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1 Characteristic Function of log of stock under Heston a.k.a. Geometric Brownian Motion with Stochastic Arrival (GBMSA)

In Heston stochastic volatility model, stock price follows the process:

$$\begin{aligned} dS_t &= (r-q)S_t dt + \sqrt{v_t} S_t dW_t^{(1)}, \\ dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_t^{(2)}, \end{aligned}$$

where the two Brownian components $W_t^{(1)}$ and $W_t^{(2)}$ are correlated with rate ρ. The parameters κ, θ, and σ have certain physical meanings: κ is the mean reversion speed, θ is the long run variance, and σ is the volatility of the volatility. The characteristic function for the log of stock price process is given by

$$\begin{aligned} \phi(u) &= \mathbb{E}(e^{iu \ln S_t}) \\ &= \frac{\exp\{iu \ln S_0 + i(r-q)tu + \frac{\kappa\theta t(\kappa - i\rho\sigma u)}{\sigma^2}\}}{(\cosh\frac{\gamma t}{2} + \frac{\kappa - i\rho\sigma u}{\gamma}\sinh\frac{\gamma t}{2})^{\frac{2\kappa\theta}{\sigma^2}}} \exp\left\{-\frac{(u^2+iu)v_0}{\gamma\coth\frac{\gamma t}{2} + \kappa - i\rho\sigma u}\right\} \end{aligned}$$

where $\gamma = \sqrt{\sigma^2(u^2+iu) + (\kappa - i\rho\sigma u)^2}$, and S_0 and v_0 are the initial values for the price process and the volatility process, respectively.

2 Characteristic Function of log of stock under Variance Gamma with Stochastic Arrival (VGSA)

To obtain VGSA, we take the VG process which is a homogeneous Lévy process and build in stochastic volatility by evaluating it at a continuous time change given by the integral of a Cox, Ingersoll and Ross (CIR) process. The mean reversion of the CIR process introduces the clustering phenomena often referred to as volatility persistence. This enables us to calibrate to market price surfaces that go across strike and maturity simultaneously. This process is tractable in the analytical expressions for its characteristic function. Formally we define the CIR process $y(t)$ as the solution to the stochastic differential equation

$$dy_t = \kappa(\eta - y_t)dt + \lambda\sqrt{y_t}dW_t$$

where $W(t)$ is a Brownian motion, η is the long-term rate of time change, κ is the rate of mean reversion, and λ is the volatility of the time change. The process $y(t)$ is the instantaneous rate of time change and so the time change is given by $Y(t)$ where

$$Y(t) = \int_0^t y(u)du$$

The SDE of the log of the market variable is the same as the VG process with the above time change. The characteristic function for the time change $Y(t)$ is given by

$$\begin{aligned} \mathbb{E}(e^{iuY(t)}) &= \phi(u, t, y(0); \kappa, \eta, \lambda) \\ &= A(t, u)e^{B(t,u)y(0)} \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} A(t, u) &= \frac{\exp\left(\frac{\kappa^2\eta t}{\lambda^2}\right)}{\left(\cosh(\gamma t/2) + \frac{\kappa}{\gamma}\sinh(\gamma t/2)\right)^{2\kappa\eta/\lambda^2}} \\ B(t, u) &= \frac{2iu}{\kappa + \gamma\coth(\gamma t/2)} \end{aligned}$$

with

$$\gamma = \sqrt{\kappa^2 - 2\lambda^2 iu}$$

The stochastic volatility Lévy process, termed the VGSA process, is defined by

$$Z_{VGSA}(t) = X_{VG}(Y(t); \sigma, \nu, \theta)$$

where σ, ν, θ, κ, η, and λ are the six parameters defining the process. Its characteristic function is given by

$$\mathbb{E}(e^{iuZ_{VGSA}(t)}) = \phi(-i\psi_{VG}(u; \sigma, \nu, \theta), t, \frac{1}{\nu}; \kappa, \eta, \lambda)$$

where Ψ_{VG} is the log characteristic function of the variance gamma process at unit time, namely,

$$\psi_{VG}(u; \sigma, \nu, \theta) = -\frac{1}{\nu}\log\left(1 - iu\theta\nu + \sigma^2\nu u^2/2\right)$$

We define the asset price process at time t as follows:

$$S(t) = S(0)\frac{e^{(r-q)t+Z(t)}}{\mathbb{E}[e^{Z(t)}]}$$

We note that

$$\mathbb{E}[e^{Z(t)}] = \phi(-i\psi_{VG}(-i;\sigma,\nu,\theta), t, \frac{1}{\nu}; \kappa, \eta, \lambda)$$

Therefore the characteristic function of the log of the asset price at time t is given by

$$\begin{aligned} \Phi(u) &= \mathbb{E}(e^{iu\ln S_t}) \\ &= \exp(iu(\log S_0 + (r-q)t)) \times \frac{\phi(-i\psi_{VG}(u;\sigma,\nu,\theta), t, \frac{1}{\nu}; \kappa, \eta, \lambda)}{\phi(-i\psi_{VG}(-i;\sigma,\nu,\theta), t, \frac{1}{\nu}; \kappa, \eta, \lambda)^{iu}} \end{aligned}$$

where $\phi(u, t, y; \kappa, \eta, \lambda)$ is provided in Equation 2.1.

3 Characteristic Function of log of stock under Normal Inverse Gaussian with Stochastic Arrival (NIGSA)

Following the argument in Section 2, it can be shown that the characteristic function for the NIGSA process at time t is explicitly given as

$$\begin{aligned} \Phi(u) &= \mathbb{E}(e^{iu\ln S_t}) \\ &= \exp(iu(\log S_0 + (r-q)t)) \times \frac{\phi(-i\psi_{NIG}(u;1,\nu,\theta), t, \sigma; \kappa, \eta, \lambda)}{\phi(-i\psi_{NIG}(-i;1,\nu,\theta), t, \sigma; \kappa, \eta, \lambda)^{iu}} \end{aligned}$$

where $\psi_{NIG}(u;\sigma,\nu,\theta)$ is the unit time log characteristic function expressed in terms of the parameters of the time-changed Brownian motion which is given by

$$\psi_{NIG}(u;\sigma,\nu,\theta) = \sigma\left(\frac{\nu}{\theta} - \sqrt{\frac{\nu^2}{\theta^2} - 2\frac{\theta iu}{\sigma^2} + u^2}\right)$$

and $\phi(u, t, y; \kappa, \eta, \lambda)$ is provided in Equation 2.1.

4 Characteristic Function of log of stock under CGMY with Stochastic Arrival (CGMYSA)

Following the argument in Section 2, it can be shown the characteristic function for the CGMYSA process at time t is explicitly given as

$$\begin{aligned}\Phi(u) &= \mathbb{E}(e^{iu\ln S_t}) \\ &= \exp(iu(\log S_0 + (r-q)t)) \times \frac{\phi(-i\psi_{CGMY}(u;1,G,M,Y),t,C;\kappa,\eta,\lambda)}{\phi(-i\psi_{CGMY}(-i;1,G,M,Y),t,C;\kappa,\eta,\lambda)^{iu}}\end{aligned}$$

where $\psi_{CGMY}(u;C,G,M,Y)$ is the unit time log characteristic function expressed in terms of the parameters is given by

$$\psi_{CGMY}(u;C,G,M,Y) = C\Gamma(-Y)((M-iu)^Y - M^Y + (G+iu)^Y - G^Y)$$

and $\phi(u,t,y;\kappa,\eta,\lambda)$ is provided in Equation 2.1.

5 Characteristic Function of log of stock under Variance Gamma Scaled Self-Decomposable (VGSSD)

Characteristic function of the log of the asset price at time t under VGSSD is given by

$$\begin{aligned}\Phi(u) &= \mathbb{E}(e^{iu\ln S_t}) \\ &= \exp(iu(\ln S_0 + (r-q)t) \times \frac{\phi_{X(t)}(u)}{\phi_{X(t)}(-i)}\end{aligned}$$

where $\phi_{X(t)}(u)$ is given by

$$\begin{aligned}\phi_{X(t)}(u) &= \phi_{VG(1)}(ut^\gamma) \\ &= \left(1 - iut^\gamma\nu\theta + \frac{1}{2}u^2t^{2\gamma}\nu\sigma^2\right)^{-1/\nu}\end{aligned}$$