Computational Mechanics by Isogeometric Analysis
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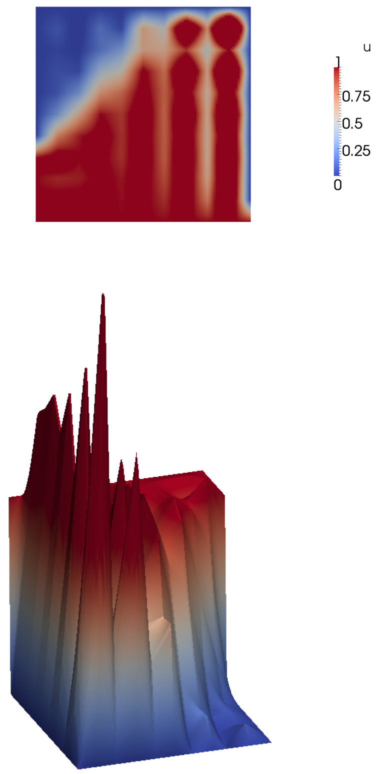
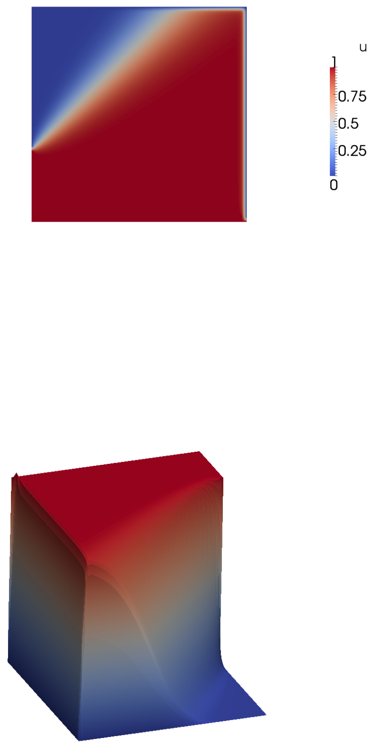
Exercises April 26, 2016: Solutions
NURBS–based Isogeometric Analysis: Galerkin method. III

1. a) We start by solving an advection–diffusion problem for which we set the following data: $\mathbf{b} = 10^2\,(1,1)^T$, $\sigma = 0$, and $g = 1$ on Γ_{D1}, where $\Gamma_{D1} := \left\{\mathbf{x} = (\mathrm{x},\mathrm{y})^T \in \Gamma_D \ : \ \mathrm{x} = 0 \text{ and } y < 1/3, \text{ or } \mathrm{y} = 0\right\}$, and $g = 0$ on $\Gamma_D \backslash \Gamma_{D1}$. The problem is advection dominated since the global Péclet number is larger than one, i.e. $\mathbb{P}e_g := \dfrac{\|\mathbf{b}\|_{L^\infty(\Omega)} L}{2\,\mu} = 50$, where $L = 1$ is the characteristic length size.
We solve the problem by considering $p = 1$ globally C^0–continuous and $p = 2$ globally C^1–continuous B–splines basis functions for different mesh sizes h ($h = h_e$ for all the mesh elements Ω_e). In particular, we firstly consider the case for which the local Péclet number in each mesh element, say $\mathbb{P}e_e := \dfrac{\|\mathbf{b}\|_{L^\infty(\Omega_e)} h_e}{2\,\mu}$, is larger than 1 ($\mathbb{P}e_e > 1$), possibly leading to numerical instabilities; then we consider the case $\mathbb{P}e_e < 1$, yielding numerically stable results. The Dirichlet boundary conditions are strongly imposed and the "interpolation at the control points" technique is used. The examples are illustrated in the following figures.

B–splines basis $p = 1$ (C^0–continuous), $\mathbf{b} = 10^2\,(1,1)^T$, $\sigma = 0$

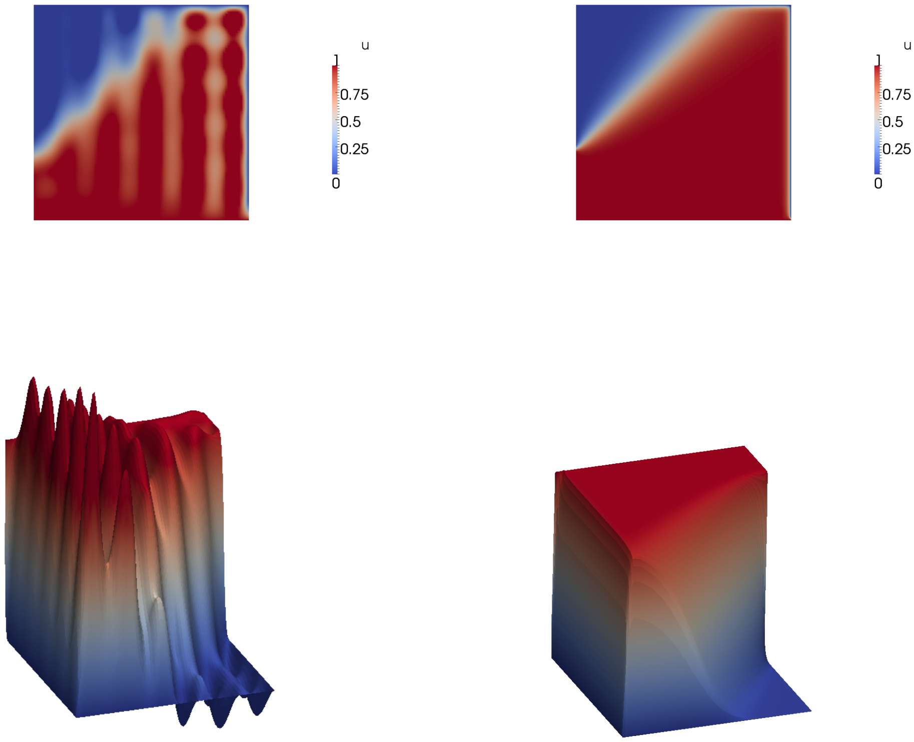


$h = 1/10$, $\mathbb{P}e_e = 5 > 1$



$h = 1/60$, $\mathbb{P}e_e = 5/6 < 1$

B–splines basis $p = 2$ *(*C^1*–continuous),* $\mathbf{b} = 10^2\,(1,1)^T$, $\sigma = 0$



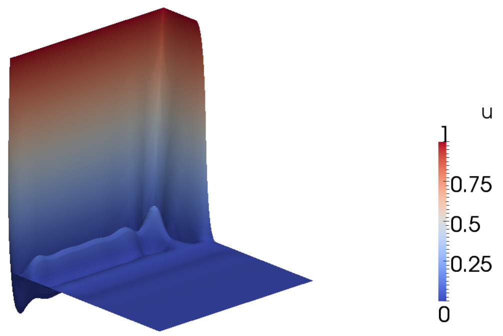
$h = 1/10$, $\mathbb{P}e_e = 5 > 1$ $\qquad$ $h = 1/60$, $\mathbb{P}e_e = 5/6 < 1$

We observe for the case $p = 1$ limited "overshooting" effects on the numerical solution appear even if $\mathbb{P}e_e = 5/6 < 1$, while for $p = 2$ the "overshooting" effects are negligible.

See the MATLAB file `ex8_1a.m` for reference.

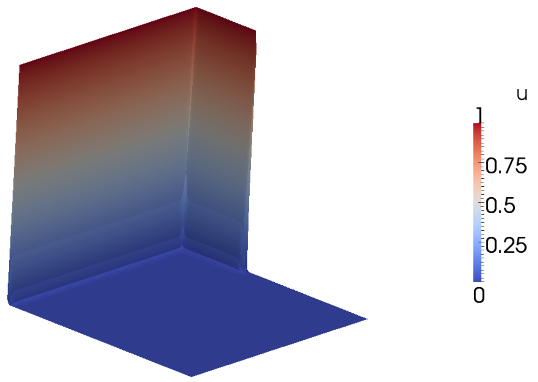
b) We solve a diffusion–reaction problem by setting $\mathbf{b} = \mathbf{0}$, $\sigma = 2 \cdot 10^4$, and g as at point 1a). The problem is reaction dominated with the global Péclet number $\mathbb{P}e_g := \dfrac{\|\sigma\|_{L^2(\Omega)} L^2}{6\,\mu} = 1/3 \cdot 10^4$. We solve the problem with B–splines basis functions of degree $p = 2$ and globally C^1–continuous. Firstly, we set $h = 1/10$ yielding the local Péclet number $\mathbb{P}e_e := \dfrac{\|\sigma\|_{L^\infty(\Omega_e)} h_e^2}{6\,\mu} = 1/3 \cdot 10^4 \gg 1$ for which we obtain the numerical solution reported in the following and exhibiting numerical instabilities.

B–splines basis $p = 2$ *(*C^1*–continuous),* $\mathbf{b} = \mathbf{0}$, $\sigma = 2 \cdot 10^4$, $h = 1/10$



We then set $h = 1/80$ yielding a numerical solution without numerical instabilities since the local Péclet number is $\mathbb{P}e \simeq 0.5208 < 1$.

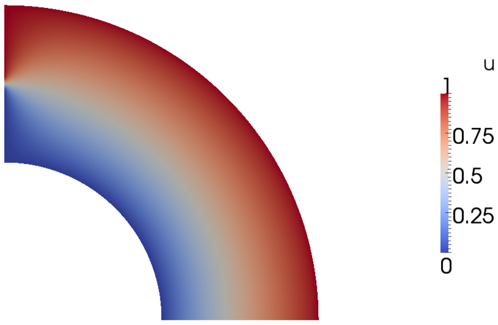
B–splines basis $p = 2$ *(*C^1*–continuous),* $\mathbf{b} = \mathbf{0}$, $\sigma = 2 \cdot 10^4$, $h = 1/80$



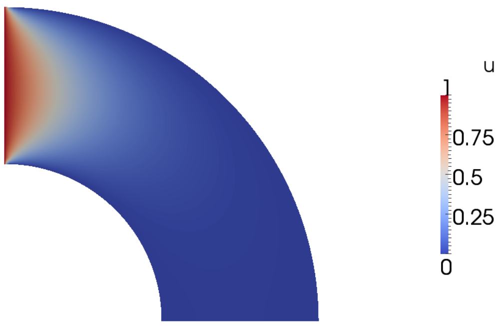
See the MATLAB file `ex8_1b.m` for reference.

2. We consider two different choices of the Dirichlet data g (the "interpolation at the control points" technique is used). For both the cases we consider NURBS basis functions of degree $p = 2$ and globally C^1–continuous with a meshes comprised of 20×50 elements.

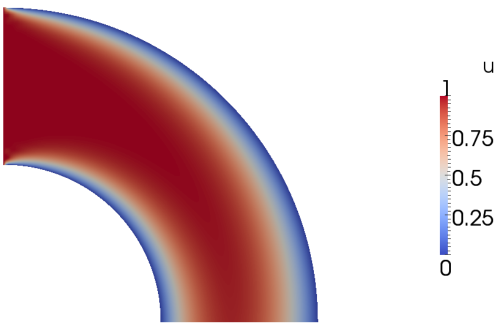
   a) We set $g = 1$ on Γ_{D1} and $g = 0$ on $\Gamma_D \backslash \Gamma_{D1}$, where the subset $\Gamma_{D1} := \left\{ \mathbf{x} = (\mathrm{x}, \mathrm{y})^T \in \Gamma_D \ : \ \mathrm{x} = 0 \text{ and } y > 3/2, \text{ or } \sqrt{\mathrm{x}^2 + \mathrm{y}^2} = 2 \right\}$. We obtain the result reported in the following for which we remark that the problem is diffusion dominated since the global Péclet number is $\mathbb{P}e_g = 0.5 < 1$.



b) We set $g = 1$ on Γ_{D1} and $g = 0$ on $\Gamma_D \backslash \Gamma_{D1}$, where $\Gamma_{D1} := \left\{ \mathbf{x} = (\mathrm{x}, \mathrm{y})^T \in \Gamma_D \ : \ \mathrm{x} = 0 \right\}$. We obtain the following result, which highlights that the problem is diffusion dominated since $\mathbb{P}e_g = 0.5 < 1$.



If we set $\mu = 10^{-2}$ we obtain an advection dominated problem with $\mathbb{P}e_g = 50 \gg 1$. The corresponding numerical solution is reported in the following.



See the MATLAB file `ex8_2.m` for reference