## Computational Mechanics by Isogeometric Analysis

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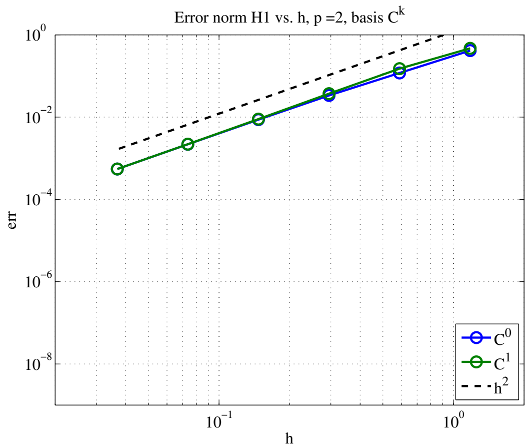
## Exercises April 14, 2015: Solutions

*NURBS–based Isogeometric Analysis: Galerkin method. II*

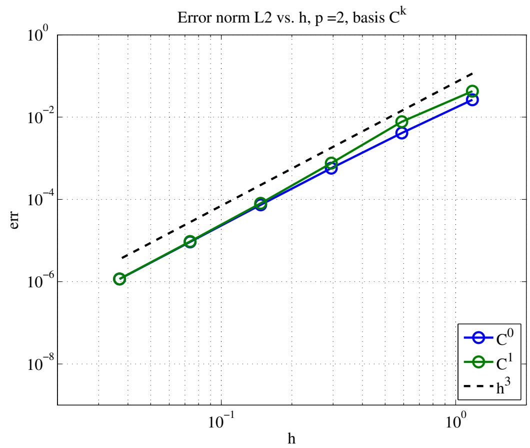
1. a) Consider the MATLAB file `ex7_1a.m`. We remark that the exact solution $u$ is infinitely differentiable, i.e. $u \in C^\infty(\Omega)$. Therefore, the expected convergence orders for the errors in norms $H^1$ and $L^2$ under $h$–refinement are $p$ and $p+1$, respectively, where $p$ is the polynomial degree used for the geometrical representation of the domain $\Omega$ and the NURBS space. We remark that the convergence orders are independent of the regularity of the NURBS basis across the mesh elements (knots).

   We report the errors in norms $H^1$ and $L^2$ vs. the characteristic mesh size $h$ and vs. the total number of degrees of freedom $n_{dof}$ in the following figures (the logarithmic scale is used for both the axis); the expected convergence orders are reported for comparison and confirmed by the numerical results. NURBS basis with different regularities are considered compatibly with the polynomial degree $p$, for which the basis functions are globally $C^k$–continuous with $k = 0, \dots, p-1$.

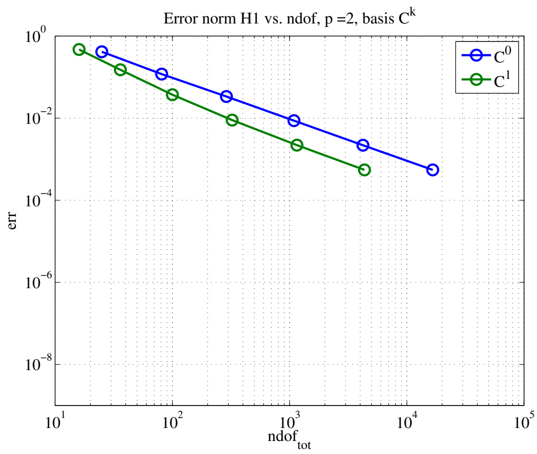
*NURBS basis $p = 2$, $C^k$–continuous with $k = 0, 1$*



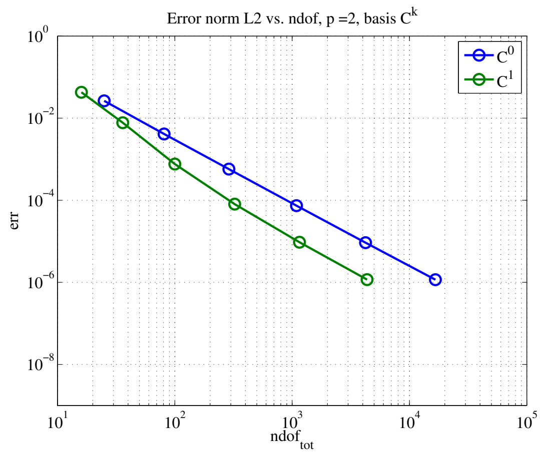

error in norm $H^1$ vs. $h$, conv. order 2




error in norm $L^2$ vs. $h$, conv. order 3

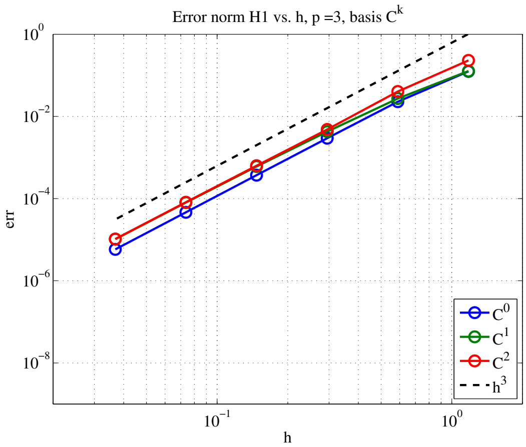



error in norm $H^1$ vs. $n_{dof}$

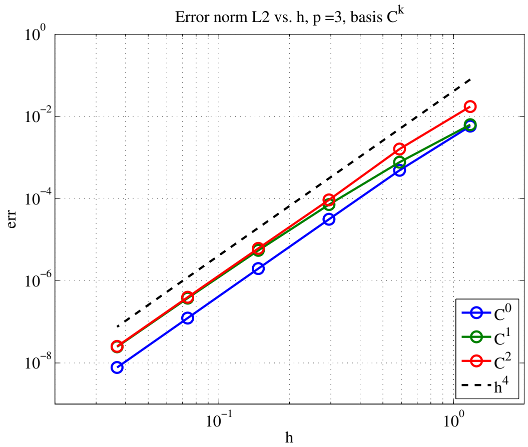



error in norm $L^2$ vs. $n_{dof}$

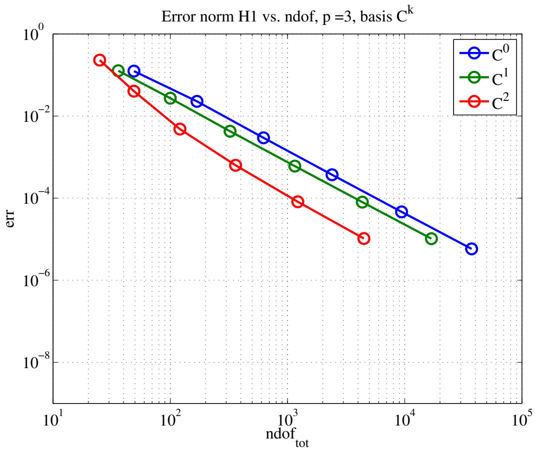
*NURBS basis $p = 3$, $C^k$–continuous with $k = 0, 1, 2$*



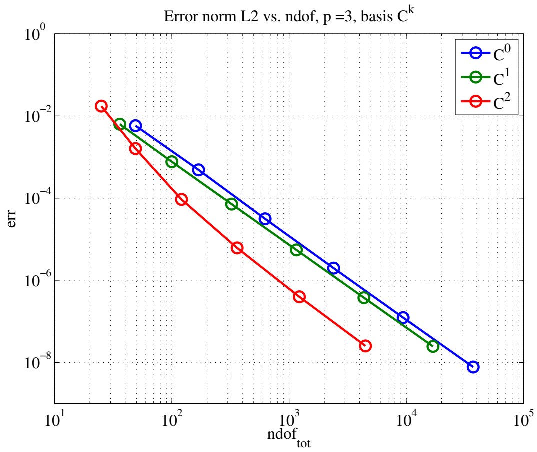

error in norm $H^1$ vs. $h$, conv. order 3




error in norm $L^2$ vs. $h$, conv. order 4




error in norm $H^1$ vs. $n_{dof}$



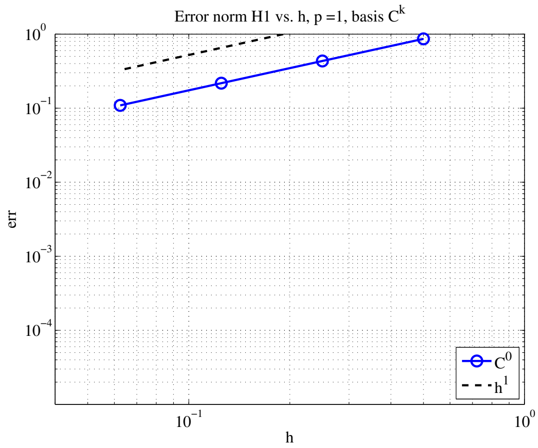

error in norm $L^2$ vs. $n_{dof}$

We remark that when using NURBS basis functions which are globally $C^{p-1}$–continuous we obtain the same convergence orders for the errors obtained with basis functions which are only $C^k$–continuous, with $k = 0, \dots, p-2$, but using a smaller number of degrees of freedom $n_{dof}$.

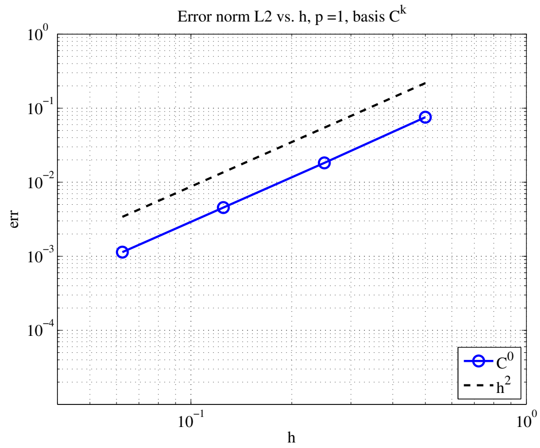
The convergence orders of the errors can be numerically estimated by assuming that the generic error (in norm $H^1$ or $L^2$), say $err$, depends on the mesh size $h$ as $err = C\, h^{\alpha}$, where $\alpha$ is the convergence order to be estimated. We consider next two different mesh sizes, say $h_1$ and $h_2$, for which the numerical simulations yield the errors $err_1$ and $err_2$, respectively. Then, we numerically estimate the convergence order as $\alpha = \log\left(err_1/err_2\right) / \log\left(h_1/h_2\right)$ since $err_1/err_2 = (h_1/h_2)^{\alpha}$.

b) Consider the MATLAB file `ex7_1b.m`. Similar considerations to point 1a) hold.

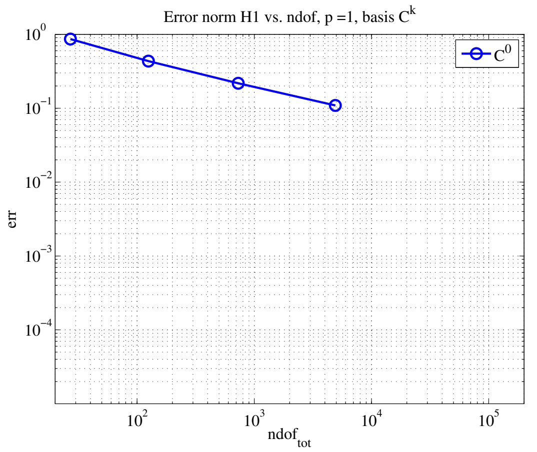
*NURBS basis* $p = 1$, $C^k$*–continuous with* $k = 0$



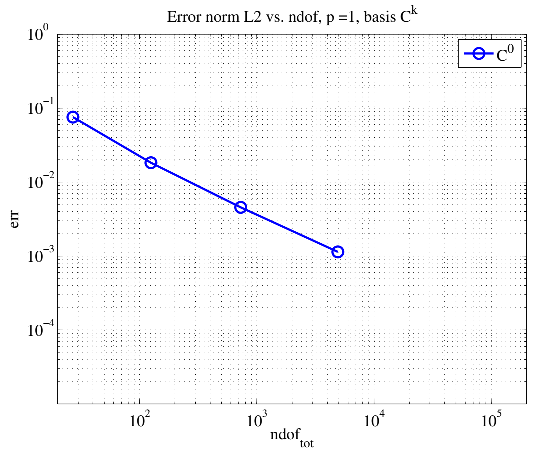

error in norm $H^1$ vs. $h$, conv. order 1




error in norm $L^2$ vs. $h$, conv. order 2

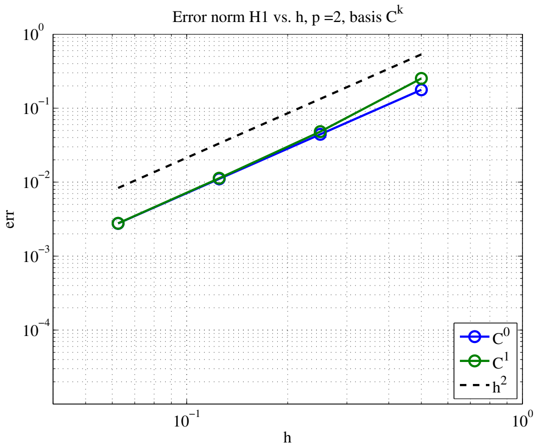



error in norm $H^1$ vs. $n_{dof}$

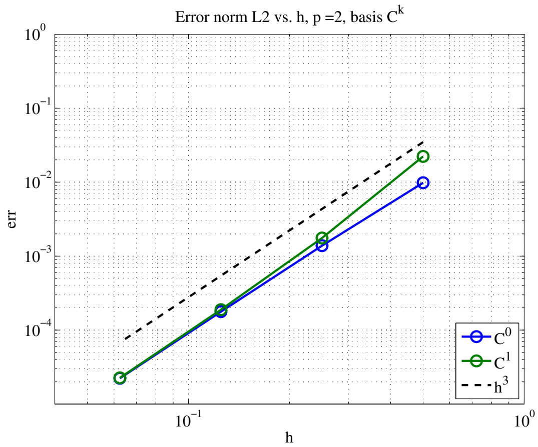



error in norm $L^2$ vs. $n_{dof}$

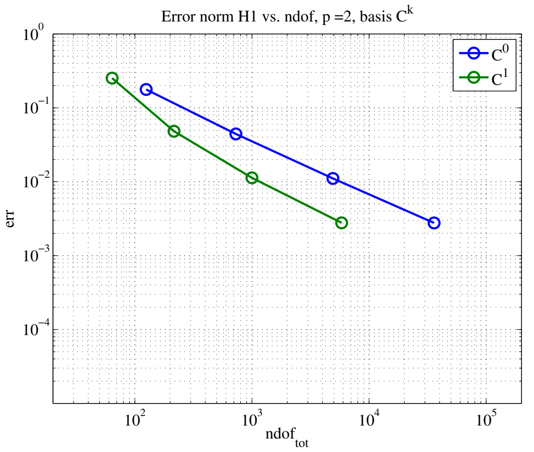
*NURBS basis* $p = 2$, $C^k$*–continuous with* $k = 0, 1$



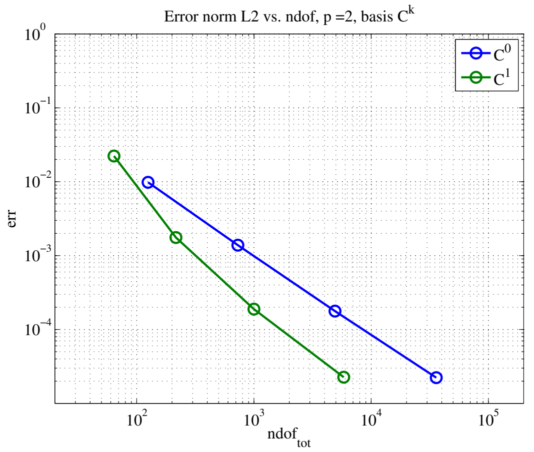

error in norm $H^1$ vs. $h$, conv. order 2




error in norm $L^2$ vs. $h$, conv. order 3

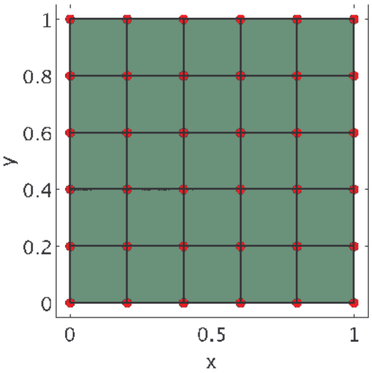



error in norm $H^1$ vs. $n_{dof}$

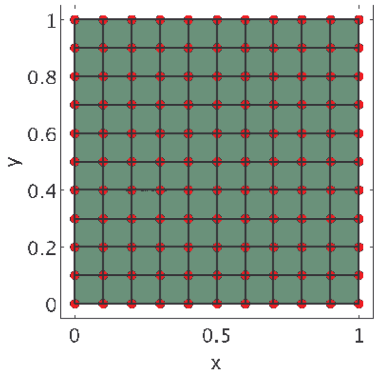



error in norm $L^2$ vs. $n_{dof}$

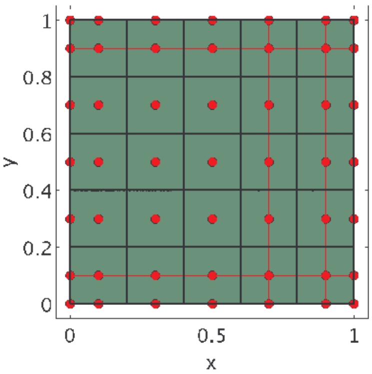
2. Consider the MATLAB file `ex7_2ab.m`. We solve the problem by considering B–splines basis of polynomial degrees $p = 1$ and $p = 2$ and the two meshes reported in the following figures; the corresponding positions of control points are also reported (red bullets). In the case $p = 2$ we consider basis functions which are globally $C^1$–continuous.



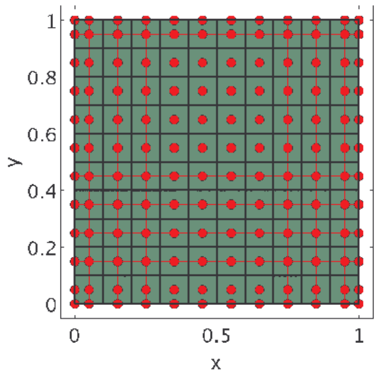

$p = 1$, $h = 1/5$




$p = 1$, $h = 1/10$



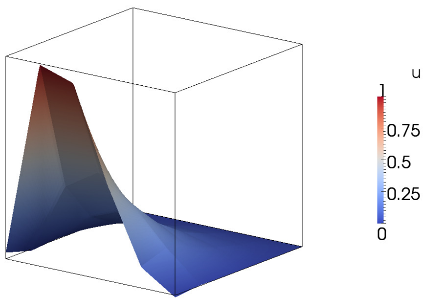

$p = 2$, $h = 1/5$



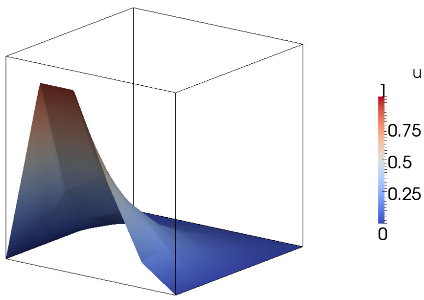

$p = 2$, $h = 1/10$

a) We report the numerical solutions $u_h$ obtained by solving the problem with the polynomial degree $p$ and the mesh sizes $h$ indicated above; the determination of the control variables for the strong imposition of the Dirichlet boundary conditions is performed by means of $L^2$ projection and the "interpolation at the control points" technique.

*B–splines basis* $p = 1$, $h = 1/5$

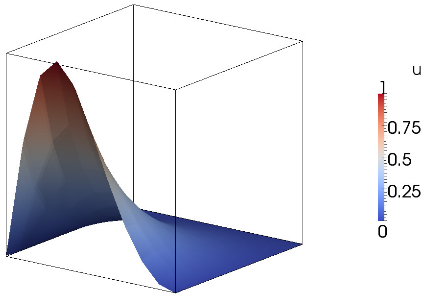



$L^2$ projection

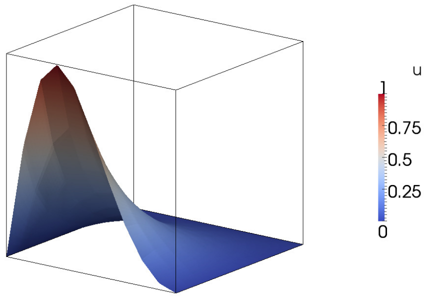



"interpolation at control points"

*B–splines basis $p = 1,\ h = 1/10$*

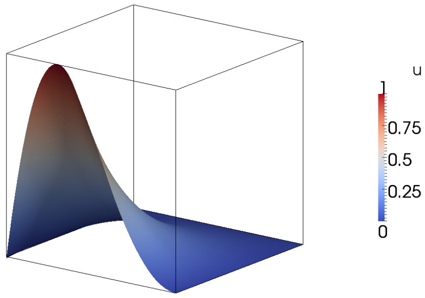



$L^2$ projection

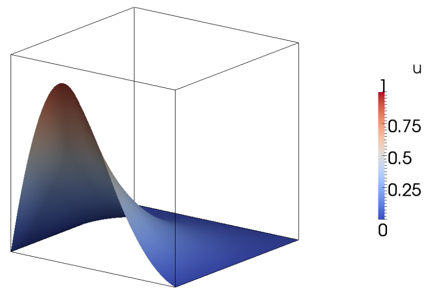



“interpolation at control points”

*B–splines basis $p = 2$ ($C^1$–continuous), $h = 1/5$*

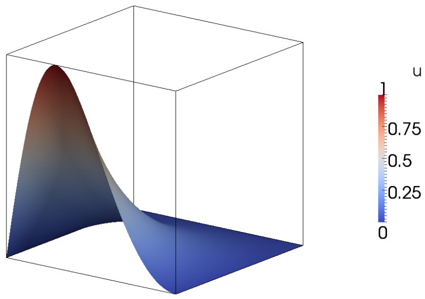



$L^2$ projection

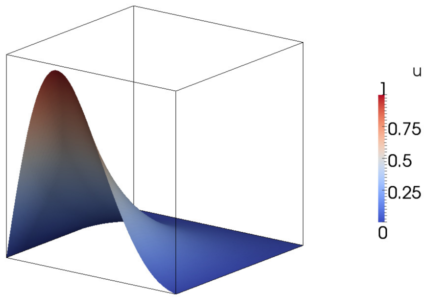



“interpolation at control points”

*B–splines basis $p = 2$ ($C^1$–continuous), $h = 1/10$*




$L^2$ projection

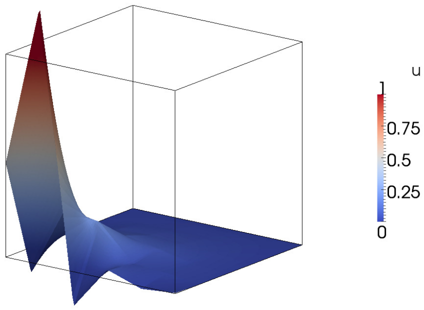



“interpolation at control points”

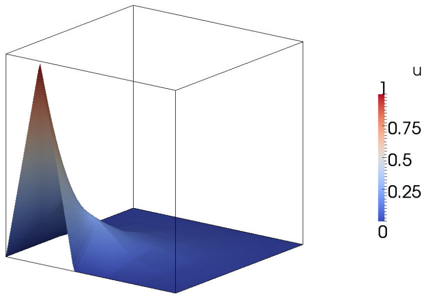
We observe that for the considered “smooth” Dirichlet data $g$ the $L^2$ projection technique generally yields better approximations of the data, say $g_h$, than the “interpolation at the control points” technique.

b) We repeat point 2a) by considering the discontinuous Dirichlet data $g$.

*B–splines basis* $p = 1,\ h = 1/5$

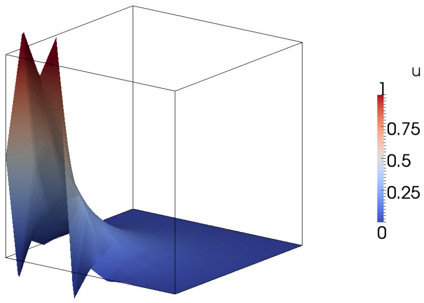



$L^2$ projection

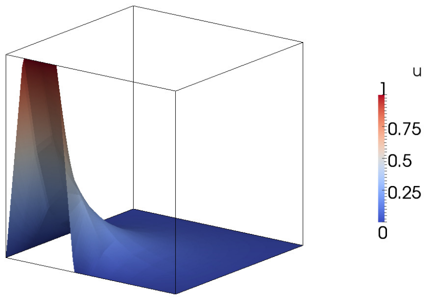



"interpolation at control points"

*B–splines basis* $p = 1,\ h = 1/10$

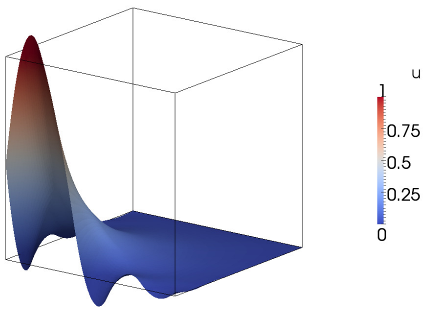



$L^2$ projection

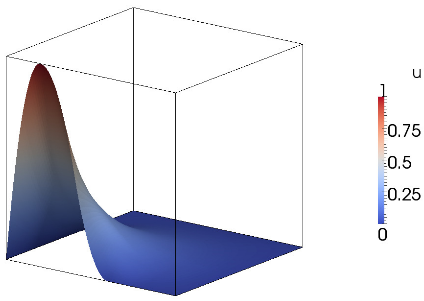



"interpolation at control points"

*B–splines basis* $p = 2$ *(*$C^1$*–continuous)*, $h = 1/5$

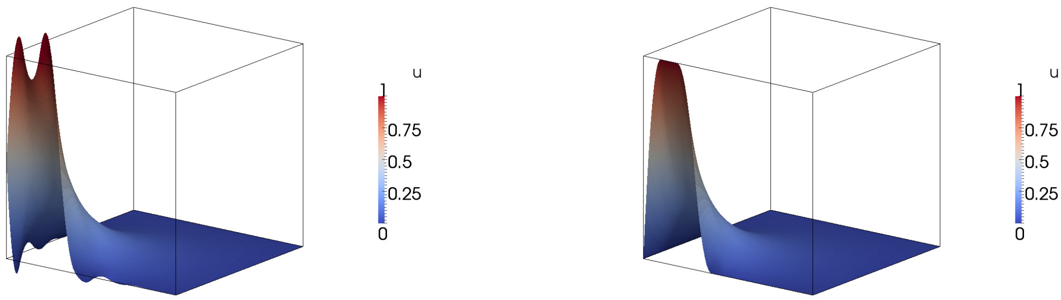



$L^2$ projection




"interpolation at control points"

*B–splines basis* $p = 2$ *(*$C^1$*–continuous),* $h = 1/10$




$L^2$ projection

"interpolation at control points"

We remark that for the considered discontinuous Dirichlet data $g$ the "interpolation at the control points" technique generally yields better approximations of the data $g_h$ than the $L^2$ projection technique. The latter exhibits significant over and undershooting and oscillations of the approximated data $g_h$ with respect to the data $g$ for all the meshes and polynomial degrees considered.