Econometric Methods: Assignment 2

Bingxue Li

1 BOOTSTRAP

(a) Consider a variant of the example of the bootstrap on slide 17 in Bootstrap.pdf, where we observe a sample of i.i.d. random variables, X_i, with $n = 3$. In particular, assume that the three observed realizations are: $X_1 = 0, X_2 = 1$, and $X_3 = 2$. Analytically, compute the bootstrap estimator (using the empirical distribution function, F_n, as an estimator of F_0) of the cdf of the sample average $G_3(\tau, F_3) \equiv P_{F_3}(T_3 \leq \tau)$, where $T_3 = \frac{1}{3}\sum_{i=1}^{3} X_i$. Then, implement this estimator using simulation, i.e., draw B times with replacement from the sample to obtain a "Monte Carlo estimate/approximation" of $G_3(\tau, F_3)$. Report your estimates for all possible realization of T_3^* with $B = 10, 100$, and 1000. [Hint: Be careful with rounding errors.]

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1. Construct the Empirical Distribution Function $F_3(x)$:

$$F_3(x) = \frac{1}{3}\sum_{i=1}^{3} \mathbb{1}\{X_i = x\} = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{3} & \text{for } 0 \leq x < 1 \\ \frac{2}{3} & \text{for } 1 \leq x < 2 \\ 1 & \text{for } x \geq 2 \end{cases} \tag{1}$$

$$\mathbb{P}_{F_3}\{X_i = 0\} = \frac{1}{3} \tag{2}$$

$$\mathbb{P}_{F_3}\{X_i = 1\} = \frac{1}{3} \tag{3}$$

$$\mathbb{P}_{F_3}\{X_i = 2\} = \frac{1}{3} \tag{4}$$

$$\mathbb{P}_{F_3}\{X_i = x\} = 0 \quad \forall x \neq 0, 1, 2 \tag{5}$$

The empirical distribution function is consistent and asymptotically normal such that $\sqrt{n}(F_3(u) - F_0(u)) \xrightarrow{d} N(0, F_0(u)(1 - F_0(u)))$.
2. Compute the pmf of T_n under F_3 to obtain $G_3(\tau, F_3) \equiv P_{F_3}(T_3 \leq \tau)$. The

support of T_3 under EDF is $\{0, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2\}$.

$$\mathbb{P}_{F_3}\{T_3 = 0\} = (\frac{1}{3})^3 \tag{6}$$

$$\mathbb{P}_{F_3}\{T_3 = \frac{1}{3}\} = (\frac{1}{3})^3 \cdot 3 \tag{7}$$

$$\mathbb{P}_{F_3}\{T_3 = \frac{2}{3}\} = (\frac{1}{3})^3 \cdot 3 + (\frac{1}{3})^3 \cdot 3 \tag{8}$$

$$\mathbb{P}_{F_3}\{T_3 = 1\} = (\frac{1}{3})^3 \cdot 6 + (\frac{1}{3})^3 \tag{9}$$

$$\mathbb{P}_{F_3}\{T_3 = \frac{4}{3}\} = (\frac{1}{3})^3 \cdot 3 + (\frac{1}{3})^3 \cdot 3 \tag{10}$$

$$\mathbb{P}_{F_3}\{T_3 = \frac{5}{3}\} = (\frac{1}{3})^3 \cdot 3 \tag{11}$$

$$\mathbb{P}_{F_3}\{T_3 = 2\} = (\frac{1}{3})^3 \tag{12}$$

$$G_3(\tau, F_3) \equiv P_{F_3}(T_3 \leq \tau) = \begin{cases} 0 & \text{for } \tau < 0 \\ (\frac{1}{3})^3 & \text{for } 0 \leq \tau < \frac{1}{3} \\ (\frac{1}{3})^3 \cdot 4 & \text{for } \frac{1}{3} \leq \tau < \frac{2}{3} \\ (\frac{1}{3})^3 \cdot 10 & \text{for } \frac{2}{3} \leq \tau < 1 \\ (\frac{1}{3})^3 \cdot 17 & \text{for } 1 \leq \tau < \frac{4}{3} \\ (\frac{1}{3})^3 \cdot 23 & \text{for } \frac{4}{3} \leq \tau < \frac{5}{3} \\ (\frac{1}{3})^3 \cdot 26 & \text{for } \frac{5}{3} \leq \tau < 1 \\ 1 & \text{for } \tau \geq 1 \end{cases} \tag{13}$$

	$T_{3,b} = 0$	$T_{3,b} = \frac{1}{3}$	$T_{3,b} = \frac{2}{3}$	$T_{3,b} = 1$	$T_{3,b} = \frac{4}{3}$	$T_{3,b} = \frac{5}{3}$	$T_{3,b} = 2$
B=10	0.1	0.2	0.2	0.2	0.3	0	0
B=100	0.04	0.13	0.22	0.31	0.17	0.08	0.05
B=1000	0.033	0.126	0.232	0.264	0.214	0.099	0.032
EDF Baseline	0.037	0.111	0.222	0.259	0.222	0.111	0.037

Table 1: Variance-Covariance Matrix of $\hat{\theta}$ for Different Estimators

Just as practices with Monte-Carlo simulation, here we see that 1000 iterations of resampling generate relatively consistent estimators.

For the remainder, we return to the model considered in assignment 1. Throughout, take $n = 300$ unless otherwise noted.

(b) Implement a bootstrap estimator of the variance of the Maximum Likelihood estimator. Report the average of the obtained estimates over 1000 Monte Carlo iterations with $B = 399$. You may use only 100 Monte Carlo iterations, but if you do, please indicate that you do! How does it compare to your answer to questions (l) and (m) of assignment 1 ?

Recall that for Assignment 1 we defined this MLE as:

$$\mathbb{P}(y_i = 0 \mid x_i) = \Phi(\alpha_{10} - \beta_0 x_i)$$
$$\mathbb{P}(y_i = 2 \mid x_i) = \Phi(\beta_0 x_i - \alpha_{20})$$
$$\mathbb{P}(y_i = 1 \mid x_i) = \Phi(\alpha_{20} - \beta_0 x_i) - \Phi(\alpha_{10} - \beta_0 x_i)$$

We write the objective function of ML estimation as log-likelihood of the sample:

$$Q_n(\theta) = -\frac{1}{n}\sum_{i=1}^{n} \mathbb{1}\{y_i = 0\} \ln \Phi(\alpha_1 - \beta x_i) + \mathbb{1}\{y_i = 1\} \ln [\Phi(\alpha_2 - \beta x_i) - \Phi(\alpha_1 - \beta x_i)]$$
$$+ \mathbb{1}\{y_i = 2\} \ln [1 - \Phi(\alpha_2 - \beta x_i)] \tag{14}$$
$$\hat{\theta} = \arg\max_{\theta\in\Theta} Q_n(\theta) \tag{15}$$

We are interested in $T_n = \hat{\theta} = \arg\max_{\theta\in\Theta} Q_n(\theta)$ Under each Monte-Carlo iteration, we conduct the following bootstrap experiments, for $b \in \{1, 2, ..., B\}$:
1. Generate a bootstrap sample of size $n = 300$, $\{X_{1,b}, ..., X_{n,b}\}$, by randomly drawing with replacement from the orginal sample.
2. Compute $T_{n,b} = T_n(X_{1,b}, ..., X_{n,b}) = \arg\max_{\theta\in\Theta} Q_{n,b}(\theta)$.
3. Repeat 1.-2. B times.

Estimation Method (True)	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\beta}$
True	$\alpha_{10} = 0$	$\alpha_{20} = 1$	$\beta_0 = 1$
MLE	0.0447	1.0119	1.0346
Bootstrapped NLLSE	-0.0826	1.0659	1.1267

Table 2: (Bootstrapped) Maximum Likelihood Estimator $\hat{\theta}$

Variance-Covaraince Matrix of MLE	θ	$\hat{\alpha_1}$	$\hat{\alpha_2}$	$\hat{\beta}$
Monte-Carlo Sample Variance (l)	$\hat{\alpha_1}$	0.0087	0.0051	0.0031
	$\hat{\alpha_2}$	0.0051	0.0109	0.0057
	$\hat{\beta}$	0.0031	0.0057	0.0089
Hessian Computed (m)	$\hat{\alpha_1}$	0.0083	0.0049	0.0030
	$\hat{\alpha_2}$	0.0049	0.0110	0.0056
	$\hat{\beta}$	0.0030	0.0056	0.0086
Bootstrapped Sample Variance	$\hat{\alpha_1}$	0.0085	0.0050	0.0031
	$\hat{\alpha_2}$	0.0050	0.0112	0.0058
	$\hat{\beta}$	0.0031	0.0058	0.0089

Table 3: Variance-Covariance Matrix of $\hat{\theta}$ for Different Estimators

The bootstrapped Maximum Likelihood Estimators approximately the same with the Maximum Likelihood estimator alone. This is due to the theorem on the consistency and bootstrap convergence in distribution of bootstrap estimator with $T_{n,b} = \hat{\theta}_b \xrightarrow{p^*} \hat{\theta}$, and $\hat{\theta} \xrightarrow{p} \theta_0$.

(c) To investigate the robustness properties of the four estimators, repeat exercise (j) of assignment 1 using the following *dgp* to generate the data: $P(y_i = 0) = \frac{1}{4}\Phi(\alpha_2 - \beta x_i) + \frac{3}{4}\Phi(\alpha_1 - \beta x_i)$, $P(y_i = 1) = \frac{1}{2}\Phi(\alpha_2 - \beta x_i) - \frac{1}{2}\Phi(\alpha_1 - \beta x_i)$, and $P(y_i = 2) = 1 - \frac{3}{4}\Phi(\alpha_2 - \beta x_i) - \frac{1}{4}\Phi(\alpha_1 - \beta x_i)$. What do you find? How can you explain your findings?

Sample Size	**MLE**	**NLLSE**	**Just-ID GMM**	**Over-ID GMM**
n=300	0.272	0.027	0.276	0.209
n=300(New)	0	0.018	0.207	0

Table 4: Consistency Performance of Applying Four Estimators to New DGP

As shown in Table 4, the MLE and Over-ID GMM perform poorly under the current DGP in terms of consistency. We therefore want to check if the current DGP fits the underlying assumptions needed for a consistent estimator for the four estimators. This failure should not be surprising as Maximum Likelihood Estimators are very sensitive to mis-specification. We find that NLLSE consistency performance is close to the case under previous DGP. One can show that the underlying assumption is recovered specifically with this new DGP. Recall that our NLLSE was defined with:

$$\begin{aligned} m(x_i, \theta_0) &:= \mathbb{E}[y_i \mid x_i] = 0 \cdot \mathbb{P}(y_i = 0 \mid x_i) + 1 \cdot \mathbb{P}(y_i = 1 \mid x_i) + 2 \cdot \mathbb{P}(y_i = 2 \mid x_i) \\ &= \Phi(\beta_0 x_i - a_{10}) + \Phi(\beta_0 x_i - a_{20}) \end{aligned} \tag{16}$$

$$Q_{n,NLLSE}(\theta) = \frac{1}{n}\sum_{i=1}^{n}\{y_i - m(x_i,\theta)\}^2 = \frac{1}{n}\sum_{i=1}^{n}\{y_i - \{\Phi(\beta x_i - a_1) + \Phi(\beta x_i - a_2)\}\}^2$$

Here, verify if the nonlinear regression function specification is satisfied:

$$\begin{aligned} \mathbb{E}[y_i \mid x_i] &= 0 \cdot \mathbb{P}(y_i = 0 \mid x_i) + 1 \cdot \mathbb{P}(y_i = 1 \mid x_i) + 2 \cdot \mathbb{P}(y_i = 2 \mid x_i) \\ &= 0 \cdot \left\{\frac{1}{4}\Phi(\alpha_2 \beta x_i) + \frac{3}{4}\Phi(\alpha_1 - \beta x_i)\right\} + 1 \cdot \left\{\frac{1}{2}\Phi(\alpha_2 - \beta x_i) - \frac{1}{2}\Phi(\alpha_1 - \beta x_i)\right\} \\ &+ 2 \cdot \left\{1 - \frac{3}{4}\Phi(\alpha_2 - \beta x_i) - \frac{1}{4}\Phi(\alpha_1 - \beta x_i)\right\} = m(x_i, \theta_0) \end{aligned} \tag{17}$$

Since our Just-Identified GMM estimator is derived with LIE with NLLSE conditions, the consistency performance of Just-ID GMM is preserved in this case as well. Both MLE and Over-ID GMM relies on precise Conditional Probability Function for fixed realization of y_i, which we deviate from under the current DGP.

2 INFERENCE

For the remainder of the assignment, use the original data generating process again. (d) Write down the score statistic for testing $H_0 : \beta_0 = \beta_{\text{null}}$ that uses $\hat{J}_1(\tilde{\theta})$ to estimate J and that uses the simplified version of the test statistic that you can find on slide 16 of Inference.pdf.

J_1 is the asymptotic variance of the maximum likelihood estimator $\hat{\theta}$ expressed with gradient elements, $J_1 = \mathbb{E}[s(w,\theta_0)s(w,\theta_0)']$ with $s(w,\theta) = \nabla_\theta \ln f(w|\theta)$, and we obtain its sample moment estimator, $\hat{J}_1(\theta) = \frac{1}{n}\sum_{i=1}^n s(w_i,\theta)s(w_i,\theta)'$.
The score statistics is defined as the following:

$$n\left(\frac{1}{n}\sum_{i=1}^n s(w_i,\theta_{\text{null}})\right)' J^{-1}\left(\frac{1}{n}\sum_{i=1}^n s(w_i,\theta_{\text{null}})\right) \xrightarrow{d} \chi^2(k) \tag{18}$$

Apply the gradient-defined asymptotic variance J_1 and replace it with a sample moment estimator $\hat{J}_1 \xrightarrow{p} J_1$, evaluated at $\tilde{\theta}$ obtained under the null constraint ($\hat{J}_1^{-1}(\tilde{\theta}) = \frac{1}{n}\sum_{i=1}^n s(w_i,\tilde{\theta})s(w_i,\tilde{\theta})'$):

$$\tilde{\theta} = \arg\max_{\theta\in\Theta} Q_n(\theta) \quad \text{s.t.} \quad r(\theta) = \beta = \beta_{null} \in \mathbb{R}^1 \quad \text{(constrained)} \tag{19}$$

$$S = n\left(\frac{1}{n}\sum_{i=1}^n s(w_i,\tilde{\theta})\right)' \hat{J}_1^{-1}(\tilde{\theta})\left(\frac{1}{n}\sum_{i=1}^n s(w_i,\tilde{\theta})\right) \xrightarrow{d} \chi^2(k) \tag{20}$$

where $k = 1$ is the number of restrictions imposed under the null. For better analytical expression, derive the score (gradient vector) of MLE as below:

$$\begin{aligned}
&\nabla_\theta \ln f(w_i|\theta) \\
&= \frac{\partial}{\partial\theta}\Big\{\mathbb{1}\{y_i=0\}\ln\Phi(\alpha_1-\beta x_i) + \mathbb{1}\{y_i=1\}\ln\left[\Phi(\alpha_2-\beta x_i)-\Phi(\alpha_1-\beta x_i)\right] \\
&+ \mathbb{1}\{y_i=2\}\ln\left[1-\Phi(\alpha_2-\beta x_i)\right]\Big\} \\
&= \begin{bmatrix} \mathbb{1}\{y_i=0\}\frac{\phi(\alpha_1-\beta x_i)}{\Phi(\alpha_1-\beta x_i)} + \mathbb{1}\{y_i=1\}\frac{-\phi(\alpha_1-\beta x_i)}{\Phi(\alpha_2-\beta x_i)-\Phi(\alpha_1-\beta x_i)} \\ \mathbb{1}\{y_i=2\}\frac{-\phi(\beta x_i-\alpha_2)}{\Phi(\beta x_i-\alpha_2)} + \mathbb{1}\{y_i=1\}\frac{\phi(\alpha_2-\beta x_i)}{\Phi(\alpha_2-\beta x_i)-\Phi(\alpha_1-\beta x_i)} \\ \mathbb{1}\{y_i=0\}\frac{\phi(\alpha_1-\beta x_i)}{\Phi(\alpha_1-\beta x_i)}\cdot(-x_i) + \mathbb{1}\{y_i=1\}\frac{\phi(\alpha_1-\beta x_i)-\phi(\alpha_2-\beta x_i)}{\Phi(\alpha_2-\beta x_i)-\Phi(\alpha_1-\beta x_i)}\cdot(x_i) + \mathbb{1}\{y_i=2\}\frac{\phi(\beta x_i-\alpha_2)}{\Phi(\beta x_i-\alpha_2)}\cdot(x_i) \end{bmatrix}
\end{aligned} \tag{21}$$

(e) Based on the maximum likelihood estimator, use the trinity of tests (together with critical values motivated by asymptotic theory) to test $H_0 : \beta_0 = 1$. For the Wald test, use $\hat{J}_2(\hat{\theta})$, i.e., the (possibly scaled) hessian spit out by optim. For the Score test, use the test statistic you obtained in (d). Report the null rejection frequency at $\alpha = 0.05$ over 1000 Monte Carlo iterations. What do you find?

First, denote the unconstrained and constrained (under H_0) estimator as the following:

$$\hat{\theta} = \arg\max_{\theta \in \Theta} Q_n(\theta) \tag{22}$$

$$\tilde{\theta} = \arg\max_{\theta \in \Theta} Q_n(\theta) \quad \text{s.t.} \quad r(\theta) = \beta = \beta_{null} \in \mathbb{R}^1 \quad \text{(constrained)} \tag{23}$$

Wald Test
Note that we conduct subvector inference and test linear hypothesis $H_0 : r(\theta_0) = 0 \cdot \alpha_{10} + 0 \cdot \alpha_{20} + \beta_0 = \beta_{null}$.

$$\begin{aligned} W &= n\left(r(\widehat{\theta}) - \beta_{null}\right)' \left(\widehat{\boldsymbol{R}}'\widehat{\boldsymbol{V}}_{\widehat{\theta}}\widehat{\boldsymbol{R}}\right)^{-1} \left(r(\widehat{\theta}) - \beta_{null}\right) \\ &= n\left(r(\widehat{\theta}) - \beta_{null}\right)' \left(\frac{\partial}{\partial\theta'} r(\hat{\theta})(\hat{J}_2(\hat{\theta}))^{-1} \frac{\partial}{\partial\theta} r(\hat{\theta})\right)^{-1} \left(r(\widehat{\theta}) - \beta_{null}\right) \\ &= n\left(\hat{\beta} - \beta_{null}\right)' \left(\frac{\partial}{\partial\theta'} r(\hat{\theta})(\hat{J}_2(\hat{\theta}))^{-1} \frac{\partial}{\partial\theta} r(\hat{\theta})\right)^{-1} \left(\hat{\beta} - \beta_{null}\right) \end{aligned} \tag{24}$$

with $\hat{J}_2(\theta) = -\frac{1}{n}\sum_{i=1}^{n} \nabla_{\theta\theta} \ln f(w|\theta)$ and $\frac{\partial}{\partial\theta'} r(\hat{\theta}) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$.
Score Test

$$S = n\left(\frac{1}{n}\sum_{i=1}^{n} s(w_i, \tilde{\theta})\right)' \hat{J}_1^{-1}(\tilde{\theta}) \left(\frac{1}{n}\sum_{i=1}^{n} s(w_i, \tilde{\theta})\right) \tag{25}$$

Likelihood Ratio Test

$$LR = 2n \cdot \left\{\frac{1}{n}\sum_{i=1}^{n} lnf(w_i|\tilde{\theta}) - \frac{1}{n}\sum_{i=1}^{n} lnf(w_i|\hat{\theta})\right\} \tag{26}$$

Asymptotic Performance of Test Statistics

$$W, S, LR \xrightarrow{d} \chi^2(\dim r) \quad \text{under } H_0 \tag{27}$$

$$W, S, LR \xrightarrow{p} \infty \quad \text{under } H_1 \tag{28}$$

Null Hypothesis	Wald Test	Score Test	Likelihood Ratio Test
$\beta_0 = 1$	0.053	0.057	0.054
$\beta_0 = 0.9$	0.190	0.267	0.199
$\beta_0 = 1.1$	0.190	0.136	0.180

Table 5: Null Rejection Frequency at $\alpha = 0.05$ over 1000 Monte Carlo Iterations

We find that for the trinity of the test, the null rejection frequencies coincide with the size of the test, $\alpha = 0.05$ under null hypothesis with true value, which is expected for consistent hypothesis testing. This corresponds to Type I error such that the probability of rejecting the null when H_0 is true.

Comparing the Tests

Indeed, for the trinity of tests, all of them are designed to be asymptotically of size α, if we want to make comparison through power function, asymptoticall , the probability of rejecting the null $\theta = 0$ as a function of the unknown θ approaches to 1 as $n \to \infty$ for $\forall \theta \in \Theta_1 = \Theta / \Theta_0$. A pointwise comparison of the trinity of tests can be uninformative as consistency is really weak criteria for us to choose among tests, especially under finite sample, one may turn to local asymptotic power investigation of the tests.

Wald test requires to calculate the unconstrained estimator, Score test involves only the constrained estimator and Likelihood Ratio Test involves both. When the constrained estimator is easier to compute, we would prefer the score test, but this is less of a concern here since we deal with linear restrictions. Wald test is sensitive to nonlinear null hypothesis specification, also not a concern here.

In terms of finite sample, Likelihood Ratio test is the most computationally costly one but very flexible as it is data-based. The potential concern with score test is that we might obtain variance matrix that is not positive definite. Also, evaluating with criteria like gradient for a global minimum(maximum) might be inconsistent if there exists several local minimum(maximum) around. We might need to check the objective function's shape to determine which tests would perform better.

(f) Redo (e) for testing $H_0 : \beta_0 = 0.9$ and $H_0 : \beta_0 = 1.1$. What do you find? [Hint: Continue to generate the data under $\beta_0 = 1$.]

We find that the null rejection frequencies are much higher than the size of the test when the null hypothesis are set to the false value, which is desired property of hypothesis testing. The score tests outcome exhibit some asymmetry when deviating from the true value, it is more likely to reject when falsely specify that $\beta_{null} < \beta_0$, this could be investigated with more iterations and larger sample size.

(g) Based on the maximum likelihood estimator, use the Wald test together with bootstrapped critical values to test $H_0 : \beta_0 = 1$. As before, use $\hat{J}_2(\hat{\theta})$ in the construction of the Wald test. Use $B = 399$. You may use only 100 Monte Carlo iterations instead of 1000 , but if you do, please indicate that you do! What do you find?

We implement the Wald-Type Bootstrap Tests with critical value calculated by bootstrap algorithm. For fixed b, given a bootstrap sample $\{X_{1,b}, ..., X_{n,b}\}$ and bootstrap MLE $\hat{\theta}_b$, we compute the wald statistics in the bootstrap universe:

$$W_b = n \left(\hat{\beta}_b - \hat{\beta} \right)' \left(\frac{\partial}{\partial \theta_b'} r(\hat{\theta}_b) (\hat{J}_2(\hat{\theta}_b))^{-1} \frac{\partial}{\partial \theta_b} r(\hat{\theta}_b) \right)^{-1} \left(\hat{\beta}_b - \hat{\beta} \right) \tag{29}$$

Note that here we center the t-statistics around $\hat{\beta}$ instead of β_0 since the sample estimator acts like true value in the bootstrap universe, otherwise we would end up with poor coverage property, like with Efron confidence intervals. With the B

bootstrap replications, we calculate the α^{th} quantile q^*_α of the distribution of W_b. In our case, targetting a 95% confidence level, we compute the 5^{th} quantile as the critical value. We reject H_0 in favor of H_1 if $W > q^*_{1-0.05}$. The percentile style intervals have good coverage property.
In general, the bootstrap Wald test is first-order correct (achieves the correct size asymptotically) as long as estimator $\hat{\theta}$ has an asymptotic distribution and under conditions for which an Edgeworth expansion exists (accurate normal approximation holds):

$$\mathbb{P}\Big\{W > q^*_{1-\alpha} \mid \ \beta_0 = \beta_{null}\Big\} = 1 - \alpha + o(n^{-1}) \tag{30}$$

and thus achieves asymptotic refinement compared with our baseline Wald Test. We find higher rejection probability under false null hypothesis for Wald-type Bootstrap Test compared with simple Wald under a sample size of 300, and such merits of Resampling approach to inference is more pronuanced with small finite sample.

Null Hypothesis	**Wald Test**	**Wald Type Bootstrap Test**
$\beta_0 = 1$	0.052	0.058
$\beta_0 = 0.9$	0.185	0.191
$\beta_0 = 1.1$	0.177	0.19

Table 6: Null Rejection Frequency at $\alpha = 0.05$ over 1000 Monte Carlo Iterations and 399 Bootstrap Resampling