,\alpha_s) f_{i/i}(N,\mu) + \mathcal{O}(1/N)\,. \tag{4.103}$$

where the moments of the splitting function takes the form

$$\begin{aligned} P_{i/i}(N,\alpha_s) &= 2\Gamma^{(i)}_{\text{cusp}}(\alpha_s) \int_0^1 dz\, z^{N-1} \left[\frac{1}{(1-z)_+}\right] + 2C^{(i)}(\alpha_s) \int_0^1 dz\, z^{N-1} \delta(1-z) \\ &= -2\Gamma^{(i)}_{\text{cusp}}(\alpha_s) \ln \bar{N} + 2C^{(i)}(\alpha_s)\,, \end{aligned} \tag{4.104}$$

where we neglect constant terms from the Mellin transform, see Eq. (D.16). These constant terms would reproduce the constant terms we neglected in Eq. (4.31), so we do the same here. Hence, the evolution equation in Mellin space take the form

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu} \ln f_{i/i}(N,\mu) = -2\Gamma^{(i)}_{\text{cusp}} \ln \bar{N} + 2C^{(i)}(\alpha_s)\,. \tag{4.105}$$

with the solution

$$f_{i/i}(N,\mu) = \exp\left\{ -\int_0^{\mu^2} \frac{d\mu'^2}{\mu'^2} \left(\Gamma^{(i)}_{\text{cusp}}(\alpha_s(\mu')) \ln \bar{N} - C^{(i)}(\alpha_s(\mu')) \right) \right\}. \tag{4.106}$$

where we have chosen the initial condition $f_{i/i}(N, \mu = 0) = 1$.

For the eikonal distributions $f^{(\text{eik})}_{i/i}$ there is a slight modification. We want them to be sum of plus distributions, so from Eq. (4.97) we must have that $C^i = 0$. The solution can then be written as

$$f^{(\text{eik})}_{i/i} = \exp\left\{ -\int_0^{\mu^2} \frac{d\mu'^2}{\mu'^2} \Gamma^{(i)}_{\text{cusp}}(\alpha_s(\mu') \ln \bar{N} \right\}. \tag{4.107}$$

Notice that the choice of the lower boundary diverges for $\mu \to 0$, which will be used later to cancel divergences coming from $\tilde{\sigma}^{(\text{eik})}(N,\epsilon)$. The equations for $J_{i/i}$ and $J^{(\text{eik})}_{i/i}$ are completely analogous.

4.6.1 Hard Virtual Gluons

With the solutions in Eq. (4.106) and Eq. (4.107) we are now ready to compute the ratios of distributions we had in Eq. (4.57). But first, we mentioned in Sec. 4.4 that the main difference between the partonic function ω_{ij} and its eikonal approximation $\omega^{(\text{eik})}_{ij}$ are contributions $G_{ij}(Q,\mu)$ from hard virtual gluons [54]. So in general, we can write the relation between the two as

$$\tilde{\omega}_{ij}(N,Q,\mu,\alpha_s(\mu)) = H_{ij}(Q,\alpha_s(\mu))\, G_{ij}(Q,\mu)\, \tilde{\omega}^{(\text{eik})}_{ij}(N,Q,\mu,\alpha_s)\,, \tag{4.108}$$

[100] Remember that $z \to 1$ corresponds to to large N.

where $H(Q)$ is the same hard process without large logarithms. If we compare this expression with Eq. (4.57), we find that the hard virtual contributions are entirely describes by the ratios

$$G_{ij}(Q,\mu) = \Big[\frac{\tilde{J}_{i/i}(N,Q,\epsilon)\tilde{J}_{j/j}(N,Q,\epsilon)}{\tilde{f}_{i/i}(N,\mu,\epsilon)\tilde{f}_{j/j}(N,\mu,\epsilon)}\Big]\Big[\frac{\tilde{f}^{(\text{eik})}_{i/i}(N,\mu,\epsilon)\,\tilde{f}^{(\text{eik})}_{j/j}(N,\mu,\epsilon)}{\tilde{J}^{(\text{eik})}_{i/i}(N,Q,\epsilon)\,\tilde{J}^{(\text{eik})}_{j/j}(N,Q,\epsilon)}\Big]\,, \tag{4.109}$$

and as mentioned in Sec. 4.4, the two ratios are free of collinear divergences and so is $G(Q,\mu)$. We can now use the solutions in Eq. (4.106) and Eq. (4.107) to compute the ratios, giving

$$\frac{\tilde{f}^{(\text{eik})}_{i/i}(N,\mu,\epsilon)\,\tilde{f}^{(\text{eik})}_{j/j}(N,\mu,\epsilon)}{\tilde{f}_{i/i}(N,\mu,\epsilon)\tilde{f}_{j/j}(N,\mu,\epsilon)} = \exp\left\{-2\int_0^{\mu^2}\frac{d\mu'^2}{\mu'^2}C^{(i)}(\alpha_s(\mu'))\right\}, \tag{4.110}$$

where the logarithm $\ln\bar{N}$ accompanied by the cusp anomalous dimension has canceled. We have also used that $C^{(i)} = C^{(j)}$ for $i = q$ and $j = \bar{q}$, giving two times $C^{(i)}$[101]. The ratio of jet distributions are equivalent, with the difference of Q instead of μ, i.e.

$$\frac{\tilde{J}_{i/i}(N,Q,\epsilon)\tilde{J}_{j/j}(N,Q,\epsilon)}{\tilde{J}^{(\text{eik})}_{i/i}(N,Q,\epsilon)\,\tilde{J}^{(\text{eik})}_{j/j}(N,Q,\epsilon)} = \exp\left\{2\int_0^{Q^2}\frac{d\mu'^2}{\mu'^2}C^{(i)}(\alpha_s(\mu'))\right\}. \tag{4.111}$$

If we insert Eq. (4.110) and Eq. (4.111) into Eq. (4.109), we get after reshuffling the ratios that the contribution from hard virtual gluons can be written as

$$G_{ij}(Q,\mu) = \exp\left\{2\int_{\mu^2}^{Q^2}\frac{d\mu'^2}{\mu'^2}C^{(i)}(\alpha_s(\mu'))\right\}, \tag{4.112}$$

where we observe that by choosing Q as an argument in the jet distributions, we have a contribution from this expression. But we have already set $Q = \mu$ on several occasions, so for the simplified result we do the same here, giving that $G_{ij}(Q,Q) = 1$. With this choice we have from Eq. (4.108) that the partonic function $\tilde{\omega}_{ij}$ is given by

$$\tilde{\omega}_{ij}(N,Q,\mu,\alpha_s(\mu)) = H_{ij}(Q,\alpha_s(\mu))\,\tilde{\omega}^{(\text{eik})}_{ij}(N,Q,\mu,\alpha_\mu)\,. \tag{4.113}$$

We have reduced the problem of resumming large logarithmic contributions to finding the eikonal function $\tilde{\omega}^{(\text{eik})}_{ij}(N)$. Hence, we will in the next section turn our attention back to the eikonal cross section $\tilde{\sigma}^{\text{eik}}_{ij}$ and how to calculate it.

4.7 The Eikonal Cross Section

In order to calculate the eikonal cross section $\tilde{\sigma}^{\text{eik}}_{ij}$, we look at the Wilson loop expectation value $\mathcal{W}_{DY}$ we found in Sec. 4.4. We will first make an $\mathcal{O}(g^2)$ expansion of the expectation value, and then use the non-Abelian eikonal exponentiation theorem to find the exponentiated form. Then we will use the factorized form of the eikonal cross section in Eq. (4.53) to find $\tilde{\omega}^{(\text{eik})}_{ij}$.

So let us start from the expectation value

$$\mathcal{W}_{DY}(y) = \langle 0|\bar{T}\,\mathcal{U}^\dagger_{DY}(y)T\,\mathcal{U}_{DY}(0)|0\rangle\,, \tag{4.114}$$

where a one-loop contribution is illustrated in Fig. 4.6. By using the momentum space expansion of Wilson lines derived in Sec. 2.3, we find that to $\mathcal{O}(g^2)$

$$\mathcal{W}_{DY} = 1 + g^2C_F\int\frac{d^dk}{(2\pi)^d}2\pi\delta^+(k^2)\frac{p_1\cdot p_2}{p_1\cdot k\,p_2\cdot k}(e^{-iy^0k^0}-1)\,, \tag{4.115}$$

[101] We choose to use generic subscripts even if we really mean $i = q$. In this way it would be easier to generalize to cases where we also have $i = g$.

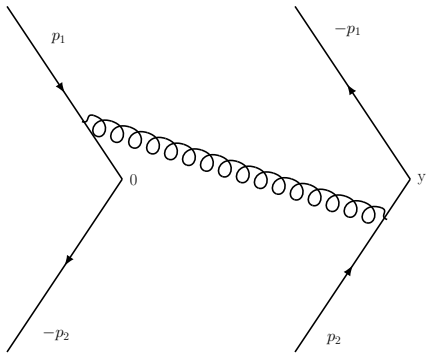


Figure 4.6: A one-loop contribution to the Drell-Yan eikonal cross section.

where we have used that $p_1^2 = p_2^2 = 0$. We have also used that in diagrams such as Fig. 4.6, we have to use a cut gluon propagator [55]

$$D^{ab}_{\mu\nu\,+}(k) = -\delta^{ab} g_{\mu\nu} 2\pi\delta^+(k^2)\,. \tag{4.116}$$

To rewrite this further, we can use that the ratio of momenta is invariant under rescaling of p_1 and p_2. With light-cone coordinates (see Appendix A.1), we have that

$$\frac{p_1\cdot p_2}{p_1\cdot k\, p_2\cdot k} = \frac{n_1\cdot n_2}{n_1\cdot k\, n_2\cdot k} = \frac{1}{k^+k^-} \tag{4.117}$$

$$k^2 = 2k^+k^- - k_\perp^2 \tag{4.118}$$

$$k^0 = (k^+ + k^-)/\sqrt{2} \tag{4.119}$$

where we used that n_1 and n_2 are light-like vectors. Let us also use the measure $d^dk = dk^+dk^-d^{d-2}k_\perp$, such that with the above rewritings the expansion can be rewritten as

$$\mathcal{W}_{DY} = 1 + \frac{\alpha_s}{\pi} C_F \int \frac{d^{d-2}k_\perp}{(2\pi)^{d-2}} \int dk^+dk^-\, 2\pi\delta(2k^+k^- - k_\perp^2) \frac{(e^{-iy^0(k^+ + k^-)/\sqrt{2}} - 1)}{k^+k^-}\,, \tag{4.120}$$

where we pulled out a factor of $(2\pi)^2$ to give the coupling. On this form, it is not obvious how to treat these integrals as the limits on the k^+ and k^- are unspecified. Instead we use the non-Abelian exponentiation theorem, where there are restrictions on the form of the exponent to preserve the exponentiation conditions. But before we give the procedure to find the eikonal cross section, we take a look at the structure of the expansion in Eq. (4.120). We observe that the factor in front of the integral looks very much like the one-loop cusp anomalous dimension in Eq. (4.94). So if we use that the scale of the coupling is $\alpha_s(k_\perp)$, we can write the expansion as[102]

$$\mathcal{W}_{DY} = 1 + \int \frac{d^{2-2\epsilon}k_\perp}{(2\pi)^{1-2\epsilon}} \Gamma_{\text{cusp}}(\alpha_s(k_\perp)) \int dk^+dk^-\, \delta(2k^+k^- - k_\perp^2) \frac{(e^{-iy^0(k^+ + k^-)/\sqrt{2}} - 1)}{k^+k^-}\,, \tag{4.121}$$

where we have inserted for $d = 4 - 2\epsilon$ and it is understood that one has to use the running coupling at one-loop order, e.g. the one-loop in Eq. (3.38). Further, by using the non-Abelian exponentiation theorem we can write $\mathcal{W}_{DY}$ as

$$\mathcal{W}_{DY} = 1 + \sum_{n=1}^{\infty} \mathcal{W}_{DY}^{(n)} = \exp\Big(\sum_{n=1}^{\infty} W_{DY}^{(n)}\Big)\,, \tag{4.122}$$

where W_{DY} are the webs we alluded to in Sec. 4.1[103]. For Drell-Yan, we have that $\mathcal{W}_{DY}^{(1)} = W_{DY}^{(1)}$, which we will use for our calculation. This is not true in general, but we are only interested in the one-loop result.

[102] It is understood that it is the cusp anomalous dimension for particles in the fundamental representation.
[103] For more details on webs, see [43, 45].

From the non-Abelian eikonal exponentiation theorem, the Mellin transformed eikonal cross section can on the most general form be written as [56][104]

$$\begin{aligned}\tilde{\sigma}_{ij}^{(\text{eik})}(N,Q,\epsilon) &= \exp\Big(2\int\frac{d^{4-2\epsilon}k}{\Omega_{1-2\epsilon}}\theta\Big(\frac{Q}{\sqrt{2}}-k^+\Big)\theta\Big(\frac{Q}{\sqrt{2}}-k^-\Big)\\ &\qquad W_{DY}\big(k^2,\frac{n_1\cdot k\, n_2\cdot k}{n_1\cdot n_2},\mu,\alpha_s,\epsilon\big)\Big(e^{-Nk^0/Q}-1\Big)\Big)\\ &= \exp\big(\tilde{E}^{(\text{eik})}(N,\epsilon)\big)\,,\end{aligned} \tag{4.123}$$

where W_{DY} is the web, and the invariance under rescaling of the momentum is applied. The theta functions are used to cut off the k^+ and k^- integrals such that they are UV-finite and resctricts the $k_\perp$ integral to have the maximum value of Q^2. Without this restriction, the exponentiation conditions would not be valid [57]. The appearance of N in the exponent can be understood from the taking the Mellin transform of Eq. (4.115), using the saddle point approximation $y^0\approx -iN/Q$ as discussed in [54]. The angular factor can in d-dimension be found in Eq. (1.233).

We can now use that $W_{DY}^{(1)}=\mathcal{W}_{DY}^{(1)}$, and use Eq. (4.121) to write the exponent as

$$\begin{aligned}\tilde{E}^{(\text{eik})}(N,Q,\epsilon) &= 2\int\frac{d^{2-2\epsilon}k_\perp}{\Omega_{1-2\epsilon}}\Gamma_{\text{cusp}}(\alpha_s(k_\perp))\int dk^+dk^-\,\theta\Big(\frac{Q}{\sqrt{2}}-k^+\Big)\theta\Big(\frac{Q}{\sqrt{2}}-k^-\Big)\\ &\qquad\times\delta(2k^+k^--k_\perp^2)\frac{1}{k^+k^-}\big(e^{-N(k^++k^-)/\sqrt{2}Q}-1\big)\\ &= 4\int\frac{d^{2-2\epsilon}k_\perp}{\Omega_{1-2\epsilon}}\frac{\Gamma_{\text{cusp}}(\alpha_s(k_\perp))}{k_\perp^2}\int\frac{dk^+}{2k^+}\theta\Big(\frac{Q}{\sqrt{2}}-k^+\Big)\theta\Big(\frac{Q}{\sqrt{2}}-\frac{k_\perp^2}{2k^+}\Big)\\ &\qquad\times\Big(e^{-N\left(k^++\frac{k_\perp^2}{2k^+}\right)/\sqrt{2}Q}-1\Big)\end{aligned} \tag{4.124}$$

where we have applied the delta function over k^-. Because of the theta functions, the lower and upper limits of the k^+ integral are finite. Hence, the exponent take the form

$$\tilde{E}^{(\text{eik})}(N,Q,\epsilon)=4\int\frac{d^{2-2\epsilon}k_\perp}{\Omega_{1-2\epsilon}}\frac{\Gamma_{\text{cusp}}(\alpha_s(k_\perp))}{k_\perp^2}\int_{k_\perp^2/\sqrt{2}Q}^{Q/\sqrt{2}}\frac{dk^+}{2k^+}\Big(e^{-N\left(k^++\frac{k_\perp^2}{2k^+}\right)/\sqrt{2}Q}-1\Big)\,. \tag{4.125}$$

One of the k^+ integrals are straightforward, i.e.

$$\int_{k_\perp^2/\sqrt{2}Q}^{Q/\sqrt{2}}\frac{dk^+}{2k^+}=-\ln\Big(\sqrt{\frac{k_\perp^2}{Q^2}}\Big)\,, \tag{4.126}$$

while the other is more tricky, it can be shown that for large N this behaves as a zeroth order modified bessel function of the second kind

$$K_0(z)=\int_0^\infty\frac{dt}{2t}e^{-t-\frac{z^2}{4t}}\,. \tag{4.127}$$

The actual rewriting is not pretty, but with a change of variable $t=Nk^+/\sqrt{2}Q$, this integral can in the large N limit be represented as

$$\int_{k_\perp^2/\sqrt{2}Q}^{Q/\sqrt{2}}\frac{dk^+}{2k^+}\,e^{-N\left(k^++\frac{k_\perp^2}{2k^+}\right)/\sqrt{2}Q}=K_0\Big(2N\sqrt{\frac{k_\perp^2}{Q^2}}\Big)\,, \tag{4.128}$$

up to terms of $\mathcal{O}(e^{-N})$.

After these considerations, we can write the exponent as

$$\tilde{E}^{(\text{eik})}(N,Q,\epsilon)=4\int\frac{d^{2-2\epsilon}k_\perp}{\Omega_{1-2\epsilon}}\frac{\Gamma_{\text{cusp}}(\alpha_s(k_\perp))}{k_\perp^2}\Big[K_0\Big(2N\sqrt{\frac{k_\perp^2}{Q^2}}\Big)+\ln\Big(\sqrt{\frac{k_\perp^2}{Q^2}}\Big)\Big]\,. \tag{4.129}$$

[104] The expression in [56] is for joint resummation, i.e. threshold and low transverse momentum of the final state. We are only considering threshold resummation, so we adjust the expression to our purpose.

We observe that the logarithm inside the bracket is divergent for $k_\perp \to 0$, but if we use the following expansion of the bessel function for z small

$$K_0(z) = -\ln\left(\frac{ze^{\gamma_E}}{2}\right) - \frac{z^4}{4}\left[\ln\left(\frac{ze^{\gamma_E}}{2}\right) - 1\right] + \mathcal{O}(z^4)\,, \tag{4.130}$$

and only keep the first term, we see that the term inside the bracket in Eq. (4.129) is given by

$$\begin{aligned} K_0\Big(2N\sqrt{\frac{k_\perp^2}{Q^2}}\Big) + \ln\Big(\sqrt{\frac{k_\perp^2}{Q^2}}\Big) &\approx -\ln\bar{N} - \ln\Big(\sqrt{\frac{k_\perp^2}{Q^2}}\Big) + \ln\Big(\sqrt{\frac{k_\perp^2}{Q^2}}\Big) \\ &= -\ln\bar{N}\,, \end{aligned} \tag{4.131}$$

i.e. the logarithm that diverges for $k_\perp \to 0$ cancels in the sum. There is still a collinear divergences, but we will soon see how to treat it.

To further rewrite the expoenent, we set $\epsilon = 0$ and use that the theta functions in Eq. (4.123) restricts the $k_\perp$ integral to maximum value of Q^2[105]. By using polar coordinates $d^2k_\perp = k_\perp dk_\perp d\Omega_1$, we can write

$$\int \frac{d^2k_\perp}{\Omega_1} = \frac{1}{2}\int_0^{Q^2} dk_\perp^2\,, \tag{4.132}$$

and we arrive at the result

$$\tilde{E}_{ij}^{(\mathrm{eik})}(N,\epsilon) = 2\int_0^{Q^2} \frac{dk_\perp^2}{k_\perp^2}\Gamma_{\mathrm{cusp}}^{(i)}(\alpha_s(k_\perp))\Big[K_0\big(2N\frac{k_\perp}{Q}\big) + \ln\Big(\frac{k_\perp}{Q}\Big)\Big)\Big]\,, \tag{4.133}$$

which is only valid up to large logarithms.

The eikonal cross section in Eq. (4.123) can then be written as

$$\tilde{\sigma}_{ij}^{(\mathrm{eik})}(N,Q,\epsilon) = \exp\left(2\int_0^{Q^2} \frac{dk_\perp^2}{k_\perp^2}\Gamma_{\mathrm{cusp}}^{(i)}(\alpha_s(k_\perp))\Big[K_0\big(2N\frac{k_\perp}{Q}\big) + \ln\Big(\frac{k_\perp}{Q}\Big)\Big]\right). \tag{4.134}$$

As previously mentioned there is still a collinear divergence in this expression. However, we factorized $\tilde{\sigma}_{ij}^{(\mathrm{eik})}(N,\epsilon)$ in such a way that $\tilde{w}_{ij}^{(\mathrm{eik})}(N)$ was to be free of these divergences. Hence, by using Eq. (4.53) we divide by the eikonal parton distributions Eq. (4.107)

$$\tilde{w}_{ij}^{(\mathrm{eik})}(N,Q,\mu,\alpha_s) = \frac{\tilde{\sigma}_{ij}^{(\mathrm{eik})}(N,Q,\epsilon)}{\tilde{f}_{i/i}^{(\mathrm{eik})}(N,\mu,\epsilon)\tilde{f}_{j/j}^{(\mathrm{eik})}(N,\mu,\epsilon)}\,, \tag{4.135}$$

giving the exponent

$$\begin{aligned} \hat{\tilde{E}}_{ij}^{(\mathrm{eik})}(N,Q,\mu) =& 2\int_0^{Q^2} \frac{dk_\perp^2}{k_\perp^2}\Gamma_{\mathrm{cusp}}^{(i)}(\alpha_s(k_\perp))\Big[K_0\big(2N\frac{k_\perp}{Q}\big) + \ln\Big(\frac{k_\perp}{Q}\Big)\Big] \\ &+ 2\int_0^{\mu^2} \frac{d\mu'^2}{\mu'^2}\Gamma_{\mathrm{cusp}}^{(i)}(\alpha_s(\mu'))\ln\bar{N}\,. \end{aligned} \tag{4.136}$$

If we add $(\ln\bar{N} - \ln\bar{N})$ inside the bracket of the first line and choose $\mu' = k_\perp$, we can group these terms as

$$\begin{aligned} \hat{\tilde{E}}_{ij}^{(\mathrm{eik})}(N,Q,\mu) =& 2\int_0^{Q^2} \frac{dk_\perp^2}{k_\perp^2}\Gamma_{\mathrm{cusp}}^{(i)}(\alpha_s(k_\perp))\Big[K_0\big(2N\frac{k_\perp}{Q}\big) + \ln\Big(\bar{N}\frac{k_\perp}{Q}\Big)\Big] \\ &- 2\int_{\mu^2}^{Q^2} \frac{dk_\perp^2}{k_\perp^2}\Gamma_{\mathrm{cusp}}^{(i)}(\alpha_s(k_\perp))\ln\bar{N}\,, \end{aligned} \tag{4.137}$$

and by choosing $\mu = Q$ can remove the last term. Hence, the eikonal function is given by

$$\tilde{w}_{ij}^{(\mathrm{eik})}(N,Q,\mu,\alpha_s) = \exp\left(2\int_0^{Q^2} \frac{dk_\perp^2}{k_\perp^2}\Gamma_{\mathrm{cusp}}^{(i)}(\alpha_s(k_\perp))\Big[K_0\big(2N\frac{k_\perp}{Q}\big) + \ln\Big(\bar{N}\frac{k_\perp}{Q}\Big)\Big]\right), \tag{4.138}$$

and we can see that when $k_\perp \to 0$ the integral is finite as the bracket perfectly cancels.

We should mention that in the general treatment, one should keep the distinction between the renormalization scale μ, factroization scale μ_F and Q. But for simplicity we have chosen them all to be the same.

[105] This had to be true for the exponentiation conditions to be valid.

4.8 Logarithmic Corrections in Drell-Yan

In this section we will use Eq. (4.138) to show how we can reproduce the large logarithm found in the NLO calculation Eq. (4.31), and also show that we find higher order logartithms without doing any higher order loop calculations.

For the current discussion we are only interested in the large logarithmic corrections, so we neglect $H(Q,\alpha_s)$ in Eq. (4.113)[106]. Then from Eq. (4.138) it follows that the partonic function $\tilde{w}_{q\bar{q}}$ has exponentiated, i.e

$$\tilde{w}_{q\bar{q}}(N,Q,\alpha_s(Q)) = \exp\left(2\int_0^{Q^2}\frac{dk_\perp^2}{k_\perp^2}\Gamma^{(q)}_{\text{cusp}}(\alpha_s(k_\perp))\Big[K_0\big(2N\frac{k_\perp}{Q}\big)+\ln\Big(\bar{N}\frac{k_\perp}{Q}\Big)\Big]\right). \tag{4.139}$$

The lower limit does not give a well defined result, but to produce large logarithms it is standard to evaluate these from $Q^2/\bar{N}$ up to Q^2 [54][107]. We can justify this by the approximation $Q^2/N \approx 0$ as $N\to\infty$. With this change of lower limit, we can make the change of variable $x = k_\perp/Q$, giving the exponent

$$\tilde{E}_{q\bar{q}}(N,Q,\alpha_s) = 4\int_{1/\bar{N}}^1\frac{dx}{x}\Gamma^{(q)}_{\text{cusp}}(\alpha_s(Qx))\Big[K_0\big(2Nx\big)+\ln\big(\bar{N}x\big)\Big], \tag{4.140}$$

which can be simplified even further by looking at the large N behaviour of the Bessel function. For large values of the argument, the modified Bessel function has the following expansion

$$K_0(z) = \Big(\frac{\pi}{z}\Big)^{1/2}e^{-z}\big(1-\mathcal{O}(z^{-1})\big)\approx 0\,, \tag{4.141}$$

and the exponent can be simplifies to

$$\tilde{E}_{q\bar{q}}(N,Q,\alpha_s) = 4\int_{1/\bar{N}}^1\frac{dx}{x}\Gamma^{(q)}_{\text{cusp}}(\alpha_s(Qx))\ln\big(\bar{N}x\big), \tag{4.142}$$

valid up to constant terms that are negligible in the large N limit.

This integral can now be solved by using the one-loop cusp anomalous dimension Eq. (4.94), and the one-loop running coupling Eq. (3.37)

$$\Gamma_{\text{cusp}}(\alpha_s(Qx)) = \frac{\alpha_s(Qx)}{\pi}C_F\,, \tag{4.143}$$

$$\alpha_s(Qx) = \frac{\alpha_s(Q)}{1+\frac{\alpha_s(Q)}{2\pi}\beta_0\ln x}\,. \tag{4.144}$$

Inserting these expression into Eq. (4.142), and with another change of variable $y=\ln x$ gives the LL (leading logarithmic) result

$$\begin{aligned}\tilde{E}^{(\text{LL})}_{q\bar{q}}(N,Q,\alpha_s) &= 4\int_{1/\bar{N}}^1\frac{dx}{x}\frac{\alpha_s(Q)}{\pi}C_F\Big(1+\frac{\alpha_s(Q)}{2\pi}\beta_0\ln x\Big)^{-1}\ln\bar{N}x\\ &= A_{q\bar{q}}\big[2\bar{\lambda}+(1-2\bar{\lambda})\ln\big(1-2\bar{\lambda}\big)\big]\big)\big]\,,\end{aligned} \tag{4.145}$$

giving

$$\tilde{w}^{(\text{LL})}_{q\bar{q}}(N,Q,\alpha_s(Q)) = \exp\Big(A_{q\bar{q}}\big[2\bar{\lambda}+(1-2\bar{\lambda})\ln\big(1-2\bar{\lambda}\big)\big]\Big)\,, \tag{4.146}$$

where we have defined $A_{q\bar{q}} = C_F/\alpha_s\pi b_0^2$ and $\bar{\lambda}=\alpha_s b_0\ln\bar{N}$, where $b_0=\beta_0/4\pi$. This result is in agreement with the LL correction for singlet ($q\bar{q}$) annihilation in Drell-Yan [58, 59].

At first sight Eq. (4.146) does not have the same form as Eq. (4.31), but if we expand the logarithm

$$\ln\big(1-2\bar{\lambda}\big) = -2\bar{\lambda}-\bar{\lambda}^2-\frac{2}{3}\bar{\lambda}^3-\frac{1}{2}\bar{\lambda}^4+\cdots\,, \tag{4.147}$$

[106] From factorization theorems this does not include logarithmic corrections.

[107] They actually solve a renormalization group equation for the Wilson line, where the integral is evaluated from $Q^2/\bar{N}$ up to Q^2.

we find the LL terms

$$\tilde{E}^{(\mathrm{LL})}_{q\bar{q}}(N,Q,\alpha_s) = \frac{\alpha_s}{\pi} 2C_F \ln^2 \bar{N} + \left(\frac{\alpha_s}{\pi}\right)^2 \frac{\beta_0}{3} C_F \ln^3 \bar{N} + \left(\frac{\alpha_s}{\pi}\right)^3 \frac{\beta_0^2}{32} \ln^4 \bar{N} + \mathcal{O}(\alpha_s^4)\,, \tag{4.148}$$

where the first term are the LL for the corresponding NLO calculation we found in Eq. (4.31)[108], and the other terms are the LL for even higher order calculations. If we wanted to compare with fixed order calculations to higher orders, we could have expanded the exponential in Eq. (4.146) and found an expanded form of $\tilde{w}_{q\bar{q}}(N)$. This expansion would give fixed NLL order terms as well, but these can be found in [60]. A last point to make is that in order to obtain resummed NLL terms, we would have to use the coupling to two-loop order and the cusp anomalous dimension up to two loop order, but we did not consider this scenario here.

4.9 Hadronic Cross Section and Inverse Mellin

We have managed to bring the partonic function on an exponentiated form, but the correct observable is the hadronic cross section. So in this section we will recover the hadronic Drell-Yan cross section in x-space, and discuss how the inverse Mellin transform can be evaluated.

From Eq. (4.27) and Eq. (4.145) the Mellin transformed hadronic cross section is given by

$$\begin{aligned}\tilde{\sigma}_{h_1h_2}(N) &= \int_0^1 d\tau \tau^{N-1} \frac{1}{\sigma_0} \frac{d\sigma_{h_1h_2}}{dQ^2} \\ &= \sum_{i,j=q,\bar{q}} \tilde{f}_{i/h_1}(N,Q)\tilde{f}_{j/h_2}(N,Q)\, \exp\left(\tilde{E}_{ij}(N,Q,\alpha_s)\right),\end{aligned} \tag{4.149}$$

where the partonic function has been exponentiated. We can now use the inverse Mellin transform Eq. (D.4) to write[109]

$$\frac{d\sigma_{h_1h_2}}{dQ^2} = \sigma_0 \sum_{i,j=q,\bar{q}} Q_q^2 \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dN\, \tau^{-N} \tilde{f}_{i/h_1}(N,Q)\tilde{f}_{j/h_2}(N,Q) \exp\left(\tilde{E}_{ij}(N,Q,\alpha_s)\right). \tag{4.150}$$

There exists numerical packages to evaluate parton distributions in Mellin space, see [61]. However, the standard method is to use parton distributions sets from the LHAPDF code existing is x-space [62, 63]. To use the x-space formalism we can do several manipulations by using the convolution properties of the Mellin transform, given in Appendix D.1. But let us first derive an expression for the inverse Mellin transform in terms of an integral over a real variable.

4.9.1 The Inverse Mellin Transform

The inverse Mellin transform for a general function is given in terms of an integral over a complex variable N, see Appendix D.1, and reads

$$h(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dN\, x^{-N}\, \tilde{h}(N)\,, \tag{4.151}$$

where

$$\tilde{h}(N) = \int_0^\infty dx\, x^{N-1}\, h(x)\,.$$

Since $h(x)$ is a real valued function

$$\tilde{h}^*(N) = \int_0^\infty dx\, x^{N^*-1}\, h(x) = \tilde{h}(N^*)\,. \tag{4.152}$$

The Mellin inversion integral Eq. (4.151) can be splitt in two, one part for the lower bound and one part for the upper bound. To manipulate the integral, we make a change of variable $N \to N^*$ in the term that is

[108] With the very important distinction that it has been exponentiated.
[109] Where we have recovered the fractional charge of the quarks.

integrated from $c - i\infty$ up to c. This makes the integration bound change accordingly $c - i\infty \to c + i\infty$, and we find

$$\begin{aligned} h(x) &= \frac{1}{2\pi i}\Big(\int_{c-i\infty}^{c} dN\, x^{-N}\,\tilde{h}(N) + \int_{c}^{c+i\infty} dN\, x^{-N}\,\tilde{h}(N)\Big) \\ &= \frac{1}{2\pi i}\Big(\int_{c+i\infty}^{c} dN^*\, x^{-N^*}\,\tilde{h}(N^*) + \int_{c}^{c+i\infty} dN\, x^{-N}\,\tilde{h}(N)\Big) \\ &= \frac{1}{2\pi i}\Big(-\int_{c}^{c+i\infty} dN^*\, x^{-N^*}\,\tilde{h}^*(N) + \int_{c}^{c+i\infty} dN\, x^{-N}\,\tilde{h}(N)\Big)\,, \end{aligned} \tag{4.153}$$

and by choosing the parametrization of the Mellin variable to be $N = c + ze^{i\phi}$, in terms of real z, the integral will take the form

$$\begin{aligned} h(x) &= \frac{1}{2\pi i}\int_0^\infty dz\, \Big(e^{i\phi}\, x^{-N}\tilde{h}(N) - e^{-i\phi}\, x^{-N^*}\tilde{h}^*(N)\Big) \\ &= \frac{1}{2\pi i}\int_0^\infty dz\, 2i\,\mathrm{Im}\left(e^{i\phi}\, x^{-N}\tilde{h}(N)\right) \\ &= \frac{1}{\pi}\int_0^\infty dz\,\mathrm{Im}\left(e^{i\phi}\, x^{-N}\tilde{h}(N)\right)\,, \end{aligned} \tag{4.154}$$

where we used Eq. (4.152) and the relation

$$\tilde{h}(N) - \tilde{h}^*(N) = 2i\,\mathrm{Im}(\tilde{h}(N))\,.$$

Writing out the parametrization, the final integral is given by

$$h(x) = \frac{1}{\pi}\int_0^\infty dz\,\mathrm{Im}\left(e^{i\phi}\, x^{-c-z\,\exp(i\phi)}\tilde{h}(c + z\,\exp(i\phi))\right)\,. \tag{4.155}$$

4.9.2 Hadronic Cross Section in x-space

In order to rewrite the inverse Mellin in x-space it is advantageous to use that Eq. (4.26) is understood to be a Mellin convolution. Let us begin with Eq. (4.26), and use Eq. (D.5) to write it as[110]

$$\frac{d\sigma_{h_1h_2}}{dQ^2} = \sigma_0 \sum_{i,j=q,\bar{q}} \int_0^1 dz\, dx_1\, dx_2\, \delta\big(\tau - x_1x_2z\big) f_{i/h_1}(x_1) f_{j/h_2}(x_2)\, \omega_{ij}(z)\,, \tag{4.156}$$

and using the delta function property

$$\int dx\, \delta(g(x)) = \left|\frac{\partial g}{\partial x}\right|^{-1}_{x=x^*} \int dx\, \delta(x - x^*)\,, \tag{4.157}$$

we find that

$$\frac{d\sigma_{h_1h_2}}{dQ^2} = \sigma_0 \sum_{i,j=q,\bar{q}} \int_\tau^1 \frac{dz}{z} \int_{\tau/z}^1 \frac{dx_1}{x_1} f_{i/h_1}(x_1) f_{j/h_2}\Big(\frac{\tau}{x_1z}\Big)\, \omega_{ij}(z)\,, \tag{4.158}$$

where the limits have changed after acting with the delta function, i.e. we have

$$x_2 = \frac{\tau}{x_1z} \le 1\,, \qquad \frac{\tau}{z} \le x_1 \le 1\,, \qquad \tau \le z \le 1\,. \tag{4.159}$$

To actually perform the inverse transform is a nontrivial exercise, so let us go through some of the general details. The difficulty originates from the singularity structure of the integrand. These singularities can be divided into two regions in the complex N-plane, see Fig. 4.7. The first ones are positioned in the left part of the

[110] For simplicity we have abbreviated some of the arguments in the integrand.

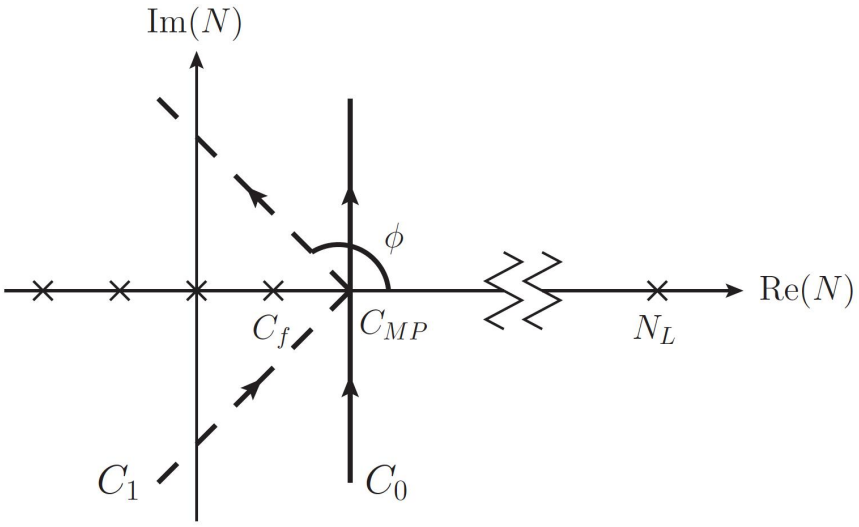


Figure 4.7: Possibilities to choose the contour for the Mellin inversion as proposed in [58]. C_0 is the vertical contour and C_1 is the contour with an angle.

complex plane. These are the result of poles in the parton distribution functions $\tilde{f}_{i/h}$. In general, the functional form of parton distributions functions at the initial scale μ_0 is given by

$$xf_{i/h}(x,\mu_0) = \sum_n A_n x^{\gamma_n}(1-x)^{\delta_n}\,, \tag{4.160}$$

where A_n, γ_n and δ_n are obtained from a fitting procedure to hard scattering data, and for small x and $\gamma_n < 0$ this becomes singular. In Mellin space, this is transformed to

$$\tilde{f}_{i/h}(N,\mu_0) = \sum_n A_n \beta(N+\gamma_n, 1+\delta_n)\,, \tag{4.161}$$

where $\beta(a,b)$ is the Euler-beta function in which the singular behaviour is well known. This first region can be avoided by choosing the contour in Fig. 4.7, as derived in [58]. We observe that the singularities from the parton distributions are avoided by choosing a constant C_{MP} that lies to the right of the rightmost singularity of $\tilde{f}_{i/h}$.

The second region is more troublesome, as it is in the right part of the complex N-plane. This singularity originates from the Landau pole that arises for small couplings, i.e. the non-perturbative regime of QCD. The Landau pole manifest itself in the exponent of Eq. (4.145), and is due to the expression $\ln\big(1-2\bar{\lambda}\big)$. Hence, for $\bar{\lambda} = 1/2$ we have

$$N_L = e^{-\gamma_E} e^{\frac{1}{2b_0\alpha_s}}\,. \tag{4.162}$$

This problem has been extensively studied in the litterature, and the most common approach is the so-called *Minimal Prescription* [58]. This method states that the contour has to be chosen such that the constant C_{MP}, satisfies

$$C_f < C_{MP} < N_L\,, \tag{4.163}$$

where C_f is the rightmost pole of $\tilde{f}_{i/h}$. In Fig. 4.7 we see the possible contours, where C_0 is the vertical line and C_1 is the bent contour. As argued in [58], the choice $C_{MP} = 2$ and C_1 with $\phi > \pi/2$, will make the integral converge faster than the choice $C_{MP} = 2$ and C_0 with $\phi = \pi/2$.

So if the hadronic cross section is to be calculated with parton distributions in Mellin space, the following integral must be implemented

$$\frac{d\sigma_{h_1h_2}}{dQ^2} = \sigma_0 \sum_{i,j=q,\bar{q}} Q_q^2 \frac{1}{\pi} \int_0^\infty dy \, \mathrm{Im}\Big[e^{i\phi} \tau^{-N} \tilde{f}_{i/h_1}(N,Q) \tilde{f}_{j/h_2}(N,Q) \, \tilde{\omega}_{ij}(N, \alpha_s(Q)) \Big] , \tag{4.164}$$

where we used Eq. (4.155) to write the cross section as an integral over a real variable and $N = C_{MP} + ye^{i\phi}$.

However, to use the x-space formalism the hadronic cross section is obtained by calculating

$$\frac{d\sigma_{h_1h_2}}{dQ^2} = \sigma_0 \sum_{i,j=q,\bar{q}} Q_q^2 \int_\tau^\infty \frac{dz}{z} \int_{\tau/z}^1 \frac{dx_1}{x_1} f_{i/h_1}(x_1,Q) f_{j/h_2}\Big(\frac{\tau}{x_1 z}, Q\Big) \, \omega_{ij}(z, Q, \alpha_s(Q)) , \tag{4.165}$$

where the parton distributions are calculated in x-space, while the hard function is found by the inverse transform

$$\omega_{ij}(z, Q, \alpha_s(Q)) = \frac{1}{\pi} \int_0^\infty dy \, \mathrm{Im}\big(e^{i\phi} z^{-N} \, \tilde{\omega}_{ij}(N, \alpha_s(Q)) \big) . \tag{4.166}$$

We observe that the upper limit of the z integral in Eq. (4.165) has changed. The reason for this change is that in the minimal prescription, the resummed cross section does not vanish for $z > 1$ due to the Landau pole [58].

In Eq. (4.165), the parton distributions are calculated in x-space, but there are problems that might occur in the hard function ω_{ij}. Close to threshold, the resummed exponent can give large oscillations, see [58, 64] for more details. One possibility to dampen the oscillations is by a simple rewriting of the integrand. We do this by using Eq. (D.12), to write

$$N \tilde{f}_{i/h}(N) = \int_0^1 dx \, x^{N-1} \mathcal{F}_{i/h}(x) , \tag{4.167}$$

where we used that parton distributions vanish for $x = 1$, and defined

$$\mathcal{F}_{i/h}(x) = -x \frac{d}{dx} f_{i/h}(x) , \tag{4.168}$$

giving the modified hadronic cross section

$$\frac{d\sigma_{h_1h_2}}{dQ^2} = \sigma_0 \sum_{i,j=q,\bar{q}} Q_q^2 \int_\tau^1 \frac{dz}{z} \int_{\tau/z}^1 \frac{dx_1}{x_1} \mathcal{F}_{i/h_1}(x_1,Q) \mathcal{F}_{j/h_2}\Big(\frac{\tau}{x_1 z}, Q\Big) \, \mathcal{S}_{ij}(z, Q, \alpha_s(Q)) , \tag{4.169}$$

where

$$\mathcal{S}_{ij}(z, Q, \alpha_s(Q)) = \frac{1}{2\pi i} \int_{C_{MP}} dN \, z^{-N} \frac{\tilde{w}_{ij}(N, Q, \alpha_s(Q))}{N^2} . \tag{4.170}$$

The derivatives of the parton distributions can be performed numerically for the common sets [65, 66]. As mentioned the point of this rewriting is to dampen the behaviour of the exponent in $\omega_{q\bar{q}}$, but as argued in [58] gluon initiated processes should have even higher powers of N as dampening factors. This would subsequently lead to higher order derivatives of the parton distributions.

Conclusion

In this thesis we have investigated IR divergences that appear in gauge theories, focusing in particular on the non-Abelian gauge theory of QCD. For theories involving massless fields these divergences are a prominent feature, and in certain regions of phase space they give rise to large logarithmic corrections to physical observables. The main focus of interest was to investigate how Wilson lines and Wilson loops can be used to resum these large corrections such that physical observables exponentiate and prevents the invalidation of perturbative expansions.

In Chapter 1 we developed the route from Green's functions to scattering amplitudes and Feynman diagrams, before going into some detail about the basic ideas and possible ways of treating divergences using regularization and renormalization. We mainly focused on the UV region of phase space, as the IR region is studied in more detail in later chapters. After this basic introduction to QFT in Chapter 1, we went on to look at the geometrical formulation of gauge theories in Chapter 2. From this formalism the concept of Wilson lines naturally appeared as fundamental building blocks for any gauge theory. We also discussed that from the Ambrose-Singer theorem it follows that physical observables can be constructed in terms of gauge invariant Wilson loops. This is the basis why we in later chapters use Wilson lines to construct Wilson loop expectation values, and from them eikonal cross sections. Further, by using Wilson lines we showed how one can construct the Yang-Mills Lagrangian from a purely geometrical standpoint. As we are mainly interested in scattering amplitudes, we focused on introducing the properties of piecewise linear Wilson lines. These naturally appear when high-energy particles meet at a point and annihilate or scatters of one another by exchanging a gauge boson.

In Chapter 3 we first introduced the QCD Lagrangian and set the stage for perturbative calculations by discussing the property of asymptotic freedom in QCD. By using the most basic experimental setup of deep inelastic scattering, we went on and studied the important concepts of factorization both in the parton model and in QCD. We also introduced parton distribution functions, which is an essential ingredient in obtaining factorization in QCD. From there we made use of Wilson lines and how these can be used to render parton distributions gauge invariant. We also derived parton-in-parton distributions and showed how these can be defined such that they incorporate the collinear divergences appearing in the partonic cross section. The significance of this is that we can use factorization theorems to group divergences into different regions and define functions that are responsible for these. Lastly we investigated the appearance of large logarithms by doing an explicit NLO calculation of a lepton pair production via the annihilation of a quark-antiquark pair. We show that higher order corrections have a significant contribution to the hadronic cross section, implying that in order to have predictive results even higher order results had to be taken into account. We also made a numerical evaluation of the NLO hadronic cross section and compared with experimental results from CMS. This is of course nothing new, but the reason we used this process is because it is the simplest one to perform resummation with. By doing the NLO calculation we found a fixed order result we could compare with the resummed Drell-Yan cross section we aimed to calculate.

In Chapter 4 we started by looking at the behaviour of scattering amplitudes using the eikonal approximation, showing that in this limit the amplitude naturally factorizes into one hard and one soft regime. The crucial point of this derivation was to show that not only does the soft and hard parts decouple, but IR divergences coming from the soft part exponentiates. Close to the final state production threshold gluon radiation from the highly energetic initial state quarks are restricted to be soft. This implies that Wilson lines on linear paths can be used to describe the soft radiation. We use factorization theorems to first refactorize the partonic cross section in terms of parton-in-parton distributions that are responsible for collinear divergences. These cross sections are explicitly factorized by using Mellin space techniques. Then we defined a close to threshold cross section, which is factorized into three parts. One part that describes the hard process without large corrections, one part that describes collinear radiation and one part that contains soft wide angle radiation. The contribution from the soft

function is found by taking the eikonal approximation giving an eikonal partonic cross section. This eikonal cross section is then constructed from a Wilson loop expectation value. The eikonal cross section contains both soft and collinear singularities, and again we used factorization theorems to group the collinear singularities into eikonal parton distributions and an infrared safe eikonal function.

We then took a closer look at the renormalization properties of Wilson lines and parton-in-parton distributions in the $x \to 1$ limit. A piecewise linear Wilson line that is constructed in terms of two semi-infinite Wilson lines on linear paths with an angle between them, contains cusp divergences. By using the renormalization properties of Wilson lines we calculate the one-loop cusp anomalous dimension for Wilson lines on the light-cone. This calculation is important as we show that parton-in-parton distributions in the $x \to 1$ obey an evolution equation in terms of this cusp anomalous dimension. The cusp anomalous dimension also appears in the eikonal cross section, showing that it is an important ingredient when doing resummation with Wilson lines. From there we show how one can calculate the Wilson loop expectation value to one-loop order and use the non-Abelian eikonal exponentiation theorem to find the exponentiated eikonal cross section. As this eikonal cross section contains IR divergences, we use the eikonal distributions to find the infrared safe eikonal function. In the end we find an exponentiated partonic hard function, given in terms of an integral over the cusp anomalous dimension. We solve this integral by using the one-loop running coupling and find that the resummed expression contains an all order series of leading logarithmic terms. Hence, we have not only reproduced the large logarithm from the fixed order NLO calculation in Chapter 3, but showed the appearance of higher order contributions without performing any fixed order calculation. We also discuss that these results are in accordance with results in the literature, apart from constant factors that we neglected throughout as we mainly focused on the limit of large N in Mellin space.

Lastly, we reconstruct the full resummed hadronic cross section and show an explicit method of how one can go about calculating the inverse Mellin transform needed for a numerical evaluation. We discuss several important features that have to be taken into account when choosing the contour in Mellin space. By using that the cross section is a real valued function, we manipulate the inverse transform and show that it reduces to an integral over a real variable.

As future work resummation techniques can be explored further by looking at physics beyond the Standard Model, for example in Supersymmetry. Supersymmetry is one of the most promising, and most studied theories for physics beyond the Standard Model. At the Large Hadron Collider there is an extensive search programme for such new physics phenomena. So far no supersymmetric particles have been found, but there is a need for higher order calculations for the production of these sparticles to precisely evaluate the current exclusion limits. As a natural extension to the work done here, we could look at the production of final state sleptons. Sleptons are colour neutral and will, as for the leptons studied here, not contain any final state gluon radiation. Hence, the radiation part of the process should not be any different from the one we have found in this thesis. The main difference will be that we are considering the production of massive particles and the threshold variable will be a function of the mass. Of course, we also have to take into account the interaction between the sleptons and the photon, leading to a different hard function. There are simplifications that can be made, and that is to look at degenerate masses. Another extension is to look at coloured final states, where one consider the production of squarks and gluinos. This extension is highly non-trivial as the final state will also contain gluon radiation.

Appendices

Appendix A

A.1 Light-Cone Coordinates

Light-cone coordinates is specifically useful in high energy scattering processes where one want to decompose the momentum of the involving particles. For a general four vector p^μ, one defines

$$p^\mu = (p^+, p^-, p_\perp)\,, \tag{A.1}$$

where

$$p^+ = \frac{1}{\sqrt{2}}(p^0 + p^3) \tag{A.2}$$

$$p^- = \frac{1}{\sqrt{2}}(p^0 - p^3) \tag{A.3}$$

$$p_\perp = (p^1, p^2)\,. \tag{A.4}$$

Scalar products are given by

$$p \cdot k = p^+ k^- + p^- k^+ - p_\perp \cdot k_\perp \tag{A.5}$$

$$p^2 = 2p^+ p^- - p_\perp^2\,, \tag{A.6}$$

where the transverse contraction $p_\perp \cdot k_\perp$ is understood from the definition of the transverse vector and must not be mistaken as the same as the four momentum contraction $p \cdot k$. We will usually parametrize our momenta in terms of plus-components and from Eq. (A.6) it follows that the minus component can be written as

$$p^- = \frac{p^2 + p_\perp^2}{2p^+}\,. \tag{A.7}$$

The d-dimensional Jacobian takes the form

$$d^d p = dp^+ dp^- d^{d-2} p_\perp\,. \tag{A.8}$$

From the above relations the light-cone metric takes the form

$$g^{\mu\nu}_{\rm LC} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\,, \tag{A.9}$$

where the index runs over $\mu = +, -, 1, 2$. We will elsewhere drop the subscript LC as it will always be clear from the context when we are using light-cone coordinates. One can also define light-like basis vectors

$$n_+^\mu = (1^+, 0^-, 0_\perp)\,, \qquad n_{+\,\mu} = (0^+, 1^-, 0_\perp)\,, \tag{A.10}$$

$$n_-^\mu = (0^+, 1^-, 0_\perp)\,, \qquad n_{-\,\mu} = (1^+, 0^-, 0_\perp)\,, \tag{A.11}$$

giving

$$n_+^2 = 0\,, \qquad n_-^2 = 0\,, \qquad n_+ \cdot n_- = 1\,. \tag{A.12}$$

These basis vectors project out the following components of a vector

$$p \cdot n_+ = p^- \,, \qquad p \cdot n_- = p^+ \,. \tag{A.13}$$

We can also construct a transversal metric

$$g_\perp^{\mu\nu} = g^{\mu\nu} - \left(n_+^\mu n_-^\nu + n_+^\nu n_-^\mu\right) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \,, \tag{A.14}$$

from which it follows that

$$g_\perp^{\mu\nu} g_{\perp\,\mu\nu} = 2 \,. \tag{A.15}$$

We can also define the gluon polarization sum in light-cone gauge, i.e $A^+ = 0$, as

$$\sum_{\text{pol}} \varepsilon_\alpha(k') \varepsilon_\beta^*(k') = -g_{\alpha\beta} + \frac{k'_\alpha \, n_{-\,\beta}}{k' \cdot n_-} + \frac{k'_\beta \, n_{-\,\alpha}}{k' \cdot n_-} \,. \tag{A.16}$$

Appendix B

B.1 Dirac Gamma Matrices

In scattering processes Dirac gamma matrices are extremely useful for calculations. Using only the Dirac algebra and trace identities they can be eliminated completely without referring to any specific representation. The convention used in this thesis is based on [67], which we refer to for a more complete treatment of gamma matrices and spinors.

The gamma matrices are defined by satisfying the Dirac algebra[111]

$$\{\gamma^\mu, \gamma^\nu\} \equiv 2g^{\mu\nu}\,, \tag{B.1}$$

where $g^{\mu\nu}$ is the usual Minkowski metric tensor.

The hermitian conjugate of a gamma matrix is given by

$$\left(\gamma^\mu\right)^\dagger = \gamma^0\gamma^\mu\gamma^0\,. \tag{B.2}$$

From the Dirac algebra we can list some useful identities in d-dimensions

$$\gamma^\mu\gamma_\mu = d \tag{B.3}$$
$$\gamma^\mu\gamma^\nu\gamma_\mu = (2-d)\gamma^\nu \tag{B.4}$$
$$\gamma^\mu\gamma^\nu\gamma^\lambda\gamma_\mu = 4g^{\nu\lambda} + (d-4)\gamma^\nu\gamma^\lambda \tag{B.5}$$
$$\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho\gamma_\mu = (d-4)\gamma^\nu\gamma^\lambda\gamma^\rho - 2\gamma^\rho\gamma^\lambda\gamma^\nu\,. \tag{B.6}$$

The trace over an odd number of gamma matrices always vanish, and for two and four gamma matrices we have the following identities

$$\mathrm{tr}(\gamma^\mu) = 0 \tag{B.7}$$
$$\mathrm{tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu} \tag{B.8}$$
$$\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho) = 4\left(g^{\mu\nu}g^{\lambda\rho} - g^{\mu\lambda}g^{\nu\rho} + g^{\mu\rho}g^{\nu\lambda}\right). \tag{B.9}$$

Often we will have contraction involving the Dirac slash notation $\not{p} = p_\mu\gamma^\mu$. In $d = 4$ we have the following identities

$$\gamma^\mu\not{p} = 2p^\mu - \not{p}\gamma^\mu \tag{B.10}$$
$$\gamma^\mu\not{p}\gamma_\mu = -2\not{p} \tag{B.11}$$
$$\gamma^\mu\not{p}\not{k}\gamma_\mu = 4p\cdot k \tag{B.12}$$
$$\gamma^\mu\not{p}\not{k}\not{q}\gamma_\mu = -2\not{p}\not{k}\not{q}\,. \tag{B.13}$$

Products of slashed vectors are given by

$$\not{p}\not{p} = p^2 \tag{B.14}$$
$$\not{p}\not{k}\not{p} = 2p\cdot k\not{p} - p^2\not{k} \tag{B.15}$$
$$\not{p}\not{k}\not{q}\not{p} = 2p\cdot q\not{p}\not{k} - 2p\cdot k\not{p}\not{q} + p^2\not{k}\not{q} \tag{B.16}$$
$$\not{p}\not{k} + \not{k}\not{p} = 2p\cdot k \tag{B.17}$$
$$\not{p}\not{k}\not{q} + \not{q}\not{k}\not{p} = 2k\cdot q\not{p} - 2p\cdot q\not{k} + 2p\cdot k\not{q}\,. \tag{B.18}$$

[111] It is implicit that there is a four by four identity matrix in this equation.

One combination that occurs often in scattering amplitudes is the following trace

$$\mathrm{tr}[\not{p}\gamma^\mu \not{k}\gamma^\nu] = 4(p^\mu k^\nu + p^\nu k^\mu - g^{\mu\nu} p \cdot k)\,. \tag{B.19}$$

Appendix C

C.1 Plus Distributions

The plus distribution is a vital mathematical construct that is widely used in both fixed order and resummation calculations. On the most general form, a plus distribution is defined as

$$f_+(x) \equiv \lim_{\alpha\to 0}\Big(f(x)\theta(1-\alpha-x) - \delta(1-\alpha-x)\int_0^{1-\alpha} dy\, f(y)\Big)\,, \tag{C.1}$$

where $f(x)$ is singular for $x = 1$. It is a distribution, so it is meant to act inside integrals. Hence, integrated with an analytic function on the domain $x \in [0, 1]$, it works as

$$\int_0^1 dx\, f_+(x)g(x) = \int_0^1 dx\, f(x)(g(x) - g(1))\,, \tag{C.2}$$

and has the useful property

$$\int_0^1 dx\, f_+(x) = 0\,. \tag{C.3}$$

In the case where the lower limit is not zero, this is evaluated as

$$\int_z^1 dx\, f_+g(x) = \int_0^1 dx\, f(x)(g(x) - g(1)) + g(1)\ln(1-x)\,. \tag{C.4}$$

These are general considerations, but let us look at a specific case we use in this thesis. In dimensional regularization, we often encounter terms like

$$(1-x)^{-1-\epsilon}\,, \tag{C.5}$$

which diverges in the limit where $z \to 1$ and $\epsilon \to 0$. But the perturbative calculable functions are not the observables we are studying directly. We are integrating perturbative functions with analytic parton distribution functions, which allows us to do several manipulations. The integrals we encounter are

$$\mathcal{I}(x,\epsilon) = \int_0^1 dx\, g(x)(1-x)^{-1-\epsilon}\,, \tag{C.6}$$

and as long as $g(x)$ does not converge towards $\mathcal{O}(1-z)$ near $z = 1$ this diverges. To treat this we simply add zero to the integral in the following way

$$\mathcal{I}(x,\epsilon) = \int_0^1 dx\, g(1)(1-x)^{-1-\epsilon} + \int_0^1 dx\, \big(g(x) - g(1)\big)(1-x)^{-1-\epsilon}\,. \tag{C.7}$$

and use the beta integral to write

$$\int_0^1 dx\,(1-x)^{-1-\epsilon} = -\frac{1}{\epsilon}\,, \tag{C.8}$$

giving the expansion in ϵ

$$\mathcal{I}(x,\epsilon) = -\frac{1}{\epsilon}g(1) + \int_0^1 dx\,\big(g(x)-g(1)\big)\Big[\frac{1}{1-z} - \frac{\ln(1-z)}{1-z}\epsilon + \mathcal{O}(\epsilon^2)\Big]\,. \tag{C.9}$$

If we use Eq. (C.2), we have that

$$\mathcal{I}(x,\epsilon) = \int_0^1 dx\,g(x)\Big[-\frac{1}{\epsilon}\delta(1-x) + \frac{1}{1-z}_+ - \epsilon\Big(\frac{\ln(1-z)}{1-z}\Big)_+ + \mathcal{O}(\epsilon^2)\Big]\,. \tag{C.10}$$

Appendix D

D.1 The Mellin Transform

For a function $f(x)$ defined on the positive real axis, the Mellin transformation $\mathcal{M}$ is the operation mapping f into the function $\tilde{f}$ defined on the complex plane. It has the following definition

$$\mathcal{M}\big[f(x):N\big] = \tilde{f}(N) = \int_0^\infty dx\, x^{N-1}\, f(x)\,, \tag{D.1}$$

where $\tilde{f}(N)$ is the Mellin transform of $f(x)$, and N is the Mellin moment conjugate to x. In general, the integral does not exist for all functions, i.e. all functions does not have a well defined Mellin transform. But even if the transform exist, it is not guaranteed to converge. The domain of N for which the integral converge for a given function is known as the fundamental strip. For a real function, the fundamental strip is denoted as $< a, b >$, given by all points on the domain $a < s < b$ such that $N = s + it$, for any t. The values of a and b is found by the asymptotic behaviour

$$a: \qquad \lim_{x\to 0^+} f(x) = \mathcal{O}(x^{-a})\,, \tag{D.2}$$

$$b: \qquad \lim_{x\to \infty} f(x) = \mathcal{O}(x^{-b})\,, \tag{D.3}$$

which implies that a function defined on the domain $0 < x < 1$, have a fundamental strip $< a, \infty >$.

The Mellin transform is closely related to the two-sided Laplace transform, with the difference of a variable change $x \to -\ln x$. Hence, the inverse Mellin is given by

$$\mathcal{M}^{-1}\big[\tilde{f}(N):x\big] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dN\, x^{-N}\,\tilde{f}(N)\,, \tag{D.4}$$

where the integration contour is along a vertical line through $Re(N) = c$, as long as c lies in the fundamental strip of the function. If the function is holomorphic in the strip and vanishes sufficiently fast when $Im(N) \to \pm\infty$, it follows from Cauchy's theorem that the contour may be deformed as long as no poles are crossed.

One very important property of the Mellin transform is its effect on convolutions

$$\big(f\star g\big)(x) = \int_0^1 dx_1 \int_0^1 dx_2\, f(x_1)g(x_2)\delta(x - x_1x_2) = \int_x^1 \frac{dx_1}{x_1}\, f(x_1)g\big(\frac{x_2}{x_1}\big)\,, \tag{D.5}$$

where $x \in (0, 1)$. To disentangle this convolution, one performs the transform

$$\begin{aligned}\mathcal{M}\big[\big(f\star g\big)(x):N\big] &= \int_0^1 dx\, x^{N-1}\int_0^1 dx_1\int_0^1 dx_2\, f(x_1)g(x_2)\delta(x - x_1x_2)\\ &= \int_0^1 dx_1\, x_1^{N-1} f(x_1)\int_0^1 dx_2\, x_2^{N-1} g(x_2)\\ &= \tilde{f}(N)\tilde{g}(N)\,,\end{aligned} \tag{D.6}$$

where the convolution in x-space has transformed into simple products in Mellin space.

D.1.1 Mellin Transforms of Functions

This section is intended to demonstrate some of the Mellin transforms that are encountered in this thesis. Since the main focus is in the domain of large N, this will not be the general treatment of these transforms.

The simplest case is the Mellin transform of a constant c,

$$\int_0^1 dx\, x^{N-1} c = \frac{c}{N}\,, \tag{D.7}$$

and a monomial

$$\int_0^1 dx\, x^{N-1} x^a = \frac{1}{N+a}\,, \tag{D.8}$$

which combined with a logarithm

$$\begin{aligned}\int_0^1 dx\, x^{N-1}(-x^a \ln x) &= -\frac{x^{N+a}}{N+a}\ln x\Big|_0^1 + \int_0^1 dx\, \frac{x^{N+a-1}}{N+a} \\ &= \frac{1}{(N+a)^2}\,,\end{aligned} \tag{D.9}$$

which can be generalized for any polynomial $P(x)$

$$\int_0^1 dx\, x^{N-1} P(x) = \mathcal{O}(1/N)\,. \tag{D.10}$$

Furthermore, by using that $x^{N-1} = e^{(N-1)\ln x}$, repeated derivatives with respect to N will give

$$\int_0^1 dx\, x^{N-1} f(x) \ln^k x = \frac{d^k}{dN^k}\tilde{f}(N), \qquad \forall\, k > 0\,. \tag{D.11}$$

Another useful property involves the derivative of a function

$$\begin{aligned}\int_0^1 dx\, x^{N-1}\Big(-x\frac{d}{dx}f(x)\Big) &= -x^N f(x)\Big|_0^1 + N\int_0^1 dx\, x^{N-1} f(x) \\ &= -f(1) + N\tilde{f}(N)\,,\end{aligned} \tag{D.12}$$

which is especially useful for functions that vanish for $x = 1$.

The Mellin transform of plus distributions are more complicated. By using Eq. (C.2), we can write

$$\int_0^1 dx\, x^{N-1}\Big[\frac{1}{1-x}\Big]_+ = \int_0^1 dx\Big(\frac{x^{N-1}}{1-x} - \frac{1}{1-x}\Big)\,. \tag{D.13}$$

The terms on the *rhs* diverge when considered separately, so we can not calculate them independently. By introducing a regulator ϵ, this can be rewritten by using beta integrals

$$\int_0^1 dx \frac{x^{N-1}}{(1-x)^{1-\epsilon}} - \int_0^1 dx \frac{1}{(1-x)^{1-\epsilon}} = \frac{\Gamma(N)\Gamma(\epsilon)}{\Gamma(N+\epsilon)} - \frac{\Gamma(1)\Gamma(\epsilon)}{\Gamma(1+\epsilon)}\,, \tag{D.14}$$

which by the recursion relation $x\Gamma(x) = \Gamma(x+1)$, can be shown to give[112]

$$\begin{aligned}\int_0^1 dx\, x^{N-1}\Big[\frac{1}{1-x}\Big]_+ &= \sum_{k=1}^{N-1}\frac{1}{k} \\ &= -\Big(\int_1^N dk\frac{1}{k} + \lim_{N\to\infty}\Big(\sum_{k=1}^{N-1}\frac{1}{k} - \int_1^N dk\frac{1}{k}\Big) + \mathcal{O}(1/N)\Big) \\ &= -\ln\bar{N} + \mathcal{O}(1/N)\,,\end{aligned} \tag{D.15}$$

[112] After the regulator has been removed.

where $\bar{N} = Ne^{\gamma_E}$.

Particularly useful moments are those of plus distributions with logarithms, see [47]

$$\int_0^1 dx\, x^{N-1}\Big[\frac{\ln(1-x)}{1-x}\Big]_+ = \frac{1}{2}\ln^2\bar{N} + \frac{1}{2}\zeta(2) + \mathcal{O}(1/N)\,, \tag{D.16}$$

$$\int_0^1 dx\, x^{N-1}\Big[\frac{\ln^2(1-x)}{1-x}\Big]_+ = -\frac{1}{3}\ln^3\bar{N} - \zeta 2\ln\bar{N} - \frac{2}{3}\zeta 3 + \mathcal{O}(1/N)\,, \tag{D.17}$$

where $\zeta(y)$ is the Riemann zeta function. In the large N limit, the following behaviour follows

$$\int_0^1 dx x^{N-1}\Big[\frac{\ln^n(1-x)}{(1-x)}\Big]_+ = \frac{(-1)^{n+1}}{n+1}\ln^{n+1}(\bar{N}) + \mathcal{O}(\ln^{n-1}(\bar{N}))\,. \tag{D.18}$$

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