The Fourier-Slice Theorem

Take the 1-D Fourier Transform of the Radon transform with respect to ρ,

$$G(\omega,\theta) = \int_{-\infty}^{+\infty} g(\rho,\theta)e^{-i2\pi\omega\rho}\,d\rho \qquad (1)$$

Expanding $g(\rho,\theta)$ we get

$$G(\omega,\ \theta) \ = \ \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\delta(x\cos\theta + y\sin\theta - \rho)e^{-j2\pi\omega\rho}\,dxdyd\rho \qquad (2)$$

$$G(\omega,\ \theta) \ = \ \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\left[\int_{-\infty}^{+\infty}\delta(x\cos\theta + y\sin\theta - \rho)e^{-j2\pi\omega\rho}\,d\rho\right]dxdy \qquad (3)$$

$$G(\omega,\ \theta) \ = \ \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)e^{-j2\pi\omega(x\cos\theta + y\sin\theta)}\,dxdy \qquad (4)$$

Letting $u \ = \omega\cos\theta$ and $v \ = \omega\sin\theta$ we get

$$G(\omega,\theta) \ = \ \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)e^{-j2\pi(ux+vy)}\,dxdy \qquad (5)$$

which is just the Fourier transform of $f(x,y)$, that is

$$G(\omega,\theta) = F(u,v) = F(\omega\cos\theta\,,\ \omega\sin\theta) \qquad (6)$$

This means, the Fourier transform of the projection which is $G(\omega,\theta)$ is the same as the Fourier transform of the object along the rotated frequency axis $(\omega\cos\theta\,,\ \omega\sin\theta)$ as illustrated in Figure 1 below.

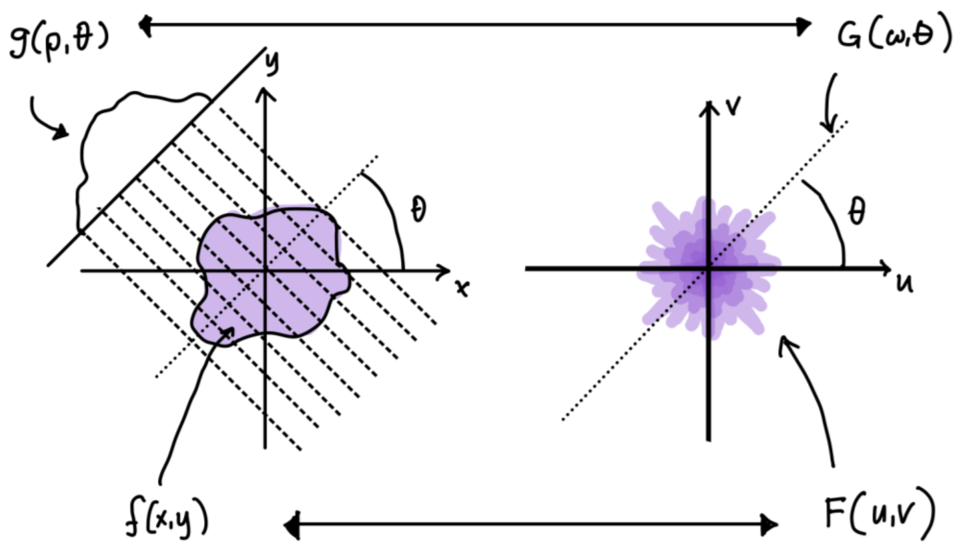

Figure 1. Illustration of the Fourier-Slice Theorem.

Reconstruction Using Parallel-Beam Filtered Back-projections

From Equation 6 if we get its inverse FT we should be able to recover f(x,y) since

$$f(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(u,v)e^{j2\pi(ux+vy)}\,dudv \qquad (7)$$

Given that $u = \omega \cos\theta$ and $v = \omega \sin\theta$,the area element $dudv$ is equal to $\omega d\omega d\theta$ in polar coordinates.

Thus expressing Equation 7 in terms of $\omega$and θ we get

$$f(x,y) = \int_0^{2\pi} \int_{-\infty}^{\infty} F(\omega\cos\theta\,, \omega\sin\theta) e^{j2\pi\omega(x\cos\theta + y\sin\theta)}\, \omega d\omega \mathrm{d}\theta \qquad (8)$$

And from the Fourier-Slice Theorem we get

$$f(x,y) = \int_0^{2\pi} \int_{-\infty}^{\infty} G(\omega,\theta) e^{j2\pi\omega(x\cos\theta + y\sin\theta)}\, \omega d\omega \mathrm{d}\theta \quad (9)$$

We can split the integral over θ to range from $0\ to\ \pi$ and $\pi\ to\ 2\pi$. Since $G(\omega,\ \theta + \pi) = G(-\omega, \theta)$ we can express Equation 9 as

$$f(x,y) = \int_0^{\pi} \int_{-\infty}^{+\infty} |\omega|\ G(\omega,\ \theta) e^{j2\pi\omega(x\cos\theta + y\sin\theta)} d\omega d\theta \qquad (10)$$

Now with $\rho = x\cos\theta + y\sin\theta$ Equation (10) can be written as

$$f(x,y) = \int_0^{\pi} \left[\int_{-\infty}^{+\infty} |\omega| G(\omega,\rho) e^{j2\pi\omega\rho}\, d\omega \right] d\theta. \qquad (11)$$

The bracketed term is not integrable because of the shape of $|\omega|$. It is a symmetric ramp extending to infinity. But if we “apodize” $|\omega|$ by multiplying it by a window function whose ends fall off to zero, we can “band-limit” $|\omega|$!

Thus the steps to recovering f(x,y) are

1. Compute the FT of each projection.
2. Multiply each FT by $|\omega|$and a windowing function.
3. Obtain the inverse 1-D transform of each filtered transform.
4. Integrate or sum up all the 1D inverse FT.

Activity

The filtered backprojection in the inverse Radon transform can be set by specifying a filter in the iradon argument.

1. Get the inverse Radon transform of your synthetic images and compare the output for different filter functions.

Reference

Gonzales and Woods, Digital Image Processing, 3rd Ed. Chapter 5.11.4