Integration of soft function at NNLO

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1 Introduction

Integration of the soft function relevant to small q_t resummation is addressed.

2 NLO graph contribution to S1bare in momentum space

$$S_d = \frac{2\pi^{(d+1)/2}}{\Gamma[(d+1)/2]} \tag{2.1}$$

$$q_T^{-2-2\alpha} \underbrace{\left(\frac{(2\pi)^{d-2}}{(2\pi)^{d/2-1}}\right)}_{\text{Fourier prefactors}} \underbrace{\left((4\pi)^2 \frac{e^{\epsilon\gamma_E}}{(4\pi)^\epsilon}\right)}_{\text{MSbar}} \underbrace{\left(\frac{2}{S_{1-2\epsilon}}\right)}_{\text{qT azimuthal average}} \mathbf{I}_{ij} \tag{2.2}$$

$$\mathbf{I}_{ij} = \frac{T_i \cdot T_j}{d_R} \equiv \int \frac{d^d k}{(2\pi)^{d-1} k_+^\alpha} \delta(k^2)\theta(k^0) \frac{-p_i \cdot p_j}{p_i \cdot k \, p_j \cdot k} \delta(k_T^2 - 1)$$

3 Double cuts over the boundary

In this section, we consider the most general integral that appear when dealing with diagrams with double cuts. These are all of the form

$$\int \frac{\frac{\mathrm{d}^d k}{(2\pi)^{d-1}} \frac{\mathrm{d}^d l}{(2\pi)^{d-1}} \left(\delta^+(k^2)\delta^+(l^2)\right) \left(\delta(|k_\perp + l_\perp|^2 - 1)\right)}{(\text{Rapidty})} \times \text{Graph part}\,, \tag{3.1}$$

where $\delta^+(k) \equiv \theta(k^0)\delta(k^2)$.

3.1 Integration of on-shell and observable condition

It is convenient to introduce the following vectors

$$n = (1,0,0,1)\,, \quad \bar{n} = (1,0,0,-1)\,, \qquad n \cdot \bar{n} = 2\,, \qquad n^2 = \bar{n}^2 = 0\,, \tag{3.2}$$

which are the directions of the colliding partons, and

$$k^\mu = n \cdot k \frac{\bar{n}^\mu}{2} + \bar{n} \cdot k \frac{n^\mu}{2} + k_\perp^\mu \tag{3.3}$$

$$\equiv k^+ \frac{\bar{n}^\mu}{2} + k^- \frac{n^\mu}{2} + k_\perp^\mu\,. \tag{3.4}$$

Then

$$k^+ = k^0 - k^z \qquad k^- = k^0 + k^z \tag{3.5}$$

$$k \cdot l = \frac{k^+ l^-}{2} + \frac{k^- l^+}{2} - l_T \cdot k_T. \tag{3.6}$$

Sebastian convention for the initial state partons is $p_1 = p_1^0 n$ and $p_2 = p_2^0 \bar{n}$. The integration measures in the transverse plane reads [1]

$$d^{d-2}k_\perp = \Omega_{1-2\epsilon}\, k_T^{1-2\epsilon} \sin^{-2\epsilon}\phi \, dk_T \, d\phi\,, \tag{3.7}$$

$$d^{d-2}l_\perp = \Omega_{2-2\epsilon}\, l_T^{1-2\epsilon} dl_T\,. \tag{3.8}$$

[1] This angle arises from $\int \mathrm{d}^{d-2}k_T = \int_0^\infty \mathrm{d}k_T k_T^{d-3} \int_{S_{d-3}} \mathrm{d}\Omega$ with $\int_{S_{d-3}} \mathrm{d}\Omega = S_{d-4} \int_{-1}^1 \mathrm{d}\cos\phi \sin^{-1-2\epsilon}$, where $S_{2\epsilon} = \frac{2(4\pi)^{-\epsilon}\Gamma[1-\epsilon]}{\Gamma[1-2\epsilon]}$. Sometimes, to integrate this angle it is convenient to change $\cos\phi \to 1 - 2\eta$, i.e. $\int_{-1}^1 \mathrm{d}\cos\phi \sin^{-1-2\epsilon} = \int_0^1 \mathrm{d}\eta\, 2(4\eta(1-\eta))^{-1/2-\epsilon}$.

where ϕ the latter is the angle between transverse components of gluons four-momenta. To integrate the observable delta function one can observe that

$$\delta\left(|k_\perp + l_\perp|^2 - 1\right) = \frac{\theta(k_\perp + l_\perp - 1)\theta(1-|k_\perp - l_\perp|)}{2k_\perp l_\perp}\delta\left(\cos\phi - \frac{1-k_\perp^2 - l_\perp^2}{2k_\perp l_\perp}\right). \tag{3.9}$$

Using this delta function to integrate ϕ and the on-shell conditions to integrate over l^- and l^+, Eq. (3.1) can be written as

$$\frac{\Omega_{1-2\epsilon}}{(2\pi)^{d-1}}\frac{\Omega_{2-2\epsilon}}{(2\pi)^{d-1}}\int k_\perp^{1-2\epsilon} dl_\perp l_\perp^{1-2\epsilon} dl_\perp \left(\frac{dk_+}{k_+^{1+\alpha}}\frac{dl_+}{l_+^{1+\alpha}}\right)\left\{\frac{1}{2k_\perp l_\perp}\left(\frac{(1-(k_\perp+l_\perp)^2)(1-(k_\perp-l_\perp)^2)}{4k_\perp^2 l_\perp^2}\right)^{-1/2-\epsilon}\right\}$$
$$\times \text{ Graph part}\,, \tag{3.10}$$

where the domain of integration is understood to satisfy the theta and delta function constrains. From now on, we shall only focus on these regions. Also remember that $\Omega_{d-1} \equiv 2\pi^{d/2}/\Gamma[d/2]$

3.2 Relevant regions

In this section, we present the regions of the integration domain in which the integrand of Eq. (3.10) vanishes.

3.2.1 Propagators of incoming partons

- Soft region: $\lambda\,(k^+, k_\perp^2/(2k^+), k_\perp)$, $\lambda \ll 1$. Within the integration domain, this constrain implies $l_\perp \to 1$, i.e. gluon l cannot be soft or collinear with the incoming partons.
- Collinear region: $(k^+, \lambda^2 k_\perp^2/(2k^+), \lambda k_\perp)$. Within the integration domain, this constrain implies $l_\perp \to 1$, i.e. gluon l cannot be soft or collinear with the incoming partons. Note that a second global re-scaling would make gluon k soft, i.e. the soft and collinear regions are not disjoint.
- "Rapidity region": $(\lambda k^+, k_\perp^2/(\lambda 2k^+), k_\perp)$. Note that parton k is not necessarily soft[2]. Also note that gluon k can not be collinear, as a collinear re-scaling would invalidate the first scaling.

The rapidity region leads to divergences in the Eikonal approximation but not for the exact theory.

3.2.2 Eikonal propagators of $(t, \bar{t})$

Clearly, these only vanish in the soft region, e.g. $(\lambda\, k^+, \lambda\, k_\perp^2/(2k^+)$ and $l_\perp \to 1$, and it is convenient to write these propagators as

$$k \cdot v_i = k^0 \frac{k \cdot v_i}{k_0} = \sqrt{2}\left(k^+ + \frac{k_\perp^2}{2k^+}\right)\frac{k \cdot v_i}{k_0}. \tag{3.11}$$

[2] A second global re-scaling would make it soft.

3.2.3 Divergences of exact propagators

After evaluation of the azimuth ϕ, this exact propagator reads

$$2k \cdot l = \left(-1 + (k_\perp^2(k^+ + l^+))/k^+ + ((k^+ + l^+)l_\perp^2)/l^+\right) \tag{3.12}$$

One can show that[3] the only points, in the domain of integration, for which this propagators identically vanish are

$$\left\{k_\perp = \frac{k^+}{k^+ + l^+}, l_\perp = \frac{l^+}{k^+ + l^+}\right\}. \tag{3.13}$$

Note that only one of the two gluons can have vanishing transverse momentum.

[3]See mathematica notebook

4 Analytical integration on the boundary

In this section, we are slopy with ep prefactor.

We perform analytical integration of the following integral over the boundary $\beta = 1$[4]:

$$I_1 = \int \mathrm{d}^d k\, \mathrm{d}^d l \left(\delta^+(k^2)\delta^+(l^2)\right)\left(\delta(|k_\perp + l_\perp|^2 - 1)\right) \times \left[\frac{1}{(n\cdot l\, n\cdot k)^\alpha\, n\cdot l\, \overline{n}\cdot k\, v_3\cdot l\, v_4\cdot k}\right]. \quad (4.1)$$

Integrating out the delta function one gets

$$\begin{aligned} I_1(v_3^0 v_4^0) = \Omega_{1-2\epsilon}\Omega_{2-2\epsilon} \int k_\perp^{1-2\epsilon}\mathrm{d}k_\perp l_\perp^{1-2\epsilon} dl_\perp \frac{\mathrm{d}k^+}{k^+}\frac{\mathrm{d}l^+}{l^+}&\left\{\frac{1}{2k_\perp\, l_\perp}\left(\frac{(1-(k_\perp+l_\perp)^2)(1-(k_\perp-l_\perp)^2)}{4k_\perp^2 l_\perp^2}\right)^{-1/2-\epsilon}\right\} \\ &\times \theta(k_\perp + l_\perp - 1)\theta(1 - |k_\perp - l_\perp|) \\ &\times 4\left[\frac{k_+^2 l_+}{(k^+ l^+)^\alpha\; l^+ k_\perp^2\, (k_+^2 + k_\perp^2)\, (l_+^2 + l_\perp^2)}\right] \end{aligned} \quad (4.2)$$

The first two lines lines are common to all scalar integrals, they correspond to measure and observable factors. After trivial re-arrangements this integral can be re-expressed as

$$\begin{aligned} &= 2^{1+2\epsilon}\Omega_{1-2\epsilon}\Omega_{2-2\epsilon}\int \mathrm{d}k_\perp \mathrm{d}l_\perp \mathrm{d}l^+ \mathrm{d}k^+ \frac{(f(k_\perp, l_\perp))^{-1/2-\epsilon}}{(k^+ l^+)^\alpha}\theta(k_\perp + l_\perp - 1)\theta(|k_\perp - l_\perp| - 1) \\ &\qquad \times 4\left(\frac{k^+}{k_\perp(k_+^2 + k_\perp^2)}\right)\left(\frac{l_\perp}{l^+(l_+^2 + l_\perp^2)}\right). \end{aligned} \quad (4.3)$$

where $f(k_\perp, l_\perp) = (1-(k_\perp + l_\perp)^2)(1-(k_\perp - l_\perp)^2)$. From this expression we can see that the integral contains divergences in the following regions

- $k_\perp \to \infty$. This "rapidity" divergence is regularised only if $\alpha > 0$
- $k_\perp, k^+ \to 0$ and $k_\perp \to 0$.
- $l_\perp, l^+ \to 0$ and $l_+ \to 0$. This latter "rapidity" divergence is regularised only if $\alpha < 0$

Hence, this integral is not a simple function of α. The integration over l^+ is straightforward by assuming $\alpha > 0$, and performing the remaining integration assuming that α, ϵ are small and such that $\alpha < 2\epsilon$. The analytic expression

$$\frac{I_1}{(p_3^0 p_4^0)\Omega_{1-2\epsilon}\Omega_{2-2\epsilon}} = \frac{\pi^3 \csc^2\left(\frac{\pi\alpha}{2}\right) 2^{\alpha+2\epsilon-1}\Gamma\left(\frac{1}{2}-\epsilon\right)\Gamma\left(-\epsilon-\frac{\alpha}{2}\right)\Gamma(\epsilon+\alpha+1)}{\Gamma\left(\frac{\alpha}{2}+1\right)^2\Gamma\left(-\epsilon-\frac{\alpha}{2}+\frac{1}{2}\right)} \quad (4.4)$$

On the other hand, this same integral can be computed by means of sector decomposion. The following mappings are needed to get rid of overlapping singularities

1. $$k_T = \frac{1 + y_T - x_T y_T}{2(1-x_T)}\,, \qquad l_T = \frac{1 - y_T + x_T y_T}{2(1-x_T)}\,, \quad (4.5)$$

   $$x_T = 1 - \frac{1}{k_T + l_T}\,, \qquad y_T = k_T - l_T\,. \quad (4.6)$$

   Where the new variables should be integrated over the domain $0 \leq x_T \in\leq 1, |y_T| \leq 1$. The points $k_T \to 0$ and $l_T \to 0$ in the new variables have moved to $x_T \to 0$ with $y_T = -1$ and $y_T = 1$.

[4] i.e., $\{v_3, v_4\}$ only have energy components.

2. In the present case, there is not manifold singularity but an analogous mapping is needed to separate the singularities at $y_T = -1$ and $y_T = 1$. For this propose, we simple break the integration domain into $y_T \in [0,1]$ and $y_T = [-1,0]$ and make the transformation $y_T \to -y_T$ for the latter case.

3. The third mapping separates regions with finite and vanishing transverse momentum, rendering four integrals For this propose, we break the integration domain at $y_T = 1/2$ and apply the transformations
$$\tilde{y}_T = 2(1 - y_T)\,, \tag{4.7}$$
and
$$\tilde{y}_T = 2y_T\,, \tag{4.8}$$
This completes the set of transformations required for the transverse components.

4. Now one needs to compress the domain of integration of the plus components, this is achieved by the transformations
$$k_+ = \frac{x}{1-x}\,, \qquad l_+ = \frac{y}{1-y}\,, \tag{4.9}$$
whose inverse reads
$$x = \frac{k_+}{1+k_+}\,, \qquad y = \frac{l_+}{1+l_+}\,. \tag{4.10}$$
The domain of integration of the new variables is $x, y \in [0,1]$.

5. Note that $x = 0$ corresponds to $k_T \to 0$ and $x = 1$ to $k_T \to \infty$. To separate this two divergences the following mapping is the same as in step 3, i.e., break the integration at x=1/2 and apply the changes of variables
$$\tilde{x} = 2(1 - x)\,, \tag{4.11}$$
and
$$\tilde{x} = 2x. \tag{4.12}$$

By doing so, Sectors.m can perform sector decomposition and Cuba perform the resulting integrals expanded up to finite terms in α and ϵ. The resulting expression is
$$\begin{aligned}\frac{I_1}{(p_3^0 p_4^0)\Omega_{1-2\epsilon}\Omega_{2-2\epsilon}} =& 7.63552 + \frac{8.71346}{\alpha^2} - \frac{8.17249}{\alpha} + \frac{1.57118}{\epsilon^3} \\ &+ \frac{2.17773}{\epsilon^2} - \frac{3.14235}{\epsilon^2\alpha} + \frac{9.25216}{\epsilon} + \frac{6.2847}{\alpha^2} - \frac{4.3568}{\alpha\epsilon}.\end{aligned} \tag{4.13}$$

Each of the poles and the finite part in this expression was computed with PresitionGoal 4 (the number of digits of relative accuracy to seek, that is, ?rel = 10?PrecisionGoal). By comparing the analytic and numeric calculation, on can see that, in the total result, the first three non-zero digits exactly agree, e.g. 3.14, and the corrections are of the order of are in the next digit.

4.1 Some comments about evaluations 5/12/16

Cuba package for numerical integration contain four independent routines (methods) for numerical integration, Suave, Vegas, Divonne and Cuhre. When, the precision goal is set as relative accuracy of the first two packages are not good in dealing with a multidimensional integral whose addends add up exactly to zero. Furthermore, Divonne is not able to do the integration as it reports that there is a divergence of the form $1/\sqrt{0}$, possibly introduced by factor $(f(k_\perp, l_\perp))^{-1/2-\epsilon}$

of the integrand. Fortunately, Cuhre is able to do all multidimensional integrals, but this library is not able to deal with single dimensional integrals which we are performing using Suave. In the future we shall work with absolute accuracy instead of relative accuarecy.

Our current version of SectorsGLogs.m has only only improvement with respect to Sectors.m: it makes the ϵ and α expansion without expanding the arguments of the resulting logarithms. At least at first sight, produces a smaller amount of integrals and the numerical evaluation seems to improve.

5 Outside of the boundary

For simplicity, from now we ommit μ, nu and a_s we will be keep all other ep prefactors correctly. The expression for a graph, with m loops and n on-shell soft partons, in momentum space is

$$I_G^{n,m} = c(\epsilon) \int_{S_1^{1-2\epsilon}} \frac{\mathrm{d}\Omega(\hat{q}_\perp)}{S_{d-3}} \left[\prod_i^n \int \frac{\mathrm{d}^d q_i}{(2\pi)^{d-1} (q_i^+)^\alpha} \delta^+(q_i) \right] \delta^{(d-2)} \left(\sum_i \vec{q}_{i\perp} - \vec{q}_\perp \right) \{\text{Graph dependent part}\} \tag{5.1}$$

where

$$c(\epsilon) = \underbrace{\left(\frac{(2\pi)^{d-2}}{(2\pi)^{d/2-1}} \right)}_{\text{Fourier prefactors}} \underbrace{\left((4\pi)^2 \frac{e^{\epsilon \gamma_E}}{(4\pi)^\epsilon} \right)^{n+m}}_{\text{MSbar}} \tag{5.2}$$

and to illustrate precisely what we mean by graph depedent part, we give to two examples. At NLO ($n = 1, m = 0$) the graph connecting i and j reads

$$\{\text{Graph dependent part}\} \to \frac{T_i \cdot T_j}{d_R} \frac{-p_i \cdot p_j}{(p_i \cdot q)(p_j \cdot q)} \tag{5.3}$$

At NNLO, a graph connecting four different hard partons reads

$$\{\text{Graph dependent part}\} \to \frac{1}{d_R} \left(T_i \cdot T_j \frac{-p_i \cdot p_j}{(p_i \cdot q)(p_j \cdot q)} \right) \left(T_l \cdot T_m \frac{-p_l \cdot p_m}{(p_l \cdot q)(p_m \cdot q)} \right) \tag{5.4}$$

6 Azimuthal integration of $q_\perp$ in $d - 3$ dims

With the help of the identity

$$\int_{S_1^{1-2\epsilon}} \mathrm{d}\Omega(\hat{q}_\perp) f(\vec{q}_\perp) = \int \mathrm{d}^{d-2} q'_\perp \delta(q'_\perp - q_\perp)(q'_\perp)^{3-d} f(\vec{q}_\perp\,') \tag{6.1}$$

$$= \int \mathrm{d}^{d-2} q'_\perp \delta((q'_\perp)^2 - q_\perp^2) 2 (q'_\perp)^{4-d} \theta(q'_\perp) f(\vec{q}_\perp\,'), \tag{6.2}$$

one can write

$$\int_{S_1^{1-2\epsilon}} \mathrm{d}\Omega(\hat{q}_\perp)\, \delta^{(d-2)}(\vec{k}_\perp + \vec{l}_\perp - \vec{q}_\perp) = \delta(|k_\perp + l_\perp|^2 - q_\perp^2) 2 (q_\perp)^{4-d} \tag{6.3}$$

Hence the form of the integral reduces to

$$I_G^{n,m} = c(\epsilon) \left(\frac{2(q_\perp)^{4-d}}{S_{d-3}} \right) \left[\prod_i^n \int \frac{\mathrm{d}^d q_i \delta^+(q_i)}{(2\pi)^{d-1} (q_i^+)^\alpha} \right] \delta \left(\left| \sum_i \vec{q}_{i\perp} \right|^2 - q_\perp^2 \right) \{\text{Graph dependent part}\}. \tag{6.4}$$

7 Double cuts for general kinematics

We shall use the following parametrisation for the heavy quarks

$$v_3 = \frac{1}{\sqrt{1-\beta^2}} (1, \beta \cos(\theta), \beta \sin(\theta) \hat{n}_\perp) \tag{7.1}$$

$$v_4 = \left(\frac{1}{\sqrt{1-\beta^2}}, -\vec{v}_3 \right). \tag{7.2}$$

where $\hat{n}_\perp$ is an unit vector in the $d-2$ dimensions transverse to the incoming partons. At we will see, at $\mathrm{N^2LO}$, the integration over the transverse momentum components of the on-shell emissions give deliver scalar expressions of the form

$$\int \mathrm{d}^{d-2} l_\perp \mathrm{d}^{d-2} k_\perp f(\vec{l}_\perp, \vec{k}_\perp, \vec{v}_{3\perp}) = g(v_{3\perp}^2). \tag{7.3}$$

This expression shows that the choice of $\hat{n}_\perp$ is completely irrelevant, see the rest of this section. Furthermore, by rotational invariance in $d-2$ dimensions, one can write this expression as

$$g(v_{3\perp}^2) = \frac{1}{S_1^{1-2\epsilon}} \int_{S_1^{1-2\epsilon}} \mathrm{d}\Omega(v_3) g(v_{3\perp}^2) \tag{7.4}$$

The reason behind of this apparently unnecessary complication is explained below. We start from

$$\int \mathrm{d}^{d-2} l_\perp \mathrm{d}^{d-2} k_\perp f(\vec{l}_\perp, \vec{k}_\perp, \vec{v}_{3\perp}) = \frac{1}{S_1^{1-2\epsilon}} \int_{S_1^{1-2\epsilon}} \mathrm{d}\Omega(\text{angles } v_3) \int \mathrm{d}^{d-2} l_\perp \mathrm{d}^{d-2} k_\perp f(\vec{l}_\perp, \vec{k}_\perp, \vec{v}_{3\perp}). \tag{7.5}$$

Since the integrand can only depend on the scalar products between $\vec{k}_\perp, \vec{l}_\perp$ and $\vec{p}_3$ and due to the rotational invariance of these scalar products, the integrand can always be rotate to a frame in which

$$\vec{k}_\perp = k_\perp (1, 0, \vec{0}_{d-4}), \tag{7.6}$$
$$\vec{l}_\perp = l_\perp (\cos\theta_1, \sin\theta_1, \vec{0}_{d-4}), \tag{7.7}$$
$$\vec{v}_{3\perp} = v_{3\perp} (\cos\theta_2, \sin\theta_2 \cos\theta_3, \sin\theta_2 \sin\theta_3, \vec{0}_{d-6}). \tag{7.8}$$

On the other hand, the differential measure of these vectors can be written as

$$\begin{aligned} \int \mathrm{d}^{d-2} k_\perp &= \int \mathrm{d}k_\perp k_\perp^{d-3} \int_{S_1^{1-2\epsilon}} \mathrm{d}\Omega(\ldots), \\ \int \mathrm{d}^{d-2} l_\perp &= \int \mathrm{d}l_\perp l_\perp^{d-3} \int_{S_1^{1-2\epsilon}} \mathrm{d}\Omega(\theta_1, \ldots), \\ \int_{S_1^{1-2\epsilon}} & \mathrm{d}\Omega(v_{3\perp}(\theta_2, \theta_3, \ldots)). \end{aligned} \tag{7.9}$$

Clearly, all the angles denoted by ellipsis can be trivially integrated. The motivation for introducing the additional integration, over the angles of $\vec{v}_{3\perp}$, is that it allow us to use the analysis infrared divergences identically to the analysis over the boundary cuts setting two soft lines on-shell. Let us show this more explicitly. Consider the expression for a graph in such case, Eq. (6.4) with $n=2$ and $m=0$, with the results of this section this can be written as

$$I_G^{2,0} = c(\epsilon) \left(\frac{2(q_\perp)^{4-d}}{S_{d-3}} \right) \int_{S_1^{1-2\epsilon}} \frac{\mathrm{d}\Omega(v_3)}{S_1^{1-2\epsilon}} \int \frac{\mathrm{d}^d k \delta^+(k)}{(2\pi)^{d-1}(l^+)^\alpha} \int \frac{\mathrm{d}^d l \delta^+(l)}{(2\pi)^{d-1}(k^+)^\alpha} \delta\left(\left| \vec{k}_\perp + \vec{l}_\perp \right|^2 - \right) \tag{7.10}$$

$$\{\text{Graph dependent part}\}. \tag{7.11}$$

where we have re-scaled $l \to q_T l$ and $k \to q_T k$ and factor the qT dependence. The details for the azimuthal integration over v_3 are presented below. As on the analysis over the boundary, we integrate the on-shell conditions using the k^-, l^- components reducing the above expression to

$$I_G^{2,0} = \frac{2q_T^{-2-2\epsilon}}{S_1^{1-2\epsilon}} \frac{1}{(2\pi)^{2d-2}} \left(\int_{S_1^{1-2\epsilon}} \frac{\mathrm{d}\Omega(v_3)}{S_1^{1-2\epsilon}} \right) \left(\int \mathrm{d}k_\perp k_\perp^{1-2\epsilon} \frac{\mathrm{d}k_+}{(k_+)^{1+\alpha}} S_1^{1-2\epsilon} \right) \left(\int \mathrm{d}l_\perp l_\perp^{1-2\epsilon} \frac{\mathrm{d}l_+}{(l_+)^{1+\alpha}} \int_{S_{1-2\epsilon}} \mathrm{d}\Omega(l_\perp) \right)$$
$$\times\, \delta\left(\left| \vec{k}_\perp + \vec{l}_\perp \right|^2 - 1 \right) \times \{\text{Graph dependent part}\}. \tag{7.12}$$

Furthermore, as it was done over the boundary, we use the θ_1 in Eq. (7.7) to integrate the delta function of the observable to get, i.e. we use

$$\delta\left(|k_\perp + l_\perp|^2 - 1 \right) = \frac{\theta(k_\perp + l_\perp - 1)\theta(1 - |k_\perp - l_\perp|)}{2k_\perp l_\perp} \delta\left(\cos\theta_1 - \frac{1 - k_\perp^2 - l_\perp^2}{2k_\perp l_\perp} \right), \tag{7.13}$$

$$\int_{S_{1-2\epsilon}} \mathrm{d}\Omega(l_\perp) = S_1^{-2\epsilon} \int_0^\pi \mathrm{d}\theta_1 \sin^{-2\epsilon}\theta_1 \tag{7.14}$$

to write

$$I_G^{2,0} = \frac{q_T^{-2-2\epsilon}}{S_{1-2\epsilon}} \frac{S_{1-2\epsilon} S_{-2\epsilon}}{(2\pi)^{2d-2}} \int_{S_{1-2\epsilon}} \frac{2\mathrm{d}\Omega(v_3)}{S_{1-2\epsilon}} \int \mathrm{d}k_\perp k_\perp \mathrm{d}l_\perp l_\perp \frac{\mathrm{d}k_+}{k_+} \frac{\mathrm{d}l_+}{l_+}$$
$$\frac{1}{2} \left(\frac{(1 - (k_\perp + l_\perp)^2)(1 - (k_\perp - l_\perp)^2)}{4} \right)^{-1/2-\epsilon}$$
$$\theta(k_\perp + l_\perp - 1)\theta(1 - |k_\perp - l_\perp|)$$
$$\times \{\text{Graph dependent part}\}. \tag{7.15}$$

We remark that the integration of the observable delta function has only been easy because the simplifications shown in Eq. (7.9). To better handle notation in the next section, it will be convenient to write the last equation as

$$I_G^{2,0} = \frac{2q_T^{-2-2\epsilon}}{S_{1-2\epsilon}} \frac{S_{1-2\epsilon} S_{-2\epsilon}}{(2\pi)^{2d-2}} \int_{S_{1-2\epsilon}} \frac{2\mathrm{d}\Omega(v_3)}{S_{1-2\epsilon}} f(\beta, \theta, \theta_1, \theta_2) \tag{7.16}$$

7.1 Details on azimuth averaging

Firstly, let us definte the solid angle:

$$S_d = \frac{2\pi^{(d+1)/2}}{\Gamma[(d+1)/2]} \tag{7.17}$$

The formula we use for the azimuthal averaged over the angles of v_3, θ_2 and θ_3, is

$$\frac{1}{S_1^{1-2\epsilon}} \int_{S_1^{1-2\epsilon}} \mathrm{d}\Omega(v_3) f(\beta, \theta, \theta_2, \theta_3) = \frac{1}{S_1^{1-2\epsilon}} \int_{-1}^{1} \mathrm{d}\cos\theta_2 \, \sin^{-1-2\epsilon}\theta_2 \times \frac{(4\pi)^{-\epsilon}\Gamma[1-\epsilon]}{\Gamma[1-2\epsilon]} \times$$
$$\left[\int_0^1 \mathrm{d}\cos\theta_3 \left(\delta(1 - \cos\theta_3) f(\beta, \theta, \theta_2, 0) - 2\epsilon \frac{4^\epsilon \Gamma[1-2\epsilon]}{\Gamma^2[1-\epsilon]} \frac{f(\beta, \theta, \theta_2, \theta_3) - f(\beta, \theta, \theta_2, 0)}{(1 - \cos^2\theta_3)^{1+\epsilon}} \right) + \right.$$
$$\left. \int_{-1}^{0} \mathrm{d}\cos\theta_3 \left(\delta(1 + \cos\theta_3) f(\beta, \theta, \theta_2, \pi) - 2\epsilon \frac{4^\epsilon \Gamma[1-2\epsilon]}{\Gamma^2[1-\epsilon]} \frac{f(\beta, \theta, \theta_2, \theta_3) - f(\beta, \theta, \theta_2, \pi)}{(1 - \cos^2\theta_3)^{1+\epsilon}} \right) \right].$$

where

$$S_{-2\epsilon} = 2\frac{(4\pi)^{-\epsilon}\Gamma[1-\epsilon]}{\Gamma[1-2\epsilon]} \tag{7.18}$$

$$S_{1-2\epsilon} = \frac{\pi 4^{\epsilon}\Gamma[1-2\epsilon]}{\Gamma^2[1-\epsilon]}S_{-2\epsilon}. \tag{7.19}$$

This was presented in Ref. [1] but we have also re-derived. To map this integral to a hypercube we use the following changes of variables in the second and thirds line respectively:

$$\begin{aligned}
&\cos\theta_2 = 1-2\cos^2(\chi\pi/2), \qquad \cos\theta_3 = 1-\eta_3,\ \sin^2\theta_3 = \eta_3(2-\eta_3)\\
&\cos\theta_2 = 1-2\cos^2(\chi\pi/2), \qquad \cos\theta_3 = \eta_3-1,\ \sin^2\theta_3 = \eta_3(2-\eta_3).
\end{aligned}$$

From these changes one gets

$$\begin{aligned}
&\int_{S_{1-2\epsilon}} \frac{\mathrm{d}\Omega(v_3)}{S_{1-2\epsilon}} f(\beta,\theta,\theta_2,\theta_3) = \frac{4^{\epsilon}S_{-2\epsilon}\pi}{S_{1-2\epsilon}2}\int_0^1 \mathrm{d}\chi\ \sin^{-2\epsilon}(\pi\chi)\times\\
&\left[\int_0^1 \mathrm{d}\eta_3 \left(4^{-\epsilon}\delta(1-\cos\theta_3)f(\beta,\theta,\theta_2,\theta_3) - \frac{\epsilon}{\frac{4^{\epsilon}S_{-2\epsilon}\pi}{S_{1-2\epsilon}2}}\frac{f(\beta,\theta,\theta_2,\theta_3)-f(\beta,\theta,\theta_2,0)}{(1-\cos^2\theta_3)^{1+\epsilon}}\right) + \right.\\
&\left.\int_0^1 \mathrm{d}\eta_3 \left(4^{-\epsilon}\delta(1+\cos\theta_3)f(\beta,\theta,\theta_2,\pi) - \frac{\epsilon}{\frac{4^{\epsilon}S_{-2\epsilon}\pi}{S_{1-2\epsilon}2}}\frac{f(\beta,\theta,\theta_2,\theta_3)-f(\beta,\theta,\theta_2,\pi)}{(1-\cos^2\theta_3)^{1+\epsilon}}\right)\right].
\end{aligned} \tag{7.20}$$

The relevant prefactor is

$$\frac{S_{-2\epsilon}\pi}{S_{1-2\epsilon}2} = \frac{4^{-\epsilon}\Gamma[1-\epsilon]^2}{2\Gamma[1-2\epsilon]}. \tag{7.21}$$

8 N approach

Using Eq. (7.20), the Eq. (7.16) for any graph reads:

$$I_G^{2,0} = q_T^{-2-2\epsilon}\underbrace{\left(\frac{(2\pi)^{d-2}}{(2\pi)^{d/2-1}}\right)}_{\text{Fourier prefactors}}\underbrace{\left((4\pi)^2\frac{e^{\epsilon\gamma_E}}{(4\pi)^{\epsilon}}\right)^2}_{\text{MSbar}}\underbrace{\left(\frac{4^{\epsilon}S_{-2\epsilon}\pi}{S_{1-2\epsilon}2}\right)}_{\text{pt azimuthal average}}\underbrace{\left(\frac{2}{S_{1-2\epsilon}}\right)}_{\text{qT azimuthal average}}\underbrace{\left(\frac{S_{1-2\epsilon}S_{-2\epsilon}}{(2\pi)^{2d-2}}\right)}_{\text{irrelevant angles}}G(\beta,\theta,\epsilon,\alpha) \tag{8.1}$$

where

$$\begin{aligned}
&G(\beta,\theta,\epsilon,\alpha) = \left(\frac{4^{\epsilon}S_{-2\epsilon}\pi}{S_{1-2\epsilon}2}\right)^{-1}\int_{S_{1-2\epsilon}} \frac{\mathrm{d}\Omega(v_3)}{S_{1-2\epsilon}} f(\beta,\theta,\theta_2,\theta_3) = \int_0^1 \mathrm{d}\chi\ \sin^{-2\epsilon}(\pi\chi)\times\\
&\left[\int_0^1 \mathrm{dcos}\,\eta_3 \left(4^{-\epsilon}\delta(1-\cos\theta_3)f(\beta,\theta,\theta_2,\theta_3) - \frac{\epsilon}{\frac{4^{\epsilon}S_{-2\epsilon}\pi}{S_{1-2\epsilon}2}}\frac{f(\beta,\theta,\theta_2,\theta_3)-f(\beta,\theta,\theta_2,0)}{(1-\cos^2\theta_3)^{1+\epsilon}}\right) + \right.\\
&\left.\int_{-1}^0 \mathrm{dcos}\,\eta_3 \left(4^{-\epsilon}\delta(1+\cos\theta_3)f(\beta,\theta,\theta_2,\pi) - \frac{\epsilon}{\frac{4^{\epsilon}S_{-2\epsilon}\pi}{S_{1-2\epsilon}2}}\frac{f(\beta,\theta,\theta_2,\theta_3)-f(\beta,\theta,\theta_2,\pi)}{(1-\cos^2\theta_3)^{1+\epsilon}}\right)\right]
\end{aligned} \tag{8.2}$$

$$f(\beta,\theta,\theta_2,\theta_3) = \int \mathrm{d}k_\perp k_\perp \mathrm{d}l_\perp l_\perp \frac{\mathrm{d}k_+}{k_+^{1+alpha}}\frac{\mathrm{d}l_+}{l_+^{1+alpha}}\frac{1}{2}\left(\frac{(1-(k_\perp+l_\perp)^2)(1-(k_\perp-l_\perp)^2)}{4}\right)^{-1/2-\epsilon}$$

$$\theta(k_\perp + l_\perp - 1)\theta(1 - |k_\perp - l_\perp|)\{\text{Graph dependent part}\}. \tag{8.3}$$

To get this result in position space, $\widetilde{I}_G^{2,0}$, we still need to fourier transform factor, $\mathrm{FT}(\epsilon, \alpha, Lp)$. of $q_T^{-2-2\epsilon-2\alpha}$,i.e.

$$\widetilde{I}_G^{2,0} = \underbrace{\mathrm{FT}(\epsilon,\alpha,Lp)\underbrace{\left(\frac{(2\pi)^{d-2}}{(2\pi)^{d/2-1}}\right)}_{\text{Fourier prefactors}}\underbrace{\left((4\pi)^2\frac{e^{\epsilon\gamma_E}}{(4\pi)^\epsilon}\right)^2}_{\text{MSbar}}\underbrace{\left(\frac{4^\epsilon S_{-2\epsilon}\pi}{S_{1-2\epsilon}2}\right)}_{\text{pt azi average}}\underbrace{\left(\frac{2}{S_{1-2\epsilon}}\right)}_{\text{qT azi average}}\underbrace{\left(\frac{S_{1-2\epsilon}S_{-2\epsilon}}{(2\pi)^{2d-2}}\right)}_{\text{irrelevant angles}}}_{C(\epsilon,\alpha,Lp)} G(\beta,\theta,\epsilon,\alpha) \tag{8.4}$$

At his point, the objecs in this formula has the expansions

$$C(\epsilon,\alpha,Lp) = \sum_{i=-1,j=0} \epsilon^i\,\alpha^j\,c_{i,j}(Lp), \tag{8.5}$$

$$G(\beta,\theta,\epsilon,\alpha) = \sum_{i=-3,j=-2} \epsilon^i\,\alpha^j\,g_{i,j}(\theta,\beta) \tag{8.6}$$

The final formula we need to define the N approach is

$$\widetilde{I}_G^{2,0} = \overbrace{\frac{\mathrm{FT}(\epsilon,\alpha,Lp)}{c_{-1,0}(0)}\underbrace{\left(\frac{(2\pi)^{d-2}}{(2\pi)^{d/2-1}}\right)}_{\text{Fourier prefactors}}\underbrace{\left((4\pi)^2\frac{e^{\epsilon\gamma_E}}{(4\pi)^\epsilon}\right)^2}_{\text{MSbar}}\underbrace{\left(\frac{4^\epsilon S_{-2\epsilon}\pi}{S_{1-2\epsilon}2}\right)}_{\text{pt azi average}}\underbrace{\left(\frac{2}{S_{1-2\epsilon}}\right)}_{\text{qT azi average}}\underbrace{\left(\frac{S_{1-2\epsilon}S_{-2\epsilon}}{(2\pi)^{2d-2}}\right)}_{\text{irrelevant angles}}}^{\equiv\text{GMaster}}\underbrace{G(\beta,\theta,\epsilon,\alpha)c_{-1,0}(0)}_{\equiv\text{SecDec}} \tag{8.7}$$

In our mathematica modules, we compute each $\widetilde{I}_G^{2,0}$ by sending $\underbrace{G(\beta,\theta,\epsilon,\alpha)c_{1,0}(0)}_{\equiv\text{SecDec}}$, to sector decomposition, then multiply by, the graph independent prefactor, GMaster to get the complete result. The normalisation $c_{-1,0}(0)$ is such the numerical integration of order ep is multiplied by 1.

9 Weight and Infrared part

It is convenient to express

$$\{\text{Graph dependent part}\} = \underbrace{(\{\text{Graph dependent part}\}|_{\beta=0})}_{\text{Infrared part}}\underbrace{\left(\frac{\{\text{Graph dependent part}\}}{\{\text{Graph dependent part}\}|_{\beta=0}}\right)}_{\text{Weight part}} \tag{9.1}$$

For any graph, the infrared part is not only independent of β but also of $\theta, \theta_2, \theta_3$, and it has divergences only associated to the boundary. The weight part has no extra divergences. Hence, only the infrared part is relevant for sector decomposition.

10 Power counting (for the infrared part)

It is convenient to map each double cut with a tripple gluon vertex into 16 mappings, and 8 otherwise, such that divergences exist only when some of the coordinates in these mappings vanish. Some of this mappings are unnecessary and limit the efficientcy of our numerics. So out of these total mappings we want to keep only those which are necessary! Let us denote by $k^+, k^-, k_T, l_T \to x_1, x_2, x_3, x_4$ one of such changes of mappings. To know whether the mapping associated to a coordinate x_r is necessary we do the following power counting which gives, sufficient but necessary condition: rescale $x_r \to \lambda x_r$, and the other variables $(i \neq r)$ in the folliwing posibilities

$$x_i \to \lambda_i^n x_i, \text{ with } n_i = 0 \text{ or } n_i = 1. \tag{10.1}$$

If for some $\{n_i\}$ one has that the integrand (without the integration measure) scales as λ^n with $n \geq 1 + n_2 + n_4 + n_5$, then this mapping is necessary because there is divergence associated to x_r! Then, according to this criteria, if there is a divergence, we should keep such mapping. For all double cuts except the gauge and the quark bubble this critira can be used to rule out uncessary mappings!

For the bubble the condition is slightly more complicated. However, we do not discuss this further as these were computed not only numerical but analytically.

11 Soft gluon double cuts

There are two ways of calculating the double cut graphs: keeping only physical polarisation for ε_i OR keeping all polarisations and introducing ghost. We do this calculation simultaneously in the light-cone gauge and in the Feynman gauge.

11.1 Amplitude

Following, add Eq., Catani-Grazzini the, tree-level, amplitude for any hard subprocess $\mathcal{M}$ with two soft emission $\{q_1, q_2$ can be written as te action of double emission tensor $\mathbf{J}^{\mu_1\mu_2}$ on the tree level hard sub-process, i.e.

$$\langle a_1 a_2 \,|\mathcal{M}_{+2}\rangle = \varepsilon_1^{\mu_1}\varepsilon_2^{\mu_2}\,\widetilde{\mathbf{J}}^{a_1a_2}_{\mu_1\mu_2}(q_1,q_2)\ |\mathcal{M}\rangle \tag{11.1}$$

In the Feynman gauge $\xi = 0$ and and in light-cone gauge $\xi = 1$ this expression is equal to

$$\begin{aligned}
\widetilde{\mathbf{J}}^{a_1a_2}_{\mu_1\mu_2} &= \left[\sum_i \left[\frac{\mathbf{T}_i^{a_1}p_{i\mu_1}}{p_i\cdot q_1}\frac{\mathbf{T}_i^{a_2}p_{i\mu_2}}{p_i\cdot(q_1+q_2)} + \frac{\mathbf{T}_i^{a_2}p_{i\mu_2}}{p_i\cdot q_2}\frac{\mathbf{T}_i^{a_1}p_{i\mu_1}}{p_i\cdot(q_1+q_2)}\right] + \sum_{i\neq j}\frac{\mathbf{T}_i^{a_1}p_{i\mu_1}}{p_i\cdot q_1}\frac{\mathbf{T}_j^{a_2}p_{j\mu_2}}{p_j\cdot q_2}\right.\\
&\left. + \sum_{i\neq j}\frac{V^{\mu_1\mu_2\nu}_{\text{complete}}(q_1,q_2)}{2q_1\cdot q_2}\left[g^{\nu\beta} - \xi\frac{(q_1+q_2)^{\{\nu}\bar{n}^{\beta\}}}{(q_1+q_2)\cdot\bar{n}}\right]\frac{if^{a_1a_2c}\mathbf{T}_i^c p_{i\beta}}{p_i\cdot(q_1+q_2)}\right],\\
V^{\text{complete}}_{\mu_1\mu_2\nu} &\equiv \frac{(-2q_2+q_1)_{\mu_1}g_{\mu_2\nu} + (2q_1-q_2)_{\mu_2}g_{\mu_1\nu} + (q_2-q_1)_\nu g_{\mu_1\mu_2}}{2q_1\cdot q_2 p_i\cdot(q_1+q_2)}\,.
\end{aligned}$$

The terms proportional to ξ cancels due to the Ward identities:

$$\begin{gathered}
\sum_i \xi\frac{(q_1+q_2)^\beta}{(q_1+q_2)\cdot\bar{n}}\frac{if^{a_1a_2c}\mathbf{T}_i^c p_{i\beta}}{p_i\cdot(q_1+q_2)} = \xi\frac{if^{a_1a_2c}}{(q_1+q_2)\cdot\bar{n}}\sum_i^n \mathbf{T}_i^c = 0,\\
\varepsilon_{1\mu_1}\varepsilon_{2\mu_2}V^{\mu_1,\mu_2,\nu}_{\text{complete}}(q_1+q_2)_\nu = 0
\end{gathered} \tag{11.2}$$

Hence, the double emission tensor reduces to tensor in the Feynman gauge. In both gauges $q_i\cdot\varepsilon_i = 0$. In particular, this means that one can neglect *logitudinal terms*, i.e. one can use $V(q_1,q_2)$ the reduced triple gluon vertex[5] Hence, both in the light-cone and Feynman gauge the amplitude is given by

$$\begin{aligned}
\mathbf{J}^{a_1a_2}_{\mu_1\mu_2}(q_1,q_2) &= \left[\sum_i \frac{\mathbf{T}_i^{a_1}p_{i\mu_1}}{p_i\cdot q_1}\frac{\mathbf{T}_i^{a_2}p_{i\mu_2}}{p_i\cdot(q_1+q_2)} + \frac{\mathbf{T}_i^{a_2}p_{i\mu_2}}{p_i\cdot q_2}\frac{\mathbf{T}_i^{a_1}p_{i\mu_1}}{p_i\cdot(q_1+q_2)} + \sum_{i\neq j}\frac{\mathbf{T}_i^{a_1}p_{i\mu_1}}{p_i\cdot q_1}\frac{\mathbf{T}_j^{a_2}p_{j\mu_2}}{p_j\cdot q_2}\right.\\
&\left. + \sum_i^n V_i^{\mu_1\mu_2\nu} if^{a_1a_2c}\mathbf{T}_i^c\right].\\
V^i_{\mu_1\mu_2} &= \frac{-2q_{2\mu_1}g_{\mu_2\nu} + 2q_{1\mu_2}g_{\mu_1\nu} + (q_2-q_1)_\nu g_{\mu_1\mu_2}}{2q_1\cdot q_2 p_i\cdot(q_1+q_2)}p_i^\nu
\end{aligned}$$

This expression is exact accordance with Eq. (101) of Ref. [2]. Thanks, to the colour conservation property of the hard-subprocess. This tensor is gauge invariant in the following sense

$$\begin{aligned}
q_1^{\mu_1}\widetilde{\mathbf{J}}^{\mu_1\mu_2} &= 0,\\
q_2^{\mu_2}\widetilde{\mathbf{J}}^{\mu_1\mu_2} &= 0\ .
\end{aligned} \tag{11.3}$$

[5]This step is NOT valid if all polarisation are included and ghost are introduced.

Following, this Ref., we write the double emission current in terms of an Abelian and a Non-Abelian part, i.e.

$$
\begin{aligned}
\widetilde{\mathbf{J}}^{a_1 a_2}_{\mu_1\mu_2}(q_1, q_2) = \sum_{i,j}^{n} \frac{1}{2}\{\mathbf{T}_i^{a_1}, \mathbf{T}_j^{a_2}\} \frac{p_{i\mu_1}}{p_i \cdot q_1} \frac{p_{j\mu_2}}{p_j \cdot q_2} \\
+ \sum_{i}^{n} i f^{a_1 a_2 c} \mathbf{T}_i^c \left(A^i_{\mu_1\mu_2} + V^i_{\mu_1\mu_2}\right) \ . \\
A^i_{\mu_1,\mu_2} \equiv \frac{p_{i\mu_1} p_{i\mu_2}}{2p_i \cdot (q_1 + q_2)} \left(\frac{1}{p_i \cdot q_1} - \frac{1}{p_i \cdot q_2}\right)
\end{aligned}
\tag{11.4}
$$

Again, this is in exact accordance with Eq. (102) of the same Ref.

11.2 Cross-section

This yields

$$
|\mathcal{M}_{+2}|^2 = \langle \mathcal{M}| \, \widetilde{\mathbf{J}}^\dagger_{\nu_1\nu_2} \left(\sum_{\sigma_1} \varepsilon_1^* \varepsilon_1\right)^{\nu_1\nu_2} \left(\sum_{\sigma_2} \varepsilon_2^* \varepsilon_2\right)^{\nu_2\mu_2} \widetilde{\mathbf{J}}_{\mu_1\mu_2} \, |\mathcal{M}\rangle \, . \tag{11.5}
$$

In principle, we should introduce the matrix elements

$$
\left(\sum_{\sigma_i} \varepsilon_2^+ \varepsilon_i\right)^{\nu_i\mu_i} = -g^{\nu_i\mu_i} + (\text{Gauge dependent terms })^{\nu_i\mu_i} \ , \tag{11.6}
$$

however we can only keep the first term because the gauge dependent terms, which should be either proportional to $q_i^{\mu_i}$ or $q_i^{\nu_i}$, give a vanishing contribution in view Eq .(11.3). Hence, the conclusion of this section is that the unpolarised double emission matrix elements yield:

$$
|\mathcal{M}_{+2}|^2 = \langle \mathcal{M}| \, \widetilde{\mathbf{J}}^\dagger_{\mu_1\mu_2} \widetilde{\mathbf{J}}^{\mu_1\mu_2} \, |\mathcal{M}\rangle \, . \tag{11.7}
$$

11.3 Symmetric and antisymmetric parts of the cross-section

In this section, we organise the different contributions of double gluon emission matrix elements to the cross-section. To obtain the cross-section, Eq. (11.7), one need to square Eq. (11.4),

$$
\begin{aligned}
\widetilde{\mathbf{J}}^\dagger_{\mu_1\mu_2} \widetilde{\mathbf{J}}^{\mu_1\mu_2} = \frac{1}{2}\left\{\mathbf{J}^2(q_1), \mathbf{J}^2(q_1)\right\} \\
+ \sum_{i\neq j}^{n} C_A \mathbf{T}_i \cdot \mathbf{T}_j \left(A^i + V^i\right)_{\mu_1\mu_2} \left(A^j + V^j\right)^{\mu_1\mu_2} \\
+ \sum_{i}^{n} C_A \mathbf{T}_i^2 \left(A^i + V^i\right)_{\mu_1\mu_2} \left(A^i + V^i\right)^{\mu_1\mu_2} \\
+ \sum_{i\neq j}^{n} \frac{C_A \mathbf{T}_i \cdot \mathbf{T}_j}{2} \left(\frac{p_i^{\mu_1} p_j^{\mu_2} - p_j^{\mu_1} p_i^{\mu_2}}{p_i \cdot q_1 p_j \cdot q_2}\right) \left(A^i + V^i - (i \to j)\right)_{\mu_1\mu_2}
\end{aligned}
\tag{11.8}
$$

Henceforth, we shall refer to the first and bottom lines of this expression as the Abelian and Non-Abelian parts. Furthermore, below, we will refer to terms $\sim V^2$ as gluon bubble contributions and to other terms as the non-bubble contributions.

11.4 The gluon bubble

In this section we express the fermion bubble contributions in terms of the gluon bubble contributions. Let start us start recalling that the integrand corresponding to the fermion bubble graph, G_{ij}^{Fermion}, connecting the wilson lines i and j yields

$$\begin{aligned} \mathbf{G}_{ij}^{\text{Fermion}} &= n_f \mathbf{T}_i \cdot \mathbf{T}_j G_{ij}^{\text{Fermion}} \\ G_{ij}^{\text{Fermion}} &= \frac{p_i \cdot q_1 p_j \cdot q_2 + p_i \cdot q_2 p_j \cdot q_1 - p_i \cdot p_j \, q_1 \cdot q_2}{p_i \cdot (q_1 + q_2) p_j \cdot (q_1 + q_2) q_1 \cdot q_2}, \end{aligned} \tag{11.9}$$

Simple colour conservation shows that[6] shows that the total contribution of the fermion bubbles is

$$G^{\text{Fermion}} = \sum_{i,j} \mathbf{G}_{ij}^{\text{Fermion}} = \sum_{i \neq j} n_f \mathbf{T}_i \cdot \mathbf{T}_j \left(G_{ij}^{\text{Fermion}} - G_{ii}^{\text{Fermion}}/2 - G_{jj}^{\text{Fermion}}/2 \right) \tag{11.10}$$

Simple algebra shows[7]

$$\begin{aligned} G_{ij}^{\text{Fermion}} - G_{ii}^{\text{Fermion}}/2 - G_{jj}^{\text{Fermion}}/2 = \frac{(p_i \cdot q_1 p_j \cdot q_2 - p_i \cdot q_2 p_j \cdot q_1)^2}{p_i \cdot (q_1 + q_2)^2 p_j \cdot (q_1 + q_2)^2 q_1 \cdot q_2^2} \\ + \frac{p_i^2}{2 p_i \cdot (q_1 + q_2)^2 q_1 \cdot q_2} + \frac{p_j^2}{2 p_j \cdot (q_1 + q_2)^2 q_1 \cdot q_2} - \frac{p_i \cdot p_j}{p_i \cdot (q_1 + q_2) p_j \cdot (q_1 + q_2) q_1 \cdot q_2}. \end{aligned} \tag{11.11}$$

Let us now denote $\mathbf{G}_{ij}^{\text{gluon}}$ the contribution of the gluon defined as

$$\begin{aligned} \mathbf{G}_{ij}^{\text{Gluon}} &= C_A \mathbf{T}_i \cdot \mathbf{T}_j G_{ij}^{\text{Gluon}} \\ G_{ij}^{\text{Gluon}} &= \frac{p_{j\beta} V^{\mu_1 \mu_2 \beta} V_{\mu_1 \mu_2 \nu} p_i^{\nu}}{(2 q_1 \cdot q_2)^2 p_i (q_1 + q_2) p_j (q_1 + q_2)}. \end{aligned} \tag{11.12}$$

Again, one can write, the total contribution of the gluon bubble as

$$G^{\text{Gluon}} = \sum_{i,j} \mathbf{G}_{ij}^{\text{Gluon}} = \sum_{i \neq j} n_f \mathbf{T}_i \cdot \mathbf{T}_j \left(G_{ij}^{\text{Gluon}} - G_{ii}^{\text{Gluon}}/2 - G_{jj}^{\text{Gluon}}/2 \right) \tag{11.13}$$

and straightforward algebra shows that[8]

$$\begin{aligned} & G_{ij}^{\text{Gluon}} - G_{ii}^{\text{Gluon}}/2 - G_{jj}^{\text{Gluon}}/2 \\ & = \frac{-d+2}{2} \Bigg(G_{ij}^{\text{Fermion}} - G_{ii}^{\text{fermion}}/2 - G_{jj}^{\text{Fermion}}/2 \\ & - \left(\frac{p_i^2}{2 p_i \cdot (q_1 + q_2)^2 q_1 \cdot q_2} + \frac{p_j^2}{2 p_j \cdot (q_1 + q_2)^2 q_1 \cdot q_2} - \frac{p_i \cdot p_j}{p_i \cdot (q_1 + q_2) p_j \cdot (q_1 + q_2) q_1 \cdot q_2} \right) \Bigg) \end{aligned} \tag{11.14}$$

I think that this is a convenient way to express the gluon bubble because in this way, up to the term in curly braces, we don't have to calculate additional integrals with complicated numerators.

[6] See section Fermion bubble section of mathematica notebook attached

[7] See the fermion bubble part in mathematica file

[8] See the gluon bubble part mathematica file

11.5 Non-bubble contributions of the non-abelian part

The gluon bubble part is given by

$$\widetilde{\mathbf{J}}^{\dagger}_{\mu_1\mu_2}\widetilde{\mathbf{J}}^{\mu_1\mu_2} = \frac{1}{2}\left\{\mathbf{J}^2(q_1), \mathbf{J}^2(q_1)\right\}$$
$$+\sum_{i,j} C_a \mathbf{T}_i\cdot\mathbf{T}_j \left(\frac{p_{i\mu_1}p_{i\mu_2}}{2p_i\cdot(q_1+q_2)}\left(\frac{1}{p_i\cdot q_1} - \frac{1}{p_i\cdot q_2}\right) + \frac{V_{\mu_1\mu_2\nu}p_i^\nu}{2q_1\cdot q_2\, p_i\cdot(q_1+q_2)}\right)(\text{same with} i\to j)_{\mu_1\mu_2} \tag{11.15}$$

12 On the equivalence between using physical polarisations or unphysical plus ghost

$$\mathbf{K}^{\mu_1\mu_2} \equiv \widetilde{\mathbf{K}}^{\mu_1\mu_2} + \mathbf{T}_{q_1}\frac{q_2^{\mu_2}g_\alpha^{\mu_1} - q_1^{\mu_1}g_\alpha^{\mu_2}}{2q_1\cdot q_2}\sum_i \mathbf{T}_i\frac{p_i^\alpha}{p_i\cdot(q_1+q_2)} \tag{12.1}$$

where

$$\begin{aligned}\widetilde{\mathbf{K}}^{a_1a_2}_{\mu_1\mu_2}(q_1,q_2) =& \sum_i \frac{\mathbf{T}_i^{a_1}p_{i\mu_1}}{p_i\cdot q_1}\frac{\mathbf{T}_i^{a_2}p_{i\mu_2}}{p_i\cdot(q_1+q_2)} + \frac{\mathbf{T}_i^{a_2}p_{i\mu_2}}{p_i\cdot q_2}\frac{\mathbf{T}_i^{a_1}p_{i\mu_1}}{p_i\cdot(q_1+q_2)} + \sum_{i\neq j}\frac{\mathbf{T}_i^{a_1}p_{i\mu_1}}{p_i\cdot q_1}\frac{\mathbf{T}_j^{a_2}p_{j\mu_2}}{p_j\cdot q_2} \\ &+ \sum_i^n V_i^{\mu_1\mu_2\nu} i f^{a_1a_2c}\mathbf{T}_i^c\,. \end{aligned} \tag{12.2}$$
$$V^i_{\mu_1\mu_2} = \frac{-2q_{2\mu_1}g_{\mu_2\nu} + 2q_{1\mu_2}g_{\mu_1\nu} + (q_2-q_1)_\nu g_{\mu_1\mu_2}}{2q_1\cdot q_2 p_i\cdot(q_1+q_2)}p_i^\nu$$

As it is shown in Catani in Eqs. (104) and (105) of Ref. [2] one has $q_1^{\mu_1}\widetilde{\mathbf{K}}_{\mu_1\mu_2} = 0$ and $q_2^{\mu_2}\widetilde{\mathbf{K}}_{\mu_1\mu_2}$. Then it follows that

$$q_{1\mu_1}\mathbf{K}^{\mu_1\mu_2} \equiv \frac{\mathbf{T}_{q_1}q_2^{\mu_2}q_1^\alpha}{2q_1\cdot q_2}\mathbf{J}_\alpha(q_1+q_2)$$
$$q_{2\mu_2}\mathbf{K}^{\mu_1\mu_2} \equiv \frac{-\mathbf{T}_{q_1}q_1^{\mu_1}q_2^\alpha}{2q_1\cdot q_2}\mathbf{J}_\alpha(q_1+q_2)$$

We shall now calculate the following quantity

$$\mathbf{K}^{\dagger\nu_1\nu_2}\left(-g_{\nu_1\mu_1} + \frac{q_1^{\nu_1}n^{\mu_1} + n^{\nu_1}q_1^{\mu_1}}{n\cdot q_1}\right)\left(-g_{\nu_2\mu_2} + \frac{q_2^{\nu_2}n^{\mu_2} + n^{\nu_2}q_2^{\mu_2}}{n\cdot q_2}\right)\mathbf{K}^{\mu_1\mu_2} \tag{12.3}$$

12.1 Ghost-less approach

It is straightforward to see that

$$\begin{aligned}&\mathbf{K}^{\dagger\nu_1\nu_2}\left(-g_{\nu_1\mu_1} + \frac{q_1^{\nu_1}n^{\mu_1} + n^{\nu_1}q_1^{\mu_1}}{n\cdot q_1}\right)\left(-g_{\nu_2\mu_2} + \frac{q_2^{\nu_2}n^{\mu_2} + n^{\nu_2}q_2^{\mu_2}}{n\cdot q_2}\right)\mathbf{K}^{\mu_1\mu_2} \\ &= \widetilde{\mathbf{K}}^{\dagger\nu_1\nu_2}\left(-g_{\nu_1\mu_1} + \frac{q_1^{\nu_1}n^{\mu_1} + n^{\nu_1}q_1^{\mu_1}}{n\cdot q_1}\right)\left(-g_{\nu_2\mu_2} + \frac{q_2^{\nu_2}n^{\mu_2} + n^{\nu_2}q_2^{\mu_2}}{n\cdot q_2}\right)\widetilde{\mathbf{K}}^{\mu_1\mu_2} \\ &= \widetilde{\mathbf{K}}^{\dagger\nu_1\nu_2}\left(-g_{\nu_1\mu_1}\right)\left(-g_{\nu_2\mu_2}\right)\widetilde{\mathbf{K}}^{\mu_1\mu_2}\,,\end{aligned} \tag{12.4}$$

where $\widetilde{\mathbf{K}}$ has a simpler expression in this gauge.

12.2 Approach with a ghosts

In this section we shall now the terms involving n a equivalent to adding a ghost contribution and keeping the complete vertex. It is useful to have explicitly the following formulae

$$\begin{aligned}
&\mathbf{K}^{\dagger\nu_1\nu_2}\left(\frac{q_2^{\nu_2}n^{\mu_2}+n^{\nu_2}q_2^{\mu_2}}{n\cdot q_2}\right)\mathbf{K}^{\mu_1\mu_2} \\
&= -\mathbf{K}^{\dagger\nu_1\nu_2}\left(\frac{q_1^{\nu_2}}{n\cdot q_2}\right)\frac{\mathbf{T}_{q_1}q_1^{\mu_1}q_2^{\alpha}}{2q_1\cdot q_2}\mathbf{J}_\alpha(q_1+q_2) - \mathbf{J}^\dagger_\alpha(q_1+q_2)\frac{\mathbf{T}_{q_1}q_1^{\nu_1}q_2^{\alpha}}{2q_1\cdot q_2}\left(\frac{q_1^{\mu_2}}{n\cdot q_2}\right)\mathbf{K}^{\mu_1\mu_2}
\end{aligned}$$

and

$$\begin{aligned}
&\mathbf{K}^{\dagger\nu_1\nu_2}\left(\frac{q_1^{\nu_1}n^{\mu_1}+n^{\nu_1}q_1^{\mu_1}}{q_1\cdot n}\right)\mathbf{K}^{\mu_1\mu_2} \\
&= \mathbf{J}^\dagger_\alpha(q_1+q_2)\frac{\mathbf{T}_{q_1}q_2^{\nu_2}q_1^{\alpha}}{2q_1\cdot q_2}\frac{n^{\mu_1}}{q_1\cdot n}\mathbf{K}^{\mu_1\mu_2} + \mathbf{K}^{\dagger\nu_1\nu_2}\frac{n^{\nu_1}}{q_1 n}\frac{\mathbf{T}_{q_1}q_2^{\mu_2}q_1^{\alpha}}{2q_1\cdot q_2}\mathbf{J}_\alpha(q_1+q_2)
\end{aligned}$$

We can now calculate

$$\begin{aligned}
&\mathbf{K}^{\dagger\nu_1\nu_2}\left(\frac{q_1^{\nu_1}n^{\mu_1}+n^{\nu_1}q_1^{\mu_1}}{q_1\cdot n}\right)\left(-g_{\nu_2\mu_2}+\frac{q_2^{\nu_2}n^{\mu_2}+n^{\nu_2}q_2^{\mu_2}}{n\cdot q_2}\right)\mathbf{K}^{\mu_1\mu_2} \\
&= \mathbf{J}^\dagger_\alpha(q_1+q_2)\frac{\mathbf{T}_{q_1}q_2^{\nu_2}q_1^{\alpha}}{2q_1\cdot q_2}\left(-g_{\nu_2\mu_2}+\frac{q_2^{\nu_2}n^{\mu_2}+n^{\nu_2}q_2^{\mu_2}}{n\cdot q_2}\right)\frac{n^{\mu_1}}{q_1\cdot n}\mathbf{K}^{\mu_1\mu_2} \\
&+ \mathbf{K}^{\dagger\nu_1\nu_2}\frac{n^{\nu_1}}{n\cdot q_2}\left(-g_{\nu_2\mu_2}+\frac{q_2^{\nu_2}n^{\mu_2}+n^{\nu_2}q_2^{\mu_2}}{n\cdot q_2}\right)\frac{\mathbf{T}_{q_1}q_2^{\mu_2}q_1^{\alpha}}{2q_1\cdot q_2}\mathbf{J}_\alpha(q_1+q_2) = 0
\end{aligned}$$

and

$$\begin{aligned}
&\mathbf{K}^{\dagger\nu_1\nu_2}\left(-g^{\mu_1\nu_1}\right)\left(\frac{q_2^{\nu_2}n^{\mu_2}+n^{\nu_2}q_2^{\mu_2}}{n\cdot q_2}\right)\mathbf{K}^{\mu_1\mu_2} \\
&= -\mathbf{K}^{\dagger\mu_1\nu_2}\left(\frac{n^{\nu_2}}{n\cdot q_2}\right)\frac{-\mathbf{T}_{q_1}q_1^{\mu_1}q_2^{\alpha}}{2q_1\cdot q_2}\mathbf{J}_\alpha(q_1+q_2) - \mathbf{J}^\dagger_\alpha(q_1+q_2)\frac{-\mathbf{T}_{q_1}q_1^{\mu_1}q_2^{\alpha}}{2q_1\cdot q_2}\left(\frac{n^{\mu_2}}{n\cdot q_2}\right)\mathbf{K}^{\mu_1\mu_2} \\
&= \mathbf{J}^\dagger_\beta(q_1+q_2)\frac{\mathbf{T}_{q_1}}{2q_1\cdot q_2}\left(q_1^{\beta}q_2^{\alpha}+q_2^{\beta}q_1^{\alpha}\right)\frac{\mathbf{T}_{q_1}}{2q_1\cdot q_2}\mathbf{J}_\alpha(q_1+q_2)
\end{aligned}$$

Together with the factor 1/2! this is the ghost contribution in the Feynman gauge. Using the above expressions one can write

$$\begin{aligned}
&\mathbf{K}^{\dagger\nu_1\nu_2}\left(-g_{\nu_1\mu_1}+\frac{q_1^{\nu_1}n^{\mu_1}+n^{\nu_1}q_1^{\mu_1}}{n\cdot q_1}\right)\left(-g_{\nu_2\mu_2}+\frac{q_2^{\nu_2}n^{\mu_2}+n^{\nu_2}q_2^{\mu_2}}{n\cdot q_2}\right)\mathbf{K}^{\mu_1\mu_2} \\
&= \mathbf{K}^{\dagger\nu_1\nu_2}\left(-g_{\nu_1\mu_1}\right)\left(-g_{\nu_2\mu_2}\right)\mathbf{K}^{\mu_1\mu_2}+ \\
&\mathbf{J}^\dagger_\beta(q_1+q_2)\frac{\mathbf{T}_{q_1}}{2q_1\cdot q_2}\left(q_1^{\beta}q_2^{\alpha}+q_2^{\beta}q_1^{\alpha}\right)\frac{\mathbf{T}_{q_1}}{2q_1\cdot q_2}\mathbf{J}_\alpha(q_1+q_2)\ .
\end{aligned} \tag{12.5}$$

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