Computational Mechanics by Isogeometric Analysis
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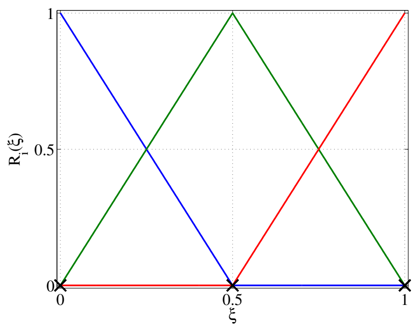
Exercises March 29, 2016: Solutions
B–splines and NURBS: function spaces

1. The B–splines function space $\mathcal{B}_h$ is determined by the following $n_{\mathcal{B}} = 3$ basis functions obtained from the knot vector $\Xi_{\mathcal{B}} = \left\{0, 0, \frac{1}{2}, 1, 1\right\}$:

$$B_1(\xi) = \begin{cases} 1 - 2\xi & \text{if } 0 \leq \xi < \dfrac{1}{2}, \\ 0 & \text{elsewhere,} \end{cases}$$

$$B_2(\xi) = \begin{cases} 2\xi & \text{if } 0 \leq \xi < \dfrac{1}{2}, \\ 2 - 2\xi & \text{if } \dfrac{1}{2} \leq \xi < 1, \\ 0 & \text{elsewhere,} \end{cases}$$

$$B_3(\xi) = \begin{cases} 2\xi - 1 & \text{if } \dfrac{1}{2} \leq \xi < 1, \\ 0 & \text{elsewhere.} \end{cases}$$



B–splines basis $\{B_i(\xi)\}_{i=1}^{n_{\mathcal{B}}}$ obtained from $\Xi_{\mathcal{B}} = \left\{0, 0, \frac{1}{2}, 1, 1\right\}$.

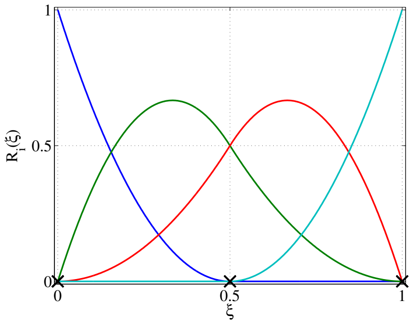
a) The B–splines function space $\mathcal{N}_h$ is determined by the following $n_{\mathcal{N}} = 4$ basis functions built from the knot vector $\Xi_{\mathcal{N}} = \left\{0, 0, 0, \frac{1}{2}, 1, 1, 1\right\}$:

$$N_1(\xi) = \begin{cases} 1 - 4\xi + 4\xi^2 & \text{if } 0 \leq \xi < \dfrac{1}{2}, \\ 0 & \text{elsewhere,} \end{cases}$$

$$N_2(\xi) = \begin{cases} 4\xi - 6\xi^2 & \text{if } 0 \leq \xi < \dfrac{1}{2}, \\ 2 - 4\xi + 2\xi^2 & \text{if } \dfrac{1}{2} \leq \xi < 1, \\ 0 & \text{elsewhere,} \end{cases}$$

$$N_3(\xi) = \begin{cases} 2\xi^2 & \text{if } 0 \leq \xi < \dfrac{1}{2}, \\ -2 + 8\xi - 6\xi^2 & \text{if } \dfrac{1}{2} \leq \xi < 1, \\ 0 & \text{elsewhere,} \end{cases}$$

$$N_4(\xi) = \begin{cases} 1 - 4\xi + 4\xi^2 & \text{if } \dfrac{1}{2} \leq \xi < 1, \\ 0 & \text{elsewhere.} \end{cases}$$



B–splines basis $\{N_i(\xi)\}_{i=1}^{n_{\mathcal{N}}}$ obtained from $\Xi_{\mathcal{N}} = \left\{0, 0, 0, \frac{1}{2}, 1, 1, 1\right\}$.

The function space $\mathcal{B}_h$ is NOT "nested" into the space $\mathcal{N}_h$ ($\mathcal{B}_h \not\subseteq \mathcal{N}_h$) since there exists at least a basis function of $\mathcal{B}_h$ that can not be expressed as linear combination of the basis functions of the space $\mathcal{N}_h$ for all $\xi \in \mathbb{R}$. Specifically, let us consider the basis function $B_2(\xi)$ by assuming that can be expressed as linear combination of the basis functions defining $\mathcal{N}_h$, that is:

$$B_2(\xi) = \sum_{j=1}^{n_{\mathcal{N}}} N_j(\xi)\, \alpha_{2,j} \quad \text{for all } \xi \in \mathbb{R},$$

where the coefficients $\alpha_{2,j} \in \mathbb{R}$ for $j = 1, \ldots, n_{\mathcal{N}}$. By enforcing the previous relation in each knot span $\left[0, \frac{1}{2}\right)$ and $\left[\frac{1}{2}, 1\right)$ (the cases $\xi < 0$ and $\xi \geq 1$ being trivial), we obtain the following linear system:

$$\left\{\begin{array}{rcr} \alpha_{2,1} & = & 0, \\ -4\alpha_{2,1} + 4\alpha_{2,2} & = & 2, \\ 4\alpha_{2,1} - 6\alpha_{2,2} + 2\alpha_{2,3} & = & 0, \\ 2\alpha_{2,2} - 2\alpha_{2,3} + \alpha_{4,2} & = & 2, \\ -4\alpha_{2,2} + 8\alpha_{2,3} - 4\alpha_{2,4} & = & -2, \\ 2\alpha_{2,2} - 6\alpha_{2,3} - 4\alpha_{2,4} & = & 0. \end{array}\right.$$

All the 6 equations in the previous linear system are linearly independent from each other. Therefore, the system is overdetermined and a set of coefficients $\alpha_{2,j}$ satisfying simultaneously all the equations does not exists. We conclude that the basis function $B_2(\xi)$ can not be expressed as a linear combination of the basis functions of the space $\mathcal{N}_h$ for all $\xi \in \mathbb{R}$ and the space $\mathcal{B}_h$ is not nested in $\mathcal{N}_h$.

The same conclusion can be deduced by observing that the basis functions of the space $\mathcal{B}_h$ are only C^0–continuous in the knot $\bar{\xi} = \frac{1}{2}$, while the basis functions of $\mathcal{N}_h$ are C^1–continuous. It follows that there exist functions $v_h \in \mathcal{B}_h$, namely those only C^0–continuous across the knot $\bar{\xi} = \frac{1}{2}$, which can not be represented in terms of the basis functions of the space $\mathcal{N}_h$, the latter being more regular.

b) In this case, the B–splines function space $\mathcal{N}_h$ is determined by the following $n_{\mathcal{N}} = 5$ basis functions built from the knot vector $\Xi_{\mathcal{N}} = \left\{0, 0, 0, \frac{1}{2}, \frac{1}{2}, 1, 1, 1\right\}$:

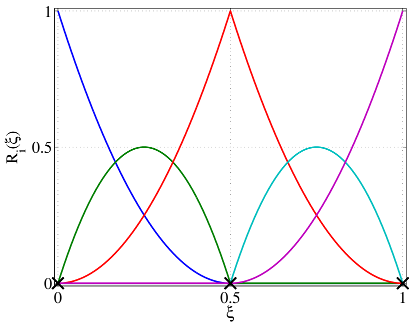
$$N_1(\xi) = \begin{cases} 1 - 4\xi + 4\xi^2 & \text{if } 0 \leq \xi < \frac{1}{2}, \\ 0 & \text{elsewhere}, \end{cases}$$

$$N_2(\xi) = \begin{cases} 4\xi - 8\xi^2 & \text{if } 0 \leq \xi < \frac{1}{2}, \\ 0 & \text{elsewhere}, \end{cases}$$

$$N_3(\xi) = \begin{cases} 4\xi^2 & \text{if } 0 \leq \xi < \frac{1}{2}, \\ 4 - 8\xi + 4\xi^2 & \text{if } \frac{1}{2} \leq \xi < 1, \\ 0 & \text{elsewhere}, \end{cases}$$

$$N_4(\xi) = \begin{cases} -4 + 12\xi - 8\xi^2 & \text{if } \frac{1}{2} \leq \xi < 1, \\ 0 & \text{elsewhere}, \end{cases}$$

$$N_5(\xi) = \begin{cases} 1 - 4\xi + 4\xi^2 & \text{if } \frac{1}{2} \leq \xi < 1, \\ 0 & \text{elsewhere}. \end{cases}$$



B–splines basis $\{N_i(\xi)\}_{i=1}^{n_{\mathcal{N}}}$ obtained from $\Xi_{\mathcal{N}} = \left\{0, 0, 0, \frac{1}{2}, \frac{1}{2}, 1, 1, 1\right\}$.

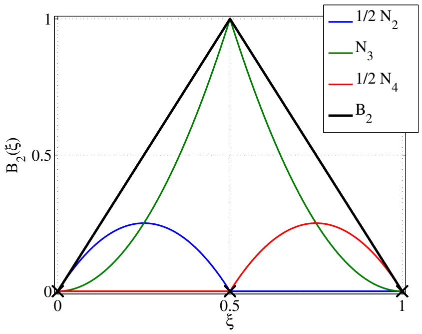
The function space $\mathcal{B}_h$ is "nested" into the space $\mathcal{N}_h$ ($\mathcal{B}_h \subseteq \mathcal{N}_h$) since all the basis functions of $\mathcal{B}_h$ can be expressed as linear combinations of the basis functions of the space $\mathcal{N}_h$ for all $\xi \in \mathbb{R}$. Indeed, by proceeding similarly to point a), we have that:

$$B_i(\xi) = \sum_{j=1}^{n_{\mathcal{N}}} N_i(\xi)\, \alpha_{i,j} \quad \text{for all } i = 1, \ldots, n_{\mathcal{B}}, \text{ for all } \xi \in \mathbb{R},$$

where the coefficients $\alpha_{i,j} \in \mathbb{R}$, with $i = 1, \ldots, n_{\mathcal{B}}$ and $j = 1, \ldots, n_{\mathcal{N}}$, are provided in the following table:

i	$\alpha_{1,j}$	$\alpha_{2,j}$	$\alpha_{3,j}$	$\alpha_{4,j}$	$\alpha_{5,j}$
1	1	$\frac{1}{2}$	0	0	0
2	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0
3	0	0	0	$\frac{1}{2}$	1

As for example, we report the basis function $B_2(\xi)$ obtained as linear combination of the basis functions $\{N_i(\xi)\}_{i=1}^{n_{\mathcal{N}}}$, specifically we have $B_2(\xi) = \frac{1}{2}\,N_2(\xi) + N_3(\xi) + \frac{1}{2}\,N_4(\xi)$.



B–splines basis function $B_2(\xi)$ obtained as linear combination of the basis functions $\{N_i(\xi)\}_{i=1}^{n_{\mathcal{N}}}$; $B_2(\xi) = \frac{1}{2}\,N_2(\xi) + N_3(\xi) + \frac{1}{2}\,N_4(\xi)$.

We observe that all the basis functions $\{B_i(\xi)\}_{i=1}^{n_{\mathcal{B}}}$ and all the basis functions $\{N_i(\xi)\}_{i=1}^{n_{\mathcal{N}}}$ are only C^0–continuous in the knot $\overline{\xi} = \frac{1}{2}$.

2. We aim at determining the function $u_h : \Omega \to \mathbb{R}$, with $u_h \in \mathcal{B}_h$, which minimizes the error in L^2 norm with respect to a prescribed function $u : \Omega \to \mathbb{R}$. By introducing the objective functional $\widetilde{J}(u_h; u) := \frac{1}{2}\|u - u_h\|^2_{L^2(\Omega)}$, the problem reads:

$$\text{given } u : \Omega \to \mathbb{R}, \text{ find } u_h \in \mathcal{B}_h \; : \; \widetilde{J}(u_h; u) \text{ is minimum.}$$

Since $u_h(\xi) = \sum_{i=1}^{n_{\mathcal{B}}} B_i(\xi)\,d_i$, with the control variables $d_i \in \mathbb{R}$ for $i = 1, \ldots, n_{\mathcal{B}}$, the problem reads:

$$\text{given } u : \Omega \to \mathbb{R}, \text{ find } \mathbf{d} \in \mathbb{R}^{n_{\mathcal{B}}} \; : \; J(\mathbf{d}; u) \text{ is minimum,}$$

where $J(\mathbf{d}; u) := \widetilde{J}(u_h(\mathbf{d}); u)$, with $u_h(\mathbf{d}) = u_h(\xi; \mathbf{d}) = \mathbf{B}(\xi)^T\,\mathbf{d}$, $\mathbf{d} := (d_1, \ldots, d_{n_{\mathcal{B}}})^T \in \mathbb{R}^{n_{\mathcal{B}}}$, and $\mathbf{B}(\xi) := (B_1(\xi), \ldots, B_{n_{\mathcal{B}}}(\xi))^T \in \mathbb{R}^{n_{\mathcal{B}}}$ for all $\xi \in \mathbb{R}$. This problem represents an unconstrained optimization problem which we solve by finding the stationary points of the objective functional $J(\mathbf{d}; u)$, that is:

$$\text{given } u : \Omega \to \mathbb{R}, \text{ find } \mathbf{d} \in \mathbb{R}^{n_{\mathcal{B}}} \; : \; \frac{\partial J}{\partial d_j}(\mathbf{d}; u) = 0 \quad \text{for all } j = 1, \ldots, n_{\mathcal{B}}.$$

This consists in solving the following linear system[1]:

$$M\,\mathbf{d} = \mathbf{f},$$

for a given function $u : \Omega \to \mathbb{R}$, where the mass matrix $M \in R^{n_{\mathcal{B}} \times n_{\mathcal{B}}}$ possesses entries:

$$M_{ij} := \int_\Omega B_i(\xi)\,B_j(\xi)\,d\Omega \quad \text{for } i, j = 1, \ldots, n_{\mathcal{B}},$$

[1] We observe that $\frac{\partial J}{\partial d_j}(\mathbf{d}; u) = \int_\Omega \left(u(\xi) - \sum_{i=1}^{n_{\mathcal{B}}} B_i(\xi)\,d_i \right) B_j(\xi)\,d\Omega = 0$ for all $j = 1, \ldots, n_{\mathcal{B}}$.

and the vector $\mathbf{f} = \mathbf{f}(u) \in R^{n_{\mathcal{B}}}$ has components:

$$f_i = f_i(u) := \int_\Omega B_i(\xi)\, u(\xi)\, d\Omega \quad \text{for } i = 1, \ldots, n_{\mathcal{B}}.$$

We observe that the matrix M only depends on the function space $\mathcal{B}_h$, while the vector $\mathbf{f}$ both on $\mathcal{B}_h$ and the given function $u : \Omega \to \mathbb{R}$.

a) The B–splines basis functions obtained from the knot vector $\Xi_{\mathcal{B}} = \{0, 0, 0, 1, 1, 1\}$ are:

$$B_1(\xi) = \begin{cases} 1 - 2\xi + \xi^2 & \text{if } 0 \le \xi < 1, \\ 0 & \text{elsewhere,} \end{cases}$$

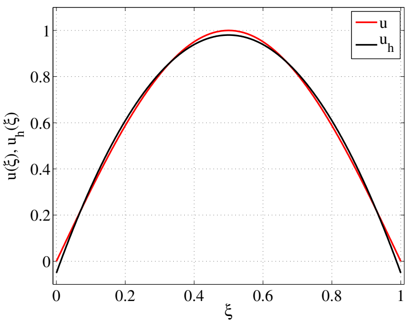
$$B_2(\xi) = \begin{cases} 2\xi - 2\xi^2 & \text{if } 0 \le \xi < 1, \\ 0 & \text{elsewhere,} \end{cases}$$

$$B_3(\xi) = \begin{cases} \xi^2 & \text{if } 0 \le \xi < 1, \\ 0 & \text{elsewhere.} \end{cases}$$

We obtain, since $\Omega = (0, 1)$:

$$M = \begin{bmatrix} \frac{1}{5} & \frac{1}{10} & \frac{1}{30} \\ \frac{1}{10} & \frac{2}{15} & \frac{1}{10} \\ \frac{1}{30} & \frac{1}{10} & \frac{1}{5} \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} \frac{1}{\pi} - \frac{4}{\pi^3} \\ \frac{8}{\pi^3} \\ \frac{1}{\pi} - \frac{4}{\pi^3} \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} \frac{12}{\pi}\left(1 - \frac{10}{\pi^2}\right) \\ \frac{6}{\pi}\left(-3 + \frac{40}{\pi^2}\right) \\ \frac{12}{\pi}\left(1 - \frac{10}{\pi^2}\right) \end{bmatrix},$$

the latter determining the function $u_h \in \mathcal{B}_h$ yielding the objective functional $J(\mathbf{d}) = 1.5030 \cdot 10^{-4}$.

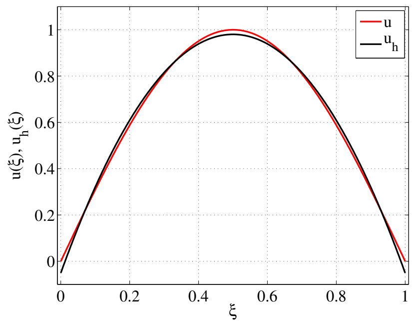


Function $u(\xi) = \sin(\pi\xi)$ and its approximation $u_h(\xi) \in \mathcal{B}_h$; the B–splines function space $\mathcal{B}_h$ is obtained from the basis functions built from the knot vector $\Xi_{\mathcal{B}} = \{0, 0, 0, 1, 1, 1\}$.

b) We consider the basis functions provided at point 1.a), from which we obtain:

$$M = \begin{bmatrix} \frac{1}{10} & \frac{7}{120} & \frac{1}{120} & 0 \\ \frac{7}{120} & \frac{1}{6} & \frac{1}{10} & \frac{1}{120} \\ \frac{1}{120} & \frac{1}{10} & \frac{1}{6} & \frac{7}{120} \\ 0 & \frac{1}{120} & \frac{7}{120} & \frac{1}{10} \end{bmatrix}, \ \mathbf{f} = \begin{bmatrix} \frac{1}{\pi} - \frac{8}{\pi^3} \\ \frac{8}{\pi^3} \\ \frac{8}{\pi^3} \\ \frac{1}{\pi} - \frac{8}{\pi^3} \end{bmatrix}, \ \mathbf{d} = \begin{bmatrix} \frac{12}{\pi}\left(1 - \frac{10}{\pi^2}\right) \\ -\frac{3}{\pi}\left(1 - \frac{20}{\pi^2}\right) \\ -\frac{3}{\pi}\left(1 - \frac{20}{\pi^2}\right) \\ \frac{12}{\pi}\left(1 - \frac{10}{\pi^2}\right) \end{bmatrix},$$

with $J(\mathbf{d}) = 1.5030 \cdot 10^{-4}$ yielding the same result of point a).

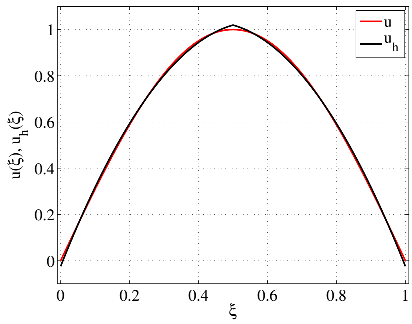


Function $u(\xi) = \sin(\pi\xi)$ and its approximation $u_h(\xi) \in \mathcal{B}_h$; the B–splines function space $\mathcal{B}_h$ is obtained from the basis functions built from the knot vector $\Xi_{\mathcal{B}} = \left\{0, 0, 0, \frac{1}{2}, 1, 1, 1\right\}$.

c) From the basis functions provided at point 1.b) we obtain:

$$M = \begin{bmatrix} \frac{1}{10} & \frac{1}{20} & \frac{1}{60} & 0 & 0 \\ \frac{1}{20} & \frac{1}{15} & \frac{1}{20} & 0 & 0 \\ \frac{1}{60} & \frac{1}{20} & \frac{1}{5} & \frac{1}{20} & \frac{1}{60} \\ 0 & 0 & \frac{1}{20} & \frac{1}{15} & \frac{1}{20} \\ 0 & 0 & \frac{1}{60} & \frac{1}{20} & \frac{1}{10} \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} \frac{1}{\pi} - \frac{8}{\pi^3} \\ -4\left(\frac{1}{\pi} - \frac{4}{\pi^3}\right) \\ 8\left(\frac{1}{\pi} - \frac{2}{\pi^3}\right) \\ -4\left(\frac{1}{\pi} - \frac{4}{\pi^3}\right) \\ \frac{1}{\pi} - \frac{8}{\pi^3} \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} \frac{6}{\pi}\left(3 + \frac{16}{\pi} - \frac{80}{\pi^2}\right) \\ -\frac{6}{\pi}\left(3 + \frac{40}{\pi} - \frac{160}{\pi^2}\right) \\ \frac{6}{\pi}\left(1 + \frac{24}{\pi} - \frac{80}{\pi^2}\right) \\ -\frac{6}{\pi}\left(3 + \frac{40}{\pi} - \frac{160}{\pi^2}\right) \\ \frac{6}{\pi}\left(3 + \frac{16}{\pi} - \frac{80}{\pi^2}\right) \end{bmatrix},$$

with $J(\mathbf{d}) = 3.5427 \cdot 10^{-5}$; we observe an improvement with respect to the results of points a) and b).



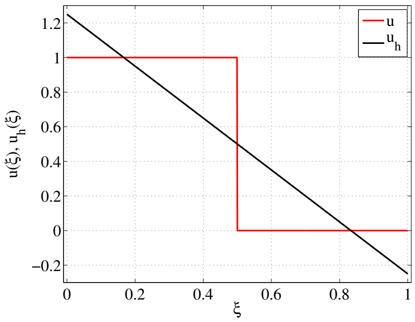
Function $u(\xi) = \sin(\pi\xi)$ and its approximation $u_h(\xi) \in \mathcal{B}_h$; the B–splines function space $\mathcal{B}_h$ is obtained from the basis functions built from the knot vector $\Xi_{\mathcal{B}} = \left\{0, 0, 0, \frac{1}{2}, \frac{1}{2}, 1, 1, 1\right\}$.

3. We repeat point 2 simply by recalculating the vectors $\mathbf{f}$ for the three different function spaces $\mathcal{B}_h$ at the points a), b), and c); the function $u(\xi) = \chi_{(0,\frac{1}{2})}(\xi)$ in $\Omega = (0,1)$ represents the characteristic function of the subdomain $\left(0, \dfrac{1}{2}\right)$.

   a) We obtain:

$$\mathbf{f}(u) = \begin{bmatrix} \frac{7}{24} \\ \frac{1}{6} \\ \frac{1}{24} \end{bmatrix} \quad \text{and} \quad \mathbf{d}(u) = \begin{bmatrix} \frac{5}{4} \\ \frac{1}{2} \\ -\frac{1}{4} \end{bmatrix},$$

   with the objective functional $J(\mathbf{d}) = 3.1281 \cdot 10^{-2}$.

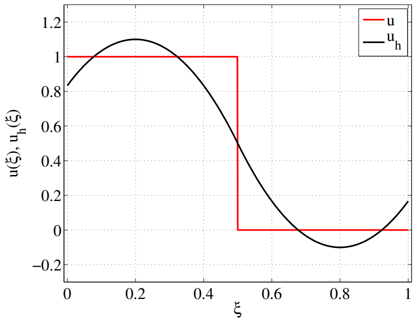


Function $u(\xi) = \chi_{(0,\frac{1}{2})}(\xi)$ and its approximation $u_h(\xi) \in \mathcal{B}_h$; the B–splines function space $\mathcal{B}_h$ is obtained from the basis functions built from the knot vector $\Xi_{\mathcal{B}} = \{0, 0, 0, 1, 1, 1\}$.

   b) We have:

$$\mathbf{f}(u) = \begin{bmatrix} \frac{1}{6} \\ \frac{1}{4} \\ \frac{1}{12} \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{d}(u) = \begin{bmatrix} \frac{5}{6} \\ \frac{3}{2} \\ -\frac{1}{2} \\ \frac{1}{6} \end{bmatrix},$$

   for which the objective functional is $J(\mathbf{d}) = 1.3903 \cdot 10^{-2}$ yielding an improved approximation with respect to point a).

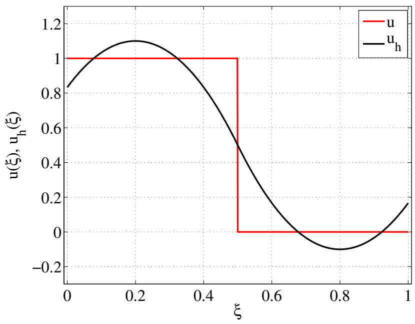


Function $u(\xi) = \chi_{(0,\frac{1}{2})}(\xi)$ and its approximation $u_h(\xi) \in \mathcal{B}_h$; the B–splines function space $\mathcal{B}_h$ is obtained from the basis functions built from the knot vector $\Xi_{\mathcal{B}} = \left\{0, 0, 0, \frac{1}{2}, 1, 1, 1\right\}$.

c) We obtain:

$$\mathbf{f}(u) = \begin{bmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{6} \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{d}(u) = \begin{bmatrix} \frac{5}{6} \\ \frac{3}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{6} \end{bmatrix},$$

with the objective functional $J(\mathbf{d}) = 1.3903 \cdot 10^{-2}$ yielding the same result of point b).



Function $u(\xi) = \chi_{(0,\frac{1}{2})}(\xi)$ and its approximation $u_h(\xi) \in \mathcal{B}_h$; the B–splines function space $\mathcal{B}_h$ is obtained from the basis functions built from the knot vector $\Xi_{\mathcal{B}} = \left\{0, 0, 0, \frac{1}{2}, \frac{1}{2}, 1, 1, 1\right\}$.