Project 1 - FYS3150

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I. THEORY

We're going to solve the differential equation

$$-u''(x) = f(x), \quad x \in (0,1), \quad u(0) = u(1) = 0. \tag{1}$$

We'll approximate this differential equation by a function $v(x) \approx u(x)$ by the approximation scheme

$$-\frac{-v_{i+1} + v_{i-1} - 2v_i}{h^2} = f_i, \quad i = 1, 2, ..., n, \tag{2}$$

which may be rearranged into

$$2v_i - v_{i+1} - v_{i-1} = f_i h^2 \equiv b_i. \tag{3}$$

From (3) we can write

$$\begin{pmatrix} 2v_1 - v_2 \\ -v_1 + 2v_2 - v_3 \\ \vdots \\ 2v_n - v_{n-1} \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & & & & \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \tag{4}$$

To this end we want to develop a general algorithm to solve the equation above. Suppose we've got a tri-diagonal matrix A and want to decompose it using LU-decomposition such that $A = LU$ where L is a lower-triangular matrix with ones on its diagonal and U is an upper-triangular matrix. Suppose that $A, L, U \in \mathbb{R}^{n \times n}$ such that

$$A = \begin{pmatrix} b_1 & c_1 & 0 & \cdots & \cdots & \cdots \\ a_1 & b_2 & c_2 & 0 & \cdots & \cdots \\ 0 & a_2 & b_3 & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n-2} & b_{n-1} & c_{n-1} \\ 0 & 0 & \cdots & 0 & a_{n-1} & b_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ l_{21} & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ l_{n1} & l_{n2} & \cdots & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & \cdots & u_{nn} \end{pmatrix} = LU. \tag{5}$$

Performing matrix multiplication yields

$$LU = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ l_{21}u_{11} + u_{21} & l_{21}u_{12} + u_{22} & \cdots & l_{21}u_{1n} + u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1}u_{11} + l_{n2}u_{21} + \cdots + u_{n1} & l_{n1}u_{12} + l_{n2}u_{22} + \cdots + u_{n2} & \cdots & l_{n1}u_{1n} + l_{n2}u_{2n} + \cdots + u_{nn} \end{pmatrix}, \tag{6}$$

thus each matrix element of A which we'll denote A_{ij} may be computed by the general formula

$$A_{ij} = u_{ji} + \sum_{k=1}^{i-1} l_{ik} u_{kj}, \tag{7}$$

which yields in combination with (6)

$$b_i = A_{ii} \qquad \text{for} \qquad i = 1, 2, ..., n \tag{8}$$

$$a_i = A_{i+1,i} \qquad \text{for} \qquad i = 1, 2, ..., n-1 \tag{9}$$

$$c_i = A_{i,i+1} \qquad \text{for} \qquad i = 1, 2, ..., n-1 \tag{10}$$

$$A_{ij} = 0 \qquad \text{otherwise} \tag{11}$$

$$A_{ij} = 0 \qquad \text{for} \qquad (i,j) \neq (i,i) \qquad \text{or} \qquad (i,j) \neq (i,i+1) \qquad \text{or} \qquad (i,j) \neq (i+1,i) \tag{12}$$