Effective Preconditioning of Uzawa Type Schemes for Generalized Stokes Problem *

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A Schur complement of a model problem is considered as a preconditioner for the Uzawa type schemes for the generalized Stokes problem (the Stokes problem with the additional term $\alpha\mathbf{u}$ in the motion equation). Implementation of the preconditioned method requires on each iteration only one extra solution of the Poisson equation with the Neumann boundary conditions. For a wide class of 2D and 3D domains a theorem on its convergence is proved. In particular, it is established that the method converges with the rate that is bounded by some constant independent of α. Some finite difference and finite element methods are discussed. Numerical results for finite difference MAC scheme are provided.

Introduction

Numerical solution of the generalized Stokes problem plays a fundamental role in simulation of viscous incompressible flows (laminar and turbulent). Although a plenty of iterative algorithms are available for solving the classic Stokes problem, their direct application to the generalized Stokes problem provokes, as a rule, growth of the convergence factor when the parameter associated with the problem tends to zero or infinity.

Thus, we need efficient iterative methods for the generalized Stokes problem, whose rate of convergence would be at least not worse in the case of varying parameters than for the well-known algorithms for the classic Stokes problem. Recently several ways to develop efficient iterative solution technique were taken. Cahouet and Chabard in [4] and Ol'shanskii in [10],[11] improved the Uzawa scheme; Pal'cev in [13] and [14] constructed algorithms based on complete and incomplete splitting of boundary conditions; Bakhvalov in [2] considered in the fictitious domain method. All these papers deal with primitive variables (pressure-velocity) formulation of the problem.

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Motivation of this work is to develop general mathematical theory that underlie the approaches from [4] and [10],[11] for preconditioning of Uzawa type schemes and to demonstrate advantages of our considerations. It means, in particular, possibility to extend ideas developed in [10] on a more wide class of domains and the rigorous proof of convergence theorems. The important result from [12] connects the method presented with ideas of Cahouet and Chabard and establishes the equivalence of these approaches in the continuous case. Two key points of the paper should be emphasized. These are the preconditioning of the Uzawa scheme with a Schur operator for the model problem (this operator is shown to be equivalent to some pseudodifferential operator) and sufficient conditions for convergence in the form of inequalities of Ladyzhenskaya-Babuška-Brezzi type. Their original proof in domains with regular boundaries is presented in section 4, and some results for Lischitz domains can be found in Appendix.

Basic considerations were taken for the continuous case although their application for finite differences and some remarks on finite elements can be found in section 5. Some numerical results we present in section 6. For further discussions on numerical performance we refer to [4] for finite element calculations with different parameters, domains, elements, etc.

1 Generalized Stokes problem, Uzawa algorithm and its preconditioning

Let Ω be a domain in $R^n,\ n = 2, 3$, with Lipschitz-continuous boundary $\partial\Omega$. Consider in Ω the system of partial differential equations

$$\begin{aligned} &-\Delta \mathbf{u} + \alpha\mathbf{u} + \nabla p = \mathbf{f}, \\ &\operatorname{div} \mathbf{u} = 0, \\ &\mathbf{u}|_{\partial\Omega} = \mathbf{0}, \\ &\textstyle\int_\Omega p\ dx = 0 \end{aligned} \tag{1.1}$$

where $\mathbf{u} = (u_1(x), \ldots, u_n(x))$ is the velocity vector, $p = p(x)$ is the pressure function, $\mathbf{f} = (f_1(x), \ldots, f_n(x))$ is the field of external forces, and $\alpha \geq 0$ is an arbitrary real parameter (constant). If $\alpha = 0$, then (1.1) turns to be the classic Stokes system. In unsteady Navier-Stokes calculations typically $\alpha \sim (\bar{\nu}\,\delta t)^{-1}$, where $\bar{\nu}$ is a kinematic viscosity and δt is a step of time integration, and hence, as a rule, $\alpha \gg 1$.

Later on we need the following functional spaces:

$$\mathbf{H}_0^1 \equiv \{\mathbf{u} \in W_2^1(\Omega)^n :\ \mathbf{u} = 0 \text{ on } \partial\Omega\}$$

with energy scalar product $(\mathbf{u}, \mathbf{v})_1 = (\nabla\mathbf{u}, \nabla\mathbf{v}),\ \mathbf{u}, \mathbf{v} \in \mathbf{H}_0^1$,

$$L_2^0 \equiv \{p \in L_2(\Omega) :\ \int_\Omega p\ dx = 0\}$$

with L_2-scalar product. Let $\mathbf{H}^{-1}$ be a dual, with respect to L_2-duality, space to $\mathbf{H}_0^1$ with the obvious norm:

$$\|\mathbf{f}\|_{-1} = \sup_{0 \neq \mathbf{u} \in \mathbf{H}_0^1} \frac{< \mathbf{f}, \mathbf{u} >}{\|\mathbf{u}\|_1}, \quad \mathbf{f} \in \mathbf{H}^{-1}$$

Solution $\{\mathbf{u}, p\}$ of *generalized Stokes problem* (1.1) (problem of the Stokes type with the parameter) exists and is unique in $\mathbf{H}_0^1 \times L_2^0$ for given $\mathbf{f} \in \mathbf{H}^{-1}$. The case of the Dirichlet boundary conditions for velocity is of fundamental interest both in theory and applications although some other conditions can be posed on the boundary (see, e.g., [16]). Nonhomogeneous conditions can be also considered without loss of generality, see [8].

Probably the simplest (but surprisingly effective [6]) method to solve the *classic* ($\alpha = 0$) Stokes problem is the iterative Uzawa algorithm, see [1]. For the generalized Stokes problem the Uzawa algorithm is described as follows: start with an arbitrary initial guess $p_0 \in L_2^0$ and for $i = 0, 1, \ldots$ do until convergence:

$$\begin{aligned} &\text{Step1. Compute } \mathbf{u}^{i+1} \text{ from} \\ &\qquad -\Delta \mathbf{u}^{i+1} + \alpha \mathbf{u}^{i+1} = \mathbf{f} - \nabla p^i \\ &\qquad \text{with } \mathbf{u}^{i+1} = 0 \text{ on } \partial\Omega. \\ &\text{Step2. Define new pressure } p^{i+1} \text{ as} \\ &\qquad p^{i+1} = p^i - \tau_0 \operatorname{div} \mathbf{u}^{i+1}. \end{aligned} \tag{1.2}$$

The algorithm converges for sufficiently small values of $\tau_0 (> 0)$.

Consider the Schur complement for system (1.1):

$$A_0(\alpha) = \operatorname{div} (\Delta - \alpha \mathrm{I})_0^{-1} \nabla$$

where $(\Delta - \alpha \mathrm{I})_0^{-1} : \mathbf{H}^{-1} \to \mathbf{H}_0^1$ denotes the solution operator for the Helmholtz problem

$$\begin{aligned} \Delta \mathbf{u} - \alpha \mathbf{u} &= \mathbf{g} \\ \mathbf{u}|_{\partial\Omega} &= \mathbf{0} \end{aligned} \tag{1.3}$$

i.e., for a given $\mathbf{g} \in \mathbf{H}^{-1}$ the vector function $\mathbf{u} = (\Delta - \alpha \mathrm{I})_0^{-1} \mathbf{g}$ is the solution of (1.3).

The operator $A_0(\alpha)$ is a self-adjoined and positive definite one from L_2^0 onto L_2^0. Now the Uzawa algorithm can be considered as a first order Richardson iterative method with a fixed iterative parameter applied to the equation

$$A_0(\alpha) p = \operatorname{div} (\Delta - \alpha \mathrm{I})_0^{-1} \mathbf{f} \tag{1.4}$$

This simple observation gives us easily obtained asymptotics for the convergence rate of (1.2), i.e., $\rho \sim 1 - O(\alpha^{-1}),\ \alpha \to \infty$ (see Remark 3.2 in section 3) and, therefore, $\rho \to 1$ for $\alpha \to \infty$. However, the same observation allows us to improve the classic Uzawa scheme by using various Krylov subspace methods (e.g., conjugate gradient or conjugate residual ones) for system (1.4), and gives us natural and fruitful idea of preconditioning.

Indeed, consider for simplicity the first order preconditioned iterative method

$$B\frac{p^{i+1}-p^i}{\tau_i} = -A_0(\alpha)p^i + \mathbf{f}, \quad i = 0, 1, \ldots \tag{1.5}$$

where $B = B^* > 0$ is a preconditioner depending probably on α and acting from L_2^0 onto L_2^0. The natural requirements are 'easy' solvability of the equation $Bp = q$ for $q \in L_2^0$ and validity of the estimate $cond(B^{-1}A_0(\alpha)) \leq c$, where c is some constant independent of α and mesh size if some discretization is considered.

As far as we know, at least two ways of preconditioning considered in literature can be candidates to satisfy these two requirements. To begin with, note that on each step of method (1.5) we have to compute $q = B^{-1}\bar{p}$ for some $\bar{p} \in L_2^0$.

Let us set

$$B^{-1}\bar{p} = \bar{p} - \alpha r$$

where r is a solution of the following boundary value problem

$$\Delta r = \bar{p}, \quad \left.\frac{\partial r}{\partial \boldsymbol{\nu}}\right|_{\partial\Omega} = 0$$

Denote by Δ_N^{-1} the solution operator for the above Poisson equation with Neumann boundary conditions. Then set formally $B_1 = (\mathrm{I} - \alpha\Delta_N^{-1})^{-1}$ and consider it as a preconditioner. This approach of Cahouet-Chabard certainly satisfies the condition of 'easy solvability' (one step of algorithm requires only one extra solving the Neumann problem). Numerous finite element calculations presented in [4] show efficiency of such preconditioning. However, there were no estimates on the condition number for $B^{-1}A_0(\alpha)$ and no an appropriate convergence theorem for (1.5) with such preconditioning.

To overview the second approach, let us consider in $\Omega = (0,1) \times (0,1)$ the generalized Stokes problem with *another boundary conditions*:

$$\begin{aligned} &-\Delta\mathbf{u} + \alpha\mathbf{u} + \nabla p = \mathbf{f}, \\ &\operatorname{div}\mathbf{u} = 0, \\ &\mathbf{u}\cdot\boldsymbol{\nu}|_{\partial\Omega} = \left.\frac{\partial(\mathbf{u}\cdot\boldsymbol{\tau})}{\partial\boldsymbol{\nu}}\right|_{\partial\Omega} = 0 \end{aligned} \tag{1.6}$$

Hereafter $\boldsymbol{\nu}$ and $\boldsymbol{\tau}$ are unit outward normal and tangent vectors to $\partial\Omega$.

The problem is well posed and the solution $\{\mathbf{u}, p\}$ of (1.6) can be *explicitly* found via Fourier series:

$$u_1(x,y) = \sum_{m,k=0}^{\infty} a_{m,k} \sin m\pi x \, \cos k\pi y$$

$$u_2(x,y) = \sum_{m,k=0}^{\infty} b_{m,k} \cos m\pi x \, \sin k\pi y$$

$$p(x,y) = \sum_{\substack{m,k=0 \\ m+n>0}}^{\infty} c_{m,k} \cos m\pi x \, \cos k\pi y$$

Now let us consider the Schur complement of (1.6) as a preconditioner

$$B_2 = \mathrm{div}\,(\Delta - \alpha \mathrm{I})_p^{-1} \nabla p$$

where the new boundary conditions are 'build in' B_2 in the same way as the Dirichlet conditions in $A_0(\alpha)$. It is easy to check that $B_2 = \mathrm{I}$ for $\alpha = 0$, hereafter I denotes the identity operator on an appropriate functions space.

In [10] the inequalities

$$cB_2 \le A_0(\alpha) \le B_2$$

were proved with c independent of α and, thus, the uniform estimate of (1.5) convergence rate was obtained for an appropriate set of τ_i (see also [11] for finite difference case). Numerical experiments with finite difference scheme demonstrate a very good convergence of (1.5) with such preconditioning. However, the method was considered only in rectangular domains. Moreover, the 'easy solvability' of the equation $B_2 p = q$, using Fast Fourier Transform, requires a rectangular domain and a uniform grid at least for one variable.

The substantial step for understanding of a mathematical theory lying behind two mentioned above approaches was made in the recent paper of Ol'shanskii [12], where the boundary conditions from (1.6) were extended onto a wide class of domains in such a way that the equivalence of the Schur complement for classic Stokes system to the identity operator on L_2^0 was preserved. We shall clarify details in the next paragraph.

2 Model boundary conditions

Again we assume Ω to be bounded Lipschitz-continuous domain in R^n, $n = 2, 3$.

The traces of vector functions from $W_1^2(\Omega)^n$ induce on $\partial\Omega$ a functional space denoted by $H^{\frac{1}{2}}(\partial\Omega)^n$ and equipped with the norm

$$\|\boldsymbol{\mu}\|_{\frac{1}{2}} = \inf_{\substack{\mathbf{v} \in W_2^1(\Omega)^n \\ \mathbf{v} = \boldsymbol{\mu} \text{ on } \partial\Omega}} \|\mathbf{v}\|_{W_2^1}, \qquad \boldsymbol{\mu} \in H^{\frac{1}{2}}(\partial\Omega)^n$$

Let $H^{-\frac{1}{2}}(\partial\Omega)^n$ be a dual space to $H^{\frac{1}{2}}(\partial\Omega)^n$ with the obvious norm

$$\|\boldsymbol{\xi}\|_{-\frac{1}{2}} = \sup_{0 \ne \boldsymbol{\mu} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)^n} \frac{<\boldsymbol{\xi}, \boldsymbol{\mu}>}{\|\boldsymbol{\mu}\|_{\frac{1}{2}}}, \qquad \boldsymbol{\xi} \in H^{-\frac{1}{2}}(\Omega)^n$$

For any vector function $\mathbf{u} \in L_2(\Omega)^n$ such that $\mathrm{div}\,\mathbf{u} \in L_2(\Omega)$, $\mathrm{curl}\,\mathbf{u} \in L_2(\Omega)^{2n-3}$, its normal and tangential components on $\partial\Omega$ ($\mathbf{u}\cdot\boldsymbol{\nu}$ and $\gamma_\tau \mathbf{u} = \mathbf{u}\cdot\boldsymbol{\tau}$ for $n = 2$, $\gamma_\tau \mathbf{u} = \mathbf{u} \times \boldsymbol{\nu}$ for $n = 3$) can be considered as elements of $H^{-\frac{1}{2}}(\Omega)^r, r = 1, 3$. Thus, the definition of the following functional space is correct:

$$\mathbf{U} \equiv \{\mathbf{u} \in L_2(\Omega)^n : \mathrm{div}\,\mathbf{u} \in L_2(\Omega),\ \mathrm{curl}\,\mathbf{u} \in L_2(\Omega)^{2n-3},\ \mathbf{u}\cdot\boldsymbol{\nu} = 0 \text{ on } \partial\Omega\}$$

$\mathbf{U}$ is a Hilbert space with respect to the scalar product $(\mathbf{u},\mathbf{v})_{\mathbf{U}} = (\mathbf{u},\mathbf{v}) + (\operatorname{div}\mathbf{u},\operatorname{div}\mathbf{v}) + (\operatorname{curl}\mathbf{u},\operatorname{curl}\mathbf{v}),\ \mathbf{u},\mathbf{v}\in\mathbf{U}$. By $\mathbf{U}^{-1}$ we denote a dual to $\mathbf{U}$ space with respect to L_2 duality.

Remark 2.1. If $\mathbf{u}\in\mathbf{U}$ and $\gamma_\tau\mathbf{u}=0$, then $\mathbf{u}\in\mathbf{H}_0^1$ (see Lemma 2.5 in [8]).

Remark 2.2. Further we need the following estimates:

$$
\begin{aligned}
&\|\mathbf{u}\cdot\boldsymbol{\nu}\|_{-\frac{1}{2}} \le \|\mathbf{u}\| + \|\operatorname{div}\mathbf{u}\| \\
&\|\gamma_\tau\mathbf{u}\|_{-\frac{1}{2}} \le \|\mathbf{u}\| + \|\operatorname{curl}\mathbf{u}\|
\end{aligned}
\tag{2.1}
$$

For more details about $\mathbf{U}$ we refer to [8].

Consider now the problem: find $\{\mathbf{u},p\}$ from $\mathbf{U}\times L_2^0$ for given $\{\mathbf{f},g\}\in\mathbf{U}^{-1}\times L_2^0$ such that

$$
\begin{aligned}
&(\operatorname{div}\mathbf{u},\operatorname{div}\mathbf{v}) + (\operatorname{curl}\mathbf{u},\operatorname{curl}\mathbf{v}) + \alpha(\mathbf{u},\mathbf{v}) - (p,\operatorname{div}\mathbf{v}) = <\mathbf{f},\mathbf{v}> \\
&(\operatorname{div}\mathbf{u},q) = (g,q), \qquad\qquad \forall\mathbf{v}\in\mathbf{U},\ q\in L_2^0
\end{aligned}
\tag{2.2}
$$

Problem (2.2) is well posed for all $\alpha\ge 0$ and results on regularity for this problem are valid (cf. [12]).

Remark 2.3. For $\alpha=0$ we should additionally require Ω to be simply connected domain. In this case the bilinear form $(\operatorname{div}\mathbf{u},\operatorname{div}\mathbf{v}) + (\operatorname{curl}\mathbf{u},\operatorname{curl}\mathbf{v})$ is coercive on $\mathbf{U}$.

It is easy to check that (2.2) is a *weak formulation* of the following generalized Stokes problem with the *model boundary conditions:*

$$
\begin{aligned}
&-\Delta\mathbf{u} + \alpha\mathbf{u} + \nabla p = \mathbf{f} \\
&\operatorname{div}\mathbf{u} = 0 \\
&\mathbf{u}\cdot\boldsymbol{\nu}|_{\partial\Omega} = \mathcal{R}\mathbf{u}|_{\partial\Omega} = 0
\end{aligned}
\tag{2.3}
$$

where

$$
\mathcal{R}\mathbf{u} = \begin{cases} \operatorname{curl}\mathbf{u}, & n=2 \\ (\operatorname{curl}\mathbf{u})\times\boldsymbol{\nu}, & n=3 \end{cases}
$$

We will also refer to this problem as a *model.*

By $A_\nu(\alpha)$ we denote the Schur complement for system (2.3)

$$
A_\nu(\alpha) \equiv \operatorname{div}(\Delta - \alpha\mathrm{I})_\nu^{-1}\nabla
$$

where $(\Delta-\alpha\mathrm{I})_p^{-1} : \mathbf{U}^{-1}\to\mathbf{U}$ denotes the solution operator for the problem

$$
\begin{aligned}
&\Delta\mathbf{u} - \alpha\mathbf{u} = \mathbf{g} \\
&\mathbf{u}\cdot\boldsymbol{\nu}|_{\partial\Omega} = \mathcal{R}\mathbf{u}|_{\partial\Omega} = 0
\end{aligned}
$$

and ∇ acts from L_2^0 to $\mathbf{U}^{-1}$, i.e., $\mathbf{w} = (\Delta-\alpha\mathrm{I})_\nu^{-1}\nabla p,\ \mathbf{w}\in\mathbf{U}$ is a solution of the problem

$$
(\operatorname{div}\mathbf{w},\operatorname{div}\mathbf{v}) + (\operatorname{curl}\mathbf{w},\operatorname{curl}\mathbf{v}) + \alpha(\mathbf{w},\mathbf{v}) = (p,\operatorname{div}\mathbf{v}), \qquad \forall\mathbf{v}\in\mathbf{U}
$$

$A_\nu(\alpha)$ is a selfadjoined positive definite operator on L_2^0.

The following important result proved in [12] concludes this paragraph.

Theorem 2.1 *For arbitrary* $\alpha \in [0, \infty)$ *and* $p \in L_2^0$ *set* $q = A_\nu(\alpha)p$, $q \in L_2^0$. *Then the following equality holds:* $p = q - \alpha\, r$, *where* $r \in W_2^1(\Omega) \cap L_2^0$ *is a solution of the Neumann problem*

$$\Delta r = q, \quad \left.\frac{\partial r}{\partial \boldsymbol{\nu}}\right|_{\partial\Omega} = 0$$

Note that in the case $\alpha = 0$ the theorem states equality of the Schur complement of the model Stokes problem to the identity operator on L_2^0. In a general case, model boundary conditions provide decoupling of pressure and velocity in the Stokes problem (see also Theorem 7 in [12] for pressure Poisson problem). If we denote by Δ_N^{-1} the solution operator for scalar Poisson problem with Neumann boundary conditions, then the statement of theorem 2.1 can be written as

$$A_\nu(\alpha)^{-1} = I - \alpha \Delta_N^{-1} \quad \text{on} \quad L_2^0.$$

Remark 2.4. Under stronger assumptions on regularity of $\partial\Omega$ (e.g., $\partial\Omega \in C^2$ or Ω is a convex polygon (polyhedron)), the following equality is valid (cf. [8]):

$$\mathbf{U} = \{\mathbf{u} \in W_2^1(\Omega)^n : \mathbf{u}{\cdot}\boldsymbol{\nu} = 0 \text{ on } \partial\Omega\}$$

and the above theory can be established in the Sobolev space. However, if the extra assumptions on $\partial\Omega$ fail, the requirement $\mathbf{u} \in W_2^1(\Omega)^n$ could be weaken and replaced by $\mathbf{u} \in \mathbf{U}$ (see example in [12]).

3 Iterative method and uniform estimate of convergence rate

For the sake of convenience we rewrite iterative algorithm (1.5) taking the Schur complement of the model problem as a preconditioner

$$A_\nu(\alpha)\frac{p^{i+1} - p^i}{\tau_i} = -A_0(\alpha)p^i + \mathbf{f}, \quad i = 0, 1, \ldots \tag{3.1}$$

Now, on the one hand method (3.1) is equivalent to (1.5) with B_1 as a preconditioner, by virtue of Theorem 2.1, and the equation $A_\nu(\alpha)p = q$ is 'easily' solved; on the other hand in rectangular domains $A_\nu(\alpha) = B_2$ and, thus, we hope to prove uniform convergence estimates in a general case.

Operators $A_\nu(\alpha)$ and $A_0(\alpha)$ differ only by the implicitly involved boundary conditions for the velocity. Therefore, we may expect these operators to be rather close. Indeed, the following theorem is valid.

Theorem 3.1 *There exists the constant $c(\Omega) > 0$ independent of $\alpha \geq 0$ such that*

$$c(\Omega) A_\nu(\alpha) \leq A_0(\alpha) \leq A_\nu(\alpha)$$

Proof. For convenience we introduce the scalar product in $\mathbf{H}_0^1$ depending on α :

$$(\mathbf{u}, \mathbf{v})_\alpha = (\nabla \mathbf{u}, \nabla \mathbf{v}) + \alpha(\mathbf{u}, \mathbf{v})$$

For any $p \in L_2^0$ the following equalities are valid

$$\begin{gathered}
(A_0(\alpha)\, p, p) = (\mathrm{div}\,(\Delta - \alpha \mathrm{I})_0^{-1} \nabla p, p) = - < (\Delta - \alpha \mathrm{I})_0^{-1} \nabla p, \nabla p >= \\
- < (\Delta - \alpha \mathrm{I})_0^{-1} \nabla p, (\Delta - \alpha \mathrm{I})(\Delta - \alpha \mathrm{I})_0^{-1} \nabla p >= \|(\Delta - \alpha \mathrm{I})_0^{-1} \nabla p\|_\alpha^2 = \\
\sup_{0 \neq \mathbf{u} \in \mathbf{H}_0^1} \frac{((\Delta - \alpha \mathrm{I})_0^{-1} \nabla p, \mathbf{u})_{1,\alpha}^2}{\|\mathbf{u}\|_\alpha^2} = \sup_{0 \neq \mathbf{u} \in \mathbf{H}_0^1} \frac{< \nabla p, \mathbf{u} >^2}{\|\mathbf{u}\|_1^2 + \alpha \|\mathbf{u}\|^2} = \\
\sup_{0 \neq \mathbf{u} \in \mathbf{H}_0^1} \frac{(p, \mathrm{div}\, \mathbf{u})^2}{\|\mathbf{u}\|_1^2 + \alpha \|\mathbf{u}\|^2} = \sup_{0 \neq \mathbf{u} \in \mathbf{H}_0^1} \frac{(p, \mathrm{div}\, \mathbf{u})^2}{\|\mathrm{div}\, \mathbf{u}\|^2 + \|\mathrm{curl}\, \mathbf{u}\|^2 + \alpha \|\mathbf{u}\|^2}
\end{gathered} \tag{3.2}$$

In a similar way we get

$$(A_\nu(\alpha)\, p, p) = \sup_{0 \neq \mathbf{u} \in \mathbf{U}} \frac{(p, \mathrm{div}\, \mathbf{u})^2}{\|\mathrm{div}\, \mathbf{u}\|^2 + \|\mathrm{curl}\, \mathbf{u}\|^2 + \alpha \|\mathbf{u}\|^2}$$

Thus, to prove the theorem it is necessary and sufficient to check validity of the inequalities:

$$\begin{aligned}
&\sup_{0 \neq \mathbf{u} \in \mathbf{H}_0^1} \frac{(p, \mathrm{div}\, \mathbf{u})^2}{\|\mathrm{div}\, \mathbf{u}\|^2 + \|\mathrm{curl}\, \mathbf{u}\|^2 + \alpha \|\mathbf{u}\|^2} \\
&\qquad \leq \sup_{0 \neq \mathbf{u} \in \mathbf{U}} \frac{(p, \mathrm{div}\, \mathbf{u})^2}{\|\mathrm{div}\, \mathbf{u}\|^2 + \|\mathrm{curl}\, \mathbf{u}\|^2 + \alpha \|\mathbf{u}\|^2}
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
&c(\Omega) \sup_{0 \neq \mathbf{u} \in \mathbf{U}} \frac{(p, \mathrm{div}\, \mathbf{u})^2}{\|\mathrm{div}\, \mathbf{u}\|^2 + \|\mathrm{curl}\, \mathbf{u}\|^2 + \alpha \|\mathbf{u}\|^2} \\
&\qquad \leq \sup_{0 \neq \mathbf{u} \in \mathbf{H}_0^1} \frac{(p, \mathrm{div}\, \mathbf{u})^2}{\|\mathrm{div}\, \mathbf{u}\|^2 + \|\mathrm{curl}\, \mathbf{u}\|^2 + \alpha \|\mathbf{u}\|^2}
\end{aligned} \tag{3.4}$$

with some $c(\Omega) > 0$ depending only on Ω.

Equality (3.3) is trivial since $\mathbf{H}_0^1 \subset \mathbf{U}$. The proof of (3.4) is a subject of section 4 (see Lemma 4.3). Checking (3.3) and (3.4), we prove the theorem.

¿From Theorem 3.1 and general theory of iterative methods with linear self-adjoint positive definite operators it immediately follows

Corollary 3.1 *For an appropriate set of τ_i (e.g. $\tau_i = 1, i = 0, 1, \ldots$) method (3.1) has geometric rate of convergence with a factor q such that $0 < q < c < 1$, where c is independent of α.*

Remark 3.1. Let us consider $\hat{\mathbf{u}} = (\Delta - \alpha \mathrm{I})_0^{-1} \nabla p$ for some $p \in L_2^0$. Then from (3.2) it follows

$$(A_0(\alpha)\, p, p) = \|\hat{\mathbf{u}}\|_{1,\alpha}^2 = \|\mathrm{div}\, \hat{\mathbf{u}}\|^2 + \|\mathrm{curl}\, \hat{\mathbf{u}}\|^2 + \alpha \|\hat{\mathbf{u}}\|^2$$

At the same time we have $(A_0(\alpha)\, p, p) = (\mathrm{div}\, \hat{\mathbf{u}}, p)$. Thus, the supremum in the expression

$$\sup_{0 \neq \mathbf{u} \in \mathbf{H}_0^1} \frac{(p, \mathrm{div}\, \mathbf{u})^2}{\|\mathrm{div}\, \mathbf{u}\|^2 + \|\mathrm{curl}\, \mathbf{u}\|^2 + \alpha \|\mathbf{u}\|^2}$$

is reached, in particular, on the function $\hat{\mathbf{u}} \in \mathbf{H}_0^1$. The same arguments are true for $\tilde{\mathbf{u}} = (\Delta - \alpha \mathrm{I})_\nu^{-1} \nabla p$ with respect to the supremum over $\mathbf{U}$. *Remark 3.2.* Since $A_\nu(0) = \mathrm{I}$ and $\|\mathbf{u}\|_1^2 = \|\mathrm{div}\, \mathbf{u}\|^2 + \|\mathrm{curl}\, \mathbf{u}\|^2$ for $\mathbf{u} \in \mathbf{H}_0^1$, inequality (3.4) with $\alpha = 0$ turns to be well known Ladyzhenskaya-Babuška-Brezzi (LBB) inequality (in a discrete case often referred also as the *infsup* condition).

Remark 3.3. Let c_0 be a constant from the LBB inequality ($c_0 \|p\| \leq \|\nabla p\|_{-1}$) and c_1 be a constant from the Friedrichs inequality ($\|\mathbf{u}\| \leq c_1 \|\mathbf{u}\|_1,\ \ \mathbf{u} \in \mathbf{H}_0^1$), then from the above arguments it follows that the equality

$$c_2 \mathrm{I} \leq A_0(\alpha) \leq c_3 \mathrm{I}$$

holds with $c_2 = c_0^2 (1 + c_1^2 \alpha)^{-1},\ c_3 = 1$.

Simple observation shows that asymptotics $c_2 = O(\alpha^{-1})$, $\alpha \to \infty$, and $c_3 = O(1)$ can not be improved. Indeed, consider in a unit square the function $\bar{p} = \cos \pi x$, $\bar{p} \in L_2^0((0,1) \times (0,1))$. Then $A_\nu(\alpha)\bar{p} = (1 + \pi^{-2}\alpha)^{-1}\bar{p}$, but $A_0(\alpha) \leq A_\nu(\alpha)$. Therefore, $c(1 + \pi^{-2}\alpha)^{-1}\|\bar{p}\|^2 = (A_0(\alpha)\bar{p}, \bar{p})$ with some $c, 0 < c \leq 1$. The similar arguments with $\bar{p} = cos\, m\pi x$ and $m \to \infty$ show the optimality of asymptotics $c_3 = O(1)$.

These asymptotics for c_2 and c_3 explain deterioration of the classic Uzawa algorithm for the generalized Stokes problem with $\alpha \gg 1$.

Remark 3.4. Necessity of computation the result of $(\Delta - \alpha \mathrm{I})_0^{-1}$ on each step of (3.1) is sometimes considered as a drawback of the Uzawa type method. To this end an *inexact* version of the Uzawa algorithm is considered in literature (cf. [7], [6]). Convergence estimates for preconditioned inexact algorithms are also heavily depend on $cond(A_\nu^{-1}(\alpha) A_0(\alpha))$.

Remark 3.5. There are variety of other preconditioned iterative methods for solving saddle point problems of (1.1) type (see e.g. [3], [15]), whose application to (1.1) requires an appropriate preconditioner for $A_0(\alpha)$ to insure a good convergence.

4 Proof of the main inequality

In this section proof of inequality (3.3) is given for domains with rather regular boundary (detailed below), more technical proof of (3.3) for Lipshitz domains we put in Appendex.

Lemma 4.1. *Fix arbitrary $\alpha \geq 0$. Let $\mathbf{u}$ be any function from $\mathbf{U}$. Then $\mathbf{u} = \nabla\psi$ for some $\psi \in W_2^1(\Omega) \cap L_2^0$ if and only if*

$$\mathbf{u} = (\Delta - \alpha \mathrm{I})_\nu^{-1} \nabla p, \tag{4.1}$$

for some $p \in L_2^0$.

Proof.

1. Assume $\mathbf{u} \in \mathbf{U}$ and $\mathbf{u} = \nabla\psi$ for some $\psi \in W_2^1(\Omega) \cap L_2^0$. It is easy to see that for such $\mathbf{u}$ the relation

$$\Delta\mathbf{u} - \alpha\mathbf{u} = \nabla p$$

holds (in a weak sense) with $p = \operatorname{div}\mathbf{u} - \alpha\psi$, $p \in L_2^0$, and since $\mathbf{u}\cdot\boldsymbol{\nu} = \mathcal{R}\mathbf{u} = 0$ equality (4.1) is valid.

2. Consider an arbitrary $p \in L_2^0$ and $\mathbf{u} \in \mathbf{U}$ such that relation (4.1) holds. According to Theorem 2.1, for $\mathbf{u}$ and p the following equality holds

$$p = \operatorname{div}\mathbf{u} - \alpha\Delta_N^{-1}\operatorname{div}\mathbf{u}.$$

Set now $\psi = \Delta_N^{-1}\operatorname{div}\mathbf{u}$, $\psi \in W_2^1(\Omega) \cap L_2^0$, and $\mathbf{v} = \nabla\psi$. We readily get $\mathbf{v}\cdot\boldsymbol{\nu} = \frac{\partial\psi}{\partial\boldsymbol{\nu}} = 0$, $\mathcal{R}\mathbf{v} = \mathcal{R}\nabla\psi = 0$ on $\partial\Omega$ and $\operatorname{div}\mathbf{v} = \Delta\psi = \operatorname{div}\mathbf{u}$. Moreover,

$$\Delta\mathbf{v} - \alpha\mathbf{v} = \nabla\operatorname{div}\mathbf{v} - \alpha\nabla\Delta_N^{-1}\operatorname{div}\mathbf{u} = \nabla(\operatorname{div}\mathbf{u} - \alpha\Delta_N^{-1}\operatorname{div}\mathbf{u}) = \nabla p$$

Hence, the function $\mathbf{w} = \mathbf{u} - \mathbf{v}, \mathbf{w} \in \mathbf{U}$ satisfies the equation

$$\begin{aligned} &\Delta\mathbf{w} - \alpha\mathbf{w} = 0 \\ &\mathbf{w}\cdot\boldsymbol{\nu}|_{\partial\Omega} = \mathcal{R}\mathbf{w}|_{\partial\Omega} = 0 \end{aligned}$$

that implies $\mathbf{w} = \mathbf{0}$. Thus, we have $\mathbf{u} = \mathbf{v} = \nabla\psi$ and so $\operatorname{curl}\mathbf{u} = 0$. Lemma is proved.

In the following two lemmas we assume the domain Ω be such that the W_2^2-regularity holds for the solution of the classic Stokes problem: for $\mathbf{f} \in L_2(\Omega)^n$ and $g \in W_2^1(\Omega) \cap L_2^0$ the solution $\{\mathbf{u}, p\}$ of the problem

$$\begin{aligned} &-\Delta\mathbf{u} + \nabla p = \mathbf{f}, \\ &\operatorname{div}\mathbf{u} = g, \\ &\mathbf{u}|_{\partial\Omega} = \mathbf{0} \end{aligned}$$

belongs to $W_2^2(\Omega)^n \times W_2^1(\Omega) \cap L_2^0$. The examples of such domains are the domains with $\partial\Omega \in C^2$ or convex (curved) polygons (polyhedrons) (cf. [17],[5]).

Lemma 4.2. *Let $\mathbf{v}$ be any function from $\mathbf{U}$ and $\mathbf{u}, p \in \mathbf{U} \times L_2^0$ be a solution of the problem*

$$\begin{aligned} &-\Delta\mathbf{u} + \nabla p = \mathbf{0}, \\ &\operatorname{div}\mathbf{u} = 0, \\ &\mathbf{u}|_{\partial\Omega} = \mathbf{v}|_{\partial\Omega} \end{aligned} \tag{4.2}$$

Then the following estimates hold

$$\begin{aligned} \|\operatorname{div}\mathbf{u}\| + \|\operatorname{curl}\mathbf{u}\| &\leq c(\Omega)(\|\operatorname{div}\mathbf{v}\| + \|\operatorname{curl}\mathbf{v}\|), \\ \|\mathbf{u}\|_{0,\Omega} &\leq c(\Omega)\|\gamma_\tau\mathbf{u}\|_{-\frac{1}{2},\partial\Omega}. \end{aligned} \tag{4.3}$$

Proof. The solution $\mathbf{u}$ of (4.2) is defined from relation $\mathbf{u} = \mathbf{v} - \mathbf{w}$, where $\mathbf{w} \in \mathbf{H}_0^1$ is a solution (together with p) of the problem

$$\begin{aligned} &(\nabla \mathbf{w}, \nabla \boldsymbol{\xi}) - (p, \operatorname{div} \boldsymbol{\xi}) = (\operatorname{div} \mathbf{v}, \operatorname{div} \boldsymbol{\xi}) + (\operatorname{curl} \mathbf{v}, \operatorname{curl} \boldsymbol{\xi}) \\ &(\operatorname{div} \mathbf{w}, \eta) = (\operatorname{div} \mathbf{v}, \eta) \qquad\qquad \forall \boldsymbol{\xi} \in \mathbf{H}_0^1,\ \eta \in L_2^0 \end{aligned} \tag{4.4}$$

¿From (4.4) and ordinary arguments (see e.g. [8]) we obtain for $\mathbf{w}$

$$\|\mathbf{w}\|_1 \le c(\|\operatorname{div} \mathbf{v}\| + \|\operatorname{curl} \mathbf{v}\|)$$

Now from the last inequality and relations $\|\operatorname{div} \mathbf{w}\|^2 + \|\operatorname{curl} \mathbf{w}\|^2 = \|\mathbf{w}\|_1^2$, $\mathbf{u} = \mathbf{v} - \mathbf{w}$ we get the first inequality from (4.3).

To prove the second estimate from (4.3), assume $\mathbf{v}$ to be an arbitrary function from $W_2^2(\Omega)^n$. The above regularity assumptions ensure that such smoothness of $\mathbf{v}$ implies $\mathbf{w}$ from (4.4) belongs to $W_2^2(\Omega)^n$. Thus, $\mathbf{u} \in W_2^2(\Omega)^n$ and $p \in W_2^1(\Omega) \cap L_2^0$. Therefore, relations (4.2) hold in $L_2(\Omega)^n$. Let us consider a scalar product of the first equality from (4.2) with an arbitrary function $\boldsymbol{\psi}$ from $W_2^2(\Omega)^n \cap \mathbf{H}_0^1$ such that $\operatorname{div} \boldsymbol{\psi} = 0$. After integration by parts we get

$$-(\mathbf{u}, \Delta \boldsymbol{\psi}) = < \gamma_\tau \mathbf{v}, \operatorname{curl} \boldsymbol{\psi} >_{\partial\Omega}$$

Assuming $\boldsymbol{\psi} \neq \mathbf{0}$, we get from the last equality

$$\frac{-(\mathbf{u}, \Delta \boldsymbol{\psi})}{\|\boldsymbol{\psi}\|_2} = \frac{< \gamma_\tau \mathbf{v}, \operatorname{curl} \boldsymbol{\psi} >_{\partial\Omega}}{\|\boldsymbol{\psi}\|_2} \tag{4.5}$$

To obtain the estimate for the right-hand side of (4.5), note that

$$\|\operatorname{curl} \boldsymbol{\psi}\|_{\frac{1}{2},\partial\Omega} \le \|\operatorname{curl} \boldsymbol{\psi}\|_{W_2^1(\Omega)} \le c_1 \|\boldsymbol{\psi}\|_2$$

and, thus,

$$\begin{aligned} &\frac{| < \gamma_\tau \mathbf{v}, \operatorname{curl} \boldsymbol{\psi} >_{\partial\Omega} |}{\|\boldsymbol{\psi}\|_2} \le c \frac{| < \gamma_\tau \mathbf{v}, \operatorname{curl} \boldsymbol{\psi} >_{\partial\Omega} |}{\|\operatorname{curl} \boldsymbol{\psi}\|_{\frac{1}{2},\partial\Omega}} \le \\ &c \sup_{\boldsymbol{\xi} \in H^{\frac{1}{2}}(\partial\Omega)^{2n-3}} \frac{< \gamma_\tau \mathbf{v}, \boldsymbol{\xi} >_{\partial\Omega}}{\|\boldsymbol{\xi}\|_{\frac{1}{2},\partial\Omega}} \equiv c\|\gamma_\tau \mathbf{v}\|_{-\frac{1}{2}} \end{aligned}$$

Now from (4.5) we get

$$\frac{|(\mathbf{u}, \Delta \boldsymbol{\psi})|}{\|\boldsymbol{\psi}\|_2} \le c\|\gamma_\tau \mathbf{v}\|_{-\frac{1}{2}} \tag{4.6}$$

As far as (4.6) holds for any $\mathbf{0} \neq \boldsymbol{\psi} \in W_2^2(\Omega)^n \cap \mathbf{H}_0^1$ with $\operatorname{div} \boldsymbol{\psi} = 0$, let us choose $\boldsymbol{\psi}$ as a solution (together with q) of the problem

$$\begin{aligned} &-\Delta \boldsymbol{\psi} + \nabla q = \mathbf{u}, \\ &\operatorname{div} \boldsymbol{\psi} = 0, \\ &\boldsymbol{\psi}|_{\partial\Omega} = \mathbf{0} \end{aligned} \tag{4.7}$$

Since $\mathbf{u} \in L_2(\Omega)^n$ and Ω assumed to be rather regular, it follows from standard regularity results for the Stokes problem that $\{\boldsymbol{\psi}, q\} \in W_2^2(\Omega)^n \times W_2^1(\Omega) \cap L_2^0$ and

$$\|q\|_{W_2^1} + \|\boldsymbol{\psi}\|_2 \leq c\|\mathbf{u}\| \tag{4.8}$$

To obtain an estimate on $\|\mathbf{u}\|$, let us consider a scalar product of the first equality from (4.7) with $\mathbf{u}$. We get

$$\|\mathbf{u}\|^2 = (\nabla q, \mathbf{u}) - (\Delta\boldsymbol{\psi}, \mathbf{u})$$

¿From this relation, using estimates (4.6), (4.8) and equality $(\nabla q, \mathbf{u}) =< q, \mathbf{u}\cdot\boldsymbol{\nu} >_{\partial\Omega}$, we deduce

$$\begin{aligned}
\|\mathbf{u}\|^2 &\leq | < q, \mathbf{u}\cdot\boldsymbol{\nu} >_{\partial\Omega} | + c\|\boldsymbol{\psi}\|_2 \, \|\gamma_\tau \mathbf{v}\|_{-\frac{1}{2},\partial\Omega} \\
&\leq \|q\|_{\frac{1}{2},\partial\Omega}\|\mathbf{u}\cdot\boldsymbol{\nu}\|_{-\frac{1}{2},\partial\Omega} + c\|\mathbf{u}\| \, \|\gamma_\tau \mathbf{v}\|_{-\frac{1}{2},\partial\Omega} \\
&\leq \|q\|_{W_2^1}\|\mathbf{u}\cdot\boldsymbol{\nu}\|_{-\frac{1}{2},\partial\Omega} + c\|\mathbf{u}\| \, \|\gamma_\tau \mathbf{v}\|_{-\frac{1}{2},\partial\Omega} \\
&\leq c_1\|\mathbf{u}\| \left(\|\mathbf{u}\cdot\boldsymbol{\nu}\|_{-\frac{1}{2},\partial\Omega} + \|\gamma_\tau \mathbf{v}\|_{-\frac{1}{2},\partial\Omega}\right)
\end{aligned}$$

¿From the last estimate it directly follows

$$\|\mathbf{u}\|_{0,\Omega} \leq c(\|\mathbf{v}\cdot\boldsymbol{\nu}\|_{-\frac{1}{2},\partial\Omega} + \|\gamma_\tau \mathbf{v}\|_{-\frac{1}{2},\partial\Omega}). \tag{4.9}$$

Assume now $\mathbf{v}$ to be an arbitrary function from $\mathbf{U}$. It can be approximate in the $\mathbf{U}$-norm by functions from W_2^2. Hence, we deduce the second estimate in (4.3) from (4.9) by passing to the limit in $\mathbf{U}$, using inequalities (2.1) and the first inequality in (4.3). The Lemma is proved.

Lemma 4.3 *For arbitrary $\alpha \in [0, \infty)$ and $p \in L_2^0$ the inequality*

$$\begin{aligned}
c(\Omega) \sup_{0\neq\mathbf{u}\in\mathbf{U}} &\frac{(p, \operatorname{div}\mathbf{u})^2}{\|\operatorname{div}\mathbf{u}\|^2 + \|\operatorname{curl}\mathbf{u}\|^2 + \alpha\|\mathbf{u}\|^2} \\
&\leq \sup_{0\neq\mathbf{u}\in\mathbf{H}_0^1} \frac{(p, \operatorname{div}\mathbf{u})^2}{\|\operatorname{div}\mathbf{u}\|^2 + \|\operatorname{curl}\mathbf{u}\|^2 + \alpha\|\mathbf{u}\|^2}
\end{aligned} \tag{4.10}$$

holds with $c(\Omega)$ independent of α and p.

Proof. Let p be an arbitrary function from L_2^0. Then we know from Remark 3.1 that the supremum in the left-hand side of (4.10) is reached on the function

$$\mathbf{u} = (\Delta - \alpha\mathrm{I})_\nu^{-1}\nabla p, \quad \mathbf{u} \in \mathbf{U} \tag{4.11}$$

¿From Lemma 4.1 and (4.11) it follows that $\operatorname{curl}\mathbf{u} = 0$, and so from (2.1) we get

$$\|\gamma_\tau \mathbf{u}\|_{-\frac{1}{2},\partial\Omega} \leq \|\mathbf{u}\| \tag{4.12}$$

To prove the Lemma it is sufficient to find for the given $\mathbf{u}$ from (4.11) such $\tilde{\mathbf{u}} \in \mathbf{H}_0^1$ that

$$\begin{aligned} &(p, \operatorname{div} \tilde{\mathbf{u}}) = (p, \operatorname{div} \mathbf{u}), \\ &\|\operatorname{div} \tilde{\mathbf{u}}\|^2 + \|\operatorname{curl} \tilde{\mathbf{u}}\|^2 + \alpha\|\tilde{\mathbf{u}}\|^2 \le c\,(\|\operatorname{div} \mathbf{u}\|^2 + \|\operatorname{curl} \mathbf{u}\|^2 + \alpha\|\mathbf{u}\|^2) \end{aligned} \tag{4.13}$$

with c independent of $\mathbf{u}$, $\tilde{\mathbf{u}}$, and α.

Let us consider the function $\tilde{\mathbf{u}} = \mathbf{u} - \mathbf{w}$, where $\mathbf{w} \in \mathbf{U}$ is a solution of the problem

$$\begin{aligned} &-\Delta \mathbf{w} + \nabla q = \mathbf{0} \\ &\operatorname{div} \mathbf{w} = 0 \\ &\mathbf{w}|_{\partial\Omega} = \mathbf{u}|_{\partial\Omega} \end{aligned} \tag{4.14}$$

It is easy to see that $(p, \operatorname{div} \tilde{\mathbf{u}}) = (p, \operatorname{div} \mathbf{u})$ and, in virtue of Remark 2.1, $\tilde{\mathbf{u}} \in \mathbf{H}_0^1$. Now we are turning to check the second condition in (4.13).

Using a priori estimates from Lemma 4.2 and (4.12), we obtain for the solution of problem (4.14)

$$\begin{aligned} &\|\operatorname{div} \mathbf{w}\|^2 + \|\operatorname{curl} \mathbf{w}\|^2 \le c(\|\operatorname{div} \mathbf{u}\|^2 + \|\operatorname{curl} \mathbf{u}\|^2) \\ &\|\mathbf{w}\|^2 \le c\|\gamma_\tau \mathbf{u}\|_{-\frac{1}{2},\partial\Omega} \le c\|\mathbf{u}\|^2 \end{aligned} \tag{4.15}$$

Inequalities (4.15) immediately give

$$\|\operatorname{div} \mathbf{w}\|^2 + \|\operatorname{curl} \mathbf{w}\|^2 + \alpha\|\mathbf{w}\|^2 \le c\left(\|\operatorname{div} \mathbf{u}\|^2 + \|\operatorname{curl} \mathbf{u}\|^2 + \alpha\|\mathbf{u}\|^2\right) \tag{4.16}$$

with c independent of α and $\mathbf{u}$.

In virtue of $\tilde{\mathbf{u}} = \mathbf{u} - \mathbf{w}$ and (4.16), the second condition in (4.14) is also valid. So the Lemma is proved.

5 Some remarks on finite differences and finite elements

The above considerations made for differential problems encourage us to expect a success of the method for discrete problems as well. Indeed, transfer of the 'main' inequality (4.6) to a discrete case is similar to checking LBB ($inf\,sup$) condition that is satisfied, as a rule, for pressure-velocity finite difference (FD) and finite element (FE) schemes. Moreover, most of the schemes admit well-posed pressure Poisson problem with Neumann boundary conditions. However, while FD or FE formulations of the model problem are rather obvious, validity of Theorem 2.1 in a discrete case is vague in general. To clarify the situation we prove below FD analogue of Theorem 2.1 in a special case of some compatibility conditions.

Let us consider MAC scheme with a staggered grids and central FD approximation of Δ, div , and grad operators (see e.g. [9], [11] for details). Further we shall use the following notations: $\mathbf{U}^h$ and L^h for FD velocity and pressure spaces defined in interior nodes of a grid domain. Let $\mathbf{U}_0^h$, $\mathbf{U}_\nu^h$, and L_0^h be their extensions such that discrete boundary conditions $\mathbf{u} =$

$\mathbf{0}$, $\mathbf{u}\cdot\boldsymbol{\nu} = \mathcal{R}\mathbf{u} = 0$, and $\dfrac{\partial p}{\partial \boldsymbol{\nu}} = 0$ are satisfied for functions from $\mathbf{U}_0^h$, $\mathbf{U}_\nu^h$, and L_0^h, respectively. By Δ_0^h, Δ_ν^h, ∇^h, div^h, and Δ_N^h we denote grid approximations of the corresponding differential operators. They acts as follows:

$\Delta_0^h : \mathbf{U}_0^h \to \mathbf{U}^h$,

$\Delta_\nu^h : \mathbf{U}_\nu^h \to \mathbf{U}^h$,

$\nabla^h : L^h \to \mathbf{U}^h$ (extended as $\nabla^h : L_0^h \to \mathbf{U}_\nu^h$),

$\operatorname{div}^h : \mathbf{U}_0^h(\mathbf{U}_\nu^h) \to L^h$,

$\Delta_N^h : L_0^h \to L^h$

FD formulations of the main problems are given below.

The generalized Stokes problem: find $\{\mathbf{u}, p\} \in \mathbf{U}_0^h \times L^h$ such that

$$\begin{aligned} -\Delta_0^h \mathbf{u} + \alpha \mathbf{u} + \nabla^h p &= \mathbf{f} \quad \text{in } \mathbf{U}^h \\ \operatorname{div}^h \mathbf{u} &= g \quad \text{in } L^h \end{aligned}$$

The vector Helmholtz problem: for $\alpha \geq 0$ find $\mathbf{u} \in \mathbf{U}_0^h$ such that

$$-\Delta_0^h \mathbf{u} + \alpha \mathbf{u} = \mathbf{f} \quad \text{in } \mathbf{U}^h$$

The pressure Poisson problem: find $p \in L_0^h$ such that

$$-\Delta_N^h p = g \quad \text{in } L^h$$

The above formulations are also considered with $\mathbf{U}_\nu^h$ instead of $\mathbf{U}_0^h$. These problems are well posed.

It is worth to note that the following compatibility conditions are satisfied:

a) $\operatorname{div}^h \nabla^h = \Delta_N^h$ on L_0^h

b) $\Delta_\nu^h \nabla^h = (\nabla^h \operatorname{div}^h) \nabla^h$ on L_0^h

For this FD scheme the following theorem holds.

Theorem 5.1. *For any $\alpha \in [0, \infty)$ and $p \in L^h$ the equality $p = q - \alpha\, (\Delta_N^h)^{-1} q$ holds with $q \in L^h$ such that*

$$q = \operatorname{div}^h (\Delta_\nu^h - \alpha \mathrm{I})^{-1} \nabla^h p \tag{5.1}$$

Proof. Fix some $p \in L^h$ and $\alpha \in [0, \infty)$. We can rewrite relation (5.1) as follows

$$\begin{aligned} -\Delta_\nu^h \mathbf{u} + \alpha \mathbf{u} + \nabla^h p &= \mathbf{0} \text{ in } \mathbf{U}^h \\ \operatorname{div}^h \mathbf{u} &= q \text{ in } L^h \end{aligned} \tag{5.2}$$

with $\mathbf{u} \in \mathbf{U}_\nu^h$.

Assume $\alpha > 0$ and consider functions $p_1 = -\alpha(\Delta_N^h)^{-1} q$ and $\tilde{\mathbf{u}} = \alpha^{-1}\nabla^h p_1$, $\tilde{\mathbf{u}} \in \mathbf{U}_\nu^h$. ¿From our definitions and compatibility conditions we have $\mathrm{div}\,^h \tilde{\mathbf{u}} = q$ and $-\Delta_\nu^h \tilde{\mathbf{u}} = -\nabla^h \mathrm{div}\,^h \tilde{\mathbf{u}} = \nabla^h q$.

Set $\tilde{p} = q + p_1$. Then for $\tilde{\mathbf{u}}$ from $\mathbf{U}_\nu^h$ and for $\tilde{p}$ from L^h the following equalities hold

$$\begin{aligned} &-\Delta_\nu^h \tilde{\mathbf{u}} + \alpha\tilde{\mathbf{u}} + \nabla\tilde{p} = \mathbf{0} \\ &\mathrm{div}\,^h \tilde{\mathbf{u}} = q \end{aligned} \tag{5.3}$$

¿From the well-posedness of the Stokes problem and relations (5.2), (5.3) we get $p = \tilde{p} = q - \alpha(\Delta_N^h)^{-1} q$. The case $\alpha = 0$ is treated in a similar manner. Theorem is proved.

There is one more reason for considering the model Schur complement as a preconditioner in (1.5). The theorem proved below shows a specific discrete reflection of the fact that $A_0(\alpha)$ and $A_\nu(\alpha)$ in (4.1) differ only up to the boundary conditions involved implicitly.

Theorem 5.2. *Under assumptions of Theorem 5.1 the eigenvalue problem*

$$A_0^h(\alpha)p = \lambda A_\nu^h(\alpha)p$$

with $A_0^h(\alpha) = \mathrm{div}\,^h(\Delta_0^h - \alpha\mathrm{I})^{-1}\nabla^h$, $A_\nu^h(\alpha) = \mathrm{div}\,^h(\Delta_\nu^h - \alpha\mathrm{I})^{-1}\nabla^h$ *has* $\lambda = 1$ *as an eigenvalue of* $O(h^{-n})$ *multiplicity, and a number of other eigenvalues is* $O(h^{-(n-1)})$, *where* h *is a typical mesh size.*

Proof. Consider a subspace $\mathbf{V}_c \subset \mathbf{U}_0^h$ of functions such that $\mathbf{v} = \nabla\psi$ for some $\psi \in L_0^h$. It is obvious that $\mathbf{V}_c \subset \mathbf{U}_\nu^h$. Fix an arbitrary $\alpha \geq 0$. Then any $\mathbf{v} \in \mathbf{V}_c$ is represented as

$$\mathbf{v} = (-\Delta_0^h + \alpha\mathrm{I})^{-1}\nabla p = (-\Delta_\nu^h + \alpha\mathrm{I})^{-1}\nabla p$$

with $p = -\mathrm{div}\,^h\mathbf{v} + \alpha\psi$. Since $\dim(\mathbf{V}_c) = O(h^{-n})$, then $\lambda = 1$ is the eigenvalue of $O(h^{-n})$ multiplicity.

Let $\mathbf{V}_0$ be a subspace of such functions $\mathbf{u}$ from $\mathbf{U}_0^h$ that $\mathbf{u} = (-\Delta_0^h + \alpha\mathrm{I})^{-1}\nabla^h q$ for some $q \in L^h$. It is easy to check that $dim(\mathbf{V}_0 \perp \mathbf{V}_c) = O(h^{-(n-1)})$. Thus, a number of eigenvalues not equal to unit is $O(h^{-(n-1)})$. The theorem is proved.

Theorem 5.2 provides us with a choice of $\tau_0 = 1$ in method (3.1). With this choice a difference between an exact solution p of (1.4) and its approximation p^i, $i = 1, 2, \ldots$ from (3.1) belongs to a subspace of a lower by order dimension. This property ensures an extra convergence of method (1.5) for discrete schemes, where the discrete preconditioner is equal to the Schur complement of the discrete model problem.

The lack of an appropriate compatibility conditions for FE schemes, as well as troubles with concerning smoothness requirements on trial functions in the proof of Theorem 5.1 make the above analysis more complicated for finite elements. However, some remarkes should be done.

Consider the matrix form of FE discretization of model problem (2.2):

$$\begin{bmatrix} -\mathrm{L}_\nu(\alpha) & \mathrm{D}^* \\ \mathrm{D} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ p \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ 0 \end{bmatrix}$$

Let I_ν^h and I_p^h be the mass matrixes of FE velocity and pressure spaces. Then the above analyses suggests a FE choice of $B^{-1} = (\mathrm{I}_p^h)^{-1} - \alpha(\mathrm{D}(\mathrm{I}_\nu^h)^{-1}\mathrm{D}^*)^{-1}$ in (1.4), which, we expect, will be better than $B^{-1} = \mathrm{I} - \alpha(\Delta_N^h)^{-1}$ (numerical results from [4] confirm this supposition). However, more theoretical investigations for finite element problems are still required, e.g., pseudodifferential 'easy invertible' forms for the Schur operator $\mathrm{D}(\mathrm{L}_\nu(\alpha))^{-1}\mathrm{D}^*$ should be found and justified.

6 Numerical experiments.

We present in this section results of numerical experiments for the equation

$$A_0^h(\alpha)p = F$$

where $A_0^h(\alpha)$ is a Shur complement for FD approximation of generalized Stokes problem (1.1). We use MAC scheme defined in section 5 and consider $\Omega = (0,1)\times(0,1)$.

Preconditioned and nonpreconditioned versions of conjugate gradients (CG) and minimal residual (MINRES) methods were tested.

Let B denote a preconditioner, p_0 an initial guess ($p_0 \equiv 0$ in all experiments), $s_i = A_0^h(\alpha)p_i - F$ denote a residual for p_i defined via iterations for $i = 1, 2, \dots$, and $r_i = B^{-1}s_i$. Then the preconditioned version of CG, which we used, is as follows.

Compute

$$a_0 = (r_0, r_0)/(B^{-1}A_0^h(\alpha)r_0, r_0),$$

$$p_1 = p_0 - a_0 r_0$$

for $i = 1, 2, \dots$ **do until convergence**

$$\sigma_{i-1} = (B^{-1}A_0^h(\alpha)r_{i-1}, r_{i-1})/(r_{i-1}, r_{i-1}),$$

$$\sigma_i = (B^{-1}A_0^h(\alpha)r_i, r_i)/(r_i, r_i),$$

$$b_i = 1./(\sigma_{i-1} + \sigma_i),$$

$$a_i = b_i\sigma_{i-1},$$

$$p_{i+1} = p_i - a_i(p_i - p_{i-1}) - b_i r_i$$

end do.

The preconditioned MINRES method, which we used, coincides with process (1.4) for $\tau_i = (B^{-1}A_0^h(\alpha)r_i, r_i)/(B^{-1}A_0^h(\alpha)r_i, B^{-1}A_0^h(\alpha)r_i)$, $i = 0, 1, \dots$.

We refer to these algorithms as preconditioned in the case $B = A_\nu^h(\alpha)$ and as nonpreconditioned in the case $B = \mathrm{I}^h$. The stopping criterion was

$$\|r_i\|_2/\|r_0\|_2 < 10^{-9}$$

We refer to the value of $(\|r_i\|_2/\|r_0\|_2)^{1/i}$ as to *the average convergence factor.*

Remark 6.1 If $\mathbf{u}_i$ is a velocity vector field corresponding to the pressure p_i via the Helmholtz equation $-\Delta^h\mathbf{u}_i + \alpha\mathbf{u}_i = \mathbf{f} - \nabla p_i$, then $s_i = \operatorname{div}\mathbf{u}_i$.

Remark 6.2 Along with the above variants of CG and MINRES methods, we will consider these methods on a subspace (see Theorem 5.2), i.e., we make one step of (3.1) with $\tau_0 = 1$ and then continue calculations according to the above algorithms.

1. *Smooth test.*

For the first test we choose smooth pressure function $p^s = x - y$ as an 'exact' solution of (6.1). Function $F = A_0^h(\alpha)p^s$ was calculated and considered as the right-hand side of (6.1). Setting $p_0 = 0$, we examine convergence of conjugate gradient method to this smooth solution. Results are presented in Tables Ia-Ic. The value 0.222 for convergence factor with $h = (512)^{-1}$ and $\alpha = 512$ means, for example, that the residual is divided approximately by 100 every 3 steps.

Table.Ia. Smooth test.
Average convergence factor for conjugate gradients.

$\alpha \backslash h$	1/16	1/32	1/64	1/128	1/256	1/512
0	0.099	0.139	0.169	0.189	0.204	0.223
16	0.134	0.175	0.207	0.225	0.245	0.244
32	0.098	0.151	0.190	0.208	0.224	0.238
64	0.083	0.121	0.166	0.188	0.200	0.217
128	0.066	0.109	0.143	0.185	0.212	0.234
256	0.049	0.096	0.131	0.171	0.215	0.244
512	0.033	0.072	0.111	0.146	0.190	0.222
1024	0.023	0.048	0.083	0.119	0.148	0.176
2048	0.012	0.035	0.063	0.095	0.133	0.163

Table.Ib. Smooth test.
Average convergence factor for conjugate gradients on subspace.

$\alpha \backslash h$	1/16	1/32	1/64	1/128	1/256	1/512
0	0.106	0.146	0.176	0.196	0.217	0.229
16	0.114	0.166	0.197	0.225	0.244	0.257
32	0.107	0.149	0.176	0.204	0.226	0.245
64	0.097	0.125	0.154	0.191	0.222	0.239
128	0.084	0.112	0.159	0.198	0.235	0.262
256	0.050	0.103	0.143	0.189	0.222	0.260
512	0.039	0.080	0.121	0.160	0.194	0.234
1024	0.026	0.058	0.092	0.130	0.162	0.192
2048	0.015	0.041	0.073	0.104	0.134	0.161

Table.Ic. Smooth test.
Average convergence factor for conjugate gradients without preconditioning.

$\alpha \backslash h$	1/16	1/32	1/64	1/128	1/256	1/512
0	0.099	0.139	0.169	0.189	0.204	0.223
16	0.150	0.201	0.226	0.243	0.263	0.271
32	0.176	0.220	0.244	0.269	0.280	0.289
64	0.199	0.250	0.271	0.290	0.309	0.318
128	0.250	0.290	0.319	0.329	0.347	0.354
256	0.313	0.353	0.380	0.397	0.406	0.417
512	0.389	0.429	0.462	0.475	0.473	0.481
1024	0.457	0.524	0.552	0.551	0.569	0.564
2048	0.512	0.605	0.631	0.649	0.651	0.649
4096	0.545	0.682	0.755	0.776	0.783	0.780

2. *Random test.*

Exact solution p^r was obtained by random numbers generator with the uniform distribution on [-1,1] and then normalized to ensure $\int_\Omega p^r dx = 0$. The values of convergence factors in Tables II, III are averaged over three random runs of program with different initializations of the random generator. There are no pronounced differences in convergence rates observed for these substantially nonsmooth solutions in compare with results for the smooth test. While averaged convergence factors for CG method on the subspace were very close to those without $\tau_0 = 1$, the MINRES method on the subspace was evidently superior to the usual one, so in the case of severe memory limitations the MINRES method with $\tau_0 = 1$ can be considered as a method alternative to CG. At any time automatic setting of $\tau_0 = 1$ saves some computations; this can be especially appreciable in unsteady simulations when only few iterations on each time step are needed to achieve a good approximation.

In all tests convergence of the preconditioned methods improved when the parameter α increased and the mesh step h was fixed. The case $\alpha = 0$ corresponds to the nonpreconditioned version of Uzawa algorithm for the classic Stokes problem ($A^h_\nu(0) = \mathrm{I}^h$), whose convergence is well known to be "independent" (estimated with some constant independent) of mesh size. The preconditioned algorithm for the generalized Stokes problem demonstrated convergence at least not worse than Uzawa algorithm for the classic Stokes problem.

If we consider a typical situation in unsteady high Reynolds simulations when $(h\, Re)(\frac{h}{\delta t}) \leq c < \infty$ with some absolute constant $c > 0$, then we have the following relation for α : $\alpha = O(h^{-2})$. In this particular case the preconditioned method demonstrates significant improving of the convergence rate with $h \to 0$.

Tables Ic and IIc show the growth of convergence factor for nonpreconditioned CG method for $\alpha \to \infty$, confirming the growth of the condition number of $A^h_0(\alpha)$.

Table.IIa. Random test.
Average convergence factor for conjugate gradients.

$\alpha \backslash h$	1/16	1/32	1/64	1/128	1/256	1/512
0	0.165	0.201	0.225	0.242	0.240	0.242
16	0.150	0.188	0.202	0.234	0.221	0.222
32	0.125	0.177	0.197	0.220	0.204	0.212
64	0.106	0.153	0.176	0.205	0.197	0.204
128	0.086	0.119	0.173	0.183	0.188	0.203
256	0.065	0.100	0.147	0.164	0.186	0.189
512	0.053	0.084	0.122	0.155	0.178	0.180
1024	0.034	0.062	0.096	0.136	0.161	0.187
2048	0.019	0.043	0.083	0.115	0.145	0.178

Table.IIb. Random test.
Average convergence factor for conjugate gradients on subspace.

$\alpha \backslash h$	1/16	1/32	1/64	1/128	1/256	1/512
0	0.166	0.200	0.225	0.236	0.237	0.242
16	0.140	0.170	0.191	0.223	0.213	0.215
32	0.114	0.155	0.177	0.209	0.203	0.204
64	0.096	0.136	0.160	0.188	0.187	0.196
128	0.079	0.114	0.157	0.170	0.182	0.194
256	0.063	0.095	0.139	0.159	0.180	0.190
512	0.049	0.083	0.119	0.149	0.174	0.184
1024	0.032	0.063	0.099	0.137	0.159	0.189
2048	0.019	0.044	0.084	0.117	0.146	0.179

Table.IIc. Random test.
Average convergence factor for conjugate gradients without preconditioning.

$\alpha \backslash h$	1/16	1/32	1/64	1/128	1/256	1/512
0	0.168	0.201	0.225	0.241	0.245	0.242
16	0.250	0.274	0.298	0.309	0.310	0.308
32	0.288	0.308	0.320	0.335	0.333	0.334
64	0.330	0.356	0.362	0.364	0.364	0.371
128	0.398	0.415	0.425	0.418	0.418	0.413
256	0.474	0.491	0.500	0.488	0.497	0.479
512	0.551	0.584	0.581	0.570	0.587	0.562
1024	0.613	0.663	0.678	0.649	0.665	0.648
2048	0.667	0.731	0.744	0.733	0.739	0.741
4096	0.709	0.795	0.790	0.763	0.787	0.769

Table.IIIa. Random test.
Average convergence factor for minimal residuales.

$\alpha \backslash h$	1/16	1/32	1/64	1/128	1/256	1/512
0	0.426	0.455	0.483	0.504	0.521	0.522
16	0.312	0.368	0.404	0.449	0.470	0.487
32	0.271	0.339	0.386	0.431	0.454	0.474
64	0.227	0.301	0.362	0.410	0.433	0.459
128	0.182	0.259	0.329	0.383	0.413	0.442
256	0.136	0.214	0.292	0.351	0.388	0.420
512	0.095	0.168	0.250	0.320	0.365	0.400
1024	0.061	0.122	0.208	0.281	0.334	0.378
2048	0.037	0.085	0.162	0.239	0.303	0.348

Table.IIIb. Random test.
Average convergence factor for minimal residuales on subspace.

$\alpha \backslash h$	1/16	1/32	1/64	1/128	1/256	1/512
0	0.363	0.396	0.428	0.448	0.471	0.474
16	0.245	0.307	0.347	0.397	0.416	0.435
32	0.200	0.273	0.328	0.377	0.399	0.420
64	0.173	0.244	0.306	0.348	0.381	0.399
128	0.143	0.215	0.284	0.324	0.364	0.372
256	0.106	0.183	0.258	0.303	0.348	0.362
512	0.074	0.143	0.222	0.283	0.331	0.358
1024	0.048	0.084	0.184	0.255	0.311	0.347
2048	0.027	0.070	0.140	0.217	0.282	0.326

Appendix

In the paper (section 4) inequality (3.4) essencial for convergence of the method was proved only for domains with sufficiently smooth boundaries or convex polygons (polyhedrons). Here we shall demonstrate that this inequality with constant $c(\Omega)$ independent of parameter α is valid for wider class of domains. Namely, we prove it's validity of for curvilinear trapezoid with Lipschitz boundary $y = g(x)$, when $|g'|$ is not too large. Unfortunatly, we have not succeeded yet in prove of inequality (3.4) for all domains with Lipschitz boundaries, but we believe this hypothesis to be true. This will be a subject of our future interest.

Further we use the notations from Section 2. We shall also use notations

$$(\mathbf{v}, \mathbf{w})_\alpha \equiv (\operatorname{div} \mathbf{v}, \operatorname{div} \mathbf{w}) + (\operatorname{curl} \mathbf{v}, \operatorname{curl} \mathbf{w}) + \alpha(\mathbf{v}, \mathbf{w}), \quad \|\mathbf{v}\|_\alpha^2 \equiv (\mathbf{v}, \mathbf{v})_\alpha$$

$$\|w_i\|_\alpha^2 \equiv \|\nabla w_i\|^2 + \alpha \|w_i\|^2, \quad \|w_i\|_{\alpha, x_j}^2 \equiv \left\| \frac{\partial w_i}{\partial x_j} \right\|^2 + \alpha \|w_i\|^2,$$

Later on, we shall denote independent variables either (x_1, x_2) or (x, y).

Let

$$\Phi(p, \mathbf{v}) \equiv \frac{(p, \operatorname{div} \mathbf{v})^2}{\|\mathbf{v}\|_\alpha^2} \tag{1}$$

The aim of this Appendix is to prove for any function $p \in L_2^0$ validity of the following inequality

$$\sup_{\mathbf{v} \in \mathbf{U}} \Phi(p, \mathbf{v}) \leq c_0 \sup_{\mathbf{w} \in \mathbf{H}_0^1} \Phi(p, \mathbf{w}) \tag{2}$$

with some constant c_0 that does not depend on $\alpha \geq 0$.

Further, we assume $\alpha \geq 1$.

At first, let $\bar{\Omega} = [0, \pi] \times [0, \pi]$. We recall that in this case $\mathbf{U}$ defined in section 2 can be represented as

$$\mathbf{U} = \left\{ \mathbf{u} = (u^1, u^2) : \quad u^i \in H^1(\Omega), \ \mathbf{u} \cdot \mathbf{n} = 0 \right\}$$

and

$$(\mathbf{v}, \mathbf{w})_\alpha = (\nabla \mathbf{v}, \nabla \mathbf{w}) + \alpha(\mathbf{v}, \mathbf{w}).$$

As for the scalar product, $(v, w) = \sum\limits_{|k|=0}^{\infty} v_k w_k$, where v_k, w_k are the Fourie series coefficients of v and w, respectively.

Lemma 1. *Let*

$$p(x, y) = \sum_{i,j=0}^{n} p_{ij} \cos ix \cos jy, \quad p_{00} = 0 \tag{3}$$

Then

$$\arg \sup_{\mathbf{v} \in \mathbf{U}} \Phi(p, \mathbf{v}) = \hat{\mathbf{u}} \tag{4}$$

where $\hat{\mathbf{u}} = (\Delta - \alpha I)_\nu^{-1} \nabla p$.

Proof directly follows from Remark 3.1.

Lemma 2. *For every vector function* $\hat{\mathbf{u}}$ *that provides maximum of* $\Phi(p, \mathbf{v})$ *over* $\mathbf{U}$*, where* p *is a trigonometric polinomial of form (3), there exists a vector function* $\mathbf{u} \in \mathbf{H}_0^1$ *satisfying the following inequalities*

$$\Phi(p, \hat{\mathbf{u}}) \leq c_0 \Phi(p, \mathbf{u})$$

where c_0 *does not depend on* n*,* p*, and* $\alpha \geq 0$.

Proof. In spite of this result was proven in [10] and follows from Lemma 4.3 of the paper (Ω satisfy here the requerments from section 4), here we prove it again because of importance of the technique used. Due to Lemma 1, there exists the vector-function $\hat{\mathbf{u}}$ that provides validity of (4). Fourier coefficients of $\hat{\mathbf{u}}$ are the following

$$\hat{u}_k^1 = \frac{k_1}{|k|^2 + \alpha} p_k, \quad \hat{u}_k^2 = \frac{k_2}{|k|^2 + \alpha} p_k \tag{5}$$

and

$$\hat{u}^1(x_1, x_2) = \sum_{|k|=1}^{n} u_k^1 \phi_k^1(x_1, x_2), \quad \hat{u}^2(x_1, x_2) = \sum_{|k|=1}^{n} u_k^2 \phi_k^2(x_1, x_2)$$

where $\phi_k^1(x_1,x_2)=\sin k_1x_1\cos k_2x_2$, $\phi_k^2(x_1,x_2)=\cos k_1x_1\sin k_2x_2$

Point out that

$$\|\hat{\mathbf{u}}\|_\alpha^2=\sum_{|k|=1}^{n}\left[\frac{|k|^4}{(|k|^2+\alpha)^2}+\frac{\alpha|k|^2}{(|k|^2+\alpha)^2}\right]p_k^2\equiv\sum_{|k|=1}^{n}\gamma_kp_k^2,\quad \Phi(p,\hat{\mathbf{u}})=\|\hat{\mathbf{u}}\|_\alpha^2$$

where $\gamma_k=\dfrac{|k|^2}{|k|^2+\alpha}\le 1$. It is obvious that γ_k is a monotone increasing function of k_i, when k_{3-i} and $\alpha, i=1,2$, are fixed.

Later on, we set $p_k=0$ for $|k|>n$ and denote all the constants in inequalities that do not depend on n and α by c.

Represent p as the orthogonal in L_2 sum $p=p^1+p^2$, where

$$p^1(x_1,x_2)=\sum_{k_1\ge k_2}^{n}p_k\psi_k(x_1,x_2),\quad p_{00}=0$$

and $\psi_k(x_1,x_2)=\cos k_1x_1\cos k_2x_2$. Then the function $q=\operatorname{div}\hat{\mathbf{u}}$ also can be represented as

$$q(x_1,x_2)=\sum_{|k|=1}^{n}q_k\psi_k(x_1,x_2)$$

and $q=q^1+q^2$, where

$$q^1(x_1,x_2)=\sum_{k_1\ge k_2}^{n}q_k\psi_k(x_1,x_2),\quad q_{00}=0$$

It can be directly checked that $q_k=\gamma_kp_k$.

Since

$$\sup_{\mathbf{u}\in\mathbf{U}}\Phi(p,\mathbf{u})=\Phi(p,\hat{\mathbf{u}})=(p,q)=(p^1,q^1)+(p^2,q^2)$$

consider two cases. In the first one we have $(p^2,q^2)\ge 0.5\Phi(p,\hat{\mathbf{u}})$. Then we consider the function

$$v=\sum_{k_1<k_2}v_k\phi_k^2,\quad \text{where } v_k=\begin{cases} q_k/k_2, & \text{if } k_1<k_2;\\ 0 & \text{in other cases}\end{cases}$$

¿From this representation we have

$$\left\|\frac{\partial v}{\partial x_2}\right\|^2=\|q^2\|^2\le\sum_{|k|=1}^{n}q_k^2=\sum_{|k|=1}^{n}\gamma_k^2p_k^2\le\|\hat{\mathbf{u}}\|_\alpha^2$$

$$\left\|\frac{\partial v}{\partial x_1}\right\|^2=\sum_{k_1<k_2}q_k^2\frac{k_1^2}{k_2^2}\le\|\hat{\mathbf{u}}\|_\alpha^2$$

$$\alpha\|v\|^2\le\alpha\sum_{k_1<k_2}\frac{q_k^2}{k_2^2}=\sum_{k_1<k_2}\frac{\alpha\gamma_k}{k_2^2}\gamma_kp_k^2\le 2\|\hat{\mathbf{u}}\|_\alpha^2$$

Thus, $\|v\|_\alpha \le c\|\hat{\mathbf{u}}\|_\alpha$, where c does not depend on α and n.

On the other hand

$$(p, \frac{\partial v}{\partial x_2}) = (p^2, q^2) = \sum_{k_1<k_2} \gamma_k p_k^2$$

Let N be some positive number that will be determined later, and l be the least even number satisfying the inequality $l \ge k_2$. It is obvious that l depends on k_2. We denote the value $m (\mathrm{mod}\, l)$ by $\bar{m}$. Introduce the function r by its Fourie coefficients r_k:

$$r_{k_1,k_2} = \begin{cases} v_{\bar{k}_1,k_2}/N, & \text{if } k_1 < lN, \\ 0, & \text{in other cases} \end{cases}$$

Then due to construction of r we have

$$\begin{aligned} \left\|\frac{\partial r}{\partial x_1}\right\|^2 &= \sum_{k_1<lN} k_1^2 r_k^2 = \frac{1}{N^2}\sum_{k_1<lN} k_1^2 v_{\bar{k}_1,k_2}^2 \\ &= \frac{1}{N^2}\sum_{k_1<lN} \frac{k_1^2}{k_2^2} q_{\bar{k}_1,k_2}^2 \le cN \sum_{k_1<k_2} q_{k_1,k_2}^2 \le cN\|\hat{\mathbf{u}}\|_\alpha^2 \end{aligned}$$

$$\begin{aligned} \left\|\frac{\partial r}{\partial x_2}\right\|^2 &= \sum_{k_1<lN} k_2^2 r_k^2 = \frac{1}{N^2}\sum_{k_1<lN} k_2^2 v_{\bar{k}_1,k_2}^2 \\ &= \frac{1}{N^2}\sum_{k_1<lN} q_{\bar{k}_1,k_2}^2 = \frac{1}{N}\sum_{k_1<k_2} q_{k_1,k_2}^2 \le \frac{1}{N}\|\hat{\mathbf{u}}\|_\alpha^2 \end{aligned}$$

$$\alpha\|r\|^2 = \alpha\sum_{k_1<lN} r_k^2 = \alpha\frac{1}{N^2}\sum_{k_1<lN} v_{\bar{k}_1,k_2}^2 \le \frac{1}{N}\sum_{k_1<k_2}\frac{\alpha q_k^2}{k_2^2} \le \frac{2}{N}\|\hat{\mathbf{u}}\|_\alpha^2$$

Besides,

$$\left(p, \frac{\partial r}{\partial x_2}\right) = \sum_{|k|=1}^{n} k_2 r_k p_k = \frac{1}{N}\sum_{k_1<lN} q_{\bar{k}_1,k_2} p_k = \frac{1}{N}\sum_{k_1<lN} \gamma_{\bar{k}_1,k_2} p_{\bar{k}_1,k_2} p_k$$

$$\le \frac{1}{2N}\sum_{k_1<lN}\left(\varepsilon\gamma_{\bar{k}_1,k_2} p_{\bar{k}_1,k_2}^2 + \frac{1}{\varepsilon}\gamma_{\bar{k}_1,k_2} p_k^2\right) \le \frac{\varepsilon}{2}\sum_{k_1<k_2}\gamma_k p_k^2 + \frac{1}{2N\varepsilon}\sum_{|k|=1}^{n}\gamma_k p_k^2$$

$$\le \frac{1}{2}\left(\varepsilon + \frac{1}{N\varepsilon}\right)\|\hat{\mathbf{u}}\|_\alpha^2 = \frac{1}{2}\left(\varepsilon + \frac{1}{N\varepsilon}\right)(p,q)$$

Taking N and ε_1 in such a way that the inequality $\left(\varepsilon + \frac{1}{N\varepsilon}\right) \le 0.5$ is valid (e.g., $N = 17$, $\varepsilon = 0.25$), we obtain the final estimate

$$\left|\left(p, \frac{\partial r}{\partial x_2}\right)\right| \le 0.25(p,q)$$

Set now $u^1 = 0$, $u^2 = v - r$. Due to construction, $\mathbf{u} = (u^1, u^2) \in \mathbf{H}_0^1$ and $u^1 \equiv 0$. Thus, we have

$$(p, \operatorname{div} \mathbf{u}) = \left(p, \frac{\partial v}{\partial x_2}\right) - \left(p, \frac{\partial r}{\partial x_2}\right) \geq (p^2, q^2) - \frac{1}{2}\left(\varepsilon + \frac{1}{N\varepsilon}\right)(p, q)$$

$$\geq 0.25(p, q)$$

In virtue of previous considerations, for $\|\mathbf{u}\|_\alpha$ the following estimate is valid

$$\|\mathbf{u}\|_\alpha \leq \|v\|_\alpha + \|r\|_\alpha \leq c\|\hat{\mathbf{u}}\|_\alpha$$

So

$$\Phi(p, \mathbf{u}) = \frac{(p, \operatorname{div} \mathbf{u})^2}{\|\mathbf{u}\|_\alpha^2} \geq c\frac{(p, q)^2}{\|\hat{\mathbf{u}}\|_\alpha^2} = c\Phi(p, \hat{\mathbf{u}})$$

Thus, in the first case Lemma is proved.

The second case, i.e., $(p^2, q^2) \leq 0.5\Phi(p, \hat{\mathbf{u}})$. can be considered in the same way. This completes the proof of Lemma 2.

Corollary 1. *For any $p \in L_2^0$ estimate (2) is valid with the constant c_0 that does not depend on α.*

Proof. The set of trigonometric polinomials of the form (3) is everywhere dense in L_2^0. Take the sequence of trigonometric polinomials p_n that converges to p. Taking into account that in virtue of Lemma 2 estimate (2) is valid for every p_n with the constant independent on α and n and passing to the limit with $n \to \infty$, we obtain the statement required.

Remark 1. From the proof of Lemma 2 it follows that estimate (2) is valid for an arbitrary square with the same constant c_0.

Remark 2. Estimate (2) is also valid for a rectangular, but in this case the constant c_0 depends on l_1/l_2, where l_i are rectangular sides lenghts.

Let us now proceed to the case when Ω is a curvilinear trapezoid, i.e.,

$$\bar{\Omega} = \{x = (x_1, x_2) : \ 0 \leq x_1 \leq \pi, \ 0 \leq x_2 \leq g(x_1)\}$$

where $\pi \leq g \leq M_1, \ |g'| \leq M_2$.

After changing of variables

$$x_1 = x, \ x_2 = yg(x)$$

our domain Ω is mapped onto the square $D = (0, \pi) \times (0, \pi)$. Since the change of variables does not depend on α, then for any function $\mathbf{v}(x_1, x_2) = \tilde{\mathbf{v}}(x, y)$ we have

$$\gamma_1\|\tilde{\mathbf{v}}\|_{\alpha,D} \leq \|\mathbf{v}\|_{\alpha,\Omega} \leq \gamma_2\|\tilde{\mathbf{v}}\|_{\alpha,D} \tag{6}$$

where γ_i do not depend on α.

The expression $\operatorname{div} \mathbf{v}$ after this changing is transformed in the following way

$$\operatorname{div} \mathbf{v}(x_1, x_2) = \frac{\partial \tilde{v}_1}{\partial x} + \frac{1}{g}\frac{\partial \tilde{v}_2}{\partial y} - \frac{yg'(x)}{g(x)}\frac{\partial \tilde{v}_1}{\partial y} \equiv \mathrm{DIV}\, \tilde{\mathbf{v}}(x, y)$$

from what follows

$$(\operatorname{div}\mathbf{v}, p)_\Omega = (\mathrm{DIV}\,\tilde{\mathbf{v}}, q)_D$$

where $q = g\tilde{p}$ and $(q,1)_D = 0$, since $p \in L_2^0(\Omega)$.

Along with the functional $\Phi(p,\mathbf{u})$, introduce the functional $\Psi(p,\mathbf{u})$:

$$\Psi(p,\mathbf{u}) \equiv \frac{(p, \mathrm{DIV}\,\mathbf{u})^2}{\|\mathbf{u}\|_\alpha^2}$$

The following statement takes place

Lemma 3. *Let*

$$p(x,y) = \sum_{|k|=1}^{n} p_k \cos k_1 x \cos k_2 y$$

and $\hat{\mathbf{u}} \in \mathbf{U}$ *be the function that provides supremum of* $\Phi(p,\mathbf{u})$ *over* $\mathbf{U}$. *Then there exists the constant* c *that does not depend on* α *and* n *such that the following inequality*

$$\sup_{\mathbf{u}\in\mathbf{U}} \Psi(p,\mathbf{u}) \le c\Phi(p,\hat{\mathbf{u}}) \tag{7}$$

is valid.

Proof. First we have the trivial estimate

$$\sup_{\mathbf{u}\in\mathbf{U}} \Psi(p,\mathbf{u}) \le 3\left(\sup_{\mathbf{u}\in\mathbf{U}} \frac{(p, \frac{\partial u^1}{\partial x})^2}{\|\mathbf{u}\|_\alpha^2} + \sup_{\mathbf{u}\in\mathbf{U}} \frac{(\frac{p}{g}, \frac{\partial u^2}{\partial y})^2}{\|\mathbf{u}\|_\alpha^2} + \sup_{\mathbf{u}\in\mathbf{U}} \frac{(p, h\frac{\partial u^1}{\partial y})^2}{\|\mathbf{u}\|_\alpha^2}\right) \tag{8}$$

where $h = \dfrac{yg'(x)}{g(x)} \equiv y\bar{g}(x)$ and $|h| \le M = \pi M_2$. It is worth to pay attention that functions g, $\bar{g}$ are the functions of the variable x only. We shall use this property later.

Let us estimate every term in the right-hand side separately:

$$\sup_{\mathbf{u}\in\mathbf{U}} \frac{\Big(p, \dfrac{\partial u^1}{\partial x}\Big)^2}{\|\mathbf{u}\|_\alpha^2} \le \sup_{\mathbf{u}\in\mathbf{U}} \frac{\Big(p, \dfrac{\partial u^1}{\partial x}\Big)^2}{\|u^1\|_\alpha^2} \le \sup_{\mathbf{u}\in\mathbf{U}} \frac{(p, \operatorname{div}\mathbf{u})^2}{\|\mathbf{u}\|_\alpha^2} \le \Phi(p,\hat{\mathbf{u}}),$$

$$\sup_{\mathbf{u}\in\mathbf{U}} \frac{\Big(\dfrac{p}{g}, \dfrac{\partial u^2}{\partial y}\Big)^2}{\|\mathbf{u}\|_\alpha^2} = \sup_{u^2:\mathbf{u}\in\mathbf{U}} \frac{\Big(p, \dfrac{\partial (u^2/g)}{\partial y}\Big)^2}{\|u^2\|_\alpha^2} \le \sup_{w:(0,w)\in\mathbf{U}} \frac{\Big(p, \dfrac{\partial w}{\partial y}\Big)^2}{\|gw\|_{\alpha,y}^2}$$

$$\le c \sup_{w:(0,w)\in\mathbf{U}} \frac{\Big(p, \dfrac{\partial w}{\partial y}\Big)^2}{\|w\|_{\alpha,y}^2} = c\sum_{|k|=1}^{n} \frac{k_2^2}{k_2^2+\alpha} p_k^2 \le c\sum_{|k|=1}^{n} \gamma_k p_k^2 = c\|\hat{\mathbf{u}}\|_\alpha^2$$

As for the third term, let us transform it before estimating. Integrating by parts, we obtain

$$\Big(p, h\frac{\partial u^1}{\partial y}\Big) = \Big(p, \frac{\partial (hu^1)}{\partial y}\Big) - \Big(p, u^1\frac{\partial h}{\partial y}\Big) = \Big(p, \frac{\partial (hu^1)}{\partial y}\Big) - (\bar{g}p, u^1) \tag{9}$$

Estimate the expression from (8) containing the second term of (9):

$$\begin{aligned}
&\sup_{\mathbf{u}\in\mathbf{U}}\frac{(\bar{g}p,u^1)^2}{\|\mathbf{u}\|_\alpha^2}\le\sup_{w:(w,0)\in\mathbf{U}}\frac{(p,\bar{g}w)^2}{\|w\|_\alpha^2}\le\sup_{w\in L_2}\frac{(p,\bar{g}w)^2}{\alpha\|w\|^2}=\sup_{w\in L_2}\frac{(\bar{g}p,w)^2}{\alpha\|w\|^2}\\
&=\frac{1}{\alpha}\|\bar{g}p\|^2\le\frac{M_2^2}{\alpha}\|p\|^2\le\frac{M_2^2}{\alpha}\sum_{|k|=1}^{n}\frac{1}{\gamma_k}\gamma_k p_k^2\le 2\Phi(p,\hat{\mathbf{u}})
\end{aligned}\tag{10}$$

To estimate the expression with the term $\Big(p,\dfrac{\partial(hu^1)}{\partial y}\Big)$, let us introduce the space $H=\Big\{w:\ w\in L_2,\ \dfrac{\partial w}{\partial y}\in L_2\Big\}$. Since

$$\|hw\|_{\alpha,y}^2=\left\|\frac{\partial(hw)}{\partial y}\right\|^2+\alpha\|hw\|^2$$

$$\le\left\|h\frac{\partial w}{\partial y}\right\|^2+\|\bar{g}w\|^2+\alpha\|hw\|^2\le M_2^2\left\|h\frac{\partial w}{\partial y}\right\|^2+(\tfrac{M_2^2}{\pi^2}+\alpha M_2^2)\|w\|^2\le 2M_2^2\|w\|_{\alpha,y}^2$$

Then

$$\begin{aligned}
&\sup_{\mathbf{u}\in\mathbf{U}}\frac{\Big(p,\dfrac{\partial(hu^1)}{\partial y}\Big)^2}{\|\mathbf{u}\|_\alpha^2}\le\sup_{u^1:\mathbf{u}\in\mathbf{U}}\frac{\Big(p,\dfrac{\partial(hu^1)}{\partial y}\Big)^2}{\|u^1\|_\alpha^2}\le\sup_{u^1:(u^1,0)\in\mathbf{U}}\frac{\Big(p,\dfrac{\partial(hu^1)}{\partial y}\Big)^2}{\|u^1\|_{\alpha,y}^2}\\
&\le 2M_2^2\sup_{u^1:(u^1,0)\in\mathbf{U}}\frac{\Big(p,\dfrac{\partial(hu^1)}{\partial y}\Big)^2}{\|hu^1\|_{\alpha,y}^2}=2M_2^2\sup_{w^1:(w^1,0)\in\mathbf{U}}\frac{\Big(p,\dfrac{\partial(w^1)}{\partial y}\Big)^2}{\|w^1\|_{\alpha,y}^2}\le c_2M_2^2\sup_{w\in H}\frac{\Big(p,\dfrac{\partial w}{\partial y}\Big)^2}{\|w\|_{\alpha,y}^2}\\
&\le c_2M_2^2\sum_{|k|=1}^{n}\frac{k_2^2}{k_2^2+\alpha}p_k^2\le c_2M_2^2\sum_{|k|=1}^{n}\frac{|k|^2}{|k|^2+\alpha}p_k^2=c_2M_2^2\|\hat{\mathbf{u}}\|_\alpha^2
\end{aligned}\tag{11}$$

Thus, the Lemma is proved. As a matter of fact, since the set of all trigonometric polinomials is everywhere dense in L_2^0, then estimate (7) holds for every function from L_2^0.

Lemma 4. *For any trigonometric polinomial*

$$p(x,y)=\sum_{|k|=1}^{n}p_k\cos k_1x\cos k_2y$$

and sufficiently small M_2 there exists the constant c that does not depend on α and n such that the following inequality

$$\Phi(p,\hat{\mathbf{u}})\le c\sup_{\mathbf{v}\in\mathbf{H}_0^1}\Psi(p,\mathbf{v})\tag{12}$$

is valid.

Proof. Remind that M_2 is the constant from the inequality $|g'| \leq M_2$. In virtue of Lemma 2, there exists the function $\mathbf{u} \in \mathbf{U}$ such that

$$\Phi(p, \hat{\mathbf{u}}) \leq c_0 \Phi(p, \mathbf{u})$$

Consider two cases: $(p^2, q^2) \geq 0.5\Phi(p, \hat{\mathbf{u}})$ and $(p^1, q^1) \geq 0.5\Phi(p, \hat{\mathbf{u}})$, where $q = \operatorname{div} \hat{\mathbf{u}}$. Due to the proof of Lemma 2, in the first case there exists the function $\mathbf{u} \in \mathbf{H}_0^1 :\ u^1 = 0,\ u^2 = v - r$ such that

$$\Phi(p, \hat{\mathbf{u}}) \leq c\Phi(p, \mathbf{u}) = c\,\frac{\left(p, \frac{\partial u^2}{\partial y}\right)^2}{\|u^2\|_\alpha^2}$$

Now we set $\mathbf{v} = (v^1, v^2)$, where $v^1 = 0$ and $v^2 = gu^2$. Then $\mathbf{v} \in \mathbf{H}_0^1$. Since

$$\|gu^2\|_\alpha^2 = \|\nabla(gu^2)\|^2 + \alpha\|gu^2\|^2 \leq \left\|\frac{\partial(gu^2)}{\partial x}\right\|^2 + M_1^2\left\|\frac{\partial u^2}{\partial y}\right\|^2$$

$$+\alpha M_1^2\|u^2\|^2 \leq c\|u^2\|_\alpha^2, \quad \text{where } c = 2(M_1^2 + M_2^2)$$

then we have

$$\Psi(p, \mathbf{v}) = \frac{\left(p, \frac{\partial u^2}{\partial y}\right)^2}{\|gu^2\|_\alpha^2} \geq c\frac{\left(p, \frac{\partial u^2}{\partial y}\right)^2}{\|u^2\|_\alpha^2} = c\,\Phi(p, \mathbf{u}) \geq c\Phi(p, \hat{\mathbf{u}})$$

So for the first case the Lemma is proved.

Consider the second case, i.e. $(p^1, q^1) \geq 0.5\Phi(p, \hat{\mathbf{u}})$. Then from the proof of Lemma 2 it follows that there exists the function $\mathbf{u} \in \mathbf{H}_0^1 :\ u^1 = v - r,\ u^2 = 0$ such that

$$\Phi(p, \hat{\mathbf{u}}) \leq c_3\Phi(p, \mathbf{u}) = c_3\,\frac{\left(p, \frac{\partial u^1}{\partial x}\right)^2}{\|u^1\|_\alpha^2}$$

Set $\mathbf{v} = \mathbf{u}$. We have to prove that

$$\Psi(p, \mathbf{v}) = \frac{\left(p, \frac{\partial v^1}{\partial x} - h\frac{\partial v^1}{\partial y}\right)^2}{\|v^1\|_\alpha^2} \geq c\Phi(p, \hat{\mathbf{u}})$$

¿From the proof of Lemma 3 we have

$$\Psi(p, \mathbf{v}) = \frac{\left(p, \frac{\partial u^1}{\partial x} + \bar{g}u^1 - \frac{\partial(hu^1)}{\partial y}\right)^2}{\|u^1\|_\alpha^2}$$

from what follows

$$\Psi(p, \mathbf{u}) \geq \frac{0.25\left(p, \frac{\partial u^1}{\partial x}\right)^2 - 4\left(p, \bar{g}u^1\right)^2 - 4\left(p, \frac{\partial(hu^1)}{\partial y}\right)^2}{\|u^1\|_\alpha^2}$$

Estimating all terms in the right-hand side of the last inequality as was done in the proof of Lemma 3, we have

$$\frac{\left(p, \frac{\partial u^1}{\partial x}\right)^2}{\|u^1\|_\alpha^2} \geq c_4 \Phi(p, \hat{\mathbf{u}}), \quad \frac{(p, \bar{g}u^1)^2}{\|u^1\|_\alpha^2} \leq c M_2^2 \Phi(p, \hat{\mathbf{u}}), \quad \frac{\left(p, \frac{\partial (hu^1)}{\partial y}\right)^2}{\|u^1\|_\alpha^2} \leq c M_2^2 \Phi(p, \hat{\mathbf{u}})$$

from what follows

$$\Psi(p, \mathbf{u}) \geq c_5 (1 - c_6 M_2^2) \Phi(p, \hat{\mathbf{u}})$$

Thus, in the case $1 - c_6 M_2^2 > 0$ the assertion of Lemma 4 is proved.

Corollary 2. *Let Ω be a curvilinear trapezoid*

$$\bar{\Omega} = \{x = (x_1, x_2) : \ 0 \leq x_1 \leq l, \ 0 \leq x_2 \leq g(x_1)\}$$

where g is a Lipschitz function. Then for sufficiently small l estimate (12) is valid for any $p \in L_2^0$.

Proof. From Lemma 4 it follows that (12) is true for any $p \in L_2^0$ if $l = \pi$ and $\max |g'|$ is sufficiently small. Making a change of variables

$$x = \frac{\pi}{l} x_1, \quad y = \frac{\pi}{l} x_2$$

we obtain the domain Ω satisfying the conditions of Lemma 4, and $\frac{\pi}{l} g'_x = g'_{x_1}$. So for sufficiently small l the derivative g'_x will satisfy the condition of Lemma 4, whence the assertion of Corollary 2 follows.

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