Soft effects in transverse spectra at hadron colliders

Antonia Adler

Masterarbeit in Physik

vorgelegt der

Fakultät für Mathematik, Informatik und Naturwissenschaften
der RWTH Aachen

im März 2015

angefertigt im

Institut für Theoretische Teilchenphysik und Kosmologie

bei

Prof. M. Czakon

Zweitgutachter: Prof. W. Bernreuther

Declaration

I declare that I have developed and written the enclosed Master's thesis completely by myself, and have not used sources or means without declaration in the text. Any thoughts from others or literal quotations are clearly marked. This Master's thesis was not used in the same or in a similar version to achieve an academic grading or is being published elsewhere.

Aachen, March 2015

Contents

1. Introduction

The main purpose of high energy, collider physics is the confirmation of the Standard Model and the search of new physics beyond the Standard Model. Experimental measurements can be performed with high precision and this demands equally precise theoretical predictions. In these calculations one has to face serious problems. A collider experiment involves several scales and this leads to logarithms of these scale ratios which are large. That's why the derivation of a factorization formula is required and one needs a resummation framework for enhanced logarithms.

A way out of these problems is served by the soft-collinear effective theory (SCET). SCET facilitates the derivation of factorization theorems and the resummation of logarithmically enhanced contributions using Renormalization Group techniques [1]. In the calculation of physical observables infrared (IR) divergences appear. These singularities cancel out in final result (KLN theorem). Infrared divergences arise from regions where massless particles are either soft or collinear to other particles. In order to understand the structure of the IR singularities in QCD, one need to study the soft and collinear limit. SCET reproduces these limits and is a tool for studying the infrared divergences of QCD [1]. The IR divergences in QCD are identical to the ultraviolet (UV) divergences in SCET. A theory with UV divergences is preferred, since UV divergences are canceled within the renormalization procedure and the corresponding Renormalization Group Equations (RGEs) are obtained whereby the resummation of enhanced contributions is achieved.

SCET is a top-down effective field theory that can be directly derived from full QCD. It describes the low-energy region. Collinear and soft particles are described by fields in SCET and therefore the same QCD field is represented by several effective theory fields. The result that is obtained when a QCD diagram is expanded in the infrared limit reproduces the result obtained in the framework of SCET. The SCET Lagrangian is derived by expanding the QCD Lagrangian in powers of an expansion parameter that is given by the ratio of the low-energy scale λ_l and the high-energy scale λ_h as $\lambda = \lambda_l/\lambda_h \ll 1$.

One major application of SCET is in collider physics. The initial state radiation and the soft radiation of a collider process can be described in the framework of SCET. In my Master's thesis I discuss the application of the SCET formalism to transverse momentum resummation for the Drell-Yan process and for top quark pair production.

The transverse momentum of the final particles with respect to the beam axis is an interesting differential observable. Especially the region with small transverse momentum is interesting. In this region the cross section is large. In a Drell-Yan production process this cross section can be used to extract the W-boson mass [2, 1]. In top quark pair production the $t\bar{t}$ charge asymmetry depends on the transverse momentum [3]. The $t\bar{t}$ charge asymmetry is studied as an indicator for new physics [4]. Thus it is important to measure this observable. In the region where the transverse momentum is much smaller than the hard scale of the process, the collinear factorization which contains the parton distribution functions (PDFs) can't be applied any more. A different factorization scheme can be proven [5]. This is the so called transverse-momentum-dependent (TMD) factorization scheme which includes the transverse parton distribution functions (TPDFs) that can be calculated in the framework of SCET. In the region where the transverse momentum is much larger than the QCD scale, the TPDFs can be matched onto the PDFs.

The structure of this thesis is as follows. In the starting chapter 2 it is shown how factorization works on parton level in the infrared limit and Wilson lines are introduced. Soft Wilson lines reproduce the soft limit and the main task of collinear Wilson lines is to ensure gauge invariance. In chapter 3 I reproduce the main ideas of SCET and show that SCET reproduces the infrared limit that was calculated in chapter 2. Hereafter, the electromagnetic current operator of SCET is derived and the matching of this operator from QCD onto SCET is performed up to order $\mathcal{O}(\alpha_s)$. This matching calculation gives the Wilson coefficient and the corresponding renormalization group equation. As a last aspect of chapter 3, I show the emergence of rapidity divergences and how they can be regulated. Rapidity divergences are a peculiarity of SCET.

In the final result those divergences have to cancel since they are unphysical. Within the framework of SCET, the factorization formula of the Drell-Yan process can be derived. An important ingredient of such a formula are the transverse parton distribution functions (TPDFs). Since they can be matched onto the PDFs, I first consider the parton distribution functions in chapter 4. In chapter 5 I focus on the TPDFs. Some essential properties of these functions are discussed, I explicitly calculate them in the framework of SCET and perform the matching onto the PDFs up to order $\mathcal{O}(\alpha_s)$. Then all ingredients are known that are necessary to derive a TMD factorization formula. In the subsequent chapter 6 I derive a factorization formula for the Drell-Yan process at small transverse momentum. The soft function is discussed and it is figured out why the soft function does not contribute in this process. After that, the resummation procedure is explained which is performed by the use of renormalization group techniques. Finally, in chapter 7 the factorization formula for top quark pair production at small transverse momentum is derived. Some aspects of the discussion of the Drell-Yan process from chapter 6 can be adopted. Nevertheless, the derivation of the top quark pair production process at small transverse momentum is more difficult due to the heavy quark final states and the additional color structure caused by the top quarks. In this process the soft function contributes to the cross section and in chapter 8 I calculate this function up to NLO. This calculation is the main aspect of my thesis. The soft function of top quark pair production at small transverse momentum is of high importance, since it is the only ingredient in the SCET cross section formula that is just known up to NLO. The NLO soft function describes the soft radiation from the initial and the final states to order $\mathcal{O}(\alpha_s)$. The calculation of the NLO soft function was performed up to $\mathcal{O}(\epsilon^0)$ in [3]. The challenge that arises in the calculation of the NLO soft function are integrals that do not seem to be solvable at a first glance. Especially the integration method that was used in [3] is unsystematic, and inappropriate to calculate even more complicated integrals such as the ones expected at NNLO. That's why I focused on the search of an alternative integration method. So, I tried to calculate the NLO soft function with the Mellin-Barnes method [6]. This method is an improvement compared to the method used in [3], but it is still not satisfactory for even harder integrals. With the derivation of differential equations [7] I figured out a sufficiently powerful method whereby the NLO integrals can be evaluated systematically. On top of that, this method can be used as a starting point for the calculation of the NNLO soft function. Within this method I calculated the bare and renormalized NLO soft functions up to $\mathcal{O}(\epsilon)$. So I enhanced the result stated in [3] for the contribution of order $\mathcal{O}(\epsilon)$. With the knowledge of this additional contribution, one finite NNLO contribution is figured out.

2. Factorization

I start with a definition of factorization. For this I quote John C. Collins. He wrote [8]

> Factorization is the property that some cross-section or amplitude is a product of two (or more) factors and that each factor depends only on physics happening on one momentum (or distance) scale. The process is supposed to involve some large momentum transfer, on a scale Q, and corrections to the factorized form are suppressed by a power of Q. (In general the product is in the sense of a matrix product or of a convolution.)

The necessity of factorization is manifested in multi-scale problems. Top quark pair production at small transverse momentum in hadronic collisions is such a multi-scale problem. The invariant mass of the top quark pair M is a large scale whereas the transverse momentum q_T and the scale of the soft gluon radiation is much smaller. In formulas one has [9]

$$M^2 = (p_3 + p_4)^2 \gg q_T^2 \gg \Lambda_{QCD}^2$$

where $\Lambda_{QCD} \sim 200$ MeV is the QCD hadronization scale below which free partons do not exist. The different scales need to be disentangled because otherwise there are logarithms of the scale ratios and so it is impossible to obtain theoretical predictions for such a process. That's why factorization theorems are necessary [10]. In general, proofs of factorization theorems are very difficult such as the one for the Drell-Yan process [11, 12].

One of the aims of collider physics is to determine the as far unknown properties of particles or even to confirm or disprove their existence. Thus, one is mainly interested in the high-energetic, short-distant processes. But it is difficult to extract these processes since they are interlaced with several other processes such as initial-state radiation, soft interactions, final-state radiation and multiparton interactions [13].

The theoretical description of the collision is getting simpler for inclusive measurements such as $pp \to l^+l^-X$ where one does not restrict the hadronic final state X. In this case the cross section can be factorized as [14]

$$d\sigma = f_i \otimes f_j \otimes H_{ij} + \mathcal{O}\left(\frac{\Lambda_{QCD}^2}{Q^2}\right) \tag{2.1}$$

where f_i denotes a parton distribution function (PDF) which states the probability to get a parton i from the hadron. The perturbative calculable physics of the collision is gathered together in the function H and $\otimes$ denotes a convolution which is given by

$$f(x) \otimes g(x) = \int_x^1 \frac{d\xi}{\xi} f(\xi) \cdot g\left(\frac{x}{\xi}\right) \tag{2.2}$$

In order to identify the high-energetic process in which one is actually interested in, one needs more differential measurements e.g. to separate jets from the hard process. If the scale of the extra measurement is much lower than the high energy scale Q the factorization formula 2.1 does not hold any longer. A new factorization formula sensitive to this low scale encoded in the observable τ is then more complicated and has the generic form [13]

$$\frac{d\sigma}{d\tau} = H \times [J_1 \otimes ... \otimes J_n \otimes S](\tau)$$

where H contains the hard process, the soft function S and the jet functions J_i describe the contributions to the measurement of τ from soft and collinear radiation. The hard process is not sensitive to the low scale.

In my thesis the low scale is identified with the transverse momentum. As already mentioned, physics sensitive to low p_T factorizes from the hard scale. This can be explicitly shown on parton level which is worked out in section 2.2. A process with small p_T can be the radiation of a soft or collinear particle. In the next section 2.1, I introduce a reasonable notation for them.

2.1. The different momentum regions

Throughout my Master's thesis I always come back to momentum regions. Those regions are classified with the help of two light-like vectors n and $\bar{n}$ which are

$$n^\mu = (1,0,0,1)$$
$$\bar{n}^\mu = (1,0,0,-1)$$

They can be identified with the directions of two colliding particles. These vectors satisfy the relations

$$n^2 = 0 = \bar{n}^2, \qquad n \cdot \bar{n} = 2.$$

With the vectors n and $\bar{n}$ any four vector can be decomposed as

$$k^\mu = n \cdot k \frac{\bar{n}^\mu}{2} + \bar{n} \cdot k \frac{n^\mu}{2} + k_\perp^\mu \equiv k^+ \frac{\bar{n}^\mu}{2} + k^- \frac{n^\mu}{2} + k_\perp^\mu$$

The vector k^μ can be written in light-cone coordinates as

$$k^\mu = (k^+, k^-, k_\perp^\mu)$$

with the components

$$k^+ \equiv n \cdot k \quad \text{and} \quad k^- \equiv \bar{n} \cdot k$$

The dot-product of two four-vectors is

$$p \cdot q = \frac{1}{2} p^+ q^- + \frac{1}{2} p^- q^+ + p_\perp q_\perp$$

and the integration measure can be written in light-cone coordinates as

$$d^d k = \frac{1}{2} dk^+ \, dk^- \, d^{d-2} k_\perp.$$

The vectors $k_\mu^\pm$ are defined as

$$k_\mu^+ = (k \cdot n) \frac{\bar{n}_\mu}{2}, \qquad k_\mu^- = (k \cdot \bar{n}) \frac{n_\mu}{2}$$

A necessary condition for an infrared divergence is that some propagators blow up [15]. This corresponds to regions of phase space where massless particles are either soft or collinear to other particles. To see what that means one takes a look at the following propagator

$$\frac{1}{(p+k)^2} = \frac{1}{2p \cdot k} = \frac{1}{2p^0 k^0 (1 - \cos\theta)} \tag{2.3}$$

This propagator blows up if $\cos\theta \to 1$ which means that the particles travel in the same direction (collinear divergence) or that $p^0 \sim 0$ or $k^0 \sim 0$ which means that at least one of the particles is soft.
The properties of soft and collinear momenta can also be described in the context of light-cone coordinates and with the introduction of a power counting parameter λ which is a measure of the dominance or importance of a momentum component or a field. The higher the power in λ, the less important is the

contribution. The power counting idea is one of the key concepts on the way to the construction of an effective field theory. In processes with small p_T one identifies

$$\lambda = \frac{p_T}{Q} \ll 1$$

where Q is a hard scale. Soft and collinear momenta are scaling as

$$p^2 = \lambda^2 Q^2$$

A soft momentum scales uniformly with λ namely

$$p_s^\mu \sim Q(\lambda, \lambda, \lambda)$$

and for a (anti-)collinear momentum one has

$$\begin{aligned} \text{collinear:} \quad & p_c^\mu \sim Q(\lambda^2, 1, \lambda) \\ \text{anti-collinear:} \quad & p_{\bar{c}}^\mu \sim Q(1, \lambda^2, \lambda) \end{aligned}$$

where collinear means that the momentum points in the direction of n^μ ($p_c^\mu \propto n^\mu$) and anti-collinear means $p_{\bar{c}} \propto \bar{n}^\mu$. With this considerations, one can figure out the scaling behavior of the propagator in eq. (2.3). In the case of both momenta p and k are collinear to each other or both soft one has

$$2p \cdot k = p^+ k^- + p^- k^+ + 2p_\perp k_\perp \sim \lambda^2 Q^2$$

and thus the propagator in eq. (2.3) blows up in this limit. A further momentum region is the ultrasoft one which is specified by

$$p_{us} \sim Q(\lambda^2, \lambda^2, \lambda^2)$$

In the next section I show that QCD matrix elements factorize for processes involving soft or collinear radiation at tree-level.

2.2. Hard-Soft-Collinear Factorization

Hard-soft-collinear factorization can be shown by studying the infrared structure of gauge theories [15]. Hard-collinear factorization can be written heuristically as [15]

$$\mathcal{M}_n \overset{p_1||...||p_m}{\cong} \mathbf{Sp}(p_1, ..., p_m) \cdot \mathcal{M}_{n-m}$$

where $\mathcal{M}_n$ indicates an n-external-particle matrix-element, $p_1^\mu ... p_m^\mu$ are the collinear external momenta and $\cong$ indicates that the two sides agree at leading power in λ. The splitting function $\mathbf{Sp}(p_1, ...p_m)$ has no dependence on the non-collinear momenta.
The factorization of soft emissions is described as

$$\mathcal{M}_n \overset{\text{q soft}}{\cong} \epsilon_\mu(q) \mathbf{J}_a^\mu \cdot \mathcal{M}_{n-1}$$

where the soft current $\mathbf{J}_a^\mu$ is an operator acting in color space.
These factorization theorems are shown in the next subsections based on generic examples.

2.2.1. Factorization of collinear emissions

The discussion starts with a single parton radiation which is collinear to one of the outgoing partons. This leads to the Altarelli-Parisi splitting functions. Afterwards the discussion is extended to the case of multiple gluon emissions. In the last part, it is shown that the radiation of a collinear gluon from an anti-collinear quark is also described via a factorized expression which leads to the collinear Wilson line.

Single collinear splitting

Considering a matrix element with n external massless particles $\mathcal{M}_n(q, p_1, ..., p_{n-1})$, where the external momenta are $q, p_1, ..., p_{n-1}$. I will show how $\mathcal{M}$ changes when the momentum q becomes collinear to p_1. These two momenta are assumed to scale as

$$q^\mu \sim p_1^\mu \sim (\lambda^2, 1, \lambda)$$

The parton with momentum q could have been radiated from any of the partons with momentum p_i. This would lead to a propagator as

$$i\frac{\not{q} + \not{p}_i}{(q + p_i)^2 + i\epsilon} = i\frac{\not{q} + \not{p}_i}{2 \cdot q \cdot p_i + i\epsilon}$$

The scaling of the denominator depends on the momentum p_i as

$$q \cdot p_i = \frac{1}{2}\left(q^+ p_i^- + q_- p_i^+\right) + q_\perp p_{i\perp} \sim \begin{cases} \lambda^2 & \text{if } q||p_i \\ 1 & \text{if } \ p_i \text{ anti-collinear or hard} \\ \lambda & \text{if } p_i \text{ soft} \end{cases}$$

One figures out that the most dominant contribution is λ^{-2} which corresponds to the case where the parton with momentum q is radiated from the parton with momentum p_1.
First I consider the case where a quark with momentum p splits into a gluon with momentum p_1 and a quark with momentum q. The vectors are parameterized as [16]

$$\begin{aligned} p_1^\mu &= zp^\mu + k_\perp^\mu - \frac{k_\perp^2 n^\mu}{z2p \cdot n} \approx z \cdot p^\mu \\ q^\mu &= (1-z)p^\mu - k_\perp^\mu - \frac{k_\perp^2 n^\mu}{(1-z)2p \cdot n} \approx (1-z) \cdot p^\mu \end{aligned}$$

where n^μ and p^μ denote the collinear direction. The corresponding Feynman diagram is shown in Fig. 2.1.

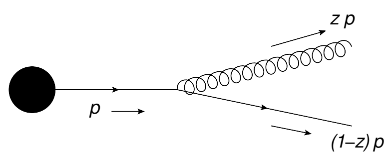


Figure 2.1.: A collinear quark is splitting into a collinear quark and a collinear gluon

The vector $k_\perp$ is orthogonal to n^μ and k^μ and it is assumed to be small. One has

$$p \cdot k_\perp = n \cdot k_\perp = 0$$

For the squared matrix element of this process one has

$$|\mathcal{M}_{g,q,...}|^2 = \mathcal{M}_{q,...}^\dagger \cdot 4\pi\alpha_s C_F \left(\frac{\not{q} + \not{p}_1}{2 \cdot q \cdot p_1}\gamma^\mu \not{q} \gamma^\nu \frac{\not{q} + \not{p}_1}{2 \cdot q \cdot p_1}\right)\left(-g_{\mu\nu} + \frac{p_{1\mu}n_\nu + p_{1\nu}n_\mu}{p_1 \cdot n}\right)\mathcal{M}_{q,...}$$

where I used axial light-cone gauge and the following identities

$$\begin{aligned} \sum_s u^s(q)\bar{u}^s(q) &= \not{q} \\ \sum_{\text{pol}} \epsilon_\mu^*(p_1)\epsilon_\nu(p_1) &= -g_{\mu\nu} + \frac{p_{1\mu}n_\nu + p_{1\nu}n_\mu}{p_1 \cdot n} \\ \sum_a t_{\alpha\beta}^a t_{\beta\gamma}^a &= C_F \delta_{\alpha\beta} \end{aligned}$$

The expression above can be simplified to

$$|\mathcal{M}_{g,q,\ldots}|^2 = \mathcal{M}^\dagger_{q,\ldots} \cdot \frac{4\pi\alpha_s}{s_{12}} \frac{C_F}{2n\cdot p_1} \left(((d-2)n\cdot p_1 + 2n\cdot q)\, \not{p}_1 + (2n\cdot p_1 + 4n\cdot q)\not{q} - 2(p_1\cdot q)\not{n} \right) \mathcal{M}_{q,\ldots}$$

The last term in this expression is negligible in the collinear limit ($k_\perp \to 0$). Namely

$$s_{12} = 2p_1 \cdot q = -\frac{k_\perp^2}{z(1-z)}$$

Replacing

$$\begin{aligned} p_1 &= z\cdot p \\ q &= (1-z)p \end{aligned}$$

one ends up with

$$|\mathcal{M}_{g,q,\ldots}|^2 = \frac{8\pi\alpha_s}{s_{12}} C_F \left(\frac{1+(1-z)^2}{z} - \epsilon z \right) |\mathcal{M}_{q,\ldots}|^2$$

In order to obtain the the quark to quark splitting one just has to change the parameterization to

$$\begin{aligned} p_1 &= (1-z)p \\ q &= zp \end{aligned}$$

One obtains

$$|\mathcal{M}_{g,q,\ldots}|^2 = \frac{8\pi\alpha_s}{s_{12}} C_F \left(\frac{1+z^2}{1-z} - \epsilon(1-z) \right) |\mathcal{M}_{q,\ldots}|^2$$

For a gluon splitting into quark and antiquark one has

$$|\mathcal{M}_{q,\bar{q},..}|^2 = \frac{4\pi\alpha_s}{s_{12}} \mathcal{M}^{\mu\dagger}_{g,\ldots} d_{\mu\rho}(p_1+q)\cdot d_{\sigma\nu}(p_1+q) \frac{T_F}{s_{12}} \mathrm{Tr}\left(\gamma^\rho \not{q} \gamma^\sigma \not{p}_1\right) \mathcal{M}^\nu_{g,..}$$

with

$$d_{\mu\nu}(p) = -g_{\mu\nu} + \frac{p_\mu n_\nu + p_\nu n_\mu}{p\cdot n}$$

The trace can be calculated which gives

$$\mathrm{Tr}\left[\gamma^\rho \not{q} \gamma^\sigma \not{p}_1\right] = 4\left(p_1^\rho q^\sigma + p_1^\sigma q^\rho - p_1\cdot q\; g^{\rho\sigma}\right)$$

the gauge dependent part of the propagator cancels since

$$p^\mu \cdot \mathcal{M}_\mu(p) = 0$$

and so one obtains

$$|\mathcal{M}_{q,\bar{q},..}|^2 = \frac{8\pi\alpha_s}{s_{12}} T_F \mathcal{M}^{\mu\dagger}_{g,\ldots} \left(-g_{\mu\nu} - 4\frac{k_{\perp\mu}k_{\perp\nu}}{s_{12}} \right) \mathcal{M}^\nu_{g,\ldots}$$

Since

$$s_{12} = 2p_1 \cdot q = -\frac{k_\perp^2}{z(1-z)}$$

one ends up with the final expression

$$|\mathcal{M}_{q,\bar{q},..}|^2 = \frac{8\pi\alpha_s}{s_{12}} T_F \mathcal{M}^{\mu\dagger}_{g,...}\mathcal{M}^{\nu}_{g,...}\left(-g_{\mu\nu} + 4z(1-z)\frac{k_{\perp\mu}k_{\perp\nu}}{k_\perp^2}\right)$$

In a similar way one obtains for a gluon splitting in two gluons

$$|\mathcal{M}_{g,g,...}|^2 = \frac{8\pi\alpha_s}{s_{12}} 2C_A \mathcal{M}^{\mu\dagger}_{g,...}\mathcal{M}^{\nu}_{g,...}\left(-g_{\mu\nu}\left(\frac{z}{1-z} + \frac{1-z}{z}\right) - 2z(1-z)(1-\epsilon)\frac{k_{\perp\mu}k_{\perp\nu}}{k_\perp^2}\right)$$

These results are described by a set of functions, called generalized Altarelli-Parisi splitting functions, $\hat{P}^{s,s'}_{a_1a_2}(z, k_\perp; \epsilon)$ [16] with the notation

$$a(p) \to a_1(zp + k_\perp + \mathcal{O}(k_\perp^2)) + a_2((1-z)p - k_\perp + \mathcal{O}(k_\perp^2))$$

$$\begin{aligned}
\hat{P}^{ss'}_{qg}(z, k_\perp; \epsilon) &= \hat{P}^{ss'}_{\bar{q}g}(z, k_\perp; \epsilon) = \delta_{ss'}C_F\left(\frac{1+z^2}{1-z} - \epsilon(1-z)\right)\\
\hat{P}^{ss'}_{gq}(z, k_\perp; \epsilon) &= \hat{P}^{ss'}_{g\bar{q}}(z, k_\perp; \epsilon) = \delta_{ss'}C_F\left(\frac{1+(1-z)^2}{z} - \epsilon z\right)\\
\hat{P}^{\mu\nu}_{q\bar{q}}(z, k_\perp; \epsilon) &= \hat{P}^{\mu\nu}_{\bar{q}q}(z, k_\perp; \epsilon) = T_F\left(-g^{\mu\nu} + 4z(1-z)\frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2}\right)\\
\hat{P}^{\mu\nu}_{gg}(z, k_\perp; \epsilon) &= 2C_A\left(-g^{\mu\nu}\left(\frac{z}{1-z} + \frac{1-z}{z}\right) - 2z(1-z)(1-\epsilon)\frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2}\right)
\end{aligned}$$

These functions depend on on the momentum fraction z, the transverse momentum $k_\perp$ and on the helicity. If the parent parton is a fermion, the splitting function is proportional to the unity matrix in the spin indices. On the other hand, if the parent parton is a gluon, the splitting function contains spin correlations which produce an azimuthal dependence.
The averaged Altarelli Parisi splitting kerns are obtained by [16]

$$\begin{aligned}
\langle P_{qg}(z;\epsilon)\rangle &= \langle P_{\bar{q}g}(z;\epsilon)\rangle = \frac{\delta_{ss'}}{2}\hat{P}^{ss'}_{qg}(z, k_\perp; \epsilon) = C_F\left(\frac{1+z^2}{1-z} - \epsilon(1-z)\right)\\
\langle P_{gq}(z;\epsilon)\rangle &= \langle P_{g\bar{q}}(z;\epsilon)\rangle = \frac{\delta_{ss'}}{2}\hat{P}^{ss'}_{gq}(z, k_\perp; \epsilon) = C_F\left(\frac{1+(1-z)^2}{z} - \epsilon z\right)\\
\langle P_{q\bar{q}}(z;\epsilon)\rangle &= \langle P_{\bar{q}q}(z;\epsilon)\rangle = \frac{d_{\mu\nu}}{d-2}\hat{P}^{\mu\nu}_{q\bar{q}}(z, k_\perp; \epsilon) = T_F\left(1 - \frac{2z(1-z)}{1-\epsilon}\right)\\
\langle P_{gg}(z;\epsilon)\rangle &= \frac{d_{\mu\nu}}{d-2}\hat{P}^{\mu\nu}_{gg}(z, k_\perp; \epsilon) = 2C_A\left(\frac{z}{1-z} + \frac{1-z}{z} + z(1-z)\right)
\end{aligned} \tag{2.4}$$

These contractions are in correspondence with an averaging over the helicity of the splitting parton. The factor $d_{\mu\nu}/(d-2)$ follows from two constraints. First it reduces to unity if contracted with the metric tensor and second it gives zero if contracted with the collinear direction.

Multiple gluon emission

Consider the case of multiple gluon emissions in which all gluons are collinear to some charged particle. One has momenta $q_{a_1}, ..., q_{b_1}$ which are collinear to p_1, momenta $q_{a_2}, ..., q_{b_2}$ collinear to p_2 and so on. The matrix element is to leading order given by the diagram in which each gluon connects directly to the charged particle collinear to it. Thus one has [17]

$$\langle p_1 \cdots p_n; q_{a_1} \cdots q_{b_n}|\,\bar{\Psi}_1 \cdots \Psi_n\,|\Omega\rangle \cong \langle p_1; q_{a_1} \cdots q_{b_1}|\,\bar{\Psi}_1\,|\Omega\rangle \cdots \langle p_n; q_{a_n} \cdots q_{b_n}|\,\Psi_n\,|\Omega\rangle$$

While the left-hand side of this equation is gauge invariant, the right-hand side is not. A gauge-invariant version of this equation is [18]

$$\langle p_1 \cdots p_n; q_{a_1} \cdots q_{b_n} | \, \bar{\Psi}_1 \cdots \Psi_n \, |\Omega\rangle \cong \langle p_1; q_{a_1} \cdots q_{b_1} | \, \bar{\Psi}_1 W_1 \, |\Omega\rangle \cdots \langle p_n; q_{a_n} \cdots q_{b_n} | \, W_n^\dagger \Psi_n \, |\Omega\rangle \tag{2.5}$$

where W_i is a Wilson line which belongs to the particle with momentum p_i. The Wilson line points in some lightlike direction t_i^μ that is not collinear to p_i. It is given by

$$W_i(\infty, x) = P \exp\left(i g_s T^a t_i^\mu \int_0^\infty ds \; A_\mu^a(x^\nu + s t_i^\nu) \right)$$

where P is the path ordering operator such that

$$P\left[A(x)A(x+s\bar{n})\right] = A(x+s\bar{n})A(x) \quad \text{for} \;\; s > 0 \tag{2.6}$$

In the discussion of infrared physics one encounters collinear and anti-collinear particles. The convention for the corresponding Wilson lines is

$$\begin{aligned} \text{collinear particle:} \qquad & W_n(\infty, x) = P \exp\left(i g_s T^a \bar{n}^\mu \int_0^\infty ds \; A_\mu^a(x^\nu + s\bar{n}^\nu) \right) \\ \text{anti-collinear particle:} \qquad & W_{\bar{n}}(\infty, x) = P \exp\left(i g_s T^a n^\mu \int_0^\infty ds \; A_\mu^a(x^\nu + s n^\nu) \right) \end{aligned}$$

The integration limits from 0 to ∞ denote that the corresponding particle is outgoing. For an incoming particle one has to adapt the integration limits from $-\infty$ to 0. The Wilson line and the quark field are taken together to form a new gauge invariant field χ_n, the so called jet field which is defined as

$$\chi_n(x) = W_n^\dagger \Psi_n(x) \tag{2.7}$$

The Wilson line has the following gauge transformation law

$$W_n^\dagger(x, y) \to U(x) W_n^\dagger(x, y) U^\dagger(y)$$

where $U(x)$ is the unitary gauge transformation matrix. Thus the field χ transforms as

$$\begin{aligned} \Rightarrow \chi_n = W_n^\dagger(\infty, x)\Psi(x) \to & \; U(\infty) W_n^\dagger(\infty, x) U^\dagger(x) \cdot U(x)\Psi(x) \\ = & \; U(\infty) W_n^\dagger(\infty, x)\Psi(x) \end{aligned} \tag{2.8}$$

without loss of generality one can assume that the fields do not transform at infinity ($U(\infty) = 1$) [19], thus the field χ_n is gauge invariant:

$$\begin{aligned} \Rightarrow \chi_n = W_n^\dagger(\infty, x)\Psi(x) \to & \; U(\infty) W_n^\dagger(\infty, x) U^\dagger(x) \cdot U(x)\Psi(x) \\ = & \; W_n^\dagger(\infty, x)\Psi(x) \end{aligned}$$

Thus both sides of equation (2.5) are gauge invariant.
So the first benefit of the collinear Wilson line is to renders the matrix element $\langle p_1; q_{a_1} \cdots q_{b_1} | \, \bar{\Psi}_1 W_1 \, |\Omega\rangle$ gauge invariant. Second, the collinear Wilson line accounts for the collinear radiation from all the other particles. This was proven with the helicity spinor formalism in [20]. Although this contribution is subleading one can calculate it and can prove the form of the collinear Wilson line. This is performed in the next section.

The collinear Wilson line

In Fig. 2.2 it is shown a vertex, which is denoted by a circled cross, from which a collinear and an anti-collinear particle come out.
An n-collinear gluon is radiated from the $\bar{n}$-collinear quark. This is shown in the following figure:

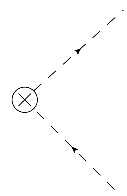


Figure 2.2.: The vertex of an $\bar{n}$- and n-collinear particle

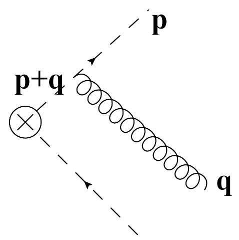


Figure 2.3.: The $\bar{n}$-collinear quark radiates an n-collinear gluon.

The matrix element of this graph is given by:

$$iT_{fi} = \bar{u}(p)(igT^a\gamma^\mu)\epsilon^*_\mu(q)i\frac{\not{p} + \not{q}}{(p+q)^2 + i\epsilon}\Gamma v(\tilde{p})$$

where Γ denotes the vertex. Since the momenta p^μ and q^μ are (anti-)collinear they can be written as

$$\begin{aligned} p^\mu &= \frac{1}{2} n \cdot p \, \bar{n}^\mu \sim \lambda^0 \\ q^\mu &= \frac{1}{2} \bar{n} \cdot q \, n^\mu \sim \lambda^0 \end{aligned}$$

The scaling behaviour of the collinear gluon can be figured out by considering the vacuum expectation value [1]

$$\langle 0| \, T \left[A^\mu(x) A^\nu(x) \right] |0\rangle = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 + i0} e^{-ipx} \left(-g^{\mu\nu} + \xi \frac{p^\mu p^\nu}{p^2} \right)$$

The second term in the backets shows that the gluon field scales as its momentum [21]. Thus for a collinear gluon one can write in leading order

$$A^\mu_n = \frac{n^\mu}{2} \bar{n} \cdot A_n \sim \lambda^0$$

Thus, the gluon polarization vector can be written in leading order of λ as

$$\epsilon^*_\mu(q) = \frac{n^\mu}{2} \bar{n} \cdot \epsilon^*(q) \sim \lambda^0$$

Thus for iT_{fi} one has [22]:

$$
\begin{aligned}
iT_{fi} &= \bar{u}(p)(igT^a\gamma^\mu)\epsilon^*_\mu(q) i\frac{\not{p}+\not{q}}{(p+q)^2+i\epsilon}\Gamma v(\tilde{p}) \\
&= \bar{u}(p)(-gT^a)\bar{n}\cdot\epsilon^*(q)\frac{\not{n}}{2}\cdot\frac{n\cdot p\not{\bar{n}}+\bar{n}\cdot q\not{n}}{2\,n\cdot p\,\bar{n}\cdot q+i\epsilon}\Gamma v(\tilde{p}) \\
&\overset{\not{n}\not{n}=0}{=} \bar{u}(p)(-gT^a)\bar{n}\cdot\epsilon^*(q)\frac{\not{n}}{4}\frac{n\cdot p\not{\bar{n}}}{n\cdot p\,\bar{n}\cdot q+i\epsilon}\Gamma v(\tilde{p}) \\
&= \bar{u}(p)(-gT^a)\bar{n}\cdot\epsilon^*(q)\frac{\not{n}}{4}\frac{\not{\bar{n}}}{\bar{n}\cdot q+i\epsilon}\Gamma v(\tilde{p}) \\
&= \bar{u}(p)\underbrace{\frac{\not{n}\not{\bar{n}}}{4}}_{\bar{P}_-=P_+}(-gT^a)\frac{\bar{n}\cdot\epsilon^*(q)}{\bar{n}\cdot q+i\epsilon}\Gamma v(\tilde{p}) \\
&= \bar{u}_{\bar{n}}(p)\left(-gT^a\frac{\bar{n}\cdot\epsilon^*(q)}{\bar{n}\cdot q+i\epsilon}\right)\Gamma v(\tilde{p})
\end{aligned}
\tag{2.9}
$$

The factor describing the collinear gluon radiation from the anti-collinear particle is independent of the anti-collinear momentum p. It factorizes from the residual matrix element and it scales as

$$
\left(-gT^a\frac{\bar{n}\cdot\epsilon^*(q)}{\bar{n}\cdot q+i\epsilon}\right)\sim\lambda^0 \tag{2.10}
$$

So this radiation is not suppressed in powers of λ and thus radiation like that can be included in arbitrary numbers without additional power suppression [23].
The projector $\bar{P}_- = P_+$ is defined as

$$
\Psi = \underbrace{\frac{\not{n}\not{\bar{n}}}{4}}_{P_+}\Psi + \underbrace{\frac{\not{\bar{n}}\not{n}}{4}}_{P_-}\Psi
$$

It is characterized by the following properties:

$$
\begin{aligned}
\bar{u}(p)\frac{\not{n}\not{\bar{n}}}{4} &= \bar{u}(p)\bar{P}_- = \bar{u}_{\bar{n}}(p) \\
\bar{u}_{\bar{n}}(p)\cdot\bar{P}_- &= \bar{u}_{\bar{n}}(p) \\
\bar{u}_{\bar{n}}(p)\cdot\not{\bar{n}} &= 0 = \bar{u}_{\bar{n}}(p)P_-
\end{aligned}
\tag{2.11}
$$

The last identity in (2.11) can be recognized as the leading term of the Dirac equation $\bar{u}(p)\cdot\not{p}=0$ when expanded in the collinear limit.
Since the factor that describes the gluon radiation is independent of the momentum of the particle it was radiated from, one can graphically interpret this process as an n-collinear gluon coming out of the vertex as shown in figure 2.4.

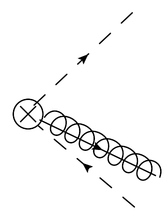


Figure 2.4.: The vertex of an $\bar{n}$- and n-collinear quark with a collinear gluon that is radiated from the anti-collinear quark

Because there was no power suppression of this process, it is reasonable to consider the situation where the

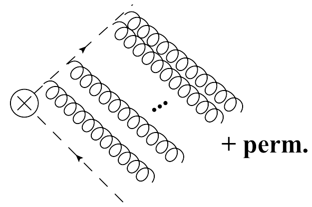


Figure 2.5.: Multiple gluon radiation

$\bar{n}$-collinear quark emits multiple n-collinear gluons as it is shown in figure 2.5.
The generalization from one gluon emission to m gluon emissions with momenta q_1,...q_m yields [22, 23]

$$\bar{u}_{\bar{n}}(p) \sum_{perm} \frac{(-g)^m}{m!} \left(\frac{\bar{n} \cdot A_{q_1} \cdots \bar{n} \cdot A_{q_m}}{(\bar{n} \cdot q_1)(\bar{n}(q_1 + q_2)) \cdots (\bar{n}(\sum_{i=1}^m q_i))} \right) \Gamma v(\tilde{p})$$

The sum over the permutations of the momenta accounts for the several multiple diagrams emerging from crossed gluon lines. The factor $m!$ accounts for the fact that all m gluons are identical. Additionally one can sum over the number of gluon emissions m. So the complete current is:

$$J = \bar{u}_{\bar{n}}(p) W_n \Gamma v(\tilde{p})$$

where W_n is the collinear Wilson line in momentum space [23]

$$W_n = \sum_m \sum_{perm} \frac{(-g)^m}{m!} \left(\frac{\bar{n} \cdot A_n(q_1) \cdots \bar{n} \cdot A_n(q_m)}{(\bar{n} \cdot q_1)(\bar{n}(q_1 + q_2)) \cdots (\bar{n}(\sum_{i=1}^m q_i))} \right)$$

A_n denotes an n-collinear gluon field, explicitly it is given by $A_n = A_n^c T^c$, where the index c denotes the color of the gluon. In position space the collinear Wilson line is given by

$$W_n(\infty, 0) = P \exp \left(ig \int_0^\infty ds\ \bar{n} \cdot A_n(\bar{n}s) \right) \tag{2.12}$$

The concept of the Wilson line is graphically shown in Fig. 2.6. In the left graph a multiple n-collinear gluon emission is shown. In the middle the factorized form of the left picture is depicted and in the right picture the factorized multiple gluon emission is replaced by an n-collinear Wilson line denoted by W. This is the conceptual idea of a Wilson line. The collinear Wilson line accounts for the collinear radiation from all other particles that are not collinear [15] and it does not point in the collinear direction.

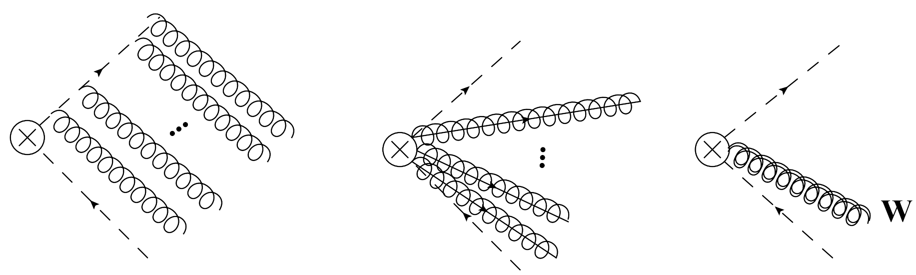


Figure 2.6.: Multiple gluon radiation

The adjoint anti-collinear Wilson line is given by

$$W_{\bar{n}}^\dagger(\infty, x) = \bar{P} \exp \left(-ig_s T^a n^\mu \int_0^\infty ds\ A^a_{\bar{c}\mu}(x^\nu + sn^\nu) \right)$$

where the operator $\bar{P}$ denotes anti-pathordering.

2.2.2. Factorization of soft emissions

In this section I will show that the emission (and absorption) of soft partons factorizes from the residual matrix element. In QCD, the coupling strength is large for an interaction with soft particles. So in scattering processes there will always be soft parton radiation and absorption. I will show that the soft interactions can be described via soft Wilson line operators. A soft Wilson line operator bears the color change of the particle it belongs to due to soft interactions. Additionally the soft Wilson line carries the information of the direction of its parent particle. With a Wilson line operator arbitrary many soft interactions can be described.

Single soft radiation

The discussion starts with an incoming quark radiating a soft gluon. This is shown in Fig. 2.7.

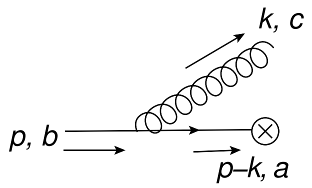


Figure 2.7.: Soft gluon-radiation from an incoming quark line

With Feynman rules one obtains

$$i\frac{\not{p}-\not{k}}{(p-k)^2+i\eta}\left(ig\gamma^\mu T^c_{ab}\right)u(p)\epsilon^*_\mu(k) = gT^c_{ab}\frac{p^\mu}{p\cdot k - i\eta}\epsilon^*_\mu(k)u(p) \tag{2.13}$$

where one used that $k^\mu \approx 0$, $\not{p}\gamma^\mu = 2p^\mu - \gamma^\mu\not{p}$ and $\not{p}\cdot u(p) = 0$. Thus, the soft emission of a gluon is just described by a prefactor in front of the spinor. The soft emission factorizes from the rest of the amplitude. This soft limit is called eikonal approximation. Since the polarization state of the gluon scales as its soft momentum [21], one obtains

$$gT^c_{ab}\frac{p^\mu}{p\cdot k - i\eta}\epsilon^*_\mu(k)u(p) \sim \frac{\lambda}{\lambda} \sim \lambda^0$$

So the radiation of soft gluons is not power suppressed.
In terms of the color space formalism (see below), a general amplitude in which one soft gluon with momentum q and color index c is radiated from one of the external legs with momentum p_i can be written as [24]

$$\langle c|\mathcal{M}(q,p_1,...,p_n)\rangle \cong\ g\ \epsilon^*_\mu(q)\cdot J^{c,\mu}(q)\cdot|\mathcal{M}(p_1,...,p_n)\rangle$$

With the eikonal current

$$\mathbf{J}^{c,\mu} = \sum_{i=1}^{n}\mathbf{T}^c_i\frac{p_i^\mu}{p_i\cdot q}$$

which describes the emission of a soft gluon from one of the external legs. The eikonal current is an operator acting in color space. Its action onto the color space is defined by [24]

$$\langle a,c_1,...,c_i,...,c_m|\,\mathbf{T}_i\,|b_1,...,b_i,...,b_m\rangle = \delta_{c_1,b_1}...T^a_{c_ib_i}...\delta_{c_m,b_m}$$

The current is conserved

$$q_\mu\mathbf{J}^{c,\mu}(q) = \sum_i\ \mathbf{T}^c_i = 0$$

Thus, the amplitude squared and summed over the polarizations of the soft gluon is (in axial gauge) given by

$$\begin{aligned}|\mathcal{M}(q,p_1,...,p_n)|^2 &\cong g^2 \left\langle \mathcal{M}^\dagger(p_1,...,p_n)\right| J^{c,\nu} \cdot \left(-g_{\mu\nu} + \frac{q_\mu n_\nu + q_\nu n_\mu}{q\cdot n}\right) \delta_{ac}\, J^{a,\mu} \left|\mathcal{M}(p_1,...,p_n)\right\rangle \\ &= -g^2 \left\langle \mathcal{M}^\dagger(p_1,...,p_n)\right| J^{c,\mu} \cdot J^c_\mu \left|\mathcal{M}(p_1,...,p_n)\right\rangle\end{aligned}$$

The squared amplitude does not factorize completely due to the color structure which leads to color correlations.

The soft Wilson line

On the other hand, soft radiation can be described by a Wilson line operator [20]. The light-like Wilson line for an incoming particle in the representation R of SU(3) with generators $\mathbf{T}^{(R)}$ is [17, 25, 26]

$$S_n^{(R)}(x,-\infty) = P\,\exp\left(ig_s\int_{-\infty}^0 dt\, n^\mu A_\mu^c(x+tn)\,\mathbf{T}^{(R)c}\, e^{\eta t}\right)$$

This operator describes the soft radiation from an incoming particle which moves in the n-direction. The integration limits going from $-\infty$ to 0 are indicating that the quark is incoming. The exponential factor $e^{\eta t}$ ensures the convergence at $t=-\infty$. η is assumed to be small. P is the path ordering operator which already appeared in the collinear Wilson lines and was defined in eq. (2.6). The soft Wilson line for an outgoing particle which travels in the direction v in the representation R is defined by [25]

$$S_v^{(R)\dagger}(\infty,x) = P\,\exp\left(ig_s\int_0^{\infty} dt\, v^\mu A_\mu^c(x+vt)\,\mathbf{T}^{(R)c}\, e^{-\eta t}\right)$$

The representation R for a quark is indicated by $\mathbf{3}$, for an antiquark by $\bar{\mathbf{3}}$, and for a gluon by $\mathbf{8}$. The corresponding color generators are given by

$$(\mathbf{T}^{(\mathbf{3})\,a})_{bc} = T^a_{bc}, \qquad (\mathbf{T}^{(\bar{\mathbf{3}})\,a})_{bc} = -T^a_{cb}, \qquad (\mathbf{T}^{(\mathbf{8})\,a})_{bc} = -if^{abc} \tag{2.14}$$

Thus, a soft gluon radiation from an incoming quark is given by the following object

$$\begin{aligned}\langle g(k)|\left[S_p^{\mathbf{3}}(0,-\infty)\right]_{ab}|0\rangle &= \langle 0|\,\alpha(k)\,\left[P\,\exp\left(ig_s\int_{-\infty}^0 dt\,p^\mu A^c_\mu(tp)T^c\,e^{\eta t}\right)\right]_{ab}|0\rangle \\ &= \langle 0|\,\alpha(k)\; ig_s\int_{-\infty}^0 dt\,p^\mu A^c_\mu(tp)T^c_{ab}\,e^{\eta t}\,|0\rangle \\ &= igT^c_{ab}\int_{-\infty}^0 dt e^{it(kp-i\eta)}p^\mu\epsilon^*_\mu(k) \\ &= igT^c_{ab}\frac{p^\mu\epsilon^*_\mu(k)}{i\,(k\cdot p-i\eta)} = g\;T^c_{ab}\;\frac{p^\mu\epsilon^*_\mu(k)}{k\cdot p-i\eta}\end{aligned}$$

So one ends up with the same result as in eq. (2.13). As the collinear Wilson line, the soft Wilson line is not power suppressed in λ. The same considerations as for the incoming quark can be done for all variations of incoming and outgoing quarks and antiquarks emitting and absorbing soft gluons which can be looked up in the appendix.

An incoming gluon radiating a soft gluon is shown in 2.8. One obtains with Feynman rules

$$\begin{aligned}&-\frac{i}{(p-k)^2+i\eta}g_{\mu_3\nu}(-gf_{cab})\Big((p+k)^{\mu_3}g^{\mu_1\mu_2}+(p-2k)^{\mu_1}g^{\mu_2\mu_3}+(k-2p)^{\mu_2}g^{\mu_1\mu_3}\Big)\epsilon^*_{\mu_2}(k)\epsilon_{\mu_1}(p) \\ &= -g\frac{1}{2pk-i\eta}(-if_{cab})\Big(-2k\epsilon(p)\cdot\epsilon^*_\nu(k)-2p\epsilon^*(k)\cdot\epsilon_\nu(p)+(p+k)_\nu\;\epsilon^*(k)\epsilon(p)\Big) \\ &\overset{k\to 0}{=} -g\frac{1}{2pk-i\eta}(-if_{cab})\Big(-2p\epsilon^*(k)\cdot\epsilon_\nu(p)+p_\nu\;\epsilon^*(k)\epsilon(p)\Big)\end{aligned}$$

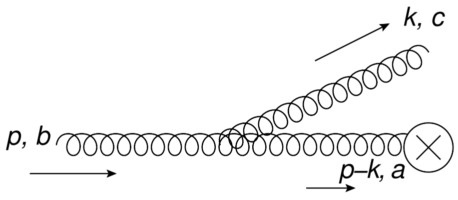


Figure 2.8.: Soft gluon-radiation from an incoming gluon line

where the transversality of the incoming and outgoing gluon were used ($k \cdot \epsilon^*(k) = 0$ and $p \cdot \epsilon(p) = 0$). The second term in the brackets, which is the longitudinal part, vanishes when multiplied with the scattering amplitude (which follows from BRST symmetry). So it remains

$$g\frac{1}{2pk - i\eta}(-if_{cab})\Big(2p\epsilon^*(k) \cdot \epsilon_\nu(p)\Big) = g\frac{1}{pk - i\eta}(-if_{cab})\Big(p\epsilon^*(k)\Big)\epsilon_\nu(p)$$

and this result exactly corresponds to the result given in equation (2.13) with the exchange

$$u(p) \rightarrow \epsilon_\nu(p)$$
$$T^c_{ab} \rightarrow -if_{cab}$$

This replacement is conform with the color generators of the Wilson lines given in eq. (2.14). Therefore, the soft radiation from a gluon line is as well described by a soft Wilson line with an appropriate change of the color generator.

Soft emission to $\mathcal{O}(\alpha_s^2)$

The correspondance between the Feynman rules in the soft limit and the soft function can also be shown to higher orders. As an example one considers the Feynman diagram shown in figure (2.9).

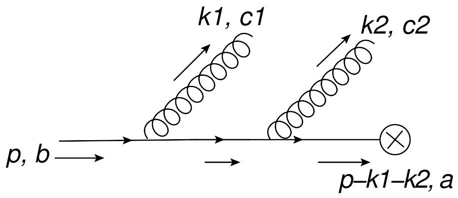


Figure 2.9.: Soft gluon-radiation from an incoming gluon line

With the Feynman rules in the soft limit one obtains

$$\mathcal{M} = g^2 u_a(p)\epsilon^*_{\nu_1}(c_1, k_1) \cdot \epsilon^*_{\nu_2}(c_2, k_2)\frac{1}{p(k_1 + k_2) - i\epsilon}p^{\nu_1}p^{\nu_2}T^{c_2}_{ac}T^{c_1}_{cb}\frac{1}{pk_1 - i\epsilon} \tag{2.15}$$

This can be compared with the expression resulting from the soft Wilson line

$$\begin{aligned}
\langle g_{c_1}(k_1)g_{c_2}(k_2)| \left[S^{\mathbf{3}}_p(0, -\infty)\right]_{ab}|0\rangle &= \langle 0|\,\alpha(c_1, k_1)\alpha(c_2, k_2)\,\left[P\exp\left(ig_s\int_{-\infty}^0 dt\, p^\mu A^c_\mu(tp)T^c\, e^{\eta t}\right)\right]_{ab}|0\rangle \\
&= -g^2\,\langle 0|\,\alpha(c_1, k_1)\alpha(c_2, k_2)\;P\left[\int_{-\infty}^0\int_{-\infty}^0 ds\;dt\;p^\mu p^\nu A^c_\mu(tp)T^c A^a_\nu(sp)T^a\;e^{\eta_2 t}\;e^{\eta_1 s}\right]_{ab}|0\rangle \\
&= -g^2 p^\mu p^\nu \cdot \epsilon^*_\mu(c_1, k_1) \cdot \epsilon_\nu(c_2, k_2)P\left[\int_{-\infty}^0\int_{-\infty}^0 ds\;dt\;T^{c_1}T^{c_2}e^{is(pk_1 - i\eta_1)}e^{it(pk_2 - i\eta_2)}\right]_{ab}
\end{aligned}$$

Assuming that $t \geq s$ which means that the gluon with momentum k_1 and color c_1 is radiated before the gluon with index 2 (which corresponds to the Feynman diagram in Fig 2.9) one obtains

$$= -g^2 p^\mu p^\nu \cdot \epsilon^*_\mu(c_1, k_1) \cdot \epsilon_\nu(c_2, k_2)T^{c_2}_{ac}T^{c_1}_{cb}\int_{-\infty}^0 dt\int_{-\infty}^0 ds\;\theta(t - s)\;e^{is(pk_1 - i\eta_1)}e^{it(pk_2 - i\eta_2)}$$

The Heaviside function can be written as

$$\theta(t-s) = \lim_{\eta \to 0} \int \frac{d\alpha}{2\pi i} e^{i\alpha(t-s)} \frac{1}{\alpha - i\eta}$$

Integrating over s and t leads to

$$-g^2 p^\mu p^\nu \cdot \epsilon^*_\mu(c_1, k_1) \cdot \epsilon_\nu(c_2, k_2) T^{c_2}_{ac} T^{c_1}_{cb} \int \frac{d\alpha}{2\pi i} \frac{1}{i(pk_1 - \alpha - i\eta_1)} \frac{1}{i(pk_2 + \alpha - i\eta_2)} \frac{1}{\alpha - i\eta}$$

This can be solved using Cauchy's theorem and this gives

$$g^2 p^\mu p^\nu \cdot \epsilon^*_\mu(c_1, k_1) \cdot \epsilon_\nu(c_2, k_2) T^{c_2}_{ac} T^{c_1}_{cb} \cdot \frac{1}{p(k_1 + k_2) - i\epsilon} \cdot \frac{1}{p_1 k - i\tilde{\epsilon}}$$

And this exactly reproduces equation (2.15).
The soft Wilson line offers the advantage that it describes the soft interaction to the required order in α_s. Thus the soft Wilson line is a convenient and shorthand notation for the soft interaction.

2.3. Summary

As it was just shown, processes with soft and collinear radiation factorize. Such a process can be written at leading power as [15]

$$\langle X_1; \cdots; X_m; X_s | \bar{\Psi}_1 \cdots \Psi_m |\Omega\rangle \cong \langle X_1 | \bar{\Psi}_1 W_1 |\Omega\rangle \cdots \langle X_m | W^\dagger_m \Psi_m |\Omega\rangle \langle X_s | S_1 \cdots S^\dagger_m |\Omega\rangle \tag{2.16}$$

where X_j contains gluons in the direction collinear to the jth jet and X_s contains the soft gluons. In this factorization formula is encoded, that the only relevant interactions at leading power are among particles going in the same direction and that soft interactions completely decouple from the rest. This factorization formula is proven in [20].
In this chapter, the necessity of factorization was mentioned and it was shown that matrix elements factorize for processes involving infrared radiation. For the description of soft and collinear radiation, the concept of Wilson lines was introduced. In the next chapter the framework and main ideas of the soft-collinear effective theory (SCET) will be introduced which are closely related to the considerations made in this chapter.

3. Soft-collinear effective theory (SCET)

The infrared structure of QCD can be studied by just expanding full QCD in the soft and collinear limit or by deriving an effective theory that reproduces these limits. The use of an effective Lagrangian provides advantages such that power corrections and the consequences of gauge invariance can be studied more easily [1]. Furthermore one has a systematic and convenient language. In this chapter a short introduction in this effective theory, called soft-collinear effective theory (SCET), is reproduced.
In the last chapter I considered the infrared limit. An important aspect of the infrared limit is the factorization of QCD in this limit. This fact is used to build SCET that reproduces QCD in the infrared limit and the associated factorization. In the framework of SCET, hard-soft-collinear decoupling is completely transparent. Therefore factorization theorems can be derived more simply than in full QCD, such as the factorization proof for Drell-Yan which is performed in SCET in [10, 27]. An other aspect is the resummation of enhanced logarithms which is achieved in SCET by renormalization group techniques [15].
In order to obtain factorization, new effective theory fields are introduced in SCET. Each relevant momentum region is represented by a different field and the original QCD fields are given by a sum of the effective theory fields. This replacement is accomplished and then the QCD Lagrangian is expanded to leading power in λ. The resulting effective theory has Feynman rules that are more complicated than those of QCD. These rules simplify after a field redefinition which moves the soft-collinear interactions from the Lagrangian into the operators and so one achieves a decoupling of the different sectors. In the first part of this chapter I reproduce the motivation for the representation of one QCD field by a sum of several effective fields. Then, the definitions of the SCET fields are given and I point out the steps which are leading to an effective Lagrangian. Hereafter, I repeat the arguments that are leading to the electromagnetic current operator in SCET and subsequently the matching of the current from QCD to SCET is performed which gives the Wilson coefficient.
In SCET, a further regularization scheme needs to be applied due to rapidity divergences [28, 29, 30](or also called light-cone divergences). The additional regulator prohibits an overlapping between soft and collinear modes. In many cases, SCET integrals are not well-defined without the additional regularization. An example for the emergence of rapidity divergences is shown in the last section of this chapter. There are multiple ways of regulating the rapidity divergences. One possibility is to employ off-shell states. Other possibilities are analytic regularization [31] and delta regularization [32]. Off-shell states are used in the matching calculation and in the last section analytic regularization is shown.

3.1. The conceptual idea

In order to derive an effective Lagrangian one has to check which momentum modes arise in a given problem and then take all relevant ones into account. This is done by writing a QCD field as a sum of effective fields with the appropriate scaling of their momenta. For example a QCD quark field can be written as

$$\Psi(x) = \Psi_c(x) + \Psi_{\bar{c}}(x) + \Psi_s(x) \tag{3.1}$$

where the momentum corresponding to the field Ψ_i scales as

$$\begin{aligned} p_s &\sim (\lambda, \lambda, \lambda)Q \\ p_c &\sim (\lambda^2, 1, \lambda)Q \\ p_{\bar{c}} &\sim (1, \lambda^2, \lambda)Q \end{aligned}$$

The motivation of eq. (3.1) can be elucidated by looking at a Feynman integral which can be expanded in the different momentum regions [1, 33]. The expansion of a Feynman integral is realized according to the following rules quoted from [1]

- Identify all regions of the integrand which lead to singularities in the limit under consideration.
- Expand the integrand in each region and integrate each expansion over the full phase space.
- Add the result of the integrations over the different regions to obtain the expansion of the original full integral.

For demonstration I reproduce the argumentation for the Sudakov problem in the scalar theory from [1]. The Feynman diagram of interest is shown in fig. 3.1.

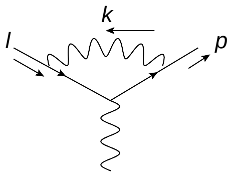


Figure 3.1.: One loop vertex correction.

Accordingly, the following loop integral needs to be calculated in dimensional regularization

$$I = i\pi^{-d/2}\mu^{4-d}\int d^dk \frac{1}{(k^2+i\epsilon)\cdot[(k+l)^2+i\epsilon]\cdot[(k+p)^2+i\epsilon]}$$

The following notation is introduced

$$P^2 \equiv -p^2 - i\epsilon, \qquad L^2 \equiv -l^2 - i\epsilon, \qquad Q^2 \equiv -(p-l)^2 - i\epsilon$$

This integral ought to be calculated in the limit $P^2 \sim L^2 \ll Q^2$. Thus the expansion parameter is stated as

$$\lambda^2 \sim \frac{P^2}{Q^2} \sim \frac{L^2}{Q^2}$$

The momenta satisfy the following scaling

$$\begin{aligned} p^\mu \sim& (\lambda^2, 1, \lambda)Q \qquad \text{and} \qquad l^\mu \sim (1, \lambda^2, \lambda)Q \\ &\Rightarrow p\cdot l \sim Q^2 + \mathcal{O}(\lambda^2) \gg l^2 \sim p^2 \end{aligned}$$

Following the steps described above one has to consider the following scaling behaviors of k

- k is a hard momentum, scaling as $k^\mu \sim (1,1,1)Q$
- k is collinear to p, scaling as $k^\mu \sim (\lambda^2, 1, \lambda)Q$
- k is collinear to l, scaling as $k^\mu \sim (1, \lambda^2, \lambda)Q$
- k is ultrasoft, scaling as $k^\mu \sim (\lambda^2, \lambda^2, \lambda^2)Q$

Next, one expands the integrand in the corresponding momentum limit of k. E.g. for k scaling collinear to p one has

$$(k+l)^2 = k^2 + 2kl + l^2 = k^2 + (k_+l_- + k_-l_+ + 2k_\perp l_\perp) + l^2 = k_-l_+ + \mathcal{O}(\lambda^2)$$

whereas

$$(p+l)^2 = \mathcal{O}(\lambda^2)$$

and thus all terms of this expression have to be taken into account. So it is to calculate

$$
\begin{aligned}
I_c &= i\pi^{-d/2}\mu^{4-d}\int d^dk\frac{1}{(k^2+i\epsilon)\cdot[k_-l_+ + i\epsilon]\cdot[(k+p)^2+i\epsilon]}\\
&= \frac{\Gamma(1+\epsilon)}{Q^2}\left(-\frac{1}{\epsilon^2}-\frac{1}{\epsilon}\ln\frac{\mu^2}{P^2}-\frac{1}{2}\ln^2\frac{\mu^2}{P^2}+\frac{\pi^2}{6}+\mathcal{O}(\lambda)\right)
\end{aligned}
$$

The same can be done for the other scaling limits of k. One obtains [1]

$$
\begin{aligned}
I_h &= \frac{\Gamma(1+\epsilon)}{Q^2}\left(\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\ln\frac{\mu^2}{Q^2}+\frac{1}{2}\ln^2\frac{\mu^2}{Q^2}-\frac{\pi^2}{6}+\mathcal{O}(\lambda)\right)\\
I_c &= \frac{\Gamma(1+\epsilon)}{Q^2}\left(-\frac{1}{\epsilon^2}-\frac{1}{\epsilon}\ln\frac{\mu^2}{P^2}-\frac{1}{2}\ln^2\frac{\mu^2}{P^2}+\frac{\pi^2}{6}+\mathcal{O}(\lambda)\right)\\
I_{\bar c} &= \frac{\Gamma(1+\epsilon)}{Q^2}\left(-\frac{1}{\epsilon^2}-\frac{1}{\epsilon}\ln\frac{\mu^2}{L^2}-\frac{1}{2}\ln^2\frac{\mu^2}{L^2}+\frac{\pi^2}{6}+\mathcal{O}(\lambda)\right)\\
I_{us} &= \frac{\Gamma(1+\epsilon)}{Q^2}\left(\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\ln\frac{\mu^2Q^2}{P^2L^2}+\frac{1}{2}\ln^2\frac{\mu^2Q^2}{P^2L^2}+\frac{\pi^2}{6}+\mathcal{O}(\lambda)\right)
\end{aligned}
$$

where the integral I_{us} only depends on a new soft scale $\Lambda^2_{soft}\sim P^2L^2/Q^2$.
As a last step the different contributions have to be added and one obtains

$$
I = I_h + I_c + I_{\bar c} + I_{us} = \frac{1}{Q^2}\left(\ln\frac{Q^2}{L^2}\ln\frac{Q^2}{P^2}+\frac{\pi^2}{3}+\mathcal{O}(\lambda)\right) \tag{3.2}
$$

and this result coincides with the result obtained by evaluating directly I and then expanding in the limit $\lambda\to 0$. Apparently, the infrared divergences of the hard region cancels against the ultraviolet divergences of the soft and collinear regions.
A further aspect is the overlapping of the soft and collinear regions. These contributions need to be subtracted in order to avoid double counting. The overlapping terms are called zero-bin contributions [15, 34] and they need to be subtracted from the final result. They are obtained by expanding the collinear integrand around the soft limit. Since the collinear integral only depends on a single scale P^2, one gets a scaleless integral performing a further expansion. Scaleless integrals vanish in dimensional regularization. Thus, in this problem zero-bin subtraction does not need to be considered.
The aim of this section was to show that a Feynman integral in dimensional regularization can be obtained in two different ways. First, the integral can be evaluated directly and then the limit one is interested in is taken and second, the integrand can be expanded before the integration is performed and then the different contributions are summed up. The second possibility corresponds to a kind of factorization, since in the expanded integrals I_i (with $i = us, c, \bar c, h$) only one scale is left over. So one has succeeded in separating the different scales. This procedure can be identified with the notation of eq. (3.1).

3.2. The SCET fields

In order to derive the effective Lagrangian of SCET, the QCD quark and gluon fields are written as a sum of a collinear field, an anti-collinear field and an ultrasoft field as

$$
\begin{aligned}
\psi(x) &\to \psi_c(x)+\psi_{\bar c}(x)+\psi_{us}(x)\\
A^\mu(x) &\to A^\mu_c(x)+A^\mu_{\bar c}(x)+A^\mu_{us}(x)
\end{aligned}
$$

This sum-notation is very similar to the identity stated in eq. (3.2). The hard region is covered by the Wilson coefficient that appears as prefactors in the Lagrangian and it is adjusted by the comparison of the full theory with the effective one. This procedure is called matching.
The (anti-)collinear quark field is further splitted as

$$
\begin{aligned}
\psi_c(x) &= \xi_c(x)+\eta_c(x)\\
\psi_{\bar c}(x) &= \xi_{\bar c}(x)+\eta_{\bar c}(x)
\end{aligned}
$$

This splitting is performed by employing projection operators. These are

$$\mathbf{1} = \frac{\not{n}\not{\bar{n}}}{4} + \frac{\not{\bar{n}}\not{n}}{4} \equiv P_+ + P_-$$

They fulfill the identities

$$P_+^2 = P_+, \qquad P_-^2 = P_-, \qquad P_+P_- = 0 = P_-P_+$$

The projector P_+ already appeared in the derivation of the collinear Wilson line in eq. (2.9). The fields ξ and η can be written as

$$\begin{aligned}
\xi_c(x) &= P_+\psi_c(x) = \frac{\not{n}\not{\bar{n}}}{4}\psi_c(x) \quad \Rightarrow \quad \not{n}\xi_c(x) = 0 \\
\eta_c(x) &= P_-\psi_c(x) = \frac{\not{\bar{n}}\not{n}}{4}\psi_c(x) \quad \Rightarrow \quad \not{\bar{n}}\eta_c(x) = 0
\end{aligned}$$

where the equations

$$\not{n}\xi_c(x) = 0 \quad \text{and} \quad \not{\bar{n}}\eta_c(x) = 0$$

are the analog of the momentum space Dirac equations for massless particle $\not{p}u(p,s) = 0$. For the anti-collinear fields one has

$$\begin{aligned}
\xi_{\bar{c}}(x) &= P_-\psi_{\bar{c}}(x) = \frac{\not{\bar{n}}\not{n}}{4}\psi_{\bar{c}}(x) \quad \Rightarrow \quad \not{\bar{n}}\xi_{\bar{c}} = 0 \\
\eta_{\bar{c}}(x) &= P_+\psi_{\bar{c}}(x) = \frac{\not{n}\not{\bar{n}}}{4}\psi_{\bar{c}}(x) \quad \Rightarrow \quad \not{n}\eta_{\bar{c}} = 0
\end{aligned}$$

In order to derive the relevant terms of the Lagrangian one needs to know how these new fields are scaling in λ. For the collinear quark field one has [1, 21]

$$\begin{aligned}
\langle 0| T\{\xi_c(x)\bar{\xi}_c(0)\} |0\rangle &= \frac{\not{n}\not{\bar{n}}}{4} \langle 0| T\{\psi_c(x)\bar{\psi}_c(0)\} |0\rangle \frac{\not{\bar{n}}\not{n}}{4} \\
&= \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2+i\epsilon}e^{-ipx}\frac{\not{n}\not{\bar{n}}}{4}\not{p}\frac{\not{\bar{n}}\not{n}}{4} \sim \lambda^4\frac{1}{\lambda^2} = \lambda^2
\end{aligned}$$

and therefore $\xi_c(x) \sim \lambda$. Similarly one finds [1, 21]

$$\begin{aligned}
\eta_c(x) &\sim \lambda^2 \\
\xi_{\bar{c}} &\sim \lambda \\
\eta_{\bar{c}} &\sim \lambda^2 \\
\psi_s &\sim \lambda^3 \\
A_s^\mu(x) &\sim p_s^\mu \\
A_c^\mu(x) &\sim p_c^\mu \\
A_{\bar{c}}^\mu(x) &\sim p_{\bar{c}}^\mu
\end{aligned}$$

Inserting the SCET fields into the QCD Lagrangian, taking the scaling of the fields into account and applying some simplifications one ends up with a nearly decoupled Lagrangian. In the leading order SCET Lagrangian, the interactions of soft gluon fields with collinear quark fields are described by eikonal vertices [35], [1]

$$\mathcal{L}_{int} = \bar{\xi}_n(x)\frac{\not{\bar{n}}}{2}g\cdot n\cdot A_s(x)\xi_n(x) + \bar{\xi}_{\bar{n}}(x)\frac{\not{n}}{2}g\cdot\bar{n}\cdot A_s(x)\xi_{\bar{n}}(x)$$

Additionally, there is a similar term for the interaction between collinear and soft gluons. All these eikonal interactions can be absorbed into Wilson lines via the field redefinitions [36, 37]

$$\xi_n^a(x) \to [S_n(x)]^{ab} \xi_n^{b\,(0)}, \qquad \xi_{\bar{\mathrm{n}}}(x) \to [S_{\bar{\mathrm{n}}}(x)]^{ab}\, \xi_{\bar{\mathrm{n}}}^{b(0)}(x) \tag{3.3}$$

with the soft Wilson lines $S_i(x)$. The fact that a soft Wilson line reproduces the soft limit was shown in section 2.2.2. A similar transformation holds for the collinear gluon field.
The superscript index (0) indicates that the corresponding field is not softly interacting. It will be left away in the following.
So one ends up with a decoupled Lagrangian of the form [17]

$$\mathcal{L}_{SCET} = \mathcal{L}_c + \mathcal{L}_{\bar{c}} + \mathcal{L}_{soft}$$

where $\mathcal{L}_c$, $\mathcal{L}_{\bar{c}}$ contains quarks and gluons in the direction n, $\bar{n}$ and $\mathcal{L}_{soft}$ contains the soft quarks and gluons.
In SCET one encounters non-local operators. These operators are products of fields at different space time points which are not gauge invariant. That's why gauge-invariant fields are used to built up operators in SCET. The gauge invariant collinear quark fields are given by

$$\chi_n(x) = W_n^\dagger(x)\xi_n(x), \qquad \bar{\chi}_{\bar{n}}(x) = \bar{\xi}_{\bar{n}}(x) W_{\bar{n}}(x)$$

This definition already appeared in eq. (2.7). The gauge invariant collinear gluon field is defined as [19]

$$A_c^\mu = \frac{1}{g} W_n^\dagger \left(iD_c^\mu W_n(x)\right) \tag{3.4}$$

with the covariant derivative D_c^μ defined as

$$iD_c^\mu(x) = i\partial^\mu + gA_c^{a\mu}(x)T^a$$

with the collinear Wilson lines

$$\begin{aligned} W_n(\infty, x) &= \mathbf{P} \exp\left(ig_s T^a \bar{n}^\mu \int_0^\infty ds\; A^a_{c\mu}(x^\nu + s\bar{n}^\nu)\right) \\ W^\dagger_{\bar{n}}(\infty, x) &= \bar{\mathbf{P}} \exp\left(-ig_s T^a n^\mu \int_0^\infty ds\; A^a_{\bar{c}\mu}(x^\nu + sn^\nu)\right) \end{aligned}$$

That the fields χ are gauge invariant was already shown above in eq. (2.8). For more details I refer to [1, 38].
So in SCET, soft Wilson lines are used to describe interactions with soft particles and the fields χ are used to describe collinear interactions. These objects are already known from section 2.2. Thus, the introduced SCET fields were already motivated in section 2.2. The concept of Wilson line operators is an important ingredient in SCET.

3.3. The electromagnetic current operator in SCET

A theory becomes powerful with the introduction of an interaction. In this section, the argumentation from [1] is repeated that is leading to the electromagnetic current in SCET. The QED current operator is given by

$$J^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$$

In order to determine a corresponding expression in the effective theory one starts with the most general form of an operator which is compatible with the symmetry of the theory. This expression is then multiplied by a Wilson coefficient. Since a vertex is not between particles going in the same direction [1], the most general form of the current in SCET is

$$J^\mu_{SCET} = C \cdot (\bar{\chi}_n \gamma^\mu \chi_{\bar{n}} + \bar{\chi}_{\bar{n}} \gamma^\mu \chi_n)$$

In addition to this expression one also has to consider operators involving derivatives of the fields [1]. The projection of the derivative of the collinear field in a given direction scales as the corresponding component of the momentum

$$n \cdot \partial \chi_n(x) \sim \lambda^2 \chi_n(x), \qquad \bar{n} \cdot \partial \chi_n(x) \sim \lambda^0 \chi_n(x), \qquad \partial^\mu_\perp \chi_n(x) \sim \lambda \chi_n(x)$$

and similar

$$n \cdot \partial \chi_{\bar{n}}(x) \sim \lambda^0 \chi_{\bar{n}}(x)$$

The derivatives scaling like λ^0 are not power suppressed. Hence, even at leading power in λ, one needs to allow for the insertion of an arbitrary number of this kind of derivatives in the current operator. One has [1]

$$\chi_n(x + t\bar{n}) = \sum_{i=0}^{\infty} \frac{t^i}{i!} \bar{n} \partial^i \chi_n(x)$$

Including terms with arbitrarily high derivatives is equivalent to allowing non-locality of the collinear fields along the collinear directions. The SCET operators are thus non-local along light-cone directions corresponding to large energies. Taking the decoupling transformation of eq. (3.3) into account, one ends up with the SCET current which is a local operator

$$\begin{aligned} J^\mu_{SCET}(x) = \int ds\, dt\, C(s,t,\mu) \Big(& \bar{\chi}_n(x + t\bar{n}) S^\dagger_n(x) S_{\bar{n}}(x) \gamma^\mu \chi_{\bar{n}}(x + sn) \\ & + \bar{\chi}_{\bar{n}}(x + sn) S^\dagger_{\bar{n}}(x) S_n(x) \gamma^\mu \chi_n(x + t\bar{n}) \Big) \end{aligned} \tag{3.5}$$

Because of charge conjugation invariance, the matching coefficients are the same for both contributions. This expression can be rewritten. Consider the Fourier decomposition of a quark field

$$q_\alpha(x) = \int \frac{d^3p}{(2\pi)^3 2p_0} \sum_{s=\pm 1/2} \Big(e^{-ipx} u_\alpha(p,s)\, a_\alpha(p,s) + e^{ipx} v_\alpha(p,s) b^\dagger_\alpha(p,s) \Big)$$

Inserting this expression for the fields χ the dependence on s and t can be written in a simpler form as

$$\begin{aligned} J^\mu_{SCET}(x) = C(\bar{n} \cdot p_1, n \cdot p_2, \mu) \Big(& \bar{\chi}_n(x) S^\dagger_n(x) S_{\bar{n}}(x) \gamma^\mu \chi_{\bar{n}}(x) \\ & + \bar{\chi}_{\bar{n}}(x) S^\dagger_{\bar{n}}(x) S_n(x) \gamma^\mu \chi_n(x) \Big) \end{aligned} \tag{3.6}$$

with

$$C(\bar{n} \cdot p_1, n \cdot p_2, \mu) = \int ds\, dt\, C(s,t,\mu) e^{itp_1\bar{n}} e^{-isp_2 n}$$

In the next section this Wilson coefficient is calculated to order $\mathcal{O}(\alpha_s)$.

3.4. One-loop anomalous dimension of the SCET current

In this chapter, the matching of the current from QCD to SCET is performed which gives the Wilson coefficient. This is achieved by comparing the result obtained in the full theory with the result obtained in SCET. A photon decaying into a quark-antiquark pair is described by the electromagnetic current and the

matching is performed to order α_s which means that QCD corrections to order α_s are taken into account. The quark is denoted as $q_n(p)$ which means that it carries momentum p which is collinear to n. The antiquark is denoted as $\bar{q}_{\bar{n}}(q)$ which indicates that it carries momentum q which is collinear to $\bar{n}$. To order α_s in full QCD one has to consider a gluon exchange between the quark and the antiquark. In SCET, the gluon exchange between the quark and the antiquark is divided into several contributions. First, one has the case of a soft gluon exchange. Then the collinear Wilson line radiates a collinear gluon interacting with the collinear quark and the anti-collinear Wilson line radiates an anti-collinear gluon which is absorbed from the anti-collinear antiquark.
The electromagnetic currents in QCD and SCET are respectively given by

$$\begin{aligned} J^\mu_{QCD}(x) &= \bar{\psi}(x)\gamma^\mu\psi(x) \\ J^\mu_{SCET}(x) &= \int ds\, dt\, C(s,t,\mu)\Big(\bar{\chi}_n(x+t\bar{n})S^\dagger_n(x)S_{\bar{n}}(x)\gamma^\mu\chi_{\bar{n}}(x+sn)\Big) \end{aligned}$$

In the SCET current only one of the two contributions is considered which describes an outgoing collinear quark and an outgoing anti-collinear antiquark. One claims that

$$\langle p;q|\, J^\mu_{QCD}\,|0\rangle \cong \langle p;q|\, J^\mu_{SCET}\,|0\rangle$$

This can be rewritten in a factorized form as

$$\langle p;q|\,\bar{\psi}(x)\gamma^\mu\psi(x)\,|0\rangle \cong \int ds\, dt\, C(s,t,\mu)\,\gamma^\mu\,\frac{\langle p|\,\bar{\chi}_n(x+t\bar{n})\,|0\rangle}{\langle 0|\,S^\dagger_n(x)W_n(x)\,|0\rangle}\,\frac{\langle q|\,\chi_{\bar{n}}(x+sn)\,|0\rangle}{\langle 0|\,W^\dagger_{\bar{n}}(x)S_{\bar{n}}(x)\,|0\rangle}\,\langle 0|\,S^\dagger_n(x)S_{\bar{n}}(x)\,|0\rangle \quad (3.7)$$

where this equality can be evaluated at any space point. For simplicity it can be chosen $x = 0$. The denominator of equation (3.7) represents the 0-bin subtraction: For external states which are both soft and collinear, one can put them either in the collinear matrix element or in the soft one. The soft-collinear region is included in both and so their overlap must be removed to avoid double counting [15]. This was already mentioned in the paragraph below eq. (3.2). In pure dimensional regularization the zero-bin subtraction terms vanish, since these are scaleless integrals. But with an additional regulator one can examine what these terms are causing.
The matching condition will be computed within dimensional regularization. Infrared divergences are controlled via off-shell states. Without using off-shell states in this problem, all SCET integrals are scaleless and thus vanish in dimensional regularization.

3.4.1. Calculation with on-shell states

To understand the concept of matching and effective theories, it is advisable to take a look at on-shell states first ($p^2 = q^2 = 0$). The computation greatly simplifies in pure dimensional regularization [39]. The on-shell graphs in SCET are scaleless integrals and vanish. In the full theory, there are infrared (IR) and ultraviolet (UV) divergences. The UV divergences in the full theory are removed by counterterms and the IR divergences of the full theory are in correspondence with the negative of the UV divergences of the effective theory (encoded in the counterterm of SCET) and thus cancel each other. The Wilson coefficient is therefore given by the finite parts of the on-shell diagram. The on-shell matrix element of the electromagnetic current is given by [39, 22]

$$\begin{aligned} \mathcal{M}_{QCD} &= \langle q_n(p)\;\bar{q}_{\bar{n}}(q)|\,J^\mu_{QCD}(0)\,|0\rangle \\ &= \bar{u}_n(p)\gamma^\mu v_{\bar{n}}(q)\left(1+\frac{\alpha_s C_F}{4\pi}\left(-\frac{2}{\epsilon^2_{IR}}-\frac{2\ln\frac{\mu^2}{-Q^2}+3}{\epsilon_{IR}}-\ln^2\frac{\mu^2}{-Q^2}-3\ln\frac{\mu^2}{-Q^2}-8+\frac{\pi^2}{6}\right)\right) \end{aligned} \quad (3.8)$$

Where Q^2 is the hard scale given by $Q^2 = (p+q)^2 \approx p^-q^+$. The contribution proportional to α_s arises from a gluon exchange between the quark and the antiquark. The matching coefficient is the finite part of eq. (3.8) which is

$$C(\mu) = 1+\frac{\alpha_s C_F}{4\pi}\left(-\ln^2\frac{\mu^2}{-Q^2}-3\ln\frac{\mu^2}{-Q^2}-8+\frac{\pi^2}{6}\right) \quad (3.9)$$

3.4.2. Calculation with off-shell states

In order to check the mechanism of the matching it is useful to work with off-shell states such that $p^2, q^2 \neq 0$ [34, 39].
In leading order the following matrix elements have to be considered. In full QCD one obtains

$$\begin{aligned} \mathcal{M}^{(0)}_{QCD} &= \langle q_n(p)\ \bar{q}_{\bar{n}}(q)|\ J^{\mu}_{QCD}(0)\ |0\rangle \\ &= \bar{u}_n(p)\gamma^{\mu} v_{\bar{n}}(q) \end{aligned}$$

and on the other hand, the SCET current produces

$$\begin{aligned} \mathcal{M}^{(0)}_{SCET} &= \langle q_n(p)\ \bar{q}_{\bar{n}}(q)|\ J^{\mu}_{SCET}(0)\ |0\rangle \\ &= C^{(0)}(\bar{n}\cdot p, n\cdot q, \mu)\cdot \bar{u}_n(p)\gamma^{\mu} v_{\bar{n}}(q) \end{aligned}$$

with the leading order Fourier transformed Wilson coefficient

$$C^{(0)}(\bar{n}\cdot p, n\cdot q, \mu) = \int ds\ dt\ C(s,t,\mu) e^{itp\bar{n}} e^{isqn}$$

Comparing the result obtained in SCET with the one obtained in QCD, one infers that

$$C^{(0)}(\bar{n}\cdot p, n\cdot q, \mu) = 1 \quad \Leftrightarrow \quad C^{(0)}(s,t,\mu) = \delta(s)\delta(t)$$

In order to determine the matching to higher orders, the Wilson coefficient and the SCET diagrams are expanded as

$$\begin{aligned} C(\bar{n}\cdot p_1, n\cdot p_2, \mu) &= C^{(0)}(\bar{n}\cdot p_1, n\cdot p_2, \mu) + C^{(1)}(\bar{n}\cdot p_1, n\cdot p_2, \mu) + C^{(2)}(\bar{n}\cdot p_1, n\cdot p_2, \mu) + ... \\ \tilde{\mathcal{M}}_{SCET} &= \tilde{\mathcal{M}}^{(0)}_{SCET} + \tilde{\mathcal{M}}^{(1)}_{SCET} + \tilde{\mathcal{M}}^{(2)}_{SCET} + ... \end{aligned}$$

where $\tilde{\mathcal{M}}^{(i)}_{SCET}$ denotes the sum of all SCET contributions to order α_s^i and $C^{(i)}$ denotes the Wilson coefficient to order α_s^i. And one has

$$\mathcal{M}_{SCET} = C(\bar{n}\cdot p_1, n\cdot p_2, \mu)\ \tilde{\mathcal{M}}_{SCET} = \mathcal{M}_{QCD}$$

The matching to order α_s^0 can be written as

$$\mathcal{M}^{(0)}_{QCD} = C^{(0)}(\bar{n}\cdot p_1, n\cdot p_2, \mu)\cdot \tilde{\mathcal{M}}^{(0)}_{SCET}$$

And the matching prescription to order α_s is written as

$$\mathcal{M}^{(1)}_{QCD} = C^{(0)}(\bar{n}\cdot p_1, n\cdot p_2, \mu)\cdot \tilde{\mathcal{M}}^{(1)}_{SCET} + C^{(1)}(\bar{n}\cdot p_1, n\cdot p_2, \mu)\cdot \tilde{\mathcal{M}}^{(0)}_{SCET}$$

with the knowledge of

$$\begin{aligned} C^{(0)}(\bar{n}\cdot p_1, n\cdot p_2, \mu) &= 1 \\ \tilde{\mathcal{M}}^{(0)}_{SCET} &= \bar{u}_n(p)\gamma^{\mu} v_{\bar{n}}(q) \end{aligned}$$

one obtains

$$\mathcal{M}^{(1)}_{QCD} = \mathcal{M}^{(1)}_{SCET} + C^{(1)}(\bar{n}p, nq, \mu)\ \bar{u}_n(p)\gamma^{\mu} v_{\bar{n}}(q) \tag{3.10}$$

where $\mathcal{M}^{(1)}_i$ is the renormalized amplitude to order α_s in the respective theory. $\mathcal{M}^{(1)}_{QCD}$ and $\mathcal{M}^{(1)}_{SCET}$ can be calculated and so one obtains the NLO matching coefficient.

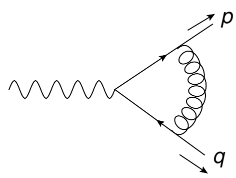


Figure 3.2.: One-loop vertex correction in QCD

QCD to $\mathcal{O}(\alpha_s)$

The one-loop vertex correction to the electromagnetic current in QCD is shown in figure 3.2.
In Feynman gauge, the contribution of this graph is given by

$$V = -ig^2 C_F \mu^{2\epsilon} \int \frac{d^dk}{(2\pi)^d} \gamma^\alpha \frac{\not{q} + \not{k}}{(k+q)^2 - i\epsilon} \gamma^\mu \frac{\not{p} - \not{k}}{(p-k)^2 - i\epsilon} \gamma_\alpha \frac{1}{k^2 + i\epsilon}$$

Now one can evaluate the integral in $d = 4 - 2\epsilon$ dimension and using off-shell states ($p^2, q^2 \neq 0$) to regulate the infrared divergences. Additionally one has to take into account the wavefunction contributions. So the vertex correction in QCD to order α_s is given by [39]:

$$\begin{aligned} \mathcal{M}^{ren}_{QCD} = -\frac{\alpha_s(\mu) C_F}{4\pi} \Bigg(& \ln \frac{-Q^2}{\mu^2} + 2 \ln \frac{p^2}{Q^2} \ln \frac{q^2}{Q^2} + 2 \ln \frac{q^2}{Q^2} \\ & + 2 \ln \frac{p^2}{Q^2} - \frac{1}{2} \ln \frac{-p^2}{\mu^2} - \frac{1}{2} \ln \frac{-q^2}{\mu^2} + 1 + \frac{2\pi^2}{3} \Bigg) \bar{u}_n(p) \gamma^\mu v_{\bar{n}}(q) \end{aligned} \tag{3.11}$$

with $Q^2 = (p+q)^2$. The superscript *ren* stands for renormalized. Nevertheless, the full QCD current is UV finite and thus there is no counterterm in QCD.

SCET to $\mathcal{O}(\alpha_s)$

In SCET, there are three contributions which have to be considered in order to calculate the correction of the electromagnetic current to $\mathcal{O}(\alpha_s)$. These are shown in figure 3.3. The final result is obtained by multiplying the ultrasoft, collinear and anti-collinear region to $\mathcal{O}(\alpha_s)$ and keeping only the terms up to $\mathcal{O}(\alpha_s)$ which corresponds to the sum of the three contributions.

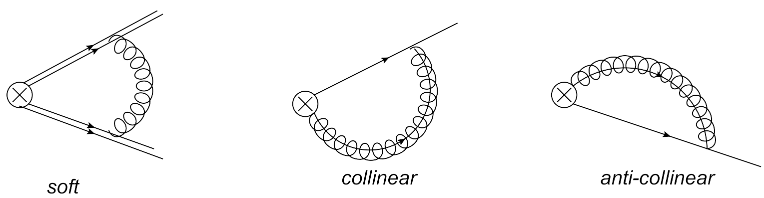


Figure 3.3.: SCET graphs

As before an expansion parameter is necessary. The hard scale is given by

$$Q^2 = (p+q)^2 \approx p^- q^+$$

and the expansion parameter is therefore

$$\lambda^2 \sim \frac{p^2}{Q^2} \sim \frac{q^2}{Q^2}$$

Ultrasoft gluon contribution

First, the ultrasoft contribution is considered. It arises from the ultrasoft gluon exchange between the n-collinear and the $\bar{n}$-collinear quark. Evaluating the SCET current at $x = 0$ one has to $\mathcal{O}(\alpha_s)$

$$I_s^{(1)} = \langle 0| S_p^\dagger(0) S_q(0) |0\rangle$$

This is the last factor of eq. (3.7). Using Feynman gauge this gives in the $\overline{MS}$-scheme

$$I_s^{(1)} = -ig^2 C_F\ (p\cdot q)\ \left(\frac{e^{\gamma_E}\mu^2}{4\pi}\right)^{\epsilon_{UV}} \int \frac{d^dk}{(2\pi)^d}\ \frac{1}{p\cdot k - i\epsilon}\cdot\frac{1}{k^2+i\epsilon}\cdot\frac{1}{q\cdot k+i\epsilon}$$

The integral I_s is scaleless and thus vanish in dimensional regularization. That's why off-shell states ($p^2, q^2 \neq 0$) are introduced. The momenta are expanded in powers of λ as

$$\begin{aligned} p^\mu &= \frac{1}{2}\bar{n}\cdot p\ n^\mu + \mathcal{O}(\lambda) \\ q^\mu &= \frac{1}{2}n\cdot q\ \bar{n}^\mu + \mathcal{O}(\lambda) \end{aligned}$$

and the integral I_s is written as

$$\begin{aligned} I_s^{(1)} &= -2ig^2 C_F\ \left(\frac{e^{\gamma_E}\mu^2}{4\pi}\right)^{\epsilon_{UV}} \int \frac{d^dk}{(2\pi)^d}\ \frac{1}{n\cdot k - \frac{p^2}{\bar{n}p} - i\epsilon}\cdot\frac{1}{k^2+i\epsilon}\cdot\frac{1}{\bar{n}k + \frac{q^2}{nq} + i\epsilon} \\ &= -ig^2 C_F\ \left(\frac{e^{\gamma_E}\mu^2}{4\pi}\right)^{\epsilon_{UV}} \int \frac{dk^+dk^-d^{d-2}k_\perp}{(2\pi)^d}\ \frac{1}{n\cdot k - \frac{p^2}{\bar{n}p} - i\epsilon}\cdot\frac{1}{k^2+i\epsilon}\cdot\frac{1}{\bar{n}k + \frac{q^2}{nq} + i\epsilon} \end{aligned}$$

The evaluation of this integral is performed in the appendix C.1. The result is

$$I_s^{(1)} = -g^2 C_F\ \left(e^{\gamma_E}\mu^2\right)^{\epsilon_{UV}}\ \frac{\Gamma(\epsilon_{UV})}{8\pi^2}\ \left(\frac{-p^2}{p^-}\right)^{-\epsilon_{UV}}\ \left(\frac{q^2}{q^+}\right)^{-\epsilon_{UV}}\ \Gamma(\epsilon_{UV})\Gamma(1-\epsilon_{UV})$$

Because there is only one divergence structure, the index UV can be omitted. Expanding $\epsilon \to 0$ one obtains

$$I_s^{(1)} = \frac{g^2}{8\pi^2}\ C_F\ \left(-\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\ln\frac{-p^2q^2}{\mu^2q^+p^-} - \frac{1}{2}\ln^2\frac{-p^2q^2}{\mu^2q^+p^-} - \frac{\pi^2}{4}\right)$$

and so up to order $\mathcal{O}(\alpha_s)$ one has

$$I_s = 1 + \frac{g^2}{8\pi^2}\ C_F\ \left(-\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\ln\frac{-p^2q^2}{\mu^2q^+p^-} - \frac{1}{2}\ln^2\frac{-p^2q^2}{\mu^2q^+p^-} - \frac{\pi^2}{4}\right)$$

Collinear gluon exchange

Next, the following contribution has to be evaluated

$$I_n = \frac{\langle q(p)|\,\bar{\chi}_n(0)\,|0\rangle}{\langle 0|\,S_n^\dagger(0)W_n(0)\,|0\rangle} = \frac{\langle q(p)|\,\bar{\chi}_n(0)\,|0\rangle}{\langle 0|\,S_n^\dagger(0)W_n(0)\,|0\rangle}$$

This is the first factor in eq. (3.7). First consider the numerator

$$I_N = \langle q(p)|\,\bar{\chi}_n(0)\,|0\rangle = \langle q(p)|\,\bar{\xi}_n(0)W_n(0)\cdot e^{i\int d^4x\mathcal{L}_{int}(x)}\,|0\rangle$$

with

$$\mathcal{L}_{int} = -g\bar{\xi}_n\gamma^\mu T^b\xi_n A^b_{n\mu}$$

One has to order α_s

$$I_N^{(1)} = \langle q(p)|\, \bar{\chi}_n(0)\, |0\rangle = \bar{u}_n(p) i\, g^2\, C_F \gamma^\mu \left(\frac{e^{\gamma_E}\mu^2}{4\pi}\right)^{\epsilon_{UV}} \int \frac{d^dk}{(2\pi)^d}\, \frac{\not{p}-\not{k}}{(p-k)^2+i\epsilon} \cdot \frac{g_{\mu\nu}}{k^2+i\epsilon} \cdot \frac{\bar{n}^\nu}{\bar{n}\cdot k+i\epsilon}$$

The momenta are expanded in powers of λ which gives

$$\not{p} = \frac{1}{2}(\bar{n}\cdot p)\,\not{n} + \mathcal{O}(\lambda)$$
$$\not{k} = \frac{1}{2}(\bar{n}\cdot k)\,\not{n} + \mathcal{O}(\lambda)$$

and thus

$$I_N^{(1)} = \bar{u}_n(p) i\, g^2\, C_F \left(\frac{e^{\gamma_E}\mu^2}{4\pi}\right)^{\epsilon_{UV}} \frac{\not{\bar{n}}\not{n}}{2} \int \frac{d^dk}{(2\pi)^d}\, \frac{p^- - k^-}{(p-k)^2+i\epsilon} \cdot \frac{1}{k^2+i\epsilon} \cdot \frac{1}{\bar{n}\cdot k+i\epsilon}$$

and because $\bar{u}_n(p) \cdot \frac{\not{\bar{n}}\not{n}}{4} = \bar{u}_n(p)$ one ends up with

$$\begin{aligned} I_N^{(1)} &= 2\bar{u}_n(p) i\, g^2\, C_F \left(\frac{e^{\gamma_E}\mu^2}{4\pi}\right)^{\epsilon_{UV}} \int \frac{d^dk}{(2\pi)^d}\, \frac{p^- - k^-}{(p-k)^2+i\epsilon} \cdot \frac{1}{k^2+i\epsilon} \cdot \frac{1}{\bar{n}\cdot k+i\epsilon} \\ &= \bar{u}_n(p) i\, g^2\, C_F \left(\frac{e^{\gamma_E}\mu^2}{4\pi}\right)^{\epsilon_{UV}} \int \frac{dk^+ dk^- d^{d-2}k_\perp}{(2\pi)^d}\, \frac{p^- - k^-}{(p-k)^2+i\epsilon} \cdot \frac{1}{k^2+i\epsilon} \cdot \frac{1}{\bar{n}\cdot k+i\epsilon} \end{aligned} \tag{3.12}$$

This integral is evaluated in the appendix C.2. The result is

$$I_N^{(1)} = -\frac{g^2}{8\pi^2}\, C_F\, \bar{u}_n(p) e^{\epsilon_{UV}\gamma_E} \mu^{2\epsilon_{UV}} \Gamma(\epsilon_{UV})(-p^2)^{-\epsilon_{UV}} \underbrace{\frac{\Gamma(-\epsilon)\Gamma(2-\epsilon)}{\Gamma(2-2\epsilon)}}_{IR} \tag{3.13}$$

The divergence encoded in $\Gamma(\epsilon_{UV})$ is due to the integration over $k_\perp$ which diverges for $k_\perp \to \infty$. Thus this divergence is an ultraviolet one.
The underlined factor in eq. (3.13) is caused by the k^--integration and this is an infrared divergence. In this case the pole in ϵ arises for the divergence when $k^- \to 0$. This is a soft divergence and hence this is an example for the soft-collinear overlapping. In order to avoid double counting, the zero-bin subtraction of equation (3.7) has to be considered. Expanding the infrared part of eq. (3.13), one has

$$\frac{\Gamma(-\epsilon_{IR})\Gamma(2-\epsilon_{IR})}{\Gamma(2-2\epsilon_{IR})} = -\frac{1}{\epsilon_{IR}} - 1 + \left(-2+\frac{\pi^2}{6}\right)\epsilon_{IR}$$

The collinear matrix element including the zero-bin subtraction is [15]

$$I_n = \frac{\langle q(p)|\, \bar{\chi}_n(0)\, |0\rangle}{\langle 0|\, S_n^\dagger(0) W_n(0)\, |0\rangle}$$

The zero-bin subtraction term can either be obtained from I_s (with $q^2 = 0$) or from I_N (in the soft limit $k^\mu \sim (\lambda^2, \lambda^2, \lambda^2)$). It is [34]

$$\langle 0|\, S_n^\dagger(0) W_n(0)\, |0\rangle = 1 - ig^2 C_F \left(\frac{e^{\gamma_E}\mu^2}{4\pi}\right)^{\epsilon_{UV}} \int \frac{dk^+ dk^- d^{d-2}k_\perp}{(2\pi)^d}\, \frac{1}{n\cdot k - \frac{p^2}{\bar{n}p} - i\epsilon} \cdot \frac{1}{k^2+i\epsilon} \cdot \frac{1}{\bar{n}k+i\epsilon}$$

and so one ends up with

$$\Rightarrow \langle 0|\, S_n^\dagger(0) W_n(0)\, |0\rangle = 1 - g^2 C_F\, \left(e^{\gamma_E}\mu^2\right)^{\epsilon_{UV}} \frac{\Gamma(\epsilon)}{8\pi^2} \left(\frac{-p^2}{p^-}\right)^{-\epsilon} \int_0^\infty dk^-\, (k^-)^{-\epsilon} \frac{1}{k^-}$$

k^- is substituted by $k^- = zp^-$ and this gives

$$\Rightarrow \langle 0| S_n^\dagger(0) W_n(0) |0\rangle = 1 - g^2 C_F \left(e^{\gamma_E}\mu^2\right)^{\epsilon_{UV}} \frac{\Gamma(\epsilon_{UV})}{8\pi^2} \left(-p^2\right)^{-\epsilon_{UV}} \int_0^\infty dz\, z^{-\epsilon-1}$$

In this z-integration there is an ultraviolet and an infrared divergence. In order to see that, the integration can be split as

$$\begin{aligned}\int_0^\infty dz\, z^{-\epsilon-1} &= \int_0^1 dz\, z^{-\epsilon-1} + \int_1^\infty dz\, z^{-\epsilon-1} \\ &= \int_0^1 dz\, z^{-\epsilon_{IR}-1} + \int_1^\infty dz\, z^{-\epsilon_{UV}-1} \\ &= -\frac{1}{\epsilon_{IR}} z^{-\epsilon_{IR}}\Big|_0^1 - \frac{1}{\epsilon_{UV}} z^{-\epsilon_{UV}}\Big|_1^\infty\end{aligned}$$

Since $\epsilon_{UV} > 0$ and $\epsilon_{IR} < 0$ the contributions at $z = 0$ and $z = \infty$ are equal to zero and thus one ends up with

$$\langle 0| S_n^\dagger(0) W_n(0) |0\rangle = 1 - g^2 C_F \left(e^{\gamma_E}\mu^2\right)^{\epsilon_{UV}} \frac{\Gamma(\epsilon_{UV})}{8\pi^2} \left(-p^2\right)^{-\epsilon_{UV}} \left(-\frac{1}{\epsilon_{IR}} + \frac{1}{\epsilon_{UV}}\right)$$

Thus, the zero-bin subtraction converts the infrared divergence into an ultraviolet one. Expanding $\epsilon \to 0$, one ends up with

$$I_n = \frac{\langle q(p)| \bar\chi_n(0) |0\rangle}{\langle 0| S_n^\dagger(0) W_n(0) |0\rangle} = \bar u_n(p) - \frac{g^2}{8\pi^2} C_F\, \bar u_n(p) \left(-\frac{1}{\epsilon^2} - \frac{1}{\epsilon} + \frac{1}{\epsilon}\ln\frac{-p^2}{\mu^2} - \frac{1}{2}\ln^2\frac{-p^2}{\mu^2} + \ln\frac{-p^2}{\mu^2} - 2 + \frac{\pi^2}{12}\right)$$

Anti-collinear gluon contribution

The $\bar n$-contribution can be obtained by simply changing $p^2 \to q^2$

$$\begin{aligned}I_{\bar n} = \frac{\langle \bar q(q)| \chi_{\bar n}(0) |0\rangle}{\langle 0| W_{\bar n}^\dagger(0) S_{\bar n}(0) |0\rangle} = v_{\bar n}(q) - \frac{g^2}{8\pi^2} C_F \Big(&-\frac{1}{\epsilon^2} - \frac{1}{\epsilon} + \frac{1}{\epsilon}\ln\frac{-q^2}{\mu^2} \\ &- \frac{1}{2}\ln^2\frac{-q^2}{\mu^2} + \ln\frac{-q^2}{\mu^2} - 2 + \frac{\pi^2}{12}\Big) v_{\bar n}(q)\end{aligned}$$

Result

The three contributions from the ultrasoft, collinear and anti-collinear sector have to be multiplied. One obtains up to order $\mathcal{O}(\alpha_s)$

$$\begin{aligned}\langle q(p)\, \bar q(q)| J^\mu_{SCET} |0\rangle &= \gamma^\mu \frac{\langle p| \bar\chi_n(0) |0\rangle}{\langle 0| S_n^\dagger(0) W_n(0) |0\rangle} \frac{\langle q| \chi_{\bar n}(0) |0\rangle}{\langle 0| W_{\bar n}^\dagger(0) S_{\bar n}(0) |0\rangle} \langle 0| S_n^\dagger(0) S_{\bar n}(0) |0\rangle \\ &= \bar u_n(p)\gamma^\mu v_{\bar n}(q) - \bar u_n(p)\gamma^\mu v_{\bar n}(q) \frac{g^2}{8\pi^2} C_F \Big(-\frac{1}{\epsilon^2} - \frac{1}{\epsilon} + \frac{1}{\epsilon}\ln\frac{-q^+p^-}{\mu^2} + \ln\frac{-q^2}{\mu^2} + \ln\frac{-p^2}{\mu^2} \\ &+ \frac{1}{2}\ln^2\frac{-q^+p^-}{\mu^2} - \ln\frac{-p^2}{\mu^2}\ln\frac{-q^+p^-}{\mu^2} - \ln\frac{-q^2}{\mu^2}\ln\frac{-q^+p^-}{\mu^2} + -\ln\frac{-p^2}{\mu^2}\ln\frac{-q^2}{\mu^2} - 4 + \frac{5\pi^2}{12}\Big)\end{aligned} \tag{3.14}$$

The poles in ϵ are ultraviolet divergences (see above). Further graphs which have to be considered are the self-energy graphs on the external legs. The soft gluon contribution to wave function renormalization vanishes in Feynman gauge because $n^2 = \bar n^2 = 0$. The collinear wavefunction renormalization graph is the same as the quark wavefunction renormalization in QCD, since the interaction of n-collinear quarks with n-collinear gluons is the same as the interaction of quarks with gluons in full QCD, which is [22]

$$\delta Z_2(p) = -\frac{C_F g^2}{16\pi^2}\left(\frac{1}{\epsilon} - \ln\frac{-p^2}{\mu^2} + 1\right) \tag{3.15}$$

Thus the whole expression to $\mathcal{O}(\alpha_s)$ of the SCET current is

$$\begin{aligned}
S_{bare} &= (3.14) + \left(\frac{1}{2}\delta Z_2(p) + \frac{1}{2}\delta Z_2(q)\right) \bar{u}_n(p)\gamma^\mu v_{\bar{n}}(q) \\
&= \bar{u}_n(p)\gamma^\mu v_{\bar{n}}(q) - \bar{u}_n(p)\gamma^\mu v_{\bar{n}}(q)\frac{\alpha_s}{4\pi}\, C_F \Big(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} + \frac{2}{\epsilon}\ln\frac{-q^+p^-}{\mu^2} + \frac{3}{2}\ln\frac{-p^2}{\mu^2} \\
&+ \frac{3}{2}\ln\frac{-q^2}{\mu^2} + \ln^2\frac{-q^+p^-}{\mu^2} - 2\ln\frac{-p^2}{\mu^2}\ln\frac{-q^+p^-}{\mu^2} - 2\ln\frac{-q^2}{\mu^2}\ln\frac{-q^+p^-}{\mu^2} + 2\ln\frac{-p^2}{\mu^2}\ln\frac{-q^2}{\mu^2} - 7 + \frac{5\pi^2}{6}\Big)
\end{aligned}$$

The renormalized SCET sum S^{ren} is defined as

$$S^{ren}(Q,\epsilon) = Z_C(\mu,Q,\epsilon)S_{bare}(\mu,Q)$$

such that the renormalization constant Z_C cancels the UV divergences which are encoded in the terms proportional to $1/\epsilon^n$. Thus, one has

$$Z_C = 1 + \frac{\alpha_s(\mu)C_F}{4\pi}\left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - \frac{2}{\epsilon}\ln\frac{\mu^2}{-Q^2}\right)$$

where $Q^2 = p^-q^+$ was inserted. The counterterm Z_C can be compared with eq. (3.8) and one finds a correspondence between the terms proportional to $1/\epsilon^i$ as it was announced. The IR divergences in QCD are identical to the UV divergences in SCET. A theory with UV divergences is preferred, since UV divergences are canceled within the renormalization procedure and the corresponding Renormalization Group Equations (RGEs) are obtained whereby the resummation of enhanced contributions is achieved. This was already mentioned in the introduction. So the same Z_C also makes the QCD scattering amplitude finite and one concludes that the IR singularities in QCD can be removed by a multiplicative factor and that the structure of these singularities is governed by a renormalization group equation [1].
So, the renormalized matrix element in SCET to $\mathcal{O}(\alpha_s)$ is

$$\begin{aligned}
\mathcal{M}^{(1)ren}_{SCET} = &-\frac{\alpha_s}{4\pi}\, C_F \Big(\frac{3}{2}\ln\frac{-p^2}{\mu^2} + \frac{3}{2}\ln\frac{-q^2}{\mu^2} + \ln^2\frac{-q^+p^-}{\mu^2} - 2\ln\frac{-p^2}{\mu^2}\ln\frac{-q^+p^-}{\mu^2} \\
&- 2\ln\frac{-q^2}{\mu^2}\ln\frac{-q^+p^-}{\mu^2} + 2\ln\frac{-p^2}{\mu^2}\ln\frac{-q^2}{\mu^2} - 7 + \frac{5\pi^2}{6}\Big)\bar{u}_n(p)\gamma^\mu v_{\bar{n}}(q)
\end{aligned} \tag{3.16}$$

This expression can be rewritten as

$$\begin{aligned}
\mathcal{M}^{(1)ren}_{SCET} = &-\frac{\alpha_s}{4\pi}\, C_F\, \bar{u}_n(p)\gamma^\mu v_{\bar{n}}(q)\Big(\frac{3}{2}\ln\frac{-p^2}{\mu^2} + \frac{3}{2}\ln\frac{-q^2}{\mu^2} \\
&\underbrace{+\ln^2\frac{-Q^2}{\mu^2} - 2\ln\frac{-p^2}{\mu^2}\ln\frac{-Q^2}{\mu^2} - 2\ln\frac{-q^2}{\mu^2}\ln\frac{-Q^2}{\mu^2} + 2\ln\frac{-p^2}{\mu^2}\ln\frac{-q^2}{\mu^2}}_{=-\ln^2\frac{-Q^2}{\mu^2}+2\ln\frac{p^2}{Q^2}\ln\frac{q^2}{Q^2}} -7 + \frac{5\pi^2}{6}\Big)
\end{aligned}$$

and in eq. (3.11) one found that

$$\begin{aligned}
\mathcal{M}^{(1)ren}_{QCD} = &-\frac{\alpha_s(\mu)C_F}{4\pi}\Bigg(\ln\frac{-Q^2}{\mu^2} + 2\ln\frac{p^2}{Q^2}\ln\frac{q^2}{Q^2} \\
&\underbrace{+2\ln\frac{q^2}{Q^2} + 2\ln\frac{p^2}{Q^2} - \frac{1}{2}\ln\frac{-p^2}{\mu^2} - \frac{1}{2}\ln\frac{-q^2}{\mu^2}}_{=\frac{3}{2}\ln\frac{-p^2}{\mu^2}+\frac{3}{2}\ln\frac{-q^2}{\mu^2}-4\ln\frac{-Q^2}{\mu^2}} +1 + \frac{2\pi^2}{3}\Bigg)\bar{u}_n(p)\gamma^\mu v_{\bar{n}}(q)
\end{aligned} \tag{3.17}$$

The infrared divergence of $\mathcal{M}^{(1)ren}_{QCD}$ and $\mathcal{M}^{(1)ren}_{SCET}$ is encoded in the terms proportional to $\ln p^2$ and $\ln q^2$. So one can see that SCET exactly reproduces the IR divergence of the full theory. Therefore, the Wilson coefficient is independent of IR divergences. As stated in eq. (3.10) the Wilson coefficient to order α_s can be obtained by

$$C^{(1)}(\bar{n}p, nq, \mu)\ \bar{u}_n(p)\gamma^\mu v_{\bar{n}}(q) = \mathcal{M}^{(1)ren}_{QCD} - \mathcal{M}^{(1)ren}_{SCET}$$

This can be simply calculated with eq. (3.17) and (3.16). One obtains

$$C^{(1)}(\bar{n}p, nq, \mu)\ \bar{u}_n(p)\gamma^\mu v_{\bar{n}}(q) = \bar{u}_n(p)\gamma^\mu v_{\bar{n}}(q)\frac{\alpha_s(\mu)C_F}{4\pi}\left(-\ln^2\frac{-Q^2}{\mu^2} + 3\ln\frac{-Q^2}{\mu^2} - 8 + \frac{\pi^2}{6}\right)$$

which agrees with eq. (3.9). In total one has

$$C(Q^2, \mu) = 1 + \frac{\alpha_s(\mu)C_F}{4\pi}\left(-\ln^2\frac{-Q^2}{\mu^2} + 3\ln\frac{-Q^2}{\mu^2} - 8 + \frac{\pi^2}{6}\right) \tag{3.18}$$

The Wilson coefficient's anomalous dimension

With the results obtained in the last sections, on can calculate the anomalous dimension of the Wilson coefficient. In general the following equality holds:

$$0 = \mu\frac{d}{d\mu}C^{bare}(\epsilon) = \mu\frac{d}{d\mu}[Z_C(\mu,\epsilon)C(\mu)] = \left[\mu\frac{d}{d\mu}Z_C(\mu,\epsilon)\right]C(\mu) + Z_C(\mu,\epsilon)\left[\mu\frac{d}{d\mu}C(\mu)\right]$$

$$\Rightarrow \mu\frac{d}{d\mu}C(\mu) = \left[-Z_C^{-1}(\mu,\epsilon)\mu\frac{d}{d\mu}Z_C(\mu,\epsilon)\right]C(\mu) = \gamma_C(\mu)C(\mu)$$

where $\gamma_C(\mu)$ is the anomalous dimension of the Wilson coefficient.
The derivative of $\alpha_s(\mu)$ is given by

$$\mu\frac{d}{d\mu}\alpha_s(\mu,\epsilon) = -2\epsilon\ \alpha_s(\mu,\epsilon) + O(\alpha_s^2)$$

Now the anomalous dimension of the Wilson coefficient can be calculated:

$$\begin{aligned}\gamma_C(\mu, Q, \epsilon) &= -Z_C^{-1}(\mu,\epsilon)\mu\frac{d}{d\mu}Z_C(\mu,\epsilon)\\ &= \mu\frac{d}{d\mu}\frac{\alpha_s(\mu)C_F}{4\pi}\left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{2}{\epsilon}\ln\frac{\mu^2}{-Q^2}\right)\\ &= \frac{\alpha_s(\mu)C_F}{4\pi}\left(-\frac{4}{\epsilon} - 6 - 4\ln\frac{\mu^2}{-Q^2} + \frac{4}{\epsilon}\right)\\ &= -\frac{\alpha_s(\mu)C_F}{4\pi}\left(6 + 4\ln\frac{\mu^2}{-Q^2}\right)\end{aligned}$$

The all order form for the anomalous dimension of the SCET current is of the form [40]

$$\gamma_C(\mu, Q, \epsilon) = -\Gamma_{cusp}\left[\alpha_s(\mu)\right]\ln\frac{\mu^2}{-Q^2} + 2\gamma_q\left[\alpha_s(\mu)\right] \tag{3.19}$$

with

$$\Gamma_{cusp}\left[\alpha_s\right] = \sum_{k=1}^{\infty}\left(\frac{\alpha_s}{4\pi}\right)^k\Gamma_{k-1}, \qquad \gamma_q\left[\alpha_s\right] = \sum_{k=1}^{\infty}\left(\frac{\alpha_s}{4\pi}\right)^k\gamma^q_{k-1}$$

where Γ_{cusp} is called cusp anomalous dimension and γ_q is the non-cusp anomalous dimension [22]. The name cusp anomalous dimension is due to the fact, that the soft matrix element can be understood as a single

Wilson line changing the direction at $x = 0$ and thus contains a cusp. Comparing the results from above with eq. (3.19) one finds

$$\Gamma_0 = 4C_F, \qquad \gamma_0^q = -3C_F$$

The renormalization group evolution equation of the Wilson coefficient is thus given by

$$\frac{d}{d\ln\mu} C_V(Q^2,\mu) = \left(\Gamma_{cusp}(\alpha_s)\ln\frac{-Q^2}{\mu^2} + 2\gamma_q(\alpha_s)\right) C_V(q^2,\mu)$$

This is further discussed in chapter 6.3.1 in which the resummation procedure of a Drell-Yan process is considered.

3.5. Regularization of rapidity divergences

In the last section off-shell states were used to regularize the infrared divergences. In SCET, a further regulator needs to be introduced [30],[28]. It serves for the separation of soft and collinear modes and for the regularization of rapidity divergences (also called light-cone divergences). These divergences are present in the separate regions, but they must cancel in total since rapidity divergences do not show up in full QCD. There are several choices of regularization schemes [28]. The only restriction is to use the same scheme in each sector in order to have a cancellation in the end. The rapidity of k is defined as

$$y = \frac{1}{2}\ln\frac{k_+}{k_-}$$

and the corresponding divergence occurs when

$$y \to \pm\infty$$

From now on I will use analytic regularization [41] to regularize the rapidity divergences. This is to insert a factor

$$\left(\frac{\nu}{\bar{n}\cdot k}\right)^\alpha \quad \text{or} \quad \left(\frac{\nu}{n\cdot k}\right)^\alpha \tag{3.20}$$

in each integral. This regularization method was not appropriate for the matching calculation of last section, because the SCET integrals of interest would have been scaleless and vanish.
As an example of light-cone divergences one considers again the collinear integral given in eq. (3.12). But now it is assumed that the virtual particle with momentum k has a mass with $m^2 \sim \lambda^2 Q^2 \gg \lambda^4 Q^2 \sim k^2$. In this limit, the collinear integral is given by

$$J_N^{(1)} = \bar{u}_n(p)i\, g^2\, C_F \left(\frac{e^{\gamma_E}\mu^2}{4\pi}\right)^\epsilon \int \frac{dk^+dk^-d^{d-2}k_\perp}{(2\pi)^d} \frac{p^- - k^-}{-2pk + k^2 + i\epsilon}\cdot\frac{1}{k^2 - m^2 + i\epsilon}\cdot\frac{1}{\bar{n}\cdot k + i\epsilon}$$

After performing the integration over k_+ and $k_\perp$, one is left with the integration over k_- or equivalently over z ($k_- = zp_-$) (same steps as before, see appendix C.2)

$$J_N^{(1)} = -\frac{g^2}{8\pi^2}\frac{1}{(4\pi)^{-\epsilon}}\, C_F\, \bar{u}_n(p)\left(\frac{e^{\gamma_E}\mu^2}{4\pi}\right)^\epsilon \Gamma(\epsilon)m^{-2\epsilon}\int_0^1 dz\frac{(1-z)^{1-\epsilon}}{z}$$

This integral is divergent which can be seen when applying the identity [42]

$$\int_0^1 dx\, x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

In order to fix this divergence, one introduces the analytic regulator $\left(\frac{\nu}{\bar{n}\cdot k}\right)^{\alpha}$ and one obtains

$$
\begin{aligned}
J_N^{(1)} &= -\frac{\nu^\alpha}{p_-^\alpha}\frac{g^2}{8\pi^2}\, C_F\, \bar{u}_n(p)\left(e^{\gamma_E}\mu^2\right)^{\epsilon}\Gamma(\epsilon)m^{-2\epsilon}\int_0^1 dz\,\frac{(1-z)^{1-\epsilon}}{z^{1+\alpha}} \\
&= -\frac{\nu^\alpha}{p_-^\alpha}\frac{g^2}{8\pi^2}\, C_F\, \bar{u}_n(p)\left(e^{\gamma_E}\mu^2\right)^{\epsilon}\Gamma(\epsilon)m^{-2\epsilon}\frac{\Gamma(-\alpha)\Gamma(2-\epsilon)}{\Gamma(2-\epsilon-\alpha)}
\end{aligned}
$$

Finally, the result is first expanded in α and then in ϵ. This gives

$$
\begin{aligned}
J_N^{(1)} = \frac{g^2}{8\pi^2}\, C_F\, \bar{u}_n(p)\Bigg(&\frac{1}{\alpha}\left(\frac{1}{\epsilon}+\ln\frac{\mu^2}{m^2}\right)+\frac{1}{\epsilon}\left(1+\ln\frac{\nu}{p_-}\right)+1-\frac{\pi^2}{6}+\ln\frac{\mu^2}{m^2} \\
&+\ln\frac{\mu^2}{m^2}\cdot\ln\frac{\nu}{p_-}+\mathcal{O}(\alpha)+\mathcal{O}(\epsilon)\Bigg)
\end{aligned}
$$

The rapidity divergence is now encoded in a pole in α.
When all contributions of the different regions are added, the dependence on the analytic regulator α and the corresponding scale ν cancels. This is shown for scalar integrals in [1].
Below, I will show a further application of the analytic regulator and their cancellation in the end of the calculation.

3.6. Summary

In this chapter the Wilson coefficient of the SCET current was calculated and the renormalization group equation (RGE) satisfied by the Wilson coefficient was encountered. On top of that, rapidity divergences emerged and different regularization schemes for them were discussed. These are important topics in SCET. The aim is to derive a factorization formula for a Drell-Yan process and for top quark pair production at small transverse momentum. In order to achieve that, it is essential to discuss the transverse momentum dependent parton distribution functions also just called transverse parton distribution functions (TPDFs). Nevertheless, in the next chapter the discussion starts with the collinear parton distribution functions (PDFs), since those have essential similarities with the TPDFs.

4. The parton distribution function

Traditional factorization theorems as well as the framework of SCET are including parton distribution functions. This was already mentioned in chapter 2. In full QCD, factorization theorems are proven for Drell-Yan processes [11, 12, 43] and for deep inelastic scattering [42]. For sufficiently inclusive measurements of a process, e.g. $pp \to L + X$, where L is the colorless final state of interest such as a W-boson, the cross section can be expressed in terms of the parton distribution functions up to corrections that are power suppressed, cf. eq. (2.1). The PDFs describe all of the initial state radiation effects [13] and they contain perturbative and non-perturbative effects. The PDFs depend on two quantities, the scale μ and the momentum fraction z. The PDFs give the probability of extracting a parton with momentum $k = zp$ from an incoming hadron with momentum p.

4.1. Probabilistic definition

Since the parton distribution function is interpreted as a probability, it can be written as a squared matrix element. Considering the case, where an incoming hadron with momentum p splits into a quark with momentum fraction $k = zp$ and into a residual jet X with momentum p_x. The probability of this process is given by [17]

$$\sum_X \left| \langle X | \psi(0) | p\rangle \right|^2 \delta^{(4)}(p - p_x - k)$$

Rewriting the delta constraint gives

$$\sum_X \int \frac{d^4x}{(2\pi)^4} e^{ix(p-p_x-k)} \langle p| \psi^\dagger(0) |X\rangle \langle X| \psi(0) |p\rangle$$

Rewriting the exponential factor in terms of the translation operator gives

$$\begin{aligned}\sum_X \int \frac{d^4x}{(2\pi)^4} e^{ix(p-p_x-k)} \langle p| \psi^\dagger(0) |X\rangle \langle X| \psi(0) |p\rangle &= \sum_X \int \frac{d^4x}{(2\pi)^4} e^{-ixk} \langle p| \psi^\dagger(x) |X\rangle \langle X| \psi(0) |p\rangle \\ &= \int \frac{d^4x}{(2\pi)^4} e^{-ixk} \langle p| \psi^\dagger(x)\psi(0) |p\rangle\end{aligned}$$

where the translation operator and its effect on a Dirac field is given by

$$U(x) = e^{i\hat{p}x} \quad \text{with} \quad U(x)\psi(y)U^{-1}(x) = \psi(x+y)$$

and

$$U(x)\,|q\rangle = e^{i\hat{p}x}\,|q\rangle = e^{ixq}\,|q\rangle = |q\rangle\, e^{ixq}$$

where the state $|q\rangle$ is an eigenstate of the momentum operator $\hat{p}$. So the expression above can be rewritten as

$$\sum_X \int \frac{d^4x}{(2\pi)^4} e^{-ixk} \langle p| \psi^\dagger(x) |X\rangle \langle X| \psi(0) |p\rangle = \int \frac{d^4x}{(2\pi)^4} e^{-ixk} \langle p| \psi^\dagger(x)\psi(0) |p\rangle \tag{4.1}$$

where the identity

$$\sum_X |X\rangle \langle X| = \mathbf{1}$$

was used. The problem can be considered in a frame in which the hadron momentum is n-collinear. So the momenta p and k are scaling as

$$p^\mu \sim (\lambda^2, 1, \lambda) \sim k^\mu$$

and thus

$$k_+ = (nk) < k_\perp \ll k_- = (\bar{n}k)$$

The most dominant contribution in powers of λ is obtained by performing a multipole expansion [44]. To this purpose, the small components k_+ and $k_\perp$ can be expanded which is equivalent to integrate over these components. Thus one has

$$\begin{aligned}&\int \frac{d^4x}{(2\pi)^4}\, dk_+\, d^2k_\perp\, e^{-i\left(\frac{1}{2}(k_+x_- + k_-x_+) + k_\perp x_\perp\right)} \langle p| \psi^\dagger(x)\psi(0) |p\rangle \\ &= \int \frac{d^4x}{(2\pi)} \delta(x_-)\delta^{(2)}(x_\perp)\, e^{-i\left(\frac{1}{2}k_-x_+\right)} \langle p| \psi^\dagger(x)\psi(0) |p\rangle\end{aligned}$$

Performing the x_- and $x_\perp$ integration one obtains

$$\frac{1}{2}\int \frac{dx_+}{2\pi}\, e^{-i\left(\frac{1}{2}k_-x_+\right)} \langle p| \psi^\dagger \left(\frac{1}{2}x_+\bar{n}^\mu\right) \psi(0) |p\rangle$$

Substituting $x_+ = 2t$ one obtains

$$\int_{-\infty}^{\infty} \frac{dt}{2\pi}\, e^{-it\bar{n}k} \langle p| \psi^\dagger (t\bar{n}^\mu)\, \psi(0) |p\rangle = \int_{-\infty}^{\infty} \frac{dt}{2\pi}\, e^{-it\bar{n}k} \langle p| \bar{\psi} (t\bar{n}^\mu)\, \gamma_0\psi(0) |p\rangle$$

Since the momentum of the splitted quark is assumed to be n-collinear, one has $\not{n}\psi \approx 0$ which means

$$\begin{aligned}\gamma^0\psi - \vec{n}\cdot\vec{\gamma}\psi &= 0 \\ \Leftrightarrow \gamma^0\psi &= -\left(\vec{\bar{n}}\cdot\vec{\gamma}\right)\psi = 0 \\ \Leftrightarrow 2\gamma^0\psi &= \not{\bar{n}}\psi\end{aligned}$$

And so one ends up with

$$f_q(z) = \int_{-\infty}^{\infty} \frac{dt}{2\pi}\, e^{-it\bar{n}k} \langle p| \bar{\psi} (t\bar{n}^\mu) \frac{\not{\bar{n}}}{2}\psi(0) |p\rangle = \int_{-\infty}^{\infty} \frac{dt}{2\pi}\, e^{-izt\bar{n}p} \langle p| \bar{\psi} (t\bar{n}^\mu) \frac{\not{\bar{n}}}{2}\psi(0) |p\rangle$$

In order to render f_q gauge invariant, a Wilson line is inserted which is stretched between the points $x = 0$ and $x^\mu = t\bar{n}^\mu$

$$W(t\bar{n}) = P\exp\left(ig\bar{n}^\mu \int_0^t ds A_\mu(s\bar{n})\right)$$

So the final expression is

$$f_q(z) = \int_{-\infty}^{\infty} \frac{dt}{2\pi}\, e^{-izt\bar{n}p} \langle p| \bar{\psi} (t\bar{n}^\mu)\; W(t\bar{n})\, \frac{\not{\bar{n}}}{2}\psi(0) |p\rangle$$

This can be rewritten in terms of the SCET fields $\bar{\chi} = \bar{\psi}_n W$

$$f_q(z) = \int_{-\infty}^{\infty} \frac{dt}{2\pi}\, e^{-izt\bar{n}p} \langle p| \bar{\chi} (t\bar{n}^\mu) \frac{\not{\bar{n}}}{2}\chi(0) |p\rangle \tag{4.2}$$

The same considerations can be done for a hadron splitting into an antiquark and this gives [45], [42]

$$f_{\bar{q}}(z) = \int_{-\infty}^{\infty} \frac{dt}{2\pi}\, e^{-izt\bar{n}p} \mathbf{Tr}\left(\frac{\bar{\not{n}}}{2} \langle p|\, \psi\,(t\bar{n}^{\mu})\, \bar{\psi}(0)\,|p\rangle\right)$$

One can apply the anti-commutation relation to the fermion fields and since the vacuum expectation value is subtracted to get the connected matrix element for the PDF one obtains [42]

$$\begin{aligned} f_{\bar{q}}(z) &= \int_{-\infty}^{\infty} \frac{dt}{2\pi}\, e^{-izt\bar{n}p} \mathbf{Tr}\left(\frac{\bar{\not{n}}}{2} \langle p|\, \psi\,(t\bar{n}^{\mu})\, \bar{\psi}(0)\,|p\rangle\right) = -\int_{-\infty}^{\infty} \frac{dt}{2\pi}\, e^{-izt\bar{n}p} \mathbf{Tr}\left(\frac{\bar{\not{n}}}{2} \langle p|\, \bar{\psi}(0)\psi\,(t\bar{n}^{\mu})\,|p\rangle\right) \\ &= -\int_{-\infty}^{\infty} \frac{dt}{2\pi}\, e^{-izt\bar{n}p} \langle p|\, \bar{\psi}(-t\bar{n}^{\mu})\frac{\bar{\not{n}}}{2}\psi\,(0)\,|p\rangle \end{aligned}$$

substituting $t = -s$ one obtains

$$\begin{aligned} f_{\bar{q}}(z) &= \int_{\infty}^{-\infty} \frac{ds}{2\pi}\, e^{izs\bar{n}p} \langle p|\, \bar{\psi}(s\bar{n}^{\mu})\frac{\bar{\not{n}}}{2}\psi\,(0)\,|p\rangle \\ &= -\int_{-\infty}^{\infty} \frac{ds}{2\pi}\, e^{izs\bar{n}p} \langle p|\, \bar{\psi}(s\bar{n}^{\mu})\frac{\bar{\not{n}}}{2}\psi\,(0)\,|p\rangle \end{aligned}$$

and in terms of the gauge invariant fields χ one has

$$f_{\bar{q}}(z) = -\int_{-\infty}^{\infty} \frac{ds}{2\pi}\, e^{izs\bar{n}p} \langle p|\, \bar{\chi}(s\bar{n}^{\mu})\frac{\bar{\not{n}}}{2}\chi\,(0)\,|p\rangle$$

Compared with eq. (4.2) one obtains the relation

$$f_{\bar{q}}(z) = -f_q(-z)$$

The PDFs $f_i(z)$ are also called "collinear PDFs" or "integrated PDFs". In these objects only the partonic momentum component along the direction of the initial hadron is kept, while all other components were integrated out. The collinear PDFs are essential in situations where the hard scale Q of the problem is of the same magnitude as q_T ($Q \sim q_T \gg \Lambda_{QCD}$). In this case the collinear factorization can be written schematically as [46], [47]

$$\frac{d^2\sigma}{dQdq_T} \sim f_{i/N_1}(x_1,\mu) \otimes f_{j/N_2}(x_2,\mu) \otimes C_{ij}(z,Q,q_T,\mu)$$

where C_{ij} describes the physics at the hard scale $Q \sim q_T$.
The gluon PDF is given by

$$f_{g/N}(z) = -g_{\perp\mu\nu}\frac{z\bar{n}\cdot p}{2\pi}\int dt e^{-izt\bar{n}\cdot p}\sum_X \langle N(p)|\, \mathcal{A}_n^{\mu\,a}(t\bar{n})\,|X\rangle \langle X|\, \mathcal{A}_n^{\nu\,a}(0)\,|N(p)\rangle \tag{4.3}$$

with the SCET gluon fields A_n^{μ} defined in eq. (3.4).

4.2. Properties of the PDFs

The parton distribution functions can be studied in the parton model of deep inelastic scattering. In this section I point out the basic ideas. For more details I refer to [17, 48, 49]. In the parton model it is assumed that the partons inside a hadron are essentially free. Furthermore, the PDF $f_i(\xi)d\xi$ gives the probability to find a parton i inside the hadron which carries momentum fraction $\xi + d\xi$. The probabilistic interpretation is justified because the parton wavefunctions within the hadron can be treated as being decoherent [17]. In the deep inelastic scattering of leptons off a hadron according to the process

$$l(k) + h(p) \to l'(k') + X \tag{4.4}$$

the limit

$$Q^2 = -q^2 = -(k-k')^2 \to \infty$$

is considered. In this limit, the fluctuations within the hadron are at time scales Λ_{QCD}^{-1} and this is much slower than the scale Q^{-1} which is probed by the exchanged photon. So because the scales are separated according to $Q \gg \Lambda_{QCD}$, the partons within the proton can be assumed to be decoherent and thus the probabilistic interpretation is justified [48].
In the study of the process given in eq. (4.4), the total cross section is decomposed into a leptonic and a hadronic part according to

$$d\sigma = L_{\mu\nu}(k,k')W^{\mu\nu}(p,q)$$

The hadron tensor is the object one is interested in. The arguments of $W^{\mu\nu}$ can be rewritten as

$$W^{\mu\nu}(p,q) = W^{\mu\nu}(x,Q^2)$$

with

$$1 \geq x = \frac{Q^2}{2pq} \geq 0 \tag{4.5}$$

On the other hand, the parton model can be used: The probability of finding a parton i with momentum fraction ξ inside the hadron is given by the PDF $f_i(\xi)$. One can define a quantity corresponding to x on parton level. One has

$$z = \frac{Q^2}{2\xi pq} = \frac{x}{\xi}$$

And so it holds

$$\begin{aligned}
W^{\mu\nu}(x,Q^2) &= \sum_i \int_0^1 dz \int_0^1 d\xi f_i(\xi)\hat{W}^{\mu\nu}(z,Q)\delta(x-z\xi) \\
&= \sum_i \int_x^1 \frac{d\xi}{\xi} f_i(\xi)\hat{W}^{\mu\nu}\left(\frac{x}{\xi},Q\right) \\
&= \sum_i f_i(x) \otimes \hat{W}^{\mu\nu}(x,Q)
\end{aligned}$$

where $\hat{W}^{\mu\nu}(z,Q)$ is the partonic version of $W^{\mu\nu}(x,Q)$. Defining the function

$$W^0(x,Q) = -g_{\mu\nu}W^{\mu\nu}$$

one obtains in the DIS limit [17]

$$W^0(x,Q) = 4\pi\sum_i Q_i^2 f_i(x)$$

Thus, W^0 can be used to define the PDFs. On parton level one has [17]

$$\hat{W}_i^0(z,Q) = 4\pi Q_i^2\delta(1-z)$$

In the next step, the next-to-leading order corrections are taken into account. In the parton model in QCD, they are given by the exchange of a virtual gluon and the real radiation of a gluon. Summing these contributions and performing a rewriting according to

$$(1-z)^{-\epsilon-1} = -\frac{1}{\epsilon}\delta(1-z) + \left(\frac{1}{1-z}\right)_+ + \mathcal{O}(\epsilon)$$

with the plus distribution defined as

$$\int_0^1 dz \left(\frac{1}{1-z}\right)_+ f(z) = \int_0^1 dz \frac{f(z)-f(1)}{1-z}$$

one ends up with [17]

$$\begin{aligned}\hat{W}_i^0 =& 4\pi Q_i^2 \Bigg(\delta(1-z) - \frac{1}{\epsilon}\frac{\alpha_s}{2\pi} P_{qq}(z) \left(\frac{\mu^2}{Q^2}\right)^\epsilon e^{\epsilon\gamma_E} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \\ &+ \frac{\alpha_s}{2\pi} C_F \left[(1+z^2)\left(\frac{\ln(1-z)}{1-z}\right)_+ - \frac{3}{2}\left(\frac{1}{1-z}\right)_+ - \frac{1+z^2}{1-z}\ln(z) + 3 - 2z - \left(\frac{9}{2}+\frac{\pi^2}{3}\right)\delta(1-z)\right]\Bigg)\end{aligned}$$

with the quark-to-quark DGLAP splitting function

$$P_{qq}^{(1)}(z) = C_F\left(\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right)$$

So the result is divergent due to the pole in ϵ. The divergence can be absorbed into the PDFs, since

$$W^0(x, Q^2) = \sum_i f_i(x) \otimes \hat{W}_i^0\,(x, Q)$$

The renormalized PDFs are defined as

$$f_i^b = \sum_j Z_{ij} \otimes f_j$$

and thus

$$\begin{aligned}\sum_i f_i^b \otimes \hat{W}_i^{0b} &= \sum_{ij} Z_{ij} \otimes f_j \otimes \hat{W}_i^{0b} \\ &= \sum_i f_i \otimes \left(\sum_j Z_{ji} \otimes \hat{W}_i^{0b}\right)\end{aligned}$$

So the finite renormalized $\hat{W}_i^0$ is given by

$$\hat{W}_i^0 = \sum_j Z_{ji} \otimes \hat{W}_i^{0b}$$

and thus

$$Z_{qq}(z) = \delta(1-z) + \frac{\alpha_s}{2\pi}\frac{1}{\epsilon} P_{qq}^{(1)}(z)$$

Similarly one finds [17, 49]

$$\begin{aligned}P_{qq}^{(1)}(z) &= C_F\left(\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right) \\ P_{qg}^{(1)}(z) &= T_F\left(z^2 + (1-z)^2\right) \\ P_{gq}^{(1)}(z) &= C_F\left(\frac{1+(1-z)^2}{z}\right) \\ P_{gg}^{(1)}(z) &= 2C_A\left(\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z)\right) + \frac{\beta_0}{2}\delta(1-z)\end{aligned} \tag{4.6}$$

where $\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F n_f$. The evolution of the PDFs is given by the DGLAP equation [17]

$$\frac{d}{d\ln\mu} f_i = \frac{\alpha_s}{\pi} \sum_j P_{ij}^{(1)} \otimes f_j$$

On parton level, to all orders in dimensional regularization the bare PDFs are given by

$$f_{i/j}^b(z) = \delta_{ij}\delta(1-z) \tag{4.7}$$

This can be easily checked with the SCET definition of the PDFs (compare eq. (4.2)), since the contributions to higher orders in α_s are scaleless integrals and thus vanish. With the renormalization described by

$$f_{i/j}^b(z) = \sum_k Z_{i/k}(z,\mu) \otimes f_{k/j}(z,\mu)$$

and with

$$Z_{i/k}(z,\mu) = \delta_{ij}\delta(1-z) + \frac{\alpha_s}{2\pi}\frac{1}{\epsilon}P_{i/j}(z)$$

one obtains the renormalized PDFs on parton level as

$$f_{i/j}(z,\mu) = \delta_{ij}\delta(1-z) - \frac{\alpha_s}{2\pi}\frac{1}{\epsilon}P_{i/j}^{(1)}(z) + \mathcal{O}(\alpha_s^2) \tag{4.8}$$

Now, after the considerations made for the PDFs in this chapter, the transverse parton distribution functions (TPDFs) are discussed in the next chapter.

5. The transverse parton distribution functions

In the first part of this chapter a definition and general properties of the TPDFs, also called beam functions, are given. One only encounters a product of a collinear and an anti-collinear beam function. In this product there is still a dependence on the hard scale Q^2 hidden. This dependence is called collinear anomaly and it is caused by the analytic regularization of the rapidity divergences. The functional form of the collinear anomaly is proven in section 5.2. The TPDFs can be matched onto the PDFs. As the PDFs the TPDFs also need to be renormalized. After the derivation of the RGEs of the relevant functions, the matching and the renormalization is performed up to order α_s.

5.1. General considerations

The collinear factorization described in eq. (2.1) breaks down if $Q \gg q_T$. The appropriate solution is "transverse momentum dependent (TMD) factorization". The scales Q and q_T needs to be factorized. For this purpose the transverse momentum dependent PDFs (TPDFs), also called unintegrated PDFs or beam functions, are useful. The quark beam functions are given in transverse position space as [46]

$$\begin{aligned}
\mathcal{B}_{q/N}(z, x_T^2, \mu) &= \frac{1}{2\pi}\int dt\, e^{-izt\bar{n}\cdot p}\sum_X \frac{\bar{\not{n}}_{\alpha\beta}}{2}\left\langle N(p)\right| \bar{\chi}^n_\alpha(t\bar{n}+x_\perp)\left|X\right\rangle \left\langle X\right| \chi^n_\beta(0)\left|N(p)\right\rangle \\
\bar{\mathcal{B}}_{q/N}(z, x_T^2, \mu) &= \frac{1}{2\pi}\int dt\, e^{-iztn\cdot p}\sum_X \frac{\not{n}_{\alpha\beta}}{2}\left\langle N(p)\right| \bar{\chi}^{\bar{n}}_\alpha(tn+x_\perp)\left|X\right\rangle \left\langle X\right| \chi^{\bar{n}}_\beta(0)\left|N(p)\right\rangle
\end{aligned} \tag{5.1}$$

where $\mathcal{B}_{q/N}$ describes a hadron that moves in the direction of n and $\bar{\mathcal{B}}_{q/N}$ describes a hadron that moves in the direction of $\bar{n}$. As for the PDFs it holds for the antiquark beam functions

$$\mathcal{B}_{q/N}(z, x_T^2, \mu) = -\mathcal{B}_{\bar{q}/N}(-z, x_T^2, \mu)$$

The gluon TPDFs are given by [50]

$$\begin{aligned}
\mathcal{B}^{\mu\nu}_{g/N}(z, x_\perp) &= \frac{-z\bar{n}\cdot p}{2\pi}\int dt e^{-izt\bar{n}\cdot p}\sum_X \left\langle N(p)\right| \mathcal{A}^{\mu\, a}_n(t\bar{n}+x_\perp)\left|X\right\rangle \left\langle X\right| \mathcal{A}^{\nu\, a}_n(0)\left|N(p)\right\rangle \\
\bar{\mathcal{B}}^{\mu\nu}_{g/N}(z, x_\perp) &= \frac{-zn\cdot p}{2\pi}\int dt e^{-iztn\cdot p}\sum_X \left\langle N(p)\right| \mathcal{A}^{\mu\, a}_{\bar{n}}(tn+x_\perp)\left|X\right\rangle \left\langle X\right| \mathcal{A}^{\nu\, a}_{\bar{n}}(0)\left|N(p)\right\rangle
\end{aligned} \tag{5.2}$$

where $\mathcal{A}^{\nu a}_n$ is the gauge invariant collinear gluon field in SCET which was given in eq. (3.4). The gluon TPDF is a Lorentz tensor in the space perpendicular to n and $\bar{n}$ [46] which means

$$n_\mu \cdot \mathcal{B}^{\mu\nu}_{g/N} = 0 \qquad \bar{n}_\mu \cdot \mathcal{B}^{\mu\nu}_{g/N} = 0$$

The gluon TPDF can be decomposed into two independent components as

$$\mathcal{B}^{\mu\nu}_{g/N} = \frac{g_\perp^{\mu\nu}}{d-2}\mathcal{B}_{g/N}(z, x_T^2) + \left(\frac{g_\perp^{\mu\nu}}{d-2} + \frac{x_\perp^\mu x_\perp^\nu}{x_T^2}\right)\mathcal{B}'_{g/N}(z, x_T^2)$$

with [46]

$$\begin{aligned}
\mathcal{B}_{g/N}(z, x_T^2) &= g_{\perp\mu\nu}\, \mathcal{B}^{\mu\nu}_{g/N}(z, x_\perp) \\
\mathcal{B}'_{g/N}(z, x_T^2) &= \frac{1}{d-3}\left[g_{\perp\mu\nu} + (d-2)\frac{x_{\perp\mu}x_{\perp\nu}}{x_T^2}\right]\mathcal{B}^{\mu\nu}_{g/N}(z, x_\perp)
\end{aligned} \tag{5.3}$$

As the collinear PDFs the TPDFs are obtained from eq. (4.1) but now only the k_+ component is integrated out

$$\int \frac{d^4x}{(2\pi)^4} e^{-ixk} \langle p| \psi^\dagger(x)\psi(0) |p\rangle \Rightarrow \int \frac{d^4x}{(2\pi)^4}\, dk_+\, e^{-i\left(\frac{1}{2}(k_+x_- + k_-x_+) + k_\perp x_\perp\right)} \langle p| \psi^\dagger(x)\psi(0) |p\rangle$$

and since one is interested in a definition in transverse position space, the Fourier integral over $x^\mu_\perp$ does not need to be taken into account and so one obtains eq. (5.1).
In the region $Q \gg q_T, \Lambda_{QCD}$ the factorization of the differential cross section has schematically the form [46]

$$\frac{d^2\sigma}{dQdq_T} \sim \mathcal{B}_{i/N_1}(z_1, x_T^2, \mu, \alpha) \otimes \bar{\mathcal{B}}_{j/N_2}(z_2, x_T^2, \mu, \alpha) \otimes \mathcal{S}_{ij}(x_T^2, \mu, \alpha) \otimes H_{ij}(Q, \mu) \tag{5.4}$$

where $\mathcal{S}$ is the soft function and H is the hard function containing the high energetic scale Q. The parameter α is the analytic regulator. Both, the PDFs and the TPDFs describe the initial state radiation. Below I will show the derivation of the TMD factorization formula in SCET for the Drell-Yan production process.
There are two scenarios which have to be distinguished. First, if $q_T \sim \Lambda_{QCD}$, the TPDFs are fully non-perturbative objects and therefore, they can't be further specified with the methods of perturbative QCD. On the other hand, if $q_T \gg \Lambda_{QCD}$, the TPDFs are semi-perturbative objects and the two scales can be further factorized [46]. In this case, the TPDFs can be matched onto the collinear PDFs and the matching coefficient is calculable in perturbation theory. The matching has the form of a convolution. Then, the non-perturbative scale dependence is encoded in the PDFs and the perturbative part is encoded in the matching coefficient. The matching takes the form [14]

$$\begin{aligned}\mathcal{B}_{i/N}(x, x_T^2, \mu) &= \sum_j \mathcal{I}_{i/j}(x, x_T^2, \mu) \otimes f_{j/N}(x, \mu) \\ &= \int_x^1 \frac{d\xi}{\xi}\, \mathcal{I}_{i/j}(x/\xi, x_T^2, \mu)\, f_{j/N}(\xi, \mu)\end{aligned} \tag{5.5}$$

In the matching described in eq. (5.5), it is possible to extract an interpretation of the TPDFs. This is done in [14]. So first consider the picture 5.1 taken from [14]. There are three distinct scales $\mu_\Lambda \ll \mu_B \ll \mu_H$.

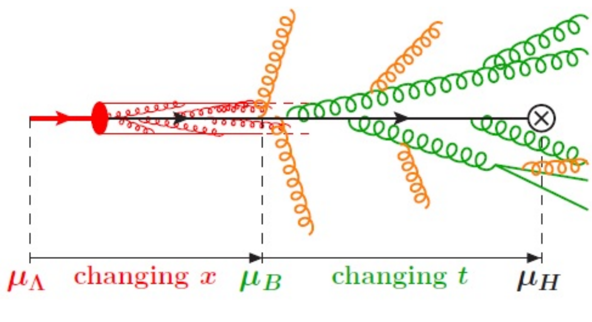


Figure 5.1.: Physics described by the TPDFs taken from [14].

At the low scale μ_Λ, a parton -with flavor k and momentum fraction x relative to the hadron momentum- which is contained in the hadron N is described by the PDF $\Phi_{k/N}(x, \mu_\Lambda)$. At the scale μ_B, the proton is probed by measuring the radiation in the final state. A parton j with momentum fraction x is identified by $\Phi_{j/N}(x, \mu_B)$. Above μ_B, the parton j radiates particles which are collinear to the incoming hadron direction up to $t = q_T^2 \ll Q^2$ (where t is the parton's virtuality). The resulting jet is described by $\mathcal{I}_{ij}(x/\xi, x_T^2, \mu)$. In figure 5.1, soft (wide-angle) radiation is shown as orange lines and the collinear radiation is depicted as green lines. A further aspect described by eq. (5.5) is that the initial state radiation changes the momentum fraction of the parton from ξ to x and the parton's flavor from j to i. At the scale μ_H, the parton i enters the hard reaction.

Thus, the beam function gives the probability of finding a parton with flavor i and collinear momentum fraction x and with transverse spread x_T^2 around the beam axis inside a hadron N.
Eq. (5.5) can be compared with the evolution equation of the PDFs which is [17]

$$\mu \frac{d}{d\mu} f_i(x,\mu) = \sum_j \frac{\alpha_s}{\pi} \int_x^1 \frac{d\xi}{\xi} P_{ij}(x/\xi,\mu) f_j(\xi,\mu)$$

Thus, the beam function can be interpreted as well as an evolution of the PDFs to a different scale.

5.2. The collinear anomaly

The cross section in eq. (5.4) contains the product of two beam functions. As described in [46, 1, 51, 50] the product of the beam functions still contains a dependence on the hard scale Q^2. In the quark case this product can be further factorized as [46]

$$\lim_{\alpha\to 0}\left(\mathcal{B}^b_{q/j}(z_1,x_T^2,\nu,\mu)\,\bar{\mathcal{B}}^b_{\bar{q}/k}(z_2,x_T^2,\nu,\mu)\right) = \underbrace{\left(\frac{x_T^2 Q^2}{4e^{-2\gamma_E}}\right)^{-F^b_{q\bar{q}}(x_T^2,\mu)}}_{\text{collinear anomaly}} B^b_{q/j}(z_1,x_T^2,\mu)\, B^b_{\bar{q}/k}(z_2,x_T^2,\mu) \tag{5.6}$$

The left-hand side of equation (5.6) is not a true factorization formula since the functions $\mathcal{B}_{i/j}$ still involve the two widely separated scales Q and q_T. To achieve a proper factorization of the two scales, one needs to extract the Q^2 dependence from the product of these functions. This was done on the right-hand side of eq. (5.6) where the $B_{i/j}$ are the true TPDFs which are universal process-independent functions and which do not depend on the hadron's direction.
The functions on the left-hand side depend on the parameter ν which is the t'Hooft scale associated with the analytic regularization of the rapidity divergences, cf. eq. (3.20). As in dimensional regularization the functions are expanded in the limit $\alpha \to 0$. The dependence on the hard scale Q^2 which is factorized explicitly on the right-hand side is caused by the analytic regularization. The right-hand side is independent of ν as it should since rapidity divergences are not physical. The form of the collinear anomaly is proven in this section.
In the case where the initial partons entering the hard reaction are gluons, one has to deal with the product of gluon TPDFs. They can be factorized in the same way such as [50]

$$\mathcal{B}^{b\mu\nu}_{g/j}(z_1,x_T^2,\nu,\mu)\,\bar{\mathcal{B}}^{b\rho\sigma}_{g/k}(z_2,x_T^2,\nu,\mu) = \left(\frac{x_T^2 Q^2}{4e^{-2\gamma_E}}\right)^{-F^b_{gg}(x_T^2,\mu)} B^{b\mu\nu}_{g/j}(z_1,x_T^2,\mu)\, B^{b\rho\sigma}_{g/k}(z_2,x_T^2,\mu) \tag{5.7}$$

In this section I will show how the dependence on the hard scale in eq. (5.6) and (5.7) arise. For that purpose, I assume that the beam functions have the following scale dependence (with the hard scale $\bar{n}p = Q = n\bar{p}$)

$$\begin{aligned} \mathcal{B}_{i/j} &= \mathcal{B}_{i/j}(z,\mu x_T,\nu\bar{n}p x_T^2) = \mathcal{B}_{i/j}(z,\mu x_T,\nu Q x_T^2) \\ \bar{\mathcal{B}}_{i/j} &= \bar{\mathcal{B}}_{i/j}\left(z,\mu x_T,\frac{\nu}{n\bar{p}}\right) = \bar{\mathcal{B}}_{i/j}\left(z,\mu x_T,\frac{\nu}{Q}\right) \end{aligned} \tag{5.8}$$

Since the soft function reduces to one in a Drell-Yan process (see below), the product of the two beam functions has to be independent of ν which means

$$\frac{d}{d\nu}\ln\left(\mathcal{B}\bar{\mathcal{B}}\right) = 0 = \frac{d}{d\nu}\left(\ln\mathcal{B} + \ln\bar{\mathcal{B}}\right)$$

Calling

$$\ln\bar{\mathcal{B}}(\nu/Q) = f_1(\nu/Q), \qquad \ln\mathcal{B}(\nu Q x_T^2) = f_2(\nu Q x_T^2)$$

one has

$$\begin{aligned} &\frac{1}{Q}f_1'(\nu/Q)+Qx_T^2 f_2'(\nu Q x_T^2)=0 \\ \Leftrightarrow\ &f_1'(\nu/Q)+Q^2x_T^2 f_2'(\nu Q x_T^2)=0 \qquad\qquad |\nu=\tau\cdot Q \\ \Leftrightarrow\ &f_1'(\tau)+Q^2x_T^2 f_2'(\tau Q^2 x_T^2)=0 \end{aligned}$$

Taking the derivative with respect to Q^2 one obtains

$$\begin{aligned} &x_T^2 f_2'(\tau Q^2x_T^2)+Q^2x_T^2\frac{df_2'(\tau Q^2 x_T^2)}{dQ^2}=0 \\ \Leftrightarrow\ &f_2'(\tau Q^2x_T^2)+Q^2\tau x_T^2\frac{df_2'(\tau Q^2 x_T^2)}{d(\tau Q^2x_T^2)}=0 \end{aligned}$$

Substituting $h(x)=f_2'(x)$ one gets

$$\begin{aligned} &h(x)+x\cdot h'(x)=0 \\ \Leftrightarrow\ &\frac{dx}{x}=-\frac{dh}{h}\Rightarrow h(x)=\frac{a}{x} \end{aligned}$$

and thus

$$f_2'(x)=\frac{a}{x}\Rightarrow f_2(x)=a\ln x+\beta$$

and since $f_1'(\nu/Q)+Q^2x_T^2f_2'(\nu Q x_T^2)=0$ one has

$$\begin{aligned} f_1'(\nu/Q)=-Q^2x_T^2\frac{\alpha}{\nu Q x_T^2}&=-Q\frac{a}{\nu} \\ \Leftrightarrow f_1'(x)&=-\frac{a}{x}\Rightarrow f_1=-a\ln x+\gamma \end{aligned}$$

So one ends up with

$$\begin{aligned} \ln\left(\mathcal{B}(\nu Q x_T^2)\cdot\bar{\mathcal{B}}(\nu/Q)\right)=f_1(\nu/Q)+f_2(\nu Q x_T^2)&=-a\ln(\nu/Q)+a\ln(\nu Q x_T^2)+\beta+\gamma \\ &=a\ln(Q^2x_T^2)+\delta \end{aligned}$$

And so

$$\mathcal{B}(\nu Q x_T^2)\cdot\bar{\mathcal{B}}(\nu/Q)\propto\left(Q^2x_T^2\right)^a$$

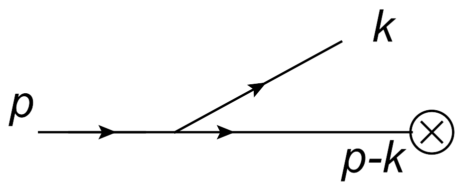
where a is not dependent on $x_T^2Q^2$, but can depend on the scale combination $x_T\mu$. Thus, these considerations reproduce the dependence on the hard scale Q of eq. (5.6) and (5.7).
As a last step it remains to show that the assumption of eq. (5.8) are met. This is done in the next section.

5.2.1. Scale combination in the beam function

Consider the collinear quark beam function on parton level which is

$$\mathcal{B}_{q/j}(z,x_T^2,\mu)=\frac{1}{2\pi}\int dt e^{-izt\bar{n}\cdot p}\sum\!\!\!\!\!\!\!\int_X \frac{\bar{\not{n}}_{\alpha\beta}}{2}\left\langle j(p)\right|\bar{\chi}^n_\alpha(t\bar{n}+x_\perp)\left|X(k)\right\rangle\left\langle X(k)\right|\chi^n_\beta(0)\left|j(p)\right\rangle$$

where the state X denotes the emitted partons and one sums over all these states.



$$p^\mu = p_0 \cdot n^\mu = (p_0, 0, 0, p_0),$$
$$k^\mu = \Big((1-z)p_0, \vec{k}_T, (1-z)p_0\Big),$$
$$p^\mu - k^\mu = (zp_0, -\vec{k}_T, zp_0)$$

For example the quark to quark splitting in NLO is given by

$$\mathcal{B}^{(1)}_{q/q}(z, x_T^2, \mu) = \frac{\mu^{2\epsilon} e^{\epsilon\gamma_E}}{(4\pi)^\epsilon} \int \frac{d^dk}{(2\pi)^{d-1}} \underbrace{\delta(k^+k^- - k_T^2)\theta(k_0)}_{\delta^+(k^2)} e^{-ix_\perp k_\perp} \underbrace{\left(\frac{\nu}{n\cdot k}\right)^\alpha}_{\text{analytic reg.}}$$
$$\underbrace{\delta(\bar{n}k - (1-z)\bar{n}p)}_{\text{splitting constraint in n-direction}} \quad \frac{\bar{\not{n}}_{\alpha\beta}}{2} \langle q(p)| \bar{\chi}^n_\alpha(0) |g(k)\rangle \langle g(k)| \chi^n_\beta(0) |q(p)\rangle$$

To fixed order, one assumes that r partons are emitted. Calling the momenta of the emitted partons l_i, $i = 1, ..., r$ so that $k = \sum_i l_i$. One obtains

$$\begin{aligned}\mathcal{B}_{q/j}(z,x_T^2, \mu) &= \left[\prod_i \int \frac{d^dl_i}{(2\pi)^{d-1}} \delta^+(l_i^2) \underbrace{\left(\frac{\nu}{n\cdot l_i}\right)^\alpha}_{\text{analytic reg.}}\right] \int d^dk\, \delta^d\Big(k - \sum_i l_i\Big) e^{-ix_\perp k_\perp} \\ &\times \delta(\bar{n}k - (1-z)\bar{n}p) \underbrace{\frac{\bar{\not{n}}_{\alpha\beta}}{2} \langle j(p)| \bar{\chi}^n_\alpha(0) |p_1(l_1)...p_r(l_r)\rangle \; \langle p_1(l_1)...p_r(l_r)| \chi^n_\beta(0) |j(p)\rangle}_{=\mathcal{M}(p,l_1,...l_r)} \\ &= \int d\Pi^n_r \cdot \mathcal{M}(p, l_1, ...l_r)\end{aligned}$$

with the phase space factor

$$\begin{aligned}\int d\Pi^n_r = &\left[\prod_i \int \frac{d^dl_i}{(2\pi)^{d-1}} \delta(l_i^+l_i^- - l_{i,T}^2)\theta(l_i^0) \left(\frac{\nu}{n\cdot l_i}\right)^\alpha\right] \\ &\times \int d^dk\, \delta^d\Big(k - \sum_i l_i\Big) e^{ix_Tk_T} \delta(\bar{n}k - (1-z)\bar{n}p)\end{aligned} \tag{5.9}$$

Which scale is combined with ν?

The collinear phase space

The collinear phase space factor $\int d\Pi^n_r$ is

$$\left[\prod_i \int \frac{d^dl_i}{(2\pi)^{d-1}} \delta(l_i^+l_i^- - l_{i,T}^2)\theta(l_i^0) \left(\frac{\nu}{n\cdot l_i}\right)^\alpha\right] \int d^dk\, \delta^d\Big(k - \sum_i l_i\Big) e^{ix_Tk_T} \delta(\bar{n}k - (1-z)\bar{n}p)$$

Rescaling to know which scale is l_i^+

$$k^- = \bar{n}\cdot k \to k^-p^- \qquad \text{and} \qquad l_i^- \to l_i^-p^-$$

achieves that k^- and l_i^- are scaleless. One obtains the delta-function $\delta(p^-l_i^+l_i^- - l_{i,T}^2)$. In order to make $\vec{l}_{i,T}$ dimensionless and to absorb the scale of l_i^- in l_i^+ one rescales

$$|\vec{l}_{i,T}| = l_{i,T} \to \frac{l_{i,T}}{x_T} \qquad \text{and} \qquad l_i^+ \to l_i^+/(p^-x_T^2)$$

Due to the rescaling one obtains

$$\delta(l_i^+ l_i^- - l_{i,T}^2)\left(\frac{\nu}{l_i^+}\right)^\alpha \to \delta\left(l_i^+ l_i^- - l_{i,T}^2\right)\left(\frac{\nu p^- x_T^2}{l_i^+}\right)^\alpha$$

And so one can conclude that

$$\left(\frac{\nu}{l_i^+}\right)^\alpha \to \left(\nu p^- x_T^2\right)^\alpha \left(\frac{1}{l_i^+}\right)^\alpha \Rightarrow \mathcal{B}_{i/j}(z, \mu x_T, \nu Q x_T^2)$$

So the arguments of the collinear beam function stated in eq. (5.8) are proven.

The anti-collinear phase space

The anti-collinear quark beam function is

$$\bar{\mathcal{B}}_{q/j}(z, x_T^2, \mu) = \frac{1}{2\pi}\int dt\, e^{-iztn\cdot p} \sum\!\!\!\!\!\!\int_X \frac{\not{n}_{\alpha\beta}}{2} \left\langle j(\bar{p})\right| \bar{\chi}_\alpha^{\bar{n}}(tn + x_\perp) \left|X\right\rangle \left\langle X\right| \chi_\beta^{\bar{n}}(0) \left|j(\bar{p})\right\rangle$$

with the phase space factor

$$\begin{aligned}\int d\Pi_r^{\bar{n}} = &\left[\prod_i \int \frac{d^d l_i}{(2\pi)^{d-1}} \delta(l_i^+ l_i^- - l_{i,T}^2)\theta(l_i^0)\left(\frac{\nu}{n\cdot l_i}\right)^\alpha\right] \\ &\times \int d^d k\, \delta^d\Big(k - \sum_i l_i\Big) e^{ix_T k_T} \cdot \delta(nk - (1-z)n\bar{p})\end{aligned}$$

The difference compared to the collinear phase space stated in eq. (5.9) is located in the splitting constraint encoded in the delta function. Rescaling

$$n\cdot k \to \bar{p}^+ k^+$$

makes k^+ scaleless. And rescaling

$$l_i^+ \to l_i^+ \bar{p}^+$$

converts

$$\left(\frac{\nu}{l_i^+}\right)^\alpha \to \left(\frac{\nu}{l_i^+ \bar{p}^+}\right)^\alpha = \left(\frac{\nu}{\bar{p}^+}\right)^\alpha \left(\frac{1}{l_i^+}\right)^\alpha \Rightarrow \bar{\mathcal{B}}_{i/j}\left(z, \mu x_T, \frac{\nu}{Q}\right)$$

Thus, the assumption that was made in order to derive the form of the collinear anomaly was correct (compare eq. (5.8)).
In this section, the collinear anomaly was treated and its functional form was proven. In the next section, the discussion continues with the matching of the TPDFs onto the PDFs.

5.3. Matching of TPDFs onto PDFs and renormalization

The index b in eq. (5.6) and (5.7) indicates that the functions still need to be renormalized. The renormalization prescription is stated as

$$\begin{aligned} B_{i/j}^b(z, x_T^2, \mu) &= Z_i^B(x_T^2, \mu) B_{i/j}(z, x_T^2, \mu) \\ F_{i\bar{i}}^b(x_T^2) &= F_{i\bar{i}}(x_T^2, \mu) + Z_i^F(\mu) \end{aligned} \tag{5.10}$$

The UV poles are absorbed in the renormalization constants Z. Thus, the renormalized functions $B_{i/j}$ and $F_{i\bar{i}}$ are UV finite. On the other hand, $B_{i/j}$ still contains IR poles. These IR poles are exactly the same as those in the PDFs. For $q_T^2 \gg \Lambda_{QCD}$ the TPDFs can be matched onto the collinear PDFs which was already given in eq. (5.5). The matching is performed on parton level with a parton j in place of the hadron N. The partonic beam functions are compared with the partonic PDFs given in eq. (4.2), (4.7) and (4.8). On parton level, and after the extraction of the collinear anomaly according to eq. (5.6) and (5.7), one has the bare and renormalized matching prescription

$$\begin{aligned} B^b_{i/j}(z, x_T^2, \mu) &= \sum_k I^b_{i/k}(z, x_T^2, \mu) \otimes \Phi^b_{k/j}(z, \mu) \\ B_{i/j}(z, x_T^2, \mu) &= \sum_k I_{i/k}(z, x_T^2, \mu) \otimes \Phi_{k/j}(z, \mu) \end{aligned} \tag{5.11}$$

where the PDFs are now called $\Phi_{i/j}$ in order to prevent confusion with other functions introduced below. The UV and IR poles are encoded in the constituent functions in the following way

$$\begin{aligned} B^b_{i/j}(z, x_T^2, \mu) &= \underbrace{Z_i^B(x_T^2, \mu)}_{\text{UV-poles}} \underbrace{B_{i/j}(z, x_T^2, \mu)}_{\text{IR-poles}} \\ &= Z_i^B(x_T^2, \mu) \sum_k \underbrace{I_{i/k}(z, x_T^2, \mu)}_{\text{free of any poles}} \otimes \Phi_{k/j}(z, \mu) \end{aligned}$$

With eq. (4.7) one concludes

$$\begin{aligned} B^b_{i/j}(z, x_T^2, \mu) &= \sum_k I^b_{i/k}(z, x_T^2, \mu) \otimes \Phi^b_{k/j}(z, \mu) = I^b_{i/j}(z, x_T^2, \mu) \\ \Rightarrow I^b_{i/j}(z, x_T^2, \mu) &= Z_i^B(x_T^2, \mu) \sum_k I_{i/k}(z, x_T^2, \mu) \otimes \Phi_{k/j}(z, \mu) \end{aligned} \tag{5.12}$$

The calculation of the beam functions is performed in light-cone gauge. In this gauge all Wilson lines do not have to be considered because they reduce to 1.

5.4. Resummation

In order to derive an evolution equation for the beam functions and the matching kernels one takes a look at the differential cross section for a Drell-Yan production process at small transverse momentum which is stated as [1, 51] (see eq. (6.4))

$$\begin{aligned} \frac{d^3\sigma}{dM^2 dq_T^2 dy} &= \frac{4\pi\alpha^2}{3N_c M^2 s} \left|\tilde{C}_V(-q^2, \mu)\right|^2 \frac{1}{4\pi} \int d^2x_\perp e^{-iq_\perp x_\perp} \\ &\times \sum_q e_q^2 \left[\left(\frac{x_T^2 q^2}{4e^{-2\gamma_E}} \right)^{-F^b_{q\bar{q}}(x_T^2, \mu)} B_{q/N_1}(z_1, x_T^2, \mu)\, B_{\bar{q}/N_2}(z_2, x_T^2, \mu) + (q \leftrightarrow \bar{q}) \right] + \mathcal{O}\left(\frac{q_T^2}{M^2} \right) \end{aligned} \tag{5.13}$$

Since the scale μ is not physical it has to drop out in the total expression which means

$$\frac{d}{d\ln\mu} \left(\frac{d^3\sigma}{dM^2 dq_T^2 dy} \right) = 0 \tag{5.14}$$

On the other hand one calculates (using $B_{q/N} = \sum_k I_{q/k} \otimes \Phi_{k/N}$)

$$\begin{aligned}
\frac{d}{d\ln\mu}\left(\frac{d^3\sigma}{dM^2 dq_T^2 dy}\right) &= \frac{4\pi\alpha_s^2}{3N_c M^2 s}|C_V(-q^2,\mu)|^2 \sum_{i,j}\sum_q e_q^2 \left(\frac{1}{4\pi}\int d^2x_\perp e^{ix_T q_T}\right. \\
&\times \left(\frac{x_T^2 q^2}{4e^{-2\gamma_E}}\right)^{-F_{q\bar{q}}(L_\perp,a_s)} \left. I_{q\leftarrow i}(z_1,L_\perp)\; I_{\bar{q}\leftarrow j}(z_2,L_\perp) + (q\leftrightarrow\bar{q})\right) \otimes \Phi_{i/N_1}(z_1,\mu) \\
&\otimes \Phi_{j/N_2}(z_2,\mu)\left(2\left(2\gamma_q(\alpha_s) + \Gamma_{cusp}(\alpha_s)\ln\frac{q^2}{\mu^2}\right) - \ln\left(\frac{x_T^2 q^2}{4e^{-2\gamma_E}}\right)\frac{dF_{q\bar{q}}(L_\perp,a_s)}{d\ln\mu}\right) \\
&+ \frac{4\pi\alpha_s^2}{3N_c M^2 s}|C_V(-q^2,\mu)|^2 \sum_{i,j}\sum_q e_q^2 \left(\frac{1}{4\pi}\int d^2x_\perp e^{ix_T q_T}\left(\frac{x_T^2 q^2}{4e^{-2\gamma_E}}\right)^{-F_{q\bar{q}}(L_\perp,a_s)}\right. \\
&\times \left.\frac{d}{d\ln\mu}\left(I_{q\leftarrow i}(z_1,L_\perp)\; I_{\bar{q}\leftarrow j}(z_2,L_\perp) + (q\leftrightarrow\bar{q})\right)\right) \otimes \Phi_{i/N_1}(z_1,\mu) \otimes \Phi_{j/N_2}(z_2,\mu) \\
&+ \frac{4\pi\alpha_s^2}{3N_c M^2 s}|C_V(-q^2,\mu)|^2 \sum_{i,j}\sum_q e_q^2 \left(\frac{1}{4\pi}\int d^2x_\perp e^{ix_T q_T}\left(\frac{x_T^2 q^2}{4e^{-2\gamma_E}}\right)^{-F_{q\bar{q}}(L_\perp,a_s)}\right. \\
&\times \left. I_{q\leftarrow i}(z_1,L_\perp)\; I_{\bar{q}\leftarrow j}(z_2,L_\perp) + (q\leftrightarrow\bar{q})\right) \\
&\times \sum_{l,k}\left(4P_{jk}(z,\mu) + 4P_{il}(z,\mu)\right) \otimes \Phi_{l/N_1}(z_1,\mu) \otimes \Phi_{k/N_2}(z_2,\mu)
\end{aligned}$$

where the already known RGEs for the Wilson coefficient C_V and the PDFs Φ were inserted. These are given by

$$\begin{aligned}
\frac{d}{d\ln\mu}C_V(-q^2,\mu) &= \left(2\gamma_q(\alpha_s) + \Gamma_{cusp}(\alpha_s)\ln\frac{-q^2}{\mu^2}\right) C_V(-q^2,\mu) \\
\frac{d}{d\ln\mu}\Phi_{j/N}(z,\mu) &= 4\sum_k P_{jk}(z,\mu)\otimes\Phi_{k/N}(z,\mu)
\end{aligned}$$

where the DGLAP splitting functions are given by $P^{(1)}_{jk}(z)$

$$P_{jk}(z,\mu) = \frac{\alpha_s}{4\pi}P^{(1)}_{jk}(z) + \left(\frac{\alpha_s}{4\pi}\right)^2 P^{(2)}_{jk}(z) + ...$$

With the constraint given by eq. (5.14), one can infer the RGEs for $F_{i\bar{i}}$ and $I_{i/j}$

$$\begin{aligned}
\frac{d}{d\ln\mu}F_{i\bar{i}}(x_T^2,\mu) &= 2\Gamma_{cusp}(\alpha_s) \\
\frac{d}{d\ln\mu}I_{i/j}(z,x_T^2,\mu) &= \left[\Gamma_{cusp}(\alpha_s)L_\perp - 2\gamma_q(\alpha_s)\right] I_{i/j}(z,x_T^2,\mu) - 4\sum_k I_{i/k}(z,x_T^2,\mu)\otimes P_{kj}(z,\mu)
\end{aligned} \tag{5.15}$$

In the next sections the beam functions and the matching kernels are calculated up to order α_s. With the RGEs the obtained results can be checked.

5.5. LO calculation

The matching of the beam functions onto the PDFs is performed on parton level. The leading order contribution is meant to be the order $\mathcal{O}(\alpha_s^0)$ which means that there is actually no splitting process. So one can conclude that one has to obtain in leading order

$$\mathcal{B}^{(0)}_{i/j}(z,x_T^2) = \delta_{ij}\delta(1-z)$$

The calculation is performed in light-cone gauge in which the Wilson lines become a factor of unity.

5.5.1. The quark TPDF

One has to calculate

$$\begin{aligned}
\mathcal{B}^{(0)}_{i/j}(z,x_T^2,\mu) &= \frac{1}{2\pi}\int dt e^{-izt\bar{n}\cdot p}\frac{\bar{\not{n}}_{\alpha\beta}}{2}\frac{1}{2}\sum_\lambda \langle j(p)|\,\bar{\chi}^i_\alpha(t\bar{n}+x_\perp)\,|0\rangle\,\langle 0|\,\chi^j_\beta(0)\,|j(p)\rangle \\
&= \delta_{ij}\frac{1}{2\pi}\int dt e^{-izt\bar{n}\cdot p}e^{it\bar{n}\cdot p}\frac{\bar{\not{n}}_{\alpha\beta}}{2}\frac{1}{2}\sum_\lambda \bar{u}_\alpha(p,\lambda)\cdot u_\beta(p,\lambda) \\
&= \delta_{ij}\frac{1}{2\pi}\int dt e^{-izt\bar{n}\cdot p}e^{it\bar{n}\cdot p}\frac{\bar{\not{n}}_{\alpha\beta}}{4}\not{p}_{\alpha\beta} \\
&= \delta_{ij}\ \delta(z-1)
\end{aligned}$$

The spin averaging according to $\frac{1}{2}\sum_\lambda$ is hidden in the $\sum_X$ in eq. (5.1).

5.5.2. The gluon TPDF

It is to calculate

$$\begin{aligned}
\mathcal{B}^{(0)\mu\nu}_{i/j}(z,x_\perp) &= \frac{-z\bar{n}\cdot p}{2\pi}\int dt e^{-izt\bar{n}\cdot p}\frac{1}{d-2}\sum_\lambda \langle j(p)|\,\mathcal{A}^{\mu\,a}_{n,\perp}(t\bar{n}+x_\perp)\mathcal{A}^{\nu\,a}_{n,\perp}(0)\,|j(p)\rangle \\
&= \delta_{ij}\frac{1}{d-2}\sum_\lambda\left(\frac{-z\bar{n}\cdot p}{2\pi}\int dt e^{-izt\bar{n}\cdot p}e^{ip(t\bar{n}+x_\perp)}\epsilon^{*\mu}(p,\lambda)\epsilon^\nu(p,\lambda)\right) \\
&= \frac{1}{d-2}\left(\frac{-z\bar{n}\cdot p}{2\pi}\int dt e^{-it\bar{n}\cdot p(z-1)}\left[-g^{\mu\nu}+\frac{p^\mu\bar{n}^\nu+p^\nu\bar{n}^\mu}{p\cdot\bar{n}}\right]\delta_{ij}\right) \\
&= \frac{1}{d-2}\left(-\delta(z-1)\left[-g^{\mu\nu}+\frac{p^\mu\bar{n}^\nu+p^\nu\bar{n}^\mu}{p\cdot\bar{n}}\right]\delta_{ij}\right)
\end{aligned}$$

Now one can calculate the two components $\mathcal{B}_{g/g'}$ and $\mathcal{B}'_{g/g'}$

$$\begin{aligned}
\mathcal{B}_{i/j}(z,x_T^2) &= g_{\perp\mu\nu}\ \mathcal{B}^{\mu\nu}_{i/j}(z,x_\perp) \\
&= \delta(z-1)\delta_{ij}
\end{aligned}$$

since $g_{\perp\mu\nu}(p^\mu n^\nu+p^\nu n^\mu)=0$ and $g_{\perp\mu\nu}g^{\mu\nu}=d-2$. And

$$\mathcal{B}'_{i/j} = \frac{1}{d-3}\left[g_{\perp\mu\nu}+(d-2)\frac{x_{\perp\mu}x_{\perp\nu}}{x_T^2}\right]\mathcal{B}^{\mu\nu}_{i/j}(z,x_\perp) = 0$$

because $x_\perp\cdot p = 0 = x_\perp\cdot\bar{n}$ and $x_\perp^2=-x_T^2$.

5.6. NLO calculation

In this section I give the result of $\mathcal{B}^{(1)}_{q/q}$ and $\mathcal{B}^{(1)}_{q/g}$ which are both calculable via the collinear quark TPDF

$$\begin{aligned}
\mathcal{B}_{q/j}(z,x_T^2,\mu) &= \frac{1}{2\pi}\int dt\ e^{-izt\bar{n}\cdot p}\sum\!\!\!\!\!\!\int_X \frac{\bar{\not{n}}_{\alpha\beta}}{2}\,\langle j(p)|\,\bar{\chi}^n_\alpha(t\bar{n}+x_\perp)\,|X(k)\rangle\,\langle X(k)|\,\chi^n_\beta(0)\,|j(p)\rangle \\
&= \delta(\bar{n}k-(1-z)\bar{n}p)\sum\!\!\!\!\!\!\int_X \frac{\bar{\not{n}}_{\alpha\beta}}{2}\,\langle j(p)|\,\bar{\chi}^n_\alpha(x_\perp)\,|X(k)\rangle\,\langle X(k)|\,\chi^n_\beta(0)\,|j(p)\rangle
\end{aligned}$$

and $\mathcal{B}^{(1)}_{g/g}$ and $\mathcal{B}^{(1)}_{g/q}$ which are calculable as

$$\begin{aligned}\mathcal{B}^{(1)}_{g/j}(z,x_T^2,\mu) &= g_{\perp\mu\nu}\mathcal{B}^{\mu\nu}_{g/j}(z,x_\perp,\mu)\\ &= g_{\perp\mu\nu}\frac{-z\bar{n}\cdot p}{2\pi}\int dt e^{-izt\bar{n}\cdot p}\sum\!\!\!\!\!\!\int_X \langle N(p)|\,\mathcal{A}^{\mu\,a}_n(t\bar{n}+x_\perp)\,|X(k)\rangle\,\langle X(k)|\,\mathcal{A}^{\nu\,a}_n(0)\,|N(p)\rangle\end{aligned}$$

The gluon TPDFs are built out of two components (see eq. (5.3)). Here only the first component is considered and matched. In a Drell-Yan production process, the gluon TPDFs do not contribute and in top quark pair production the function $\mathcal{B}'^{(1)}_{g/i}$ does not contribute to NNLL. This is shown above. So for these applications, it is sufficient to evaluate the function $\mathcal{B}^{(1)}_{g/i}$. Comparing eq. (4.3) with eq. (5.3), it is reasonable to match the first component onto the PDFs due to their similarity. In [50], the result for $I'^{(1)}_{g/i}$ is stated.
The bare NLO results are given by [46]

$$\begin{aligned}\mathcal{B}^{(1)}_{i/j}(z,x_T^2,\mu,\nu) &= \frac{g^2}{16\pi^2}e^{\alpha L_c+\epsilon L_\perp}e^{-(\epsilon+2\alpha)\gamma_E}\frac{\Gamma(-\epsilon-\alpha)}{\Gamma(1+\alpha)}(1-z)^{\alpha}f^{(1)}_{i/j}(z)\\ \bar{\mathcal{B}}^{(1)}_{i/j}(z,x_T^2,\mu,\nu) &= \frac{g^2}{16\pi^2}e^{\alpha L_a+\epsilon L_\perp}e^{-\epsilon\gamma_E}\Gamma(-\epsilon)(1-z)^{-\alpha}f^{(1)}_{i/j}(z)\end{aligned} \tag{5.16}$$

with

$$\begin{aligned}L_c &= \ln\frac{\nu p_- x_T^2}{4e^{-2\gamma_E}}\\ L_\perp &= \ln\frac{x_T^2\mu^2}{4e^{-2\gamma_E}}\\ L_a &= \ln\frac{\nu}{n\cdot\bar{p}} = \ln\frac{\nu}{\bar{p}^+}\end{aligned}$$

where p denotes the collinear momentum $p^\mu \sim n^\mu$ and $\bar{p}$ denotes the anti-collinear momentum $\bar{p}^\mu \sim \bar{n}^\mu$. And the functions $f^{(1)}_{i/j}$ are

$$\begin{aligned}f^{(1)}_{g/g}(z) &= 4C_A\left(\frac{1-z}{z}+\frac{z}{1-z}+z(1-z)\right)\\ f^{(1)}_{g/q}(z) &= 2C_F\left(\frac{1+(1-z)^2}{z}-\epsilon z\right)\\ f^{(1)}_{q/g}(z) &= 2T_F\left(1-\frac{2}{1-\epsilon}z(1-z)\right)\\ f^{(1)}_{q/q}(z) &= 2C_F\left(\frac{1+z^2}{1-z}-\epsilon(1-z)\right)\end{aligned}$$

The notation $f_{i/j}$ means that parton j with momentum p is splitting into parton i with momentum zp, where i, j stands for quark, antiquark or gluon. These functions are in correspondence with the the splitting functions in eq. (2.4) where a different naming convention is used.
As an example I show the calculation of the quark to quark beam function. One starts with (using $\overline{\text{MS}}$ scheme)

$$\begin{aligned}\mathcal{B}^{(1)}_{q/q}(z,x_T^2,\mu) = \frac{\mu^{2\epsilon}e^{\epsilon\gamma_E}}{(4\pi)^\epsilon}&\int\frac{d^dk}{(2\pi)^{d-1}}\delta(k^+k^- - k_T^2)\theta(k_0)e^{-ix_\perp k_\perp}\left(\frac{\nu}{n\cdot k}\right)^{\alpha}\\ &\frac{1}{2N_C}\sum_{spin,color}\delta(\bar{n}k-(1-z)\bar{n}p)\frac{\bar{\not{n}}_{\alpha\beta}}{2}\langle q(p)|\,\bar{\chi}^n_\alpha(0)\,|g(k)\rangle\,\langle g(k)|\,\chi^n_\beta(0)\,|q(p)\rangle\end{aligned}$$

The matrix element can be calculated using FORM [52]

$$\frac{1}{2N_C}\sum_{spin,color}\frac{\bar{\not{n}}_{\alpha\beta}}{2}\langle q(p)|\,\bar{\chi}^n_\alpha(0)\,|g(k)\rangle\,\langle g(k)|\,\chi^n_\beta(0)\,|q(p)\rangle$$
$$=\frac{g^2C_F}{2}\mathrm{Tr}\left(\frac{\bar{\not{n}}}{2}\frac{(\not{p}-\not{k})}{(p-k)^2}\gamma^\nu\;\not{p}\gamma^\mu\frac{(\not{p}-\not{k})}{(p-k)^2}\right)\cdot\left(-g_{\mu\nu}+\frac{k_\mu\bar{n}_\nu+k_\nu\bar{n}_\mu}{k\cdot\bar{n}}\right)$$
$$=g^2C_F\left(2(1-\epsilon)\frac{k_-}{p_-k_+}+4\frac{p_--k_-}{k_+k_-}\right)$$

where the SCET interaction Lagrangian was inserted which is done as usual in the form

$$\exp\left(i\int d^dx\,\mathcal{L}_I\right)=\exp\left(-ig\int d^dx\,\bar{\xi}_n(x)\gamma^\mu T^b\xi_n(x)A^b_{c\mu}(x)\right)$$
$$\overset{NLO}{=}-ig\int d^dx\,\bar{\xi}_n(x)\gamma^\mu T^b\xi_n(x)A^b_{c\mu}(x)$$

and in lightcone gauge one has

$$\sum_\lambda\epsilon^{*b}_\mu(k,\lambda)\epsilon^c_\nu(k,\lambda)=\left(-g_{\mu\nu}+\frac{k_\mu\bar{n}_\nu+k_\nu\bar{n}_\mu}{k\cdot\bar{n}}\right)\delta^{bc}$$

The integral of interest is thus

$$\mathcal{B}^{(1)}_{q/q}(z,x_T^2,\mu)=\frac{\mu^{2\epsilon}e^{\epsilon\gamma_E}}{(4\pi)^\epsilon}\int\frac{d^dk}{(2\pi)^{d-1}}\delta(k^+k^--k_T^2)\theta(k_0)e^{-ix_\perp k_\perp}\left(\frac{\nu}{n\cdot k}\right)^\alpha$$
$$\delta(\bar{n}k-(1-z)\bar{n}p)g^2C_F\left(2(1-\epsilon)\frac{k_-}{p_-k_+}+4\frac{p_--k_-}{k_+k_-}\right)$$

The k^+ and k^- integration can be performed using the delta functions.

$$\mathcal{B}^{(1)}_{q/q}(z,x_T^2,\mu)=\frac{\mu^{2\epsilon}e^{\epsilon\gamma}}{(4\pi)^\epsilon}g^2C_F(\nu p_-)^\alpha\int\frac{d^{(d-2)}k_T}{(2\pi)^{d-1}}e^{ik_Tx_T}\left[(1-\epsilon)\left(\frac{(1-z)}{k_T^2}\right)^{\alpha+1}+2z(1-z)^{\alpha-1}\frac{1}{k_T^{2(\alpha+1)}}\right]$$

This integral can be solved using the following formula [46] which can be checked using MATHEMATICA

$$\int\frac{d^{d-2}k_T}{k_T^{2+2\delta}}e^{ik_Tx_T}=\frac{\pi^{1-\epsilon}}{\mu^{2\epsilon+2\delta}}\frac{\Gamma(-\epsilon-\delta)}{\Gamma(1+\delta)}\left(\frac{x_T^2\mu^2}{4e^{-2\gamma_E}}\right)^{\epsilon+\delta}\tag{5.17}$$

So one ends up with

$$\mathcal{B}^{(1)}_{i/j}(z,x_T^2,\mu)=\frac{g^2C_F}{8\pi^2}e^{\alpha L_c+\epsilon L_\perp}e^{-(\epsilon+2\alpha)\gamma_E}\frac{\Gamma(-\alpha-\epsilon)}{\Gamma(1+\alpha)}(1-z)^\alpha\left(\frac{2z}{1-z}+(1-z)(1-\epsilon)\right)$$

This can be rewritten, so that one obtains

$$\mathcal{B}^{(1)}_{q/q}(z,x_T^2,\mu)=\frac{g^2}{16\pi^2}e^{\alpha L_c+\epsilon L_\perp}e^{-(\epsilon+2\alpha)\gamma_E}\frac{\Gamma(-\alpha-\epsilon)}{\Gamma(1+\alpha)}(1-z)^\alpha f^{(1)}_{q/q}(z)$$

with

$$f^{(1)}_{q/q}(z)=\frac{2C_F}{1-z}\left(1+z^2-\epsilon(1-z)^2\right)$$

The same calculation can be done for the anti-collinear case. For that one has to relabel the momenta $\bar{n} \sim n$ and $p \sim \bar{p}$ whereas the analytic regulator remains the same. So one has

$$\begin{aligned}\bar{\mathcal{B}}^{(1)}_{i/j}(z, x_T^2, \mu) &= g^2 C_F \frac{\mu^{2\epsilon} e^{\epsilon\gamma}}{(4\pi)^\epsilon} \int \frac{d^d k}{(2\pi)^{d-1}} \delta\left(n(k-(1-z)\bar{p})\right) e^{ik_T x_T} \delta(k_+ k_- - k_T^2)\theta(k_0) \\ &\qquad \cdot \left(2(1-\epsilon)\frac{k_+}{\bar{p}_+ k_-} + 4\frac{\bar{p}_+ - k_+}{k_+ k_-}\right)\left(\frac{\nu}{k_+}\right)^\alpha \\ &= \frac{g^2}{16\pi^2} e^{\alpha L_a + \epsilon L_\perp} e^{-\epsilon\gamma_E}\, \Gamma(-\epsilon)\, (1-z)^{-\alpha} f^{(1)}_{q/q}(z)\end{aligned}$$

In this way, the result given in eq. (5.16) is reproduced. The other matrix elements for the collinear TPDFs are given in the appendix D.

5.7. The matching calculation

In this section I perform the matching steps to order $\mathcal{O}(\alpha_s)$ which were described in section 5.3.

5.7.1. Quark-to-quark splitting

First, the quark-to-quark splitting is considered again. One found

$$\begin{aligned}\mathcal{B}^{(1)}_{i/j}(z, x_T^2, \mu, \nu) &= \frac{g^2}{16\pi^2} e^{\alpha L_c + \epsilon L_\perp} e^{-(\epsilon+2\alpha)\gamma_E} \frac{\Gamma(-\epsilon-\alpha)}{\Gamma(1+\alpha)} (1-z)^\alpha f^{(1)}_{i/j}(z) \\ \bar{\mathcal{B}}^{(1)}_{i/j}(z, x_T^2, \mu, \nu) &= \frac{g^2}{16\pi^2} e^{\alpha L_a + \epsilon L_\perp} e^{-\epsilon\gamma_E} \Gamma(-\epsilon)(1-z)^{-\alpha} f^{(1)}_{i/j}(z)\end{aligned}$$

with

$$f^{(1)}_{q/q}(z) = 2C_F\left(\frac{1+z^2}{1-z} - \epsilon(1-z)\right)$$

The factor $(1-z)^{\pm\alpha}$ is needed in order to regularize the limit $z \to 1$. This corresponds to the limit where the radiated parton gets soft, i.e. $\bar{n}k = (1-z)\bar{n}p \to 0$. Thus, the analytic regulator affects a separation of the soft and collinear region. The contribution singular in $z \to 1$ can be rewritten using the plus distribution. One has

$$(1-z)^{-1\pm\alpha} = \pm\frac{1}{\alpha}\delta(1-z) + \left[\frac{1}{1-z}\right]_+ + \mathcal{O}(\alpha)$$

So one has

$$\begin{aligned}\mathcal{B}_{q/q}(z_1, x_T^2, \mu) = \delta(1-z_1) &+ \frac{\alpha_s}{4\pi} e^{\alpha L_c + \epsilon L_\perp} e^{-(\epsilon+2\alpha)\gamma_E} \frac{\Gamma(-\epsilon-\alpha)}{\Gamma(1+\alpha)} \\ &\times \left(2C_F\left(\frac{1+z_1^2}{[1-z_1]_+} + \frac{1}{\alpha}\delta(1-z_1) - \epsilon(1-z_1)\right)\right)\end{aligned}$$

and

$$\begin{aligned}\bar{\mathcal{B}}_{q/q}(z, x_T^2, \mu, \nu) &= \delta(1-z_2) + \frac{\alpha_s}{4\pi} e^{\alpha L_a + \epsilon L_\perp} e^{-\epsilon\gamma_E}\Gamma(-\epsilon)\left(2C_F\left(\frac{1+z_2^2}{[1-z_2]_+} - \frac{1}{\alpha}\delta(1-z_2) - \epsilon(1-z_2)\right)\right) \\ &= \bar{\mathcal{B}}_{\bar{q}/\bar{q}}(z, x_T^2, \mu, \nu)\end{aligned}$$

With the notation

$$\mathcal{B}_{q/q}(z_1, x_T^2, \mu) = \delta(1-z_1) + \mathcal{B}^{(1)}_{q/q}(z_1, x_T^2, \mu)$$

one obtains for the product of the two beam functions to order α_s

$$\mathcal{B}_{q/q}(z_1,x_T^2,\mu)\ \cdot\bar{\mathcal{B}}_{\bar{q}/\bar{q}}(z_2,x_T^2,\mu)=\mathcal{B}^{(1)}_{q/q}(z_1,x_T^2,\mu)\delta(1-z_2)+\bar{\mathcal{B}}^{(1)}_{\bar{q}/\bar{q}}(z_2,x_T^2,\mu)\delta(1-z_1)+\delta(1-z_1)\delta(1-z_2)$$

One has to expand this sum first in the limit $\alpha\to 0$ and then in $\epsilon\to 0$. Then one obtains an expression which is independent of α

$$\begin{aligned}
&\lim_{\alpha,\epsilon\to 0}\mathcal{B}^b_{q/q}(z_1,x_T^2,\mu)\ \bar{\mathcal{B}}^b_{\bar{q}/\bar{q}}(z_2,x_T^2,\mu)\\
&=\delta(1-z_1)\delta(1-z_2)\left[1+\frac{\alpha_s C_F}{4\pi}\left(\frac{4}{\epsilon^2}-\underbrace{\left(\frac{4}{\epsilon}+4L_\perp\right)\ln\frac{q^2x_T^2}{4e^{-2\gamma_E}}}_{\text{Collinear anomaly}}+\frac{4}{\epsilon}L_\perp-\frac{\pi^2}{3}+2L_\perp^2\right)\right]\\
&\quad-\frac{C_F\alpha_s}{4\pi}\left[2\delta(1-z_1)\left[\left(\frac{1}{\epsilon}+L_\perp\right)\frac{1+z_2^2}{[1-z_2]_+}-(1-z_2)\right]+(z_1\leftrightarrow z_2)\right]
\end{aligned}$$

Thus one can determine the anomalous exponent $F^b_{q\bar{q}}(x_T^2,\mu)$ which is

$$F^b_{q\bar{q}}(x_T^2,\mu)=\frac{\alpha_s C_F}{4\pi}\left(\frac{4}{\epsilon}+4L_\perp\right)$$

The renormalzation prescription is given in eq. (5.10). Thus, the renormalized $F_{q\bar{q}}$ is

$$F_{q\bar{q}}(x_T^2,\mu)=\frac{\alpha_s C_F}{\pi}L_\perp$$

and

$$Z_q^{F\ (1)}=\frac{\alpha_s C_F}{\pi}\frac{1}{\epsilon}$$

After the extraction of the collinear anomaly one is left with

$$\begin{aligned}
B^b_{q/q}(z_1,x_T^2,\mu)\cdot B^b_{\bar{q}/\bar{q}}(z_2,x_T^2,\mu)=&\delta(1-z_1)\delta(1-z_2)\cdot\left[1+\frac{\alpha_s C_F}{4\pi}\left(\frac{4}{\epsilon^2}+\frac{4}{\epsilon}L_\perp-\frac{\pi^2}{3}+2L_\perp^2\right)\right]\\
&-\frac{C_F\alpha_s}{4\pi}\left[2\delta(1-z_1)\left[\left(\frac{1}{\epsilon}+L_\perp\right)\frac{1+z_2^2}{[1-z_2]_+}-(1-z_2)\right]+(z_1\leftrightarrow z_2)\right]\\
=&Z_q^B(x_T^2,\mu)B_{q/q}(z,x_T^2,\mu)\cdot Z_{\bar{q}}^B(x_T^2,\mu)B_{\bar{q}/\bar{q}}(z,x_T^2,\mu)
\end{aligned}$$

Due to $B_{\bar{q}/\bar{q}}=B_{q/q}$ one has

$$\begin{aligned}
B^b_{q/q}(z_1,x_T^2,\mu)=&\delta(1-z_1)\left[1+\frac{\alpha_s C_F}{4\pi}\left(\frac{2}{\epsilon^2}+\frac{2}{\epsilon}L_\perp-\frac{\pi^2}{6}+L_\perp^2\right)\right]\\
&-\frac{C_F\alpha_s}{2\pi}\left(\left(\frac{1}{\epsilon}+L_\perp\right)\frac{1+z_1^2}{[1-z_1]_+}-(1-z_1)\right)
\end{aligned}$$

The aim is to determine the regularized expressions for $B_{q/q}$ and $I_{q/q}$. The transverse and collinear PDFs are related by matching kernels via

$$B_{i/j}(z,x_T^2,\mu)=\sum_k I_{i/k}(z,x_T^2,\mu)\otimes\Phi_{k/j}(z,\mu)$$

The definition of the collinear PDFs was stated in eq. (4.3). The only difference to the TPDFs is that $x_\perp=0$. Thus, compared to the TPDFs, the exponential factor $e^{ix_Tk_T}$ is missing. So, for the PDFs one ends up with a k_T integration where the integrand is scaleless (compare (5.17)). Thus, one has to all orders in dimensional regularization

$$\Phi^b_{i/j}(z)=\delta_{ij}\delta(1-z)$$

The PDFs are renormalized with the DGLAP splitting functions. For the renormalized PDFs one has to $\mathcal{O}(\alpha_s)$

$$\Phi_{i/j}(z,\mu) = \delta_{ij}\delta(1-z) - \frac{\alpha_s}{2\pi}\frac{1}{\epsilon}P^{(1)}_{i/j}(z)$$

Consider again the quark to quark splitting. We found that

$$\begin{aligned} B^b_{q/q}(z_2, x_T^2, \mu) =& \delta(1-z_2)\left[1 + \frac{\alpha_s C_F}{4\pi}\left(\frac{2}{\epsilon^2} + \frac{2}{\epsilon}L_\perp - \frac{\pi^2}{6} + L_\perp^2\right)\right] \\ &- \frac{C_F\alpha_s}{2\pi}\left(\left(\frac{1}{\epsilon} + L_\perp\right)\frac{1+z_2^2}{[1-z_2]_+} - (1-z_2)\right) \end{aligned}$$

Above one found that (5.12)

$$\begin{aligned} &B^b_{i/j}(z, x_T^2, \mu) = I^b_{i/j}(z, x_T^2, \mu) \\ &\Rightarrow I^b_{i/j}(z, x_T^2) = Z_i^B(x_T^2, \mu)\sum_k I_{i/k}(z, x_T^2, \mu) \otimes \Phi_{k/j}(z,\mu) \end{aligned}$$

where the renormalized matching kernel $I_{i/k}$ is free of any poles. Thus, one has to $\mathcal{O}(\alpha_s)$

$$B^b_{q/q}(z, x_T^2) = I^b_{q/q}(z, x_T^2) = Z_q^B(x_T^2,\mu)\; I_{q/q}(z, x_T^2, \mu) \otimes \left(\delta_{qq}\delta(1-z) - \frac{\alpha_s}{2\pi}\frac{1}{\epsilon}P^{(1)}_{q/q}(z)\right)$$

with quark to quark DGLAP splitting function

$$P^{(1)}_{q/q}(z) = C_F\frac{1+z^2}{[1-z]_+} + \frac{3}{2}C_F\delta(1-z)$$

Replacing $I^b_{q/q}$ and $P^{(1)}_{q/q}$ and assuming that $I_{q/q}$ is independent of any poles one can compare the two sides of the equation which yields

$$\begin{aligned} B^b_{q/q}(z, x_T^2, \mu) =& \delta(1-z)\left[1 + \frac{\alpha_s C_F}{4\pi}\left(\frac{2}{\epsilon^2} + \frac{2}{\epsilon}L_\perp - \frac{\pi^2}{6} + L_\perp^2\right)\right] - \frac{C_F\alpha_s}{2\pi}\left(\left(\frac{1}{\epsilon} + L_\perp\right)\frac{1+z^2}{[1-z]_+} - (1-z)\right) \\ =& Z_q^B(x_T^2,\mu)\int_z^1 \frac{dy}{y} I_{q/q}(y, x_T^2,\mu)\left(z\delta(y-z) - \frac{\alpha_s}{2\pi}\frac{1}{\epsilon}P^{(1)}_{q/q}(z/y)\right) \\ =& Z_q^B(x_T^2,\mu)\;\left(\delta(1-z)\left[1 + \frac{\alpha_s C_F}{4\pi}\left(-\frac{\pi^2}{6} + L_\perp^2\right)\right] - \frac{C_F\alpha_s}{2\pi}\left(L_\perp\frac{1+z^2}{[1-z]_+} - (1-z)\right)\right) \\ &- Z_q^B(x_T^2,\mu)\frac{C_F\alpha_s}{2\pi\epsilon}\left(\frac{1+z^2}{[1-z]_+} + \frac{3}{2}\delta(1-z)\right) \end{aligned}$$

and thus

$$Z_q^B(x_T^2,\mu) = Z_{\bar{q}}^B(x_T^2,\mu) = 1 + \frac{\alpha_s C_F}{4\pi}\left(\frac{2}{\epsilon^2} + \frac{2}{\epsilon}L_\perp + \frac{3}{\epsilon}\right)$$

and for the renormalized $I_{q/q}$ we set

$$I_{q/q}(z, x_T^2,\mu) = \delta(1-z)\left[1 + \frac{\alpha_s C_F}{4\pi}\left(-\frac{\pi^2}{6} + L_\perp^2\right)\right] - \frac{C_F\alpha_s}{2\pi}\left(L_\perp\frac{1+z^2}{[1-z]_+} - (1-z)\right)$$

With the renormalization constant Z_q^B one can calculate the renormalized beam function $B_{q/q}$. One obtains

$$\begin{aligned} B_{q/q}(z_2, x_T^2,\mu) &= \delta(1-z_2)\left[1 + \frac{\alpha_s C_F}{4\pi}\left(-\frac{3}{\epsilon} - \frac{\pi^2}{6} + L_\perp^2\right)\right] - \frac{C_F\alpha_s}{2\pi}\left(\left(\frac{1}{\epsilon} + L_\perp\right)\frac{1+z_2^2}{[1-z_2]_+} - (1-z_2)\right) \\ &= \delta(1-z_2)\left[1 + \frac{\alpha_s C_F}{4\pi}\left(-\frac{\pi^2}{6} + L_\perp^2\right)\right] - \frac{C_F\alpha_s}{2\pi}\left(L_\perp\frac{1+z_2^2}{[1-z_2]_+} - (1-z_2)\right) - \frac{1}{\epsilon}\frac{\alpha_s}{2\pi}P^{(1)}_{q/q}(z_2) \end{aligned}$$

So the infrared poles in $B_{q/q}$ are in correspondence with the poles in the PDFs described by the splitting function.

5.7.2. Gluon-to-quark splitting

A real splitting described by $j \to i + X$ with $i \neq j$ does not contain a contribution to $\mathcal{O}(1)$. So one has schematically

$$\mathcal{B}_{i/j}(z, x_T^2, \mu) = \mathcal{B}_{i/j}^{(1)}(z, x_T^2, \mu)$$
$$I_{i/j}(z, x_T^2, \mu) = I_{i/j}^{(1)}(z, x_T^2, \mu)$$

The gluon-to-quark splitting can be evaluated by considering one of the following equations

$$\mathcal{B}^b_{q/g}(z_1, x_T^2, \mu)\, \bar{\mathcal{B}}^b_{\bar{q}/\bar{q}}(z_2, x_T^2, \mu) = \left(\frac{x_T^2 q^2}{4e^{-2\gamma_E}}\right)^{-F^b_{q\bar{q}}(x_T^2,\mu)} B^b_{q/j}(z_1, x_T^2, \mu)\, B^b_{\bar{q}/k}(z_2, x_T^2, \mu) \tag{5.18}$$

$$\mathcal{B}^b_{q/q}(z_1, x_T^2, \mu)\, \bar{\mathcal{B}}^b_{\bar{q}/g}(z_2, x_T^2, \mu) = \left(\frac{x_T^2 q^2}{4e^{-2\gamma_E}}\right)^{-F^b_{q\bar{q}}(x_T^2,\mu)} B^b_{q/j}(z_1, x_T^2, \mu)\, B^b_{\bar{q}/k}(z_2, x_T^2, \mu) \tag{5.19}$$

A product as

$$\mathcal{B}^b_{q/g}(z_1, x_T^2, \mu)\, \bar{\mathcal{B}}^b_{\bar{q}/g}(z_2, x_T^2, \mu) = \left(\frac{x_T^2 q^2}{4e^{-2\gamma_E}}\right)^{-F^b_{q\bar{q}}(x_T^2,\mu)} B^b_{q/j}(z_1, x_T^2, \mu)\, B^b_{\bar{q}/k}(z_2, x_T^2, \mu)$$

can't be considered, since there is no $\mathcal{O}(\alpha_s)$ contribution. Choosing (5.18), the beam function $\bar{\mathcal{B}}_{\bar{q}/\bar{q}}$ only contribute to leading order

$$\bar{\mathcal{B}}_{\bar{q}/\bar{q}}(z, x_T^2, \mu) = \delta(1-z)$$

One has

$$\begin{aligned}
&\lim_{\alpha,\epsilon\to 0} \mathcal{B}_{q/g}(z, x_T^2, \mu)\bar{\mathcal{B}}_{\bar{q}/\bar{q}}(y, x_T^2, \mu) \\
&= \lim_{\alpha,\epsilon\to 0}\left[\delta(1-y)\frac{\alpha_s}{4\pi} e^{\alpha L_c + \epsilon L_\perp} e^{-(\epsilon+2\alpha)\gamma_E}\frac{\Gamma(-\alpha-\epsilon)}{\Gamma(1+\alpha)}(1-z)^\alpha 2T_F\left(1 - \frac{2z}{1-\epsilon}(1-z)\right)\right] \\
&= \frac{\alpha_s}{4\pi}\delta(1-y)\left(-\frac{2}{\epsilon}T_F(1+2z(z-1)) - 2T_F\left[2z(z-1) + (1+2z(z-1))L_\perp\right]\right)
\end{aligned}$$

This has no dependence on the hard scale q^2. In the quark-to-quark splitting case, one found that

$$F^b_{q\bar{q}}(x_T^2, \mu) = \frac{\alpha_s C_F}{4\pi}\left(\frac{4}{\epsilon} + 4L_\perp\right) \sim \mathcal{O}(\alpha_s)$$

The expansion in α_s gives

$$\left(\frac{x_T^2 Q^2}{4e^{-2\gamma_E}}\right)^{-F^b_{q\bar{q}}(x_T^2,\mu)} = 1 - \underbrace{F^b_{q\bar{q}}(x_T^2, \mu)\ln\left(\frac{x_T^2 Q^2}{4e^{-2\gamma_E}}\right)}_{\sim\mathcal{O}(\alpha_s)}$$

Because $\mathcal{B}_{q/g} \sim \mathcal{O}(\alpha_s)$ only the first term equal to one contributes to the product of the beam functions. One obtains

$$[\mathcal{B}_{q/g}(z_1, x_T^2, \mu)\, \bar{\mathcal{B}}_{\bar{q}/\bar{q}}(z_2, x_T^2, \mu)]_{Q^2} = 1 \cdot B^b_{q/g}(z_1, x_T^2, \mu)\, B^b_{\bar{q}/\bar{q}}(z_2, x_T^2, \mu)$$

As justified above, the bare quantities $B^b_{i/j}$ and $I^b_{i/j}$ are equal. Thus one has

$$I^b_{q/g}(z, x_T^2) = \frac{\alpha_s}{4\pi}\left(-\frac{2}{\epsilon}T_F(1+2z(z-1)) - 2T_F\left[2z(z-1) + (1+2z(z-1))L_\perp\right]\right)$$

This expression can be rewritten in terms of the DGLAP splitting function $P^{(1)}_{q/g}$

$$I^b_{q/g}(z,x_T^2) = -\frac{\alpha_s}{2\pi}P^{(1)}_{q/g}(z) + \frac{\alpha_s}{4\pi}\left(-2T_F 2z(z-1) - 2P^{(1)}_{q/g}(z)L_\perp\right)$$

This expression still has to be renormalized as

$$\begin{aligned}I^b_{q/g}(z,x_T^2) &= Z_q^B(x_T^2,\mu)\sum_k I_{q/k}(z,x_T^2,\mu)\otimes\Phi_{k/g}(z,\mu)\\ &= Z_q^B(x_T^2,\mu)I_{q/q}(z,x_T^2,\mu)\otimes\Phi_{q/g}(z,\mu) + Z_q^B(x_T^2,\mu)I_{q/g}(z,x_T^2,\mu)\otimes\Phi_{g/g}(z,\mu)\end{aligned}$$

with

$$\begin{aligned}\Phi_{q/g}(z,\mu) &= -\frac{\alpha_s}{2\pi}\frac{1}{\epsilon}T_F\left(z^2+(1-z)^2\right) = -\frac{\alpha_s}{2\pi}\frac{1}{\epsilon}P^{(1)}_{qg}(z)\\ \Phi_{g/g}(z,\mu) &= \delta_{gg}\delta(1-z) - \frac{\alpha_s}{2\pi}\frac{1}{\epsilon}P^{(1)}_{g/g}(z)\end{aligned}$$

Thus to order α_s $I^b_{q/g}$ reduces to

$$\begin{aligned}I^b_{q/g}(z,x_T^2) &= Z_q^B(x_T^2,\mu)\delta(1-z)\otimes\Phi_{q/g}(z,\mu) + Z_q^B(x_T^2,\mu)I_{q/g}(z,x_T^2,\mu)\otimes\delta(1-z)\\ &= Z_q^B(x_T^2,\mu)\Phi_{q/g}(z,\mu) + Z_q^B(x_T^2,\mu)I_{q/g}(z,x_T^2,\mu)\\ &= Z_q^B(x_T^2,\mu)\left(\Phi_{q/g}(z,\mu) + I_{q/g}(z,x_T^2,\mu)\right)\\ &= \Phi_{q/g}(z,\mu) + I_{q/g}(z,x_T^2,\mu)\end{aligned}$$

where it was used, that Z_q^B only contributes with the term equal to 1. In the last section I computed

$$Z_q^B(x_T^2,\mu) = 1 + \frac{\alpha_s C_F}{4\pi}\left(\frac{2}{\epsilon^2} + \frac{2}{\epsilon}L_\perp + \frac{3}{\epsilon}\right)$$

Thus one has

$$\begin{aligned}I_{q/g}(z,x_T^2,\mu) &= I^b_{q/g}(z,x_T^2) + \frac{\alpha_s}{2\pi}\frac{1}{\epsilon}P^{(1)}_{qg}(z)\\ &= \frac{\alpha_s}{4\pi}\left(4T_F z(1-z) - 2P^{(1)}_{q/g}(z)L_\perp\right)\end{aligned}$$

5.7.3. Quark-to-gluon splitting

With the same arguments as in section 5.7.2, one has

$$\begin{aligned}\mathcal{B}_{g/q}(z,x_T^2)\cdot\mathcal{B}_{g/g}(y,x_T^2) &= \mathcal{B}^{(1)\,b}_{g/q}(z,x_T^2)\cdot\delta(1-y)\\ &= \delta(1-y)\frac{\alpha_s}{4\pi}2C_F\left(\frac{1}{\epsilon}\left(2-\frac{2}{z}-z\right) - \left(-z+\frac{2}{z}L_\perp - 2L_\perp + zL_\perp\right)\right)\end{aligned}$$

Thus to order α_s we find

$$\begin{aligned}B^b_{g/q}(z,x_T^2) &= \frac{\alpha_s}{4\pi}2C_F\left(\frac{1}{\epsilon}\left(2-\frac{2}{z}-z\right) - \left(-z+\frac{2}{z}L_\perp - 2L_\perp + zL_\perp\right)\right)\\ &= -\frac{\alpha_s}{4\pi}2C_F\left(\frac{1+(1-z)^2}{\epsilon z} + \left(-z+\frac{2}{z}L_\perp - 2L_\perp + zL_\perp\right)\right)\\ &= -\frac{\alpha_s}{2\pi}\frac{1}{\epsilon}P^{(1)}_{g/q}(z) + \frac{\alpha_s}{4\pi}2\left(C_F z - P^{(1)}_{g/q}(z)L_\perp\right)\end{aligned}$$

For the calculation of $I_{g/q}$ we need

$$\begin{aligned}
I^b_{g/q}(z,x_T^2) &= Z_g^B(x_T^2,\mu)\sum_k I_{g/k}(z,x_T^2,\mu)\otimes\Phi_{k/q}(z,\mu)\\
&= Z_g^B(x_T^2,\mu)I_{g/g}(z,x_T^2,\mu)\otimes\Phi_{g/q}(z,\mu)+Z_g^B(x_T^2,\mu)I_{g/q}(z,x_T^2,\mu)\otimes\Phi_{q/q}(z,\mu)
\end{aligned}$$

with

$$\begin{aligned}
\Phi_{g/q}(z,\mu) &= -\frac{\alpha_s}{2\pi}\frac{1}{\epsilon}C_F\left(\frac{1+(1-z)^2}{z}\right) = -\frac{\alpha_s}{2\pi}\frac{1}{\epsilon}P_{g/q}(z)\\
\Phi_{q/q}(z,\mu) &= \delta(1-z)-\frac{\alpha_s}{2\pi}\frac{1}{\epsilon}P^{(1)}_{q/q}(z)
\end{aligned}$$

With the same arguments as in section 5.7.2, one has to order α_s $I^b_{q/g}$

$$I^b_{g/q}(z,x_T^2) = \Phi_{g/q}(z,\mu)+I_{g/q}(z,x_T^2,\mu)$$

and so

$$I_{g/q}(z,x_T^2,\mu) = I^b_{g/q}(z,x_T^2)+\frac{\alpha_s}{2\pi}\frac{1}{\epsilon}P^{(1)}_{g/q}(z) = \frac{\alpha_s}{4\pi}2\left(C_F z-P^{(1)}_{g/q}(z)L_\perp\right)$$

5.7.4. Gluon-to-gluon splitting

The calculation for the gluon-to-gluon splitting follows the same arguments as the quark-to-quark splitting calculation. First one has to rewrite the singularity in the limit $z\to 1$ in terms of the plus distribution

$$\begin{aligned}
4C_A(1-z)^{\pm\alpha}\left(\frac{1-z}{z}+\frac{z}{1-z}+z(1-z)\right) &= 4C_A\left(\frac{1-z}{z}+\frac{z}{[1-z]_+}+z(1-z)\pm\frac{1}{\alpha}\delta(1-z)\right)\\
&\equiv 2P^r_{gg}(z)\pm\frac{4C_A}{\alpha}\delta(1-z)
\end{aligned}$$

So one has

$$\begin{aligned}
\mathcal{B}^{(1)}_{g/g}(z,x_T^2,\mu,\nu) &= \frac{\alpha_s}{4\pi}e^{\alpha L_c+\epsilon L_\perp}e^{-(\epsilon+2\alpha)\gamma_E}\frac{\Gamma(-\epsilon-\alpha)}{\Gamma(1+\alpha)}\left(2P^r_{gg}(z)+\frac{4C_A}{\alpha}\delta(1-z)\right)\\
\bar{\mathcal{B}}^{(1)}_{g/g}(z,x_T^2,\mu,\nu) &= \frac{\alpha_s}{4\pi}e^{\alpha L_a+\epsilon L_\perp}e^{-\epsilon\gamma_E}\Gamma(-\epsilon)\left(2P^r_{gg}(z)-\frac{4C_A}{\alpha}\delta(1-z)\right)
\end{aligned}$$

So one has to $\mathcal{O}(\alpha_s)$

$$\begin{aligned}
\mathcal{B}_{g/g}(z_1,x_T^2,\mu)\ \cdot\bar{\mathcal{B}}_{g/g}(z_2,x_T^2,\mu) = {}&\mathcal{B}^{(1)}_{g/g}(z_1,x_T^2,\mu)\delta(1-z_2)+\bar{\mathcal{B}}^{(1)}_{g/g}(z_2,x_T^2,\mu)\delta(1-z_1)\\
&+\delta(1-z_1)\delta(1-z_2)
\end{aligned}$$

This gives

$$\begin{aligned}
&\mathcal{B}^b_{g/g}(z,x_T^2,\mu)\ \cdot\bar{\mathcal{B}}^b_{g/g}(y,x_T^2,\mu)\\
&\quad=\delta(1-z)\delta(1-y)\left(1+\frac{\alpha_s}{4\pi}C_A\left(\frac{4}{\epsilon^2}-\left(\frac{4}{\epsilon}+4L_\perp\right)\ln\frac{q^2x_T^2}{4e^{-2\gamma_E}}+\frac{4}{\epsilon}L_\perp+2L_\perp^2-\frac{\pi^2}{3}\right)\right)\\
&\qquad+\frac{\alpha_s}{4\pi}\left[\left(-\left(\frac{2}{\epsilon}+2L_\perp\right)P^r_{gg}(z)\right)\delta(1-y)+(z\leftrightarrow y)\right]
\end{aligned}$$

The collinear anomaly becomes evident and one can immediately read out

$$F^b_{gg}(x_T^2,\mu) = \frac{\alpha_s C_A}{4\pi}\left(\frac{4}{\epsilon}+4L_\perp\right)$$

and thus

$$Z_g^F = \frac{\alpha_s C_A}{\pi}\frac{1}{\epsilon}$$

One is left with

$$\begin{aligned} B^b_{g/g}(z, x_T^2, \mu) \ \cdot B^b_{g/g}(y, x_T^2, \mu) =&\delta(1-z)\delta(1-y)\left(1+\frac{\alpha_s}{4\pi}C_A\left(\frac{4}{\epsilon^2}+\frac{4}{\epsilon}L_\perp+2L_\perp^2-\frac{\pi^2}{3}\right)\right) \\ &+\frac{\alpha_s}{4\pi}\left[\left(-\left(\frac{2}{\epsilon}+2L_\perp\right)P^r_{gg}(z)\right)\delta(1-y)+(z\leftrightarrow y)\right] \end{aligned}$$

and one concludes

$$B^b_{g/g}(z, x_T^2, \mu) =\delta(1-z)\left(1+\frac{\alpha_s}{4\pi}C_A\left(\frac{2}{\epsilon^2}+\frac{2}{\epsilon}L_\perp+L_\perp^2-\frac{\pi^2}{6}\right)\right)-\frac{\alpha_s}{4\pi}\left(\frac{2}{\epsilon}+2L_\perp\right)P^r_{gg}(z)$$

With

$$\begin{aligned} Z_g^B(x_T^2, \mu) &= 1 + Z_g^{(1)B}(x_T^2, \mu) \\ I_{g/g}(z, x_T^2, \mu) &= \delta(1-z) + I^{(1)}_{g/g}(z, x_T^2, \mu) \end{aligned}$$

one obtains up to $\mathcal{O}(\alpha_s)$

$$\begin{aligned} B^b_{g/g}(z, x_T^2) = I^b_{g/g}(z, x_T^2) &= Z_g^B(x_T^2, \mu)\ I_{g/g}(z, x_T^2, \mu) \otimes \left(\delta(1-z) - \frac{\alpha_s}{2\pi}\frac{1}{\epsilon}P^{(1)}_{g/g}(z)\right) \\ &= I^{(1)}_{g/g}(z, x_T^2, \mu) + Z_g^{(1)B}(x_T^2, \mu)\ - \frac{\alpha_s}{2\pi}\frac{1}{\epsilon}P^{(1)}_{g/g}(z) + \delta(1-z) \end{aligned}$$

where $P^{(1)}_{g/g}$ is the gluon-to-gluon DGLAP splitting function:

$$P^{(1)}_{g/g}(z) = 2C_A\left(\frac{z}{(1-z)_+}+\frac{1-z}{z}+z(1-z)\right)+\frac{1}{2}\delta(1-z)\left(\frac{11}{3}C_A-\frac{4}{3}T_F N_f\right)$$

Thus, one gets

$$Z_g^B(x_T^2, \mu) = 1+\frac{\alpha_s}{4\pi}\left[C_A\frac{2}{\epsilon^2}+C_A\frac{2}{\epsilon}L_\perp+\frac{1}{\epsilon}\left(\frac{11}{3}C_A-\frac{4}{3}T_F N_f\right)\right]$$

and the renormalized kernel is

$$I_{g/g}(z, x_T^2, \mu) =\delta(1-z)\left(1+\frac{\alpha_s}{4\pi}C_A\left(L_\perp^2-\frac{\pi^2}{6}\right)\right)-\frac{\alpha_s}{4\pi}2L_\perp P^r_{gg}(z)$$

with

$$P^r_{gg}(z) = 2C_A\left(\frac{1-z}{z}+\frac{z}{[1-z]_+}+z(1-z)\right)$$

5.7.5. Overview of the results

One found

$$\begin{aligned} I_{g/g}(z, x_T^2, \mu) =&\delta(1-z)\left(1+\frac{\alpha_s}{4\pi}C_A\left(L_\perp^2-\frac{\pi^2}{6}\right)\right)-\frac{\alpha_s}{4\pi}2L_\perp P^r_{gg}(z) \\ I_{g/q}(z, x_T^2, \mu) =&\frac{\alpha_s}{4\pi}2\left(C_F z - P^{(1)}_{g/q}(z)L_\perp\right) \\ I_{q/g}(z, x_T^2, \mu) =&\frac{\alpha_s}{4\pi}\left(4T_F z(1-z) - 2P^{(1)}_{q/g}(z)L_\perp\right) \\ I_{q/q}(z, x_T^2, \mu) =&\delta(1-z)\left[1+\frac{\alpha_s C_F}{4\pi}\left(-\frac{\pi^2}{6}+L_\perp^2\right)\right]-\frac{C_F\alpha_s}{2\pi}\left(L_\perp\frac{1+z^2}{[1-z]_+}-(1-z)\right) \end{aligned}$$

It is possible to write the above results in a closed form as [53]

$$I_{i/j}(z, L_\perp, \mu) = \delta(1-z)\delta_{ij}\left[1 + \left(\Gamma^i_{cusp}\frac{L_\perp^2}{4} - \gamma^i L_\perp\right)\right] + \frac{\alpha_s}{4\pi}\Big(-2P^{(1)}_{ij}(z)L_\perp + \mathcal{R}^{(1)}_{ij}(z)\Big) \tag{5.20}$$

and with the expansion

$$\begin{aligned}\Gamma^i_{cusp} &= \frac{\alpha_s}{4\pi}\Gamma^i_0 + \left(\frac{\alpha_s}{4\pi}\right)^2\Gamma^i_1 + ...\\ \gamma^i &= \frac{\alpha_s}{4\pi}\gamma^i_0 + \left(\frac{\alpha_s}{4\pi}\right)^2\gamma^i_1 + ...\end{aligned} \tag{5.21}$$

one finds

$$\begin{aligned}\frac{\Gamma^q_0}{C_F} &= \frac{\Gamma^g_0}{C_A} = 4\\ \gamma^q_0 &= -3C_F\\ \gamma^g_0 &= -\frac{11}{3}C_A + \frac{4}{3}T_F N_f\end{aligned} \tag{5.22}$$

and the one loop remainder functions are

$$\begin{aligned}\mathcal{R}^{(1)}_{qq}(z) &= C_F\left(2(1-z) - \frac{\pi^2}{6}\delta(1-z)\right)\\ \mathcal{R}^{(1)}_{qg}(z) &= 4T_F z(1-z)\\ \mathcal{R}^{(1)}_{gq}(z) &= 2C_F z\\ \mathcal{R}^{(1)}_{gg}(z) &= -\delta(1-z)C_A\frac{\pi^2}{6}\end{aligned}$$

and $P^{(1)}_{ij}$ are the DGLAP splitting functions given by

$$\begin{aligned}P^{(1)}_{qq}(z) &= C_F\left(\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right)\\ P^{(1)}_{qg}(z) &= T_F\left(z^2 + (1-z)^2\right)\\ P^{(1)}_{gq}(z) &= C_F\left(\frac{1+(1-z)^2}{z}\right)\\ P^{(1)}_{gg}(z) &= 2C_A\left(\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z)\right) + \frac{1}{2}\left(\frac{11}{3}C_A - \frac{4}{3}T_F N_f\right)\delta(1-z)\end{aligned}$$

with the renormalization factors

$$\begin{aligned}Z^B_q(x_T^2, \mu) &= 1 + \frac{\alpha_s C_F}{4\pi}\left(\frac{2}{\epsilon^2} + \frac{2}{\epsilon}L_\perp + \frac{3}{\epsilon}\right)\\ Z^B_g(x_T^2, \mu) &= 1 + \frac{\alpha_s}{4\pi}\left[C_A\frac{2}{\epsilon^2} + C_A\frac{2}{\epsilon}L_\perp + \frac{1}{\epsilon}\left(\frac{11}{3}C_A - \frac{4}{3}T_F N_f\right)\right]\end{aligned} \tag{5.23}$$

And the anomalous exponents were

$$\frac{F^b_{gg}(x_T^2, \mu)}{C_A} = \frac{\alpha_s}{4\pi}\left(\frac{4}{\epsilon} + 4L_\perp\right) = \frac{F^b_{q\bar{q}}(x_T^2, \mu)}{C_F}$$

and thus

$$\frac{Z^F_g}{C_A} = \frac{\alpha_s}{\pi}\frac{1}{\epsilon} = \frac{Z^F_q}{C_F}$$

and the renormalized anomalous exponents are given by

$$\frac{F_{q\bar{q}}(x_T^2,\mu)}{C_F} = \frac{\alpha_s}{\pi} L_\perp = \frac{F_{gg}(x_T^2,\mu)}{C_A} \tag{5.24}$$

For completeness I give the results of $I'^{(1)}_{g/i}$ which can be evaluated by matching $\mathcal{B}'^{(1)}_{j/i}$ onto the PDFs. The result is taken from [50] and is stated as

$$I'^{(1)}_{g/g}(z,\mu) = -4\frac{\alpha_s}{4\pi} C_A \frac{1-z}{z}, \qquad I'^{(1)}_{g/q}(z,\mu) = -4\frac{\alpha_s}{4\pi} C_F \frac{1-z}{z}$$

so they are independent of x_T.

5.8. Check via RGEs

In section 5.4 the RGEs were derived. These were

$$\begin{aligned}
\frac{d}{d\ln\mu} C_V(-q^2,\mu) &= \left(2\gamma_q(\alpha_s) + \Gamma_{cusp}(\alpha_s)\ln\frac{-q^2}{\mu^2}\right) C_V(-q^2,\mu) \\
\frac{d}{d\ln\mu} \Phi_{j/N}(z,\mu) &= 4\sum_k P_{jk}(z,\mu) \otimes \Phi_{k/N}(z,\mu) \\
\frac{d}{d\ln\mu} F_{i\bar{i}}(x_T^2,\mu) &= 2\Gamma^i_{cusp}(\alpha_s) \\
\frac{d}{d\ln\mu} I_{i/j}(z,x_T^2,\mu) &= \left[\Gamma^i_{cusp}(\alpha_s)L_\perp - 2\gamma^i(\alpha_s)\right] I_{i/j}(z,x_T^2,\mu) - 4\sum_k I_{i/k}(z,x_T^2,\mu) \otimes P_{kj}(z,\mu)
\end{aligned} \tag{5.25}$$

In this section the obtained results for the anomalous exponent and the matching kernels are checked with the help of the RGEs.

5.8.1. The anomalous exponent

It is simple to check that the result (5.24) obeys the RGE described in eq. (5.25). The correctness of the renormalization constant Z_i^F can also be checked.
Since the bare functions do not depend on μ each renormalization constant obeys a RGE which exactly compensates the μ dependence of the corresponding renormalized function. Solving these equations one can express the renormalization constants in terms of the corresponding anomalous dimensions. First I consider the renormalization constant of Z^F. Remember:

$$F^b_{i\bar{i}}(x_T^2) = F_{i\bar{i}}(x_T^2,\mu) + Z_i^F(\mu)$$

Taking the derivative with respect to $\ln\mu$ and taking into account the RGE one obtains

$$0 = 2\Gamma^i_{cusp}(\alpha_s) + \frac{d}{d\ln\mu} Z_i^F(\mu)$$

This equation can be rewritten and with the expansion of the cusp anomalous dimension (eq. (5.21)) one has to order α_s

$$dZ_i^F(\mu) = -2\,\frac{\alpha_s}{4\pi}\,\Gamma_0^i\, d\ln\mu \tag{5.26}$$

In order to expand in series of α one rewrites $d\ln\mu$ in terms of $d\alpha$. So one considers

$$\begin{aligned}
\frac{d\alpha_s}{d\ln\mu} = \mu\frac{d\alpha_s}{d\mu} &= \mu\frac{d}{d\mu}\left(\frac{4\pi}{\mu^2 e^{\epsilon\gamma_E}}\right)^\epsilon \frac{1}{Z_\alpha(\mu)}\alpha_s^b \\
&= -2\epsilon\alpha_s(\mu) - \frac{1}{Z_\alpha(\mu)}\mu\frac{dZ_\alpha}{d\mu}\alpha_s(\mu)
\end{aligned}$$

and thus one has to order α_s

$$\frac{d}{d\ln\mu}\alpha_s = -2\epsilon\alpha_s + \mathcal{O}(\alpha_s^2) \tag{5.27}$$

So one ends up with

$$d\ln\mu = -\frac{d\alpha_s}{2\epsilon\alpha_s}$$

inserting this in eq. (5.26) one has

$$\begin{aligned} dZ_i^F(\mu) &= 2\,\frac{\alpha_s}{4\pi}\,\Gamma_0^i\,\frac{d\alpha_s}{2\epsilon\alpha_s} \\ \Rightarrow Z_i^F(\mu) &= \frac{1}{\epsilon}\Gamma_0^i\frac{\alpha_s}{4\pi} \end{aligned}$$

Thus, the results given in (5.24) are reproduced.

5.8.2. The matching kernels

The matching kernels were rewritten in the closed form given in eq. (5.20). One can immediately check that this form obeys the RGE stated in eq. (5.25). The dependence on μ is hidden in $L_\perp$. Taking the derivative with respect to $\ln\mu$ of eq. (5.20) and taking into account

$$\frac{d}{d\ln\mu}L_\perp = \frac{d}{d\ln\mu}\;\ln\frac{\mu^2 x_T^2}{4e^{-2\gamma_E}} = 2, \qquad \frac{d}{d\ln\mu}\alpha_s \sim \mathcal{O}(\alpha_s^2)$$

one obtains

$$\frac{d}{d\ln\mu}I_{i/j}(z,L_\perp,\mu) = \delta(1-z)\delta_{ij}\left[\left(\Gamma^i_{cusp}L_\perp - 2\gamma^i\right)\right] + \frac{\alpha_s}{4\pi}\Big(-4P_{ij}^{(1)}(z)\Big) \tag{5.28}$$

On the other hand, the RGE is given to $\mathcal{O}(\alpha_s)$

$$\begin{aligned} \frac{d}{d\ln\mu}I_{i/j}(z,L_\perp,\mu) &= \left[\Gamma^i_{cusp}(\alpha_s)L_\perp - 2\gamma^i(\alpha_s)\right]\delta_{ij}\delta(1-z) - 4\frac{\alpha_s}{4\pi}\sum_k \delta_{ik}\delta(1-z)\otimes P_{kj}^{(1)}(z) \\ &= \left[\Gamma^i_{cusp}(\alpha_s)L_\perp - 2\gamma^i(\alpha_s)\right]\delta_{ij}\delta(1-z) - 4\frac{\alpha_s}{4\pi}P_{ij}^{(1)}(z) \end{aligned} \tag{5.29}$$

The results obtained in eq. (5.28) and (5.29) coincides. Thus, the result obtained by the matching is verified by the RGEs.
As a last step one can check if the renormalization factors Z_i^B obey an RGE such that the derivative with respect to $\ln\mu$ of the unrenormalized matching kernels yields zero. So one has

$$0 = \frac{d}{d\ln\mu}I^b_{i/j}(z,L_\perp,\mu) = \frac{d}{d\ln\mu}\left(Z_i^B(x_T^2,\mu)\sum_k I_{i/k}(z,x_T^2,\mu)\otimes\Phi_{k/j}(z,\mu)\right)$$

and with (5.25) obtains

$$\Rightarrow \frac{d}{d\ln\mu}Z_i^B(x_T^2,\mu) = -\left(\Gamma^i_{cusp}(\alpha_s)L_\perp - 2\gamma^i(\alpha_s)\right)Z_i^B(x_T^2,\mu) \tag{5.30}$$

In order to check the above results the expressions of eq. (5.23) are inserted and eq. (5.27) is taken into account. Thus one obtains to order α_s

$$\begin{aligned} \frac{d}{d\ln\mu}Z_q^B(x_T^2,\mu) &= \frac{\alpha_s C_F}{4\pi}\left(-2\frac{2}{\epsilon} - 4L_\perp - 6 + \frac{4}{\epsilon}\right) = \frac{\alpha_s C_F}{4\pi}\left(-4L_\perp - 6\right) \longrightarrow \checkmark \\ \frac{d}{d\ln\mu}Z_g^B(x_T^2,\mu) &= \frac{\alpha_s}{4\pi}\left[-4C_A L_\perp - 2\left(\frac{11}{3}C_A - \frac{4}{3}T_F N_f\right)\right] \longrightarrow \checkmark \end{aligned}$$

where "$\longrightarrow \checkmark$" means that the RGE of eq. (5.30) is fullfilled.
Now all ingredients are known that occur in the factorization formula for a Drell-Yan process at small transverse momentum. The derivation of the factorization formula is reproduced in the next chapter.

6. The Drell-Yan process at small transverse momentum

This chapter is all about the Drell-Yan process at small transverse momentum. In the first section the derivation of the factorized cross section is shown. After that the soft function is considered and then the resummation procedure is pointed out.

6.1. The factorization formula

The derivation of a factorization formula is reproduced from [1, 51]. The production process is given by

$$N_1(P_1) + N_2(P_2) \to l^+(p_1) + l^-(p_2) + X(p_X)$$

which is shown in Fig. 6.1.

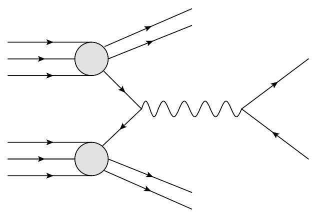


Figure 6.1.: The Drell-Yan process

On parton level, one has the reaction

$$q(p_1) + \bar{q}(p_2) \to \gamma^*/Z^* \to l^+(\tilde{p}_1) + l^-(\tilde{p}_2)$$

where $p_1 = x_1 P_1$ and $p_2 = x_2 P_2$. Considering the kinematic region ($q = p_1 + p_2$)

$$M^2 = q^2 \gg q_T^2 \gg \Lambda_{QCD}^2$$

only soft and collinear emissions can contribute. The expansion parameter is

$$\lambda = \frac{q_T}{M} \ll 1$$

In leading power in λ one has

$$q^2 = (p_1 + p_2)^2 = 2p_1 p_2 \approx p_1^- p_2^+$$

since p_1 and p_2 are assumed to be collinear and anti-collinear. The scaling of the soft and collinear momenta are

$$\begin{aligned} \text{collinear:} \quad & k^\mu \sim M(\lambda^2, 1, \lambda) \\ \text{anti-collinear:} \quad & k^\mu \sim M(1, \lambda^2, \lambda) \\ \text{ultrasoft:} \quad & k^\mu \sim M(\lambda^2, \lambda^2, \lambda^2) \end{aligned}$$

Adding an ultrasoft momentum to the (anti-)collinear momentum does not change its scaling behavior. A particle with soft momentum scaling $k^\mu \sim M(\lambda, \lambda, \lambda)$ is not allowed to interact with the (anti-)collinear

partons due to momentum conservation. Thus, those particles do not contribute to this problem.
For simplicity one can restrict the analysis on the case where the virtual particle is a photon. The cross section is computed as a function of the lepton pair momentum $q = p_1 + p_2$. The cross section is

$$\begin{aligned}\frac{d\sigma}{d^4q\, dM^2} =& \frac{1}{2s}\int \frac{d^3p_1}{(2\pi)^3 2E_1}\frac{d^3p_2}{(2\pi)^3 2E_2}\delta^{(4)}(q-p_1-p_2) \\ &\times \sum\!\!\!\!\!\!\!\int_X |\langle l^+l^-X|N_1N_1\rangle|^2(2\pi)^4\delta^4(P_1+P_2-q-p_X)\delta(M^2-q^2)\end{aligned}$$

where $s = (P_1+P_2)^2$. The leptonic part factorizes from the hadronic part of the amplitude

$$\langle l^+l^-X|N_1N_1\rangle = \frac{e^2}{q^2}\bar{u}(p_2)\gamma_\mu v(p_1)\langle X|\, J^\mu(0)\,|N_1N_2\rangle$$

The state $|X\rangle$ is a tensor product compound of the different momentum region contributions

$$|X\rangle = |X_s\rangle \otimes |X_c\rangle \otimes |X_{\bar{c}}\rangle$$

The leptonic tensor is defined as

$$\begin{aligned}L_{\mu\nu} &= \int \frac{d^3p_1}{(2\pi)^3 2E_1}\frac{d^3p_2}{(2\pi)^3 2E_2}\delta^{(4)}(q-p_1-p_2)\sum_s \bar{u}(p_2)\gamma_\mu v(p_1)\cdot\bar{v}(p_1)\gamma_\nu u(p_2) \\ &= \int \frac{d^3p_1}{(2\pi)^3 2E_1}\frac{d^3p_2}{(2\pi)^3 2E_2}\delta^{(4)}(q-p_1-p_2)\mathrm{Tr}(\not{p}_1\gamma_\nu\not{p}_2\gamma_\mu) \\ &= \frac{1}{(2\pi)^4\cdot 6\pi}(q_\mu q_\nu - g_{\mu\nu}q^2)\end{aligned}$$

The tensor structure was fixed by current conservation $q^\mu L_{\mu\nu} = q^\nu L_{\mu\nu} = 0$. The overall prefactor was determined by computing $g^{\mu\nu}L_{\mu\nu}$. Due to transversality of the hadron tensor the whole cross section is given by

$$\frac{d\sigma}{d^4q\, dM^2} = \frac{4\pi\alpha^2}{3sq^2}\frac{1}{(2\pi)^4}(-g_{\mu\nu})W^{\mu\nu}\delta(M^2-q^2)$$

The partonic cross section is encoded in this expression. The cross section for the s-channel reaction $l^+l^- \to \gamma^* \to \tilde{l}^+\tilde{l}^-$ with massless leptons ($\hat{s} \gg m_l^2$) is given by [49]

$$\hat{\sigma} = \frac{4\pi\alpha^2}{3\hat{s}}$$

The hadron tensor is given by

$$\begin{aligned}W_{\mu\nu} &= \sum_{X_c,X_{\bar{c}},X_s}\langle N_1N_2|\, J^\dagger_\mu(0)\,|X\rangle\langle X|\, J_\nu(0)\,|N_1N_2\rangle(2\pi)^4\delta^4(P_1+P_2-q-p_X) \\ &= \int d^4x\, e^{-iqx}\langle N_1N_2|\, J^\dagger_\mu(x)J_\nu(0)\,|N_1N_2\rangle\end{aligned}$$

where the momentum conservation delta constraint was rewritten in a Fourier integral and then the translation operator was extracted and applied on J^μ. For J_μ the SCET current is inserted which was

$$\begin{aligned}J^\mu(x) = \int ds\int dt\, C_V(s,t)\Big(&\bar{\chi}_n(x+t\bar{n})S^\dagger_n(x)S_{\bar{n}}(x)\gamma^\mu\chi_{\bar{n}}(x+sn) \\ &+\bar{\chi}_{\bar{n}}(x+tn)S^\dagger_{\bar{n}}(x)S_n(x)\gamma^\mu\chi_n(x+s\bar{n})\Big)\end{aligned}$$

In order to see what exactly happens when squaring the amplitude, one explicitly writes the spinor indices of the Dirac fields and one only considers the first contribution of the SCET current

$$J_1^\mu(x) = \bar{\chi}^{a_2}_{\bar{n}\alpha}(x+t_2 n)\gamma^\nu_{\alpha\beta}\chi^{a_1}_{n\beta}(x+t_1\bar{n})$$

J_1^μ describes an incoming collinear quark and an incoming anti-collinear antiquark. This can be seen on parton level where one has

$$\langle 0|\, J_1^\mu \,|q\bar{q}\rangle \sim \langle 0|\,(b_{\bar{n}}\bar{v}_{\bar{n}})\,\gamma^\mu\,(u_n a_n)\, a^\dagger b^\dagger\,|0\rangle \sim \bar{v}_{\bar{n}}\gamma^\mu u_n$$

where the Fourier decomposition of the Dirac fields were inserted. On hadron level, one has

$$\begin{aligned}
&\langle N_1N_2|\, J_1^{\mu\dagger}(x)J_1^\nu(0)\,|N_1N_2\rangle \\
&= \langle N_1N_2|\left(\bar{\chi}^{b_1}_{n\rho}(x+t_1'\bar{n})\gamma^\mu_{\rho\sigma}\chi^{b_2}_{\bar{n}\sigma}(x+t_2'n)\right)\left(\bar{\chi}^{a_2}_{\bar{n}\alpha}(t_2n)\gamma^\nu_{\alpha\beta}\chi^{a_1}_{n\beta}(t_1\bar{n})\right)|N_1N_2\rangle \\
&= \gamma^\mu_{\rho\sigma}\gamma^\nu_{\alpha\beta}\,\langle N_1N_2|\,\bar{\chi}^{b_1}_{n\rho}(x+t_1'\bar{n})\chi^{a_1}_{n\beta}(t_1\bar{n})\chi^{b_2}_{\bar{n}\sigma}(x+t_2'n)\bar{\chi}^{a_2}_{\bar{n}\alpha}(t_2n)\,|N_1N_2\rangle \\
&= \gamma^\mu_{\rho\sigma}\gamma^\nu_{\alpha\beta}\langle N_1|\,\bar{\chi}^{b_1}_{n\rho}(x+t_1'\bar{n})\chi^{a_1}_{n\beta}(t_1\bar{n})\,|N_1\rangle\langle N_2|\,\chi^{b_2}_{\bar{n}\sigma}(x+t_2'n)\bar{\chi}^{a_2}_{\bar{n}\alpha}(t_2n)\,|N_2\rangle
\end{aligned}$$

where it was assumed, that N_1 is collinear and thus only the collinear fields interact with this particle, whereas N_2 was assumed to be anti-collinear. These matrix elements can be further specified. First, because the matrix elements do not interact with one another, one infers that the both matrix element must be a color singlet. Second, one can conclude, that the collinear matrix element must be proportional to $\not{n}_{\beta\rho}$, since this is the only 4-vector appearing in this matrix element. This follows from Lorentz invariance. The collinear matrix element can be written as

$$\langle N_1|\,\bar{\chi}^{b_1}_{n\rho}(x+t_1'\bar{n})\chi^{a_1}_{n\beta}(x+t_1\bar{n})\,|N_1\rangle = \frac{\delta^{b_1a_1}}{N_c}\,\langle N_1|\,\bar{\chi}^{b_1}_{n\rho}(x+t_1'\bar{n})\chi^{a_1}_{n\beta}(x+t_1\bar{n})\,|N_1\rangle \propto \not{n}_{\beta\rho}$$

where $N_c = 3$ is the number of colors. To determine the proportionality constant, both sides are multiplied with $\not{\bar{n}}_{\rho\beta}$ and one ends up with

$$\langle N_1|\,\bar{\chi}^{b_1}_{n\rho}(x+t_1'\bar{n})\chi^{a_1}_{n\beta}(x+t_1\bar{n})\,|N_1\rangle = \frac{\delta^{b_1a_1}}{N_c}\frac{\not{n}_{\beta\rho}}{4}\,\langle N_1|\,\bar{\chi}^{b_1}_{n}(x+t_1'\bar{n})\frac{\not{\bar{n}}}{2}\chi^{a_1}_{n}(x+t_1\bar{n})\,|N_1\rangle$$

The same considerations are done for the anti-collinear matrix element. One has

$$\langle N_2|\,\chi^{b_2}_{\bar{n}\sigma}(x+t_2'n)\bar{\chi}^{a_2}_{\bar{n}\alpha}(t_2n)\,|N_2\rangle \propto \not{\bar{n}}_{\sigma\alpha}$$

Thus one obtains

$$\begin{aligned}
&\frac{\delta^{b_1a_1}}{N_c}\frac{\delta^{b_2a_2}}{N_c}\frac{\not{n}_{\beta\rho}}{4}\frac{\not{\bar{n}}_{\sigma\alpha}}{4}\gamma^\mu_{\rho\sigma}\gamma^\nu_{\alpha\beta}\,\langle N_1|\,\bar{\chi}^{b_1}_{n}(x+t_1'\bar{n})\frac{\not{\bar{n}}}{2}\chi^{a_1}_{n}(x+t_1\bar{n})\,|N_1\rangle\,\frac{\not{n}_{\alpha\sigma}}{2}\,\langle N_2|\,\chi^{b_2}_{\bar{n}\sigma}(x+t_2'n)\bar{\chi}^{a_2}_{\bar{n}\alpha}(t_2n)\,|N_2\rangle \\
&= \frac{\delta^{b_1a_1}\delta^{b_2a_2}}{16N_c^2}\mathbf{Tr}\left(\not{n}\gamma^\mu\not{\bar{n}}\gamma^\nu\right)\langle N_1|\,\bar{\chi}^{b_1}_{n}(x+t_1'\bar{n})\frac{\not{\bar{n}}}{2}\chi^{a_1}_{n}(x+t_1\bar{n})\,|N_1\rangle\,\frac{\not{n}_{\alpha\sigma}}{2}\,\langle N_2|\,\chi^{b_2}_{\bar{n}\sigma}(x+t_2'n)\bar{\chi}^{a_2}_{\bar{n}\alpha}(t_2n)\,|N_2\rangle
\end{aligned} \tag{6.1}$$

The collinear and anti-collinear matrix elements can be expressed by the transverse momentum dependent parton distribution functions. The trace can be further calculated. One obtains

$$\mathbf{Tr}\left(\not{n}\gamma^\mu\not{\bar{n}}\gamma^\nu\right) = -8\left(g^{\mu\nu} - \frac{1}{2}\left(n^\mu\bar{n}^\nu + n^\nu\bar{n}^\mu\right)\right) = -8g_\perp^{\mu\nu}$$

with $g_{\mu\nu}g_\perp^{\mu\nu} = 2$ one ends up with

$$\begin{aligned}
&-g_{\mu\nu}\,\langle N_1N_2|\,J_1^{\mu\dagger}(x)J_1^\nu(0)\,|N_1N_2\rangle \\
&= \frac{\delta^{b_1a_1}\delta^{b_2a_2}}{N_c^2}\,\langle N_1|\,\bar{\chi}^{b_1}_{n}(x+t_1'\bar{n})\frac{\not{\bar{n}}}{2}\chi^{a_1}_{n}(x+t_1\bar{n})\,|N_1\rangle\,\frac{\not{n}_{\alpha\sigma}}{2}\,\langle N_2|\,\chi^{b_2}_{\bar{n}\sigma}(x+t_2'n)\bar{\chi}^{a_2}_{\bar{n}\alpha}(t_2n)\,|N_2\rangle
\end{aligned}$$

Returning to the cross section formula, one has

$$\frac{d\sigma}{dM^2} = \frac{1}{N_c}\frac{4\pi e_q^2\alpha^2}{3sq^2}\delta(M^2-q^2)\frac{d^4q}{(2\pi)^4}\int ds\int ds'\int dt\int dt'\int d^4x\, e^{-iqx} \\ \times \hat{S}_{q\bar{q}}(x)\cdot C_V(s,t)\cdot C_V(s',t')\cdot \mathcal{M}_c\cdot\mathcal{M}_{\bar{c}}$$

with the soft matrix element

$$\hat{S}_{q\bar{q}}(x) = \frac{1}{N_c}\mathrm{Tr}\,\langle 0|\,\bar{\mathbf{T}}\left(S_n^\dagger(x)S_{\bar{n}}(x)\right)\mathbf{T}\left(S_{\bar{n}}^\dagger(0)S_n(0)\right)|0\rangle$$

The collinear and anti-collinear matrix elements are

$$\begin{aligned}\text{collinear:}\quad \mathcal{M}_c &= \langle N_1|\,\bar{\chi}_n^{b_1}(x+t_1'\bar{n})\frac{\bar{\not{n}}}{2}\chi_n^{a_1}(x+t_1\bar{n})\,|N_1\rangle\\ \text{anti-collinear:}\quad \mathcal{M}_{\bar{c}} &= \frac{\not{n}_{\alpha\sigma}}{2}\langle N_2|\,\chi_{\bar{n}\sigma}^{b_2}(x+t_2'n)\bar{\chi}_{\bar{n}\alpha}^{a_2}(t_2n)\,|N_2\rangle\end{aligned}$$

The matrix elements need to be multipole expanded and only the leading terms that scale as $\sim\mathcal{O}(1)$ are kept. Since $q^\mu\sim(1,1,\lambda)$ one has

$$\begin{aligned} x^\mu\sim(1,1,\lambda^{-1})\qquad &\Rightarrow\qquad P_1\cdot x = \frac{1}{2}\underbrace{P_1^-x^+}_{\sim 1}+\frac{1}{2}\underbrace{P_1^+x^-}_{\sim\lambda^2}+\underbrace{P_{1\perp}x_\perp}_{\sim 1}\approx\frac{1}{2}P_1^-x^++P_{1\perp}x_\perp\\ &\Rightarrow\qquad P_2\cdot x\approx\frac{1}{2}P_2^+x^-+P_{2\perp}x_\perp\\ &\Rightarrow\qquad P^s\cdot x\approx P_\perp^s x_\perp\end{aligned}$$

where P_1 is collinear and P_2 is anti-collinear and P^s is soft.
So one ends up with

$$\begin{aligned}\mathcal{M}_c &= \langle N_1(P_1)|\,\bar{\chi}_n^{b_1}(t_1'\bar{n}_\mu+x_\mu^++x_{\perp\mu})\frac{\bar{\not{n}}}{2}\chi_n^{a_1}(t_1\bar{n}_\mu)\,|N_1(P_1)\rangle\\ \mathcal{M}_{\bar{c}} &= \frac{\not{n}_{\alpha\sigma}}{2}\langle N_2|\,\chi_{\bar{n}\sigma}^{b_2}(x_\mu^-+x_{\perp\mu}+t_2'n_\mu)\bar{\chi}_{\bar{n}\alpha}^{a_2}(t_2n_\mu)\,|N_2\rangle\\ \hat{S}_{q\bar{q}} &= \hat{S}_{q\bar{q}}(x_\perp) = \frac{1}{N_c}\mathrm{Tr}\,\langle 0|\,\bar{\mathbf{T}}\big(S_n^\dagger(x_\perp)S_{\bar{n}}(x_\perp)\big)\mathbf{T}\left(S_{\bar{n}}^\dagger(0)S_n(0)\right)|0\rangle\end{aligned}$$

with

$$x_\mu^+ = (n\cdot x)\,\frac{\bar{n}^\mu}{2}\,,\qquad x_\mu^- = (\bar{n}\cdot x)\,\frac{n^\mu}{2}$$

First, I take a look at the collinear matrix element. It can be expressed via the collinear TPDFs given by

$$\mathcal{B}_{i/N}(z,x_T^2) = \frac{1}{2\pi}\int dt e^{-izt\bar{n}\cdot p}\sum_X\frac{\bar{\not{n}}_{\alpha\beta}}{2}\langle N(p)|\,\bar{\chi}_\alpha^{a,n}(t\bar{n}+x_\perp)\,|X\rangle\,\langle X|\,\chi_\beta^{a,n}(0)\,|N(p)\rangle$$

where the index a is the color index and n denotes the collinear direction. The collinear matrix element can be rewritten in terms of the TPDFs as

$$\begin{aligned}\Rightarrow\mathcal{M}_c &= \langle N_1(P_1)|\,\bar{\chi}_n^a(t_1'\bar{n}^\mu+x_\perp^\mu+x_+^\mu)\frac{\bar{\not{n}}}{2}\chi_n^b(t_1\bar{n}^\mu)\,|N_1(P_1)\rangle\\ &= (\bar{n}\cdot P_1)\int_{-1}^1 dz\, e^{ip_1\left(\bar{n}(t_1'-t_1)+x_+\right)}\mathcal{B}_{q/N_1}(z,x_T^2,\mu)\end{aligned}\tag{6.2}$$

The integration range from -1 to 1 is justified with the following property of the beam functions

$$\mathcal{B}_{i/N}(z,x_T^2) = -\mathcal{B}_{\bar{i}/N}(-z,x_T^2)\tag{6.3}$$

and since the TPDFs vanish for $z > 1$ (with the definition of z as p_1^-/P_1^- and momentum conservation), one can immediately conclude from eq. (6.3), that the TPDFs vanish as well for $z < -1$.
Actually the integration range is even restricted to positive values of z [1]. This restriction arises from the fact that the final state in the hard scattering, which consists of the Drell-Yan pair and soft radiation, has positive energy and invariant mass. Therefore, it has positive light-cone momenta in the plus and minus direction. By momentum conservation these are equal to the incoming light-cone components $x_1 P_1^-$ and $x_2 P_2^+$ which forces positive momentum fractions. Thus one has

$$\Rightarrow \mathcal{M}_c = (\bar{n}\cdot P_1)\int_0^1 dz\; e^{izP_1(\bar{n}(t'-t)+x_+)}\mathcal{B}_{q/N_1}(z, x_T^2, \mu)$$

For the anti-collinear matrix element one needs the anti-collinear antiquark beam function

$$\mathcal{B}^{\bar{n}}_{\bar{q}/N}(z, x_T) = \int_{-\infty}^{\infty}\frac{dt}{2\pi}\, e^{-iztnp}\left(\frac{\not{n}_{\alpha\sigma}}{2}\langle N(p)|\,\chi^{\bar{n}}_{\sigma}(x_\perp + tn)\bar{\chi}^{\bar{n}}_{\alpha}(0)\,|N(p)\rangle\right)$$

and with the same reflections as above, one obtains

$$\begin{aligned}
\mathcal{M}_{\bar{c}} &= \frac{\not{n}_{\alpha\sigma}}{2}\langle N_2(P_2)|\,\chi^{b_2}_{\bar{n}\sigma}(x^-_\mu + x_{\perp\mu} + t'_2 n_\mu)\bar{\chi}^{a_2}_{\bar{n}\alpha}(t_2 n_\mu)\,|N_2(P_2)\rangle \\
&= n\cdot P_2\int_0^1 dz\;\; e^{izP_2\left(n(t'_2-t_2)+x_-\right)}\mathcal{B}^{\bar{n}}_{\bar{q}/N}(z, x_T)
\end{aligned}$$

This gives to leading power

$$\begin{aligned}
\frac{d\sigma}{dM^2} =& \frac{d^4q}{(2\pi)^4}\frac{4\pi\alpha^2}{3sq^2N_c}\left|\tilde{C}_V(-q^2,\mu)\right|^2\delta(M^2-q^2)\int_0^1 dz_1 P_{1-}\int_0^1 dz_2 P_{2+}\int\frac{dx_+dx_-}{2}e^{i\frac{x_+}{2}(z_1P_{1-}-q_-)}e^{i\frac{x_-}{2}(z_2P_{2+}-q_+)} \\
&\times\int d^2x_\perp e^{-iq_\perp x_\perp}\sum_q e_q^2\Big(\mathcal{B}_{q/N_1}(z_1, x_T^2,\mu)\bar{\mathcal{B}}_{\bar{q}/N_2}(z_1, x_T^2,\mu)\mathcal{S}(x_\perp,\mu) + (q\leftrightarrow\bar{q})\Big)
\end{aligned}$$

where the second contribution which is indicated as $q \leftrightarrow \bar{q}$ comes from the other contribution in the SCET current with $n \leftrightarrow \bar{n}$. The integration over x_+ and x_- can be performed such that

$$\int\frac{dx_+dx_-}{2}e^{i\frac{x_+}{2}(z_1P_{1-}-q_-)}e^{i\frac{x_-}{2}(z_2P_{2+}-q_+)} = \frac{2(2\pi)^2}{P_{1-}P_{2+}}\delta\left(z_1-\frac{q_-}{P_{1-}}\right)\delta\left(z_2-\frac{q_+}{P_{2+}}\right)$$

With the variable transformation

$$\begin{aligned}
q^- &= q^+e^{2y} \\
q^+ &= q^+
\end{aligned}$$

where y is the rapidity of q defined by

$$y = \frac{1}{2}\ln\frac{q^-}{q^+}$$

and the Jacobian determinant is given by $2e^{2y}q^+ dy\, dq^+$. The integral is then

$$\int\frac{1}{2}dq_+dq_-d^2q_T\delta(q^2-M^2)\frac{1}{q^2} = \frac{1}{2}\frac{1}{M^2}dy\; d^2q_T = \frac{\pi}{2}\frac{1}{M^2}dy\; dq_T^2$$

So one obtains the factorization formula

$$\begin{aligned}
\frac{d^3\sigma}{dM^2dq_T^2dy} &= \frac{4\pi\alpha^2}{3N_cM^2s}\left|\tilde{C}_V(-q^2,\mu)\right|^2\frac{1}{4\pi}\int d^2x_\perp e^{-iq_\perp x_\perp} \\
&\times\sum_q e_q^2\left[S(x_\perp,\mu)\mathcal{B}_{q/N_1}(z_1, x_T^2,\mu)\cdot\bar{\mathcal{B}}_{\bar{q}/N_2}(z_2, x_T^2,\mu) + (q\leftrightarrow\bar{q})\right] + \mathcal{O}\left(\frac{q_T^2}{M^2}\right)
\end{aligned} \tag{6.4}$$

$\tilde{C}_V(-q^2,\mu)$ is the momentum space Wilson coefficient

$$C_V(-q^2,\mu) = \int ds \int dt\, C_V(s,t)\; e^{-i\bar{n}\cdot p_1 t}\; e^{-in\cdot p_2 s}$$

With the collinear anomaly the complete factorized expression for a Drell-Yan process is

$$\begin{aligned}\frac{d^3\sigma}{dM^2 dq_T^2 dy} &= \frac{4\pi\alpha^2}{3N_c M^2 s}\left|\tilde{C}_V(-q^2,\mu)\right|^2 \frac{1}{4\pi}\int d^2x_\perp e^{-iq_\perp x_\perp} \\ &\times \sum_q e_q^2 \left[\left(\frac{x_T^2 q^2}{4e^{-2\gamma_E}}\right)^{-F^b_{q\bar{q}}(x_T^2,\mu)} B_{i/j}(z_1,x_T^2,\mu)\; B_{\bar{i}/k}(z_2,x_T^2,\mu) + (q\leftrightarrow\bar{q})\right] + \mathcal{O}\left(\frac{q_T^2}{M^2}\right)\end{aligned} \tag{6.5}$$

where the soft function does not appear any more. The reason for that is that the soft function is given by a scaleless integral and thus vanish. This is shown in the next section.

6.2. The soft function

The soft function was defined as

$$\hat{S}_{q\bar{q}}(x_T^2,\mu) = \frac{1}{N_c}\sum\!\!\!\!\!\!\int_X \mathrm{Tr}\,\langle 0|\,\bar{\mathbf{T}}\big(S_n^\dagger(x_\perp)S_{\bar{n}}(x_\perp)\big)\,|X(k)\rangle\,\langle X(k)|\,\mathbf{T}\left(S_{\bar{n}}^\dagger(0)S_n(0)\right)|0\rangle$$

with the soft Wilson lines

$$\begin{aligned} S_n(x) &= \mathbf{P}\exp\left[ig_sT^a n^\mu \int_{-\infty}^0 dt\; A^a_{s,\mu}(x^\nu + tn^\nu)\right] \\ S_{\bar{n}}^\dagger(x) &= \bar{\mathbf{P}}\exp\left[-ig_sT^a \bar{n}^\mu \int_{-\infty}^0 dt\; A^a_{s,\mu}(x^\nu + t\bar{n}^\nu)\right]\end{aligned}$$

The soft function describes virtual and real contributions. In the following it is shown that both vanish and the soft function is therefore equal to 1.

- **Virtual** contributions vanish since the corresponding integrals are obviously **scaleless**.
- **Real** contributions are not scaleless at a first glance, because the integrals contain the factor $\exp(ik_Tx_T)$. But with rescaling it can be shown that they vanish. Schematically I will show in the following that

  $$I_s = r\cdot I_s \quad\Rightarrow\quad I_s = 0$$

  where I_s denotes a soft integral.

As an example one can consider the radiation of r gluons. Then one has the following phase space integral

$$\begin{aligned} I_s =& \mu^{2\epsilon}\left[\prod_{i=1}^r \int \frac{d^dl_i}{(2\pi)^{d-1}}\delta\left(l_i^+l_i^- - l_{i,T}^2\right)\left(\frac{\nu}{l_i^+}\right)^\alpha\right]\int d^dk\;\delta^d\Big(k-\sum_i l_i\Big)e^{ix_Tk_T}(n\cdot\bar{n})^r \\ &\times \frac{1}{l_1 n}\frac{1}{n(l_1+l_2)}\cdots\frac{1}{n(l_1+...+l_r)}\cdots\frac{1}{l_1\bar{n}}\cdots\frac{1}{\bar{n}(l_1+...+l_r)}\end{aligned}$$

where the factor $\mu^{2\epsilon}$ denotes the $\overline{MS}$-scheme. Rescaling

$$k_T \to \frac{k_T}{x_T} \qquad \text{and} \qquad l_{i,T}\to\frac{l_{i,T}}{x_T}$$

one obtains the delta-function

$$\delta\left(l_i^+l_i^- - l_{i,T}^2/x_T^2\right)$$

which leads to a further rescaling

$$l_i^- \to l_i^-/x_T^2 \qquad \text{and} \qquad k^- \to k^-/x_T^2$$

and

$$l_i^+ \to \nu l_i^+ \qquad \text{and} \qquad l_i^- \to l_i^-/\nu$$

and so one obtains

$$\begin{aligned} I_s = &\left[\prod_{i=1}^{r}\int \frac{d^d l_i}{(2\pi)^{d-1}}\,(\mu\cdot x_T)^{2\epsilon}\,\delta\left(l_i^+ l_i^- - l_{i,T}^2\right)\left(\frac{1}{l_i^+}\right)^{\alpha}\right]\int d^d k\;\delta^d\Big(k-\sum_i l_i\Big)e^{ik_T} \\ &\times (n\cdot\bar{n})^r\frac{1}{l_1 n}\frac{1}{n(l_1+l_2)}\cdots\frac{1}{n(l_1+...+l_r)}\cdots\frac{1}{l_1\bar{n}}\cdots\frac{1}{\bar{n}(l_1+...+l_r)} = \frac{x_T^{2\epsilon}}{\nu^\alpha}I_s \end{aligned}$$

and thus

$$I_s = 0$$

The emergence of the factor $(\mu x_T)^{2\epsilon}$ is because of the following behavior

$$\mu^{2\epsilon}\;d^d l\;\delta(l^+l^- - l_T^2) \to \mu^{2\epsilon}\,\left(\frac{1}{x_T^2}\right)^d x_T^4\;d^d\tilde{l}\;\delta(\tilde{l}^+\tilde{l}^- - \tilde{l}_T^2) = (\mu x_T)^{2\epsilon}\;\;d^d\tilde{l}\;\delta(\tilde{l}^+\tilde{l}^- - \tilde{l}_T^2)$$

where the components $\tilde{l}_i$ are the rescaled quantities and $d = 4 - 2\epsilon$ was inserted.

6.3. Resummation

The factorization formula for a Drell-Yan production process with small transverse momentum is

$$\begin{aligned} \frac{d^3\sigma}{dM^2 dq_T^2 dy} =& \frac{4\pi\alpha_s^2}{3N_c M^2 s}\underbrace{|C_V(-q^2,\mu)|^2}_{\sim\ln\frac{-q^2}{\mu^2}} \\ &\times\sum_{i,j}\sum_q e_q^2\;\underbrace{\left[C_{q\bar{q}\leftarrow ij}(z_1,z_2,q_T^2,q^2,\mu)+(q\leftrightarrow\bar{q})\right]}_{\sim\ln\left(x_T^2\mu^2\right)}\otimes\Phi_{i/N_1}(z_1,\mu)\otimes\Phi_{j/N_2}(z_2,\mu) \end{aligned} \tag{6.6}$$

with the scattering kernel

$$C_{q\bar{q}\leftarrow ij}(z_1,z_2,q_T^2,q^2,\mu) = \frac{1}{4\pi}\int d^2x_\perp e^{ix_T q_T}\left(\frac{x_T^2 q^2}{4e^{-2\gamma_E}}\right)^{-F_{q\bar{q}}(L_\perp,a_s)} I_{q\leftarrow i}(z_1,L_\perp)\;I_{\bar{q}\leftarrow j}(z_2,L_\perp)$$

where the beam functions are expressed via a matching onto the PDFs. As indicated in eq. (6.6), there is a problem with the setting of the scale μ. This scale appears in two combinations, namely

$$\ln\frac{-q^2}{\mu^2} \qquad \text{and} \qquad \ln\left(x_T^2\mu^2\right)$$

These two logs can't be controlled at the same time. Setting $\mu^2 \sim q^2$, the second logarithm gets large and vise versa. An expedient to this problem is served by the Renormalization Group Equations that allow to evolve the Wilson coefficient C_V to a scale at which $C_{q\bar{q}}$ can be calculated using a perturbative expansion and setting $x_T^2\mu^2 \sim 1$. The evolution of the Wilson coefficient is shown in the next section.

6.3.1. The evolution of the Wilson coefficient

The Wilson coefficient is already known as (cf. eq. (3.18))

$$C_V(-q^2,\mu) = 1 + \frac{\alpha_s(\mu)C_F}{4\pi}\left(-\ln^2\frac{q^2}{\mu^2} + 3\ln\frac{q^2}{\mu^2} - 8 + \frac{\pi^2}{6}\right)$$

with the RGE

$$\frac{d}{d\ln\mu}C_V(-q^2,\mu) = \left(\Gamma^i_{cusp}(\alpha_s)\ln\frac{q^2}{\mu^2} + 2\gamma^i(\alpha_s)\right)C_V(-q^2,\mu)$$

where $i = q, g$ distinguishes between the gluon and quark case. This RGE can be solved as

$$C_V(q^2,\mu) = U(\mu_h,\mu)C_V(q^2,\mu_h)$$

with the evolution matrix $U(\mu_h,\mu)$ which can be written as [1]

$$U(\mu_h,\mu) = \exp\left(2S(\mu_h,\mu) - 2A_{\gamma^i}(\mu_h,\mu)\right)\left(\frac{q^2}{\mu_h^2}\right)^{-A_{\Gamma^i_{cusp}}(\mu_h,\mu)}$$

with the functions

$$\begin{aligned}
S(\nu,\mu) &= -\int_{\alpha_s(\nu)}^{\alpha_s(\mu)} d\alpha\frac{\Gamma^i_{cusp}(\alpha)}{\beta(\alpha)}\int_{\alpha_s(\nu)}^{\alpha}\frac{d\alpha'}{\beta(\alpha')}\\
A_{\gamma^i}(\nu,\mu) &= -\int_{\alpha_s(\nu)}^{\alpha_s(\mu)} d\alpha\frac{\gamma^i(\alpha)}{\beta(\alpha)}
\end{aligned}$$

The explicit expression of these functions can be obtained by expanding the beta function and the anomalous dimensions in powers of α_s as [1]

$$\begin{aligned}
\beta(\alpha_s) &= -2\alpha_s\left(\beta_0\left(\frac{\alpha_s}{4\pi}\right) + \beta_1\left(\frac{\alpha_s}{4\pi}\right)^2 + \mathcal{O}(\alpha_s^3)\right)\\
\Gamma^i_{cusp}(\alpha_s) &= \Gamma^i_0\left(\frac{\alpha_s}{4\pi}\right) + \Gamma^i_1\left(\frac{\alpha_s}{4\pi}\right)^2 + \mathcal{O}(\alpha_s^3)\\
\gamma^i(\alpha_s) &= \gamma^i_0\left(\frac{\alpha_s}{4\pi}\right) + \gamma^i_1\left(\frac{\alpha_s}{4\pi}\right)^2 + \mathcal{O}(\alpha_s^3)
\end{aligned}$$

The coefficients of the expansions are given in appendix A. Large logarithm counts as $1/\alpha_s$. This can be checked with

$$\ln\frac{\nu}{\mu} = \int_{\alpha_s(\mu)}^{\alpha_s(\nu)}\frac{d\alpha}{\beta(\alpha)} = \int_{\alpha_s(\mu)}^{\alpha_s(\nu)}\frac{d\alpha}{-2\beta_0\alpha_s}4\pi = \frac{2\pi}{\beta_0}\left(\frac{1}{\alpha_s(\nu)} - \frac{1}{\alpha_s(\mu)}\right) \sim \frac{1}{\alpha_s} \tag{6.7}$$

In Renormalization Group Improved Perturbation Theory, one eliminates large logarithms in favor of coupling constants at the different scales and expands in these couplings [1].
Using the expanded expressions for β, γ^i and Γ_{cusp} one obtains [54]

$$\begin{aligned}
S(\mu_h,\mu) &= \frac{\Gamma^i_0}{4\beta_0^2}\left(\frac{4\pi}{\alpha_s(\mu_h)}\left(1 - \frac{1}{r} - \ln r\right) + \left(\frac{\Gamma^i_1}{\Gamma^i_0} - \frac{\beta_1}{\beta_0}\right)(1 - r + \ln r) + \frac{\beta_1}{2\beta_0}\ln^2 r\right) + \mathcal{O}(\alpha_s)\\
A_{\Gamma^i_{cusp}}(\mu_h,\mu) &= \frac{\Gamma^i_0}{2\beta_0}\ln r + \mathcal{O}(\alpha_s)\\
A_{\gamma^i}(\mu_h,\mu) &= \frac{\gamma^i_0}{2\beta_0}\ln r + \mathcal{O}(\alpha_s)
\end{aligned}$$

where $r = \alpha_s(\mu)/\alpha_s(\mu_h)$.
The function $S(\mu_h, \mu)$ resums double logarithms of the form $\alpha_s^n \ln^{2n}(\mu/\mu_h)$. This can be seen by taking a closer look at the contribution proportional to $-\ln r/\alpha_s(\mu_h)$. In eq. (6.7) one found

$$\frac{1}{\alpha_s(\mu_h)} = \frac{1}{\alpha_s(\mu)} + \frac{\beta_0}{2\pi} \ln \frac{\mu_h}{\mu}$$

Expanding in $\alpha_s(\mu)$ gives

$$\begin{aligned}
-\frac{1}{\alpha_s(\mu_h)} \ln r &= -\frac{1}{\alpha_s(\mu_h)} \ln \frac{\alpha_s(\mu)}{\alpha_s(\mu_h)} \\
&= -\left(\frac{1}{\alpha_s(\mu)} - \frac{\beta_0}{2\pi} \ln \frac{\mu}{\mu_h}\right) \ln \left(1 - \frac{\alpha_s(\mu)\beta_0}{2\pi} \ln \frac{\mu}{\mu_h}\right) \\
&= \left(\frac{1}{\alpha_s(\mu)} - \frac{\beta_0}{2\pi} \ln \frac{\mu}{\mu_h}\right) \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{\alpha_s(\mu)\beta_0}{2\pi} \ln \frac{\mu}{\mu_h}\right)^n \\
&\sim \alpha_s^n L^{n+1} \qquad \text{with} \qquad L = \ln \frac{\mu}{\mu_h}
\end{aligned}$$

Such that

$$C \sim \exp\left(-\alpha_s L^2 - \alpha_s^2 L^3 - \alpha_s^3 L^4 - ...\right)$$

So the function S sums the leading-logarithmic terms. Because $\alpha_s L^2$ has the form $\alpha_s^n \ln^{2n}(\mu/\mu_h)$, it reproduces the double logarithmic series by expanding the exponential.
For the resummation of large logarithms one chooses

$$\mu_h = q, \qquad \text{and} \qquad \mu \sim q_T$$

$$\Rightarrow |C_V(-q^2, q_T)|^2 = \exp\left(4S(q, q_T) - 4A_{\gamma_q}(q, q_T)\right) |C_V(-q^2, q)|^2$$

In order to derive a resummation formula, the RGE solution of the anomalous exponent has to be considered. This is done in the next section.

6.3.2. RGE solution of the anomalous exponent

One can write the perturbative expansion of $F_{q\bar{q}}$ in the form

$$F_{q\bar{q}}(L_\perp, \alpha_s) = \sum_{n=1}^{\infty} d_n^q(L_\perp) \left(\frac{\alpha_s}{4\pi}\right)^n$$

where $d_1^q(L_\perp) = 4C_F L_\perp$. The evolution equation

$$\frac{dF_{q\bar{q}}(x_T^2, \mu)}{d\ln\mu} = 2\Gamma_{cusp}^q(\alpha_s)$$

implies the relation [51]

$$\frac{d}{dL_\perp} d_n^q(L_\perp) = \Gamma_{n-1}^q + \sum_{m=1}^{n-1} m\, \beta_{n-1-m}\, d_m^q(L_\perp)$$

where it was used

$$\begin{aligned}
\frac{d}{d\ln\mu} &= \frac{2\partial}{\partial L_\perp} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} \\
\Gamma_{cusp}^i(\alpha_s) &= \sum_{n=1}^{\infty} \Gamma_{n-1}^i \left(\frac{\alpha_s}{4\pi}\right)^n \\
\beta(\alpha_s) &= -2\alpha_s \sum_{n=1}^{\infty} \beta_{n-1} \left(\frac{\alpha_s}{4\pi}\right)^n
\end{aligned}$$

And one obtains

$$d_1^q(L_\perp) = \Gamma_0^q L_\perp + d_1^0, \qquad d_2^q(L_\perp) = \frac{1}{2}\Gamma_0^q \beta_0 L_\perp^2 + \Gamma_1^q L_\perp + d_2^{q,0}$$

with $d_1^0 = 0$ and in general $d_n^0 = d_n(0)$. The two-loop coefficient is given by [51]

$$d_2^{q,0} = 4C_F \left(\left(\frac{202}{27} - 7\zeta(3) \right) C_A - \frac{56}{27} T_F n_f \right) \tag{6.8}$$

In the next section the scattering kernel and the remaining integration over $x_\perp$ is investigated.

6.3.3. The scattering kernel

The scattering kernel was

$$C_{q\bar{q}\leftarrow ij}(z_1, z_2, q_T^2, q^2, \mu) = \frac{1}{4\pi} \int d^2 x_\perp e^{i x_T q_T} \left(\frac{x_T^2 q^2}{4e^{-2\gamma_E}} \right)^{-F_{q\bar{q}}(L_\perp, a_s)} I_{q\leftarrow i}(z_1, L_\perp)\, I_{\bar{q}\leftarrow j}(z_2, L_\perp)$$

First it is to mention that due to the collinear anomaly one can not assume that x_T and q_T are conjugate variables behaving $x_T q_T \sim 1$. Second, in the scattering kernel there is left a large logarithm. This can be seen by rewriting

$$\left(\frac{x_T^2 q^2}{4e^{-2\gamma_E}} \right)^{-F_{q\bar{q}}} = \exp\left(-F_{q\bar{q}} \ln \frac{q^2}{\mu^2} - F_{q\bar{q}} L_\perp \right)$$

with

$$F_{q\bar{q}}(L_\perp, \alpha_s) = d_1^q(L_\perp) \left(\frac{\alpha_s}{4\pi} \right) + d_2^q(L_\perp) \left(\frac{\alpha_s}{4\pi} \right)^2 + \mathcal{O}(\alpha_s^3)$$

the collinear anomaly is

$$\left(\frac{x_T^2 q^2}{4e^{-2\gamma_E}} \right)^{-F_{q\bar{q}}} = \exp\left(-\left(\Gamma_0^q L_\perp \left(\frac{\alpha_s}{4\pi} \right) + d_2^q(L_\perp) \left(\frac{\alpha_s}{4\pi} \right)^2 + \mathcal{O}(\alpha_s^3) \right) \ln \frac{q^2}{\mu^2} - F_{q\bar{q}} L_\perp \right)$$

Counting $\ln(q^2/\mu^2)$ as $1/\alpha_s$, a new variable of order $\mathcal{O}(1)$ can be defined as

$$\eta_F(q^2, \mu^2) = \Gamma_0^q \frac{\alpha_s}{4\pi} \ln \frac{q^2}{\mu^2} \sim \mathcal{O}(1)$$

So the collinear anomaly can be written in terms of η as [51]

$$\begin{aligned} \left(\frac{x_T^2 q^2}{4e^{-2\gamma_E}} \right)^{-F_{q\bar{q}}} &= \left(\frac{x_T^2 \mu^2}{4e^{-2\gamma_E}} \right)^{-\eta_F(q^2,\mu^2)} \cdot \exp\left(-L_\perp F_{q\bar{q}}(L_\perp, \alpha_s) - \frac{\alpha_s}{4\pi} \eta_F(q^2, \mu^2) \frac{d_2^q(L_\perp)}{\Gamma_0^q} \right) \\ &= \left(\frac{x_T^2 \mu^2}{4e^{-2\gamma_E}} \right)^{-\eta_F(q^2,\mu^2)} \cdot E_{q\bar{q}}(L_\perp, a_s, \eta) \end{aligned}$$

$E_{q\bar{q}}$ is expanded to $\mathcal{O}(a_s^2)$ which gives

$$E_{q\bar{q}}(L_\perp, a_s, \eta) = 1 - \frac{\alpha_s}{4\pi} \left(\Gamma_0^q L_\perp^2 + \eta_F \left(\frac{1}{2}\beta_0 L_\perp^2 + \frac{\Gamma_1^q}{\Gamma_0^q} L_\perp + \frac{d_2^q}{\Gamma_0^q} \right) \right)$$

with the following formula taken from [51]

$$\frac{1}{4\pi} \int d^2 x_\perp e^{-i q_\perp x_\perp} L_\perp^n \left(\frac{x_T^2 \mu^2}{4e^{-2\gamma_E}} \right)^{-\eta} = (-\partial_\eta)^n \frac{1}{q_T^2} \left(\frac{q_T^2}{\mu^2} \right)^\eta \frac{\Gamma(1-\eta)}{e^{2\eta\gamma_E}\Gamma(\eta)}$$

a closed-form expression for the resummed scattering kernel can be stated as

$$C_{q\bar{q}\leftarrow ij} = I_{q\leftarrow i}(z_1, -\partial_\eta)\, I_{\bar{q}\leftarrow j}(z_2, -\partial_\eta) E_{q\bar{q}}(-\partial_\eta, a_s, \eta) \frac{1}{q_T^2} \left(\frac{q_T^2}{\mu^2} \right)^\eta \frac{\Gamma(1-\eta)}{e^{2\eta\gamma_E}\Gamma(\eta)}$$

The integral converges in the UV region ($q_T \to \infty$) only as long as $\eta < 1$. A vivid explanation of this phenomenon is given in [51] or [55].

6.3.4. Improved resummation

In this section I reproduce the main ideas given in [55] and [50].
In order to obtain a finite result for all values of η one has to keep $\mathcal{O}(\alpha_s L_\perp^2)$ terms in the exponent. One rewrites the matching kernels as

$$I_{i/j}(z, L_\perp, a_s) = e^{h_F(L_\perp, a_s)} \bar{I}_{i/j}(z, L_\perp, a_s)$$

such that terms of order $\sim \alpha_s L_\perp^2$ are encoded in the exponent. One obtains the modified evolution equations

$$\begin{aligned}
\frac{d}{d\ln\mu} h_F(L_\perp, a_s) &= \Gamma_{cusp}(\alpha_s) L_\perp - 2\gamma_q(\alpha_s) \\
\frac{d}{d\ln\mu} \bar{I}_{i/j}(z, L_\perp, a_s) &= -2\sum_k \bar{I}_{i/j}(z, x_T^2, \mu) \otimes P_{kj}(z, \mu)
\end{aligned}$$

and it is chosen that $h_F(0, a_s) = 0$. h_F contains all double-logarithmic terms.
The scattering kernels can be rewritten in the form

$$\begin{aligned}
C_{q\bar{q}\leftarrow ij}(z_1, z_2, q_T^2, M^2, \mu) = \frac{1}{2}\int_0^\infty & dx_T\, x_T\, J_0(x_T q_T) \exp\left[g_F(M^2, \mu, L_\perp, a_s\right] \\
& \times \bar{I}_{q\leftarrow i}(z_1, L_\perp, a_s) \bar{I}_{\bar{q}\leftarrow j}(z_2, L_\perp, a_s)
\end{aligned} \tag{6.9}$$

where the Bessel function $J_0(x_T q_T)$ is obtained by the angular integration and with $q^2 = M^2$ and $a_s = \alpha_s/4\pi$.
A new function is introduced which is

$$\begin{aligned}
g_F(M^2, \mu, L_\perp, a_s) &= -\left(\ln\frac{M^2}{\mu^2} + L_\perp\right) F_{q\bar{q}}(L_\perp, a_s) + 2h_F(L_\perp, a_s) \\
&= -\eta_F L_\perp - a_s\left((\Gamma_0^q + \eta_F\beta_0)\frac{L_\perp^2}{2} + (2\gamma_0^q + \eta K)L_\perp + \eta_F d_2\right) + \mathcal{O}(a_s^2)
\end{aligned} \tag{6.10}$$

with $d_2 = d_2^{q,0}/\Gamma_0^q$ which is (compare eq. (6.8))

$$d_2 = \left(\frac{202}{27} - 7\zeta(3)\right) C_A - \frac{56}{27} T_F n_f \tag{6.11}$$

and

$$K = \frac{\Gamma_1^q}{\Gamma_0^q} = \left(\frac{67}{9} - \frac{\pi^2}{3}\right) C_A - \frac{20}{9} T_F n_f \tag{6.12}$$

The contribution in eq. (6.10) which is quadratic in $L_\perp$ provides a regulator for the infrared and the ultraviolet region.
Consider the region where $\eta \to 1$, but with μ in the perturbative domain. This prescription defines a scale $\mu = q_*$ which is given by

$$q_* = \sqrt{q^2} \exp\left(-\frac{2\pi}{\Gamma_0 \alpha_s(q_*)}\right)$$

For $q^2 = M_Z^2$ one has $q_* \approx 1.88$ GeV which is still in the perturbative domain. In g_F, the term proportional to $L_\perp^2$ provides a gaussian weight to the integral. At the same time the quadratic term also provides a regulator for the IR region of very large x_T. Since $L_\perp$ diverges for $x_T \to 0$ and $x_T \to \infty$ the exponential factor let the integrand goes to zero for these values and so the integrand is regulated for the IR and the UV domain of x_T.
For small $q_T < q_*$, this gaussian fall-off is the dominating factor, which prevents that x_T can become

arbitrarily large! Remarkably this implies that $\langle x_T \rangle$ decouples from q_T^{-1} and stays in a short-distance domain even in the extreme case where q_T is taken to zero. This motivates the choice of the scale as [55]

$$\mu \sim \langle x_T^{-1} \rangle \sim max(q_T, q_*)$$

Evaluating the kernel $C_{q\bar{q}\leftarrow ij}$ for $q_T \to 0$ (changing the variables from x_T to $l = L_\perp$) gives [55]

$$\begin{aligned} C_{q\bar{q}\leftarrow ij}|_{q_T\to 0} = \frac{b_0^2}{4\mu^2} \int_{-\infty}^{\infty} dl \exp\left[(1-\eta_F)l - a_s\left((\Gamma_0^q + \eta_F\beta_0)\frac{l^2}{2} + (2\gamma_0^q + \eta_F K)l + \eta_F d_2\right) + \mathcal{O}(\alpha_s^2)\right] \\ \times \bar{I}_{q\leftarrow i}(z_1, l, a_s)\bar{I}_{\bar{q}\leftarrow j}(z_2, l, a_s) \end{aligned}$$

where one has a gaussian peak at

$$l_{peak} = \frac{1 - \eta - a_s(2\gamma_0 + \eta K)}{a_s(\Gamma_0 + \eta\beta_0)}$$

with a width proportional to $1/\sqrt{a_s}$. So, for $q_T \ll q_*$ the scale choice $\mu = q_*$ ensures that $L_\perp = \mathcal{O}(1)$ at the peak of the integrand, but the gaussian weight factor allows for significant contributions to the integral over a range of $L_\perp$ values with width proportional to $1/\sqrt{a_s}$. Thus one has a modified power counting

$$\begin{aligned} L_\perp &\sim 1/\sqrt{a_s} \\ (a_s L_\perp)^n &\sim a_s^{n/2} \\ (a_s L_\perp^2)^n &\sim 1 \end{aligned}$$

As one can see the double-logarithmic terms are unsuppressed and must be resummed to all orders. One introduces an auxiliary expansion parameter ϵ, so that $a_s \sim \epsilon$ and $L_\perp \sim \epsilon^{-1/2}$. Thus the exponent g_F in eq. (6.9) and (6.10) can be reordered in orders of ϵ as [55]

$$\begin{aligned} g_F(\eta, L_\perp, a_s) = &- [\eta_F L_\perp]_{\epsilon^{-1/2}} - \left[a_s(\Gamma_0^q + \eta_F\beta_0)\frac{L_\perp^2}{2}\right]_{\epsilon^0} \\ &- \left[a_s(2\gamma_0^q + \eta_F K)L_\perp + a_s^2(\Gamma_0^q + \eta_F\beta_0)\beta_0\frac{L_\perp^3}{3}\right]_{\epsilon^{1/2}} \\ &- \left[a_s\eta_F d_2 + a_s^2(K\Gamma_0^q + 2\gamma_0^q\beta_0 + \eta_F(\beta_1 + 2K\beta_0)\frac{L_\perp^2}{2} + a_s^3(\Gamma_0^q + \eta_F\beta_0)\beta_0^2\frac{L_\perp^4}{4}\right]_{\epsilon} \\ &- \mathcal{O}(\epsilon^{3/2}) \end{aligned} \tag{6.13}$$

The parameter ϵ counts the orders in a_s resulting after the x_T integral has been performed.
The scattering kernel in eq. (6.9) with g_F given in eq. (6.13) is the expression for the resummed scattering kernel.
So in this chapter, it was shown how to evaluate the factorization formula with the help of the RGEs and how to set the scale μ in order to avoid large logarithms.
In the next chapter top quark pair production at small transverse momentum is considered.

7. Top quark pair production at small transverse momentum

In the last chapter the factorization formula for a Drell-Yan production process at small transverse momentum was considered. This discussion can be transferred to the top quark pair production at small transverse momentum. What is different compared to Drell-Yan are the heavy quark final states and the additional color structure of the final particles. Both aspects complicate the calculation. The SCET particles are not appropriate to describe the heavy top quarks. Thus, this chapter starts with the introduction of an effective theory called *Heavy-Quark Effective Theory* (HQET). In this framework heavy tops can be described. Then, all ingredients are known that are necessary to derive a factorization formula for top quark pair production at small transverse momentum which is performed in section 7.2. The starting point of the discussion is the derivation of the effective Hamiltonian. In the Drell-Yan process an effective Hamiltonian was not needed, since the final leptonic part factorizes from the hadronic part and since the hard process is not a QCD process. Physical quantities in the effective theory are described by matrix elements of the effective Hamiltonian. So after the clarification of the effective Hamiltonian, the factorized cross section formula is derived. Hereafter, the resummation formula for top quark pair production is stated which is very similar to the one obtained for Drell-Yan. In top quark pair production a hard function appears which is calculated in the last section of this chapter to order α_s.

7.1. Heavy quark physics

Light quarks (up, down, strange quarks) have masses with $m \leq \Lambda_{QCD} \sim 200$ MeV. Consider a meson made up of a light and a heavy quark which is bound by the nonperturbative gluon dynamics. Such a system is of order $\sim \Lambda_{QCD}^{-1}$ [56]. The momentum transfer between the heavy and the light quark is of order Λ_{QCD}. Thus, the velocity of the heavy quark is almost unchanged and can be interpreted as a conserved quantity [17]. The limit $m_Q \to \infty$ (where m_Q is the heavy quark mass) leads to the heavy quark flavor symmetry: the dynamics is unchanged under the exchange of heavy quark flavors. The only strong interaction of a heavy quark is with gluons. In the limit $m_Q \to \infty$ the interaction of the static heavy quark with gluons is spin independent. This leads to heavy quark spin symmetry: the dynamics is unchanged under arbitrary transformations on the spin of the heavy quark. For further details see [17, 56, 57]. In HQET the heavy-quark symmetries are exact.
In the framework of HQET the momentum of a softly interacting heavy quark can be written as [58]

$$p_Q^\mu = m_Q v^\mu + k^\mu$$

where v^μ (with $v^2 = 1$) is the velocity of the quark and k^μ is the so called residual momentum with $k \sim \Lambda_{soft}$. In this way the scales m_Q and Λ_{soft} are separated. In the heavy quark limit the quark propagator can be written as

$$i\frac{\not{p} + m_Q}{p^2 - m_Q^2 + i\epsilon} = \frac{i}{v \cdot k + i\epsilon}\frac{1 + \not{v}}{2} + \mathcal{O}\left(\frac{k}{m_Q}\right) \to \frac{i}{v \cdot k + i\epsilon} P_+$$

with the projection operators $P_\pm$ defined as

$$P_\pm = \frac{1 \pm \not{v}}{2}$$

And the HQET gluon vertex is

$$ig\gamma^\mu T^a \to igT^a \frac{1+\slashed{v}}{2}\gamma^\mu \frac{1+\slashed{v}}{2} \to igv^\mu T^a \frac{1+\slashed{v}}{2}$$

The additional projectors on both sides of γ^μ are caused by the propagators of the heavy quark field. The projectors fulfill the relations

$$P_+ + P_- = 1\,, \qquad P_\pm^2 = 1\,, \qquad P_\pm P_\mp = 0.$$

So, the heavy quark field can be decomposed as [9, 1]

$$Q(x) = P_+Q(x) + P_-Q(x) = e^{-im_Q vx}\left(h_v(x) + H_v(x)\right) \tag{7.1}$$

where $Q(x)$ denotes a heavy quark spinor field. The phase factor subtracts the large part of the heavy quark momentum and thus only fluctuations of the order k^μ are present in $h_v(x)$ [1]. The phase factor can also be interpreted as taking the non-relativistic limit [17]. By implication the fields H_v and h_v are

$$h_v(x) = e^{im_Q v\cdot x}P_+Q(x) \quad \text{and} \quad H_v(x) = e^{im_Q v\cdot x}P_-Q(x)$$

Due to their definition the fields h_v and H_v obey the identities

$$\begin{aligned} P_-h_v(x) &= 0\,, \qquad & P_-H_v(x) &= H_v(x) \\ P_+h_v(x) &= h_v(x)\,, \qquad & P_+H_v(x) &= 0 \end{aligned}$$

From the Dirac equation one can conclude that the component $H_v(x)$ has to vanish. In order to show that one starts with the Dirac eq. for a heavy quark which is given by

$$\begin{aligned} \left(\slashed{p} - m_Q\right)Q(x) &= 0 \\ \Leftrightarrow (1-\slashed{v})\,Q(x) &= 0 \\ \Leftrightarrow P_-Q(x) &= 0 \end{aligned}$$

For $Q(x)$ one inserts the fields of eq. (7.1). This gives

$$\begin{aligned} P_-e^{-im_Q vx}\left(h_v(x) + H_v(x)\right) &= 0 \\ \Rightarrow H_v(x) &= 0 \end{aligned}$$

And thus, the field $h_v(x)$ is identified with the heavy quark field.

7.2. The factorization formula

Within the effective theories of SCET and HQET it is possible to derive a factorization formula for the cross section for the top quark pair production with small transverse momentum. The top quark pair is produced in the following hadronic process

$$N_1(P_1) + N_2(P_2) \to t(p_3) + \bar{t}(p_4) + X(p_X) \tag{7.2}$$

with two collinear directions

$$\begin{aligned} P_1^\mu &\propto n^\mu \\ P_2^\mu &\propto \bar{n}^\mu \end{aligned}$$

At leading order two partonic processes contribute. These are

$$\begin{aligned} q(p_1) + \bar{q}(p_2) &\to t(p_3) + \bar{t}(p_4) \\ g(p_1) + g(p_2) &\to t(p_3) + \bar{t}(p_4) \end{aligned}$$

where $p_1 = \xi_1 P_1$ and $p_2 = \xi_2 P_2$. The following kinematic variables are

$$\begin{aligned} s &= (P_1+P_2)^2, & \hat{s} &= (p_1+p_2)^2, \\ M^2 &= (p_3+p_4)^2, & t_1 &= (p_1-p_3)^2 - m_t^2, \\ u_1 &= (p_1-p_4)^2 - m_t^2, & \tau &= \frac{M^2+q_T^2}{s} \end{aligned}$$

where q_T is the transverse momentum of the $t\bar{t}$ pair and m_t is the top quark mass. Momentum conservation on Born level implies $u_1 + t_1 + \hat{s} = 0$. A kinematic region is considered where only soft or collinear emissions can contribute. This is

$$\hat{s}, M^2, |t_1|, |u_1|, m_t^2 \gg q_T^2 \gg \Lambda_{QCD}^2$$

The power counting parameter is

$$\lambda = \frac{q_T}{M}$$

In order to describe top quark pair production one needs SCET fields for the incoming partons and HQET fields for the outgoing heavy top quarks. As above, the QCD quark and gluon fields are replaced by (anti-)collinear gauge invariant SCET fields. Those were

$$\chi_n(x) = W_n^\dagger(x)\xi_n(x), \qquad A_n^\mu(x) = \frac{1}{g}W_n^\dagger(x)(iD_n^\mu W_n(x)) = \frac{1}{g}W_n^\dagger(i\partial^\mu + gA_n^\mu)W_n$$

The top quark fields are described by the HQET fields. All these fields are interacting with soft gluons via eikonal vertices. The soft interaction is decoupled via soft Wilson lines as it was already described in section 3.2. The top quark momenta can be written as

$$p_i^\mu = m_t v_i^\mu + k_i^\mu, \qquad \text{and} \qquad k_i^\mu \sim M(\lambda,\lambda,\lambda), \qquad i = 3,4$$

After the decoupling of soft gluons, the heavy-quark fields are effectively free fields. The next step is the clarification of the effective Hamiltonian, since the scattering amplitude is expressed by the matrix element of the effective Hamiltonian. The following ingredients are necessary that build up the most general Hamiltonian that describes the process of eq. (7.2)

- 1 collinear field for each direction $\to \chi_n$, $\bar{\chi}_{\bar{n}}$ or A_n, $A_{\bar{n}}$ (depending on the partonic production process)
- 2 HQET fields for the top quarks $\to \bar{h}_{v_3}$, h_{v_4}
- Soft Wilson lines due to soft decoupling transformation $\to S_n$, $S_{\bar{n}}$, $S_{v_3}^\dagger$, $S_{v_4}^\dagger$
- Wilson coefficient that is determined in a matching computation

The effective Hamiltonian describing the top quark pair production is thus given by [3]

$$\mathcal{H}_{\text{eff}}(x) = \sum_{I,m}\int dt_1\, dt_2\, e^{im_t(v_3+v_4)\cdot x}[\tilde{C}_{Im}^{q\bar{q}}(t_1,t_2)O_{Im}^{q\bar{q}}(x,t_1,t_2) + \tilde{C}_{Im}^{gg}(t_1,t_2)O_{Im}^{gg}(x,t_1,t_2)]$$

where I labels the color structure and m labels the Dirac structure. The exponential factor is justified below. The first term corresponds to the top-antitop production via the quark-antiquark annihilation process and the second term via the gluon fusion process. Similar to the SCET current that was matched onto the electromagnetic current, the effective Hamiltonian is as well defined via a convolution along light-like directions. $\tilde{C}_{Im}^{q\bar{q}}$ and $\tilde{C}_{Im}^{gg}$ are the corresponding Wilson coefficients. The convolution is between the collinear operators and the Wilson coefficients, compare eq. (7.4) and (7.5). Due to the decoupling of the different sectors the operator in the $q\bar{q}$ channel can be written as

$$O_{Im}^{q\bar{q}}(x,t_1,t_2) = \sum_{\{a\},\{b\}} (c_I)_{\{a\}}^{q\bar{q}}\; [O_m^h(x)]^{b_3 b_4}\; [O_m^c(x,t_1,t_2)]^{b_1 b_2}\; [O^s(x)]^{\{a\},\{b\}} \tag{7.3}$$

The partons entering the hard reaction carry color b_i. The soft radiation is decoupled from the collinear and hard sector. The soft function describes the color change of the incoming and outgoing partons.
The tensors $(c_I)^{q\bar{q}}_{\{a\}}$ define a basis in color space. They are orthogonal, but not normalized. In $[O^s(x)]$ all the soft interactions are contained. The operator $[O^c_m(x,t_1,t_2)]$ contains the incoming (anti-)collinear light partons and with the operator $[O^h_m(x)]$ the outgoing heavy top quarks are described. These operators are

$$\begin{aligned} \left[O^h_m(x)\right]^{b_3b_4} &= \bar{h}^{b_3}_{v_3}(x)\Gamma''_m h^{b_4}_{v_4}(x), \qquad [O^c_m(x,t_1,t_2)]^{b_1b_2} = \bar{\chi}^{b_2}_{\bar{n}}(x+t_2n)\Gamma'_m\chi^{b_1}_n(x+t_1\bar{n}) \\ [O^s(x)]^{\{a\},\{b\}} &= [S^{\mathbf{3}\dagger}_{v_3}(x)]^{a_3b_3}[S^{\bar{\mathbf{3}}\dagger}_{v_4}(x)]^{a_4b_4}[S^{\bar{\mathbf{3}}}_{\bar{n}}(x)]^{b_2a_2}[S^{\mathbf{3}}_n(x)]^{b_1a_1} \end{aligned} \tag{7.4}$$

where the notation of the soft Wilson lines was explained in chapter 2.2.2 and appendix B.4.
And for the gluon channel one has

$$O^{gg}_{Im}(x,t_1,t_2) = \sum_{\{a\},\{b\}} (c_I)^{gg}_{\{a\}}\ [O^h_m(x)]^{\mu\nu}_{b_3b_4}\ [O^c_m(x,t_1,t_2)]^{b_1b_2}_{\mu\nu}\ [O^s(x)]^{\{a\},\{b\}}$$

where

$$\begin{aligned} \left[O^h_m(x)\right]^{\mu\nu}_{b_3b_4} &= \bar{h}^{b_3}_{v_3}(x)\Gamma^{\mu\nu}_m h^{b_4}_{v_4}(x), \qquad [O^c_m(x,t_1,t_2)]^{b_1b_2}_{\mu\nu} = \mathcal{A}^{b_1}_{n\mu\perp}(x+t_1\bar{n})\mathcal{A}^{b_2}_{\bar{n}\nu\perp}(x+t_2n) \\ [O^s(x)]^{\{a\},\{b\}} &= [S^{\mathbf{3}\dagger}_{v_3}(x)]^{a_3b_3}[S^{\bar{\mathbf{3}}\dagger}_{v_4}(x)]^{a_4b_4}[S^{\mathbf{8}}_{\bar{n}}(x)]^{b_2a_2}[S^{\mathbf{8}}_n(x)]^{b_1a_1} \end{aligned} \tag{7.5}$$

The matrices Γ'_m, Γ''_m and $\Gamma^{\mu\nu}_m$ are combinations of Dirac matrices and the external vectors n, $\bar{n}$, v_3 and v_4. They describe the hard reaction on parton level. They are matrices in color space and they are chosen such that the QCD result is reproduced. At tree level they are adjusted by claiming that the QCD scattering amplitude corresponds to the one obtained in the effective theory which is given by

$$\mathcal{M}^{q\bar{q}}_{\{a\}} = \langle t^{a_3}\bar{t}^{a_4}|\,\mathcal{H}_{\text{eff}}(0)\,|q^{a_1}(p_1)\bar{q}^{a_2}(p_2)\rangle$$

In order to deal with the color structure it is reasonable to work within the color-space formalism which was already mentioned in section 2.2.2. In this formalism, scattering amplitudes are treated as vectors and color generators and any object involving them are treated as matrices in this color space. Boldface letters are used to denote color space matrices and the bra-ket-notation is used to denote vectors. The color basis $(c_I)_{\{a\}}$ are vectors $|c_I\rangle$. The amplitude for the process with fixed color indices $\{a\}$ for the external particles is indicated with $\mathcal{M}_{\{a\}}$. The abstract vector representing the process in color space is $|\mathcal{M}\rangle$ which is related to $\mathcal{M}_{\{a\}}$ by

$$\mathcal{M}_{\{a\}} = \langle a_1,a_2,a_3,a_4|\mathcal{M}\rangle = \langle\{a\}|\mathcal{M}\rangle$$

A color generator $\mathbf{T}^c_i$ acts on the color index of the i-th parton in the basis vectors as follows

$$\mathbf{T}^c_i\,|\cdots,a_i,\cdots\rangle = (\mathbf{T}^c_i)_{b_ia_i}\,|\cdots,b_i,\cdots\rangle$$

Since the amplitudes of interest are color singlets, the QCD amplitude can be decomposed into a set of color structures called color basis. This is done by writing

$$|\mathcal{M}\rangle = \sum_I \mathcal{M}_I \sum_{\{a\}} (c_I)_{\{a\}}\,|\{a\}\rangle \equiv \sum_I \mathcal{M}_I\,|c_I\rangle$$

$\mathcal{M}_I$ is obtained via projecting the scattering amplitude onto the color basis

$$\mathcal{M}_I = \frac{1}{\langle c_I|c_I\rangle}\,\langle c_I|\mathcal{M}\rangle$$

The color basis which is also used in [3, 35, 25, 9] is the singlet-octet bases. This choice can be justified considering the color structure of the final top quark pair. It has the representation $3 \otimes \bar{3} = 1 \oplus 8$. The singlet-octet basis for the quark-antiquark annihilation process is given by

$$\begin{aligned} \text{singlet:} \quad & (c_1^{q\bar{q}})_{\{a\}} = \delta_{a_1 a_2} \delta_{a_3 a_4} \\ \text{octet:} \quad & (c_2^{q\bar{q}})_{\{a\}} = t^c_{a_2 a_1} t^c_{a_3 a_4} \end{aligned}$$

For the production of a $3 \otimes \bar{3}$ final state from gluon fusion, there are three possible combinations of initial and final state representations. The final singlet state is produced via the initial singlet state and the final octet state is either produced by a symmetric or antisymmetric octet initial state [25, 26]. These are

$$\begin{aligned} \text{singlet:} \quad & (c_1^{gg})_{\{a\}} = \delta^{a_1 a_2} \delta_{a_3 a_4}, \\ \text{antisymmetric octet:} \quad & (c_2^{gg})_{\{a\}} = f^{a_1 a_2 c} t^c_{a_3 a_4}, \\ \text{symmetric octet:} \quad & (c_3^{gg})_{\{a\}} = d^{a_1 a_2 c} t^c_{a_3 a_4} \end{aligned}$$

The color structure is orthogonal which means

$$\langle c_I | c_J \rangle \propto \delta_{IJ}$$

In order to obtain an explicit expression for the matrix Γ'_m of the quark-channel one can first take a look on the parton matrix element which should coincides in full QCD and in the effective theory. The scattering amplitude in QCD is given by

$$\mathcal{M}^{QCD} = \frac{g^2}{(p_1 + p_2)^2} T^c_{a_3 a_4} T^c_{a_2 a_1} \cdot (\bar{u}(p_3) \gamma^\mu v(p_4)) \cdot (\bar{v}(p_2) \gamma_\mu u(p_1))$$

This matrix element can be projected onto the color basis one obtains

$$|\mathcal{M}^{QCD}\rangle = \frac{g^2}{(p_1 + p_2)^2} (\bar{u}(p_3) \gamma^\mu v(p_4)) \cdot (\bar{v}(p_2) \gamma_\mu u(p_1)) |c_2\rangle$$

In the effective theory one has on leading order

$$\mathcal{M}^{q\bar{q}}_{\{a\}} = \langle t^{a_3} \bar{t}^{a_4} | \mathcal{H}_{\text{eff}}(0) | q^{a_1}(p_1) \bar{q}^{a_2}(p_2) \rangle = \big(\bar{u}(p_3) \Gamma''_m v(p_4) \big) \cdot \big(\bar{v}(p_2) \Gamma'_m u(p_1) \big)$$

Due to the demand that they coincide, one first assumes

$$\Gamma''_m \sim \gamma^\mu \qquad \Gamma'_m = \gamma^\mu$$

Actually the matrices Γ_m are matrices in color space, but since the QCD matrix element only has one color structure and the color basis is orthogonal, the matrices Γ_m only have one entry and the assumption from above is justified. The problem becomes evident in the gluon case. The fine-tuning due to the right prefactor will be done later. First, one takes a look at the collinear operator. Inserting γ^μ in the operator one obtains a similar expression as for the current operator derived above

$$[O^c_m(x, t_1, t_2)]^{a_1 a_2} = \bar{\chi}^{a_2}_{\bar{n}}(x + t_2 n) \gamma^\mu \chi^{a_1}_n(x + t_1 \bar{n}) \equiv I^\mu_{a_1 a_2}(x)$$

Since the different sectors decouple, one has

$$\sigma \sim \langle N_1 N_2 | I^{\dagger\mu}_{b_1 b_2} I^\nu_{a_1 a_2} | N_1 N_2 \rangle$$

This was already calculated in the derivation of the Drell-Yan factorization formula, compare eq. (6.1). It is

$$\begin{aligned} &\langle N_1 N_2 | \Big(\bar{\chi}^{b_1}_{n\rho}(x + t'_1 \bar{n}) \gamma^\mu_{\rho\sigma} \chi^{b_2}_{\bar{n}\sigma}(x + t'_2 n) \Big) \Big(\bar{\chi}^{a_2}_{\bar{n}\alpha}(x + t_2 n) \gamma^\nu_{\alpha\beta} \chi^{a_1}_{n\beta}(x + t_1 \bar{n}) \Big) | N_1 N_2 \rangle \\ &= -\frac{\delta^{b_1 a_1} \delta^{b_2 a_2}}{2 N_c^2} g^{\mu\nu}_\perp \langle N_1 | \bar{\chi}^{b_1}_n(x + t'_1 \bar{n}) \frac{\not{\bar{n}}}{2} \chi^{a_1}_n(x + t_1 \bar{n}) | N_1 \rangle \frac{\not{n}_{\alpha\sigma}}{2} \langle N_2 | \chi^{b_2}_{\bar{n}\sigma}(x + t'_2 n) \bar{\chi}^{a_2}_{\bar{n}\alpha}(t_2 n) | N_2 \rangle \end{aligned}$$

The factor $-g_\perp^{\mu\nu}/2$ will be absorbed in the hard operator. The matrix elements can be expressed by the TPDFs.
In the effective theory, the hadronic differential cross section for $t\bar{t}$ production is given by

$$d\sigma = \frac{1}{2s}\frac{d^3\vec{p}_3}{(2\pi)^3 2E_3}\frac{d^3\vec{p}_4}{(2\pi)^3 2E_4}\sum_X \int d^4x\, e^{i(P_1+P_2-p_3-p_4-p_X)}\, \langle\mathcal{M}(0)|\mathcal{M}(0)\rangle$$

with

$$\begin{aligned}
|\mathcal{M}(0)\rangle &= \sum_{I,m}\int dt_1\, dt_2\, \langle t\bar{t}X|\, [\boldsymbol{O}_m^h(0)]\, [\boldsymbol{O}_m^c(0,t_1,t_2)]\, [\boldsymbol{O}^s(0)]\, |N_1(P_1)N_2(P_2)\rangle\, |\tilde{C}_m(t_1,t_2)\rangle \\
|\tilde{C}_m(t_1,t_2)\rangle &= \sum_I \tilde{C}_{Im}(t_1,t_2)\, |c_I\rangle
\end{aligned}$$

The integral over d^4x and the the exponential factor are caused through the transcription of the momentum conservation delta function

$$(2\pi)^4\delta^4(P_1+P_2-p_3-p_4-p_X) = \int d^4x e^{i(P_1+P_2-p_3-p_4-p_X)}$$

Performing the same steps as in the Drell-Yan case, the field operators can be translated by rewriting the exponential in terms of the translation operator. So one obtains

$$\begin{aligned}
&\sum_{X_c,X_{\bar{c}},X_s} e^{i(P_1+P_2-p_3-p_4-p_c-p_{\bar{c}}-p_s)\cdot x}\, \langle\mathcal{M}(0)|\mathcal{M}(0)\rangle \\
&\quad= \sum_{m,m'}\int dt_1\, dt_2\, dt_1' dt_2' e^{-i(p_3+p_4)x}\, \langle 0|\, [\boldsymbol{O}_{m'}^{h\dagger}(0)]\, |t(p_3)\bar{t}(p_4)\rangle\, \langle t(p_3)\bar{t}(p_4)|\, [\boldsymbol{O}_m^h(0)]\, |0\rangle \\
&\quad\times \frac{\delta^{b_1a_1}}{N_c}\sum_{X_c} e^{i(P_1-p_c)x}\frac{\bar{\not{n}}_{\alpha\beta}}{2}\, \langle N_1|\, \bar{\chi}_{n\alpha}^{b_1}(t_1'\bar{n})\, |X_c\rangle\, \langle X_c|\, \chi_{n\beta}^{a_1}(t_1\bar{n})\, |N_1\rangle \\
&\quad\times \frac{\delta^{b_2a_2}}{N_c}\sum_{X_{\bar{c}}} e^{i(P_2-p_{\bar{c}})x}\frac{\not{n}_{\rho\sigma}}{2}\, \langle N_2|\, \chi_{\bar{n}\sigma}^{b_2}(t_2'n)\, |X_{\bar{c}}\rangle\, \langle X_{\bar{c}}|\, \bar{\chi}_{\bar{n}\rho}^{a_2}(t_2n)\, |N_2\rangle \\
&\quad\times \sum_{X_s} e^{-ip_sx}\, \langle\tilde{C}_{m'}(t_1,t_2)|\, \langle 0|\, \boldsymbol{O}_s^\dagger(0)\, |X_s\rangle\, \langle X_s|\, \boldsymbol{O}_s(0)\, |0\rangle\, |\tilde{C}_m(t_1,t_2)\rangle
\end{aligned} \tag{7.6}$$

These matrix elements need to be multipole expanded and only the leading terms that scale as $\sim\mathcal{O}(1)$ are kept. Since $q^\mu\sim(1,1,\lambda)$ one has

$$\begin{aligned}
x^\mu\sim(1,1,\lambda^{-1}) \qquad &\Rightarrow \qquad P_1\cdot x = \frac{1}{2}\underbrace{P_1^-x^+}_{\sim 1}+\frac{1}{2}\underbrace{P_1^+x^-}_{\sim\lambda^2}+\underbrace{P_{1\perp}x_\perp}_{\sim 1}\approx\frac{1}{2}P_1^-x^+ + P_{1\perp}x_\perp \\
&\Rightarrow \qquad P_2\cdot x\approx\frac{1}{2}P_2^+x^- + P_{2\perp}x_\perp \\
&\Rightarrow \qquad P^s\cdot x\approx P_\perp^s x_\perp
\end{aligned}$$

where P_1 is collinear and P_2 is anti-collinear and P^s is soft. $x_\perp$ has the meaning of an impact factor in a scattering process. And so one obtains

$$\begin{aligned}
&= \sum_{m,m'}\int dt_1\, dt_2\, dt_1' dt_2' e^{-i(p_3+p_4)x}\, \langle 0|\, [\boldsymbol{O}_{m'}^{h\dagger}(0)]\, |t(p_3)\bar{t}(p_4)\rangle\, \langle t(p_3)\bar{t}(p_4)|\, [\boldsymbol{O}_m^h(0)]\, |0\rangle \\
&\times \frac{\delta^{b_1a_1}}{N_c}\frac{\bar{\not{n}}_{\alpha\beta}}{2}\, \langle N_1(P_1)|\, \bar{\chi}_{n\alpha}^{b_1}(x_+^\mu + x_\perp^\mu + t_1'\bar{n}^\mu)\chi_{n\beta}^{a_1}(t_1\bar{n})\, |N_1(P_1)\rangle \\
&\times \frac{\delta^{b_2a_2}}{N_c}\frac{\not{n}_{\rho\sigma}}{2}\, \langle N_2(P_2)|\, \chi_{\bar{n}\sigma}^{b_2}(x_-^\mu + x_\perp^\mu + t_2'n^\mu)\bar{\chi}_{\bar{n}\rho}^{a_2}(t_2n)\, |N_2(P_2)\rangle \\
&\times \langle\tilde{C}_{m'}(t_1,t_2)|\, \langle 0|\, \boldsymbol{O}_s^\dagger(x_\perp^\mu)\boldsymbol{O}_s(0)\, |0\rangle\, |\tilde{C}_m(t_1,t_2)\rangle
\end{aligned} \tag{7.7}$$

Following the same steps as before for Drell-Yan, the (anti)collinear matrix elements are expressed via the TPDFs. As in eq. (6.2), one has

$$\begin{aligned}\mathcal{M}_c &\equiv \frac{\delta_{ab}}{N_c} \langle N_1(P_1)| \bar{\chi}^a_n(t'\bar{n}^\mu + x^\mu_\perp + x^\mu_+)\frac{\bar{\not{n}}}{2}\chi^b_n(t\bar{n}^\mu) |N_1(P_1)\rangle \\ &= \frac{\delta_{ab}}{N_c} (\bar{n}\cdot P_1) \int_{-1}^{1} dz\, e^{iz\bar{n}P_1(t'-t+x_+)} \mathcal{B}_{q/N_1}(z, x_T^2, \mu) \\ &= \frac{\delta_{ab}}{N_c} (\bar{n}\cdot p_1) \int_0^1 \frac{dz}{z}\, e^{i\bar{n}p_1(t'-t+x_+)} \mathcal{B}_{q/N_1}(z, x_T^2, \mu) - \frac{\delta_{ab}}{N_c} (\bar{n}\cdot p_1) \int_0^1 \frac{ds}{s}\, e^{i\bar{n}p_1(t'-t+x_+)} \mathcal{B}_{q/N_1}(-s, x_T^2, \mu) \\ &= \frac{\delta_{ab}}{N_c} (\bar{n}\cdot p_1) \int_0^1 \frac{dz}{z}\, e^{i\bar{n}p_1(t'-t+x_+)} \left(\mathcal{B}_{q/N_1}(z, x_T^2, \mu) + \mathcal{B}_{\bar{q}/N_1}(z, x_T^2, \mu)\right)\end{aligned}$$

where a substitution was performed $z = -s$. The same steps can be applied to the anti-collinear matrix element which gives

$$\begin{aligned}\mathcal{M}_{\bar{c}} &\equiv \frac{\delta_{ab}}{N_c} \frac{\not{n}_{\rho\sigma}}{2} \langle N_2(P_2)| \chi^b_{\bar{n}\sigma}(x^\mu_- + x^\mu_\perp + t_2' n^\mu)\bar{\chi}^a_{\bar{n}\rho}(t_2 n) |N_2(P_2)\rangle \\ &= \frac{\delta_{ab}}{N_c} (n\cdot P_2) \int_{-1}^{1} dz\, e^{iznP_2(t_2'-t_2+x_+)} \bar{\mathcal{B}}_{\bar{q}/N_2}(z, x_T^2, \mu) \\ &= \frac{\delta_{ab}}{N_c} \int_0^1 \frac{dz}{z}\, n\cdot p_2\, e^{inp_2(t_2'-t_2+x_-)} \left(\bar{\mathcal{B}}_{q/N_2}(z, x_T^2, \mu) + \bar{\mathcal{B}}_{\bar{q}/N_2}(z, x_T^2, \mu)\right)\end{aligned}$$

For the gluon TPDFs one has the identity

$$\mathcal{B}_{g/N}(z, x_T^2) = -\mathcal{B}_{g/N}(-z, x_T^2)$$

and thus

$$\begin{aligned}\langle N_1(P_1)| \mathcal{A}^a_{n\mu\perp}(t_1'\bar{n}^\mu + x^\mu_\perp + x^\mu_+)\mathcal{A}^b_{n\nu\perp}(t_1\bar{n}^\mu) |N_1(P_1)\rangle &= -\frac{2\delta_{ab}}{N_g} \int_0^1 \frac{dz}{z} \mathcal{B}^{\mu\nu,n}_{g/N}(z, x_\perp, \mu) e^{i(x^\mu_+ + (t_1'-t_1)\bar{n})zP_1} \\ \langle N_2(P_2)| \mathcal{A}^a_{\bar{n}\mu\perp}(t_2' n^\mu + x^\mu_\perp + x^\mu_-)\mathcal{A}^b_{\bar{n}\nu\perp}(t_2 n^\mu) |N_2(P_2)\rangle &= -\frac{2\delta_{ab}}{N_g} \int_0^1 \frac{dz}{z} \mathcal{B}^{\mu\nu,\bar{n}}_{g/N}(z, x_\perp, \mu) e^{i(x^\mu_- + (t_2'-t_2)n)zP_2}\end{aligned}$$

where $N_g = N_c^2 - 1 = 8$ is the possible number of color combinations a gluon can have.
Now the whole dependence on the integration variables t_1, t_2, t_1' and t_2' is encoded in the exponential terms - which arose when the collinear matrix elements were expressed by the TPDFs - and the Wilson coefficient. Thus, the integrals over t_1, t_2, t_1' and t_2' produce the Fourier transformed Wilson coefficients. One has

$$\begin{aligned}|C_m\rangle &= \int dt_1\, dt_2\, e^{-it_1\bar{n}p_1 - it_2 np_2} |\tilde{C}_m(t_1, t_2)\rangle \\ \langle C_{m'}| &= \int dt_1'\, dt_2'\, e^{it_2' np_2 + it_1'\bar{n}p_1} \langle \tilde{C}_{m'}(t_1', t_2')|\end{aligned}$$

The hard function is defined as

$$\begin{aligned}\mathbf{H}^{\mu\nu\rho\sigma}_{gg}(M, m_t, v_3, \mu) &= \frac{3}{8}\frac{1}{(4\pi)^2}\frac{1}{4N_g} \sum_{spin} |C_m\rangle \langle C_{m'}| \langle 0| [\boldsymbol{O}^{h\dagger}_{m'}(0)]^{\rho\sigma} |t(p_3)\bar{t}(p_4)\rangle \langle t(p_3)\bar{t}(p_4)| [\boldsymbol{O}^h_m(0)]^{\mu\nu} |0\rangle \\ \mathbf{H}_{q\bar{q}}(M, m_t, v_3, \mu) &= \frac{3}{8}\frac{1}{(4\pi)^2}\frac{1}{4N_c} \sum_{spin} |C_m\rangle \langle C_{m'}| \langle 0| [\boldsymbol{O}^{h\dagger}_{m'}(0)] |t(p_3)\bar{t}(p_4)\rangle \langle t(p_3)\bar{t}(p_4)| [\boldsymbol{O}^h_m(0)] |0\rangle\end{aligned}$$

The factor 1/4 accounts for the averaging over the polarization of the initial state partons [35]. In both channels this gives a factor of 1/4. The factor $3/8(4\pi)^2$ is a convention used in [3]. The factor $1/N_c$ is one of the color averaging factors arising when the collinear matrix elements are expressed by the TPDFs. The position space soft functions are defined as

$$\mathbf{W}(x_\perp,\mu) = \frac{1}{N_R}\langle 0|\,\bar{\mathbf{T}}[\boldsymbol{O}_s^\dagger(x_\perp)]\mathbf{T}[\boldsymbol{O}_s(0)]\,|0\rangle \qquad \text{with} \quad N_R = N_c, N_g \tag{7.8}$$

where $\mathbf{T}$ is the time ordering operator. The soft function depends on color, because it describes soft gluon radiation. The soft function in Drell-Yan reduced to unity. The difference here is that one has two more soft Wilson lines due to color charged final states that do not point in light-like directions.
The elements of these matrices in the chosen color basis are defined as

$$H_{IJ} = \frac{1}{\langle c_I|c_I\rangle\,\langle c_J|c_J\rangle}\,\langle c_I|\,\mathbf{H}\,|c_J\rangle\,, \qquad W_{IJ} = \langle c_I|\,\mathbf{W}\,|c_J\rangle$$

so that the soft and hard function in the differential cross section are proportional to

$$\mathrm{Tr}[\mathbf{HW}] = \sum_{I,J} H_{IJ}W_{JI}$$

Taking all ingredients for the cross section together, one obtains for the gluon channel

$$\begin{aligned} d\sigma_{gg} =& \frac{1}{2s}\frac{d^3\vec{p}_3}{(2\pi)^3 2E_3}\frac{d^3\vec{p}_4}{(2\pi)^3 2E_4}\int d^4x\; e^{i(-p_3-p_4)\cdot x}\,\frac{8}{3}\,(4\pi)^2\int_0^1\int_0^1 \frac{dz_1}{z_1}\frac{dz_2}{z_2}e^{\frac{i}{2}(x^+p_1^-+x^-p_2^+)} \\ &\times\left(\frac{x_T^2M^2}{4e^{-2\gamma_E}}\right)^{-F_{gg}} 4B^{\mu\nu}_{g/N_1}(z_1,x_T^2,\mu)\;B^{\rho\sigma}_{g/N_2}(z_2,x_T^2,\mu)\mathrm{Tr}\left[\mathbf{H}^{\mu\nu\rho\sigma}_{gg}(M,m_t,v_3,\mu)\mathbf{W}_{gg}(x_\perp,\mu)\right] \end{aligned}$$

where the collinear anomaly was already taken into account. In order to obtain the invariant mass kinematics, one inserts the following identity into the differential cross section, defining $q = p_3 + p_4$

$$1 = \int d^4q\; dM^2\;\delta^{(4)}(q-p_3-p_4)\delta(M^2-q^2)$$

Performing the $\vec{p_4}$ integration using the 4-dim delta function one obtains

$$\begin{aligned} d\sigma_{gg} =& \frac{1}{12\pi^4 s}\frac{d^3\vec{p}_3}{E_3\;E_4}\int d^4x\; e^{-iqx}\int_0^1\int_0^1 \frac{dz_1}{z_1}\frac{dz_2}{z_2}e^{\frac{i}{2}(x^+p_1^-+x^-p_2^+)}\int d^4q\;dM^2\;\delta(q^0-E_3-E_4)\delta(M^2-q^2) \\ &\times\left(\frac{x_T^2M^2}{4e^{-2\gamma_E}}\right)^{-F_{gg}} 4B^{\mu\nu}_{g/N_1}(z_1,x_T^2,\mu)\;B^{\rho\sigma}_{g/N_2}(z_2,x_T^2,\mu)\mathrm{Tr}\left[\mathbf{H}^{\mu\nu\rho\sigma}_{gg}(M,m_t,v_3,\mu)\mathbf{W}_{gg}(x_\perp,\mu)\right] \end{aligned}$$

It is reasonable, to evaluate first the x_+ and x_- integration. That gives δ constraints in z_1 and z_2 which makes the z_1, z_2 integration simple. These steps are (remembering $p_1 = z_1P_1$ and $p_2 = z_2P_2$)

$$\begin{aligned} &\int d^4x\int_0^1\int_0^1 \frac{dz_1}{z_1}\frac{dz_2}{z_2}\;e^{\frac{i}{2}(x^+p_1^-+x^-p_2^+)}\;e^{-iqx}\delta(M^2-q^2) \\ &= 2(2\pi)^2\delta(p_1^- - q^-)\delta(p_2^+ - q^+)\int_0^1\int_0^1\frac{dz_1}{z_1}\frac{dz_2}{z_2}\;d^2\vec{x}_T\;e^{i\vec{x}_T\vec{q}_T}\delta(M^2-q^+q^--q_T^2) \\ &= 8\pi^2\frac{1}{P_1^-P_2^+}\int_0^1\int_0^1\frac{dz_1}{z_1}\frac{dz_2}{z_2}\;\delta\left(z_1-\frac{q^-}{P_1^-}\right)\delta\left(z_2-\frac{q^+}{P_2^+}\right)d^2\vec{x}_T\;e^{i\vec{x}_T\vec{q}_T}\delta(M^2-q^+q^--q_T^2) \\ &= 8\pi^2\frac{1}{q^+q^-}\int d^2\vec{x}_T\;e^{i\vec{x}_T\vec{q}_T}\delta(M^2-q^+q^--q_T^2) \end{aligned}$$

with $q_T^2 \equiv \vec{q}_T^{\,2} = -q_\perp^2$. And one identifies

$$\frac{q^-}{P_1^-} = \xi_1 = \sqrt{\tau}e^{y}, \qquad \frac{q^+}{P_2^+} = \xi_2 = \sqrt{\tau}e^{-y},$$

where

$$\tau = \frac{M^2 + q_T^2}{s}, \qquad y = \frac{1}{2}\ln\frac{q^-}{q^+}$$

Now the integration variables are transformed as

$$\begin{aligned} q^- &= q^+ e^{2y} \\ q^+ &= q^+ \end{aligned}$$

which gives the Jacobian $2e^{2y}q^+ dy\, dq^+$ and thus the following integral is

$$\int \frac{1}{2} dq_+ dq_- d^2\vec{q}_T \delta(q^2 - M^2)\frac{1}{q_+ q_-} = \frac{1}{2}\frac{1}{q_T^2 + M^2} dy\, d^2 q_T$$

Inserting this in the above calculation yields

$$4\pi^2\, dy\, d^2\vec{q}_T\, \frac{1}{q_T^2 + M^2} d^2\vec{x}_T\, e^{i\vec{x}_T\vec{q}_T} = 2\pi^2\, dy\, \frac{1}{q_T^2 + M^2}\, dq_T^2\, d\Phi_q\, d^2\vec{x}_T\, e^{i\vec{x}_T\vec{q}_T}$$

where the pair transverse momentum is supposed to be small and y is the rapidity of q^μ given by

$$y = \frac{1}{2}\ln\frac{q^-}{q^+}$$

This can be inserted in the differential cross section yielding

$$\begin{aligned} d\sigma_{gg} &= \frac{1}{6\pi^2 s} dy\, \frac{1}{q_T^2 + M^2} dM^2 \int dq_T^2\, d\Phi_q\, d^2\vec{x}_T\, e^{i\vec{x}_T\vec{q}_T} \frac{d^3\vec{p}_3}{E_3\, E_4}\delta(M - E_3 - E_4) \\ &\times \left(\frac{x_T^2 M^2}{4e^{-2\gamma_E}}\right)^{-F_{gg}} 4B^{\mu\nu}_{g/N_1}(\xi_1, x_T^2, \mu)\, B^{\rho\sigma}_{g/N_2}(\xi_2, x_T^2, \mu)\mathrm{Tr}\left[\mathbf{H}^{\mu\nu\rho\sigma}_{gg}(M, m_t, v_3, \mu)\mathbf{W}_{gg}(x_\perp, \mu)\right] \end{aligned}$$

The $|\vec{p_3}|$ integration can be performed using the remaining delta function. In the partonic center of mass frame we have $\vec{q} \approx 0$ which means that the absolute value of $\vec{p}_3$ and $\vec{p}_4$ are the same and thus $E_3 \approx E_4$. Next I calculate

$$\int \frac{d^3\vec{p}_3}{E_3^2}\delta(M - 2E_3) = \int \frac{|\vec{p}_3|^2}{E_3^2}\, d|\vec{p}_3|\, d\cos\theta_3\, d\Phi_3\, \delta\left(M - 2\sqrt{m_t^2 + |\vec{p}_3|^2}\right) = \frac{\beta_t}{2} d\cos\theta_3\, d\Phi_3$$

where β_t is

$$\beta_t = \sqrt{1 - \frac{4m_t^2}{M^2}}$$

So in total one has

$$\begin{aligned} d\sigma_{gg} &= \frac{1}{12\pi^2 s} dy\, \frac{\beta_t}{M^2 + q_T^2}\, dq_T^2 dM^2\, d\cos\theta_3 \int d\Phi_q\, d^2\vec{x}_T\, e^{i\vec{x}_T\vec{q}_T}\, d\Phi_3 \\ &\times \left(\frac{x_T^2 M^2}{4e^{-2\gamma_E}}\right)^{-F_{gg}} 4B^{\mu\nu}_{g/N_1}(\xi_1, x_T^2, \mu)\, B^{\rho\sigma}_{g/N_2}(\xi_2, x_T^2, \mu)\mathrm{Tr}\left[\mathbf{H}^{\mu\nu\rho\sigma}_{gg}(M, m_t, v_3, \mu)\mathbf{W}_{gg}(x_\perp, \mu)\right] \\ &= \frac{1}{6\pi^2 s} dy\, \beta_t \frac{M}{q_T^2 + M^2}\, dq_T^2\, dM\, d\cos\theta_3 \int d\Phi_q\, d|\vec{x}_T||\vec{x}_T|\, d\Phi_x\, e^{i\vec{x}_T\vec{q}_T}\, d\Phi_3 \\ &\times \left(\frac{x_T^2 M^2}{4e^{-2\gamma_E}}\right)^{-F_{gg}} 4B^{\mu\nu}_{g/N_1}(\xi_1, x_T^2, \mu)\, B^{\rho\sigma}_{g/N_2}(\xi_2, x_T^2, \mu)\mathrm{Tr}\left[\mathbf{H}^{\mu\nu\rho\sigma}_{gg}(M, m_t, v_3, \mu)\mathbf{W}_{gg}(x_\perp, \mu)\right] \end{aligned}$$

The same calculation can be performed for the quark channel. Due to the definition of the beam function, one obtains an additional factor of M^2 in the cross section. This factor is absorbed into the hard function. So in total one has (with the notation $d|\vec{x}_T||\vec{x}_T| = dx_T\, x_T$)

$$\begin{aligned} d\sigma = &\frac{\beta_t}{6\pi^2 s}\frac{M}{q_T^2+M^2} dy\, dq_T^2\, dM\, d\cos\theta_3 \int dx_T\, x_T\, d\Phi_q\, d\Phi_x\, d\Phi_3\, e^{i\vec{x}_T\vec{q}_T} \\ &\times \Bigg(\left(\frac{x_T^2 M^2}{4e^{-2\gamma_E}}\right)^{-F_{gg}(x_T^2,\mu)} 4B^{\mu\nu}_{g/N_1}(\xi_1, x_T^2,\mu)\, B^{\rho\sigma}_{g/N_2}(\xi_2, x_T^2,\mu) \mathrm{Tr}\left[\mathbf{H}^{\mu\nu\rho\sigma}_{gg}(M, m_t, v_3,\mu)\mathbf{W}_{gg}(x_\perp,\mu)\right] \\ &+ \left(\frac{x_T^2 M^2}{4e^{-2\gamma_E}}\right)^{-F_{q\bar{q}}(x_T^2,\mu)} B_{q/N_1}(\xi_1, x_T^2,\mu)\, B_{\bar{q}/N_2}(\xi_2, x_T^2,\mu) \mathrm{Tr}\left[\mathbf{H}_{q\bar{q}}(M, m_t, \cos\theta,\mu)\mathbf{W}_{q\bar{q}}(x_\perp,\mu)\right] + (q \leftrightarrow \bar{q}) \Bigg) \end{aligned}$$

Where θ_3 is the scattering angle of the top quark in the $t\bar{t}$ rest frame, Φ_3, Φ_q and Φ_x are the azimuthal angles of v_3, $q_\perp$ and $x_\perp$ respectively. The azimuthal integrals can be simplified by noting that the integrand only depends on the two differences $\Phi_q - \Phi_x$ and $\Phi_3 - \Phi_x$. Where the exponential factor depends on $\Phi_q - \Phi_x$ and the collinear, soft and hard functions can depend on $\Phi_3 - \Phi_x$. One can transform the integration variables as

$$\begin{aligned} \Phi &= \Phi_3 - \Phi_x \\ \chi &= \Phi_q - \Phi_x \\ \Phi_x &= \Phi_x \end{aligned}$$

One can perform the integration over χ and Φ_x which gives

$$\int d\Phi_x\, d\chi\, d\Phi\, e^{iq_T x_T \cos(\chi)} = 8\pi^3 J_0(q_T x_T) \int \frac{d\Phi}{2\pi}$$

where J_0 is a bessel function. This leads to the differential cross section

$$\begin{aligned} \frac{d^4\sigma}{dy\, dq_T^2\, dM\, d\cos\theta_3} = &\frac{8\pi\beta_t M}{6s(M^2+q_T^2)} \int dx_T\, x_T \frac{d\Phi}{2\pi} J_0(x_T q_T) \\ &\times \Bigg(\left(\frac{x_T^2 M^2}{4e^{-2\gamma_E}}\right)^{-F_{gg}(x_T^2,\mu)} 4B^{\mu\nu}_{g/N_1}(\xi_1, x_T^2,\mu)\, B^{\rho\sigma}_{g/N_2}(\xi_2, x_T^2,\mu) \mathrm{Tr}\left[\mathbf{H}^{\mu\nu\rho\sigma}_{gg}(M, m_t, v_3,\mu)\mathbf{W}_{gg}(x_\perp,\mu)\right] \\ &+ \left(\frac{x_T^2 M^2}{4e^{-2\gamma_E}}\right)^{-F_{q\bar{q}}(x_T^2,\mu)} B_{q/N_1}(\xi_1, x_T^2,\mu)\, B_{\bar{q}/N_2}(\xi_2, x_T^2,\mu) \mathrm{Tr}\left[\mathbf{H}_{q\bar{q}}(M, m_t, \cos\theta,\mu)\mathbf{W}_{q\bar{q}}(x_\perp,\mu)\right] \\ &+ (q \leftrightarrow \bar{q}) \Bigg) \end{aligned} \tag{7.9}$$

This is the all order factorization formula. Power corrections to this formula are of the order q_T^2/M^2 and Λ^2_{QCD}/q_T^2. Up to NNLL accuracy, this formula can be further simplified by rewriting the tensor structure. At NNLL, the second Lorentz structure in the $B^{\mu\nu}_{g/N}$ functions does not contribute. Remembering that

$$\mathcal{B}^{\mu\nu}_{g/N} = \frac{g_\perp^{\mu\nu}}{2}\mathcal{B}_{g/N}(z, x_T^2) + \left(\frac{g_\perp^{\mu\nu}}{2} + \frac{x_\perp^\mu x_\perp^\nu}{x_T^2}\right)\mathcal{B}'_{g/N}(z, x_T^2).$$

In section 5.5.2 I showed that $B'_{g/N}$ vanishes at leading order. Additionally, one has

$$\int_0^{2\pi} d\Phi\, g_\perp^{\mu\rho}\left(\frac{g_\perp^{\nu\sigma}}{2} + \frac{x_\perp^\nu x_\perp^\sigma}{x_T^2}\right)\mathbf{H}^{(0),\mu\nu\rho\sigma}_{gg}(M, m_t, \cos\theta,\mu) = 0.$$

where $\mathbf{H}_{gg}^{(0),\mu\nu\rho\sigma}$ is the leading order coefficient of $\mathbf{H}_{gg}^{\mu\nu\rho\sigma}$ in the perturbative expansion of α_s. This can be proven assuming that $\mathbf{H}_{gg}$ has a constant v_3 which means that the integrand reduces to the expression in brackets which is

$$\int_0^{2\pi} d\Phi \left(\frac{g_\perp^{\nu\sigma}}{2} + \frac{x_\perp^\nu x_\perp^\sigma}{x_T^2} \right) = \left(\pi g_\perp^{\nu\sigma} + \pi \left(-g_\perp^{\nu\sigma} \right) \right) = 0$$

Thus one can replace

$$B_{g/N}^{\mu\nu} \to \frac{1}{2} g_\perp^{\mu\nu} B_{g/N}$$

The transverse metric tensor is rewritten in the hard function.
So if the second Lorentz structure does not contribute the collinear functions do not depend on Φ and thus this dependence resides only in the soft functions. This leads to the new definition of the soft functions as

$$S_{i\bar{i}}(L_\perp, M, m_t, \cos\theta, \mu) = \int \frac{d\Phi}{2\pi} \, \mathbf{W}(x_\perp, \mu) \tag{7.10}$$

Thus the simplified factorization formula, valid up to the NNLL accuracy, now reads [35]

$$\begin{aligned} \frac{d^4\sigma}{dy\, dq_T^2\, dM\, d\cos\theta_3} = \sum_{i=q,\bar{q},g} & \frac{8\pi\beta_t M}{6s(M^2+q_T^2)} \int x_T \, dx_T J_0(x_T q_T) \left(\frac{x_T^2 M^2}{4e^{-2\gamma_E}} \right)^{-F_{i\bar{i}}(x_T^2,\mu)} \\ & \times B_{i/N_1}(\xi_1, x_T^2, \mu) \; B_{\bar{i}/N_2}(\xi_2, x_T^2, \mu) \mathrm{Tr} \left[\mathbf{H}_{i\bar{i}}(M, m_t, \cos\theta, \mu) \mathbf{S}_{i\bar{i}}(x_\perp, \mu) \right] \end{aligned} \tag{7.11}$$

The TPDFs are process independent and thus they can be written as in eq. (5.11) with the same result as in section 5.7.5.

7.3. Resummation formula

The same assumptions as for a Drell-Yan process, see chapter 6.3.4, are leading to [3]

$$\begin{aligned} \frac{d^4\sigma}{dy\, dq_T^2\, dM\, d\cos\theta_3} = & \sum_{i=q,\bar{q},g} \sum_{a,b} \frac{8\pi\beta_t M}{3s(M^2+q_T^2)} \\ & \times \; C_{i\bar{i}\leftarrow ab}(z_1, z_2, q_T^2, q^2, m_t, \mu) \otimes \Phi_{a/N_1}(z_1, \mu) \otimes \Phi_{b/N_2}(z_2, \mu) \end{aligned} \tag{7.12}$$

with the scattering kernels [3]

$$\begin{aligned} C_{i\bar{i}\leftarrow ab}(z_1, z_2, q_T^2, M^2, m_t, \mu) = & \frac{1}{2} \int_0^\infty dx_T \; x_T \; J_0(x_T q_T) \exp\left[g_i(\eta_i, L_\perp, \alpha_s) \right] \\ \times \bar{I}_{i\leftarrow a}(z_1, L_\perp, a_s) \bar{I}_{\bar{i}\leftarrow b}(z_2, L_\perp, a_s) & \times \mathrm{Tr} \left(\mathbf{H}_{i\bar{i}}(M, m_t, \cos\theta, \mu) \mathbf{S}_{i\bar{i}}(L_\perp, M, m_t, \cos\theta, \mu) \right) \end{aligned}$$

where

$$\eta_i = \frac{C_i \alpha_s}{\pi} \ln \frac{M^2}{\mu^2}$$

with $C_q = C_F$ and $C_g = C_A$ and with

$$\begin{aligned} g_i(\eta_i, L_\perp, \alpha_s) = & - \left[\eta_i L_\perp \right]_{\epsilon^{-1/2}} - \left[a_s (\Gamma_0^i + \eta_i \beta_0) \frac{L_\perp^2}{2} \right]_{\epsilon^0} \\ & - \left[a_s (2\gamma_0^i + \eta_i K) L_\perp + a_s^2 (\Gamma_0^i + \eta_i \beta_0) \beta_0 \frac{L_\perp^3}{3} \right]_{\epsilon^{1/2}} \\ & - \left[a_s \eta_i d_2 + a_s^2 (K\Gamma_0^i + 2\gamma_0^i \beta_0 + \eta_i(\beta_1 + 2K\beta_0) \frac{L_\perp^2}{2} + a_s^3 (\Gamma_0 + \eta_i \beta_0) \beta_0^2 \frac{L_\perp^4}{4} \right]_\epsilon \\ & - \mathcal{O}(\epsilon^{3/2}) \end{aligned}$$

with $a_s = \alpha_s / 4\pi$ and d_2 and K given in eq. (6.11) and (6.12).
The TPDFs were already treated and the soft function is going to be treated in the next chapter. What is left is the hard function. This is the topic of the next section.

7.4. The hard function

The hard function was defined as

$$\mathbf{H}_{gg}^{\mu\nu\rho\sigma}(M, m_t, v_3, \mu) = \frac{3}{8}\frac{1}{(4\pi)^2}\frac{1}{4N_g}\sum_{spin} |C_m\rangle \langle C_{m'}| \langle 0| [\boldsymbol{O}_{m'}^{h\dagger}(0)]^{\rho\sigma} |t(p_3)\bar{t}(p_4)\rangle \langle t(p_3)\bar{t}(p_4)| [\boldsymbol{O}_m^h(0)]^{\mu\nu} |0\rangle$$

$$\mathbf{H}_{q\bar{q}}(M, m_t, v_3, \mu) = \frac{3}{8}\frac{1}{(4\pi)^2}\frac{1}{4N_c}\sum_{spin} |C_m\rangle \langle C_{m'}| \langle 0| [\boldsymbol{O}_{m'}^{h\dagger}(0)] |t(p_3)\bar{t}(p_4)\rangle \langle t(p_3)\bar{t}(p_4)| [\boldsymbol{O}_m^h(0)] |0\rangle$$

The hard functions are matrices in color space. The matrix entry H_{IJ} is given by

$$\mathrm{H}_{IJ} = \frac{1}{\langle c_I|c_I\rangle \langle c_J|c_J\rangle} \langle c_I| \mathbf{H} |c_J\rangle$$

The LO hard function is just the tree level amplitude squared and decomposed into the color basis. The LO and NLO calculation was performed in [3]. Up to NLO, the hard function can be written as

$$\mathbf{H}_{i\bar{i}} = \frac{3\alpha_s^2}{8N_i}\left(\mathbf{H}_{i\bar{i}}^{(0)} + \frac{\alpha_s}{4\pi}\mathbf{H}_{i\bar{i}}^{(1)}\right)$$

with

$$\mathbf{H}_{IJ}^{(0)} = \frac{1}{4}\frac{1}{\langle c_I|c_I\rangle \langle c_J|c_J\rangle} \langle c_I|\mathcal{M}_{ren}^{QCD(0)}\rangle \langle \mathcal{M}_{ren}^{QCD(0)}|c_J\rangle$$

Since loop integrals in SCET are scaleless and vanish, the NLO hard functions are obtained by evaluating the one-loop virtual QCD diagrams [3].

7.4.1. The LO quark channel

The quark channel is easier to calculate than the gluon channel since there is only one graph contributing to the cross section (see Fig. 7.1) and only one color structure.

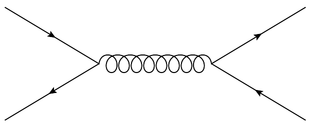


Figure 7.1.: The tree level matrix element contributing to the top quark pair production in the quark channel

In the comparison of the full QCD result with the result obtained in the effective theory, the matrix Γ_m'' can be determined. In full QCD, one obtains in Feynman gauge at tree level

$$iT_{fi} = i\frac{g^2}{(p_1+p_2)^2}T_{a_3a_4}^c T_{a_2a_1}^c \cdot (\bar{u}(p_3)\gamma^\mu v(p_4)) \cdot (\bar{v}(p_2)\gamma_\mu u(p_1))$$

For this one considers the squared matrix element summed over spin which is

$$\sum_{spin}\frac{1}{4}|T_{fi}|^2 = \frac{g^4}{4M^4}T_{a_3a_4}^c T_{a_2a_1}^c T_{a_4a_3}^{c'} T_{a_1a_2}^{c'} \mathrm{Tr}\left(\gamma_\nu(\not{p}_3 + m_t)\gamma_\mu(\not{p}_4 - m_t)\right) \mathrm{Tr}\left(\gamma^\nu \not{p}_2 \gamma^\mu \not{p}_1\right)$$

With $M^2 = (p_1+p_2)^2$ and $p_1^\mu = (M/2)n^\mu$ and $p_2^\mu = (M/2)\bar{n}^\mu$ and one obtains

$$\sum_{spin}\frac{1}{4}|T_{fi}|^2 = \frac{g^4}{16M^2}T_{a_3a_4}^c T_{a_2a_1}^c T_{a_4a_3}^{c'} T_{a_1a_2}^{c'} \mathrm{Tr}\left(\gamma_\nu(\not{p}_3 + m_t)\gamma_\mu(\not{p}_4 - m_t)\right) \mathrm{Tr}\left(\gamma^\nu \not{\bar{n}} \gamma^\mu \not{n}\right)$$

The color structure given by the factor $T^c_{a_3a_4}T^c_{a_2a_1}$ is equal to $|c_2\rangle$. The only non-vanishing entry in the color matrix is H_{22}, since $\mathrm{Tr}(T^c) = 0$. Replacing the momentum products with the kinematic variables

$$\begin{aligned} p_3 \cdot p_4 &= \frac{M^2}{2} - m_t^2 \\ p_4 \cdot \bar{n} = p_3 \cdot n &= -\frac{t_1}{M} \\ p_4 \cdot n = p_3 \cdot \bar{n} &= -\frac{u_1}{M} \end{aligned} \tag{7.13}$$

one obtains

$$\sum_{spin} \frac{1}{4}|T_{fi}|^2 = 2g^4 \left(\frac{t_1^2 + u_1^2}{M^4} + \frac{2m_t^2}{M^2} \right) |c_2\rangle \langle c_2| \tag{7.14}$$

One needs the same color structure in the effective theory and thus, one has

$$\mathbf{H}^{(0)}_{q\bar{q}}(M, m_t, v_3, \mu) = \frac{1}{4} |c_2\rangle \langle c_2| \sum_{spin} \langle 0| \bar{h}_{v_3}(x)\Gamma''_m h_{v_4}(x) |t(p_3)\bar{t}(p_4)\rangle \langle t(p_3)\bar{t}(p_4)| \bar{h}_{v_4}\gamma_0 \left(\Gamma''_{m'}\right)^\dagger \gamma_0 h_{v_3} |0\rangle$$

above one found that

$$\Gamma''_m \sim \gamma^\mu$$

Remembering that $-g_\perp^{\mu\nu} \cdot M^2/2$ was shifted into the hard quark tensor, one obtains

$$\begin{aligned} \mathbf{H}^{(0)}_{q\bar{q}}(M, m_t, v_3, \mu) &= -\frac{1}{4} |c_2\rangle \langle c_2| \,\mathrm{Tr}\left(\gamma_\nu(\not{v}_3 + 1)\gamma_\mu(\not{v}_4 - 1)\right) \frac{M^2}{2} g_\perp^{\mu\nu} \\ &= \frac{1}{8} |c_2\rangle \langle c_2| \left(8M^2 + 4\frac{u_1^2}{m_t^2} + 4\frac{t_1^2}{m_t^2} \right) \end{aligned}$$

In order to obtain the same result as in eq (7.14), one chooses

$$\Gamma''_m = 2\frac{m_t}{M^2} g^2 \gamma^\mu$$

and so one has

$$\mathbf{H}^{(0)}_{q\bar{q}} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \left(\frac{t_1^2 + u_1^2}{M^4} + \frac{2m_t^2}{M^2} \right)$$

The quark case is especially easy, since there is only one entry in the color matrix. In the gluon case, one will have different contributions of Γ''_m in each matrix entry.

7.4.2. The LO gluon channel

In the gluon fusion channel three Feynman diagrams contribute to the tree level amplitude which are shown in figure 7.2.
With the QCD Feynman rules one obtains

$$\begin{aligned} iT_{fi} = g^2 \bar{u}(p_3) \bigg(&\frac{1}{\hat{s}} T^a_{a_3a_4} f^a_{a_1a_2} \cdot p(\not{n} - \not{\bar{n}}) g^{\mu\nu} + i T^{a_1}_{a_3c} T^{a_2}_{ca_4} \frac{1}{t_1} \gamma^\mu \left(\not{p}_1 - \not{p}_3 - m_t \right) \gamma^\nu \\ &- i T^{a_2}_{a_3c} T^{a_1}_{ca_4} \frac{1}{u_1} \gamma^\nu \left(\not{p}_1 - \not{p}_4 + m_t \right) \gamma^\mu \bigg) \cdot v(p_4) \cdot \epsilon_\mu(p_1) \cdot \epsilon_\nu(p_2) \end{aligned}$$

where I used that $p_1^\mu = pn^\mu$ and $p_2 = p\bar{n}^\mu$ and due to the transversality of real gluons one has

$$n \cdot \epsilon = 0 \,, \qquad \bar{n} \cdot \epsilon = 0.$$

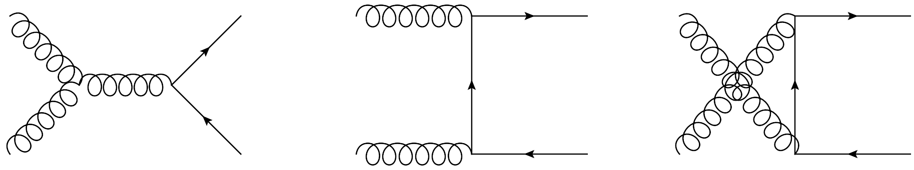


Figure 7.2.: The three tree level matrix elements contributing to the top quark pair production in the gluon channel

The incoming gluon polarization states $\epsilon_\mu(p_1)$ and $\epsilon_\nu(p_2)$ are produced from the collinear matrix element and thus they are left away. The spinors from the top quarks are poduced from the HQET spinors. Thus, one can conclude that matrix $\Gamma_m^{\mu\nu}$ is given by

$$\Gamma_m^{\mu\nu}\left|C_m\right\rangle = g^2\Bigg(\frac{1}{\hat{s}}T^a_{a_3a_4}f^a_{a_1a_2}\cdot p(\not{n}-\bar{\not{n}})g^{\mu\nu} + iT^{a_1}_{a_3c}T^{a_2}_{ca_4}\frac{1}{t_1}\gamma^\mu\left(\not{p}_1-\not{p}_3-m_t\right)\gamma^\nu \\ - iT^{a_2}_{a_3c}T^{a_1}_{ca_4}\frac{1}{u_1}\gamma^\nu\left(\not{p}_1-\not{p}_4+m_t\right)\gamma^\mu\Bigg) = \sum_I \Gamma^{\mu\nu}_{mI}\left|c_I\right\rangle$$

where it was used in the last step, that the Wilson coefficient is 1 at leading order and thus, it can be left away in this calculation. The color factor $T^a_{a_3a_4}f^a_{a_1a_2}$ corresponds to $(c_2^{gg})_{\{a\}}$. And the color factor $T^{a_1}T^{a_2}$ can be rewritten as

$$\begin{aligned} T^{a_1}_{a_3c}T^{a_2}_{ca_4} &= \frac{1}{2}\{T^{a_1},T^{a_2}\}_{a_3a_4} + \frac{1}{2}[T^{a_1},T^{a_2}]_{a_3a_4} \\ &= \frac{1}{2}d_{a_1a_2c}T^c_{a_3a_4} + \frac{1}{2N}\delta_{a_1a_2}\delta_{a_3a_4} + \frac{i}{2}T^c_{a_3a_4}f^c_{a_1a_2} \\ &= \frac{1}{2}\left|c_3^{gg}\right\rangle + \frac{1}{2N}\left|c_1^{gg}\right\rangle + \frac{i}{2}\left|c_2^{gg}\right\rangle \end{aligned}$$

and for the other factor one has

$$\begin{aligned} T^{a_2}_{a_3c}T^{a_1}_{ca_4} &= \frac{1}{2}d_{a_1a_2c}T^c_{a_3a_4} + \frac{1}{2N}\delta_{a_1a_2}\delta_{a_3a_4} - \frac{i}{2}T^c_{a_3a_4}f^c_{a_1a_2} \\ &= \frac{1}{2}\left|c_3^{gg}\right\rangle + \frac{1}{2N}\left|c_1^{gg}\right\rangle - \frac{i}{2}\left|c_2^{gg}\right\rangle \end{aligned}$$

Thus one can write

$$\begin{aligned} \sum_I \Gamma^{\mu\nu}_{mI}\left|c_I\right\rangle = g^2\Bigg(&\frac{1}{\hat{s}}\left|c_2^{gg}\right\rangle\cdot p(\not{n}-\bar{\not{n}})g^{\mu\nu} \\ &+ i\left(\frac{1}{2}\left|c_3^{gg}\right\rangle + \frac{1}{2N}\left|c_1^{gg}\right\rangle + \frac{i}{2}\left|c_2^{gg}\right\rangle\right)\frac{1}{t_1}\gamma^\mu\left(\not{p}_1-\not{p}_3-m_t\right)\gamma^\nu \\ &- i\left(\frac{1}{2}\left|c_3^{gg}\right\rangle + \frac{1}{2N}\left|c_1^{gg}\right\rangle - \frac{i}{2}\left|c_2^{gg}\right\rangle\right)\frac{1}{u_1}\gamma^\nu\left(\not{p}_1-\not{p}_4+m_t\right)\gamma^\mu\Bigg) \end{aligned}$$

The matrix can be rearranged as

$$\begin{aligned} \sum_I \Gamma^{\mu\nu}_{mI}\left|c_I\right\rangle = &\left|c_1^{gg}\right\rangle\frac{ig^2}{2N}\left(\frac{1}{t_1}\gamma^\mu\left(\not{p}_1-\not{p}_3-m_t\right)\gamma^\nu - \frac{1}{u_1}\gamma^\nu\left(\not{p}_1-\not{p}_4+m_t\right)\gamma^\mu\right) \\ &+ \left|c_2^{gg}\right\rangle g^2\left(\frac{1}{\hat{s}}p(\not{n}-\bar{\not{n}})g^{\mu\nu} - \frac{1}{2u_1}\gamma^\nu\left(\not{p}_1-\not{p}_4+m_t\right)\gamma^\mu - \frac{1}{2t_1}\gamma^\mu\left(\not{p}_1-\not{p}_3-m_t\right)\gamma^\nu\right) \\ &+ \left|c_3^{gg}\right\rangle\frac{ig^2}{2}\left(\frac{1}{t_1}\gamma^\mu\left(\not{p}_1-\not{p}_3-m_t\right)\gamma^\nu - \frac{1}{u_1}\gamma^\nu\left(\not{p}_1-\not{p}_4+m_t\right)\gamma^\mu\right) \end{aligned}$$

One the other hand, one has

$$\begin{aligned}\sum_I \langle c_I|\,\Gamma^{\rho\sigma}_{m'I} =& - \langle c_1^{gg}|\,\frac{ig^2}{2N}\left(\frac{1}{t_1}\gamma^\sigma\left(\not p_1-\not p_3-m_t\right)\gamma^\rho-\frac{1}{u_1}\gamma^\rho\left(\not p_1-\not p_4+m_t\right)\gamma^\sigma\right)\\ &+\langle c_2^{gg}|\,g^2\left(\frac{1}{\hat s}p(\not n-\not \bar n)g^{\rho\sigma}-\frac{1}{2u_1}\gamma^\rho\left(\not p_1-\not p_4+m_t\right)\gamma^\sigma-\frac{1}{2t_1}\gamma^\sigma\left(\not p_1-\not p_3-m_t\right)\gamma^\rho\right)\\ &-\langle c_3^{gg}|\,\frac{ig^2}{2}\left(\frac{1}{t_1}\gamma^\sigma\left(\not p_1-\not p_3-m_t\right)\gamma^\rho-\frac{1}{u_1}\gamma^\rho\left(\not p_1-\not p_4+m_t\right)\gamma^\sigma\right)\end{aligned}$$

Up to NNLL, one can simplify the calculation and one can shift the transverse metric tensor from the beam functions into the hard gluon function. Thus, one has

$$\begin{aligned}\mathbf{H}_{gg}=g_{\perp\mu\rho}\cdot g_{\perp\sigma\nu}\mathbf{H}^{\mu\nu\rho\sigma}_{gg}&=\frac{3}{8}\frac{1}{(4\pi)^2}\frac{1}{4d_g}\sum_{I,J,m,m'}|c_I\rangle\langle c_J|\;g_{\perp\mu\rho}\cdot g_{\perp\sigma\nu}\;\bar v(v_4)\Gamma^{\rho\sigma}_{m'}u(v_3)\bar u(v_3)\Gamma^{\mu\nu}_m v(v_4)\\ &=\frac{3}{8}\frac{1}{(4\pi)^2}\frac{1}{4d_g}\sum_{i,j,m,m'}|c_i\rangle\langle c_j|\;g_{\perp\mu\rho}\cdot g_{\perp\sigma\nu}\;\bar v(v_4)\Gamma^{\rho\sigma}\left(\not v_3+1\right)\Gamma^{\mu\nu}v(v_4)\end{aligned}$$

With

$$\mathrm{H}_{IJ}=\frac{1}{\langle c_I|c_I\rangle\langle c_J|c_J\rangle}\langle c_I|\,\mathbf{H}\,|c_J\rangle$$

and

$$\mathbf{H}_{i\bar i}=\frac{3\alpha_s^2}{8d_i}\mathbf{H}^{(0)}_{i\bar i}+...$$

one obtains for the 11-entry of the hard function color matrix

$$\begin{aligned}\mathrm{H}^{(0)}_{11}&=g_{\perp\mu\rho}\cdot g_{\perp\sigma\nu}\,\frac{1}{4}\sum_{spin}\bar v(p_4)\Gamma_1^{\rho\sigma}v(p_3)\bar v(p_3)\Gamma_1^{\mu\nu}v(p_4)\\ &=g_{\perp\mu\rho}\cdot g_{\perp\sigma\nu}\,\frac{1}{4}\mathrm{Tr}\left(\left(\not p_4-m_t\right)\Gamma_1^{\rho\sigma}\left(\not p_3+m_t\right)\Gamma_1^{\mu\nu}\right)\end{aligned}$$

where one transformed $\Gamma^{\alpha\beta}\to m_t\Gamma^{\alpha\beta}$. The matrix $\Gamma_1^{\mu\nu}$ was given by

$$\Gamma_1^{\mu\nu}=\frac{ig^2}{2N}\left(\frac{1}{t_1}\gamma^\mu\left(\not p_1-\not p_3-m_t\right)\gamma^\nu-\frac{1}{u_1}\gamma^\nu\left(\not p_1-\not p_4+m_t\right)\gamma^\mu\right)$$

and $\Gamma_1^{\rho\sigma}$ respectively. $\mathrm{H}^{(0)}_{11}$ can be calculated using FORM and the kinematic variables of eq. (7.13) and momentum conservation given by

$$\hat s+u_1+t_1=0\qquad\Rightarrow\qquad u_1+t_1=-\hat s=-M^2$$

One obtains

$$\begin{aligned}\mathrm{H}^{(0)}_{11}&=\frac{1}{16N^2}\frac{M^4}{u_1t_1}\left(-32\frac{m_t^4}{t_1u_1}+8\frac{t_1^2+u_1^2}{M^4}+32\frac{m_t^2}{M^2}\right)\\ &=\frac{1}{N^2}\frac{M^4}{2u_1t_1}\left(-4\frac{m_t^4}{t_1u_1}+\frac{t_1^2+u_1^2}{M^4}+4\frac{m_t^2}{M^2}\right)\end{aligned}$$

From this result one can derive the result for $\mathrm{H}^{(0)}_{33}$, $\mathrm{H}^{(0)}_{13}$ and $\mathrm{H}^{(0)}_{31}$. In the same way as $\mathrm{H}^{(0)}_{11}$ one can calculate $\mathrm{H}^{(0)}_{22}$ using FORM which gives

$$\mathrm{H}^{(0)}_{22}=\frac{1}{2t_1u_1}\left(\frac{t_1^2+u_1^2}{M^4}(t_1-u_1)^2+\frac{4m_t^2}{M^2}(t_1-u_1)^2-\frac{4m_t^2}{t_1u_1}(t_1-u_1)^2\right)$$

and equivalently one can calculate the interference term $\mathrm{H}_{12}^{(0)}$ which is

$$\mathrm{H}_{12}^{(0)} = \frac{1}{N}\frac{1}{2u_1 t_1}\left(\frac{t_1^2+u_1^2}{M^2}(t_1-u_1) + 4m_t^2(t_1-u_1) - M^2\frac{4m_t^4}{u_1 t_1}(t_1-u_1)\right)$$

with this knowledge the terms $\mathrm{H}_{21}^{(0)}$, $\mathrm{H}_{23}^{(0)}$ and $\mathrm{H}_{32}^{(0)}$ can be deduced. Thus the color matrix of the hard gluon function is

$$\mathbf{H}_{gg}^{(0)} = \begin{pmatrix} \frac{1}{N^2} & \frac{1}{N}\frac{t_1-u_1}{M^2} & \frac{1}{N} \\ \frac{1}{N}\frac{t_1-u_1}{M^2} & \frac{(t_1-u_1)^2}{M^4} & \frac{t_1-u_1}{M^2} \\ \frac{1}{N} & \frac{t_1-u_1}{M^2} & 1 \end{pmatrix} \cdot \frac{M^4}{2t_1 u_1}\left(\frac{t_1^2+u_1^2}{M^4} + \frac{4m_t^2}{M^2} - \frac{4m_t^4}{u_1 t_1}\right)$$

And so one has the result for the LO hard function matrices. The NLO hard function is obtained by computing the one-loop virtual diagrams for top quark pair production. This was performed in [35]. For the cross section formula of eq. (7.11) the soft function is the only ingredient which is yet unknown. It will be evaluated in the next chapter.

8. The soft function of top quark pair production at small transverse momentum

The soft function of top quark pair production is of high importance, since it is the only ingredient in the cross section formula given in eq. (7.9) which is only known up to NLO which is given in [3]. The TPDFs are given up to NNLO in [46] and the hard function is given up to NNLO in [59, 60]. In this chapter I show the calculation of this soft function up to NLO which was performed in [3]. In the appendix of this paper the explicit way of the calculation is stated. Within this method one encounters integrals which seem to be unsolvable. In the correspondence with the authors of [3] I found out how they managed to solve these integrals. I will present this method. However more difficult integrals do not seem to be solvable within this method. Since the aim is to calculate the soft function up to NNLO one needs to employ a different method. So after the method of "brute force", I will present the calculation of the soft function with two alternative methods. The first method is the Mellin-Barnes method [61, 6]. The integration within this method is much simpler, but nevertheless it seems inappropriate for the solution of even more complicated integrands such as the ones that will appear at NNLO. So finally I will perform the integration with the derivation of differential equations [61]. This method is powerful and it seems to be advisable to employ this method for the NNLO calculation. In [3] the soft function is stated up to order $\mathcal{O}(\epsilon^0)$. In this chapter I will evaluate the integrals of the soft function up to order $\mathcal{O}(\epsilon)$. The renormalization procedure leads to an explicit expression of the renormalization constant up to $\mathcal{O}(\alpha_s)$. The renormalization constant contains $1/\epsilon$ poles. Together with the $\mathcal{O}(\epsilon)$ contributions of the renormalized soft function one obtains a prediction for one of the finite contributions for NNLO.

The soft function was defined in eq. (7.8) and (7.10) as

$$\mathbf{S}_{i\bar{i}}(L_\perp, M, m_t, \cos\theta, \mu) = \int \frac{d\Phi}{2\pi}\mathbf{W}(x_\perp, \mu) = \int \frac{d\Phi}{2\pi}\frac{1}{N_i}\langle 0|\,\bar{\mathbf{T}}[\boldsymbol{O}_s^\dagger(x_\perp)]\mathbf{T}[\boldsymbol{O}_s(0)]\,|0\rangle$$

The perturbative expansion of the soft function is represented as

$$\mathbf{S}_{i\bar{i}}(L_\perp, M, m_t, \cos\theta, \mu) = \sum_{n=0}^{\infty}\mathbf{S}_{i\bar{i}}^{(n)}\left(\frac{\alpha_s}{4\pi}\right)^n \tag{8.1}$$

Again, the entries of the soft function color matrix are given by

$$S_{IJ} = \langle c_I|\,\mathbf{S}\,|c_J\rangle$$

The LO soft function

The leading order soft function is just given by the contraction between the color basis which is

$$\left(\mathbf{S}_{i\bar{i}}^{(0)}\right)_{IJ} = \frac{1}{N_i}\langle c_I|c_J\rangle$$

It corresponds to to the case without any soft radiation. The LO soft functions for the $q\bar{q}$ and gg channels are given by

$$\mathbf{S}_{q\bar{q}}^{(0)} = \begin{pmatrix} N & 0 \\ 0 & \frac{C_F}{2}\end{pmatrix}, \qquad \mathbf{S}_{gg}^{(0)} = \begin{pmatrix} N & 0 & 0 \\ 0 & \frac{N}{2} & 0 \\ 0 & 0 & \frac{N^2-4}{2N}\end{pmatrix}$$

In the appendix A.3 of [49] there are listed useful color identies which help to evaluate the matrix elements. The most tedious contribution is $\langle c_3|c_3\rangle$ which is evaluated with the help of the identities

$$\begin{aligned}\{T^a, T^b\} &= d_{abc}T^c + \frac{1}{N}\delta_{ab}\mathbf{1}\\ T^a_{ij}T^a_{kl} &= \frac{1}{2}\left(\delta_{il}\delta_{kj} - \frac{1}{N}\delta_{ij}\delta_{kl}\right)\end{aligned}$$

The NLO soft function

The NLO soft function describes the real radiation of a soft gluon from one of the external legs. Virtual diagrams do not contribute since they correspond to scaleless integrals. The bare soft function at NLO is given by the matrix element

$$\mathbf{S}^{(1),bare}_{i\bar{i}} = \frac{(4\pi)^2}{g^2}\frac{1}{N_i}\int \frac{d\Phi}{2\pi}\langle 0|\,\bar{\mathbf{T}}\left[\mathbf{O}^\dagger_s(x_\perp)\right]|g(k)\rangle\langle g(k)|\,\mathbf{T}\left[\mathbf{O}_s(0)\right]|O\rangle$$

where the fractor $(4\pi)^2/g^2$ arises due to the expansion given in eq. (8.1). $\mathbf{S}^{(1),bare}_{i\bar{i}}$ can be expressed via the eikonal current given by

$$\mathbf{J}^{c\mu}(k) = \sum_j \mathbf{T}^c_j \frac{p^\mu_j}{p_j\cdot k}$$

where $\mathbf{T}^c_j$ is the color generator that belongs to the j-th leg. Then one has (using dimensional regularization)

$$\begin{aligned}\mathbf{S}^{(1),bare}_{i\bar{i}} &= (4\pi)^2\frac{1}{N_i}\int\frac{d\Phi}{2\pi}\int\frac{d^dk}{(2\pi)^{d-1}}\frac{\nu^\alpha}{k^\alpha_+}\delta(k^2)\theta(k^0)\mathbf{J}^{c_1\mu}(k)\sum_{\lambda=\pm1}\epsilon^{c_1*}_\mu(k)\epsilon^{c_2}_\nu(k)\mathbf{J}^{c_2\nu}(k)e^{-ix_\perp k_\perp}\\ &= (4\pi)^2\frac{1}{N_i}\int\frac{d\Phi}{2\pi}\int\frac{d^dk}{(2\pi)^{d-1}}\frac{\nu^\alpha}{k^\alpha_+}\delta(k^2)\theta(k^0)\mathbf{J}^{c\mu}(k)\left(-g_{\mu\nu}\right)\mathbf{J}^{c\nu}(k)e^{-ix_\perp k_\perp}\end{aligned}$$

where it was used that the current is transverse $k_\mu J^\mu(k) = 0$ and the phase space was analytically regulated with ν^α/k^α_+. At next to leading order the bare soft function can be written as

$$\mathbf{S}^{(1),bare}_{i\bar{i}} = \sum_{j,k}\boldsymbol{w}^{i\bar{i}}_{jk}I_{jk}$$

The color structure is encoded in the color matrices $\boldsymbol{w}_{jk}$ which are

$$\left(\boldsymbol{w}^{i\bar{i}}_{jk}\right)_{IJ} = \frac{1}{N_i}\langle c_I|\,\mathbf{T}_j\cdot\mathbf{T}_k\,|c_J\rangle \tag{8.2}$$

and the integrals I_{jk} are given in the $\overline{MS}$-scheme as

$$I_{jk} = -\,(4\pi)^2\left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^\epsilon\int_0^{2\pi}\frac{d\Phi}{2\pi}\int\frac{d^dk}{(2\pi)^{d-1}}\frac{\nu^\alpha}{k^\alpha_+}\delta(k^2)\theta(k^0)\frac{v_j\cdot v_k e^{-ix_\perp k_\perp}}{v_j\cdot k v_k\cdot k} \tag{8.3}$$

where the vector v^μ_j corresponds to n^μ for $j=1$, to $\bar{n}^\mu$ for $j=2$, and to $v^\mu_{3,4}$ for $j=3,4$. The integral I_{jk} describes the interference between the diagram where a soft gluon is radiated from leg j with the diagram where the soft gluon is radiated from leg k. This is shown in Fig. 8.1 that is taken from [3]. The contribution I_{12} is not depicted since it does not contribute because it is a scaleless integral and thus vanish in dimensional regularization (see section 6.2).

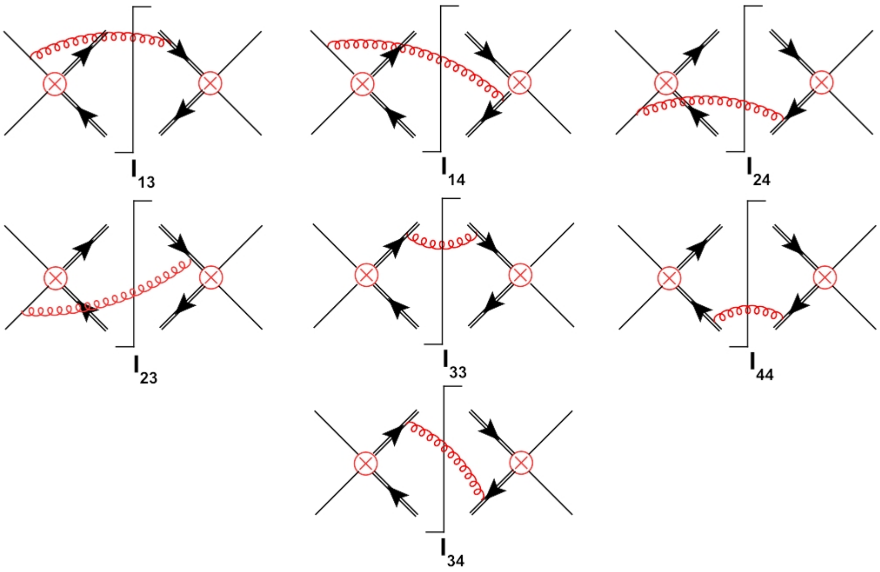


Figure 8.1.: Feynman diagrams contributing to the NLO soft functions. The double lines represent the Wilson lines in the time-like directions v_3 and v_4, the single lines are the Wilson lines in the light-like direction. Figure taken from [3]

The calculation of the NLO color matrices

In this section, the NLO color matrices given in eq. (8.2) are calculated. The color generator $\mathbf{T}_j$ describes the gluon radiation from leg j. The matrices $w_{jk}^{i\bar{i}}$ contain the sum over all color indices. The basis vectors in color space for the quark channel were stated as

$$\begin{aligned}
|c_1^{q\bar{q}}\rangle &= \delta_{a_1a_2}\delta_{a_3a_4} \\
|c_2^{q\bar{q}}\rangle &= T^c_{a_2a_1}T^c_{a_3a_4} \\
\langle c_1^{q\bar{q}}| &= \delta_{a_1a_2}\delta_{a_3a_4} \\
\langle c_2^{q\bar{q}}| &= T^c_{a_4a_3}T^c_{a_1a_2}
\end{aligned}$$

When a soft parton is radiated, the color index is changed. The inner color index is labeled b_i and the outer color index a_i. This is shown in Fig. 8.2.

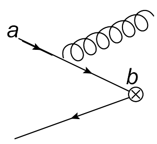


Figure 8.2.: Radiation of a soft gluon from the incoming quark leg

In the quark channel, one has the following color generators with the appropriate sign (see appendix soft radiation)

$$\begin{aligned}
\text{radiation from incoming quark line:} \quad & T^c_{b_1a_1} \\
\text{radiation from incoming antiquark line:} \quad & -T^c_{a_2b_2} \\
\text{radiation from outgoing top line:} \quad & -T^c_{a_3b_3} \\
\text{radiation from outgoing anti-top line:} \quad & T^c_{b_4a_4}
\end{aligned} \tag{8.4}$$

The action of the color generator is given by

$$\mathbf{T}_i^c \, |..., a_i, ...\rangle = (\mathbf{T}_i^c)_{b_ia_i} \, |..., b_i, ...\rangle$$

and with the considerations from eq. (8.4) the color generators can be further specified as

$$(\mathbf{T}_i^c)_{b_i a_i} = \begin{cases} T_{b_i a_i}, & \text{if i-th parton is an initial-state quark or final state antiquark} \\ \bar{T}_{b_i a_i} = -T_{a_i b_i}, & \text{if i-th parton is an initial-state antiquark or final state quark} \end{cases}$$

For the cross section the squared matrix element is needed, so these color generators are multiplied by

$$\begin{aligned} \text{radiation from incoming quark line:} \quad & T^c_{a_1 b_1} \\ \text{radiation from incoming antiquark line:} \quad & -T^c_{b_2 a_2} \\ \text{radiation from outgoing top line:} \quad & -T^c_{b_3 a_3} \\ \text{radiation from outgoing anti-top line:} \quad & T^c_{a_4 b_4} \end{aligned}$$

As an example the interference of a real gluon radiation from the incoming quark with the incoming antiquark is considered. For this case the basis vectors are

$$\begin{aligned} |c_1^{q\bar{q}}\rangle &= \delta_{b_1 a_2}\delta_{a_3 a_4} \\ |c_2^{q\bar{q}}\rangle &= T^c_{a_2 b_1} T^c_{a_3 a_4} \end{aligned}$$

and

$$\begin{aligned} \langle c_1^{q\bar{q}}| &= \delta_{a_1 b_2}\delta_{a_3 a_4} \\ \langle c_2^{q\bar{q}}| &= T^c_{a_4 a_3} T^c_{a_1 b_2} \end{aligned}$$

In this case the product of the color generators is $T^d_{b_1 a_1}(-T^d_{b_2 a_2})$. So one has

$$\begin{aligned} \left(\boldsymbol{w}_{12}^{q\bar{q}}\right)_{11} &= -\sum_{a_1,\ldots,a_4}\sum_{b_1,b_2} \frac{1}{N}\langle c_1| T^d_{b_1 a_1} T^d_{b_2 a_2} |c_1\rangle \\ &= -\sum_{a_1,\ldots,a_4}\sum_{b_1,b_2} \frac{1}{N}\delta_{a_1 b_2}\delta_{a_3 a_4}\, T^d_{b_1 a_1}\cdot T^d_{b_2 a_2}\, \delta_{b_1 a_2}\delta_{a_3 a_4} \\ &= -\sum_{b_1,b_2}\frac{N}{N} T^d_{b_2 b_1}\cdot T^d_{b_1 b_2} \\ &= -\sum_{b_2} C_F\, \delta_{b_2 b_2} = -C_F N \end{aligned}$$

where this was evaluated with the help of the identity

$$T^a_{ik} T^a_{kl} = C_F \delta_{il}$$

From now on I will not write the sum over colors explicitly. For the matrix element $\left(\boldsymbol{w}_{12}^{q\bar{q}}\right)_{12}$ one has

$$\begin{aligned} \left(\boldsymbol{w}_{12}^{q\bar{q}}\right)_{12} &= -\frac{1}{N}\langle c_1| T^d_{b_1 a_1}\cdot T^d_{b_2 a_2} |c_2\rangle \\ &= -\frac{1}{N}\delta_{a_1 b_2}\delta_{a_3 a_4}\, T^d_{b_1 a_1}\cdot T^d_{b_2 a_2}\, T^c_{a_2 b_1} T^c_{a_3 a_4} \\ &= 0 \end{aligned}$$

since $\mathrm{Tr}\,(T^a) = 0$. With the same argument the matrix element $\left(\boldsymbol{w}_{12}^{q\bar{q}}\right)_{21}$ vanishes. For the entry $\left(\boldsymbol{w}_{12}^{q\bar{q}}\right)_{22}$ one has

$$\begin{aligned}
\left(\boldsymbol{w}_{12}^{q\bar{q}}\right)_{22} &= -\frac{1}{N}\left\langle c_2\right| T^d_{b_1a_1}\cdot T^d_{b_2a_2}\left|c_2\right\rangle \\
&= -\frac{1}{N}\,T^c_{a_4a_3}T^c_{a_1b_2}T^d_{b_1a_1}T^d_{b_2a_2}\,T^a_{a_2b_1}T^a_{a_3a_4} \\
&= -\frac{1}{N}T_F\,T^c_{a_1b_2}T^d_{b_1a_1}T^d_{b_2a_2}\,T^c_{a_2b_1} \\
&= \frac{1}{2N^2}T_F\mathrm{Tr}\left(T^cT^c\right) \\
&= \frac{1}{2N^2}T_FC_FN = \frac{C_F}{4N}
\end{aligned}$$

where one has used that

$$T^a_{ij}T^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{kj} - \frac{1}{N}\delta_{ij}\delta_{kl}\right)$$

$$\mathrm{Tr}\left(T^{a(R)}T^{b(R)}\right) = T_R\delta_{ab}$$

$$T_F = 1/2$$

So one has

$$\boldsymbol{w}_{12}^{q\bar{q}} = -\frac{C_F}{4N}\begin{pmatrix}4N & 0\\ 0 & -1\end{pmatrix}$$

All color matrices are computed like that. One obtains

$$\boldsymbol{w}_{34}^{q\bar{q}} = \boldsymbol{w}_{12}^{q\bar{q}} = -\frac{C_F}{4N}\begin{pmatrix}4N & 0\\ 0 & -1\end{pmatrix}$$

$$\boldsymbol{w}_{44}^{q\bar{q}} = \boldsymbol{w}_{33}^{q\bar{q}} = \frac{C_F}{2}\begin{pmatrix}2N & 0\\ 0 & C_F\end{pmatrix}$$

$$\boldsymbol{w}_{24}^{q\bar{q}} = \boldsymbol{w}_{13}^{q\bar{q}} = -\frac{C_F}{2}\begin{pmatrix}0 & 1\\ 1 & 2C_F-\frac{N}{2}\end{pmatrix}$$

$$\boldsymbol{w}_{23}^{q\bar{q}} = \boldsymbol{w}_{14}^{q\bar{q}} = -\frac{C_F}{2N}\begin{pmatrix}0 & -N\\ -N & 1\end{pmatrix}$$

The color generators of the soft gluon radiation from gluon legs are for all legs given by

$$\mathbf{T}^c_{ba} = -if^{cba}$$

Thus, in the squared amplitude there are contributions as

$$\left(-if^{cb_ia_i}\right)\left(if^{ca_jb_j}\right) = f^{cb_ia_i}f^{ca_jb_j}$$

With tedious color algebra given in [49] one obtains

$$\boldsymbol{w}_{12}^{gg} = -\frac{1}{4}\begin{pmatrix}4N^2 & 0 & 0\\ 0 & N^2 & 0\\ 0 & 0 & N^2-1\end{pmatrix}$$

$$\boldsymbol{w}_{34}^{gg} = -\begin{pmatrix}C_FN & 0 & 0\\ 0 & -\frac{1}{4} & 0\\ 0 & 0 & -\frac{N^2-4}{4N^2}\end{pmatrix}$$

$$
\boldsymbol{w}_{44}^{gg} = \boldsymbol{w}_{33}^{gg} = \frac{C_F}{2N} \begin{pmatrix} 2N^2 & 0 & 0 \\ 0 & N^2 & 0 \\ 0 & 0 & N^2 - 4 \end{pmatrix}
$$

$$
\boldsymbol{w}_{13}^{gg} = \boldsymbol{w}_{24}^{gg} = -\frac{1}{8} \begin{pmatrix} 0 & 4N & 0 \\ 4N & N^2 & N^2 - 4 \\ 0 & N^2 - 4 & N^2 - 4 \end{pmatrix}
$$

$$
\boldsymbol{w}_{23}^{gg} = \boldsymbol{w}_{14}^{gg} = -\frac{1}{8} \begin{pmatrix} 0 & -4N & 0 \\ -4N & N^2 & -N^2 + 4 \\ 0 & -N^2 + 4 & N^2 - 4 \end{pmatrix}
$$

Calculation of I_{jk}

Now one is left with the NLO soft integrals I_{jk} of eq. (8.3). The integrals are given by

$$
\begin{aligned}
I_{jk} &= -(4\pi)^2 \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^{\epsilon} \int_0^{2\pi} \frac{d\Phi}{2\pi} \int \frac{d^d k}{(2\pi)^{d-1}} \frac{\nu^\alpha}{k_+^\alpha} \delta(k^2)\theta(k^0) \frac{v_j \cdot v_k e^{-ix_\perp k_\perp}}{v_j \cdot k v_k \cdot k} \\
&= -\frac{\mu^{2\epsilon} e^{\epsilon\gamma_E}}{\pi^{2-\epsilon}} \int_0^{2\pi} d\Phi \int d^d k \, \frac{\nu^\alpha}{k_+^\alpha} \, \delta(k^2)\theta(k^0) \frac{v_j \cdot v_k e^{-ix_\perp k_\perp}}{v_j \cdot k v_k \cdot k}
\end{aligned}
$$

Due to the exponential term, it is easier to calculate these integrals in momentum space. One performs a Fourier transformation as

$$
\begin{aligned}
\tilde{I}_{jk} &= \frac{1}{2\pi} \int d^2 x_\perp e^{ix_\perp q_\perp} I_{jk} \\
&= -(2\pi) \frac{\mu^{2\epsilon} e^{\epsilon\gamma_E}}{\pi^{2-\epsilon}} \int_0^{2\pi} d\Phi \int d^d k \, \frac{\nu^\alpha}{k_+^\alpha} \, \delta(k^2)\theta(k^0) \frac{v_j \cdot v_k}{v_j \cdot k v_k \cdot k} \delta^{(2)} \left(q_\perp - k_\perp \right) \\
&= -2 \frac{\mu^{2\epsilon} e^{\epsilon\gamma_E}}{\pi^{1-\epsilon}} \int_0^{2\pi} d\Phi \int d^d k \, \frac{\nu^\alpha}{k_+^\alpha} \, \delta(k^2)\theta(k^0) \frac{v_j \cdot v_k}{v_j \cdot k v_k \cdot k} \delta^{(2)} \left(q_\perp - k_\perp \right)
\end{aligned} \tag{8.5}
$$

where Φ is now the azimuthal angle of $q_\perp$. Compared to [3], I obtained an additional factor 2 in the expression of $\tilde{I}_{jk}$ in eq. (8.5). This cancels in the retransformation to position space. So I keep in mind that there is an additional factor 2, but in order to avoid confusion with the paper [3], I leave it away in the calculation. So the difference is caused by a different convention of the Fourier transformation. The delta function $\delta^{(2)} \left(q_\perp - k_\perp \right)$ can be rewritten in spherical coordinates. For this purpose the vector $k_\perp$ is assumed to point in the x-direction of a coordinate system, so that

$$
k_\perp = (0, k_T, 0, 0)
$$

The vector $q_\perp$ can be written as

$$
\begin{aligned}
q_x &= q_T \cos\Phi \\
q_y &= q_T \sin\Phi
\end{aligned}
$$

So the delta function is given by

$$
\begin{aligned}
\delta^{(2)} \left(q_\perp - k_\perp \right) &= \delta(q_x - k_x)\delta(q_y - k_y) = \delta(q_x - k_T)\delta(q_y) = \delta(q_T \cos\Phi - k_T)\delta(q_T \sin\Phi) \\
&= \frac{1}{q_T} \delta(k_T - q_T \cos\Phi)\delta(\Phi) = \frac{1}{q_T} \delta(k_T - q_T)\delta(\Phi) = 2\delta(k_T^2 - q_T^2)\delta(\Phi)
\end{aligned}
$$

Now the Φ integral can be performed with the help of the δ function. One obtains

$$
\tilde{I}_{jk} = -2 \frac{\mu^{2\epsilon} e^{\epsilon\gamma_E}}{\pi^{1-\epsilon}} \int d^d k \, \frac{\nu^\alpha}{k_+^\alpha} \, \delta(k^2)\theta(k^0) \frac{v_j \cdot v_k}{v_j \cdot k v_k \cdot k} \delta \left(k_T^2 - q_T^2 \right)
$$

In d dimensions the vector k can be parametrized as

$$k = k_0(1,\ \sin\theta_1 \sin\theta_2 ... \sin\theta_{d-2}\cos\theta_{d-1},\ \sin\theta_1\sin\theta_2...\sin\theta_{d-2}\sin\theta_{d-1},\ ..., \\ \sin\theta_1\sin\theta_2\cos\theta_3,\ \sin\theta_1\cos\theta_2,\ \cos\theta_1) \tag{8.6}$$

such that $k^2 = 0$ and

$$k_T^2 = k_0^2 \sin^2\theta_1$$

Now a parametrization of the vectors v_3 and v_4 has to be chosen. Assuming that $q_\perp$ is small and considering the partonic center of mass frame one has

$$M^2 = (p_3 + p_4)^2 \approx 4E^2 \qquad \Rightarrow E = E_3 = E_4 = \frac{M}{2}$$

Since $p_3^2 = p_4^2 = m_t^2$ one has

$$p_3^2 = m_t^2 = E^2 - |\vec{p}_3|^2 \qquad \Rightarrow |\vec{p}_3| = E\sqrt{1 - \frac{4m_t^2}{M^2}} = E\cdot\beta_t$$

assuming that p_3 and p_4 are lying in the $y-z-$plane one has

$$p_3^\mu = (E, 0, ..., 0, E\beta_t\sin\theta, E\beta_t\cos\theta) = \frac{M}{2}(1, 0, ..., 0, \beta_t\sin\theta, \beta_t\cos\theta)$$

where θ is the scattering angle. Since $p_{3,4} \approx m_t v_{3,4}$ and $p_{3\perp} + p_{4\perp} \approx 0$ one has

$$\begin{aligned} v_3^\mu &= \frac{M}{2m_t}(1, 0, ..., 0, \beta_t\sin\theta, \beta_t\cos\theta) = \frac{1}{\sqrt{1-\beta_t^2}}(1, 0, ..., 0, \beta_t\sin\theta, \beta_t\cos\theta) \\ v_4^\mu &= \frac{1}{\sqrt{1-\beta_t^2}}(1, 0, ..., 0, -\beta_t\sin\theta, -\beta_t\cos\theta) \end{aligned}$$

So the scalar products appearing in $\tilde{I}_{jk}$ are

$$\begin{aligned} v_3\cdot v_4 &= \frac{1+\beta_t^2}{1-\beta_t^2} \\ n\cdot v_3 = \bar{n}\cdot v_4 &= \frac{1-\beta_t\cos\theta}{\sqrt{1-\beta_t^2}} \\ \bar{n}\cdot v_3 = n\cdot v_4 &= \frac{1+\beta_t\cos\theta}{\sqrt{1-\beta_t^2}} \\ n\cdot k &= k_0\left(1-\cos\theta_1\right) \\ \bar{n}\cdot k &= k_0\left(1+\cos\theta_1\right) \\ v_3\cdot k &= \frac{1}{\sqrt{1-\beta_t^2}}k_0\left(1-\beta_t\sin\theta\sin\theta_1\cos\theta_2 - \beta_t\cos\theta\cos\theta_1\right) \\ v_4\cdot k &= \frac{1}{\sqrt{1-\beta_t^2}}k_0\left(1+\beta_t\sin\theta\sin\theta_1\cos\theta_2 + \beta_t\cos\theta\cos\theta_1\right) \end{aligned}$$

The integration measure can be rewritten as

$$d^dk\delta(k^2)\theta(k_0) = \frac{k_0^{1-2\epsilon}}{2}\ dk_0\ \sin^{1-2\epsilon}\theta_1 d\theta_1\ \sin^{-2\epsilon}\theta_2 d\theta_2\ d\Omega_{d-3}$$

and [49]

$$\Omega_{d-3} = \frac{2\pi^{\frac{1}{2}-\epsilon}}{\Gamma(\frac{1}{2}-\epsilon)} = 2^{1-2\epsilon}\pi^{-\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}$$

In the next section I present the integration method that is employed in [3]. I call it the method of "brute force".

8.1. Method of brute force

As it is performed in [3], the integrals I_{jk} can be explicitly computed as

$$\tilde{I}_{13} = -\frac{\mu^{2\epsilon} e^{\epsilon\gamma_E}}{\pi^{1-\epsilon}} n \cdot v_3 \, \Omega_{d-3} \int k_0^{1-2\epsilon} \, dk_0 \, \sin^{1-2\epsilon}\theta_1 \, d\theta_1 \sin^{-2\epsilon}\theta_2 \, d\theta_2 \, \delta(k_0^2 \sin^2\theta_1 - q_T^2)$$
$$\times \frac{\nu^\alpha}{k_0^{2+\alpha}} \frac{1}{(1-\cos\theta_1)^{1+\alpha}} \frac{\sqrt{1-\beta^2}}{1-\beta\sin\theta_1\cos\theta_2\sin\theta - \beta\cos\theta_1\cos\theta}$$

The integration over k_0 can be performed using the δ function and subsequently one can integrate over θ_2 which gives

$$\tilde{I}_{13} = \frac{1}{q_T^2}\left(\frac{\mu^2}{q_T^2}\right)^{\epsilon+\alpha/2}\left(\frac{\nu^2}{\mu^2}\right)^{\alpha/2} \tilde{I}'_{13}$$

$$\tilde{I}'_{13} = -\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} \int_0^\pi d\theta_1 \left(\cot\frac{\theta_1}{2}\right)^{1+\alpha} \frac{1-\beta\cos\theta}{1-\beta\cos(\theta+\theta_1)} \, {}_2F_1\left(1, \frac{1}{2}-\epsilon, 1-2\epsilon, \frac{2\beta\sin\theta\sin\theta_1}{1-\beta\cos(\theta+\theta_1)}\right) \tag{8.7}$$

The integration variable is redefined as

$$\theta_1 = 2\arctan\left(\frac{1-y}{y}\right)$$

With

$$\sin x = \frac{\tan x}{\sqrt{1+\tan^2 x}}$$
$$\cos x = \frac{1}{\sqrt{1+\tan^2 x}}$$

and

$$\sin x = 2\sin\frac{x}{2}\cdot\cos\frac{x}{2}$$

one obtains

$$\sin\theta_1 = 2\frac{t}{1+t^2}$$

where $t = \frac{1-y}{y}$. For the integration measure one finds

$$\int_0^\pi d\theta_1 = \int_0^1 \frac{2}{1-2y+y^2} dy$$

with this substitution it remains the integral

$$\tilde{I}'_{13} = -\frac{2e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}(1-\beta\cos\theta)$$
$$\times \int_0^1 dy \, y^{1+\alpha}(1-y)^{-1-\alpha} \frac{{}_2F_1\left(1, \frac{1}{2}-\epsilon, 1-2\epsilon, \frac{4y(1-y)\beta\sin\theta}{1-2y(1-y)+(1-2y)\beta\cos\theta+2y(1-y)\beta\sin\theta}\right)}{1-2y(1-y)+(1-2y)\beta\cos\theta+2y(1-y)\beta\sin\theta}$$

The hypergeometric function is expanded with the Mathematica package HypExp [62] as a series in ϵ. The factor $(1-y)^{-1-\alpha}$ has to be rewritten as

$$(1-y)^{-1-\alpha} = -\frac{1}{\alpha}\delta(1-y) + \left(\frac{1}{1-y}\right)_+ + \mathcal{O}(\alpha)$$

Once again the analytic regulator is necessary to regulate the region $y \to 1$, which corresponds to $\theta_1 = 0$ and this implies that $k^\mu \sim n^\mu$ (cf. eq. (8.6)) which is the collinear region. Thus, the analytic regulator is again necessary to separate the soft from the collinear region. The expansion of the hypergeometric function gives

$$ {}_2F_1\left(1, \frac{1}{2} - \epsilon, 1 - 2\epsilon, z\right) = \frac{1}{\sqrt{1-z}} - \frac{2\epsilon}{\sqrt{1-z}}\left(\ln\left(1 - \frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right) + \ln\left(1 + \frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right)\right) $$

Evaluating the contribution which contains the delta function $\delta(1-y)$ one obtains

$$ \tilde{I}'_{13,a} = \frac{2}{\alpha}\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} $$

Evaluating the plus distribution one has to solve to order $\mathcal{O}(\epsilon^0)$

$$ I = \int_0^1 dy \left(-\frac{1}{1-y} + \frac{y}{1-y} \cdot (1 - \beta\cos\theta) \frac{1}{\sqrt{(1 - 2y(1-y) + (1-2y)\beta\cos\theta)^2 - (2y(1-y)\beta\sin\theta)^2}} \right) $$

In order to evaluate this integral one first has to take the derivative of the expression with respect to β. According to the Leibniz integral rule one may change integration and differentiation and one obtains

$$ \begin{aligned} \frac{d}{d\beta} I &= \int_0^1 dy \frac{d}{d\beta} \left(\frac{y}{1-y} \cdot (1 - \beta\cos\theta) \frac{1}{\sqrt{(1 - 2y(1-y) + (1-2y)\beta\cos\theta)^2 - (2y(1-y)\beta\sin\theta)^2}} \right) \\ &= \int_0^1 dy \Bigg((-1+y)\, y \left[(2 + 4y(y-1))\cos\theta + \beta(1 - 2y(1+y) + (1 + 2y(y-1))\cos(2\theta))\right] \\ &\quad \times \frac{1}{(-1 - 2(-1+y)y + \beta(-1+2y)\cos\theta + 2y\beta\sin\theta(-1+y))^2} \\ &\quad \times \frac{1}{(1 + 2(-1+y)y + \beta(1-2y)\cos\theta + 2y\beta\sin\theta(-1+y))} \\ &\quad \times \frac{1}{\sqrt{1 - \frac{4(-1+y)y\beta\sin\theta}{-1-2y(-1+y)+\beta(-1+2y)\cos\theta + 2y\beta\sin\theta(-1+y)}}} \Bigg) \end{aligned} $$

Integrating this expression over y from 0 to 1 one obtains

$$ \frac{d}{d\beta} I = \frac{\beta - \cos\theta}{(1 - \beta\cos\theta)(1 - \beta^2)} $$

And finally integrating over β gives

$$ I = \ln\frac{1 - \beta\cos\theta}{\sqrt{1-\beta^2}} + const $$

The constant term can be calculated via a boundary constraint. Taking as a boundary constraint the value at $\beta = 0$. So the original integral is

$$ I = \int_0^1 dy \, \frac{-1+2y}{1 + 2y(y-1)} = 0 $$

This can be compared with

$$ 0 = I = \ln\frac{1}{1} + const $$

and thus the constant term is equal to zero. So one obtains as a final result

$$I = \ln \frac{1-\beta\cos\theta}{\sqrt{1-\beta^2}}$$

This expression can be rewritten. Remember that

$$t_1 = (p_1 - p_3)^2 - m_t^2$$

and

$$M^2 = (p_3 + p_4)^2$$

and thus

$$\frac{-t_1}{m_t\,M} = \frac{2p_1 \cdot p_3}{m_t\,M} = n \cdot v_3 = \frac{1-\beta\cos\theta}{\sqrt{1-\beta^2}}$$

$$\Rightarrow I = \ln\left(\frac{-t_1}{m_t\,M}\right)$$

For the result to order ϵ one takes the formula (8.7) as a starting point. With the Mathematica package HypExp the hypergeometric function is expanded as a series in ϵ. To order ϵ the plus distribution reduces to the normal function with $\alpha = 0$, since for $y = 1$ the argument of the hypergeometric function is equal to zero and thus

$${}_2F_1\left(1, \frac{1}{2} - \epsilon, 1 - 2\epsilon, 0\right) = 1 - 0 \cdot \epsilon$$

To order ϵ it therefore remains to solve

$$\tilde{I}_{13}^{(\epsilon)} = -\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} \int_0^\pi d\theta_1\ \cot\left(\frac{\theta_1}{2}\right) \frac{1-\beta\cos\theta}{1-\beta\cos(\theta+\theta_1)}\ {}_2F_1^{(\epsilon)}\left(1, \frac{1}{2} - \epsilon, 1-2\epsilon, \frac{2\beta\sin\theta\sin\theta_1}{1-\beta\cos(\theta+\theta_1)}\right)$$

So to order ϵ one has the integral

$$I^{(\epsilon)} = \epsilon \int_0^\pi d\theta_1\ \cot\left(\frac{\theta_1}{2}\right) \frac{1-\beta\cos\theta}{1-\beta\cos(\theta+\theta_1)} \frac{\ln\left(16\frac{-1+\beta\cos(\theta_1-\theta)}{-1+\beta\cos(\theta_1+\theta)}\right) - 4\ln\left(1+\sqrt{\frac{-1+\beta\cos(\theta_1-\theta)}{-1+\beta\cos(\theta_1+\theta)}}\right)}{\sqrt{\frac{-1+\beta\cos(\theta_1-\theta)}{-1+\beta\cos(\theta_1+\theta)}}}$$

Solving the integral $I^{(\epsilon)}$ for all θ and β seems impossible, thus one starts by evaluating the integral for special values of θ where the integrand looks pretty simple. For $\theta = 0, \pi$ the integrand reduces to zero. For $\theta = \pi/2$ one has

$$I^\epsilon_{\pi/2} = \epsilon \int_0^\pi d\theta_1\ \cot\left(\frac{\theta_1}{2}\right) \frac{1}{1+\beta\sin(\theta_1)} \frac{\ln\left(16\frac{2}{1+\beta\sin(\theta_1)} - 16\right) - 4\ln\left(1+\sqrt{\frac{2}{1+\beta\sin(\theta_1)} - 1}\right)}{\sqrt{\frac{2}{1+\beta\sin(\theta_1)} - 1}}$$

with the boundary condition at $\beta = 0$

$$I^\epsilon_{\pi/2}(\beta = 0) = 0$$

To calculate this integral on first takes the derivative with respect to β and then one can integrate over θ_1 which gives

$$\frac{d}{d\beta} I^\epsilon_{\pi/2} = \epsilon \cdot 2\frac{\ln\left(1-\beta^2\right)}{\beta - \beta^3}$$

This can be integrated over β and one obtains

$$I^{\epsilon}_{\pi/2} = \epsilon \text{Li}_2\left(\frac{-\beta^2}{1-\beta^2}\right) \tag{8.8}$$

where the polylogarithm is defined as

$$\begin{aligned}\text{Li}_{n+1}(z) &= \int_0^z dt \frac{\text{Li}_n(t)}{t} \qquad \text{with} \qquad n \in \{0,1,2,...\}\\ \text{Li}_1(z) &= -\ln(1-z)\end{aligned}$$

The result for $I^{\epsilon}_{\pi/2}$ in eq. (8.8) also respects the boundary condition at $\beta = 0$. Since the integral I^{ϵ} reduces to zero for $\theta = 0, \pi$ one can guess, that the argument of the dilog is proportional to $\sin\theta$. This guess can be verified by comparing a series expansion in β. The integrand of $I^{(\epsilon)}$ is expanded in $\beta \to 0$ and afterwards it is possible to integrate the coefficients over θ_1. The guess $I_g = \text{Li}_2\left(\frac{-\beta^2 \sin^2\theta}{1-\beta^2}\right)$ is as well expanded in $\beta \to 0$ and the result coincides with the expansion obtained from $I^{(\epsilon)}$. Thus, the expression I_g is the solution to order ϵ. So the final result for all θ and β is

$$I^{\epsilon} = \epsilon \cdot \text{Li}_2\left(-\frac{\beta^2 \sin^2\theta}{1-\beta^2}\right)$$

So the whole result is

$$\tilde{I}_{13} = \frac{1}{q_T^2}\left(\frac{\mu^2}{q_T^2}\right)^{\epsilon+\alpha/2}\left(\frac{\nu^2}{\mu^2}\right)^{\alpha/2}\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\left[\frac{2}{\alpha} - 2\ln\frac{1-\beta\cos\theta}{\sqrt{1-\beta^2}} + \epsilon\cdot\text{Li}_2\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)\right]$$

This was the method which was employed in [3] to evaluate I_{jk}. This method was really tedious and it seems impractical for even more difficult integrands such as the interference term I_{34}. So one needs to employ other methods.

8.2. The Mellin-Barnes methods

In this section I show how the integral I_{13} can be calculated using the Mellin-Barnes method which was described by G. Somogyi [6] and implemented in Mathematica by M. Czakon [63]. The integral which is under consideration is given in eq. (B3) of reference [3], namely

$$\tilde{I}_{13} = -\frac{2\mu^{2\epsilon}e^{\epsilon\cdot\gamma_E}}{\pi^{1-\epsilon}}\int d^dk\left(\frac{\nu}{n\cdot k}\right)^{\alpha}\delta(k^2)\theta(k^0)\delta(k_T^2-q_T^2)\cdot\frac{v_1\cdot v_3}{v_1\cdot k\; v_3\cdot k}$$

The integral in invariant under the scaling of v_3 and thus one can choose the momenta to be

$$\begin{aligned}v_1^{\mu} &= n^{\mu} = (1, 0_{d-2}, 1)\\ v_2^{\mu} &= \bar{n}^{\mu} = (1, 0_{d-2}, -1)\\ v_3^{\mu} &= (1, 0, ..., 0, \beta\sin\theta, \beta\cos\theta)\\ v_4^{\mu} &= (1, 0, ..., 0, -\beta\sin\theta, -\beta\cos\theta)\\ k^{\mu} &= k^0(1, ..., \sin\theta_1\sin\theta_2, \sin\theta_1\cos\theta_2, \cos\theta_1)\end{aligned} \tag{8.9}$$

First the k^0 integral can be evaluated using the delta function in k_T:

$$d^dk\delta(k^2)\theta(k_0)\delta(k_T^2-q_T^2) = \frac{k_0^{1-2\epsilon}}{2}\,dk_0\;\delta(k_0^2\sin^2\theta_1 - q_T^2)\;d\Omega(k)$$

Due to the denominator there is an additional factor of $k_0^{-2-\alpha}$ and thus one has

$$\int dk_0\,\frac{k_0^{-1-2\epsilon-\alpha}}{2}\;\delta(k_0^2\sin^2\theta_1 - q_T^2) = \frac{1}{4}\left(\frac{1}{q_T^2}\right)^{1+\epsilon+\alpha/2}\sin^{2\epsilon+\alpha}\theta_1$$

In the Mellin-Barnes procedure the whole dependence of k^μ has to be expressed through scalar products of the momenta mentioned above and one finds that

$$\sin^2\theta_1 = n\cdot\tilde{k}\,\bar{n}\cdot\tilde{k}$$

with

$$\tilde{k}^\mu = (1, ..., \sin\theta_1\sin\theta_2, \sin\theta_1\cos\theta_2, \cos\theta_1).$$

And so the whole integral gives

$$\tilde{I}_{13} = -\frac{\mu^{2\epsilon}e^{\epsilon\cdot\gamma_E}}{2\pi^{1-\epsilon}}\nu^\alpha \int d\Omega(k)\left(\frac{1}{q_T^2}\right)^{1+\epsilon+\alpha/2} \frac{n\cdot v_3}{\left(n\cdot\tilde{k}\right)^{1-\epsilon+\alpha/2}\left(\bar{n}\cdot\tilde{k}\right)^{-\epsilon-\alpha/2} v_3\cdot\tilde{k}}$$

One can apply the Mellin-Barnes method [6] to this integral. The basic tool of this method is the following formula [61]

$$\frac{1}{(X+Y)^\lambda} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dz \frac{Y^z}{X^{\lambda+z}}\Gamma(\lambda+z)\Gamma(-z)$$

Following the same steps as in [6] one can rewrite this integral as

$$\begin{aligned}\tilde{I}_{13} = &-\frac{\mu^{2\epsilon}e^{\epsilon\cdot\gamma_E}}{2\pi^{1-\epsilon}}\nu^\alpha\left(\frac{1}{q_T^2}\right)^{1+\epsilon+\alpha/2} n\cdot v_3\; 2^{2-j-2\epsilon}\pi^{1-\epsilon}\frac{1}{\prod_{k=1}^3\Gamma(j_k)\Gamma(2-j-2\epsilon)}\\ &\times\int_{-i\infty}^{i\infty}\frac{dz_{12}}{2\pi i}\frac{dz_{13}}{2\pi i}\frac{dz_{23}}{2\pi i}\frac{dz_{33}}{2\pi i} v_{12}^{z_{12}}v_{13}^{z_{13}}v_{23}^{z_{23}}v_{33}^{z_{33}}\Gamma(j_1-z_{12}-z_{13})\Gamma(-z_{12})\Gamma(-z_{13})\Gamma(-z_{23})\\ &\times\Gamma(-z_{33})\Gamma(j_2-z_{23}-z_{12})\Gamma(j_3-z_{13}-z_{23}-2z_{33})\Gamma(1-j-\epsilon-z_{12}-z_{13}-z_{23}-z_{33})\end{aligned}$$

which corresponds to the case "three denominators, one mass" according to [6]. And v_{ij} is given by

$$v_{ij} = \begin{cases}\frac{v_i\cdot v_j}{2} & \text{for } v_i\neq v_j\\ \frac{v_i^2}{4} & \text{for } v_i = v_j\end{cases} \tag{8.10}$$

The j_i is the exponent of the the scalar product of vector v_i with $\tilde{k}$. So one has

$$\begin{aligned}j_1 &= 1-\epsilon+\alpha/2\\ j_2 &= -\epsilon-\alpha/2\\ j_3 &= 1\\ j &= j_1+j_2+j_3 = 2-2\epsilon\end{aligned}$$

There is a problem considering the gamma function $\Gamma(2-j-2\epsilon)$ since the argument becomes zero. To solve this problem one can introduce a further variable δ and set

$$j_2 \to \delta-\epsilon-\alpha/2$$

and thus one has

$$\begin{aligned}j_1 &= 1-\epsilon+\alpha/2\\ j_2 &= \delta-\epsilon-\alpha/2\\ j_3 &= 1\\ j &= 2-2\epsilon+\delta.\end{aligned}$$

In the end δ has to be set equal to zero.
With the Mellin-Barnes package [63] implemented in Mathematica one can solve the integral $\tilde{I}_{13}$. The variables ϵ, α and δ have to be analytically continued and one obtains

$$\tilde{I}_{13} = \mu^{2\epsilon}\left(\frac{1}{q_T^2}\right)^{1+\epsilon+\alpha/2}\nu^{\alpha}\left(\frac{2}{\alpha} - 2\ln\left(\frac{v_{13}}{\sqrt{v_{33}}}\right) + \epsilon\frac{\pi^2}{6} + \epsilon\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}\left(\frac{v_{33}}{v_{23}v_{13}}\right)^z\Gamma^2(-z)\Gamma(z)\Gamma(1+z)\right)$$

It remains to solve the order ϵ^1 contribution which is

$$I_\epsilon(x) = \epsilon\frac{\pi^2}{6} + \epsilon\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}x^z\Gamma^2(-z)\Gamma(z)\Gamma(1+z)$$

with

$$\left(\frac{v_{33}}{v_{23}v_{13}}\right) = x$$

It holds

$$I_\epsilon(0) = \epsilon\frac{\pi^2}{6}$$

The integral can be solved by deriving a differential equation with respect to x, namely

$$\begin{aligned}\frac{d}{dx}I_\epsilon(x) &= \frac{1}{x}\epsilon\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}x^z z\Gamma^2(-z)\Gamma(z)\Gamma(1+z)\\ &= \frac{1}{x}\epsilon\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}x^z\Gamma^2(-z)\Gamma(1+z)\Gamma(1+z)\end{aligned}$$

Applying formula (D.73) of reference [61] to this expression one gets

$$\frac{d}{dx}I_\epsilon(x) = \frac{1}{x}\epsilon\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}(x-1)^z\frac{\Gamma(-z)\Gamma(1+z)\Gamma(1+z)}{\Gamma(2+z)}$$

and applying formula (D.71) of the same reference afterwards one ends up with

$$\begin{aligned}\frac{d}{dx}I_\epsilon(x) &= \frac{\epsilon}{x}\,{}_2F_1(1,1,2,1-x)\\ &= -\frac{\epsilon}{x}\frac{\ln(x)}{1-x}\end{aligned}$$

Integrating now with respect to x one obtains

$$I_\epsilon(x) = -\epsilon\left(\frac{1}{2}\ln^2(x) + \mathrm{Li}_2(1-x)\right) + c = \epsilon\cdot\mathrm{Li}_2(1-x^{-1}) + c$$

and now one fixes c via the boundary condition at $x=0$. One has

$$I_\epsilon(0) = \epsilon\underbrace{\mathrm{Li}_2(1)}_{=\frac{\pi^2}{6}} + c \equiv \epsilon\frac{\pi^2}{6} \quad \Rightarrow c = 0$$

Calculating the argument of the polylogarithm one gets

$$I_\epsilon(x) = \epsilon\cdot\mathrm{Li}_2\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)$$

and all together one has

$$\tilde{I}_{13} = \mu^{2\epsilon}\left(\frac{1}{q_T^2}\right)^{1+\epsilon+\alpha/2}\nu^{\alpha}\left(\frac{2}{\alpha} - 2\ln\left(\frac{v_{13}}{\sqrt{v_{33}}}\right) + \epsilon\cdot \mathrm{Li}_2\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)\right)$$

This result is in correspondence with the result of [3]. Remembering that

$$\frac{v_{13}}{\sqrt{v_{33}}} = \frac{1-\beta\cos\theta}{1-\beta^2} = \frac{-t_1}{m_t\cdot M}$$

This method is an improvement compared to the method of "brute force" in the last chapter. But even this integral gets impractical for more difficult integrands as for I_{34}. This is shown in the next section.

Calculation of $\tilde{I}_{34}$ with the Mellin-Barnes Method

The integral of interest is

$$\tilde{I}_{34} = -\frac{2\mu^{2\epsilon}e^{\epsilon\cdot\gamma_E}}{\pi^{1-\epsilon}}\int d^dk\left(\frac{\nu}{n\cdot k}\right)^{\alpha}\delta(k^2)\theta(k^0)\delta(k_T^2-q_T^2)\cdot\frac{v_4\cdot v_3}{v_4\cdot k\; v_3\cdot k}$$

The vectors v_i are chosen as in eq. (8.9). Again, the k^0 integral is first evaluated with the use of the delta function in k_T and one ends up with

$$\tilde{I}_{34} = -\frac{\mu^{2\epsilon}e^{\epsilon\cdot\gamma_E}}{2\pi^{1-\epsilon}}\nu^{\alpha}\int d\Omega(k)\left(\frac{1}{q_T^2}\right)^{1+\epsilon+\alpha/2}\frac{v_4\cdot v_3}{\left(n\cdot\tilde{k}\right)^{-\epsilon+\alpha/2}\left(\bar{n}\cdot\tilde{k}\right)^{-\epsilon-\alpha/2}v_3\cdot\tilde{k}\;v_4\cdot\tilde{k}}$$

Following the same steps as in [6] one can rewrite the angular integral as

$$\begin{aligned}\tilde{I}_{34}^{\text{ang}} =& v_{34}\int_{-i\infty}^{i\infty}\frac{dz_{12}}{2\pi i}\frac{dz_{13}}{2\pi i}\frac{dz_{14}}{2\pi i}\frac{dz_{23}}{2\pi i}\frac{dz_{24}}{2\pi i}\frac{dz_{33}}{2\pi i}\frac{dz_{34}}{2\pi i}\frac{dz_{44}}{2\pi i}\frac{2^{-2\epsilon-j+2}}{\Gamma(j_1)\Gamma(j_2)\Gamma(j_3)\Gamma(j_4)\Gamma(-2\epsilon-j+2)}\\ &\times\Gamma(-z_{12})\Gamma(-z_{13})\Gamma(-z_{14})\Gamma(-z_{23})\Gamma(-z_{24})\Gamma(-z_{33})\Gamma(-z_{34})\Gamma(-z_{44})v_{34}^{z_{34}}v_{13}^{z_{13}+z_{24}}v_{14}^{z_{14}+z_{23}}v_{33}^{z_{33}+z_{44}}\\ &\times\Gamma(j_1+z_{12}+z_{13}+z_{14})\Gamma(j_2+z_{12}+z_{23}+z_{24})\Gamma(j_3+z_{13}+z_{23}+2z_{33}+z_{34})\\ &\times\Gamma(j_4+z_{14}+z_{24}+z_{34}+2z_{44})\\ &\times\Gamma(-\epsilon-j-z_{12}-z_{13}-z_{14}-z_{23}-z_{24}-z_{33}-z_{34}-z_{44}+1)\end{aligned}$$

with

$$v_{ij} = \begin{cases}\frac{v_i\cdot v_j}{2} & \text{for } v_i\neq v_j\\ \frac{v_i^2}{4} & \text{for } v_i = v_j\end{cases} \tag{8.11}$$

and

$$\begin{aligned}j_1 &= -\epsilon+\alpha/2\\ j_2 &= \delta-\epsilon-\alpha/2\\ j_3 &= 1\\ j_4 &= 1\\ j &= \sum_i j_i = 2-2\epsilon+\delta\end{aligned}$$

This 8 dimensional Mellin-Barnes integral is not solvable with the Mellin-Barnes package [63]. Although the Mellin-Barnes method is an improvement compared to the method of brute force, it is still not suitable to solve the difficult integrand of $\tilde{I}_{34}$. Since the aim is to calculate the soft function at NNLO, which is assumed to be even more complicated, it is necessary to look for a further method. In the next section the integrals $\tilde{I}_{jk}$ are solved by working out differential equations.

8.3. The differential equation method

For the calculation of the integrals of the soft function to order α_s one can perform the following steps: The delta constraints are expressed as cut propagators via reverse unitarity [17, 64, 65, 66]. This is

$$\delta(x) \to \frac{1}{2\pi i}\left(\frac{1}{x-i0}-\frac{1}{x+i0}\right) \tag{8.12}$$

Then with integration by parts relations [61, 67] the integrals are reduced to master integrals. This is achieved by the Laporta algorithm [68] which is implemented in MAPLE by Anastasiou and Lazapoulos [69]. The corresponding program is called A.I.R. (**A**utomatic **I**ntegral **R**eduction). In order to solve these master integrals one derives differential equations for them [7, 70, 71].
The integrals of the NLO soft function can be written in the form

$$I(a_1,a_2,a_3,a_4,a_5,a_6) = \Sigma\int d^dk \frac{1}{(n\cdot k)^{a_1+\alpha}}\cdot\frac{1}{(\bar{n}\cdot k)^{a_2}}\cdot\frac{1}{(\tilde{v}_3\cdot k)^{a_3}}\cdot\frac{1}{(\tilde{v}_4\cdot k)^{a_4}}\cdot\frac{1}{(k^2)^{a_5}}\cdot\frac{1}{\left(n\cdot k\;\bar{n}\cdot k-q_T^2\right)^{a_6}}$$

with

$$\Sigma = -\frac{2\mu^{2\epsilon}e^{\epsilon\cdot\gamma_E}\nu^\alpha}{\pi^{1-\epsilon}}$$

where the two last terms arise due to delta functions. One has

$$\begin{aligned}
\tilde{I}_{13} &= (n\cdot\tilde{v}_3)\cdot I(1,0,1,0,1,1) \equiv (n\cdot\tilde{v}_3)\cdot K_{13}\\
\tilde{I}_{14} &= (n\cdot\tilde{v}_4)\cdot I(1,0,0,1,1,1) \equiv (n\cdot\tilde{v}_4)\cdot K_{14}\\
\tilde{I}_{23} &= (\bar{n}\cdot\tilde{v}_3)\cdot I(0,1,1,0,1,1) \equiv (\bar{n}\cdot\tilde{v}_3)\cdot K_{23}\\
\tilde{I}_{24} &= (\bar{n}\cdot\tilde{v}_4)\cdot I(0,1,0,1,1,1) \equiv (\bar{n}\cdot\tilde{v}_4)\cdot K_{24}\\
\tilde{I}_{33} &= (\tilde{v}_3\cdot\tilde{v}_3)\cdot I(0,0,2,0,1,1) \equiv (\tilde{v}_3\cdot\tilde{v}_3)\cdot K_{33}\\
\tilde{I}_{44} &= (\tilde{v}_4\cdot\tilde{v}_4)\cdot I(0,0,0,2,1,1) \equiv (\tilde{v}_4\cdot\tilde{v}_4)\cdot K_{44}\\
\tilde{I}_{34} &= (\tilde{v}_4\cdot\tilde{v}_3)\cdot I(0,0,1,1,1,1) \equiv (\tilde{v}_3\cdot\tilde{v}_4)\cdot K_{34}
\end{aligned} \tag{8.13}$$

Since the integrals I_{ij} are invariant under a rescaling of v_3 and v_4 one chooses

$$\begin{aligned}
&\tilde{v}_3^\mu = (1,0,...,0,\beta\sin\theta,\beta\cos\theta) \quad\Rightarrow\quad \tilde{v}_3^2 = 1-\beta^2\\
&\tilde{v}_4^\mu = (1,0,...,0,-\beta\sin\theta,-\beta\cos\theta) \quad\Rightarrow\quad \tilde{v}_4^2 = 1-\beta^2 \quad\text{and}\quad \tilde{v}_3\cdot\tilde{v}_4 = 1+\beta^2
\end{aligned}$$

The two independent kinematic variables are called a and c and they are given by

$$\begin{aligned}
a &= n\cdot\tilde{v}_3 = \bar{n}\cdot\tilde{v}_4 = 1-\beta\cos\theta\\
b &= n\cdot\tilde{v}_4 = \bar{n}\cdot\tilde{v}_3 = 2-a = 1+\beta\cos\theta\\
c &= \tilde{v}_3^2 = \tilde{v}_4^2 = 1-\beta^2\\
\tilde{v}_3\cdot\tilde{v}_4 &= 2-c = 1+\beta^2
\end{aligned} \tag{8.14}$$

and d is the dimension, namely $d = 4-2\epsilon$.
For the reduction the IBP relations have to be derived. These are

$$\begin{aligned}
&\int d^dk\frac{\partial}{\partial k^\mu}k^\mu\cdot\frac{1}{(n\cdot k)^{a_1+\alpha}}\cdot\frac{1}{(\bar{n}\cdot k)^{a_2}}\cdot\frac{1}{(\tilde{v}_3\cdot k)^{a_3}}\cdot\frac{1}{(\tilde{v}_4\cdot k)^{a_4}}\cdot\frac{1}{(k^2)^{a_5}}\cdot\frac{1}{\left(n\cdot k\;\bar{n}\cdot k-q_T^2\right)^{a_6}} = 0\\
&\Leftrightarrow (d-\alpha-a_1-a_2-a_3-a_4-2\cdot a_5)\cdot I(a_1,a_2,a_3,a_4,a_5,a_6)\\
&\quad -2\cdot a_6\cdot I(a_1-1,a_2-1,a_3,a_4,a_5,a_6+1) = 0
\end{aligned}$$

$$\int d^dk \frac{\partial}{\partial k^\mu} n^\mu \cdot \frac{1}{(n\cdot k)^{a_1+\alpha}} \cdot \frac{1}{(\bar{n}\cdot k)^{a_2}} \cdot \frac{1}{(\tilde{v}_3\cdot k)^{a_3}} \cdot \frac{1}{(\tilde{v}_4\cdot k)^{a_4}} \cdot \frac{1}{(k^2)^{a_5}} \cdot \frac{1}{\left(n\cdot k\,\bar{n}\cdot k - q_T^2\right)^{a_6}} = 0$$
$$\begin{aligned}\Leftrightarrow\ & 2\cdot a_2 \cdot I(a_1, a_2+1, a_3, a_4, a_5, a_6) + a\cdot a_3 \cdot I(a_1, a_2, a_3+1, a_4, a_5, a_6)\\ & + (2-a)\cdot a_4 \cdot I(a_1, a_2, a_3, a_4+1, a_5, a_6) + 2\cdot a_5 \cdot I(a_1-1, a_2, a_3, a_4, a_5+1, a_6)\\ & + 2\cdot a_6 \cdot I(a_1-1, a_2, a_3, a_4, a_5, a_6+1) = 0\end{aligned}$$

$$\int d^dk \frac{\partial}{\partial k^\mu} \bar{n}^\mu \cdot \frac{1}{(n\cdot k)^{a_1+\alpha}} \cdot \frac{1}{(\bar{n}\cdot k)^{a_2}} \cdot \frac{1}{(\tilde{v}_3\cdot k)^{a_3}} \cdot \frac{1}{(\tilde{v}_4\cdot k)^{a_4}} \cdot \frac{1}{(k^2)^{a_5}} \cdot \frac{1}{\left(n\cdot k\,\bar{n}\cdot k - q_T^2\right)^{a_6}} = 0$$
$$\begin{aligned}\Leftrightarrow\ & 2\cdot(\alpha+a_1) \cdot I(a_1+1, a_2, a_3, a_4, a_5, a_6) + (2-a)\cdot a_3 \cdot I(a_1, a_2, a_3+1, a_4, a_5, a_6)\\ & + a\cdot a_4 \cdot I(a_1, a_2, a_3, a_4+1, a_5, a_6) + 2\cdot a_5 \cdot I(a_1, a_2-1, a_3, a_4, a_5+1, a_6)\\ & + 2\cdot a_6 \cdot I(a_1, a_2-1, a_3, a_4, a_5, a_6+1) = 0\end{aligned}$$

$$\int d^dk \frac{\partial}{\partial k^\mu} \tilde{v}_3^\mu \cdot \frac{1}{(n\cdot k)^{a_1+\alpha}} \cdot \frac{1}{(\bar{n}\cdot k)^{a_2}} \cdot \frac{1}{(\tilde{v}_3\cdot k)^{a_3}} \cdot \frac{1}{(\tilde{v}_4\cdot k)^{a_4}} \cdot \frac{1}{(k^2)^{a_5}} \cdot \frac{1}{\left(n\cdot k\,\bar{n}\cdot k - q_T^2\right)^{a_6}} = 0$$
$$\begin{aligned}\Leftrightarrow\ & a\cdot(\alpha+a_1) \cdot I(a_1+1, a_2, a_3, a_4, a_5, a_6) + (2-a)\cdot a_2 \cdot I(a_1, a_2+1, a_3, a_4, a_5, a_6)\\ & + a_3\cdot c \cdot I(a_1, a_2, a_3+1, a_4, a_5, a_6) + a_4\cdot(2-c) \cdot I(a_1, a_2, a_3, a_4+1, a_5, a_6)\\ & + 2\cdot a_5 \cdot I(a_1, a_2, a_3-1, a_4, a_5+1, a_6) + a_6\cdot(2-a) \cdot I(a_1-1, a_2, a_3, a_4, a_5, a_6+1)\\ & + a_6\cdot a \cdot I(a_1, a_2-1, a_3, a_4, a_5, a_6+1) = 0\end{aligned}$$

and there is an additional relation between the cut propagator and the normal propagators, namely

$$\begin{aligned} I(a_1, a_2, a_3, a_4, a_5, a_6) &= I(a_1, a_2, a_3, a_4, a_5, a_6) \cdot \frac{n\cdot k\,\bar{n}\cdot k - q_T^2}{n\cdot k\,\bar{n}\cdot k - q_T^2}\\ &= I(a_1, a_2, a_3, a_4, a_5, a_6+1) \cdot \left(n\cdot k\,\bar{n}\cdot k - q_T^2\right)\\ \Leftrightarrow\ I(a_1, a_2, a_3, a_4, a_5, a_6) &- I(a_1-1, a_2-1, a_3, a_4, a_5, a_6+1) + q_T^2 \cdot I(a_1, a_2, a_3, a_4, a_5, a_6+1) = 0\end{aligned}$$

and the last relation is due to momentum conservation

$$\begin{aligned} I(a_1, a_2, a_3, a_4, a_5, a_6)\cdot(\tilde{v}_3\cdot k + \tilde{v}_4\cdot k - n\cdot k - \bar{n}\cdot k) = 0\\ \Leftrightarrow\ I(a_1, a_2, a_3, a_4-1, a_5, a_6) + I(a_1, a_2, a_3-1, a_4, a_5, a_6)\\ -I(a_1-1, a_2, a_3, a_4, a_5, a_6) - I(a_1, a_2-1, a_3, a_4, a_5, a_6) = 0\end{aligned}$$

The reduction is automated in Maple by the routine A.I.R. [69]. The IBP relations, optionally a list of known master integrals and "zero topologies" are given in an input file. In the first run the list of master integrals is left empty. In the list of the "zero topologies" it is stated for which powers of a_i the integrals vanish. In our case, the integrals vanish if the cut propagators appear with power $a_i \leq 0$ for $i = 5, 6$. This is justified by eq. (8.12). This list is therefore given by

$$\begin{aligned}\text{ZERO TOPOLOGIES: } =&[\text{ThetaF}(a_6) = 0,\\ &\text{ThetaF}(a_5) = 0,\\ &\text{NULL}]:\end{aligned} \tag{8.15}$$

In [65] it is defined $\Theta(x) = 1$ for $x > 1$ and $\Theta(x) = 0$ for $x \leq 0$. With the IBP relations and the "zero topologies" one obtains the following master integrals

$$\begin{aligned} M_1 &= I(0,-1,1,0,1,1) = I(1,0,1,0,1,1)\cdot q_T^2 = K_{13}\cdot q_T^2\\ M_2 &= I(0,-1,0,1,1,1) = I(1,0,0,1,1,1)\cdot q_T^2 = K_{14}\cdot q_T^2\\ M_3 &= I(0,0,1,-1,1,1) = (K_{23}+K_{13})\cdot q_T^2\\ M_4 &= I(0,0,-1,1,1,1) = (K_{24}+K_{14})\cdot q_T^2\\ M_5 &= I(0,0,1,1,1,1) = K_{34}\end{aligned}$$

So the integral K_{34} itself is a master integral. Since it is nicer to obtain K_{23} and K_{24} as master integrals instead of M_3 and M_4 one states in the second run of Maple the master integrals as

$$\begin{aligned}\text{MASTERS} :=[&I(0,-1,1,0,1,1),\\ &I(0,-1,0,1,1,1),\\ &I(-1,0,0,1,1,1),\\ &I(-1,0,1,0,1,1),\\ &I(0,0,1,1,1,1)]:\end{aligned} \tag{8.16}$$

I also tried to obtain K_{13} to K_{24} as master integrals. But Maple wants to find the "easiest" integrals as master integrals and in the A.I.R. algorithm, integrals like $I(0,-1,1,0,1,1)$ are preferred compared to $I(1,0,1,0,1,1)$. The simplicity of an integral is for example measured via the smallest number of denominators via

$$\mathcal{N}_{prop} = \sum_i \theta(a_i)$$

And a master integral is thus given by the smallest $\mathcal{N}_{prop}$. For that reason I also tried to leave away one or both of the last two relations. But this could not achieve to obtain the desired master integrals. The best master integrals are thus given by (8.16).

8.3.1. Notation

There are seven integrals which have to be calculated. These are given in equation (8.13) by K_{13} to K_{34}. From now on the master integrals are called M_{ij} and they are

$$\begin{aligned}M_{13} &= I(0,-1,1,0,1,1) = q_T^2 \cdot K_{13} = \frac{q_T^2}{n\cdot\tilde{v}_3}\tilde{I}_{13}\\ M_{23} &= I(-1,0,1,0,1,1) = q_T^2 \cdot K_{23} = \frac{q_T^2}{\bar{n}\cdot\tilde{v}_3}\tilde{I}_{23}\\ M_{14} &= I(0,-1,0,1,1,1) = q_T^2 \cdot K_{14} = \frac{q_T^2}{n\cdot\tilde{v}_4}\tilde{I}_{14}\\ M_{24} &= I(-1,0,0,1,1,1) = q_T^2 \cdot K_{24} = \frac{q_T^2}{\bar{n}\cdot\tilde{v}_4}\tilde{I}_{24}\\ M_{34} &= I(0,0,1,1,1,1) = K_{34}\end{aligned} \tag{8.17}$$

The general ansatz to solve a differential equation is the evaluation from order to order. Thus a master integral is expanded in the following way

$$M_{ij} = \frac{1}{\alpha}M_{ij}^{\alpha} + M_{ij}^{1} + \epsilon M_{ij}^{\epsilon} + \epsilon^2 M_{ij}^{\epsilon^2} \tag{8.18}$$

8.3.2. M_{13}, M_{23} and M_{33}

I start by the evaluation of M_{13} via a differential equation. Maple reduces the integrals K_{13} and K_{23} to

$$\begin{aligned}K_{13} &= I(1,0,1,0,1,1) = \frac{1}{q_T^2}\cdot I(0,-1,1,0,1,1) = \frac{1}{q_T^2}\cdot M_{13}\\ K_{23} &= I(0,1,1,0,1,1) = \frac{1}{q_T^2}\cdot I(-1,0,1,0,1,1) = \frac{1}{q_T^2}\cdot M_{23}\end{aligned}$$

Taking the derivative with respect to β gives

$$\begin{aligned}
\frac{d}{d\beta}M_{13} &= -I(0,-1,2,0,1,1)\cdot k_0\left(-\sin\theta\sin\theta_1\cos\theta_2 - \cos\theta\cos\theta_1\right)\\
&= -I(0,-1,2,0,1,1)\cdot\frac{1}{\beta}k_0\left(1-\beta\sin\theta\sin\theta_1\cos\theta_2 - \beta\cos\theta\cos\theta_1 - 1\right)\\
&= -\frac{1}{\beta}I(0,-1,1,0,1,1) + \frac{1}{2\beta}\left(n\cdot k + \bar{n}\cdot k\right)I(0,-1,2,0,1,1)\\
&= -\frac{1}{\beta}M_{13} + \frac{1}{2\beta}\Big(I(-1,-1,2,0,1,1) + I(0,-2,2,0,1,1)\Big)
\end{aligned}$$

The last two integrals in the last line can also be reduced to master integrals which gives

$$I(-1,-1,2,0,1,1) = \frac{1}{2}M_{23}\cdot b\cdot\frac{4-d+\alpha}{c} - \frac{1}{2}M_{13}\frac{a(d+\alpha-4)}{c}$$

and

$$I(0,-2,2,0,1,1) = \frac{1}{2}M_{23}\cdot b^2\frac{4-d+\alpha}{a\cdot c} + \frac{1}{2}M_{13}\frac{(-b\cdot(d+\alpha-4)a + 4c(1+\alpha))}{a\cdot c}$$

where a, b, c are given in eq. (8.14). As one can see the only master integrals appearing in the derivative of M_{13} is M_{13} itself and M_{23}. So one has

$$\begin{aligned}
\frac{d}{d\beta}M_{13} = &\left(-\frac{1}{\beta} + \frac{1}{2\beta}\left(\frac{1}{2}\frac{(-b\cdot(d+\alpha-4)a + 4c(1+\alpha))}{a\cdot c} - \frac{1}{2}\frac{a(d+\alpha-4)}{c}\right)\right)M_{13}\\
&+\frac{1}{2\beta}\left(\frac{1}{2}b\frac{4-d+\alpha}{c} + \frac{1}{2}b^2\frac{4-d+\alpha}{a\cdot c}\right)M_{23}\\
&\equiv c_{11}(\beta)\;M_{13} + c_{12}(\beta)\;M_{23}
\end{aligned}$$

In order to obtain a system of differential equations consider the derivative of M_{23} with respect to β

$$\frac{d}{d\beta}M_{23} = -\frac{1}{\beta}M_{23} + \frac{1}{2\beta}\Big(I(-2,0,2,0,1,1) + I(-1,-1,2,0,1,1)\Big)$$

and converting this expression into master integrals one has

$$\begin{aligned}
\frac{d}{d\beta}M_{23} = &\left(-\frac{1}{\beta} + \frac{1}{2\beta}\left(\frac{1}{2}\frac{(-a(d-\alpha-4)b - 4c(\alpha-1)}{c\cdot b} + \frac{1}{2}\cdot b\cdot\frac{4-d+\alpha}{c}\right)\right)M_{23}\\
&+\frac{1}{2\beta}\left(-\frac{1}{2}\frac{a(d+\alpha-4)}{c} - \frac{1}{2}\frac{a^2(d+\alpha-4)}{c\cdot b}\right)M_{13}\\
&\equiv c_{21}\cdot M_{13} + c_{22}\cdot M_{23}
\end{aligned}$$

and

$$\begin{aligned}
c_{11} &= \frac{\frac{2\left(\beta^2+\epsilon-1\right)-\alpha}{\beta^2-1} + \frac{2(\alpha+1)}{\beta\cos\theta-1}}{2\beta} = \frac{1}{\sec(\theta)-\beta} + \frac{\epsilon}{(\beta-\beta^3)} + \frac{(1-2\beta^2+\beta\cos\theta)}{2(-\beta+\beta^3)(-1+\beta\cos\theta)}\alpha\\
c_{12} &= \frac{(\alpha+2\epsilon)(\beta\cos(\theta)+1)}{2\beta\left(\beta^2-1\right)\left(\beta\cos(\theta)-1\right)} = \frac{\beta\cos(\theta)+1}{2\beta\left(\beta^2-1\right)\left(\beta\cos(\theta)-1\right)}\alpha + \frac{\beta\cos(\theta)+1}{\beta\left(\beta^2-1\right)\left(\beta\cos\theta-1\right)}\epsilon\\
c_{21} &= -\frac{(\alpha-2\epsilon)(\beta\cos\theta-1)}{2\beta\left(\beta^2-1\right)\left(\beta\cos(\theta)+1\right)} = -\frac{\beta\cos(\theta)-1}{2\beta\left(\beta^2-1\right)\left(\beta\cos(\theta)+1\right)}\alpha + \frac{\beta\cos(\theta)-1}{\beta\left(\beta^2-1\right)\left(\beta\cos(\theta)+1\right)}\epsilon\\
c_{22} &= -\frac{1}{\sec(\theta)+\beta} + \frac{\epsilon}{(\beta-\beta^3)} + \frac{\left(\frac{1}{1-\beta^2} - \frac{2}{\beta\cos(\theta)+1}\right)}{2\beta}\alpha
\end{aligned}$$

and so one obtains the differential equation system

$$\frac{d}{d\beta}\vec{M} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \vec{M}$$

and

$$\vec{M} = \begin{pmatrix} M_{13} \\ M_{23} \end{pmatrix}, \qquad C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

In leading order in α and ϵ the matrix C is in a diagonal form namely

$$C_0 = \begin{pmatrix} \frac{1}{\sec(\theta)-\beta} & 0 \\ 0 & -\frac{1}{\sec(\theta)+\beta} \end{pmatrix}$$

and so the leading order can be calculated easily with the differential equation

$$\frac{d}{d\beta}M_{13}^{\alpha} = \frac{1}{\sec(\theta)-\beta}M_{13}^{\alpha}$$

and thus

$$\ln M_{13}^{\alpha} = -\ln(1-\beta\cos\theta) \qquad \Rightarrow \quad M_{13}^{\alpha} = R_1 \cdot \frac{1}{1-\beta\cos\theta} \tag{8.19}$$

and R_1 has to be determined via a boundary condition. For this one evaluates the integral $M_{13}(\beta = 0)$. This can be done via the Mellin-Barnes method. After the k_0-integration one has to evaluate

$$K_{13}(\beta=0) = -\frac{1}{2}\frac{1}{q_T^2}\left(\frac{\mu^2}{q_T^2}\right)^{\epsilon+\alpha/2}\left(\frac{\nu^2}{\mu^2}\right)^{\alpha/2} e^{\epsilon\gamma_E}\int d^{d-1}k\frac{1}{(n\cdot k)^{1-\epsilon+\alpha/2}}\cdot\frac{1}{(\bar{n}\cdot k)^{-\epsilon-\alpha/2}}$$

For the angular integral one obtains with Mellin Barnes

$$\begin{aligned}
K_{angular} =& -\frac{1}{2}e^{\epsilon\gamma_E}\int d^{d-1}k\frac{1}{(n\cdot k)^{1-\epsilon+\alpha/2}}\cdot\frac{1}{(\bar{n}\cdot k)^{-\epsilon-\alpha/2}} \\
=& M_{Bint}\left(\frac{2}{\alpha}+\frac{\pi^2\epsilon}{6},\{\{\alpha\to 0,\epsilon\to 0\},\{\}\}\right), \\
& M_{Bint}\left(\epsilon\Gamma^2(-\mathrm{z})\Gamma(\mathrm{z})\Gamma(\mathrm{z}+1),\left\{\{\alpha\to 0,\epsilon\to 0\},\left\{\mathrm{z}\to -\frac{1}{4}\right\}\right\}\right)
\end{aligned}$$

where this is the output of Mathematica using the Mellin-Barnes package [63] and it was taken into account that at $\beta = 0$

$$n\cdot\tilde{v}_3 = 1$$

Applying the routine "DoAllBarnes" one obtains the result

$$K_{angular} = \frac{2}{\alpha} - \frac{\pi^2\epsilon^2}{6\alpha} + ... \tag{8.20}$$

which can be identified with

$$K_{angular} = \frac{2}{\alpha}\cdot\frac{e^{\epsilon\cdot\gamma_E}}{\Gamma(1-\epsilon)} \tag{8.21}$$

and so

$$K_{13}(\beta=0) = \frac{1}{q_T^2}\cdot\left(\frac{\mu^2}{q_T^2}\right)^{\epsilon+\alpha/2}\cdot\left(\frac{\nu^2}{\mu^2}\right)^{\alpha/2}\frac{2}{\alpha}\cdot\frac{e^{\epsilon\cdot\gamma_E}}{\Gamma(1-\epsilon)}$$

And since

$$M_{13}(\beta=0) = q_T^2 \cdot K_{13}(\beta=0)$$

one has

$$M_{13}(\beta=0) = \left(\frac{\mu^2}{q_T^2}\right)^{\epsilon+\alpha/2} \cdot \left(\frac{\nu^2}{\mu^2}\right)^{\alpha/2} \frac{2}{\alpha} \cdot \frac{e^{\epsilon\cdot\gamma_E}}{\Gamma(1-\epsilon)}$$

Comparing eq. (8.19) with the expression for $M_{13}(\beta=0)$ one obtains

$$\begin{aligned} R_1 &= \frac{2}{\alpha} \cdot \left(\frac{\mu^2}{q_T^2}\right)^{\epsilon+\alpha/2} \left(\frac{\nu^2}{\mu^2}\right)^{\alpha/2} \cdot \frac{e^{\epsilon\cdot\gamma_E}}{\Gamma(1-\epsilon)} \\ \Rightarrow M_{13}^{\alpha} &= \frac{2}{\alpha} \cdot \left(\frac{\mu^2}{q_T^2}\right)^{\epsilon+\alpha/2} \left(\frac{\nu^2}{\mu^2}\right)^{\alpha/2} \frac{e^{\epsilon\cdot\gamma_E}}{\Gamma(1-\epsilon)} \cdot \frac{1}{1-\beta\cos\theta} \equiv R \cdot \frac{2}{\alpha} \cdot \frac{1}{1-\beta\cos\theta} \end{aligned}$$

with the notation

$$R = \left(\frac{\mu^2}{q_T^2}\right)^{\epsilon+\alpha/2} \left(\frac{\nu^2}{\mu^2}\right)^{\alpha/2} \frac{e^{\epsilon\cdot\gamma_E}}{\Gamma(1-\epsilon)}. \tag{8.22}$$

Thus one obtains

$$\tilde{I}_{13}^{(\alpha)} = \frac{(n\cdot\tilde{v}_3)}{q_T^2} \cdot M_{13}^{\alpha} = \frac{(n\cdot\tilde{v}_3)}{q_T^2} R \cdot \frac{2}{\alpha} \cdot \frac{1}{1-\beta\cos\theta} = \frac{1}{q_T^2} \cdot \left(\frac{\mu^2}{q_T^2}\right)^{\epsilon+\alpha/2} \cdot \left(\frac{\nu^2}{\mu^2}\right)^{\alpha/2} \frac{e^{\epsilon\cdot\gamma_E}}{\Gamma(1-\epsilon)} \frac{2}{\alpha}$$

The same can be done for the other master integral M_{23}^{α}. One finds

$$\frac{d}{d\beta} M_{23}^{\alpha} = -\frac{1}{\sec(\theta)+\beta} M_{23}^{\alpha}$$

and thus

$$\ln M_{23}^{\alpha} = -\ln(1+\beta\cos\theta) \qquad \Rightarrow \qquad M_{23}^{\alpha} = R_2 \cdot \frac{1}{1+\beta\cos\theta}$$

and via the boundary condition one obtains the full solution

$$M_{23}^{\alpha} = -R \cdot \frac{2}{\alpha} \cdot \frac{1}{1+\beta\cos\theta}$$

For the derivation of the next order of M_{13} and and M_{23} the coefficients of the matrix C is denoted as

$$\begin{aligned} c_{11} &= a_1 + \epsilon\cdot a_2 + \alpha\cdot a_3 \\ c_{12} &= \alpha\cdot b_1 + \epsilon\cdot b_2 \end{aligned}$$

one can infer from the differential equation derived above the differential equation in M_{13}^1

$$\frac{d}{d\beta} M_{13}^1 = a_1 M_{13}^1 + a_3 M_{13}^{\alpha} + b_1 M_{23}^{\alpha}$$

with

$$a_3 M_{13}^{\alpha} + b_1 M_{23}^{\alpha} = R \cdot \frac{2(\beta-\cos\theta)}{(\beta^2-1)(\beta\cos\theta-1)^2}$$

one has

$$\begin{aligned}\frac{d}{d\beta}M_{13}^{1} &= \frac{1}{\sec(\theta)-\beta}M_{13}^{1} + R\frac{2(\beta-\cos\theta)}{(\beta^2-1)(\beta\cos\theta-1)^2}\\ &= \frac{\cos\theta}{1-\beta\cos\theta}M_{13}^{1} + R\frac{2(\beta-\cos\theta)}{(\beta^2-1)(\beta\cos\theta-1)^2}\end{aligned}$$

In order to solve this differential equation one can substitute

$$M_{13}^{1} = \frac{z}{1-\beta\cos\theta} \qquad \Rightarrow \frac{d}{d\beta}M_{13}^{1} = \frac{\cos\theta}{(1-\beta\cos\theta)^2}z + \frac{1}{1-\beta\cos\theta}\frac{d}{d\beta}z$$

Rewriting the differential equation in M_{13}^{1} in terms of this substitution one obtains

$$\frac{\cos\theta}{(1-\beta\cos\theta)^2}z + \frac{1}{1-\beta\cos\theta}\frac{d}{d\beta}z = \frac{\cos\theta}{(1-\beta\cos\theta)^2}z + R\frac{2(\beta-\cos\theta)}{(\beta^2-1)(\beta\cos\theta-1)^2}$$

which reduces to

$$\frac{d}{d\beta}z = R\frac{2(\beta-\cos\theta)}{(\beta^2-1)(1-\beta\cos\theta)}$$

and this can be integrated and thus one obtains

$$z = -2\cdot R\cdot\ln\frac{1-\beta\cos\theta}{\sqrt{1-\beta^2}} + const$$

and so one has for M_{13}^{1}

$$M_{13}^{1} = \frac{z}{1-\beta\cos\theta} = -2\cdot R\cdot\ln\left(\frac{1-\beta\cos\theta}{\sqrt{1-\beta^2}}\right)\cdot\frac{1}{1-\beta\cos\theta} + \frac{const}{1-\beta\cos\theta}$$

The constant term can be evaluated by the boundary condition at $\beta = 0$. Above I found that

$$M_{13}(\beta=0) = R\cdot\frac{2}{\alpha}$$

and so one finds that this constant term has to be equal to zero and thus

$$M_{13}^{1} = \frac{e^{\epsilon\cdot\gamma_E}}{\Gamma(1-\epsilon)}\left(\frac{\mu^2}{q_T^2}\right)^{\epsilon+\alpha/2}\left(\frac{\nu^2}{\mu^2}\right)^{\alpha/2}\left(-2\ln\left(\frac{1-\beta\cos\theta}{\sqrt{1-\beta^2}}\right)\cdot\frac{1}{1-\beta\cos\theta}\right)$$

In the same way M_{23}^{1} can be calculated which gives

$$M_{23}^{1} = \frac{e^{\epsilon\cdot\gamma_E}}{\Gamma(1-\epsilon)}\left(\frac{\mu^2}{q_T^2}\right)^{\epsilon+\alpha/2}\left(\frac{\nu^2}{\mu^2}\right)^{\alpha/2}\left(-2\ln\left(\frac{1+\beta\cos\theta}{\sqrt{1-\beta^2}}\right)\cdot\frac{1}{1+\beta\cos\theta}\right)$$

with this knowledge a differential equation for the order ϵ contribution can be derived which gives

$$\begin{aligned}\frac{d}{d\beta}M_{13}^{\epsilon} &= a_2\cdot M_{13}^{1} + b_2\cdot M_{23}^{1} + a_1\cdot M_{13}^{\epsilon}\\ &= -\frac{2\left(\ln\left(\frac{\beta\cos\theta+1}{\sqrt{1-\beta^2}}\right)+\ln\left(-\frac{\beta\cos\theta-1}{\sqrt{1-\beta^2}}\right)\right)}{(\beta^3-\beta)(\beta\cos\theta-1)}\cdot R + \frac{\cos\theta}{1-\beta\cos\theta}M_{13}^{\epsilon}\end{aligned}$$

with the same substitution as for M_{13}^1 one obtains the differential equation

$$\frac{d}{d\beta}y = \frac{2\left(\ln\left(\frac{\beta\cos\theta+1}{\sqrt{1-\beta^2}}\right)+\ln\left(-\frac{\beta\cos\theta-1}{\sqrt{1-\beta^2}}\right)\right)}{(\beta^3-\beta)}\cdot R = \frac{2\ln\left(1+\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)}{\beta^3-\beta}\cdot R$$

and so one obtains

$$y = R\cdot \mathrm{Li}_2\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)$$

where the function $\mathrm{Li}_2(z)$ is the dilogarithm defined as

$$\mathrm{Li}_2(z) = -\int_0^z dt\,\frac{\ln(1-t)}{t}$$

And thus one obtains

$$M_{13}^{\epsilon} = \frac{y}{1-\beta\cos\theta} = \frac{1}{1-\beta\cos\theta}\cdot R\cdot \mathrm{Li}_2\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)$$

M_{23}^{ϵ} can be derived with the second differential equation, namely

$$\frac{d}{d\beta}M_{23} = c_{21}\cdot M_{13} + c_{22}\cdot M_{23}$$

and the coefficients c_{21} and c_{22}. One obtains

$$M_{23}^{\epsilon} = R\cdot\frac{1}{1+\beta\cos\theta}\mathrm{Li}_2\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)$$

For the $\mathcal{O}(\epsilon^2)$ terms the following equations are holding

$$\begin{aligned}\frac{d}{d\beta}M_{13}^{\epsilon^2} &= a_1 M_{13}^{\epsilon^2} + a_2 M_{13}^{\epsilon} + b_2 M_{23}^{\epsilon}\\ &= \frac{1}{\sec(\theta)-\beta}M_{13}^{\epsilon^2} + \frac{1}{(\beta-\beta^3)}M_{13}^{\epsilon} + \frac{\beta\cos(\theta)+1}{\beta\left(\beta^2-1\right)\left(\beta\cos\theta-1\right)}M_{23}^{\epsilon}\end{aligned}$$

and

$$\frac{d}{d\beta}M_{23}^{\epsilon^2} = -\frac{1}{\sec(\theta)+\beta}M_{23}^{\epsilon^2} + \frac{1}{(\beta-\beta^3)}M_{23}^{\epsilon} + \frac{\beta\cos(\theta)-1}{\beta\left(\beta^2-1\right)\left(\beta\cos(\theta)+1\right)}M_{13}^{\epsilon}$$

applying the substitution

$$\begin{aligned}M_{13}^{\epsilon^2} &= \frac{1}{1-\beta\cos\theta}A\\ M_{23}^{\epsilon^2} &= \frac{1}{1+\beta\cos\theta}B\end{aligned}$$

one has the two differential equations

$$\begin{aligned}\frac{d}{d\beta}A &= \frac{2}{\beta\left(1-\beta^2\right)}\cdot R\cdot \mathrm{Li}_2\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)\\ \frac{d}{d\beta}B &= \frac{2}{\beta\left(1-\beta^2\right)}\cdot R\cdot \mathrm{Li}_2\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)\end{aligned}$$

This can be calculated in the following way

$$A = B = R\int_0^\beta d\tilde{\beta}\ \frac{2}{\tilde{\beta}(1-\tilde{\beta}^2)}\mathrm{Li}_2\left(-\frac{\tilde{\beta}^2}{1-\tilde{\beta}^2}\sin^2\theta\right)$$

in order to solve this, one performs the following substitution

$$z = -\frac{\tilde{\beta}^2}{1-\tilde{\beta}^2}\sin^2\theta \quad \Rightarrow \quad \frac{dz}{d\tilde{\beta}} = -\frac{2\tilde{\beta}}{(1-\tilde{\beta}^2)^2}\sin^2\theta \quad \Rightarrow \quad d\tilde{\beta} = -\frac{(1-\tilde{\beta}^2)^2}{2\tilde{\beta}\sin^2\theta}dz$$

and so it remains to solve

$$\begin{aligned}
A &= -R\int_0^{\tilde{z}} dz\ \frac{(1-\tilde{\beta}^2)^2}{2\tilde{\beta}\sin^2\theta}\ \frac{2}{\tilde{\beta}(1-\tilde{\beta}^2)}\mathrm{Li}_2(z) \\
&= -R\int_0^{\tilde{z}} dz\frac{1-\tilde{\beta}^2}{\tilde{\beta}^2\sin^2\theta}\mathrm{Li}_2(z) \\
&= R\int_0^{\tilde{z}} dz\frac{1}{z}\mathrm{Li}_2(z) = R\cdot\mathrm{Li}_3(\tilde{z}) \\
&= R\cdot\mathrm{Li}_3\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)
\end{aligned}$$

and one has

$$A(\beta = 0) = 0$$

which coincides with the boundary condition from above.
With R given in eq. (8.22) one has in total

$$M_{13} = R\cdot\frac{1}{1-\beta\cos\theta}\left(\frac{2}{\alpha} - 2\ln\frac{1-\beta\cos\theta}{\sqrt{1-\beta^2}} + \epsilon\cdot\mathrm{Li}_2\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right) + \epsilon^2\mathrm{Li}_3\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)\right)$$

and therefore

$$\begin{aligned}
\tilde{I}_{13} &= \frac{n\cdot\tilde{v}_3}{q_T^2}\cdot M_{13} = R\cdot\frac{1}{q_T^2}\left(\frac{2}{\alpha} - 2\ln\frac{1-\beta\cos\theta}{\sqrt{1-\beta^2}} + \epsilon\cdot\mathrm{Li}_2\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right) + \epsilon^2\mathrm{Li}_3\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)\right) \\
&= \frac{e^{\epsilon\cdot\gamma_E}}{\Gamma(1-\epsilon)}\frac{1}{q_T^2}\left(\frac{\mu^2}{q_T^2}\right)^{\epsilon+\alpha/2}\left(\frac{\nu^2}{\mu^2}\right)^{\alpha/2}\left(\frac{2}{\alpha} - 2\ln\frac{1-\beta\cos\theta}{\sqrt{1-\beta^2}} + \epsilon\cdot\mathrm{Li}_2\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right) + \epsilon^2\mathrm{Li}_3\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)\right)
\end{aligned}$$

and

$$M_{23} = R\cdot\frac{1}{1+\beta\cos\theta}\left(-\frac{2}{\alpha} - 2\ln\frac{1+\beta\cos\theta}{\sqrt{1-\beta^2}} + \epsilon\cdot\mathrm{Li}_2\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right) + \epsilon^2\mathrm{Li}_3\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)\right)$$

and thus

$$\begin{aligned}
\tilde{I}_{23} &= \frac{\bar{n}\cdot\tilde{v}_3}{q_T^2}\cdot M_{23} = R\cdot\frac{1}{q_T^2}\left(-\frac{2}{\alpha} - 2\ln\frac{1+\beta\cos\theta}{\sqrt{1-\beta^2}} + \epsilon\cdot\mathrm{Li}_2\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right) + \epsilon^2\mathrm{Li}_3\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)\right) \\
&= \frac{e^{\epsilon\cdot\gamma_E}}{\Gamma(1-\epsilon)}\cdot\frac{1}{q_T^2}\left(\frac{\mu^2}{q_T^2}\right)^{\epsilon+\alpha/2}\left(\frac{\nu^2}{\mu^2}\right)^{\alpha/2}\left(-\frac{2}{\alpha} - 2\ln\frac{1+\beta\cos\theta}{\sqrt{1-\beta^2}} + \epsilon\cdot\mathrm{Li}_2\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right) + \epsilon^2\mathrm{Li}_3\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)\right)
\end{aligned}$$

As a test of consistency the results of M_{13} and M_{23} can be inserted in the differential equation system and one can check whether the equation is fulfilled. The differential equation system was given by

$$\begin{aligned}\frac{d}{d\beta}M_{13} &= c_{11}M_{13} + c_{12}M_{23} \\ \frac{d}{d\beta}M_{23} &= c_{21}M_{13} + c_{22}M_{23}\end{aligned}$$

The left hand side has the following structure

$$\frac{d}{d\beta}M_{i3} = D1_i \cdot \frac{1}{\alpha} + D2_i + D3_i \cdot \epsilon + D4_i \cdot \epsilon^2$$

and the differential system has to be fulfilled order by order. To check that the coefficients are written in the form

$$\begin{aligned}c_{11} &= a_1 + \epsilon \cdot a_2 + \alpha \cdot a_3 \\ c_{12} &= \alpha \cdot b_1 + \epsilon \cdot b_2 \\ c_{21} &= \alpha \cdot c_1 + \epsilon \cdot c_2 \\ c_{22} &= d_1 + \epsilon \cdot d_2 + \alpha \cdot d_3\end{aligned}$$

and we had

$$\begin{aligned}M_{13} &= M_{13}^{\alpha}\frac{1}{\alpha} + M_{13}^{1} + \epsilon \cdot M_{13}^{\epsilon} + \epsilon^2 \cdot M_{13}^{\epsilon^2} \\ M_{23} &= M_{23}^{\alpha}\frac{1}{\alpha} + M_{23}^{1} + \epsilon \cdot M_{23}^{\epsilon} + \epsilon^2 \cdot M_{23}^{\epsilon^2}\end{aligned}$$

so the following equations have to satisfied

$$\begin{aligned}\mathcal{O}\left(\alpha^{-1}\right) &\begin{cases} D1_1 - M_{13}^{\alpha} \cdot a_1 & = 0 \\ D1_2 - M_{23}^{\alpha} \cdot d_1 & = 0 \end{cases} \\ \mathcal{O}\left(1\right) &\begin{cases} D2_1 - M_{13}^{\alpha} \cdot a_3 - M_{13}^{1} \cdot a_1 - M_{23}^{\alpha} \cdot b_1 & = 0 \\ D2_2 - M_{23}^{\alpha} \cdot d_3 - M_{23}^{1} \cdot d_1 - M_{13}^{\alpha} \cdot c_1 & = 0 \end{cases} \\ \mathcal{O}\left(\epsilon\right) &\begin{cases} D3_1 - a_1 \cdot M_{13}^{\epsilon} - a_2 \cdot M_{13}^{1} - b_2 \cdot M_{23}^{1} & = 0 \\ D3_2 - d_1 \cdot M_{23}^{\epsilon} - d_2 \cdot M_{23}^{1} - c_2 \cdot M_{13}^{1} & = 0 \end{cases} \\ \mathcal{O}\left(\epsilon^2\right) &\begin{cases} D4_1 - a_1 \cdot M_{13}^{\epsilon^2} - a_2 \cdot M_{13}^{\epsilon} - b_2 \cdot M_{23}^{\epsilon} & = 0 \\ D4_2 - d_1 \cdot M_{23}^{\epsilon^2} - d_2 \cdot M_{23}^{\epsilon} - c_2 \cdot M_{13}^{\epsilon} & = 0 \end{cases}\end{aligned}$$

and with the above expressions these equations are fulfilled. So I haven't done any mistake in the derivation. As a second check, the differential equation system

$$\frac{d}{d\beta}\vec{M} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}\vec{M}$$

can be used to derive a second order differential equation for M_{13} or M_{23}. One obtains for example

$$M_{13}'' - \underbrace{\left(c_{11} + \frac{c_{12}'}{c_{12}} + c_{22}\right)}_{h_1} M_{13}' - \underbrace{\left(c_{11}' - \frac{c_{11}c_{12}'}{c_{12}} + c_{12}c_{21} - c_{22}c_{11}\right)}_{h_0} M_{13} = 0$$

with

$$\begin{aligned}
h_1 =& \frac{-5\beta^3\cos\theta + 3\beta^2 + 3\beta\cos\theta - 1}{\beta\left(\beta^2-1\right)\left(\beta\cos\theta-1\right)} - \frac{2\epsilon}{\beta\left(\beta^2-1\right)} - \frac{2\alpha\cos\theta}{(\beta\cos\theta-1)(\beta\cos\theta+1)} \\
=& u_1 + u_2\cdot\epsilon + u_3\cdot\alpha \\
h_0 =& \frac{\cos\theta - 3\beta^2\cos\theta}{\left(\beta^3-\beta\right)\left(\beta\cos\theta-1\right)} - \frac{2\epsilon\cos\theta}{\left(\beta^3-\beta\right)\left(\beta\cos\theta-1\right)} \\
&+ \left(\frac{-3\beta^2\cos\theta - 2\beta + \cos\theta}{\left(\beta^3-\beta\right)\left(\beta^2\cos^2\theta-1\right)} - \frac{2\epsilon\cos\theta}{\left(\beta^3-\beta\right)\left(\beta^2\cos^2\theta-1\right)}\right)\alpha + \frac{\alpha^2}{-\left(\beta^2-1\right)\beta^2\cos^2\theta + \beta^2 - 1} \\
=& v_1 + v_2\cdot\epsilon + v_3\cdot\alpha + v_4\cdot\alpha\cdot\epsilon + v_5\cdot\alpha^2
\end{aligned}$$

and writing as well M''_{13} and M'_{13} in powers of α and ϵ as

$$\begin{aligned}
M'_{13} &= D1_1\cdot\frac{1}{\alpha} + D2_1 + D3_1\cdot\epsilon + D4_1\cdot\epsilon^2 \\
M''_{13} &= E1_1\cdot\frac{1}{\alpha} + E2_1 + E3_1\cdot\epsilon + E4_1\cdot\epsilon^2
\end{aligned}$$

So according to the second order differential equation the following identities must hold

$$\begin{aligned}
\mathcal{O}\left(\alpha^{-1}\right):\Big\{\ E1_1 - u_1\cdot D1_1 - v_1\cdot M^{\alpha}_{13} &= 0 \\
\mathcal{O}(1):\Big\{\ E2_1 - u_1\cdot D2_1 - u_3\cdot D1_1 - v_3\cdot M^{\alpha}_{13} - v_1\cdot M^{1}_{13} &= 0 \\
\mathcal{O}(\epsilon):\Big\{\ E3_1 - u_1\cdot D3_1 - u_2\cdot D2_1 - v_2\cdot M^{1}_{13} - v_4\cdot M^{\alpha}_{13} - v_1\cdot M^{\epsilon}_{13} &= 0 \\
\mathcal{O}\left(\epsilon^2\right):\Big\{\ E4_1 - u_1 D4_1 - u_2\cdot D3_1 - v_1 M^{\epsilon^2}_{13} - v_2 M^{\epsilon}_{13} &= 0
\end{aligned}$$

which are confirmed by inserting the above expressions. Of course the same procedure can be done for M_{23}. Knowing the expressions for M_{13} and M_{23} it is easy to calculate I_{33} which is reduced by A.I.R. to

$$\begin{aligned}
\tilde{I}_{33} =& \tilde{v}_3^2\cdot I(0,0,2,0,1,1) \\
=& \frac{\tilde{v}_3^2}{q_T^2}\left(\frac{(\alpha+2\epsilon)(\beta\cos(t)+1)}{2\left(1-\beta^2\right)}M_{23} - \frac{(\alpha-2\epsilon)(1-\beta\cos(t))}{2\left(1-\beta^2\right)}M_{13}\right) \\
=& \frac{1}{q_T^2}\left(\frac{(\alpha+2\epsilon)(\beta\cos(t)+1)}{2}M_{23} - \frac{(\alpha-2\epsilon)(1-\beta\cos(t))}{2}M_{13}\right) \\
=& \frac{1}{q_T^2}\cdot R\cdot\frac{(\alpha+2\epsilon)}{2}\left(-\frac{2}{\alpha} - 2\ln\frac{1+\beta\cos\theta}{\sqrt{1-\beta^2}} + \epsilon\cdot\mathrm{Li}_2\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)\right) \\
&- \frac{1}{q_T^2}\cdot R\cdot\frac{(\alpha-2\epsilon)}{2}\left(\frac{2}{\alpha} - 2\ln\frac{1-\beta\cos\theta}{\sqrt{1-\beta^2}} + \epsilon\cdot\mathrm{Li}_2\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)\right) \\
=& \frac{e^{\epsilon\cdot\gamma_E}}{\Gamma(1-\epsilon)}\frac{1}{q_T^2}\left(\frac{\mu^2}{q_T^2}\right)^{\epsilon+\alpha/2}\left(\frac{\nu^2}{\mu^2}\right)^{\alpha/2}\left(-2 + 2\epsilon\cdot\ln\frac{1-\beta^2}{1-\beta^2\cos^2\theta} + 2\epsilon^2\mathrm{Li}_3\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right) + \mathcal{O}(\epsilon^3)\right)
\end{aligned}$$

8.3.3. M_{14}, M_{24} and M_{44}

The same procedure can be done for M_{14}, M_{24} and M_{44}. As before one has two master integrals called M_{14} and M_{24} which are

$$\begin{aligned}
\tilde{I}_{14} &\equiv \frac{(n\cdot\tilde{v}_4)}{q_T^2}M_{14} \\
\tilde{I}_{24} &\equiv \frac{(\bar{n}\cdot\tilde{v}_4)}{q_T^2}M_{24}
\end{aligned} \tag{8.23}$$

as before these integrals are determined via a differential equation. One has

$$\frac{d}{d\beta}M_{14} = -\frac{1}{\beta}M_{14} + \frac{1}{2\beta}\left(I(-1,-1,0,2,1,1) + I(0,-2,0,2,1,1)\right)$$

with A.I.R. one obtains

$$\begin{aligned}
I(-1,-1,0,2,1,1) &= \frac{M_{14}(\alpha-2\epsilon)(-\beta\cos(\theta)-1)}{2\left(1-\beta^2\right)} + \frac{M_{24}(\alpha+2\epsilon)(1-\beta\cos(\theta))}{2\left(1-\beta^2\right)} \\
I(0,-2,0,2,1,1) &= \frac{-\alpha\beta^2\cos^2(\theta) + 2\alpha\beta\cos(\theta) - \alpha - 2\beta^2\epsilon\cos^2(\theta) + 4\beta\epsilon\cos(\theta) - 2\epsilon}{2\left(\beta^2-1\right)\left(\beta\cos(\theta)+1\right)}M_{24} \\
&+ \frac{-\alpha\beta^2\cos^2(\theta) + 4\alpha\beta^2 - 3\alpha + 4\beta^2 + 2\beta^2\epsilon\cos^2(\theta) - 2\epsilon - 4}{2\left(\beta^2-1\right)\left(\beta\cos(\theta)+1\right)}M_{14}
\end{aligned}$$

So one has in total

$$\begin{aligned}
\frac{d}{d\beta}M_{14} &= M_{14}\left(\frac{\frac{2(\alpha+1)}{\beta\cos(\theta)+1} + \frac{\alpha-2\left(\beta^2+\epsilon-1\right)}{\beta^2-1}}{2\beta}\right) + M_{24}\left(\frac{(\alpha+2\epsilon)(\beta\cos(\theta)-1)}{2\beta\left(\beta^2-1\right)\left(\beta\cos(\theta)+1\right)}\right) \\
&\equiv c_{33}M_{14} + c_{34}M_{24}
\end{aligned}$$

and

$$\frac{d}{d\beta}M_{24} = c_{44}M_{24} + c_{43}M_{14}$$

with

$$\begin{aligned}
c_{33} &= -\frac{\cos(\theta)}{\beta\cos(\theta)+1} + \alpha\left(\frac{1}{2\beta\left(\beta^2-1\right)} + \frac{1}{\beta(\beta\cos(\theta)+1)}\right) - \frac{\epsilon}{\beta\left(\beta^2-1\right)} \\
c_{34} &= \frac{\alpha(\beta\cos(\theta)-1)}{2\beta\left(\beta^2-1\right)\left(\beta\cos(\theta)+1\right)} + \frac{\epsilon(\beta\cos(\theta)-1)}{\beta\left(\beta^2-1\right)\left(\beta\cos(\theta)+1\right)} \\
c_{43} &= \frac{\alpha(-\beta\cos(\theta)-1)}{2\beta\left(\beta^2-1\right)\left(\beta\cos(\theta)-1\right)} + \frac{\epsilon(\beta\cos(\theta)+1)}{\beta\left(\beta^2-1\right)\left(\beta\cos(\theta)-1\right)} \\
c_{44} &= \alpha\left(\frac{1}{\beta(\beta\cos(\theta)-1)} - \frac{1}{2\beta\left(\beta^2-1\right)}\right) - \frac{\cos(\theta)}{\beta\cos(\theta)-1} - \frac{\epsilon}{\beta\left(\beta^2-1\right)}
\end{aligned}$$

The expressions c_{33} to c_{44} can be compared with c_{11} to c_{22} and one finds a similarity between those! M_{14} and M_{24} is written as

$$\begin{aligned}
M_{14} &= \frac{1}{\alpha}M_{14}^{\alpha} + M_{14}^{1} + \epsilon M_{14}^{\epsilon} + \epsilon^2 M_{14}^{\epsilon^2} \\
M_{24} &= \frac{1}{\alpha}M_{24}^{\alpha} + M_{24}^{1} + \epsilon M_{24}^{\epsilon} + \epsilon^2 M_{24}^{\epsilon^2}
\end{aligned}$$

To leading order one has

$$\begin{aligned}
M_{14}^{\alpha} &= C_3 \cdot \frac{1}{\beta\cos(\theta)+1} \\
M_{24}^{\alpha} &= C_4 \cdot \frac{1}{1-\beta\cos(\theta)}
\end{aligned}$$

with Mellin Barnes one obtains

$$\begin{aligned}
C_3 &= \frac{2}{\alpha}\cdot\left(\frac{\mu^2}{q_T^2}\right)^{\epsilon+\alpha/2}\left(\frac{\nu^2}{\mu^2}\right)^{\alpha/2}\frac{e^{\epsilon\cdot\gamma_E}}{\Gamma(1-\epsilon)} = \frac{2}{\alpha}\cdot R \\
C_4 &= -\frac{2}{\alpha}\cdot\left(\frac{\mu^2}{q_T^2}\right)^{\epsilon+\alpha/2}\left(\frac{\nu^2}{\mu^2}\right)^{\alpha/2}\frac{e^{\epsilon\cdot\gamma_E}}{\Gamma(1-\epsilon)} = -\frac{2}{\alpha}\cdot R
\end{aligned}$$

As for M_{13} and M_{23} the full solution of M_{14} and M_{24} is evaluated step by step. In the end one has

$$M_{14} = R \cdot \frac{1}{\beta\cos(\theta)+1}\left(\frac{2}{\alpha} - 2\ln\frac{1+\beta\cos\theta}{\sqrt{1-\beta^2}} + \epsilon\cdot\mathrm{Li}_2\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right) + \epsilon^2\mathrm{Li}_3\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)\right)$$

and

$$M_{24} = R \cdot \frac{1}{1-\beta\cos(\theta)}\left(-\frac{2}{\alpha} - 2\ln\frac{1-\beta\cos\theta}{\sqrt{1-\beta^2}} + \epsilon\cdot\mathrm{Li}_2\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right) + \epsilon^2\mathrm{Li}_3\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)\right)$$

Via the eq. (8.23) and (8.22) the expressions for $\tilde{I}_{14}$ and $\tilde{I}_{24}$ are obtained. With the knowledge of M_{14} and M_{24} it is easy to calculate I_{44} which is given by

$$\begin{aligned}
\tilde{I}_{44} &= \tilde{v}_4^2 I(0,0,0,2,1,1)\\
&= \frac{\tilde{v}_4^2}{q_T^2}\left(\frac{(\alpha-2\epsilon)(-\beta\cos(\theta)-1)}{2\,(1-\beta^2)}M_{14} + \frac{(\alpha+2\epsilon)(1-\beta\cos(\theta))}{2\,(1-\beta^2)}M_{24}\right)\\
&= \frac{e^{\epsilon\cdot\gamma_E}}{\Gamma(1-\epsilon)}\frac{1}{q_T^2}\left(\frac{\mu^2}{q_T^2}\right)^{\epsilon+\alpha/2}\left(\frac{\nu^2}{\mu^2}\right)^{\alpha/2}\left(-2+2\epsilon\ln\frac{1-\beta^2}{1-\beta^2\cos^2\theta} + 2\epsilon^2\mathrm{Li}_3\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)\right) = \tilde{I}_{33}
\end{aligned}$$

8.3.4. M_{34}

As a last step it remains to solve the integral $\tilde{I}_{34}$ which is given by

$$\tilde{I}_{34} = (\tilde{v}_3\cdot\tilde{v}_4)\; I(0,0,1,1,1,1) \equiv (\tilde{v}_3\cdot\tilde{v}_4)\cdot M_{34}$$

Again one can derive the solution via a differential equation. First consider the derivative with respect to β

$$\frac{d}{d\beta}M_{34} = \frac{2}{\beta}I(0,0,2,0,1,1) - \frac{1}{\beta}I(0,0,2,1,1,1)\cdot k_0 - \frac{1}{\beta}I(0,0,1,2,1,1)\cdot k_0$$

k_0 can be replaced by

$$k_0 = \frac{\tilde{v}_3\cdot k + \tilde{v}_4\cdot k}{2}$$

and so one ends up with

$$\frac{d}{d\beta}M_{34} = \frac{1}{\beta}I(0,0,2,0,1,1) - \frac{1}{\beta}M_{34}$$

now transforming

$$M_{34} = \frac{1}{\beta}J$$

one obtains the differential equation in J which is

$$\begin{aligned}
\frac{d}{d\beta}J = I(0,0,2,0,1,1) = &\underbrace{\frac{e^{\epsilon\cdot\gamma_E}}{\Gamma(1-\epsilon)}\frac{1}{q_T^2}\left(\frac{\mu^2}{q_T^2}\right)^{\epsilon+\alpha/2}\left(\frac{\nu^2}{\mu^2}\right)^{\alpha/2}}_{\tilde{R}}\frac{1}{1-\beta^2}\\
&\times\left(-2+2\epsilon\ln\frac{1-\beta^2}{1-\beta^2\cos^2\theta} + 2\epsilon^2\mathrm{Li}_3\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)\right)
\end{aligned}$$

with

$$\tilde{R} = \frac{1}{q_T^2}\cdot R = \frac{e^{\epsilon\cdot\gamma_E}}{\Gamma(1-\epsilon)}\frac{1}{q_T^2}\left(\frac{\mu^2}{q_T^2}\right)^{\epsilon+\alpha/2}\left(\frac{\nu^2}{\mu^2}\right)^{\alpha/2} \tag{8.24}$$

and the solutions of this differential equation is to order $\mathcal{O}(\epsilon^0)$

$$\begin{aligned} J^{(0)} &= \tilde{R} \int \frac{-2}{1-\beta^2}\, d\beta = 2\left(\frac{1}{2}\ln(1-\beta) - \frac{1}{2}\ln(\beta+1)\right) \\ &= \tilde{R} \ln \frac{1-\beta}{1+\beta} \end{aligned}$$

To order $\mathcal{O}(\epsilon)$ one has to calculate

$$J^{(1)} = \tilde{R} \int d\beta \; 2 \ln\left(\frac{1-\beta^2}{1-\beta^2\cos^2\theta}\right) \frac{1}{1-\beta^2}$$

To integrate this one can rewrite the logarithm as

$$\frac{1-\beta^2\cos^2\theta}{1-\beta^2} = \frac{(2\beta\cos(\theta)+2)}{(\beta\cos(\theta)+\beta+\cos(\theta)+1)} \cdot \frac{\sec^2\left(\frac{\theta}{2}\right)\cdot(\beta\cos(\theta)-1)}{(\beta-1)} \cdot \frac{1}{\sec^4\left(\frac{\theta}{2}\right)}$$

So one has to evaluate

$$\begin{aligned} J^{(1)} = -2\tilde{R} \int d\beta \frac{1}{1-\beta^2} \Bigg(&\ln\left(\frac{(2\beta\cos(\theta)+2)}{(\beta\cos(\theta)+\beta+\cos(\theta)+1)}\right) \\ &+ \ln\left(\frac{\sec^2\left(\frac{\theta}{2}\right)\cdot(\beta\cos(\theta)-1)}{(\beta-1)}\right) - 4\ln\left(\sec\left(\frac{\theta}{2}\right)\right)\Bigg) \end{aligned}$$

The last contribution of this equation is easy to calculate which gives

$$J_3^{(1)} = 8\tilde{R} \int d\beta \; \ln\left(\sec\left(\frac{\theta}{2}\right)\right) \frac{1}{1-\beta^2} = 4 \cdot \ln\sec\left(\frac{\theta}{2}\right) \ln\frac{1+\beta}{1-\beta}$$

The first part is

$$J_1^{(1)} = -2\tilde{R} \int_0^\beta d\beta \ln\left(\frac{2\beta\cos(\theta)+2}{\beta\cos(\theta)+\beta+\cos(\theta)+1}\right) \cdot \frac{1}{1-\beta^2}$$

This expression is rewritten as

$$\frac{2\beta\cos(\theta)+2}{\beta\cos(\theta)+\beta+\cos(\theta)+1} = 1 - \frac{\beta-1}{\beta+1}\tan^2\frac{\theta}{2}$$

and then one substitutes

$$y = \frac{\beta-1}{\beta+1}\tan^2\frac{\theta}{2} \quad \Rightarrow \beta = -\frac{(y-1)\cos(\theta)+y+1}{(y+1)\cos(\theta)+y-1} \quad \Rightarrow d\beta = \left(\frac{2\tan^2\left(\frac{\theta}{2}\right)}{(\beta+1)^2}\right)^{-1} dy$$

and performing this substitution one obtains

$$\begin{aligned} J_1^{(1)} &= -2\tilde{R} \int_0^\beta d\beta \, \frac{1}{1-\beta^2} \ln\left(1 - \frac{\beta-1}{\beta+1}\tan^2\frac{\theta}{2}\right) \\ &= \tilde{R} \int_{-\tan^2\frac{\theta}{2}}^{\frac{\beta-1}{\beta+1}\tan^2\frac{\theta}{2}} dy \, \frac{1}{y} \; \ln(1-y) \\ &= -\tilde{R}\cdot \mathrm{Li}_2\left(\frac{\beta-1}{\beta+1}\tan^2\frac{\theta}{2}\right) + \tilde{R}\cdot\mathrm{Li}_2\left(-\tan^2\frac{\theta}{2}\right) \end{aligned}$$

The last integral is

$$J_2^{(1)} = -2\tilde{R} \cdot \int_0^\beta d\beta \; \ln \left(\frac{\sec^2\left(\frac{\theta}{2}\right) \cdot (\beta \cos(\theta) - 1)}{(\beta - 1)} \right) \cdot \frac{1}{1 - \beta^2}$$

which is rewritten as

$$\ln \left(\frac{\sec^2\left(\frac{\theta}{2}\right) \cdot (\beta \cos(\theta) - 1)}{(\beta - 1)} \right) = \ln \left(1 - \frac{(\beta + 1) \tan^2\left(\frac{\theta}{2}\right)}{\beta - 1} \right)$$

substituting

$$z = \frac{(\beta + 1) \tan^2\left(\frac{\theta}{2}\right)}{\beta - 1} \quad \Rightarrow \quad \frac{dz}{d\beta} = -\frac{2 \tan^2\left(\frac{\theta}{2}\right)}{(\beta - 1)^2}$$

and

$$\beta = \frac{(z - 1) \cos(\theta) + z + 1}{(z + 1) \cos(\theta) + z - 1}$$

one obtains

$$\begin{aligned} J_2^{(1)} &= -2\tilde{R} \cdot \int_0^\beta d\beta \; \ln \left(\frac{\sec^2\left(\frac{\theta}{2}\right) \cdot (\beta \cos(\theta) - 1)}{(\beta - 1)} \right) \cdot \frac{1}{1 - \beta^2} \\ &= -\tilde{R} \cdot \int_{-\tan^2 \frac{\theta}{2}}^{\frac{\beta+1}{\beta-1} \tan^2 \frac{\theta}{2}} dz \; \frac{1}{z} \; \ln(1 - z) \\ &= \tilde{R} \cdot \mathrm{Li}_2\left(\frac{\beta + 1}{\beta - 1} \tan^2 \frac{\theta}{2} \right) - \tilde{R} \cdot \mathrm{Li}_2\left(-\tan^2 \frac{\theta}{2} \right) \end{aligned}$$

and thus all together one ends up with

$$\begin{aligned} J^{(1)} &= J_1^{(1)} + J_2^{(1)} + J_3^{(1)} \\ &= \tilde{R} \cdot \left(\mathrm{Li}_2\left(\frac{\beta + 1}{\beta - 1} \tan^2 \frac{\theta}{2} \right) - \mathrm{Li}_2\left(-\tan^2 \frac{\theta}{2} \right) \right. \\ &\qquad \left. - \mathrm{Li}_2\left(\frac{\beta - 1}{\beta + 1} \tan^2 \frac{\theta}{2} \right) + \mathrm{Li}_2\left(-\tan^2 \frac{\theta}{2} \right) + 4 \cdot \ln \sec\left(\frac{\theta}{2} \right) \ln \frac{1 + \beta}{1 - \beta} \right) \\ &= \tilde{R} \cdot \left(\mathrm{Li}_2\left(\frac{\beta + 1}{\beta - 1} \tan^2 \frac{\theta}{2} \right) - \mathrm{Li}_2\left(\frac{\beta - 1}{\beta + 1} \tan^2 \frac{\theta}{2} \right) + 4 \cdot \ln \sec\left(\frac{\theta}{2} \right) \ln \frac{1 + \beta}{1 - \beta} \right) \end{aligned}$$

So one obtains up to $\mathcal{O}(\epsilon)$

$$\begin{aligned} \tilde{I}_{34} &= \frac{1 + \beta^2}{\beta} J \\ &= \frac{e^{\epsilon \cdot \gamma_E}}{\Gamma(1 - \epsilon)} \frac{1}{q_T^2} \left(\frac{\mu^2}{q_T^2} \right)^{\epsilon + \alpha/2} \left(\frac{\nu^2}{\mu^2} \right)^{\alpha/2} \frac{1 + \beta^2}{\beta} \left(\ln \frac{1 - \beta}{1 + \beta} + \epsilon f_{34} + C \right) \end{aligned}$$

with

$$f_{34} = \left(\mathrm{Li}_2\left(\frac{\beta + 1}{\beta - 1} \tan^2 \frac{\theta}{2} \right) - \mathrm{Li}_2\left(\frac{\beta - 1}{\beta + 1} \tan^2 \frac{\theta}{2} \right) + 4 \cdot \ln \sec\left(\frac{\theta}{2} \right) \ln \frac{1 + \beta}{1 - \beta} \right)$$

To determine the constant C one expands the expression in $\beta \to 0$. One obtains

$$\frac{1}{\beta} f_{34}(\beta \to 0) = 0$$

and

$$\frac{1}{\beta}\ln\frac{1-\beta}{1+\beta} = -2 + \mathcal{O}(\beta^2)$$

So one has

$$\tilde{I}_{34}(\beta\to 0) = -2\frac{1}{q_T^2}\left(\frac{\mu^2}{q_T^2}\right)^{\epsilon+\alpha/2}\left(\frac{\nu^2}{\mu^2}\right)^{\alpha/2} + C$$

To determine C one can calculate $I_{34}(\beta = 0)$ with Mellin-Barnes. This gives

$$\begin{aligned}\tilde{I}_{34}(\beta = 0) &= -\frac{1}{2}\frac{1}{q_T^2}\left(\frac{\mu^2}{q_T^2}\right)^{\epsilon+\alpha/2}\left(\frac{\nu^2}{\mu^2}\right)^{\alpha/2} e^{\epsilon\gamma_E}\int d^{d-1}k\frac{1}{(n\cdot\tilde{k})^{-\epsilon+\alpha/2}}\cdot\frac{1}{(\bar{n}\cdot\tilde{k})^{-\epsilon-\alpha/2}}\\ &= -2\frac{1}{q_T^2}\left(\frac{\mu^2}{q_T^2}\right)^{\epsilon+\alpha/2}\left(\frac{\nu^2}{\mu^2}\right)^{\alpha/2} + \mathcal{O}(\epsilon^2)\end{aligned}$$

with

$$\tilde{k}^\mu = \frac{1}{k_0}\cdot k^\mu$$

and so

$$C = 0$$

and one obtains

$$\tilde{I}_{34} = \frac{1}{q_T^2}\frac{e^{\epsilon\cdot\gamma_E}}{\Gamma(1-\epsilon)}\left(\frac{\mu^2}{q_T^2}\right)^{\epsilon+\alpha/2}\left(\frac{\nu^2}{\mu^2}\right)^{\alpha/2}\frac{1+\beta^2}{\beta}\left(\ln\frac{1-\beta}{1+\beta} + \epsilon f_{34}\right)$$

The order $\mathcal{O}(\epsilon^2)$ term for K_{34} is more difficult to calculate. Due to the dependence on β and θ it is possible to derive two differential equations; one in β and the other one in θ. These equations are

$$\frac{d}{d\beta}J^{\epsilon^2} = 2\cdot\tilde{R}\cdot\frac{1}{1-\beta^2}\cdot\mathrm{Li}_3\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)$$

$$\frac{d}{d\theta}M_{34}^{\epsilon^2} = \frac{2\cot(\theta)\left(4\ln\left(\frac{\beta+1}{1-\beta}\right)\ln\left(\sec\left(\frac{\theta}{2}\right)\right) + \mathrm{Li}_2\left(\frac{(\beta+1)\tan^2\left(\frac{\theta}{2}\right)}{\beta-1}\right) - \mathrm{Li}_2\left(\frac{(\beta-1)\tan^2\left(\frac{\theta}{2}\right)}{\beta+1}\right)\right)}{\beta}$$

The differential equation in θ is derived in section 8.3.5.
Since the boundary value for $\beta = 0$ is especially easy to calculate it is better to evaluate the differential equation in β. It is

$$\frac{d}{d\beta}J^{\epsilon^2} = \tilde{R}\cdot\frac{1}{1-\beta^2}\cdot 2\cdot\epsilon^2\cdot\mathrm{Li}_2\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)$$

with the boundary condition

$$J^{\epsilon^2}(\beta = 0) = 0$$

First $\sin\theta$ is rewritten in terms of $t = \tan\frac{\theta}{2}$ or $u = -t^2$. Namely

$$\sin\theta = \frac{2t}{1+t^2} \quad\Rightarrow\quad \sin^2\theta = \frac{4t^2}{(1+t)^2} = -\frac{4u}{(1-u)^2}$$

So one has

$$J^{\epsilon^2} = \tilde{R} \int d\beta \ \frac{2}{1-\beta^2} \mathrm{Li}_2 \left(\frac{4u\beta^2}{(1-u)^2 \, (1-\beta^2)} \right)$$

and the integration over β can be rewritten in terms of $x_s = \frac{1-\beta}{\beta+1}$

$$J^{\epsilon^2} = -\tilde{R} \int dx_s \ \frac{\mathrm{Li}_2 \left(\frac{u(x_s-1)^2}{(u-1)^2 x_s} \right)}{x_s} \tag{8.25}$$

In order to solve this one can first take the derivative with respect to u. As a boundary condition one can extract from eq. (8.25)

$$J^{\epsilon^2}(u=0) = 0$$

Taking the derivative of J with respect to u one gets

$$\begin{aligned}
\frac{d}{du} J^{\epsilon^2} &= \tilde{R} \int dx_s \left(- \frac{(u+1) \ln \left(\frac{(u-x_s)(ux_s-1)}{(u-1)^2 x_s} \right)}{(u-1)u x_s} \right) = \int dx_s \ \frac{u+1}{u(1-u)} \frac{1}{x_s} \left(\ln \left(\frac{u-x_s}{x_s} \right) + \ln \left(\frac{ux_s-1}{(u-1)^2} \right) \right) \\
&= \frac{u+1}{(1-u)u} \left(\ln(-1) \ln(x_s) + \mathrm{Li}_2 \left(\frac{u}{x_s} \right) + \ln \left(- \frac{1}{(u-1)^2} \right) \ln(x_s) - \mathrm{Li}_2(ux_s) \right) \\
&= \frac{u+1}{(1-u)u} \left(\ln(x_s) \cdot \ln \left(\frac{1}{(1-u)^2} \right) + \mathrm{Li}_2 \left(\frac{u}{x_s} \right) - \mathrm{Li}_2(ux_s) \right)
\end{aligned}$$

This exactly reproduces the differential equation in θ rewritten in terms of u, compare eq. 8.26. Finally, the integration over u gives with MATHEMATICA

$$\begin{aligned}
\frac{J^{\epsilon^2}}{\tilde{R}} =& 2\mathrm{Li}_3 \left(\frac{u-x_s}{u-1} \right) + 2\mathrm{Li}_3 \left(\frac{ux_s-1}{u-1} \right) + 2\mathrm{Li}_3 \left(1 - \frac{u}{x_s} \right) + \mathrm{Li}_3 \left(\frac{u}{x_s} \right) - \mathrm{Li}_3(ux_s) - 2\mathrm{Li}_3 \left(\frac{u-x_s}{(u-1)x_s} \right) \\
&- 2\mathrm{Li}_3(1-ux_s) - 2\mathrm{Li}_3 \left(\frac{1-ux_s}{x_s-ux_s} \right) \\
&+ 2 \left(-\mathrm{Li}_2 \left(\frac{ux_s-1}{u-1} \right) + \mathrm{Li}_2(1-ux_s) + \mathrm{Li}_2 \left(\frac{1-ux_s}{x_s-ux_s} \right) \right) \ln \left(\frac{ux_s-1}{u-1} \right) \\
&+ 2\mathrm{Li}_2(u) \ln(x_s) + 2 \left(-\mathrm{Li}_2 \left(\frac{u-x_s}{u-1} \right) + \mathrm{Li}_2 \left(\frac{u}{x_s} \right) + \mathrm{Li}_2 \left(\frac{u-x_s}{(u-1)x_s} \right) \right) \ln \left(\frac{u-x_s}{(u-1)x_s} \right) \\
&- \ln(x_s) \left(\ln^2 \left(\frac{ux_s-1}{u-1} \right) + \ln^2(1-ux_s) + \ln^2 \left(1 - \frac{u}{x_s} \right) + \ln^2 \left(\frac{u-x_s}{(u-1)x_s} \right) \right) \\
&+ 2 \left(\ln(x_s) \ln(1-ux_s) + \ln(1-u) \ln(u) \right) \ln \left(\frac{ux_s-1}{u-1} \right) \\
&+ \frac{1}{3}\pi^2 \ln(1-u) + \frac{1}{3} \ln \left(1 - \frac{u}{x_s} \right) \left(6\ln(u) \left(\ln \left(\frac{x_s-u}{x_s} \right) \right) - \pi^2 \right) \\
&- \ln \left(\frac{u-x_s}{(u-1)x_s} \right) 2\ln(1-u)\ln(u) - 2\ln(1-u)\ln(u)\ln(1-ux_s) \\
&- \ln(x_s) \left(2\ln(1-u)(\ln(u-1) - \ln(u)) + \ln \left(\frac{1}{(u-1)^2} \right) \left(\ln(u-1) + 2\tanh^{-1}(1-2u) \right) \right) + C
\end{aligned}$$

where the last term C is a constant not dependent on u. To evaluate C one can check the boundary condition at $u = 0$. One obtains

$$\begin{aligned}
J^{\epsilon^2}(u=0) &= \left(2 \cdot \mathrm{Li}_3(x_s) - 2 \cdot \mathrm{Li}_3 \left(\frac{1}{x_s} \right) \right) + C \stackrel{!}{=} 0 \\
\Rightarrow C &= \left(-2 \cdot \mathrm{Li}_3(x_s) + 2 \cdot \mathrm{Li}_3 \left(\frac{1}{x_s} \right) \right)
\end{aligned}$$

with $\tilde{R}$ is given in eq. (8.24). Now the boundary condition for $\beta = 0$ can be checked. With Mellin-Barnes I found that

$$I_{34}(\beta = 0) = -2\left(\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\right) \overset{\epsilon\to 0}{=} -2 + \frac{\pi^2\epsilon^2}{6} + \mathcal{O}(\epsilon^3)$$

and thus

$$J^{\epsilon^2}(\beta = 0) \overset{!}{=} 0$$

Instead of taking the boundary condition at $\beta = 0$ one can take the limit at $x_s = 1$. It is

$$I(0,0,1,1,1,1) = \frac{1}{\beta}\cdot J^{\epsilon^2} = \frac{1}{\frac{1-x_s}{x_s+1}} J^{\epsilon^2} = \frac{x_s+1}{1-x_s} J^{\epsilon^2} \overset{x_s\to 1}{=} 0$$

and thus the result stated above is the final solution for J^{ϵ^2} and for I_{34} one has

$$\tilde{I}_{34} = \frac{1}{q_T^2}\frac{e^{\epsilon\cdot\gamma_E}}{\Gamma(1-\epsilon)}\left(\frac{\mu^2}{q_T^2}\right)^{\epsilon+\alpha/2}\left(\frac{\nu^2}{\mu^2}\right)^{\alpha/2}\frac{1+\beta^2}{\beta}\left(\ln\frac{1-\beta}{1+\beta} + \epsilon f_{34} + \epsilon^2 J^{\epsilon^2}\right)$$

8.3.5. Check via differential equations in θ

To be sure that one has obtained the right θ dependence via the differential equations in β, one can check if the solutions also fulfill the differential equations in θ.

Differential equation for M_{13}

The differential equation in M_{13} with respect to θ is

$$\begin{aligned}
\frac{d}{d\theta}M_{13} &= -I(1,0,2,0,1,1)\cdot\beta\left(-k_0\sin\theta_1\ \cos\theta_2\ \cos\theta + \cos\theta_1\ \sin\theta\right)\\
&= -I(1,0,2,0,1,1)\cdot\frac{\cos\theta}{\sin\theta}\left(\tilde{v}_3\cdot k - k_0 + \frac{k_0\beta\cos\theta_1}{\cos\theta}\right)\\
&= -\frac{\cos\theta}{\sin\theta}M_{13} + \frac{\cos\theta}{\sin\theta}I(1,0,2,0,1,1)\cdot\left(\frac{nk+\bar{n}k}{2}\right) - \frac{\beta}{\sin\theta}I(1,0,2,0,1,1)\left(\frac{\bar{n}k-nk}{2}\right)\\
&= -\frac{\cos\theta}{\sin\theta}M_{13} + \left(\frac{\cos\theta}{2\sin\theta} + \frac{\beta}{2\sin\theta}\right)I(0,0,2,0,1,1) + \left(\frac{\cos\theta}{2\sin\theta} - \frac{\beta}{2\sin\theta}\right)I(1,-1,2,0,1,1)
\end{aligned}$$

and via A.I.R. one found that

$$\begin{aligned}
I(1,-1,2,0,1,1) =& M_{13}\left(\frac{\alpha\beta^2\cos^2(\theta) - 4\alpha\beta^2 + 3\alpha - 4\beta^2 - 2\beta^2\epsilon\cos^2(\theta) + 2\epsilon + 4}{2\left(\beta^2-1\right)\left(\beta\cos(\theta)-1\right)}\right)\\
&+ M_{23}\left(\frac{\alpha\beta^2\cos^2(\theta) + 2\alpha\beta\cos(\theta) + \alpha + 2\beta^2\epsilon\cos^2(\theta) + 4\beta\epsilon\cos(\theta) + 2\epsilon}{2\left(\beta^2-1\right)\left(\beta\cos(\theta)-1\right)}\right)
\end{aligned}$$

and

$$I(0,0,2,0,1,1) = \left(-\frac{(\alpha-2\epsilon)(\beta\cos(\theta)-1)}{2\left(\beta^2-1\right)}\right)\cdot M_{13} + \left(-\frac{(\alpha+2\epsilon)(\beta\cos(\theta)+1)}{2\left(\beta^2-1\right)}\right)M_{23}$$

So one has

$$
\begin{aligned}
\frac{d}{d\theta}M_{13} =& M_{13}\Bigg(-\frac{\cos\theta}{\sin\theta} - \frac{(\alpha-2\epsilon)(\beta\cos(\theta)-1)}{2\left(\beta^2-1\right)}\left(\frac{\beta}{2\sin(\theta)}+\frac{\cos(\theta)}{2\sin(\theta)}\right) \\
&+\left(\frac{\cos\theta}{2\sin\theta}-\frac{\beta}{2\sin\theta}\right)\cdot\frac{\alpha\beta^2\cos^2(\theta)-4\alpha\beta^2+3\alpha-4\beta^2-2\beta^2\epsilon\cos^2(\theta)+2\epsilon+4}{2\left(\beta^2-1\right)\left(\beta\cos(\theta)-1\right)}\Bigg) \\
&+M_{23}\Bigg(\left(\frac{\cos\theta}{2\sin\theta}-\frac{\beta}{2\sin\theta}\right)\left(\frac{\alpha\beta^2\cos^2(\theta)+2\alpha\beta\cos(\theta)+\alpha+2\beta^2\epsilon\cos^2(\theta)+4\beta\epsilon\cos(\theta)+2\epsilon}{2\left(\beta^2-1\right)\left(\beta\cos(\theta)-1\right)}\right) \\
&-\frac{(\alpha+2\epsilon)(\beta\cos(\theta)+1)}{2\left(\beta^2-1\right)}\left(\frac{\beta}{2\sin(\theta)}+\frac{\cos(\theta)}{2\sin(\theta)}\right)\Bigg) \\
\equiv& c_1\cdot M_{13}+c_2 M_{23}
\end{aligned}
$$

and one can decompose the coefficients in orders of α and ϵ as

$$
\begin{aligned}
c_1 =& \frac{\beta\sin(\theta)}{\beta\cos(\theta)-1}+\frac{-2\beta\csc(\theta)+\beta\cos(\theta)\cot(\theta)+\cot(\theta)}{2-2\beta\cos(\theta)}\cdot\alpha+\cot(\theta)\epsilon \\
=& a_{10}+a_{11}\cdot\alpha+a_{12}\cdot\epsilon
\end{aligned}
$$

and

$$
\begin{aligned}
c_2 =& \frac{\cot(\theta)(\beta\cos(\theta)+1)}{2-2\beta\cos(\theta)}\cdot\alpha+\frac{\cot(\theta)(\beta\cos(\theta)+1)}{1-\beta\cos(\theta)}\cdot\epsilon \\
=& a_{21}\cdot\alpha+a_{22}\cdot\epsilon
\end{aligned}
$$

and as before one decomposes the integrals as

$$
M_{13} = M_{13}^{\alpha}\cdot\frac{1}{\alpha}+M_{13}^{1}+M_{13}^{\epsilon}\cdot\epsilon+M_{13}^{\epsilon^2}\cdot\epsilon^2
$$

with

$$
\begin{aligned}
M_{13}^{\alpha} &= R\frac{2}{(1-\beta\cos(\theta))} \\
M_{13}^{1} &= -R\frac{2}{1-\beta\cos(\theta)}\ln\left(\frac{1-\beta\cos(\theta)}{\sqrt{1-\beta^2}}\right) \\
M_{13}^{\epsilon} &= R\frac{1}{1-\beta\cos(\theta)}\mathrm{Li}_2\left(-\frac{\beta^2\sin^2(\theta)}{1-\beta^2}\right) \\
M_{13}^{\epsilon^2} &= R\frac{1}{1-\beta\cos\theta}\mathrm{Li}_3\left(-\frac{\beta^2\sin^2(\theta)}{1-\beta^2}\right)
\end{aligned}
$$

with R given in eq. (8.22) and with the same notation for M_{23}. So one obtains to order $\mathcal{O}(\alpha^{-1})$

$$
\begin{aligned}
&\frac{d}{d\theta}M_{13}^{\alpha} = a_{10}\cdot M_{13}^{\alpha} \\
\Leftrightarrow\ & -\frac{2\beta\sin(\theta)}{(\beta\cos(\theta)-1)^2} = -\frac{2\beta\sin(\theta)}{(\beta\cos(\theta)-1)^2}
\end{aligned}
$$

and to order $\mathcal{O}(1)$ one obtains

$$
\begin{aligned}
&\frac{d}{d\theta}M_{13}^{1} = a_{10}\cdot M_{13}^{1}+a_{11}\cdot M_{13}^{\alpha}+a_{21}\cdot M_{23}^{\alpha} \\
\Leftrightarrow\ & \frac{2\beta\sin(\theta)\left(\ln\left(\frac{1-\beta\cos(\theta)}{\sqrt{1-\beta^2}}\right)-1\right)}{(\beta\cos(\theta)-1)^2} = \frac{2\beta\sin(\theta)\left(\ln\left(\frac{1-\beta\cos(\theta)}{\sqrt{1-\beta^2}}\right)-1\right)}{(\beta\cos(\theta)-1)^2} \qquad \checkmark
\end{aligned}
$$

and to order $\mathcal{O}(\epsilon)$

$$\begin{aligned}\frac{d}{d\theta}M_{13}^{\epsilon} &= a_{12}\cdot M_{13}^{1} + a_{10}\cdot M_{13}^{\epsilon} + a_{22}\cdot M_{23}^{1}\\ \Leftrightarrow \frac{d}{d\theta}M_{13}^{\epsilon} &- \left(a_{12}\cdot M_{13}^{1} + a_{10}\cdot M_{13}^{\epsilon} + a_{22}\cdot M_{23}^{1}\right) = 0 \qquad \checkmark\end{aligned}$$

which is fulfilled!
To $\mathcal{O}(\epsilon^2)$ one has to check the equation

$$\frac{d}{d\theta}M_{13}^{\epsilon^2} = a_{10}\cdot M_{13}^{\epsilon^2} + a_{12}\cdot M_{13}^{\epsilon} + a_{22}M_{23}^{\epsilon} \qquad \checkmark$$

and since this equation is as well fulfilled, the function M_{13} reproduces the correct dependence on θ and thus it is the function I was looking for.

Differential equation for M_{23}

$$\begin{aligned}\frac{d}{d\theta}M_{23} &= -I(0,1,2,0,1,1)\cdot\beta\left(-k_0\sin\theta_1\ \cos\theta_2\ \cos\theta + \cos\theta_1\ \sin\theta\right)\\ &= -I(0,1,2,0,1,1)\cdot\frac{\cos\theta}{\sin\theta}\left(\tilde{v}_3\cdot k - k_0 + \frac{k_0\beta\cos\theta_1}{\cos\theta}\right)\\ &= -\frac{\cos\theta}{\sin\theta}M_{23} + \frac{\cos\theta}{\sin\theta}I(0,1,2,0,1,1)\cdot\left(\frac{nk+\bar{n}k}{2}\right) - \frac{\beta}{\sin\theta}I(0,1,2,0,1,1)\left(\frac{\bar{n}k-nk}{2}\right)\\ &= -\frac{\cos\theta}{\sin\theta}M_{23} + \left(\frac{\cos\theta}{2\sin\theta} - \frac{\beta}{2\sin\theta}\right)I(0,0,2,0,1,1) + \left(\frac{\cos\theta}{2\sin\theta} + \frac{\beta}{2\sin\theta}\right)I(-1,1,2,0,1,1)\end{aligned}$$

The same procedure as before...

$$I(0,0,2,0,1,1) = \left(-\frac{(\alpha-2\epsilon)(\beta\cos(\theta)-1)}{2\left(\beta^2-1\right)}\right)\cdot M_{13} + \left(-\frac{(\alpha+2\epsilon)(\beta\cos(\theta)+1)}{2\left(\beta^2-1\right)}\right)M_{23}$$

and

$$I(-1,1,2,0,1,1) = \frac{(\alpha-2\epsilon)(\beta\cos(\theta)-1)^2}{2\left(\beta^2-1\right)(\beta\cos(\theta)+1)}\cdot M_{13} + \frac{-4(\alpha-1)\beta^2+\beta^2(\alpha+2\epsilon)\cos^2(\theta)+3\alpha-2\epsilon-4}{2\left(\beta^2-1\right)(\beta\cos(\theta)+1)}\cdot M_{23}$$

and so

$$\begin{aligned}\frac{d}{d\theta}M_{23} =& M_{13}\Bigg(\left(\frac{\cos\theta}{2\sin\theta} - \frac{\beta}{2\sin\theta}\right)\cdot\left(-\frac{(\alpha-2\epsilon)(\beta\cos(\theta)-1)}{2\left(\beta^2-1\right)}\right)\\ &\quad + \left(\frac{\cos\theta}{2\sin\theta} + \frac{\beta}{2\sin\theta}\right)\Bigg)\frac{(\alpha-2\epsilon)(\beta\cos(\theta)-1)^2}{2\left(\beta^2-1\right)(\beta\cos(\theta)+1)}\\ &+ M_{23}\Bigg(-\frac{\cos\theta}{\sin\theta} + \left(\frac{\cos\theta}{2\sin\theta} - \frac{\beta}{2\sin\theta}\right)\cdot\left(-\frac{(\alpha+2\epsilon)(\beta\cos(\theta)+1)}{2\left(\beta^2-1\right)}\right)\\ &\quad + \left(\frac{\cos\theta}{2\sin\theta} + \frac{\beta}{2\sin\theta}\right)\frac{-4(\alpha-1)\beta^2+\beta^2(\alpha+2\epsilon)\cos^2(\theta)+3\alpha-2\epsilon-4}{2\left(\beta^2-1\right)(\beta\cos(\theta)+1)}\Bigg)\\ \equiv& d_1M_{13} + d_2M_{23}\end{aligned}$$

with

$$\begin{aligned}d_1 &= \frac{(\alpha-2\epsilon)\cot(\theta)(\beta\cos(\theta)-1)}{2\beta\cos(\theta)+2}\\ &= \frac{\cot(\theta)(\beta\cos(\theta)-1)}{2\beta\cos(\theta)+2}\cdot\alpha + \frac{\cot(\theta)(1-\beta\cos(\theta))}{\beta\cos(\theta)+1}\cdot\epsilon\\ &\equiv b_{11}\alpha + b_{12}\epsilon\end{aligned}$$

and

$$\begin{aligned}
d_2 &= \frac{\cot(\theta)(\beta(\alpha+2\epsilon-2)\cos(\theta)-\alpha+2\epsilon)-2(\alpha-1)\beta\csc(\theta)}{2\beta\cos(\theta)+2} \\
&= \frac{\beta\sin(\theta)}{\beta\cos(\theta)+1} + \frac{\cot(\theta)(\beta\cos(\theta)-1)-2\beta\csc(\theta)}{2\beta\cos(\theta)+2}\cdot\alpha + \cot(\theta)\cdot\epsilon \\
&\equiv b_{20} + b_{21}\alpha + b_{22}\epsilon
\end{aligned}$$

with the same notation as before one has to check to order $\mathcal{O}(\alpha^{-1})$

$$\begin{aligned}
\frac{d}{d\theta}M_{23}^{\alpha} &= b_{20}\cdot M_{23}^{\alpha} \\
\Leftrightarrow -\frac{2\beta\sin(\theta)}{(\beta\cos(\theta)+1)^2} &= -\frac{2\beta\sin(\theta)}{(\beta\cos(\theta)+1)^2} \qquad \checkmark
\end{aligned}$$

to order $\mathcal{O}(1)$ one has

$$\begin{aligned}
\frac{d}{d\theta}M_{23}^{1} &= b_{21}\cdot M_{23}^{\alpha} + b_{20}\cdot M_{23}^{1} + b_{11}\cdot M_{13}^{\alpha} \\
\Leftrightarrow \frac{d}{d\theta}M_{23}^{1} - \left(b_{21}\cdot M_{23}^{\alpha} + b_{20}\cdot M_{23}^{1} + b_{11}\cdot M_{13}^{\alpha}\right) &= 0 \qquad \checkmark
\end{aligned}$$

and to order $\mathcal{O}(\epsilon)$:

$$\frac{d}{d\theta}M_{23}^{\epsilon} - \left(b_{20}\cdot M_{23}^{\epsilon} + b_{22}\cdot M_{23}^{1} + b_{12}\cdot M_{13}^{1}\right) = 0 \qquad \checkmark$$

and to order $\mathcal{O}(\epsilon^2)$:

$$\frac{d}{d\theta}M_{23}^{\epsilon^2} - \left(b_{20}\cdot M_{23}^{\epsilon^2} + b_{22}\cdot M_{23}^{\epsilon} + b_{12}\cdot M_{13}^{\epsilon}\right) = 0 \qquad \checkmark$$

Since M_{23} obeys the differential equations in θ, M_{23} is the function I was looking for.

Differential equation for M_{33}

Since M_{33} is composed of M_{23} and M_{13}, the θ-dependence of M_{33} is already confirmed via the confirmation of M_{23} and M_{13} which was done in the previous sections.

Differential equation for M_{34}

Taking the derivative with respect to θ of M_{34} gives

$$\begin{aligned}
\frac{d}{d\theta}I(0,0,1,1,1,1) =& \frac{d}{d\theta}M_{34} = \frac{2\cos\theta}{\sin\theta}I(0,0,2,0,1,1) + I(-1,0,2,1,1,1)\left(-\frac{\cos\theta}{2\sin\theta}+\frac{\beta}{2\sin\theta}\right) \\
&+ I(0,-1,2,1,1,1)\left(-\frac{\cos\theta}{2\sin\theta}-\frac{\beta}{2\sin\theta}\right) \\
&+ I(-1,0,1,2,1,1)\left(-\frac{\cos\theta}{2\sin\theta}-\frac{\beta}{2\sin\theta}\right) + I(0,-1,1,2,1,1)\left(-\frac{\cos\theta}{2\sin\theta}+\frac{\beta}{2\sin\theta}\right)
\end{aligned}$$

The integrals appearing in this expression are reduced to the master integrals stated above which are already known as

$$I(0,0,2,0,1,1) = \left(-\frac{(\alpha-2\epsilon)(\beta\cos(\theta)-1)}{2\left(\beta^2-1\right)}\right)\cdot M_{13} + \left(-\frac{(\alpha+2\epsilon)(\beta\cos(\theta)+1)}{2\left(\beta^2-1\right)}\right)M_{23}$$

$$L_1 = I(-1,0,2,1,1,1) = \frac{1}{8\beta^2}\Bigg(\frac{\left(\beta^2+1\right)M_{13}(\alpha-2\epsilon)(\beta\cos(\theta)-1)^2}{\beta^2-1} + M_{14}(\alpha-2\epsilon)(\beta\cos(\theta)+1)^2$$
$$+\frac{M_{23}(\alpha+2\epsilon)(\beta\cos(\theta)+1)\left(\left(\beta^3+\beta\right)\cos(\theta)-3\beta^2+1\right)}{\beta^2-1}$$
$$+M_{24}(\alpha+2\epsilon)(\beta\cos(\theta)-1)^2+4\beta M_{34}(-\alpha\beta+\beta+2\epsilon\cos(\theta)+\cos(\theta))\Bigg)$$

$$L_2 = I(0,-1,2,1,1,1) = -\frac{1}{8\beta^2}\Bigg(\frac{M_{13}(\alpha-2\epsilon)\left(\left(\beta^3+\beta\right)\cos(\theta)+3\beta^2-1\right)(\beta\cos(\theta)-1)}{\beta^2-1}$$
$$+M_{14}(\alpha-2\epsilon)(\beta\cos(\theta)+1)^2$$
$$+\frac{\left(\beta^2+1\right)M_{23}(\alpha+2\epsilon)(\beta\cos(\theta)+1)^2}{\beta^2-1}+M_{24}(\alpha+2\epsilon)(\beta\cos(\theta)-1)^2$$
$$-4\beta M_{34}((\alpha+1)\beta-(2\epsilon+1)\cos(\theta))\Bigg)$$

$$L_3 = I(-1,0,1,2,1,1) = \frac{1}{8\beta^2}\Bigg(M_{13}(\alpha-2\epsilon)(\beta\cos(\theta)-1)^2+\frac{\left(\beta^2+1\right)M_{14}(\alpha-2\epsilon)(\beta\cos(\theta)+1)^2}{\beta^2-1}$$
$$+M_{23}(\alpha+2\epsilon)(\beta\cos(\theta)+1)^2$$
$$+\frac{M_{24}(\alpha+2\epsilon)\left(\left(\beta^3+\beta\right)\cos(\theta)+3\beta^2-1\right)(\beta\cos(\theta)-1)}{\beta^2-1}$$
$$-4\beta M_{34}((\alpha-1)\beta+(2\epsilon+1)\cos(\theta))\Bigg)$$

$$L_4 = I(0,-1,1,2,1,1) = -\frac{1}{8\beta^2}\Bigg(M_{13}(\alpha-2\epsilon)(\beta\cos(\theta)-1)^2$$
$$+\frac{M_{14}(\alpha-2\epsilon)(\beta\cos(\theta)+1)\left(\left(\beta^3+\beta\right)\cos(\theta)-3\beta^2+1\right)}{\beta^2-1}$$
$$+M_{23}(\alpha+2\epsilon)(\beta\cos(\theta)+1)^2+\frac{\left(\beta^2+1\right)M_{24}(\alpha+2\epsilon)(\beta\cos(\theta)-1)^2}{\beta^2-1}$$
$$-4\beta M_{34}(\beta+\beta+2\epsilon\cos(\theta)+\cos(\theta))\Bigg)$$

Inserting these expressions one obtains the following structure

$$q_T^2\cdot\frac{d}{d\theta}M_{34} \equiv f_1M_{13}+f_2M_{23}+f_3M_{24}+f_4M_{14}+q_T^2\cdot f_5M_{34}$$

The coefficients f_i are

$$f_1 = \frac{(\alpha-2\epsilon)\csc(\theta)\left(\left(\beta^3+\beta\right)\cos(\theta)+\beta^2(-\cos(2\theta))-2\beta^2+1\right)}{4\beta\left(\beta^2-1\right)} \equiv f_{11}\alpha+f_{12}\epsilon$$

$$f_2 = -\frac{(\alpha+2\epsilon)\csc(\theta)\left(\left(\beta^3+\beta\right)\cos(\theta)+\beta^2\cos(2\theta)+2\beta^2-1\right)}{4\beta\left(\beta^2-1\right)} \equiv f_{21}\alpha+f_{22}\epsilon$$

$$f_3 = -\frac{(\alpha+2\epsilon)\csc(\theta)\left(\beta^2\cos(2\theta)+\left(\beta^2-3\right)\beta\cos(\theta)+1\right)}{4\beta\left(\beta^2-1\right)} \equiv f_{31}\alpha + f_{32}\epsilon$$

$$f_4 = -\frac{(\alpha-2\epsilon)\csc(\theta)\left(\beta^2\cos(2\theta)-\left(\beta^2-3\right)\beta\cos(\theta)+1\right)}{4\beta\left(\beta^2-1\right)} \equiv f_{41}\alpha + f_{42}\epsilon$$

$$f_5 = 2\epsilon\cot(\theta) \equiv f_{52}\cdot\epsilon$$

Since

$$\frac{d}{d\theta}M_{34}^1 = 0$$

one obtains the following equation to order $\mathcal{O}(1)$

$$0 = f_{11}\cdot M_{13}^\alpha + f_{21}\cdot M_{23}^\alpha + f_{31}\cdot M_{24}^\alpha + f_{41}\cdot M_{14}^\alpha \qquad \checkmark$$

which is fulfilled! And to the order $\mathcal{O}(\epsilon)$ one has

$$0 = f_{12}\cdot M_{13}^1 + f_{22}\cdot M_{23}^1 + f_{32}\cdot M_{24}^1 + f_{42}\cdot M_{14}^1 + q_T^2 f_{52} M_{34}^1 - q_T^2\frac{d}{d\theta}M_{34}^\epsilon \qquad \checkmark$$

And finally one has to $\mathcal{O}(\epsilon^2)$

$$0 = f_{12}M_{13}^\epsilon + f_{22}M_{23}^\epsilon + f_{32}M_{24} + f_{42}\cdot M_{14} + q_T^2 f_{52}\cdot M_{34}^\epsilon - q_T^2\frac{d}{d\theta}M_{34}^{\epsilon^2} \qquad \checkmark$$

So, M_{34} which was calculated via a differential equation in β also obeys the differential equation in θ and thus the above result turns out to be correct. The differential equation to $\mathcal{O}(\epsilon^2)$ can be further specified

$$\frac{d}{d\theta}M_{34}^{\epsilon^2} = \tilde{R}\frac{2\cot(\theta)\left(4\ln\left(\frac{\beta+1}{1-\beta}\right)\ln\left(\sec\left(\frac{\theta}{2}\right)\right)+\mathrm{Li}_2\left(\frac{(\beta+1)\tan^2\left(\frac{\theta}{2}\right)}{\beta-1}\right)-\mathrm{Li}_2\left(\frac{(\beta-1)\tan^2\left(\frac{\theta}{2}\right)}{\beta+1}\right)\right)}{\beta}$$

Instead of integrating over θ it seems to be easier to integrate over $\tan\frac{\theta}{2}$. One has

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1-2\sin^2\frac{\theta}{2}}{2\cos\frac{\theta}{2}\cos\frac{\theta}{2}}$$

with

$$t \equiv \tan\frac{\theta}{2}$$

one has

$$\cos\frac{\theta}{2} = \frac{1}{\sqrt{1+t^2}}$$
$$\sin\frac{\theta}{2} = \frac{t}{\sqrt{1+t^2}}$$

and so

$$\cot\theta = \frac{1-2\frac{t^2}{1+t^2}}{\frac{2t}{1+t^2}} = \frac{1-t^2}{2t}$$

and for the integration measure one has

$$\frac{d}{d\theta}\tan\frac{\theta}{2} = \frac{1}{2}\sec^2\left(\frac{\theta}{2}\right) = \frac{1}{2}(1+t^2)$$

So one has

$$\begin{aligned} M_{34}^{\epsilon^2} &= \tilde{R}\int dt\,\frac{1}{\beta}2\frac{1-t^2}{2t}\frac{2}{1+t^2}\left(2\ln\left(\frac{\beta+1}{1-\beta}\right)\ln\left(1+t^2\right)+\mathrm{Li}_2\left(\frac{(\beta+1)t^2}{\beta-1}\right)-\mathrm{Li}_2\left(\frac{(\beta-1)t^2}{\beta+1}\right)\right) \\ &= \tilde{R}\int dt\,\frac{2}{\beta}\frac{1-t^2}{t(1+t^2)}\left(2\ln\left(\frac{\beta+1}{1-\beta}\right)\ln\left(1+t^2\right)+\mathrm{Li}_2\left(\frac{(\beta+1)t^2}{\beta-1}\right)-\mathrm{Li}_2\left(\frac{(\beta-1)t^2}{\beta+1}\right)\right) \end{aligned}$$

with the substitution

$$u=-t^2 \qquad \Rightarrow dt = -\frac{1}{2t}du$$

one ends up with

$$M_{34}^{\epsilon^2} = \tilde{R}\cdot\frac{1}{\beta}\int\ du\ \frac{1}{1-u}\cdot\frac{1+u}{u}\left(-2\ln(x_s)\ln(1-u)-\mathrm{Li}_2(x_s u)+\mathrm{Li}_2\left(\frac{u}{x_s}\right)\right) \tag{8.26}$$

with

$$x_s = \frac{1-\beta}{1+\beta}$$

which already appeared in the derivation of $M_{34}^{\epsilon^2}$.

8.3.6. Overview of the results in momentum space

So with this method it was possible to evaluate the integrals of the NLO soft function even in one order higher in ϵ than in [3]. The results are given by

$$\tilde{I}_{13} = \frac{e^{\epsilon\cdot\gamma_E}}{\Gamma(1-\epsilon)}\frac{1}{q_T^2}\left(\frac{\mu^2}{q_T^2}\right)^{\epsilon+\alpha/2}\left(\frac{\nu^2}{\mu^2}\right)^{\alpha/2}\left(\frac{2}{\alpha}-2\ln\frac{1-\beta\cos\theta}{\sqrt{1-\beta^2}}+\epsilon\mathrm{Li}_2\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)+\epsilon^2\mathrm{Li}_3\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)\right)$$

$$\tilde{I}_{23} = \frac{e^{\epsilon\cdot\gamma_E}}{\Gamma(1-\epsilon)}\frac{1}{q_T^2}\left(\frac{\mu^2}{q_T^2}\right)^{\epsilon+\alpha/2}\left(\frac{\nu^2}{\mu^2}\right)^{\alpha/2}\left(-\frac{2}{\alpha}-2\ln\frac{1+\beta\cos\theta}{\sqrt{1-\beta^2}}+\epsilon\mathrm{Li}_2\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)+\epsilon^2\mathrm{Li}_3\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)\right)$$

$$\tilde{I}_{33} = \tilde{I}_{44} = \frac{e^{\epsilon\cdot\gamma_E}}{\Gamma(1-\epsilon)}\frac{1}{q_T^2}\left(\frac{\mu^2}{q_T^2}\right)^{\epsilon+\alpha/2}\left(\frac{\nu^2}{\mu^2}\right)^{\alpha/2}\left(-2+2\epsilon\ln\frac{1-\beta^2}{1-\beta^2\cos^2\theta}+2\epsilon^2\mathrm{Li}_3\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)\right)$$

$$\tilde{I}_{14} = \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\frac{1}{q_T^2}\left(\frac{\mu^2}{q_T^2}\right)^{\epsilon+\alpha/2}\left(\frac{\nu^2}{\mu^2}\right)^{\alpha/2}\left(\frac{2}{\alpha}-2\ln\frac{1+\beta\cos\theta}{\sqrt{1-\beta^2}}+\epsilon\mathrm{Li}_2\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)+\epsilon^2\mathrm{Li}_3\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)\right)$$

$$\tilde{I}_{24} = \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\frac{1}{q_T^2}\left(\frac{\mu^2}{q_T^2}\right)^{\epsilon+\alpha/2}\left(\frac{\nu^2}{\mu^2}\right)^{\alpha/2}\left(-\frac{2}{\alpha}-2\ln\frac{1-\beta\cos\theta}{\sqrt{1-\beta^2}}+\epsilon\mathrm{Li}_2\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)+\epsilon^2\mathrm{Li}_3\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)\right)$$

8.4. The results in position space

In the last chapter the integrals of the soft function were calculated in momentum space. In order to obtain the results in position space, one has to perform a Fourier transformation. This is given by

$$I_{jk} = \frac{1}{2\pi}\int d^2\vec{q}_T\; e^{-i\vec{q}_T\vec{x}_T}\,\tilde{I}_{jk}$$

where $\tilde{I}_{jk}$ are the just calculated functions. Taking into account that an additional factor of 2 was suppressed in the results of the last chapters (see the comment after eq. (8.5)), it is to calculate

$$I_{jk} = \frac{1}{\pi}\int d^2\vec{q}_T\; e^{-i\vec{q}_T\vec{x}_T}\,\tilde{I}_{jk}$$

In order to obtain the result in position space one has to perform the Fourier transform. For this purpose I first regard the following function

$$\tilde{I}_{test} = \frac{\left(\frac{\nu^2}{\mu^2}\right)^{\alpha/2}\left(\frac{\mu^2}{q_T^2}\right)^{\frac{\alpha}{2}+\epsilon}}{q_T^2}\cdot\frac{e^{\epsilon\cdot\gamma_E}}{\Gamma(1-\epsilon)}$$

where this function corresponds to the prefactor of e.g I_{13}. The Fourier transform of this function is given by

$$\begin{aligned}
I_{test} =& \frac{1}{\pi}\int d^2\vec{q}_T\; e^{-i\vec{q}_T\vec{x}_T}\,\tilde{I}_{test}\\
=& \frac{e^{-\gamma_E(\alpha+2\epsilon)}e^{L_\perp\left(\frac{\alpha}{2}+\epsilon\right)}\Gamma\left(-\frac{\alpha}{2}-\epsilon\right)\left(\frac{\nu^2}{\mu^2}\right)^{\alpha/2}}{\Gamma\left(\frac{\alpha}{2}+\epsilon+1\right)}\cdot\frac{e^{\epsilon\cdot\gamma_E}}{\Gamma(1-\epsilon)}\\
=& \frac{1}{12}\epsilon\left(\pi^2-6L_\perp^2\right)-\frac{1}{\epsilon}-L_\perp\\
&+\left(\frac{1}{2\epsilon^2}+\frac{1}{24}\epsilon\left(\left(\pi^2-6L_\perp^2\right)\log\left(\frac{\nu^2}{\mu^2}\right)-4\left(L_\perp^3+5\zeta(3)\right)\right)-\frac{\log\left(\frac{\nu^2}{\mu^2}\right)}{2\epsilon}\right.\\
&\left.+\frac{1}{24}\left(-6L_\perp^2-12L_\perp\log\left(\frac{\nu^2}{\mu^2}\right)-\pi^2\right)\right)\alpha+\mathcal{O}(\epsilon^2)+\mathcal{O}(\alpha^2)\\
\equiv& \frac{1}{12}\epsilon\left(\pi^2-6L_\perp^2\right)-\frac{1}{\epsilon}-L_\perp+\Xi\cdot\alpha
\end{aligned}$$

with

$$\Xi = \frac{1}{2\epsilon^2}+\frac{1}{24}\epsilon\left(\left(\pi^2-6L_\perp^2\right)\log\left(\frac{\nu^2}{\mu^2}\right)-4\left(L_\perp^3+5\zeta(3)\right)\right)-\frac{\log\left(\frac{\nu^2}{\mu^2}\right)}{2\epsilon}+\frac{1}{24}\left(-6L_\perp^2-12L_\perp\log\left(\frac{\nu^2}{\mu^2}\right)-\pi^2\right)$$

The results in momentum space were given in section 8.3.6. They can be written as

$$\tilde{I}_{jk} = \tilde{I}_{test}\cdot\tilde{i}_{jk}$$

with e.g.

$$\tilde{i}_{13} = \left(\frac{2}{\alpha}-2\ln\frac{1-\beta\cos\theta}{\sqrt{1-\beta^2}}+\epsilon\cdot\mathrm{Li}_2\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)+\epsilon^2\mathrm{Li}_3\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)\right)$$

the expressions $\tilde{i}_{jk}$ are independent on $\vec{q}_T$. Thus the Fourier transformed, expanded expressions are given by

$$I_{jk} = \lim_{\epsilon\to 0}\lim_{\alpha\to 0}\left(I_{test}\cdot\tilde{i}_{jk}\right)$$

So the Fourier transform of $\tilde{I}_{13}$ is given by the following expression which needs to be expanded up to $\mathcal{O}(\epsilon)$

$$
\begin{aligned}
I_{13} =& \left(\frac{1}{12}\epsilon\left(\pi^2-6L_\perp^2\right)-\left(\frac{1}{\epsilon}+L_\perp\right)+\Xi\cdot\alpha\right)\\
&\times\left(\frac{2}{\alpha}-2\ln\frac{1-\beta\cos\theta}{\sqrt{1-\beta^2}}+\epsilon\cdot\mathrm{Li}_2\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)+\epsilon^2\mathrm{Li}_3\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)\right)\\
=&2\Xi+\frac{2}{\alpha}\left(\frac{1}{12}\epsilon\left(\pi^2-6L_\perp^2\right)-\left(\frac{1}{\epsilon}+L_\perp\right)\right)\\
&-2\ln\frac{1-\beta\cos\theta}{\sqrt{1-\beta^2}}\left(\frac{1}{12}\epsilon\left(\pi^2-6L_\perp^2\right)-\left(\frac{1}{\epsilon}+L_\perp\right)\right)\\
&+\epsilon\cdot\mathrm{Li}_2\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)\left(-\left(\frac{1}{\epsilon}+L_\perp\right)\right)\\
&+\epsilon^2\cdot\mathrm{Li}_3\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)\left(-\frac{1}{\epsilon}\right)+\mathcal{O}(\epsilon^2)+\mathcal{O}(\alpha)
\end{aligned}
$$

And one has

$$
\begin{aligned}
I_{14} =&2\Xi+\frac{2}{\alpha}\left(\frac{1}{12}\epsilon\left(\pi^2-6L_\perp^2\right)-\left(\frac{1}{\epsilon}+L_\perp\right)\right)\\
&-2\ln\frac{1+\beta\cos\theta}{\sqrt{1-\beta^2}}\left(\frac{1}{12}\epsilon\left(\pi^2-6L_\perp^2\right)-\left(\frac{1}{\epsilon}+L_\perp\right)\right)\\
&+\epsilon\cdot\mathrm{Li}_2\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)\left(-\left(\frac{1}{\epsilon}+L_\perp\right)\right)\\
&+\epsilon^2\cdot\mathrm{Li}_3\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)\left(-\frac{1}{\epsilon}\right)+\mathcal{O}(\epsilon^2)+\mathcal{O}(\alpha)
\end{aligned}
$$

and

$$
\begin{aligned}
I_{23} =&-2\Xi-\frac{2}{\alpha}\left(\frac{1}{12}\epsilon\left(\pi^2-6L_\perp^2\right)-\left(\frac{1}{\epsilon}+L_\perp\right)\right)\\
&-2\ln\frac{1+\beta\cos\theta}{\sqrt{1-\beta^2}}\left(\frac{1}{12}\epsilon\left(\pi^2-6L_\perp^2\right)-\left(\frac{1}{\epsilon}+L_\perp\right)\right)\\
&+\epsilon\cdot\mathrm{Li}_2\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)\left(-\left(\frac{1}{\epsilon}+L_\perp\right)\right)\\
&+\epsilon^2\cdot\mathrm{Li}_3\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)\left(-\frac{1}{\epsilon}\right)+\mathcal{O}(\epsilon^2)+\mathcal{O}(\alpha)
\end{aligned}
$$

and

$$
\begin{aligned}
I_{24} =&-2\Xi-\frac{2}{\alpha}\left(\frac{1}{12}\epsilon\left(\pi^2-6L_\perp^2\right)-\left(\frac{1}{\epsilon}+L_\perp\right)\right)\\
&-2\ln\frac{1-\beta\cos\theta}{\sqrt{1-\beta^2}}\left(\frac{1}{12}\epsilon\left(\pi^2-6L_\perp^2\right)-\left(\frac{1}{\epsilon}+L_\perp\right)\right)\\
&+\epsilon\cdot\mathrm{Li}_2\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)\left(-\left(\frac{1}{\epsilon}+L_\perp\right)\right)\\
&+\epsilon^2\cdot\mathrm{Li}_3\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)\left(-\frac{1}{\epsilon}\right)+\mathcal{O}(\epsilon^2)+\mathcal{O}(\alpha)
\end{aligned}
$$

And for I_{33} one obtains

$$
\begin{aligned}
I_{33} =& \left(\frac{1}{12}\epsilon\left(\pi^2 - 6L_\perp^2\right) - \left(\frac{1}{\epsilon} + L_\perp\right) + \Xi\cdot\alpha\right)\left(-2 + 2\epsilon\cdot\ln\frac{1-\beta^2}{1-\beta^2\cos^2\theta} + 2\epsilon^2 \mathrm{Li}_3\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)\right) \\
=& -2\left(\frac{1}{12}\epsilon\left(\pi^2 - 6L_\perp^2\right) - \left(\frac{1}{\epsilon} + L_\perp\right)\right) \\
&+ 2\epsilon\cdot\ln\frac{1-\beta^2}{1-\beta^2\cos^2\theta}\left(-\left(\frac{1}{\epsilon} + L_\perp\right)\right) \\
&+ 2\epsilon^2\mathrm{Li}_3\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)\left(-\frac{1}{\epsilon}\right) + \mathcal{O}(\alpha) + \mathcal{O}(\epsilon^2)
\end{aligned}
$$

And

$$
\begin{aligned}
I_{34} =& \left(\frac{1}{12}\epsilon\left(\pi^2 - 6L_\perp^2\right) - \left(\frac{1}{\epsilon} + L_\perp\right) + \Xi\cdot\alpha\right)\frac{1+\beta^2}{\beta}\left(\ln\frac{1-\beta}{1+\beta} + \epsilon f_{34} + \epsilon^2 J^{\epsilon^2}\right) \\
=& \frac{1+\beta^2}{\beta}\ln\frac{1-\beta}{1+\beta}\left(\frac{1}{12}\epsilon\left(\pi^2 - 6L_\perp^2\right) - \left(\frac{1}{\epsilon} + L_\perp\right)\right) \\
&+ \frac{1+\beta^2}{\beta}\epsilon f_{34}\left(-\left(\frac{1}{\epsilon} + L_\perp\right)\right) \\
&+ \frac{1+\beta^2}{\beta}\epsilon^2 J^{\epsilon^2}\left(-\frac{1}{\epsilon}\right) + \mathcal{O}(\alpha) + \mathcal{O}(\epsilon^2)
\end{aligned}
$$

And since

$$
\begin{aligned}
\boldsymbol{w}_{i\bar{i}}^{13} &= \boldsymbol{w}_{i\bar{i}}^{24} \\
\boldsymbol{w}_{i\bar{i}}^{14} &= \boldsymbol{w}_{i\bar{i}}^{23} \\
\boldsymbol{w}_{i\bar{i}}^{33} &= \boldsymbol{w}_{i\bar{i}}^{44}
\end{aligned}
$$

one has

$$
\begin{aligned}
\mathbf{S}_{i\bar{i}}^{(1),bare} &= \sum_{j,k}\boldsymbol{w}_{i\bar{i}}^{jk} I_{jk} \\
&= 2\cdot\boldsymbol{w}_{i\bar{i}}^{13} I_{13} + 2\cdot\boldsymbol{w}_{i\bar{i}}^{14} I_{23} + 2\cdot\boldsymbol{w}_{i\bar{i}}^{14} I_{14} + 2\cdot\boldsymbol{w}_{i\bar{i}}^{13} I_{24} + 2\boldsymbol{w}_{i\bar{i}}^{33} I_{33} + 2\boldsymbol{w}_{i\bar{i}}^{34} I_{34} \\
&= 2\cdot\boldsymbol{w}_{i\bar{i}}^{13}\left(I_{13} + I_{24}\right) + 2\cdot\boldsymbol{w}_{i\bar{i}}^{14}\left(I_{23} + I_{14}\right) + 2\boldsymbol{w}_{i\bar{i}}^{33} I_{33} + 2\boldsymbol{w}_{i\bar{i}}^{34} I_{34}
\end{aligned}
$$

The additional factor 2 arises because the contributions are symmetric which means $I_{jk} = I_{kj}$ and it was $I_{33} = I_{44}$. Due to the similarities between the functions I_{jk} one gets for example

$$
\begin{aligned}
2\cdot\boldsymbol{w}_{i\bar{i}}^{13}\left(I_{13} + I_{24}\right) = 2\cdot\boldsymbol{w}_{i\bar{i}}^{13}\Bigg(& -4\ln\frac{1-\beta\cos\theta}{\sqrt{1-\beta^2}}\left(\frac{1}{12}\epsilon\left(\pi^2 - 6L_\perp^2\right) - \left(\frac{1}{\epsilon} + L_\perp\right)\right) \\
& - 2\epsilon\cdot\mathrm{Li}_2\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)\left(\left(\frac{1}{\epsilon} + L_\perp\right)\right) \\
& - 2\epsilon\cdot\mathrm{Li}_3\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)\Bigg)
\end{aligned}
$$

So one has

$$
\begin{aligned}
\mathbf{S}_{i\bar{i}}^{(1),bare} =& 4\boldsymbol{w}_{i\bar{i}}^{13}\Bigg(\left(L_\perp + \frac{1}{\epsilon} - \frac{1}{12}\epsilon\left(\pi^2 - 6L_\perp^2\right)\right) 2\ln\left(\frac{-t_1}{m_t\cdot M}\right) \\
&\qquad - \epsilon \mathrm{Li}_2\left(\frac{\beta^2\sin^2(t)}{\beta^2-1}\right)\left(L_\perp + \frac{1}{\epsilon}\right) - \epsilon\mathrm{Li}_3\left(\frac{\beta^2\sin^2(t)}{\beta^2-1}\right)\Bigg) \\
&+ 4\boldsymbol{w}_{i\bar{i}}^{14}\Bigg(\left(L_\perp + \frac{1}{\epsilon} - \frac{1}{12}\epsilon\left(\pi^2 - 6L_\perp^2\right)\right) 2\ln\left(\frac{-u_1}{m_t\cdot M}\right) \\
&\qquad - \epsilon \mathrm{Li}_2\left(\frac{\beta^2\sin^2(t)}{\beta^2-1}\right)\left(L_\perp + \frac{1}{\epsilon}\right) - \epsilon\mathrm{Li}_3\left(\frac{\beta^2\sin^2(t)}{\beta^2-1}\right)\Bigg) \\
&+ 2\boldsymbol{w}_{i\bar{i}}^{33}\Bigg(-2\left(\frac{1}{12}\epsilon\left(\pi^2 - 6L_\perp^2\right) - \left(\frac{1}{\epsilon} + L_\perp\right)\right) \\
&\qquad - 2\epsilon\left(L_\perp + \frac{1}{\epsilon}\right)\ln\left(\frac{\left(\beta^2-1\right)}{\beta^2\cos^2(\theta)-1}\right) - 2\epsilon\ \mathrm{Li}_3\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)\Bigg) \\
&+ 2\boldsymbol{w}_{i\bar{i}}^{34}\Bigg(-\frac{\left(\beta^2+1\right)\ln\left(x_s\right)}{\beta}\left(L_\perp + \frac{1}{\epsilon} - \frac{1}{12}\epsilon\left(\pi^2 - 6L_\perp^2\right)\right) \\
&\qquad - \epsilon f_{34}\frac{\left(\beta^2+1\right)}{\beta}\left(L_\perp + \frac{1}{\epsilon}\right) - \epsilon J^{\epsilon^2}\frac{\left(\beta^2+1\right)}{\beta}\Bigg) + \mathcal{O}(\epsilon^2)
\end{aligned}
$$

with

$$
\frac{-t_1}{m_T\cdot M} = \frac{1-\beta\cos\theta}{\sqrt{1-\beta^2}}, \qquad\qquad \frac{-u_1}{m_T\cdot M} = \frac{1+\beta\cos\theta}{\sqrt{1-\beta^2}}
$$

So the unphysical dependence on α and ν dropped out as it was expected.
The renormalization prescription is given by

$$
\mathbf{S}_{i\bar{i}}^{ren} = \mathbf{Z}_{i\bar{i}}\cdot\mathbf{S}_{i\bar{i}}^{bare}
$$

In order to shift the divergent parts of $\mathbf{S}$ in $\mathbf{Z}$ one determines $\mathbf{Z}$ to be

$$
\mathbf{Z}_{i\bar{i}} = 1 - \frac{\alpha_s}{4\pi}\left(8\cdot\boldsymbol{w}_{i\bar{i}}^{13}\frac{1}{\epsilon}\ln\left(\frac{-t_1}{m_t\cdot M}\right) + 8\boldsymbol{w}_{i\bar{i}}^{14}\frac{1}{\epsilon}\ln\left(\frac{-u_1}{m_t\cdot M}\right) + \frac{4\boldsymbol{w}_{i\bar{i}}^{33}}{\epsilon} - 2\boldsymbol{w}_{i\bar{i}}^{34}\frac{\left(\beta^2+1\right)\ln\left(x_s\right)}{\beta}\frac{1}{\epsilon}\right)\cdot\left(\mathbf{S}_{i\bar{i}}^{(0)}\right)^{-1}
$$

where $\left(\mathbf{S}_{i\bar{i}}^{(0)}\right)^{-1}$ is the inversed leading order soft matrix which is

$$
\left(\mathbf{S}_{q\bar{q}}^{(0)}\right)^{-1} = \begin{pmatrix} \frac{1}{N} & 0 \\ 0 & \frac{2}{C_F} \end{pmatrix}
$$

$$
\left(\mathbf{S}_{gg}^{(0)}\right)^{-1} = \begin{pmatrix} \frac{1}{N} & 0 & 0 \\ 0 & \frac{2}{N} & 0 \\ 0 & 0 & \frac{2N}{N^2-4} \end{pmatrix}
$$

and so the renormalized soft function is given by

$$
\begin{aligned}
\mathbf{S}_{i\bar{i}}^{(1),ren} =&4\boldsymbol{w}_{i\bar{i}}^{13}\Bigg(\left(L_\perp - \frac{1}{12}\epsilon\left(\pi^2-6L_\perp^2\right)\right)2\ln\left(\frac{-t_1}{m_t\cdot M}\right) \\
&\qquad - \mathrm{Li}_2\left(\frac{\beta^2\sin^2(t)}{\beta^2-1}\right)(L_\perp\epsilon+1) - \epsilon\mathrm{Li}_3\left(\frac{\beta^2\sin^2(t)}{\beta^2-1}\right)\Bigg) \\
&+ 4\boldsymbol{w}_{i\bar{i}}^{14}\Bigg(\left(L_\perp - \frac{1}{12}\epsilon\left(\pi^2-6L_\perp^2\right)\right)2\ln\left(\frac{-u_1}{m_t\cdot M}\right) \\
&\qquad - \mathrm{Li}_2\left(\frac{\beta^2\sin^2(t)}{\beta^2-1}\right)(L_\perp\epsilon+1) - \epsilon\mathrm{Li}_3\left(\frac{\beta^2\sin^2(t)}{\beta^2-1}\right)\Bigg) \\
&+ 2\boldsymbol{w}_{i\bar{i}}^{33}\Bigg(-2\left(\frac{1}{12}\epsilon\left(\pi^2-6L_\perp^2\right)-(L_\perp)\right) \\
&\qquad - 2\epsilon\left(L_\perp+\frac{1}{\epsilon}\right)\ln\left(\frac{(\beta^2-1)}{\beta^2\cos^2(\theta)-1}\right) - 2\epsilon\,\mathrm{Li}_3\left(-\frac{\beta^2\sin^2\theta}{1-\beta^2}\right)\Bigg) \\
&+ 2\boldsymbol{w}_{i\bar{i}}^{34}\Bigg(-\frac{(\beta^2+1)\ln(x_s)}{\beta}\left(L_\perp - \frac{1}{12}\epsilon\left(\pi^2-6L_\perp^2\right)\right) \\
&\qquad - f_{34}\frac{(\beta^2+1)}{\beta}(L_\perp\epsilon+1) - \epsilon J^{\epsilon^2}\frac{(\beta^2+1)}{\beta}\Bigg)
\end{aligned}
\tag{8.27}
$$

The renormalization group equation of the soft function is given in [3].
The contribution to order $\mathcal{O}(\epsilon)$ provides a prediction for the NNLO calculation. This can be explained by schematically expanding the soft function and its renormalization constant in order of α_s as

$$
\begin{aligned}
S &= S_0 + \alpha_s S_1 + \alpha_s^2 S_2 + ... \\
Z &= Z_0 + \alpha_s Z_1 + \alpha_s^2 Z_2 + ...
\end{aligned}
$$

Schematically one has to order α_s^2 the contributions

$$\mathcal{O}(\alpha_s^2) \sim Z_2 S_0 + Z_0 S_2 + Z_1 S_1$$

since S^1 was calculated up to $\mathcal{O}(\epsilon)$ and the renormalization constant Z_1 contains $1/\epsilon$ poles, one can predict one finite contribution that appears in the NNLO calculation. This is the finite term of $Z_1 S_1$.
So in this chapter the soft function of top quark pair production at small transverse momentum in the framework of SCET was calculated up to NLO. The NLO result is stated in eq. (8.27). The integration of the tedious NLO integrals was performed with the differential equation method. This method provides an improvement compared to the original method employed in [3]. With the differential equation method a systematization of the integration procedure was achieved. This method can be used as a starting point for the NNLO calculation. On top of that, the differential equation method offers the possibility to perform the NLO calculation in a higher order in ϵ than in [3]. In [3] the NLO soft function was stated up to $\mathcal{O}(\epsilon^0)$. I succeeded to calculate the renormalized and bare NLO soft function up to $\mathcal{O}(\epsilon)$. With the additional NLO result of order ϵ and the renormalization constant one obtains a prediction for one of the finite NNLO contributions.

9. Conclusion

The aim of my Master's thesis was to study the soft function of top quark pair production at small transverse momentum. For that purpose it is necessary to study the soft-collinear effective theory (SCET). SCET has been proven to be an efficient tool to deal with soft and collinear radiation and it provides a solution for factorization and resummation problems. In SCET, the different momentum regions are completely decoupled and this is how factorization of the different scales is achieved. In QCD soft and collinear radiation factorizes on parton level and it was shown that SCET reproduces the results obtained in full QCD.
I derived a factorization formula for the Drell-Yan process at small transverse momentum and for top quark pair production at small transverse momentum. These factorization formulas are composed of a hard, a collinear and a soft component. The hard part of the factorization formula reproduces the hard dynamics such as for Drell-Yan the partonic process $q\bar{q} \to l^+l^-$. This hard part depends on the large momentum transfer $Q^2 \gg q_T^2$. The collinear functions can be identified with the transverse parton distribution functions (TPDFs). I showed how to calculate them in SCET and I performed the matching calculation onto the collinear PDFs up to order α_s. The collinear part describes the initial state collinear radiation, it depends on the low-energy scales q_T and Λ_{QCD}. Finally the soft function describes the soft radiation from the initial and final states and it is also dependent on the low scale q_T. It was shown that the soft function is described by the vacuum expectation value of soft Wilson lines.
The soft function is of high interest because it is the only ingredient in the SCET factorization formula that is not yet known at NNLO. However the soft function even at NLO turns out to be very tricky to calculate. In order to calculate the NNLO soft function it is first necessary to improve the calculation method stated in [3]. This is what I managed in my thesis. I worked out the application of the Mellin-Barnes method and the differential equation method in order to calculate the NLO soft function. Finally the derivation of differential equations turns out to be sufficiently powerful to calculate the NLO soft function. In my Master's thesis I explicitly show all the necessary steps. Furthermore, the differential equation method can be used as a starting point for the calculation of the NNLO soft function. On top of that I extended the existing results. In [3] the NLO soft function was stated up to $\mathcal{O}(\epsilon^0)$. In my thesis I calculated the renormalized and bare NLO soft function up to $\mathcal{O}(\epsilon)$. With the knowledge of this additional contribution, one finite NNLO contribution is figured out.
The results for top quark pair production at small transverse momentum that are obtained in the framework of SCET reproduce the experimental measurements [3].
Transverse-momentum resummation for heavy-quark production in hadronic collisions can also be studied beyond SCET. This was performed in [72] where the results obtained in SCET were reproduced.

A. Anomalous dimensions

The anomalous dimensions are expanded in powers of α_s as [1]

$$\begin{aligned}
\Gamma^i_{cusp} &= \frac{\alpha_s}{4\pi}\Gamma^i_0 + \left(\frac{\alpha_s}{4\pi}\right)^2 \Gamma^i_1 + ... \\
\gamma^i &= \frac{\alpha_s}{4\pi}\gamma^i_0 + \left(\frac{\alpha_s}{4\pi}\right)^2 \gamma^i_1 + ...
\end{aligned} \tag{A.1}$$

with the coefficients

$$\begin{aligned}
\frac{\Gamma^q_0}{C_F} &= \frac{\Gamma^g_0}{C_A} = 4 \\
\frac{\Gamma^q_1}{C_F} &= \frac{\Gamma^g_1}{C_A} = 4\left(\frac{67}{9} - \frac{\pi^2}{3}\right) C_A - \frac{80}{9} T_F n_f \\
\gamma^q_0 &= -3C_F \\
\gamma^g_0 &= -\frac{11}{3}C_A + \frac{4}{3}T_F N_f
\end{aligned} \tag{A.2}$$

where n_f is the number of quark flavors. The β function is expanded as

$$\beta(\alpha_s) = -2\alpha_s\left(\beta_0\left(\frac{\alpha_s}{4\pi}\right) + \beta_1\left(\frac{\alpha_s}{4\pi}\right)^2 + \mathcal{O}(\alpha_s^3)\right)$$

with

$$\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F n_f$$

The anomalous exponent was written as

$$F_{i\bar{i}}(L_\perp, \alpha_s) = \sum_{n=1}^{\infty} d^i_n(L_\perp)\left(\frac{\alpha_s}{4\pi}\right)^n$$

with

$$\begin{aligned}
d^i_1(L_\perp) &= \Gamma^i_0 L_\perp \\
d^i_2(L_\perp) &= \frac{1}{2}\Gamma^i_0\beta_0 L^2_\perp + \Gamma^i_1 L_\perp + d^{i,0}_2 \\
d^{q,0}_2 &= 4C_F\left(\left(\frac{202}{27} - 7\zeta(3)\right) C_A - \frac{56}{27}T_F n_f\right)
\end{aligned}$$

B. The soft Wilson line

The soft radiation of a particle can be described by a so called Wilson line operator $S_n^R(x;a,b)$:

$$S_n^{(R)}(b,a) = P\ \exp\left(ig_s \int_a^b dt\, n^\mu A_\mu^c(x+tn)\mathbf{T}^{(R)\ c} e^{\eta t}\right) \tag{B.1}$$

For an outgoing particle which travels in the direction v in the representation R the Wilson line is defined by [25]

$$S_v^{(R)\dagger}(\infty,x) = P\ \exp\left(ig_s \int_0^\infty dt\, v^\mu A_\mu^c(x+vt)\mathbf{T}^{(R)c} e^{-\eta t}\right)$$

B.1. Incoming antiquark line

The same considerations as for the incoming quark can be done for an incoming antiquark that radiates a soft gluon.

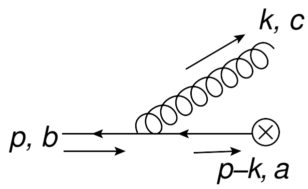


Figure B.1.: Soft gluon-radiation from an incoming antiquark line

Applying again the Feynman rules one has

$$\bar{v}(p)\ ig\gamma^\mu T_{ba}^c \left(-i\frac{\not{p}-\not{k}}{(p-k)^2+i\eta}\right)\epsilon_\mu^*(k) = -gT_{ba}^c \frac{p^\mu}{p\cdot k - i\eta}\epsilon_\mu^*(k)\bar{v}(p) \tag{B.2}$$

The corresponding Wilson line is

$$\begin{aligned} S_{n,ab}^{\bar{\mathbf{3}}}(0,-\infty) &= \left[P\ \exp\left(ig_s \int_{-\infty}^0 dt\, n^\mu A_\mu^c(x+tn)\bar{T}^c e^{\eta t}\right)\right]_{ab} \\ &= \left[\bar{P}\ \exp\left(-ig_s \int_{-\infty}^0 dt\, n^\mu A_\mu^c(x+tn)T^c e^{\eta t}\right)\right]_{ba} \end{aligned}$$

The index $\bar{\mathbf{3}}$ indicates that the radiating parton is an antiquark. And thus

$$\begin{aligned} \langle g(k)| \left[S_{p,ab}^{\bar{\mathbf{3}}}(0,-\infty)\right]|0\rangle &= \langle 0|\,\alpha(k)\left(-ig_s\int_{-\infty}^0 dt\, p^\mu A_\mu^c(tp)T_{ba}^c e^{\eta t}\right)|0\rangle \\ &= -igT_{ba}^c \int_{-\infty}^0 dt\ e^{iktp}e^{\eta t}\ p^\mu \epsilon_\mu^*(k) \\ &= -igT_{ba}^c \frac{p^\mu\epsilon_\mu^*(k)}{i\,(k\cdot p - i\eta)} = -g\ T_{ba}^c\ \frac{p^\mu\epsilon_\mu^*(k)}{k\cdot p - i\eta} \end{aligned}$$

which coincides with the result given in eq. (B.2).

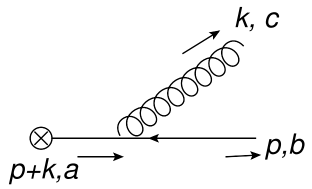


Figure B.2.: Soft gluon-radiation from an outgoing antiquark line

B.2. Outgoing antiquark

Fig. B.2 corresponds to the amplitude

$$-i\frac{\not{p}+\not{k}}{(p+k)^2+i\eta}ig\gamma^\mu T^c_{ab}v(p)\epsilon^*_\mu(k) = g\,T^c_{ab}\frac{p^\mu}{p\cdot k+i\eta}\epsilon^*_\mu(k)\;v(p) \tag{B.3}$$

The Wilson line describing soft radiation from an outgoing antiquark is $S^{\bar{\mathbf{3}}\dagger}_{v,ba}(\infty,0)$ which is

$$\begin{aligned}S^{\bar{\mathbf{3}}\dagger}_{v,ba}(\infty,0) &= \left[P\exp\left(ig_s\int_0^\infty dt\,v^\mu A^c_\mu(x+tv)\bar{T}^c e^{-\eta t}\right)\right]_{ba}\\ &= \left[\bar{P}\exp\left(-ig_s\int_0^\infty dt\,v^\mu A^c_\mu(x+tv)T^c e^{-\eta t}\right)\right]_{ab}\end{aligned}$$

and so one obtains

$$\begin{aligned}\langle g(k)|\left[S^{\bar{\mathbf{3}}\dagger}_{p,ba}(\infty,0)\right]|0\rangle &= \langle 0|\,\alpha(k)\left(-ig_s\int_0^\infty dt\,p^\mu A^c_\mu(tp)T^c_{ab}e^{-\eta t}\right)|0\rangle\\ &= g\,T_{ab}\,p^\mu\epsilon^*_\mu(k)\frac{1}{p\cdot k+i\eta}\end{aligned}$$

which corresponds to the result stated in eq. (B.3).

B.3. Outgoing quark

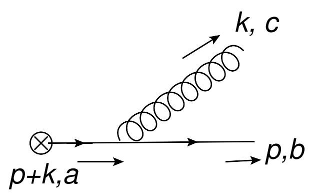


Figure B.3.: Soft gluon-radiation from an outgoing quark line

With the help of the Feynman rules one obtains

$$\bar{u}(p)\;ig\gamma^\mu T^c_{ba}\;i\frac{\not{p}+\not{k}}{(p+k)^2+i\eta}\epsilon^*_\mu(k) = -\bar{u}(p)\;g\;T^c_{ba}\frac{p^\mu}{p\cdot k+i\eta}\epsilon^*_\mu(k) \tag{B.4}$$

The Wilson line describing an outgoing quark is given by

$$[S^{\mathbf{3}\dagger}_v(\infty),0]_{ba} = \left[P\exp\left(ig_s\int_0^\infty dt\,n^\mu A^c_\mu(x+tn)T^c e^{-\eta t}\right)\right]_{ba}$$

So one obtains

$$\begin{aligned}\langle g(k)|\,[S_p^{\mathbf{3}\dagger}(\infty,0)]_{ba}\,|0\rangle &= \langle 0|\,\alpha(k)\left(ig_s\int_0^\infty dt\,p^\mu A_\mu^c(tp)T_{ba}^c e^{-\eta t}\right)|0\rangle \\ &= ig\,T_{ba}^c\,p^\mu \epsilon_\mu^*(k)\int_0^\infty dt\,e^{iktp}e^{-\eta t} \\ &= -ig\,T_{ba}^c\,p^\mu\epsilon_\mu^*(k)\frac{1}{i\,(kp+i\eta)} = -g\,T_{ba}^c\frac{p^\mu}{p\cdot k+i\eta}\epsilon_\mu^*(k)\end{aligned}$$

which coincides with eq. (B.4).

B.4. General concept

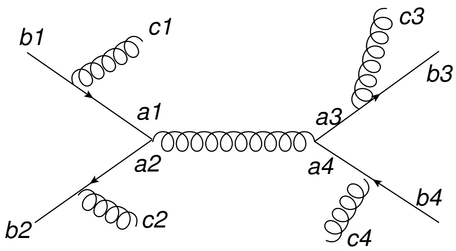


Figure B.4.: Soft gluon-radiation

- The radiation of soft gluons from the incoming quark is given by

$$\langle g(k_1)...g(k_n)|\left[S_p^{\mathbf{3}}(0;-\infty,0)\right]_{a_1b_1}|0\rangle$$

- The radiation of soft gluons from the incoming antiquark is given by

$$\langle g(k_1)...g(k_n)|\left[S_p^{\bar{\mathbf{3}}}(0;-\infty,0)\right]_{a_2b_2}|0\rangle$$

- The radiation of soft gluons from the outgoing quark is given by

$$\langle g(k_1)...g(k_n)|\left[S_p^{\mathbf{3}\dagger}(0;0,\infty)\right]_{b_3a_3}|0\rangle$$

- The radiation of soft gluons from the outgoing antiquark is given by

$$\langle g(k_1)...g(k_n)|\left[S_p^{\bar{\mathbf{3}}\dagger}(0;0,\infty)\right]_{b_4a_4}|0\rangle$$

B.5. Absorption of a soft gluon

Soft Wilson lines also describe the absorption of a soft gluon, e.g. the incoming quark absorbs a soft gluon which is described by the matrix element

$$\begin{aligned}\langle 0|\left[S_p^{\mathbf{3}}(0;-\infty,0)\right]_{a_1b_1}|g(k)\rangle &= \langle 0|\left[P\exp\left(ig_s\int_{-\infty}^0 dt\,p^\mu A_\mu^c(tp)T^c\right)\right]_{a_1b_1}\alpha^\dagger(k)\ |0\rangle \\ &= ig\int_{-\infty}^0 dt\,p^\mu\epsilon_\mu(k)T_{a_1b_1}^c e^{-iktp} = -g\,T_{a_1b_1}^c\ \frac{p^\mu\epsilon_\mu(k)}{k\cdot p+i\eta}\end{aligned}$$

This result is appropriate in the sense, that the only difference compared to Fig. (2.7) is that the denominator of the propagator is given by $(p+k)^2$.

B.6. The squared amplitude

In the end, one is interested in the cross section which is the squared matrix element. For this one has to consider for contributions like

$$\left[\langle g(k)|\,[S_i^{\mathbf{3}}(0;0,\infty)]\,|0\rangle\right]^{\dagger} = \langle 0|\left[S_i^{\mathbf{3}}(0;0,\infty)\right]^{\dagger}|g(k)\rangle$$

For illustration I compute the contribution of the incoming quark which is

$$\begin{aligned}
\langle 0|\left[S_p^{\mathbf{3}}(0;-\infty,0)\right]^{\dagger}_{ab}|g(k)\rangle &= \langle 0|\left[S_p^{\bar{\mathbf{3}}}(0;-\infty,0)\right]_{ba}|g(k)\rangle \\
&= \langle 0|\left[-igT^c_{ba}\int_{-\infty}^{0} dt\, p^{\mu}A^c_{\mu}(tp)\right]|g(k)\rangle \\
&= -ig\int_{-\infty}^{0} dt\, p^{\mu}\epsilon_{\mu}(k)T^c_{ba}e^{-iktp} = g\; T^c_{ba}\;\frac{p^{\mu}\epsilon_{\mu}(k)}{k\cdot p+i\eta}
\end{aligned}$$

In a similar way one computes the contributions from the other lines which are

- Incoming antiquark

$$\left(\langle g(k)|\left[S_p^{\bar{\mathbf{3}}}(0;-\infty,0)\right]_{b_2a_2}|0\rangle\right)^{\dagger} = g\; T^c_{a_2b_2}\;\frac{p^{\mu}\epsilon_{\mu}(k)}{k\cdot p+i\eta}$$

- Outgoing quark

$$\left(\langle g(k)|\,[S_p^{\mathbf{3}\dagger}(0;0,\infty)]_{a_3b_3}\,|0\rangle\right)^{\dagger} = g\; T^c_{b_3a_3}\frac{p^{\mu}}{p\cdot k-i\eta}\epsilon_{\mu}(k)$$

- Outgoing antiquark

$$\left(\langle g(k)|\left[S_p^{\bar{\mathbf{3}}\dagger}(0;0,\infty)\right]_{b_4a_4}|0\rangle\right)^{\dagger} = -g\; T_{b_4a_4}\; p^{\mu}\epsilon_{\mu}(k)\frac{1}{p\cdot k-i\eta}$$

C. The calculation of the integrals contributing to the one loop SCET current

C.1. Ultrasoft integral

In this section it is shown how the integral describing an ultrasoft gluon exchange in SCET (compare section 3.4.2) is evaluated. The integral under consideration is

$$\begin{aligned} I_s^{(1)} &= -2ig^2 C_F \left(\frac{e^{\gamma_E}\mu^2}{4\pi}\right)^{\epsilon_{UV}} \int \frac{d^dk}{(2\pi)^d} \frac{1}{n\cdot k - \frac{p^2}{\bar{n}p} - i\epsilon} \cdot \frac{1}{k^2 + i\epsilon} \cdot \frac{1}{\bar{n}k + \frac{q^2}{nq} + i\epsilon} \\ &= -ig^2 C_F \left(\frac{e^{\gamma_E}\mu^2}{4\pi}\right)^{\epsilon_{UV}} \int \frac{dk^+ dk^- d^{d-2}k_\perp}{(2\pi)^d} \frac{1}{n\cdot k - \frac{p^2}{\bar{n}p} - i\epsilon} \cdot \frac{1}{k^2 + i\epsilon} \cdot \frac{1}{\bar{n}k + \frac{q^2}{nq} + i\epsilon} \end{aligned}$$

Employing the contour integration method, the k^+ integration can be evaluated. For this purpose, the poles of the integrand have to be determined. We consider the k^+ dependent part of I_s.

$$\int dk^+ \; f(k^+) = \int dk^+ \frac{1}{k^+ - \frac{p^2}{\bar{n}p} - i\epsilon} \cdot \frac{1}{k^+k^- - k_\perp^2 + i\epsilon}$$

- 1. pole:

$$k_1^+ = \frac{p^2}{p^-} + i\epsilon$$

- 2. pole:

$$k_2^+ = \frac{k_\perp^2}{k^-} - \frac{i\epsilon}{k^-} \tag{C.1}$$

The cases $k^- > 0$ and $k^- < 0$ have to be considered separately. In the case of $k^- < 0$ the two residues have to be added and this gives zero.
In the case of $k^- > 0$ only one of the residues are inside the contour. Choosing the contour to be in the positive imaginary plane one has to consider only the residue of the pole k_1^+. The result is:

$$\oint dk^+ \; f(k^+) = \oint dk^+ \frac{1}{k^+ - \frac{p^2}{\bar{n}p} - i\epsilon} \cdot \frac{1}{k^+k^- - k_\perp^2 + i\epsilon} = 2\pi i \; \mathrm{Res}(f(k_1^+))$$

The residue of k_1^+ is:

$$\mathrm{Res}(f(k_1^+)) = \lim_{k^+ \to k_1^+} \left(f(k^+) \cdot (k^+ - k_1^+)\right) = ... = \frac{1}{p^2 \frac{k^-}{p^-} - k_\perp^2 - i\epsilon}$$

Because the original integral was not a closed loop integral, one still has to subtract the arc of the integration contour

$$\oint_{\mathcal{C}} f(z)\; dz = \int_{-a}^{a} dz \; f(z) + \int_{\mathrm{arc}} dz \; f(z)$$

One has

$$\left|\int_{\mathrm{arc}} dk^+ \; f(k^+)\right| \leq \frac{1}{k^+} \xrightarrow{k^+ \to \infty} 0$$

So the integration over k^+ can be written as

$$\int dk^+ \; f(k^+) = 2\pi i \, \frac{1}{p^2 \frac{k^-}{p^-} - k_\perp^2 - i\epsilon}$$

Taking into account that the $k_\perp$ integration is already defined in Euclidean space, one can apply the following general formula [49]

$$\int \frac{d^D l}{(2\pi)^D} \frac{1}{(l^2 + R + i\eta)} = \frac{1}{(4\pi)^{\frac{D}{2}}} \Gamma\left(1 - \frac{D}{2}\right) (R + i\eta)^{\frac{D}{2}-1} \tag{C.2}$$

with $D = d - 2 = 2 - 2\epsilon$ one has

$$i \int \frac{d^{d-2} k_\perp}{(2\pi)^{d-2}} \frac{1}{p^2 \frac{k^-}{p^-} - k_\perp^2 - i\epsilon} = -i \, \frac{\Gamma(\epsilon)}{(4\pi)^{1-\epsilon}} (-p^2 \frac{k^-}{p^-})^{-\epsilon}$$

and this gives the result

$$I_s^{(1)} = -g^2 C_F \, \left(e^{\gamma_E} \mu^2\right)^{\epsilon_{UV}} \int \frac{dk^-}{(2\pi)} \frac{\Gamma(\epsilon)}{(4\pi)} (-p^2 \frac{k^-}{p^-})^{-\epsilon} \frac{1}{k^- + \frac{q^2}{q^+}}$$

With the constraint of the k^+ integration, namely $k^- > 0$ one has to adjust the integration limits

$$I_s^{(1)} = -g^2 C_F \, \left(e^{\gamma_E} \mu^2\right)^{\epsilon_{UV}} \frac{\Gamma(\epsilon)}{8\pi^2} \left(\frac{-p^2}{p^-}\right)^{-\epsilon} \int_0^\infty dk^- \; (k^-)^{-\epsilon} \frac{1}{k^- + \frac{q^2}{q^+}}$$

In the preceding equation the divergence structure can be seen very clearly. In the limit $\epsilon \to 0$ the integral is logarithmically divergent $\int_0^\infty dk^- \; \frac{1}{k^- + \frac{q^2}{q^+}} \sim \ln \left(\frac{1}{k^- + \frac{q^2}{q^+}}\right)\Bigg|_0^\infty$. The infrared divergence $k^- \to 0$ is regulated by q^2/q^+. Thus this integral is purely ultraviolet divergent.
Considering the identity [42]

$$\int_0^\infty dx \; \frac{x^{\alpha-1}}{(A + x)^\beta} = A^{\alpha-\beta} \frac{\Gamma(\alpha)\Gamma(\beta - \alpha)}{\Gamma(\beta)}$$

one gets the following result:

$$I_s^{(1)} = -g^2 C_F \, \left(e^{\gamma_E} \mu^2\right)^{\epsilon_{UV}} \frac{\Gamma(\epsilon_{UV})}{8\pi^2} \left(\frac{-p^2}{p^-}\right)^{-\epsilon_{UV}} \left(\frac{q^2}{q^+}\right)^{-\epsilon_{UV}} \frac{\Gamma(\epsilon_{UV})\Gamma(1 - \epsilon_{UV})}{\Gamma(1)}$$

C.2. Collinear integral

The integral describing the contraction between the collinear Wilson line and the collinear quark was

$$I_N^{(1)} = \bar{u}_n(p) i \; g^2 \; C_F \left(\frac{e^{\gamma_E} \mu^2}{4\pi}\right)^{\epsilon_{UV}} \int \frac{dk^+ dk^- d^{d-2} k_\perp}{(2\pi)^d} \; \frac{p^- - k^-}{(p-k)^2 + i\epsilon} \cdot \frac{1}{k^2 + i\epsilon} \cdot \frac{1}{\bar{n} \cdot k + i\epsilon}$$

With the method of contour integration the k^+ integral can be evaluated. Consider:

$$\begin{aligned} I &= \int dk^+ \frac{(p^- - k^-)}{(p-k)^2 + i\epsilon} \cdot \frac{1}{k^2 + i\epsilon} \cdot \frac{1}{k^- + i\epsilon} \\ &= \int dk^+ \frac{(p^- - k^-)}{p^2 - p^+ k^- + k^+(k^- - p^-) + \vec{p}_\perp \vec{k}_\perp - \vec{k}_\perp^2 + i\epsilon} \cdot \frac{1}{k^+ k^- - \vec{k}_\perp^2 + i\epsilon} \cdot \frac{1}{k^- + i\epsilon} \end{aligned}$$

Since $p^\mu \sim n^\mu$ the vector $p_\perp$ is zero. The component p^+ has to be kept in order to have $p^2 = p^+ p^- \neq 0$. So one has:

$$I = \int dk^+ \frac{(p^- - k^-)}{p^2 - p^+ k^- + k^+(k^- - p^-) - \vec{k}_\perp^2 + i\epsilon} \cdot \frac{1}{k^+ k^- - \vec{k}_\perp^2 + i\epsilon} \cdot \frac{1}{k^- + i\epsilon}$$

In the expression of I there are two poles:

- 1. pole:

$$k_1^+ = \frac{p^+k^- + \vec{k}_\perp^2 - p^2}{k^- - p^-} - \frac{i\epsilon}{k^- - p^-}$$

- 2. pole:

$$k_2^+ = \frac{\vec{k}_\perp^2}{k^-} - \frac{i\epsilon}{k^-} \tag{C.3}$$

  This second pole in k^+ is the same as in the soft integral (see eq. (C.1)).

For the decision of the contour and the contained poles one has to distinguish two cases, namely $k^- > 0$ and $k^- < 0$.
Then one can apply the residue theorem which is

$$\oint_{\mathcal{C}} f(z)\,dz = 2\pi i \cdot \sum_{j=1}^{k} \mathrm{Res}(f(z_j))$$

where $\mathcal{C}$ is a closed, positive orientated, regular curve and the poles $z_1, ...z_k$ are in the area surrounded by $\mathcal{C}$.

1. Starting with $k^- < 0$ both poles k_1^+ and k_2^+ are in the positive imaginary plane. So one choose the integration contour to be a semi-cycle in the positive imaginary plane and one has to add the two residues of the poles. This exactly gives zero:

$$\mathrm{Res}(f(k_1^+)) + \mathrm{Res}(f(k_2^+)) = 0$$

   where $f(k^+)$ is the integrand of I.
   Because the original integral was not a closed loop integral, one still has to subtract the arc of the integration contour:

$$\oint_{\mathcal{C}} f(z)\,dz = \int_{-a}^{a} dz\ f(z) + \int_{\mathrm{arc}} dz\ f(z)$$

   To determine the contribution of $\int_{\mathrm{arc}} dz\ f(z)$ one has to examine the absolute value of $f(z)$:

$$|f(z)| \sim \frac{1}{(k^+)^2}$$

   so one has:

$$\left|\int_{\mathrm{arc}} dz\ f(z)\right| \leq \frac{1}{k^+} \xrightarrow{k^+\to\infty} 0$$

   That is why one can write for this integrand:

$$\oint_{\mathcal{C}} f(z)\,dz = \int_{-\infty}^{\infty} dz\ f(z) = 2\pi i \cdot \sum_{j=1}^{k} \mathrm{Res}(f(z_j))$$

2. Now the case $k^- > 0$ is considered. One has to distinguish whether $(k^- - p^-)$ is greater or less than zero.

   - For $k^- - p^- > 0$, both poles are in the negative imaginary plane and as before both residues have to be added and this yields zero.

- So as a last step one has to consider the case where $k^- - p^- < 0$. So k^- is bounded as

$$0 \le k^- \le p^-$$

So one can substitute $k^- = zp^-$ where z is in the domain $[0,1]$. In this case the pole k_1^+ is in the positive imaginary plane and k_2^+ is in the negative imaginary plane. The integration contour can be chosen to be in the positive imaginary plane. So for I one obtains

$$I = 2\pi i \, \text{Res}(f(k_1^+)) = 2\pi i \, \frac{1-z}{zp^-} \frac{1}{\vec{k}_\perp^2 - z(1-z)p^2 - i\epsilon}$$

And for the remaining integral one has

$$I_N^{(1)} = -g^2 \, C_F \, \bar{u}_n(p) \left(\frac{e^{\gamma_E}\mu^2}{4\pi} \right)^{\epsilon_{UV}} \int_0^1 \frac{dz}{2\pi} \int \frac{d^{d-2}k_\perp}{(2\pi)^{d-2}} \frac{(1-z)}{z(k_\perp^2 - z(1-z)p^2)}$$

By use of Eq. (C.2) the $k_\perp$ integration can be evaluated

$$\int \frac{d^D l}{(2\pi)^D} \frac{1}{(l^2 + R + i\eta)} = \frac{1}{(4\pi)^{\frac{D}{2}}} \Gamma\left(1 - \frac{D}{2}\right) (R + i\eta)^{\frac{D}{2}-1}$$

with $D = d - 2 = 2 - 2\epsilon$ one has

$$\begin{aligned} I_N^{(1)} &= -\frac{g^2}{8\pi^2} \frac{1}{(4\pi)^{-\epsilon}} \, C_F \, \bar{u}_n(p) \left(\frac{e^{\gamma_E}\mu^2}{4\pi} \right)^{\epsilon_{UV}} \int_0^1 dz \Gamma(\epsilon) \frac{(1-z)(-p^2 z(1-z))^{-\epsilon}}{z} \\ &= -\frac{g^2}{8\pi^2} \, C_F \, \bar{u}_n(p) e^{\epsilon_{UV}\gamma_E} \mu^{2\epsilon_{UV}} \int_0^1 dz \Gamma(\epsilon) \frac{(1-z)(-p^2 z(1-z))^{-\epsilon}}{z} \\ &= -\frac{g^2}{8\pi^2} \, C_F \, \bar{u}_n(p) e^{\epsilon_{UV}\gamma_E} \mu^{2\epsilon_{UV}} \int_0^1 dz \Gamma(\epsilon) \frac{(1-z)}{z} (-p^2)^{-\epsilon} \frac{1}{(z(1-z))^\epsilon} \end{aligned}$$

This can be evaluated with the help of the formule [42]

$$\int_0^1 dx \, x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

and one obtains

$$I_N^{(1)} = -\frac{g^2}{8\pi^2} \, C_F \, \bar{u}_n(p) e^{\epsilon_{UV}\gamma_E} \mu^{2\epsilon_{UV}} \Gamma(\epsilon)(-p^2)^{-\epsilon} \frac{\Gamma(-\epsilon)\Gamma(2-\epsilon)}{\Gamma(2-2\epsilon)}$$

D. The matrix elements of the TPDFs

I give the solutions of the matrix elements of the collinear TPDFs that are denoted as $\mathcal{M}_{i/j}(z,k)$ at next-to-leading order, compare chapter 5.6. These are defined as

$$\mathcal{B}^{(1)}_{i/j}(z,x_\perp) = \frac{\mu^{2\epsilon}e^{\epsilon\gamma}}{(4\pi)^\epsilon}\int \frac{d^dk}{(2\pi)^{d-1}}\delta\left(\bar{n}(k-(1-z)p)\right)e^{ik_Tx_T}\delta(k_+k_- - k_T^2)\theta(k_0)\cdot\mathcal{M}_{i/j}(z,k)$$

One obtains

$$\begin{aligned}
\mathcal{M}_{g/q}(z,k) &= \frac{4zp_-g^2C_F}{(p_-\cdot k_+)^2}\left(-(\epsilon-1)p\cdot k + \frac{k_T^2}{(p-k)\bar{n}}p\cdot\bar{n} + \frac{k_T^2}{((p-k)\bar{n})^2}(p\bar{n})\,(k\bar{n})\right)\\
\mathcal{M}_{q/g}(z,k) &= \frac{g^2T_F}{d-2}\frac{1}{2p^-k^+}\left(16k^-\left(\frac{k^-}{p^-}-1\right)+8(1-\epsilon)p^-\right)\\
\mathcal{M}_{g/g}(z,k) &= 4zp^-g^2\,C_A\,\frac{1}{k_T^2}\frac{1}{z^2}\left(1-z+z^2\right)^2
\end{aligned}$$

The matrix elements contain the average factor of spin and color. That is the origin of the factor $1/(d-2)$ in the gluon to quark splitting case.

Bibliography

[1] Thomas Becher, Alessandro Broggio, and Andrea Ferroglia. Introduction to Soft-Collinear Effective Theory. 2014.

[2] Giuseppe Bozzi, Luca Citelli, and Alessandro Vicini. PDF uncertainties on the W boson mass measurement from the lepton transverse momentum distribution. 2015.

[3] Hai Tao Li, Chong Sheng Li, Ding Yu Shao, Li Lin Yang, and Hua Xing Zhu. Top quark pair production at small transverse momentum in hadronic collisions. *Phys.Rev.*, D88:074004, 2013.

[4] Johann H. Kuhn and German Rodrigo. Charge asymmetries of top quarks at hadron colliders revisited. *JHEP*, 1201:063, 2012.

[5] John C. Collins, Davison E. Soper, and George F. Sterman. Transverse Momentum Distribution in Drell-Yan Pair and W and Z Boson Production. *Nucl.Phys.*, B250:199, 1985.

[6] Gabor Somogyi. Angular integrals in d dimensions. *J.Math.Phys.*, 52:083501, 2011.

[7] Ettore Remiddi. Differential equations for Feynman graph amplitudes. *Nuovo Cim.*, A110:1435–1452, 1997.

[8] John C. Collins. Sudakov form-factors. *Adv.Ser.Direct.High Energy Phys.*, 5:573–614, 1989.

[9] Valentin Ahrens. *Top-Quark Pair Production at Hadron Colliders.* PhD thesis, Johannes Guttenberg-Universitt Mainz, 2011.

[10] F. Liu and J.P. Ma. Glauber Gluons in Soft Collinear Effective Theory and Factorization of Drell-Yan Processes. 2008.

[11] John C. Collins, Davison E. Soper, and George F. Sterman. Soft Gluons and Factorization. *Nucl.Phys.*, B308:833, 1988.

[12] John C. Collins, Davison E. Soper, and George F. Sterman. All Order Factorization for Drell-Yan Cross-sections. *Phys.Lett.*, B134:263, 1984.

[13] Jonathan R. Gaunt, Maximilian Stahlhofen, and Frank J. Tackmann. The Quark Beam Function at Two Loops. *JHEP*, 1404:113, 2014.

[14] Iain W. Stewart, Frank J. Tackmann, and Wouter J. Waalewijn. The Quark Beam Function at NNLL. *JHEP*, 1009:005, 2010.

[15] Ilya Feige and Matthew D. Schwartz. Hard-Soft-Collinear Factorization to All Orders. *Phys.Rev.*, D90(10):105020, 2014.

[16] Stefano Catani and Massimiliano Grazzini. Infrared factorization of tree level QCD amplitudes at the next-to-next-to-leading order and beyond. *Nucl.Phys.*, B570:287–325, 2000.

[17] Matthew D. Schwartz. *Quantum Field Theory and the Standard Model.* Cambridge University Press, 2014.

[18] Christian W. Bauer and Iain W. Stewart. Invariant operators in collinear effective theory. *Phys.Lett.*, B516:134–142, 2001.

[19] Richard J. Hill and Matthias Neubert. Spectator interactions in soft collinear effective theory. *Nucl.Phys.*, B657:229–256, 2003.

[20] Ilya Feige and Matthew D. Schwartz. An on-shell approach to factorization. *Phys.Rev.*, D88(6):065021, 2013.

[21] Christian W. Bauer, Dan Pirjol, and Iain W. Stewart. Power counting in the soft collinear effective theory. *Phys.Rev.*, D66:054005, 2002.

[22] Ian W. Stewart. Lecture notes on the soft-collinear effective theory. `http://ocw.mit.edu/courses/physics/8-851-effective-field-theory-spring-2013/lecture-notes/MIT8_851S13_scetnotes.pdf`, 2013.

[23] Christian W. Bauer, Sean Fleming, Dan Pirjol, and Iain W. Stewart. An Effective field theory for collinear and soft gluons: Heavy to light decays. *Phys.Rev.*, D63:114020, 2001.

[24] Stefano Catani and Massimiliano Grazzini. The soft gluon current at one loop order. *Nucl.Phys.*, B591:435–454, 2000.

[25] M. Beneke, P. Falgari, and C. Schwinn. Soft radiation in heavy-particle pair production: All-order colour structure and two-loop anomalous dimension. *Nucl.Phys.*, B828:69–101, 2010.

[26] M. Czakon and P. Fiedler. The soft function for color octet production at threshold. *Nucl.Phys.*, B879:236–255, 2014.

[27] Christian W. Bauer, Sean Fleming, Dan Pirjol, Ira Z. Rothstein, and Iain W. Stewart. Hard scattering factorization from effective field theory. *Phys.Rev.*, D66:014017, 2002.

[28] Jui-Yu Chiu, Ambar Jain, Duff Neill, and Ira Z. Rothstein. A Formalism for the Systematic Treatment of Rapidity Logarithms in Quantum Field Theory. *JHEP*, 1205:084, 2012.

[29] Jui-yu Chiu, Ambar Jain, Duff Neill, and Ira Z. Rothstein. The Rapidity Renormalization Group. *Phys.Rev.Lett.*, 108:151601, 2012.

[30] Miguel G. Echevarra, Ahmad Idilbi, and Ignazio Scimemi. Soft and Collinear Factorization and Transverse Momentum Dependent Parton Distribution Functions. *Phys.Lett.*, B726:795–801, 2013.

[31] Thomas Becher and Guido Bell. Analytic Regularization in Soft-Collinear Effective Theory. *Phys.Lett.*, B713:41–46, 2012.

[32] Jui-yu Chiu, Andreas Fuhrer, Andre H. Hoang, Randall Kelley, and Aneesh V. Manohar. Soft-Collinear Factorization and Zero-Bin Subtractions. *Phys.Rev.*, D79:053007, 2009.

[33] M. Beneke and Vladimir A. Smirnov. Asymptotic expansion of Feynman integrals near threshold. *Nucl.Phys.*, B522:321–344, 1998.

[34] Aneesh V. Manohar and Iain W. Stewart. The Zero-Bin and Mode Factorization in Quantum Field Theory. *Phys.Rev.*, D76:074002, 2007.

[35] Valentin Ahrens, Andrea Ferroglia, Matthias Neubert, Ben D. Pecjak, and Li Lin Yang. Renormalization-Group Improved Predictions for Top-Quark Pair Production at Hadron Colliders. *JHEP*, 1009:097, 2010.

[36] Thomas Becher, Richard J. Hill, and Matthias Neubert. Soft collinear messengers: A New mode in soft collinear effective theory. *Phys.Rev.*, D69:054017, 2004.

[37] Christian W. Bauer, Dan Pirjol, and Iain W. Stewart. Soft collinear factorization in effective field theory. *Phys.Rev.*, D65:054022, 2002.

[38] Christian W. Bauer, Dan Pirjol, and Iain W. Stewart. On Power suppressed operators and gauge invariance in SCET. *Phys.Rev.*, D68:034021, 2003.

[39] Aneesh V. Manohar. Deep inelastic scattering as x $\to$ 1 using soft collinear effective theory. *Phys.Rev.*, D68:114019, 2003.

[40] I.A. Korchemskaya and G.P. Korchemsky. On lightlike Wilson loops. *Phys.Lett.*, B287:169–175, 1992.

[41] Vladimir A. Smirnov. Asymptotic expansions of two loop Feynman diagrams in the Sudakov limit. *Phys.Lett.*, B404:101–107, 1997.

[42] John Collins. *Foundation of Perturbative QCD*. Cambridge University Press, 2011.

[43] Geoffrey T. Bodwin. Factorization of the Drell-Yan Cross-Section in Perturbation Theory. *Phys.Rev.*, D31:2616, 1985.

[44] M. Beneke and T. Feldmann. Multipole expanded soft collinear effective theory with nonAbelian gauge symmetry. *Phys.Lett.*, B553:267–276, 2003.

[45] John C. Collins and Davison E. Soper. Parton Distribution and Decay Functions. *Nucl.Phys.*, B194:445, 1982.

[46] Thomas Gehrmann, Thomas Luebbert, and Li Lin Yang. Calculation of the transverse parton distribution functions at next-to-next-to-leading order. *JHEP*, 1406:155, 2014.

[47] Iain W. Stewart, Frank J. Tackmann, and Wouter J. Waalewijn. Factorization at the LHC: From PDFs to Initial State Jets. *Phys.Rev.*, D81:094035, 2010.

[48] B.R. Webber R.K. Ellis, W.J. Stirling. *QCD and Collider Physics*. Cambridge University Press, 1996.

[49] Daniel V. Schroeder Michael P. Peskin. *An Introduction to Quantum Field Theory*. Westview Press, 1995.

[50] Thomas Becher, Matthias Neubert, and Daniel Wilhelm. Higgs-Boson Production at Small Transverse Momentum. *JHEP*, 1305:110, 2013.

[51] Thomas Becher and Matthias Neubert. Drell-Yan production at small q_T, transverse parton distributions and the collinear anomaly. *Eur.Phys.J.*, C71:1665, 2011.

[52] J.A.M. Vermaseren. The Symbolic manipulation program FORM. 1992.

[53] Thomas Becher and Matthias Neubert. Factorization and NNLL Resummation for Higgs Production with a Jet Veto. *JHEP*, 1207:108, 2012.

[54] Thomas Becher, Matthias Neubert, and Ben D. Pecjak. Factorization and Momentum-Space Resummation in Deep-Inelastic Scattering. *JHEP*, 0701:076, 2007.

[55] Thomas Becher, Matthias Neubert, and Daniel Wilhelm. Electroweak Gauge-Boson Production at Small q_T: Infrared Safety from the Collinear Anomaly. *JHEP*, 1202:124, 2012.

[56] Aneesh V. Manohar and Mark B. Wise. Heavy quark physics. *Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol.*, 10:1–191, 2000.

[57] K.A. Olive et al. Review of Particle Physics. *Chin.Phys.*, C38:090001, 2014.

[58] Antonio Pich. Effective field theory: Course. pages 949–1049, 1998.

[59] P. Fiedler. Top pair cross section at NNLO. pages 161–164, 2014.

[60] Michal Czakon, Paul Fiedler, Alexander Mitov, and Juan Rojo. Further exploration of top pair hadroproduction at NNLO. 2013.

[61] Vladimir A. Smirnov. *Feynman Integral Calculus*. Springer-Verlag Berlin Heidelberg, 2006.

[62] T. Huber and Daniel Maitre. HypExp: A Mathematica package for expanding hypergeometric functions around integer-valued parameters. *Comput.Phys.Commun.*, 175:122–144, 2006.

[63] M. Czakon. Automatized analytic continuation of Mellin-Barnes integrals. *Comput.Phys.Commun.*, 175:559–571, 2006.

[64] Charalampos Anastasiou, Lance J. Dixon, Kirill Melnikov, and Frank Petriello. High precision QCD at hadron colliders: Electroweak gauge boson rapidity distributions at NNLO. *Phys.Rev.*, D69:094008, 2004.

[65] Charalampos Anastasiou, Lance J. Dixon, and Kirill Melnikov. NLO Higgs boson rapidity distributions at hadron colliders. *Nucl.Phys.Proc.Suppl.*, 116:193–197, 2003.

[66] Charalampos Anastasiou and Kirill Melnikov. Higgs boson production at hadron colliders in NNLO QCD. *Nucl.Phys.*, B646:220–256, 2002.

[67] K.G. Chetyrkin and F.V. Tkachov. Integration by Parts: The Algorithm to Calculate beta Functions in 4 Loops. *Nucl.Phys.*, B192:159–204, 1981.

[68] S. Laporta. High precision calculation of multiloop Feynman integrals by difference equations. *Int.J.Mod.Phys.*, A15:5087–5159, 2000.

[69] Charalampos Anastasiou and Achilleas Lazopoulos. Automatic integral reduction for higher order perturbative calculations. *JHEP*, 0407:046, 2004.

[70] A.V. Kotikov. Differential equation method. The Calculation of n point diagrams. 1993.

[71] A.V. Kotikov. The Differential equation method: Evaluation of complicated Feynman integrals. 2000.

[72] Stefano Catani, Massimiliano Grazzini, and A. Torre. Transverse-momentum resummation for heavy-quark hadroproduction. *Nucl.Phys.*, B890:518–538, 2014.