Computational Mechanics by Isogeometric Analysis
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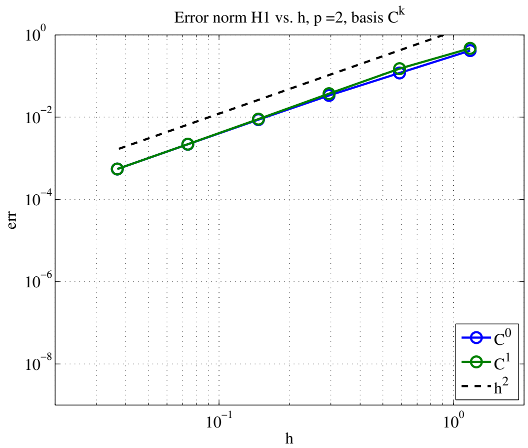
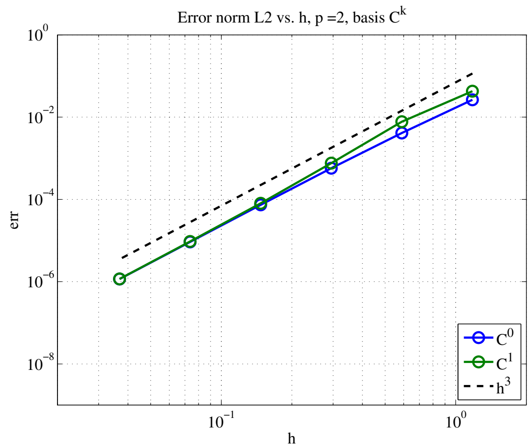
Exercises April 19, 2016: Solutions
NURBS–based Isogeometric Analysis: Galerkin method. II

1. a) Consider the MATLAB file `ex7_1a.m`. We remark that the exact solution u is infinitely differentiable, i.e. $u \in C^\infty(\Omega)$. Therefore, the expected convergence orders for the errors in norms H^1 and L^2 under h–refinement are p and $p+1$, respectively, where p is the polynomial degree used for the geometrical representation of the domain Ω and the NURBS space. We remark that the convergence orders are independent of the regularity of the NURBS basis across the mesh elements (knots).
We report the errors in norms H^1 and L^2 vs. the characteristic mesh size h and vs. the total number of degrees of freedom n_{dof} in the following figures (the logarithmic scale is used for both the axis); the expected convergence orders are reported for comparison and confirmed by the numerical results. NURBS basis with different regularities are considered compatibly with the polynomial degree p, for which the basis functions are globally C^k–continuous with $k = 0, \ldots, p-1$.

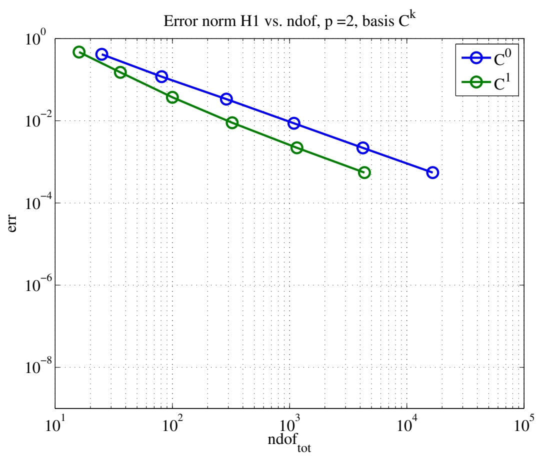
NURBS basis $p = 2$, C^k–continuous with $k = 0, 1$



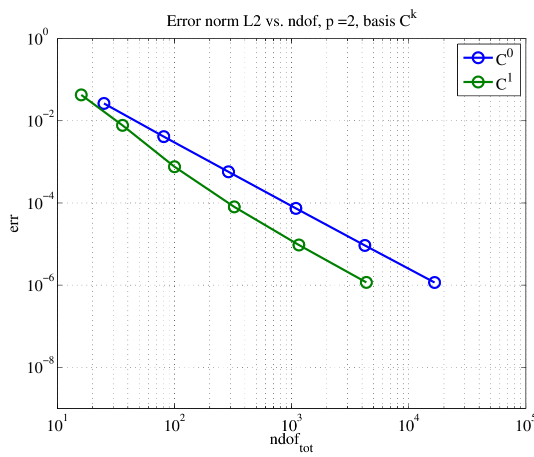
error in norm H^1 vs. h, conv. order 2



error in norm L^2 vs. h, conv. order 3

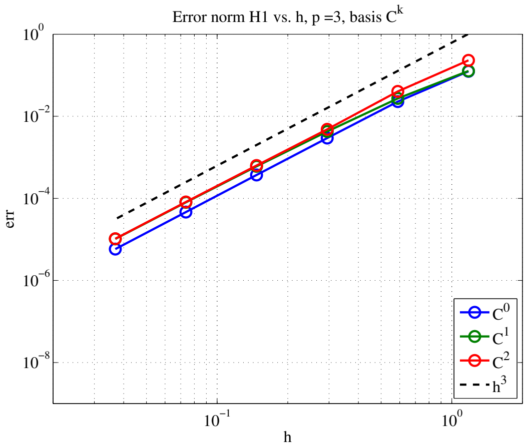


error in norm H^1 vs. n_{dof}

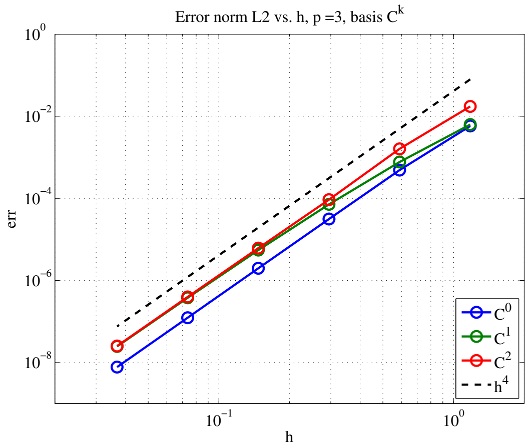


error in norm L^2 vs. n_{dof}

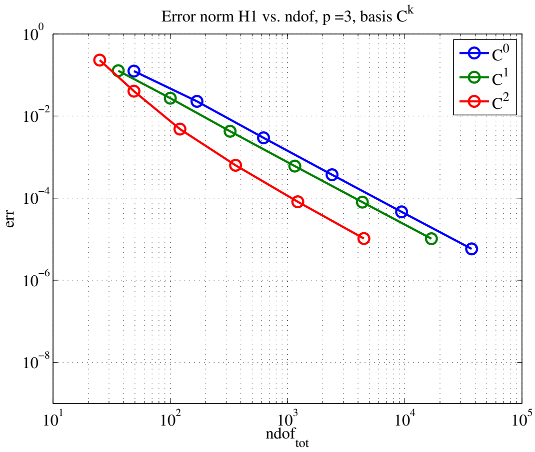
NURBS basis $p = 3$, C^k–continuous with $k = 0, 1, 2$



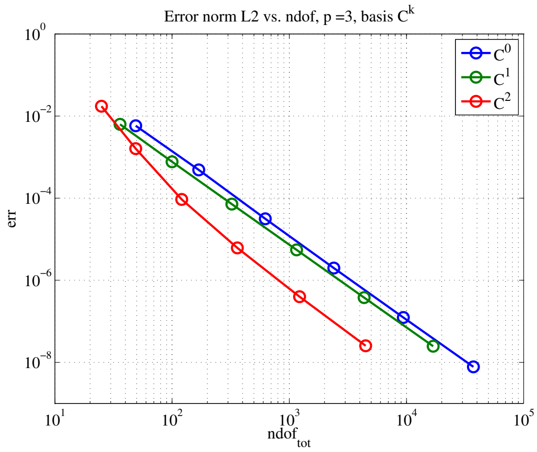
error in norm H^1 vs. h, conv. order 3



error in norm L^2 vs. h, conv. order 4



error in norm H^1 vs. n_{dof}



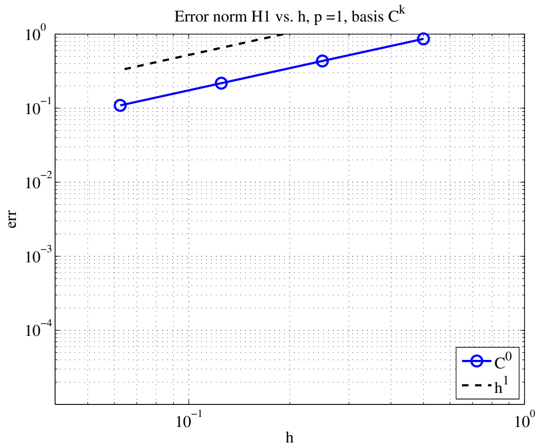
error in norm L^2 vs. n_{dof}

We remark that when using NURBS basis functions which are globally C^{p-1}–continuous we obtain the same convergence orders for the errors obtained with basis functions which are only C^k–continuous, with $k = 0, \ldots, p-2$, but using a smaller number of degrees of freedom n_{dof}.

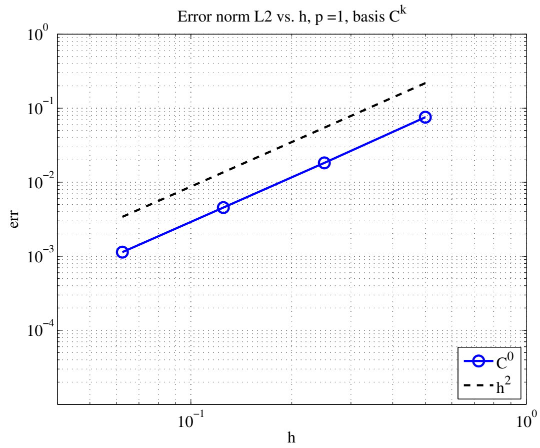
The convergence orders of the errors can be numerically estimated by assuming that the generic error (in norm H^1 or L^2), say err, depends on the mesh size h as $err = C\, h^{\alpha}$, where α is the convergence order to be estimated. We consider next two different mesh sizes, say h_1 and h_2, for which the numerical simulations yield the errors err_1 and err_2, respectively. Then, we numerically estimate the convergence order as $\alpha = \log\left(err_1/err_2\right) / \log\left(h_1/h_2\right)$ since $err_1/err_2 = (h_1/h_2)^{\alpha}$.

b) Consider the MATLAB file `ex7_1b.m`. Similar considerations to point 1a) hold.

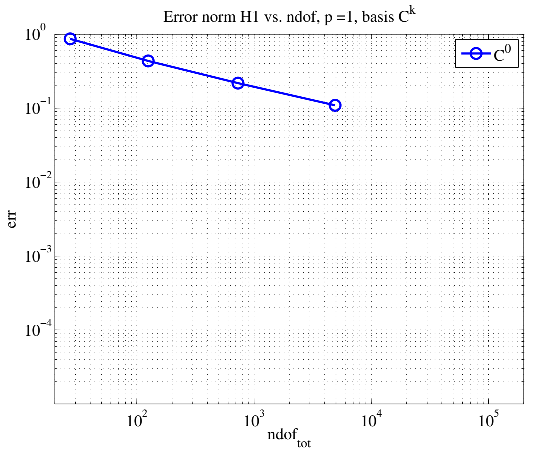
NURBS basis $p = 1$, C^k*–continuous with* $k = 0$



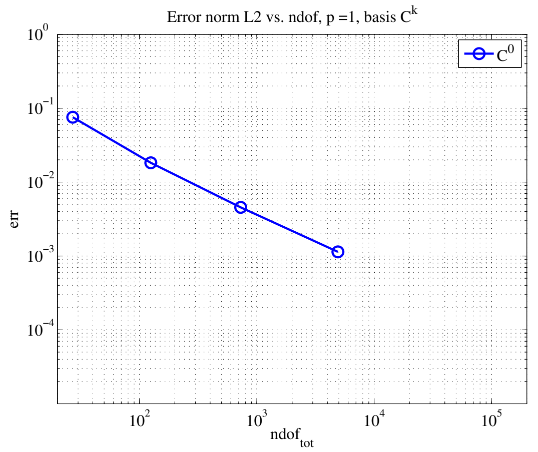
error in norm H^1 vs. h, conv. order 1



error in norm L^2 vs. h, conv. order 2

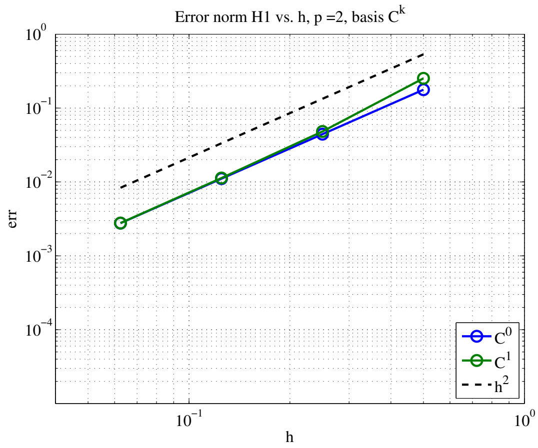


error in norm H^1 vs. n_{dof}

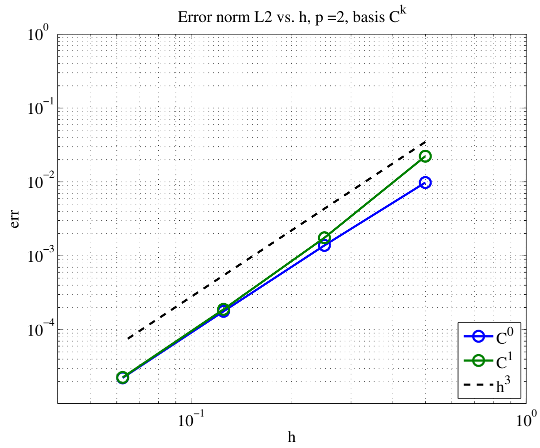


error in norm L^2 vs. n_{dof}

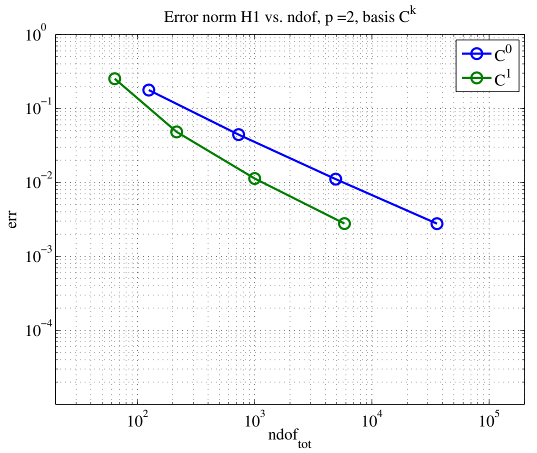
NURBS basis $p = 2$, C^k–continuous with $k = 0, 1$



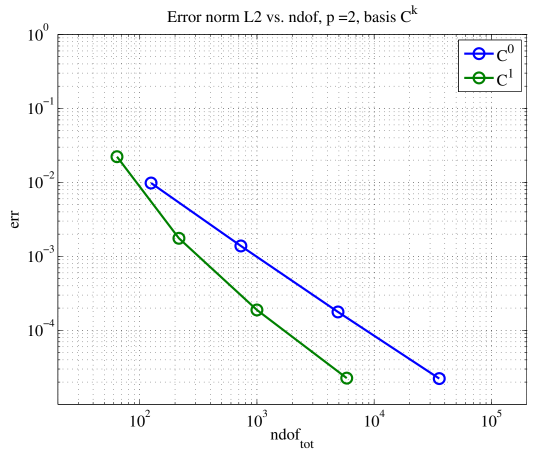
error in norm H^1 vs. h, conv. order 2



error in norm L^2 vs. h, conv. order 3

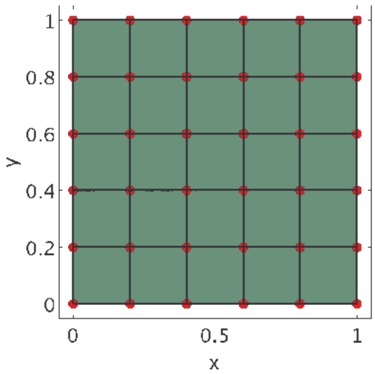


error in norm H^1 vs. n_{dof}

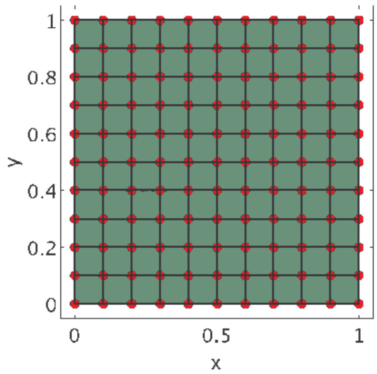


error in norm L^2 vs. n_{dof}

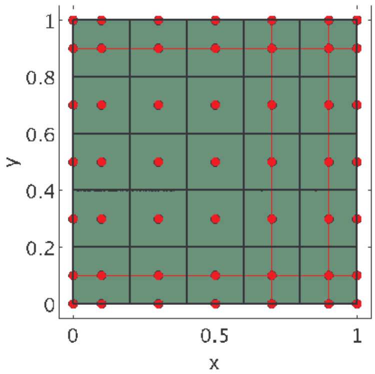
2. Consider the MATLAB file `ex7_2ab.m`. We solve the problem by considering B–splines basis of polynomial degrees $p = 1$ and $p = 2$ and the two meshes reported in the following figures; the corresponding positions of control points are also reported (red bullets). In the case $p = 2$ we consider basis functions which are globally C^1–continuous.



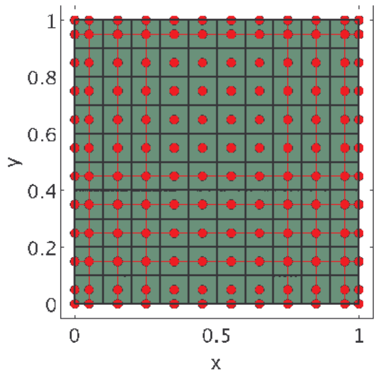
$p = 1,\ h = 1/5$



$p = 1,\ h = 1/10$



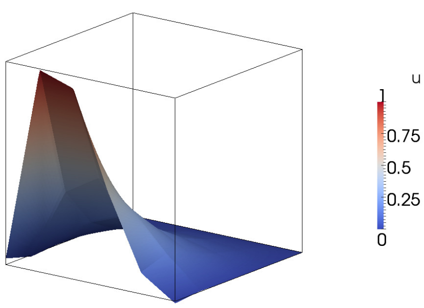
$p = 2,\ h = 1/5$



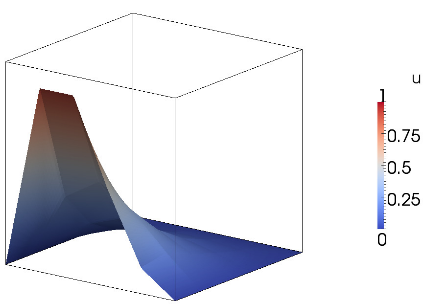
$p = 2,\ h = 1/10$

a) We report the numerical solutions u_h obtained by solving the problem with the polynomial degree p and the mesh sizes h indicated above; the determination of the control variables for the strong imposition of the Dirichlet boundary conditions is performed by means of L^2 projection and the "interpolation at the control points" technique.

B–splines basis $p = 1,\ h = 1/5$

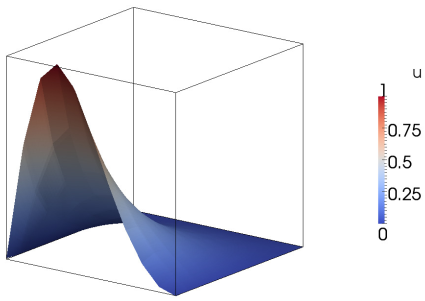


L^2 projection

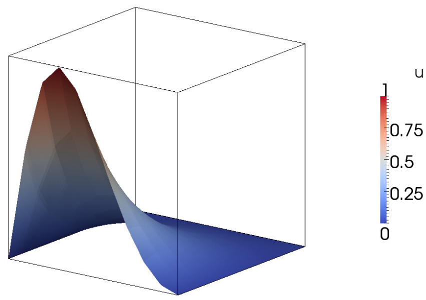


"interpolation at control points"

B–splines basis $p = 1,\ h = 1/10$

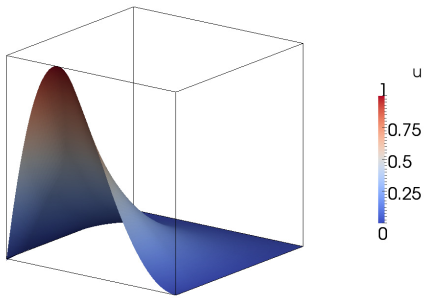


L^2 projection

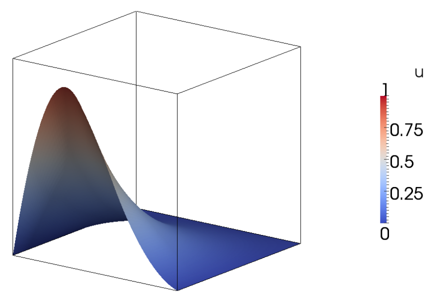


"interpolation at control points"

B–splines basis $p = 2$ *(*C^1*–continuous),* $h = 1/5$

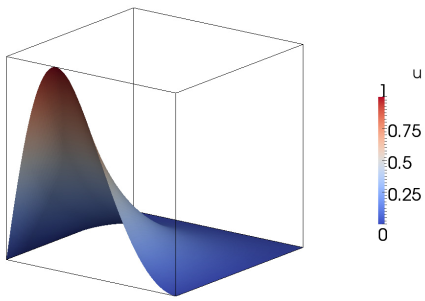


L^2 projection

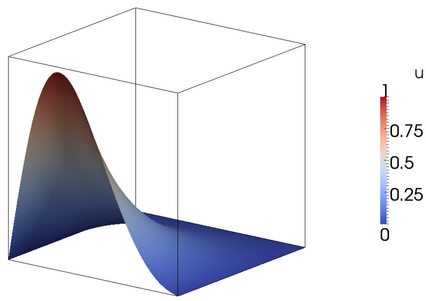


"interpolation at control points"

B–splines basis $p = 2$ *(*C^1*–continuous),* $h = 1/10$



L^2 projection

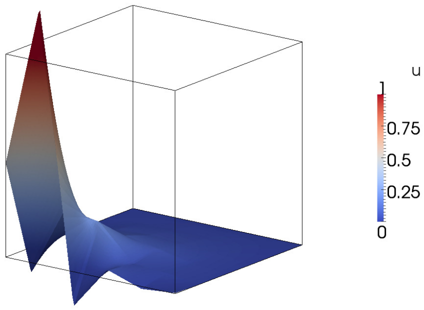


"interpolation at control points"

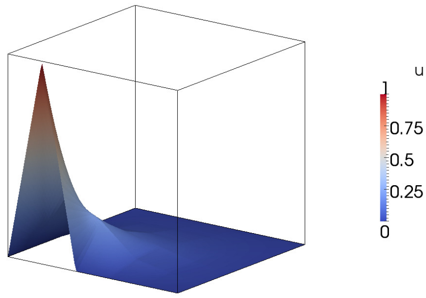
We observe that for the considered "smooth" Dirichlet data g the L^2 projection technique generally yields better approximations of the data, say g_h, than the "interpolation at the control points" technique.

b) We repeat point 2a) by considering the discontinuous Dirichlet data g.

B–splines basis $p = 1,\ h = 1/5$

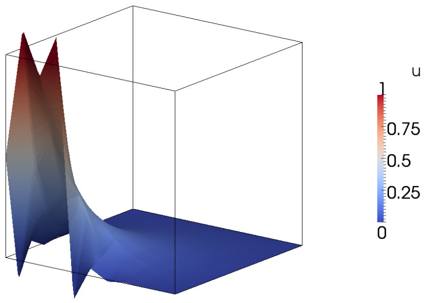


L^2 projection

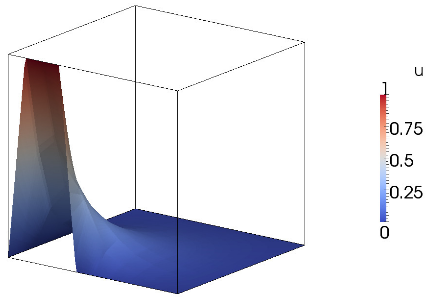


"interpolation at control points"

B–splines basis $p = 1,\ h = 1/10$

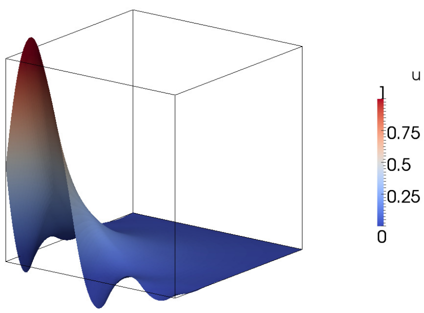


L^2 projection

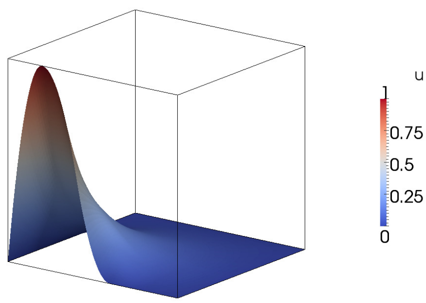


"interpolation at control points"

B–splines basis $p = 2$ *(C^1–continuous)*, $h = 1/5$

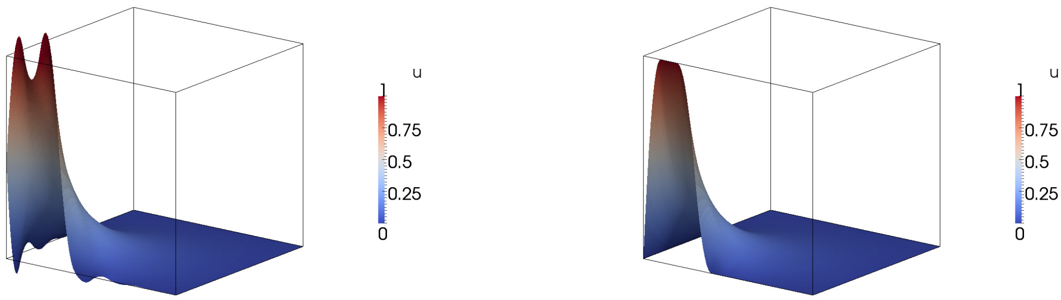


L^2 projection



"interpolation at control points"

B–splines basis $p = 2$ *(C^1–continuous),* $h = 1/10$



L^2 projection "interpolation at control points"

We remark that for the considered discontinuous Dirichlet data g the "interpolation at the control points" technique generally yields better approximations of the data g_h than the L^2 projection technique. The latter exhibits significant over and undershooting and oscillations of the approximated data g_h with respect to the data g for all the meshes and polynomial degrees considered.