## Project 1 - FYS3150

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## **METHOD**

We're going to solve the differential equation

$$-u''(x) = f(x), \quad x \in (0,1), \quad u(0) = u(1) = 0. \tag{1}$$

We'll approximate this differential equation by a function  $v(x) \approx u(x)$  by the approximation scheme

$$-\frac{-v_{i+1} + v_{i-1} - 2v_i}{h^2} = f_i, \quad i = 1, 2, ..., n,$$
(2)

which may be rearranged into

$$2v_i - v_{i+1} - v_{i-1} = f_i h^2 \equiv \tilde{b_i}. \tag{3}$$

From (3) we can write

$$\begin{pmatrix} 2v_1 - v_2 \\ -v_1 + 2v_2 - v_3 \\ \vdots \\ 2v_n - v_{n-1} \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & & & & \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \tilde{b_1} \\ \tilde{b_2} \\ \vdots \\ \tilde{b_n} \end{pmatrix}$$
(4)

To this end we will develop an algorithm based on the LU-decomposition A = LU:

$$A = \begin{pmatrix} b_{1} & c_{1} & 0 & \cdots & \cdots & \cdots \\ a_{1} & b_{2} & c_{2} & 0 & \cdots & \cdots \\ 0 & a_{2} & b_{3} & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n-2} & b_{n-1} & c_{n-1} \\ 0 & 0 & \cdots & 0 & a_{n-1} & b_{n} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \ell_{2} & 1 & \cdots & \cdots & 0 \\ 0 & \ell_{3} & 1 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \ell_{n-1} & 1 & 0 \\ 0 & 0 & \cdots & \ell_{n-1} & 1 & 0 \\ 0 & 0 & \cdots & \cdots & \ell_{n} & 1 \end{pmatrix} \begin{pmatrix} d_{1} & u_{1} & \cdots & \cdots & 0 \\ 0 & d_{2} & u_{2} & \cdots & \cdots & 0 \\ 0 & 0 & d_{3} & u_{3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \ell_{n-1} & 1 & 0 \\ 0 & 0 & \cdots & \cdots & \ell_{n-1} & 1 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & d_{n} \end{pmatrix} = LU, \quad (5)$$

and performing matrix multiplication we get

$$LU = \begin{pmatrix} d_1 & u_1 & \cdots & \cdots & \cdots & 0 \\ \ell_2 d_1 & \ell_2 u_1 + d_2 & u_2 & \cdots & \cdots & 0 \\ 0 & \ell_3 d_2 & \ell_3 u_2 + d_3 & u_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \ell_{n-1} d_{n-2} & \ell_{n-1} u_{n-2} + d_{n-1} & u_{n-1} \\ 0 & 0 & \cdots & \ell_n d_{n-1} & \ell_n u_{n-1} + d_n \end{pmatrix},$$
(6)

which yields the following general relations:

$$b_i = d_i, \qquad c_i = u_i, \qquad \text{for} \qquad i = 1, \tag{7}$$

$$\ell_i = \frac{a_{i-1}}{d_{i-1}}, \quad \text{for} \quad 1 < i < n,$$
 (8)

$$d_i = b_i - \ell_i u_{i-1}, \qquad \text{for} \qquad 1 < i < n, \tag{9}$$

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$$d_{i} = b_{i} - \ell_{i} u_{i-1}, \quad \text{for} \quad 1 < i < n,$$

$$\ell_{n} = \frac{a_{n-1}}{d_{n-1}}, \quad d_{n} = b_{n} - \ell_{n} u_{n-1}, \quad \text{for} \quad i = n.$$

$$(10)$$

Now, assuming that  $b_1 = b_2 = \cdots = b_n$  and  $a_1 = c_1$ ,  $a_2 = c_2$ ,  $\cdots$ ,  $a_{n-1} = c_{n-1}$ , we can simplify the relations as