Physics 265 Problem Set 1

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1 Problem 1.1

Derive explicitly (step-by-step) the wave equations for the electric field vector **E** (Equation 5, Section 1.2, Born & Wolf) and the magnetic field vector **H** (Equation 6) from Equations 1 to 11 in Section 1.1.

We begin with the Maxwell-Faraday Equation on a system with no charges or currents, that is, $\mathbf{J} \to 0$ and $\rho \to 0$. Faraday's law is given by

$$\begin{aligned} \nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} &= 0. \end{aligned} \tag{1.1}$$

We then apply the material equation $\mathbf{B} = \mu \mathbf{H}$ as follows,

$$\begin{aligned} \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} &= 0 \\ \nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial (\mu \mathbf{H})}{\partial t} &= 0. \end{aligned} \tag{1.2}$$

Since μ is just a constant, we take it out of the partial derivative, allowing us to divide all the terms with μ as shown below

$$\begin{aligned} \nabla \times \mathbf{E} + \frac{\mu}{c}\frac{\partial \mathbf{H}}{\partial t} &= 0 \\ \frac{1}{\mu}\left(\nabla \times \mathbf{E} + \frac{\mu}{c}\frac{\partial \mathbf{H}}{\partial t}\right) &= 0 \\ \frac{1}{\mu}\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} &= 0. \end{aligned} \tag{1.3}$$

Taking the curl and expanding the curl of a sum, we obtain

$$\begin{aligned} \nabla \times \left(\frac{1}{\mu}\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}\right) &= 0 \\ \nabla \times \left(\frac{1}{\mu}\nabla \times \mathbf{E}\right) + \frac{1}{c}\left(\nabla \times \frac{\partial \mathbf{H}}{\partial t}\right) &= 0. \end{aligned} \tag{1.4}$$

In resolving the second term for Eq. (1.4), we start with the fourth Maxwell's equation, the Ampere-Maxwell law that is

$$
\begin{aligned}
\nabla \times \mathbf{H} &= \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} \\
\nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} &= \frac{4\pi}{c}\mathbf{J}, \quad \text{(where } \mathbf{J} \to 0\text{)} \\
\nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} &= 0,
\end{aligned}
$$

in which $\mathbf{D} = \varepsilon\mathbf{E}$, hence

$$
\nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial(\varepsilon\mathbf{E})}{\partial t} = 0. \tag{1.5}
$$

Taking the time derivative and rearranging the terms, we obtain

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial(\varepsilon\mathbf{E})}{\partial t}\right) &= 0 \\
\nabla \times \frac{\partial \mathbf{H}}{\partial t} - \frac{\varepsilon}{c}\frac{\partial^2 \mathbf{E}}{\partial t^2} &= 0 \\
\nabla \times \frac{\partial \mathbf{H}}{\partial t} &= \frac{\varepsilon}{c}\frac{\partial^2 \mathbf{E}}{\partial t^2}
\end{aligned} \tag{1.6}
$$

We then substitute Eq. (1.6) back to Eq. (1.4)

$$
\begin{aligned}
\nabla \times \left(\frac{1}{\mu}\nabla \times \mathbf{E}\right) + \frac{1}{c}\left(\frac{\varepsilon}{c}\frac{\partial^2 \mathbf{E}}{\partial t^2}\right) &= 0 \\
\nabla \times \left(\frac{1}{\mu}\nabla \times \mathbf{E}\right) + \left(\frac{\varepsilon}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}\right) &= 0.
\end{aligned} \tag{1.7}
$$

Recalling the curl identities and applying it to the equation above, we get

$$
\begin{aligned}
\nabla \times (u\mathbf{v}) &= u(\nabla \times \mathbf{v}) + \nabla u \times \mathbf{v} \\
\nabla \times \left(\frac{1}{\mu}(\nabla \times \mathbf{E})\right) &= \frac{1}{\mu}\left(\nabla \times (\nabla \times \mathbf{E})\right) + \nabla\left(\frac{1}{\mu}\right) \times (\nabla \times \mathbf{E}))
\end{aligned} \tag{1.8}
$$

Simplifying further, we rewrite the gradient of a scalar quantity from Eq. (1.8) into

$$
\begin{aligned}
\nabla\left(\frac{1}{\mu}\right) &= -\left(\frac{1}{\mu^2}\right)\nabla\mu = -\frac{1}{\mu}\left(\frac{1}{\mu}\nabla\mu\right) \\
\nabla\left(\frac{1}{\mu}\right) &= -\frac{1}{\mu}\nabla(\ln\, \mu)
\end{aligned} \tag{1.9}
$$

and then, the following curl identity is applied to the second term on Eq. (1.8)

$$
\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E}. \tag{1.10}
$$

These identities turns Eq. (1.7) into

$$\begin{aligned}
\nabla\left(\frac{1}{\mu}\right)\times(\nabla\times\mathbf{E})+\frac{1}{\mu}\left(\nabla\times(\nabla\times\mathbf{E})\right)+\left(\frac{\varepsilon}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}\right)&=0\\
-\frac{1}{\mu}\nabla(\ln\,\mu)\times(\nabla\times\mathbf{E})+\frac{1}{\mu}\left(\nabla\times(\nabla\times\mathbf{E})\right)+\left(\frac{\varepsilon}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}\right)&=0\\
\nabla(\ln\,\mu)\times(\nabla\times\mathbf{E})-\left(\nabla\times(\nabla\times\mathbf{E})\right)-\left(\frac{\varepsilon\mu}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}\right)&=0\\
\nabla(\ln\,\mu)\times(\nabla\times\mathbf{E})-\left(\nabla(\nabla\cdot\mathbf{E})-\nabla^2\mathbf{E}\right)-\left(\frac{\varepsilon\mu}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}\right)&=0\\
\nabla(\ln\,\mu)\times(\nabla\times\mathbf{E})-\nabla(\nabla\cdot\mathbf{E})+\nabla^2\mathbf{E}-\left(\frac{\varepsilon\mu}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}\right)&=0. \qquad (1.11)
\end{aligned}$$

This almost looks like the final form but we still need to smooth out the and simplify the term $\nabla(\nabla\cdot\mathbf{E})$ from Eq. (1.11). We begin with the first Maxwell's equation, Gauss' law that is

$$\nabla\cdot D=4\pi\rho,\quad (\text{where }\rho\to 0)$$

in which $\mathbf{D}=\varepsilon\mathbf{E}$, hence

$$\nabla\cdot D=\nabla(\varepsilon\mathbf{E})=0. \qquad (1.12)$$

Recalling the identity $\nabla\times(u\mathbf{v})=u(\nabla\times\mathbf{v})+\nabla u\times\mathbf{v}$, Eq. (1.12) becomes

$$\begin{aligned}
\varepsilon\nabla\cdot\mathbf{E}+\mathbf{E}\cdot\nabla\varepsilon&=0\\
\nabla\cdot\mathbf{E}+\mathbf{E}\cdot\frac{1}{\varepsilon}\nabla\varepsilon&=0\\
\nabla\cdot\mathbf{E}&=-\mathbf{E}\cdot\frac{1}{\varepsilon}\nabla\varepsilon.
\end{aligned}$$

Using Eq. (1.23), we can rewrite the expression as

$$\nabla\cdot\mathbf{E}=-\mathbf{E}\cdot\nabla(\ln\,\varepsilon.. \qquad (1.13)$$

Substituting Eq. (1.13) back in Eq. (1.11), we now have

$$\nabla(\ln\,\mu)\times(\nabla\times\mathbf{E})+\nabla(\mathbf{E}\cdot\nabla(\ln\,\varepsilon))+\nabla^2\mathbf{E}-\left(\frac{\varepsilon\mu}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}\right)=0,$$

and further rearranging the terms, we obtain a final expression of

$$\boxed{\nabla^2\mathbf{E}-\left(\frac{\varepsilon\mu}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}\right)+\nabla(\ln\,\mu)\times(\nabla\times\mathbf{E})+\nabla(\mathbf{E}\cdot\nabla(\ln\,\varepsilon))=0. \qquad (1.14)}$$

For the magnetic field wave equation, we start with the Maxwell-Ampere Equation on a system with no charges or currents as well.

$$\nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} = 0. \tag{1.15}$$

Applying the material equation $\mathbf{D} = \varepsilon \mathbf{E}$:

$$\nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \varepsilon \mathbf{E}}{\partial t} = 0. \tag{1.16}$$

Taking ε out of the partial derivative:

$$\begin{aligned}
\nabla \times \mathbf{H} - \frac{\varepsilon}{c}\frac{\partial \mathbf{E}}{\partial t} &= 0 \\
\frac{1}{\varepsilon}\left(\nabla \times \mathbf{H} - \frac{\varepsilon}{c}\frac{\partial \mathbf{E}}{\partial t}\right) &= 0 \\
\frac{1}{\varepsilon}\nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} &= 0.
\end{aligned} \tag{1.17}$$

Taking the curl and expanding the curl of a sum:

$$\begin{aligned}
\nabla \times \left(\frac{1}{\varepsilon}\nabla \times \mathbf{H} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\right) &= 0 \\
\nabla \times \left(\frac{1}{\varepsilon}\nabla \times \mathbf{H}\right) + \frac{1}{c}\left(\nabla \times \frac{\partial \mathbf{E}}{\partial t}\right) &= 0.
\end{aligned} \tag{1.18}$$

Resolving the second term for Eq. (1.18) using the Maxwell-Faraday law:

$$\begin{aligned}
\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} &= 0, \\
\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mu \mathbf{H}}{\partial t} &= 0.
\end{aligned} \tag{1.19}$$

Taking the time derivative and rearranging the terms:

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mu \mathbf{H}}{\partial t}\right) &= 0 \\
\nabla \times \frac{\partial \mathbf{E}}{\partial t} - \frac{\mu}{c}\frac{\partial^2 \mathbf{H}}{\partial t^2} &= 0 \\
\nabla \times \frac{\partial \mathbf{E}}{\partial t} &= \frac{\mu}{c}\frac{\partial^2 \mathbf{H}}{\partial t^2}
\end{aligned} \tag{1.20}$$

Substituting Eq. (1.20) back into Eq. (1.18):

$$\begin{aligned}
\nabla \times \left(\frac{1}{\varepsilon}\nabla \times \mathbf{H}\right) + \frac{1}{c}\left(\frac{\mu}{c}\frac{\partial^2 \mathbf{H}}{\partial t^2}\right) &= 0 \\
\nabla \times \left(\frac{1}{\varepsilon}\nabla \times \mathbf{H}\right) + \left(\frac{\mu}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2}\right) &= 0
\end{aligned} \tag{1.21}$$

Recalling the curl identities and applying them to Eq. (1.21)

$$\begin{aligned}
\nabla \times (u\mathbf{v}) &= u(\nabla \times \mathbf{v}) + \nabla u \times \mathbf{v} \\
\nabla \times \left(\frac{1}{\varepsilon}\nabla \times \mathbf{H}\right) &= \nabla\left(\frac{1}{\varepsilon}\right) \times (\nabla \times \mathbf{H})) + \frac{1}{\varepsilon}\left(\nabla \times (\nabla \times \mathbf{H})\right)
\end{aligned} \tag{1.22}$$

Rewriting the gradient of a scalar quantity from Eq. (1.22):

$$\begin{aligned}
\nabla\left(\frac{1}{\varepsilon}\right) &= -\left(\frac{1}{\varepsilon^2}\right)\nabla\varepsilon = -\frac{1}{\varepsilon}\left(\frac{1}{\varepsilon}\nabla\varepsilon\right) \\
\nabla\left(\frac{1}{\varepsilon}\right) &= -\frac{1}{\varepsilon}\nabla(\ln\,\varepsilon)
\end{aligned} \tag{1.23}$$

Applying the curl identity to the second term in Eq. (1.8):

$$\nabla \times \nabla \times \mathbf{H} = \nabla(\nabla \cdot \mathbf{H}) - \nabla^2\mathbf{H}. \tag{1.24}$$

These identities turn Eq. (1.7) into:

$$\begin{aligned}
\nabla\left(\frac{1}{\varepsilon}\right) \times (\nabla \times \mathbf{H}) + \frac{1}{\varepsilon}\left(\nabla \times (\nabla \times \mathbf{H})\right) + \left(\frac{\mu}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2}\right) &= 0 \\
-\frac{1}{\varepsilon}\nabla(\ln\,\varepsilon) \times (\nabla \times \mathbf{H}) + \frac{1}{\varepsilon}\left(\nabla \times (\nabla \times \mathbf{H})\right) + \left(\frac{\mu}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2}\right) &= 0 \\
\nabla(\ln\,\varepsilon) \times (\nabla \times \mathbf{H}) - \left(\nabla \times (\nabla \times \mathbf{H})\right) - \left(\frac{\varepsilon\mu}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2}\right) &= 0 \\
\nabla(\ln\,\varepsilon) \times (\nabla \times \mathbf{H}) - \left(\nabla(\nabla \cdot \mathbf{H}) - \nabla^2\mathbf{H}\right) - \left(\frac{\varepsilon\mu}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2}\right) &= 0 \\
\nabla(\ln\,\varepsilon) \times (\nabla \times \mathbf{H}) - \nabla(\nabla \cdot \mathbf{H}) + \nabla^2\mathbf{H} - \left(\frac{\varepsilon\mu}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2}\right) &= 0.
\end{aligned} \tag{1.25}$$

To simplify the term $\nabla(\nabla\cdot\mathbf{H})$ from Eq. (1.25), we use Gauss' law (the first Maxwell's equation):

$$\nabla \cdot \mathbf{B} = 0.$$

Substituting $\mathbf{B} = \mu\mathbf{H}$ into Gauss' law:

$$\nabla \cdot \mu\mathbf{H} = 0. \tag{1.26}$$

Using the identity $\nabla \times (u\mathbf{v}) = u(\nabla \times \mathbf{v}) + \nabla u \times \mathbf{v}$, Eq. (1.26) becomes:

$$\begin{aligned}
\mu\nabla \cdot \mathbf{H} + \mathbf{H} \cdot \nabla\mu &= 0 \\
\nabla \cdot \mathbf{H} + \mathbf{H} \cdot \frac{1}{\mu}\nabla\mu &= 0 \\
\nabla \cdot \mathbf{H} &= -\mathbf{H} \cdot \frac{1}{\mu}\nabla\mu.
\end{aligned}$$

Using Eq. (1.25), we can rewrite the expression as:

$$\nabla \cdot \mathbf{H} = -\mathbf{H} \cdot \nabla(\ln\,\varepsilon). \tag{1.27}$$

Substituting Eq. (1.27) back into Eq. (1.25), we obtain the final expression:

$$\nabla(\ln\,\varepsilon) \times (\nabla \times \mathbf{H}) + \nabla(\mathbf{H} \cdot \nabla(\ln\,\varepsilon)) + \nabla^2\mathbf{H} - \left(\frac{\varepsilon\mu}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2}\right) = 0.$$

Rearranging the terms, we obtain the final form:

$$\boxed{\nabla^2\mathbf{H} - \left(\frac{\varepsilon\mu}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2}\right) + \nabla(\ln\,\varepsilon) \times (\nabla \times \mathbf{H}) + \nabla(\mathbf{H} \cdot \nabla(\ln\,\mu)) = 0. \qquad (1.28)}$$

2 Problem 1.2

Derive the respective scalar wave equations for the Cartesian components (E_x, E_y, and E_z) of $\mathbf{E}$ in a medium where the dielectric constant is given by ε(z) and the magnetic permeability $\mu \sim 1$.

$$\begin{aligned}
\nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E} &= -\frac{\partial}{\partial t}\nabla \times \mathbf{B} \\
\nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E} &= -\frac{\partial}{\partial t}\nabla \times (\mu\mathbf{H}) \\
\nabla(\nabla\varepsilon \cdot \mathbf{E}) + \nabla^2\mathbf{E} &= -\frac{\partial}{\partial t}\nabla \times (\mu\mathbf{H}) \\
\nabla(\nabla\varepsilon \cdot \mathbf{E}) + \nabla^2\mathbf{E} &= \nabla\mu \times \frac{\partial\mathbf{H}}{\partial t} + \mu\frac{\partial\nabla \times \mathbf{H}}{\partial t} \\
\nabla(\nabla\varepsilon \cdot \mathbf{E}) + \nabla^2\mathbf{E} &= \nabla\mu \times \frac{\partial\mathbf{H}}{\partial t} + \mu\frac{\partial^2 D}{\partial t^2} \\
\nabla(\nabla\varepsilon \cdot \mathbf{E}) + \nabla^2\mathbf{E} &= \nabla\mu \times \frac{\partial\mathbf{H}}{\partial t} + \mu\frac{\partial^2 \varepsilon\mathbf{E}}{\partial t^2} \\
\nabla(\nabla\varepsilon \cdot \mathbf{E}) + \nabla^2\mathbf{E} &= \nabla\mu \times \frac{\partial\mathbf{H}}{\partial t} + \mu\varepsilon\frac{\partial^2 \mathbf{E}}{\partial t^2} \\
\nabla^2\mathbf{E} - \mu\varepsilon\frac{\partial^2 \mathbf{E}}{\partial t^2} &= \nabla(\nabla\varepsilon \cdot \mathbf{E}) + \nabla\mu \times \frac{\partial\mathbf{H}}{\partial t}
\end{aligned}$$

Employing this identity onto $\nabla(\nabla\varepsilon \cdot \mathbf{E})$,

$$\begin{aligned}
\nabla(\mathbf{A} \cdot \mathbf{B}) &= (\mathbf{B} \cdot \nabla)\mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{B} \times (\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla \times \mathbf{B}) \\
\nabla(\nabla\varepsilon \cdot \mathbf{E}) &= -(\mathbf{E} \cdot \nabla)\nabla\varepsilon - (\nabla\varepsilon \cdot \nabla)\mathbf{E} - \nabla(\varepsilon) \times (\nabla \times \mathbf{E})
\end{aligned}$$

and resolving the second term,

$$
\begin{aligned}
-\nabla\mu \times \frac{\partial \mathbf{H}}{\partial t} &= \nabla\mu \times \frac{1}{\mu}\frac{\partial \mathbf{B}}{\partial t} \\
&= \nabla\mu \times \frac{1}{\mu}(-\nabla \times \mathbf{E})
\end{aligned}
$$

$$
\begin{aligned}
\nabla^2\mathbf{E} - \mu\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2} &= [-(\mathbf{E}\cdot\nabla)\nabla\varepsilon - (\nabla\varepsilon\cdot\nabla)\mathbf{E} - \nabla(\varepsilon)\times(\nabla\times\mathbf{E})] - \nabla\mu\times(\nabla\times\mathbf{E}) \\
\nabla^2\mathbf{E} - \mu\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2} &= -(\mathbf{E}\cdot\nabla)\nabla\varepsilon - (\nabla\varepsilon\cdot\nabla)\mathbf{E} - \nabla(\varepsilon+\mu)\times(\nabla\times\mathbf{E})
\end{aligned}
$$

Since the dielectric constant is specified as $\varepsilon(z)$, which acts on z-direction only, our equation reduces into

$$
\frac{\partial^2\mathbf{E}}{\partial z^2} - \mu\varepsilon(z)\frac{\partial^2\mathbf{E}}{\partial t^2} = -(\mathbf{E}\cdot\nabla)\varepsilon'\hat{k} - \left(\varepsilon'\frac{\partial}{\partial z}\right)\mathbf{E} - (\varepsilon'+\mu')\hat{k}\times(\nabla\times\mathbf{E})
$$

Since $\mu = 1$, and $\mu' = 0$, the equations becomes

$$
\begin{aligned}
\left(\frac{\partial^2}{\partial z^2} - \varepsilon\frac{\partial^2}{\partial t^2}\right)E_x &= \mu'\frac{\partial E_x}{\partial z} = 0 \\
\left(\frac{\partial^2}{\partial z^2} - \varepsilon\frac{\partial^2}{\partial t^2}\right)E_y &= \mu'\frac{\partial E_y}{\partial z} = 0 \\
\left(\frac{\partial^2}{\partial z^2} - \varepsilon\frac{\partial^2}{\partial t^2}\right)E_z &= -\varepsilon' E_z' - \varepsilon'' E_z.
\end{aligned}
$$

We assume a solution of

$$
\mathbf{E}(z,t) = \mathbf{E}(z)e^{i\omega t} \tag{2.1}
$$

$$
\begin{aligned}
\left(\frac{\partial^2}{\partial z^2} - \mu\varepsilon\omega^2\right)\mathbf{E}(z) &= \mu'\frac{\partial E_x}{\partial z} = 0 \\
\left(\frac{\partial^2}{\partial z^2} - \mu\varepsilon\omega^2\right)\mathbf{E}(z) &= \mu'\frac{\partial E_y}{\partial z} = 0
\end{aligned}
$$