METHODS AND APPLICATIONS OF WHITE NOISE ANALYSIS

1st Semester, AY 2023-2024

Reinabelle Reyes

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Written Report

(Due date: *January 4, 2024*)

Write a Report (8 – 12 pages) on data accessible to you which could be analyzed as possibly exhibiting a stochastic process with memory. The Report should contain:

(1) Background/description of source of data (*Do not go beyond 4 pages, but figures and pictures would help. Raw data may be attached as Appendix if not too long, but excluded from the 12 pages*).

(2) A plot of the Displacement Probability Distribution based on the data.

(3) A plot of the MSD (could be log-log plot) based on your empirical data.

(4) From the references (book/papers), propose a candidate memory function by comparing/matching empirical and theoretical MSD *(just try your best).*

(5) Conclusion: remarks, thoughts, or plans.

(6) References (*Journals, books, or websites*).

Preliminary Report: Proposed Datasets

(Due date: *September 21, 2023*)

Write a preliminary report (2-4 pages) describing 2-3 different datasets accessible to you which could be analyzed as possibly exhibiting a stochastic process with memory.

The Report should contain:

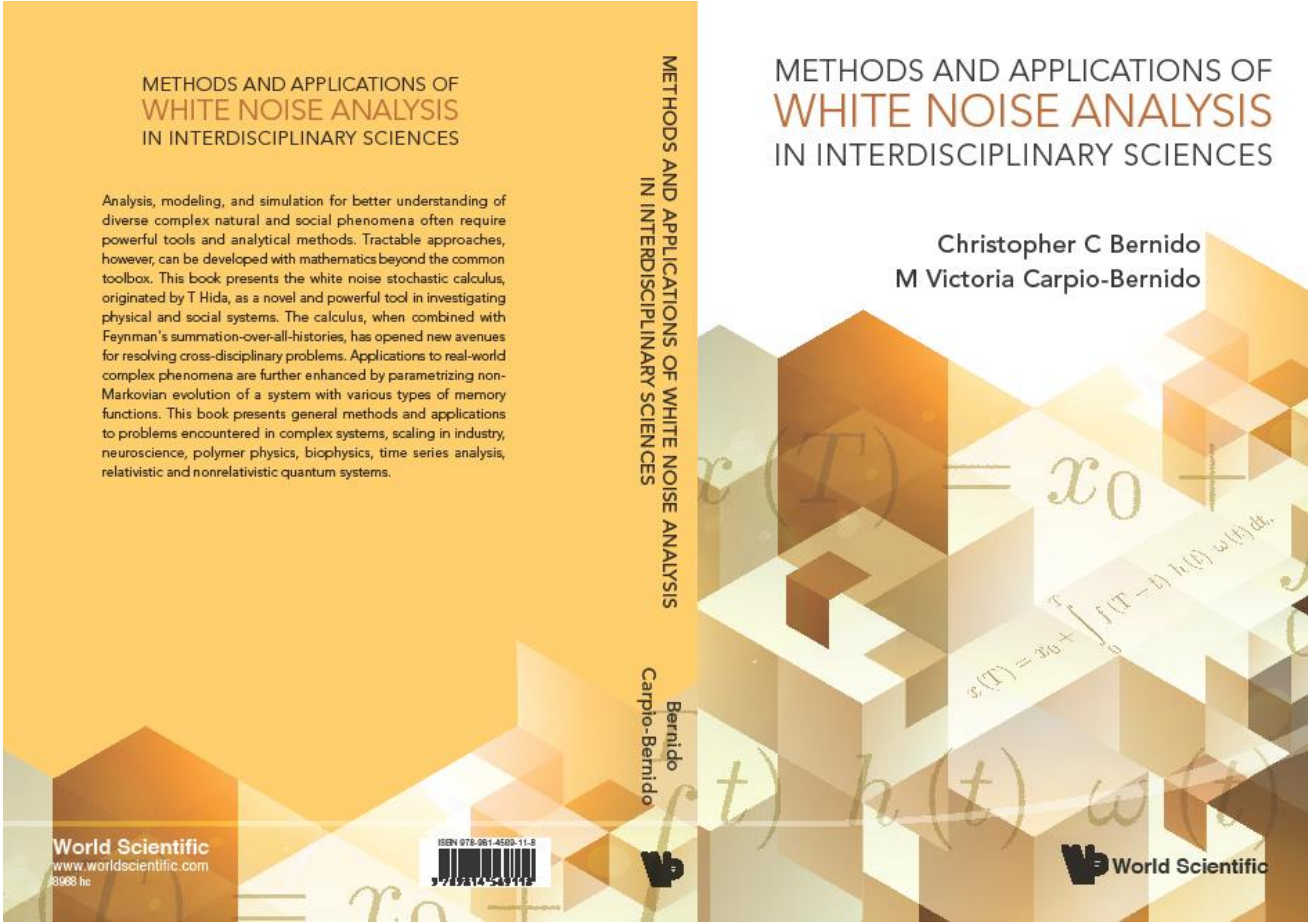
(1) List of 2-3 datasets (Indicate which one is 1st choice, 2nd choice, etc.)

(2) Background/description of dataset (*Figures and pictures would help*).

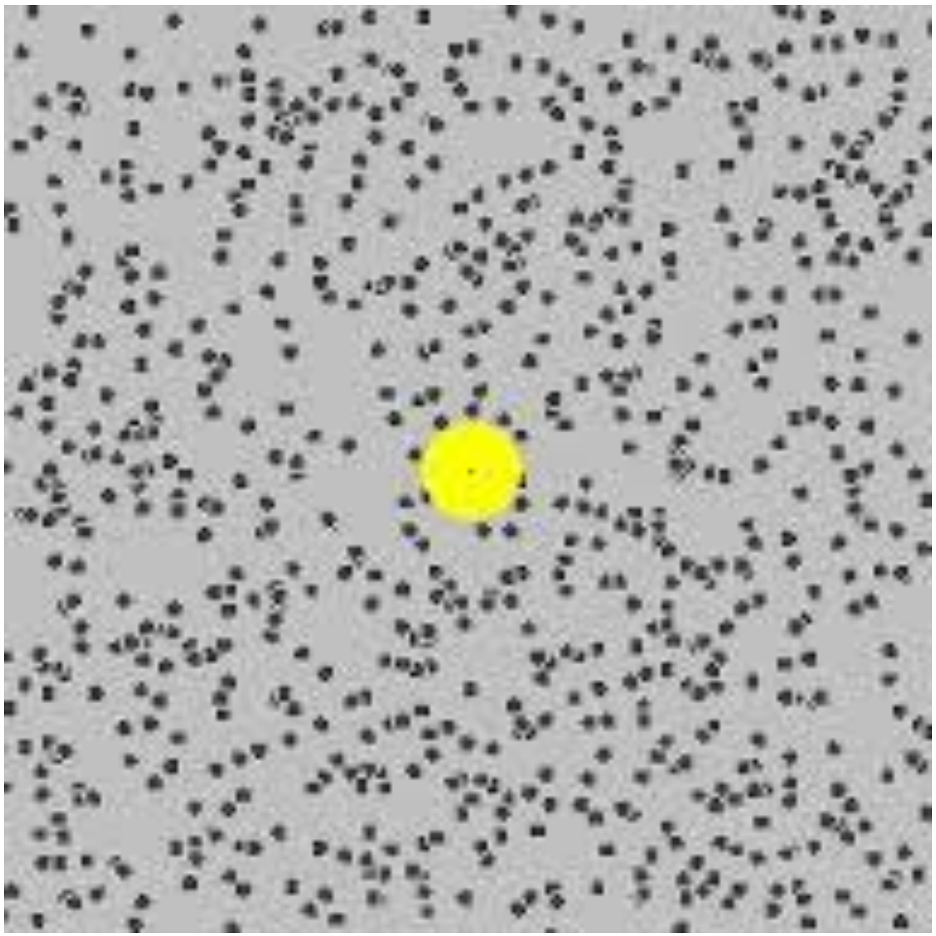
(3) Indicate source of dataset (*Indicate URL if applicable*)

(4) List at least 2 references per dataset (*Journals, books, or websites; must include peer-reviewed journal article published within the last 2 years*).

Reference for the Course (*available at Amazon and World Scientific*):



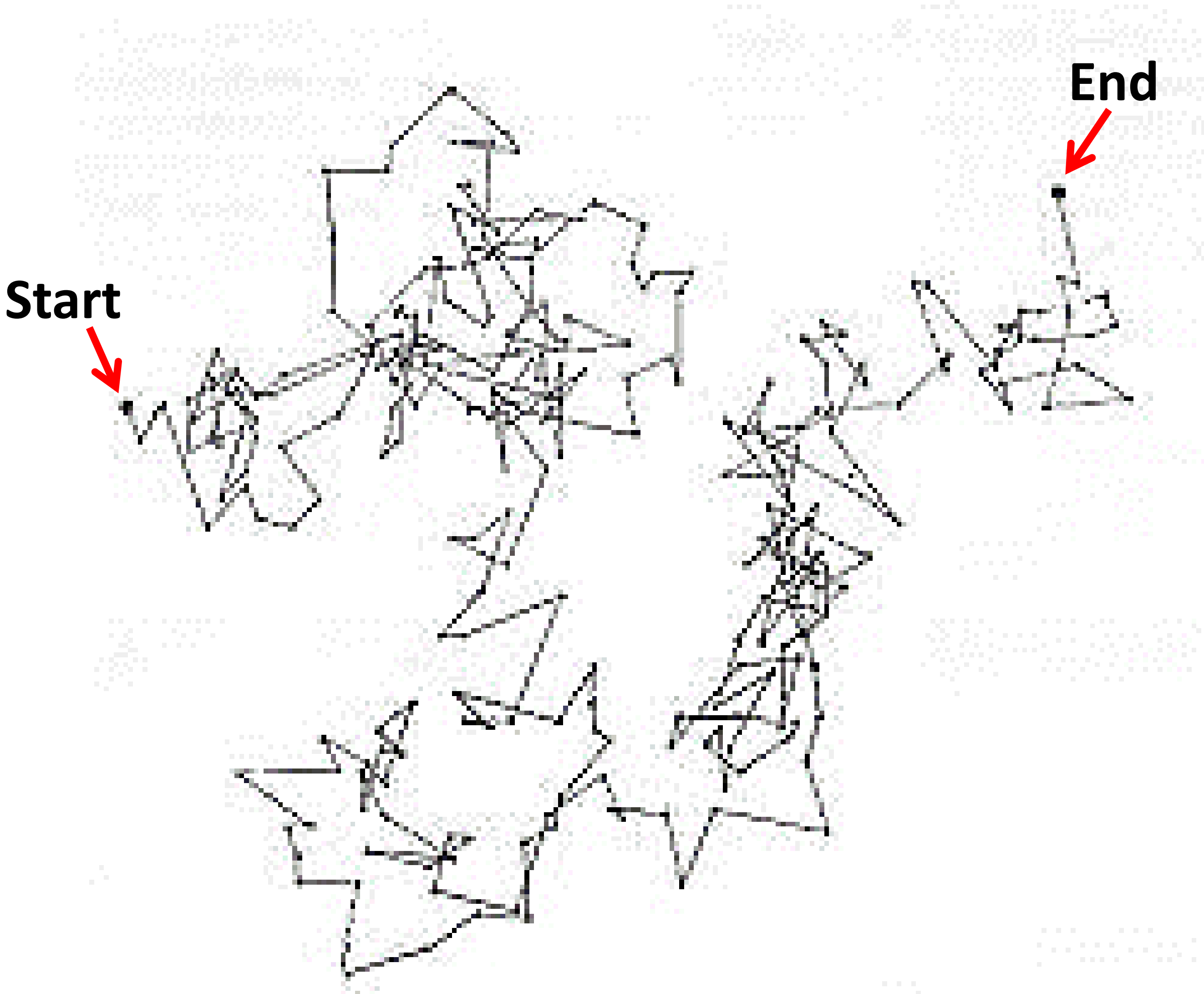
1827: The botanist Robert Brown observed a jittery motion of a particle in a fluid.



Particle in a Fluid

https://www.google.com.ph/search?q=Images+Brownian+motion&tbm=isch&imgil=tPFFqxL-3jc8kM%253A%253BYSbYechQHNVs2M%253Bhttp%25253A%25252F%25252Fwww.tutorvista.com%25252Fcontent%25252Fphysics%25252Fphysics-i%25252Fmatter%25252Fbrownian-motion.php&source=iu&usg=__3NI8UiQQxifPTCMez8FfN6iS2Hc%3D&sa=X&ei=EifkU_PPE4bd8AWq24GIAw&ved=0CBwQ9QEwAA&biw=1093&bih=468#facrc=_&imgdii=_&imgrc=tPFFqxL-3jc8kM%253A%3BYSbYechQHNVs2M%3Bhttp%253A%252F%252Fimages.tutorvista.com%252Fcontent%252Fmatter%252Fbrownian-motion.jpeg%3Bhttp%253A%252F%252Fwww.tutorvista.com%252Fcontent%252Fphysics%252Fphysics-i%252Fmatter%252Fbrownian-motion.php%3B430%3B305

1827: R. Brown observed jittery motion of particle in fluid.



https://www.google.com.ph/search?q=Images+Brownian+motion&tbm=isch&imgil=tPFFqxL-3jc8kM%253A%253BYSbYechQHNVs2M%253Bhttp%25253A%25252F%25252Fwww.tutorvista.com%25252Fcontent%25252Fphysics%25252Fphysics-i%25252Fmatter%25252Fbrownian-motion.php&source=iu&usg=__3NI8UiQQxifPTCMez8FfN6iS2Hc%3D&sa=X&ei=EifkU_PPE4bd8AWq24GIAw&ved=0CBwQ9QEwAA&biw=1093&bih=468#facrc=_&imgdii=_&imgrc=tPFFqxL-3jc8kM%253A%3BYSbYechQHNVs2M%3Bhttp%253A%252F%252Fimages.tutorvista.com%252Fcontent%252Fmatter%252Fbrownian-motion.jpeg%3Bhttp%253A%252F%252Fwww.tutorvista.com%252Fcontent%252Fphysics%252Fphysics-i%252Fmatter%252Fbrownian-motion.php%3B430%3B305

Theory on Brownian Motion

(1905 Doctoral Thesis)

https://www.pinterest.com/pin/333688653613483844/

NO MEMORY OF THE PAST

http://www.riskencyclopedia.com/articles/brownian_motion/

Pictures of Louis Bachelier - MacTutor History of Mathematics (st-andrews.ac.uk)

Louis Bachelier

French Mathematician

1900: PhD Thesis: *The Theory of Speculation*

Stock Prices are subjected to a random movement

Birth of Mathematical Finance

Aged 15

He is credited as the first person to model the stochastic process now called **Brownian motion.**

http://en.wikipedia.org/wiki/Louis_Bachelier

Correspondence between Finite and Infinite Dimensions:

FINITE DIMENSIONS	INFINITE DIMENSION
Independent variable: x_j	***Independent random variable***: $\omega(t)$
Coordinate system: $(x_1, \dots, x_n)$	***Coordinate system***: $\{\omega(t); t \in \mathbf{R}\}$
Function: $f(x_1, \dots, x_n)$	***Functional***: $\Phi(\omega(t);\ t \in \mathbf{R})$
Space: R^n	***Space of Hida distributions***: S^*
Lebesgue measure: dx	***Gaussian measure***: $d\mu(\omega)$

White Noise Analysis works with the Gelfand triple: $S \subset L^2 \subset S^*$

Space of test functions: S

Hilbert space of square integrable functions: L^2

T. Hida, H. H. Kuo, J. Potthoff, and L. Streit, *White Noise. An Infinite Dimensional Calculus* (Kluwer, 1993)

Get the *T*-transform of the following:

(1) $\Phi = 1$

(2) $\Phi(\omega) = \exp(i\langle\omega,\eta\rangle)$

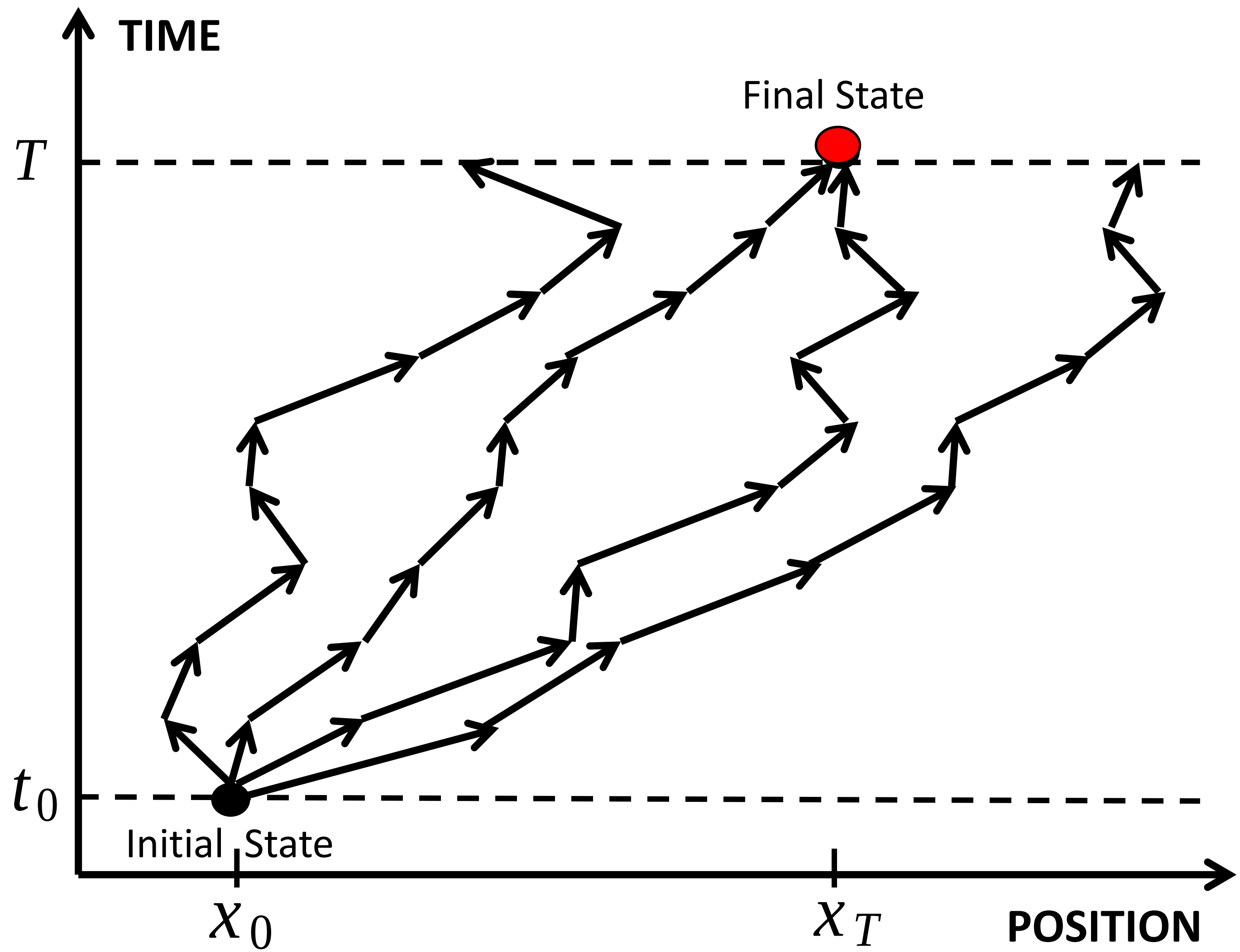
(3) $\Phi(\omega) = \exp\left(-i\langle\omega,\eta\rangle\sqrt{2}\,y\right)$

(4) Express your answer in number (3) in terms of the Hermite polynomials, $H_k(x)$. Take the norm, $\int \eta^2 d\tau = 1$.

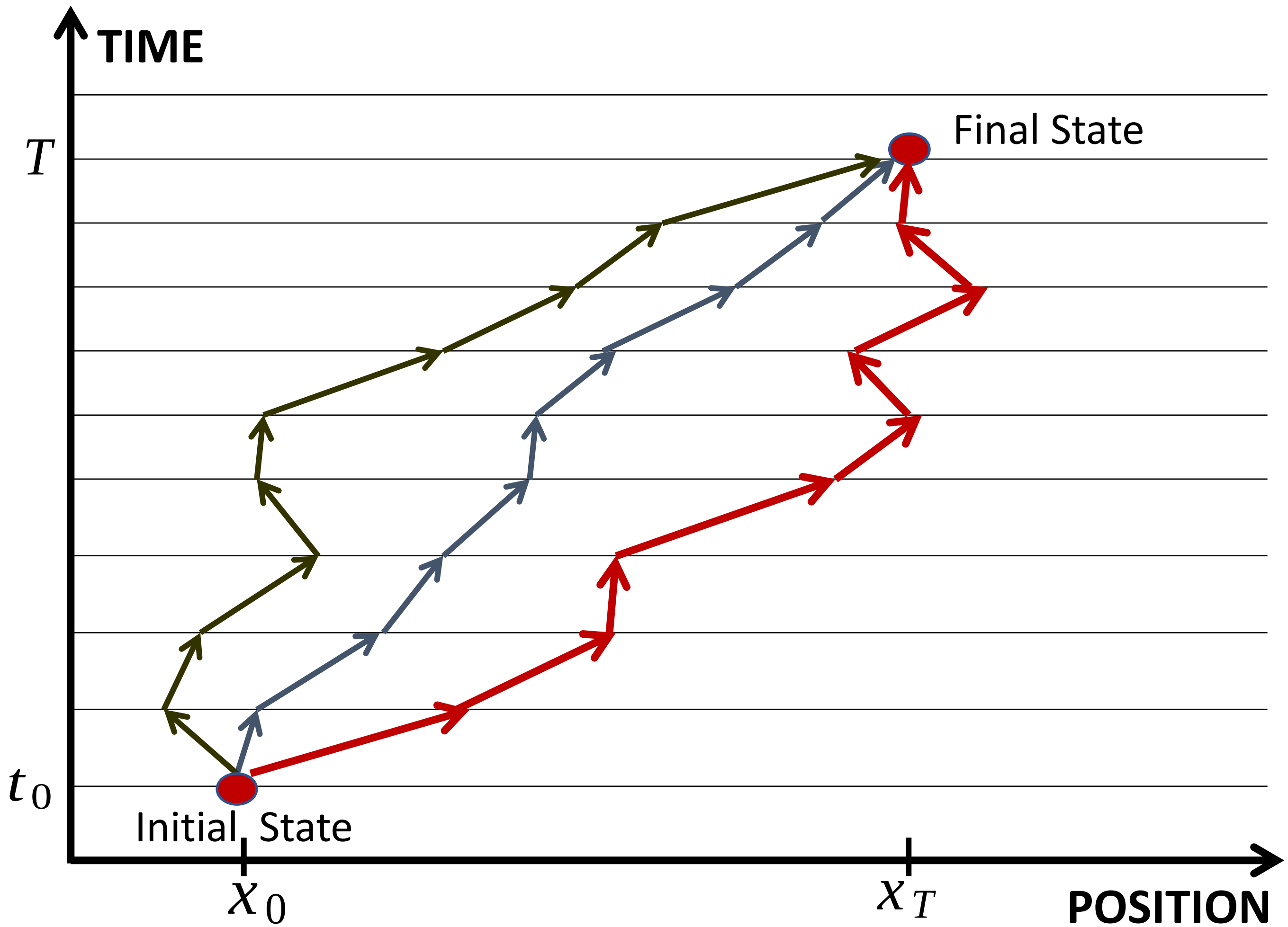
Note: The generating function for Hermite polynomials is:

$$\exp\left[\sqrt{2}\,y\,\langle\xi,\eta\rangle - y^2\right] = \sum_{k=0}^{\infty} \frac{y^k}{k!} H_k\left(\langle\xi,\eta\rangle/\sqrt{2}\right)$$

$$x(T) = x_0 + \sqrt{2D}\, B(T)$$



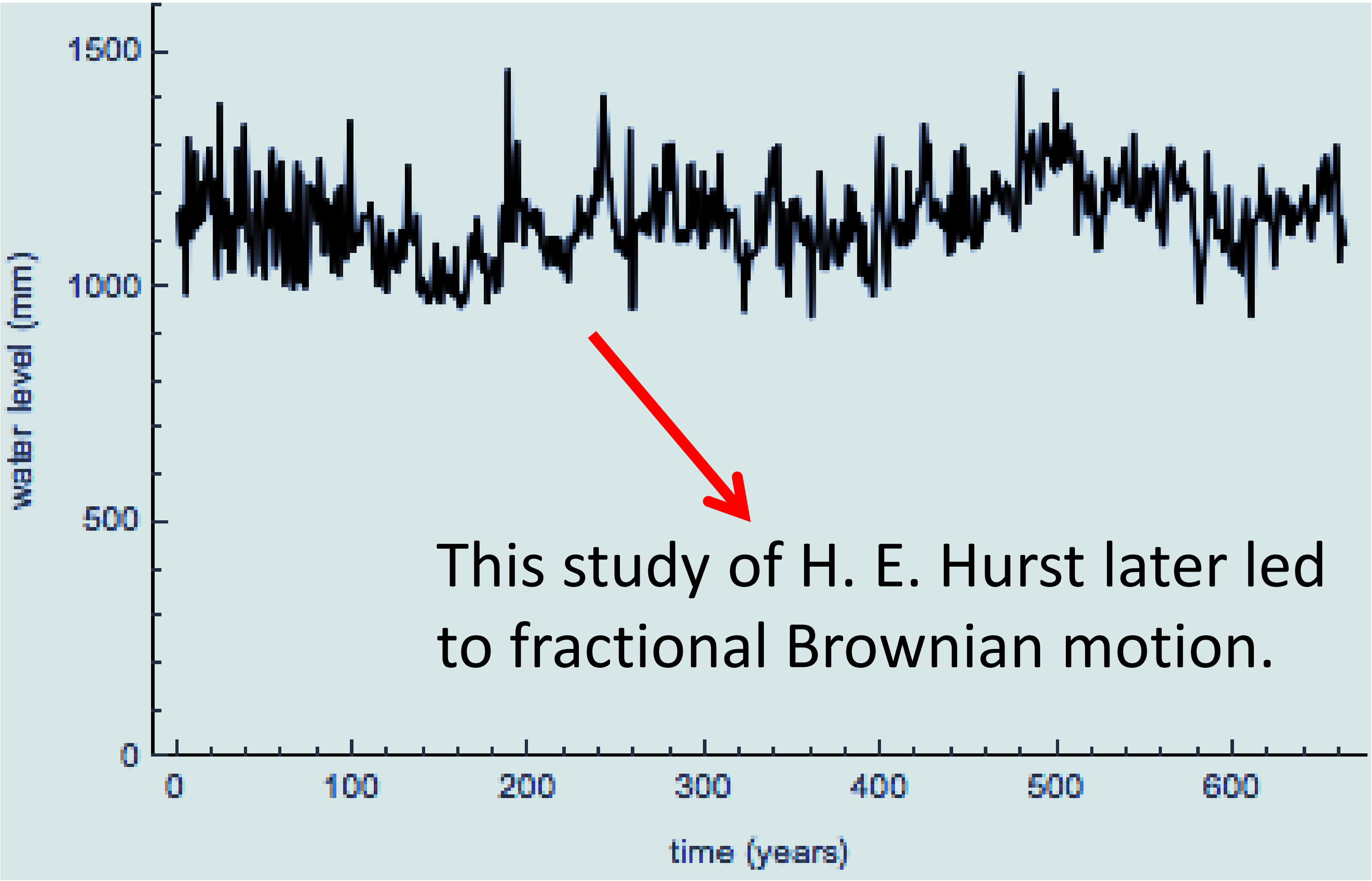
Feynman's Sum-Over-All Paths



$$\delta(x(T) - x_T)$$

TIME SERIES

Example of Big Data: **Nile River**



Graph courtesy of M. G. O. Escobido

Let $B(t)$ be the ordinary Brownian Motion.

Fractional Brownian Motion:

$$B^H(T) = \frac{1}{\Gamma\left(H + \frac{1}{2}\right)} \int_0^T (T - t)^{H-(1/2)} dB(t)$$

Riemann-Liouville representation

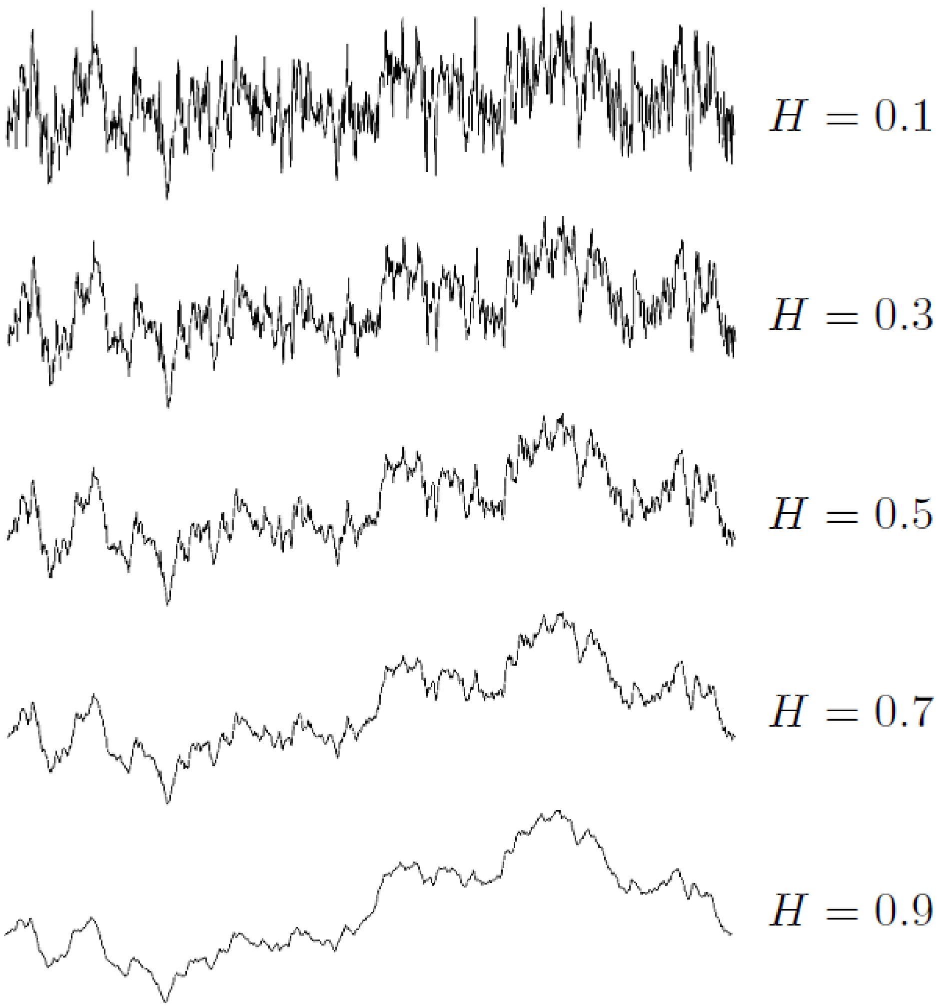
H **is the Hurst Exponent**

$0 < H < ½$: *Subdiffusion*

$½ < H < 1$: *Superdiffusion*

$H = ½$: *Normal Diffusion*

What is Memory?



Short Memory Domain
(negatively correlated)

No Memory ;
(*Brownian Motion*)

Long Memory Domain
(positively correlated)

Graphs by S. Shevchenko
Jagna Workshop (2014)

Fractional Brownian Motion:

$$x(T) = x_0 + \frac{1}{\Gamma(H + 1/2)} \int_0^T (T - t)^{H - \frac{1}{2}} \, dB(t)$$

Biophysical Journal Vol. **103** (2012)1839– 1847

Universal Algorithm for Identification of Fractional Brownian Motion. A Case of Telomere Subdiffusion

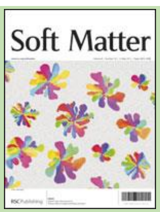
K. Burnecki, E. Kepten, J. Janczura, I. Bronshtein, Y. Garini, and A. Weron

MATHEMATICAL FINANCE

An International Journal of Mathematics, Statistics and Financial Economics

NO ARBITRAGE UNDER TRANSACTION COSTS, WITH FRACTIONAL BROWNIAN MOTION AND BEYOND

Paolo Guasoni, *Mathematical Finance* **16** (2006) 569-582.



Fractional Brownian motion in crowded fluids

D. Ernst, M. Hellmann, J. Köhler, and M. Weiss

Soft Matter (2012)

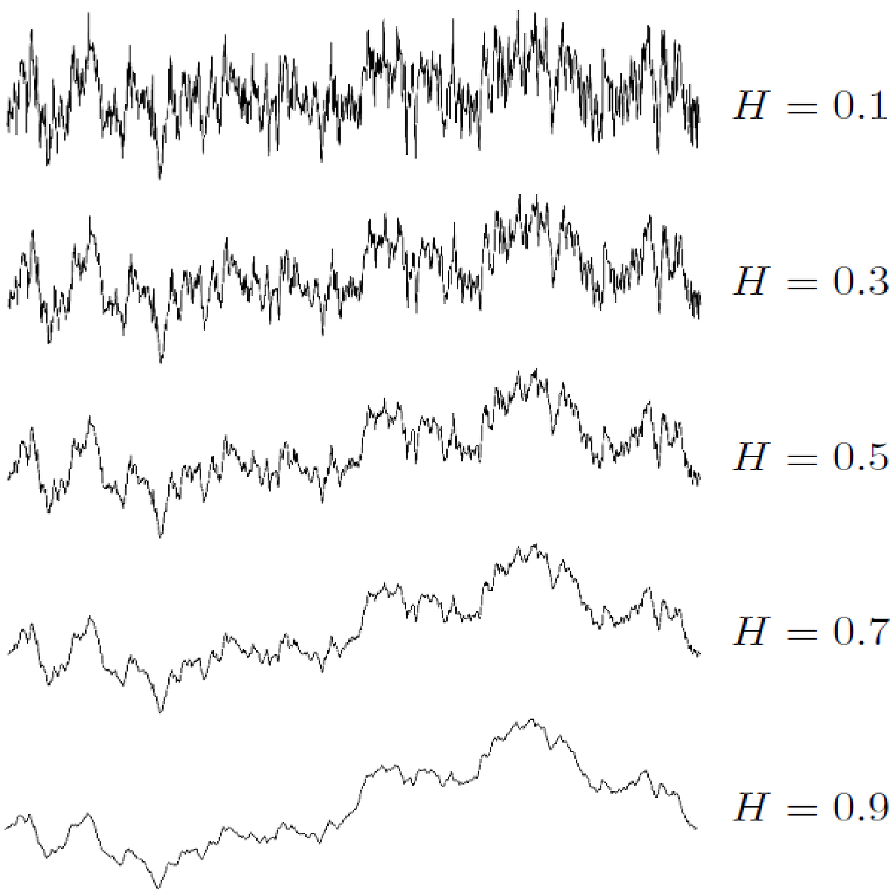
A note on the use of fractional Brownian motion for financial modeling

S. Rostek and R. Schöbel ,

Economic Modelling **30** (2013) 30-35.

We get the PDF for **fractional Brownian motion**: $0 < H < 1$

$$P(x_T, T; x_0, 0) = \sqrt{\frac{H\,\Gamma^2\big(H + (1/2)\big)}{\pi\, T^{2H}}}\; exp\left\{-\frac{H\,\Gamma^2\big(H + (1/2)\big)(x_0 - x_T)^2}{T^{2H}}\right\}$$



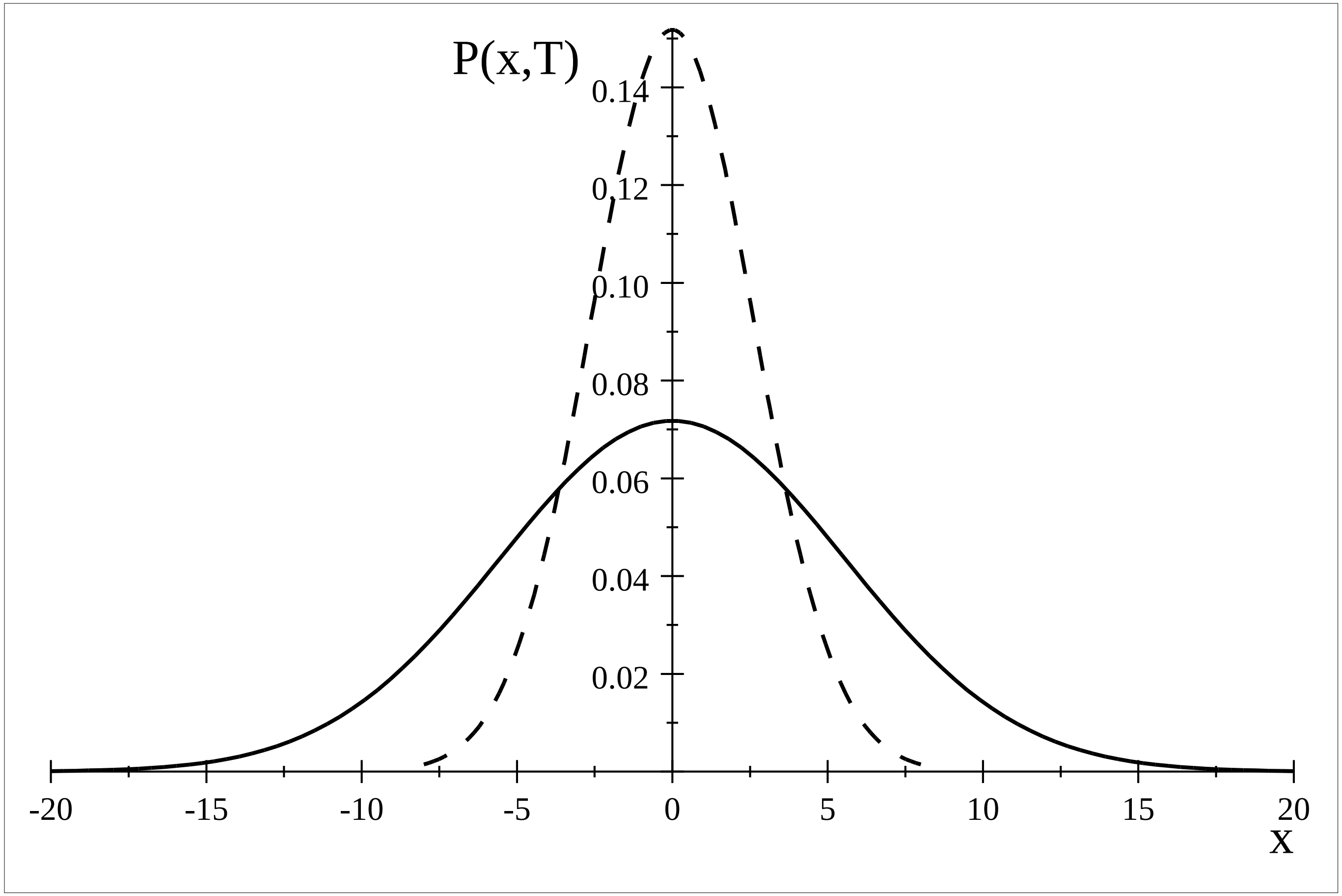
Subdiffusion
(negatively correlated)

No Memory ;
(*Brownian Motion*)

Superdiffusion
(positively correlated)

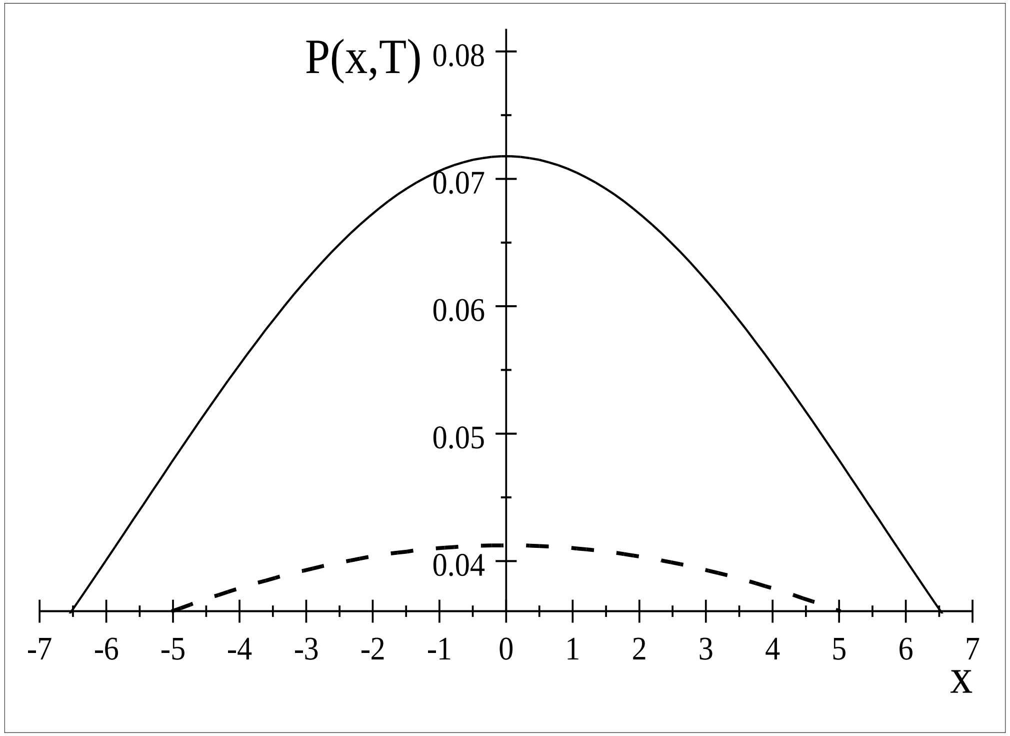
Graphs by S. Shevchenko
Jagna Workshop (2014)

FRACTIONAL BROWNIAN MOTION

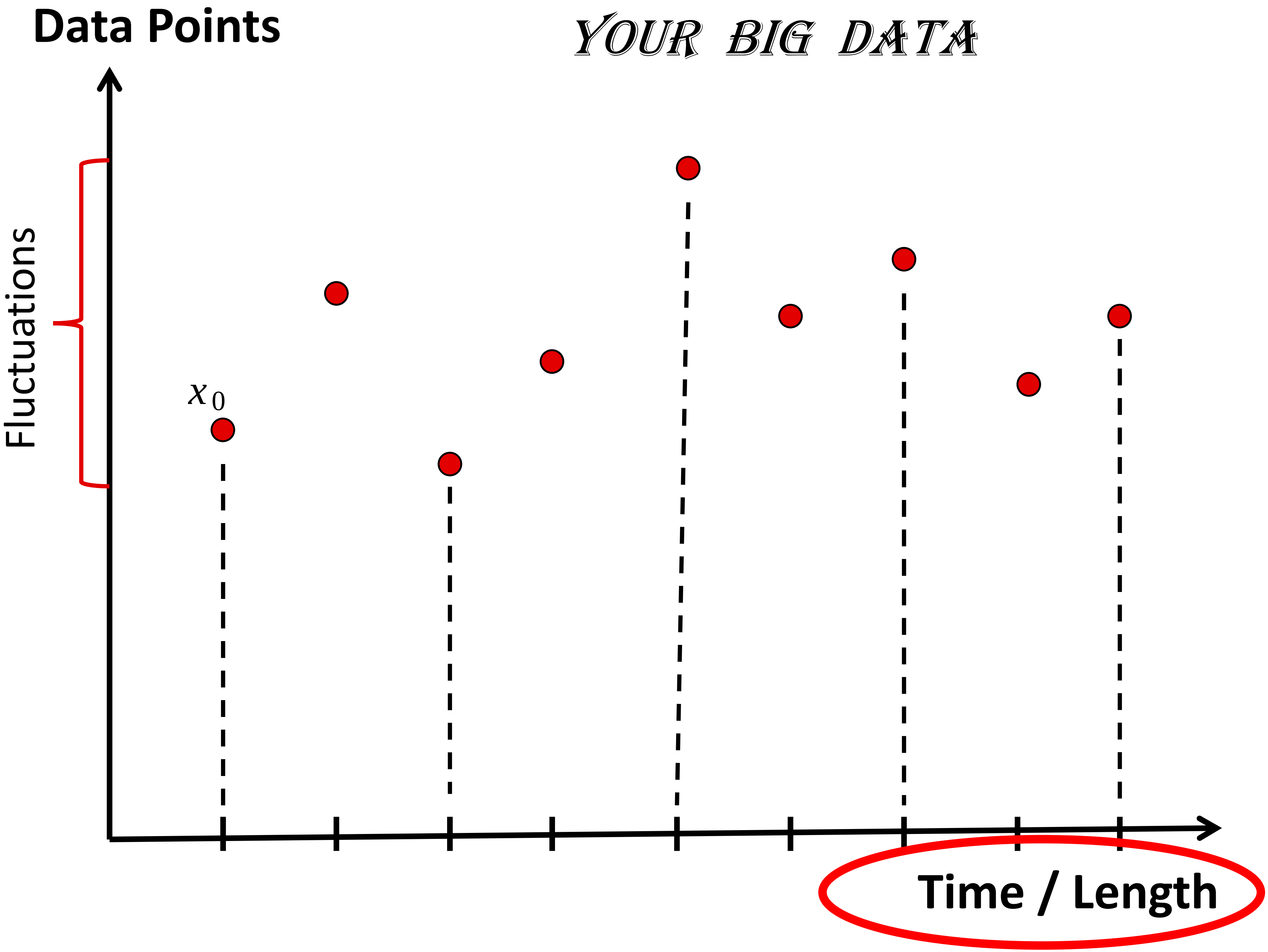


Plots for fractional Brownian motion for fixed time T = 10. Solid line: H = 0.8 ; Dash line: H = 0.4

FRACTIONAL BROWNIAN MOTION:



Behavior of fractional Brownian motion as time increases for Hurst exponent, H = 0.8. Time T = 10 (solid line); T = 20 (dash line).



Data Points
YOUR BIG DATA
Fluctuations
x_0
Time / Length

Parametrizing Fluctuating Observables

$$x(T) = x_0 + Fluctuations$$

$$= \underset{\text{initial value at time } t=0}{x_0} + g(T) \int_0^T \underbrace{f(T-t)}_{\textbf{\textit{Memory Function}}} h(t) \, \underbrace{dB(t)}_{\textbf{\textit{Ordinary Brownian Motion}}}$$

initial value
at time $t = 0$

Memory Function

Ordinary Brownian Motion

where $x(T)$ is a fluctuating variable representing:

- *positions of probe particle in complex fluid*
- *geomagnetic fluctuations*
- *DNA distances in a genome*
- *rise and fall of stock prices*
- *positions of typhoons*
- *atmospheric* CO_2 *levels , etc ...*

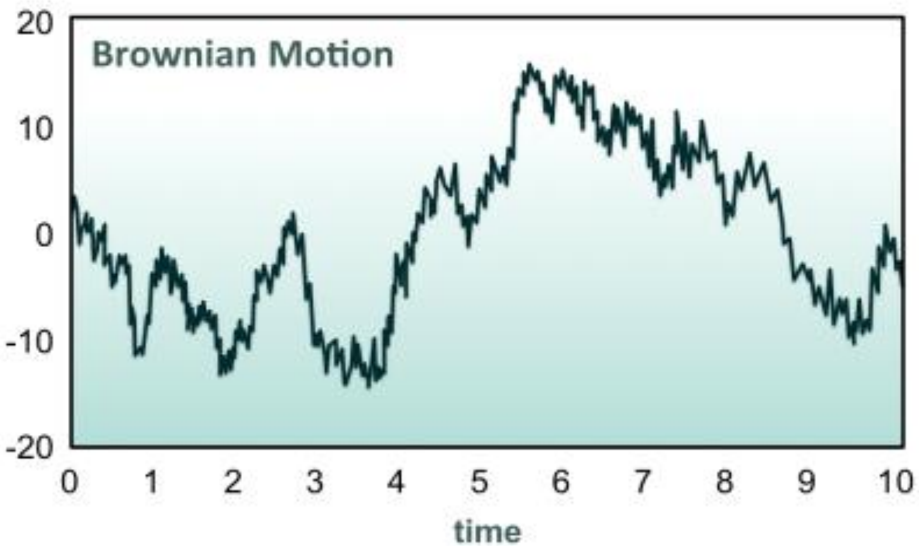
Observable variable: $$\boxed{x(T) = x_0 + g(T) \int_0^T f(T-t)\, h(t)\, dB(t)}$$

Example 1: Choosing, $g(T) = f(T-t) = h(t) = 1$

$$x(T) = x_0 + \int_0^T dB(t)$$

$$= x_0 + B(T)$$

We get **ordinary Brownian motion** $B(T)$.



NO MEMORY OF THE PAST

http://www.riskencyclopedia.com/articles/brownian_motion/

Observable variable: $$\boxed{x(T) = x_0 + g(T) \int_0^T f(T-t)\, h(t)\, dB(t)}$$

Example 2:

Choosing, $g(T) = h(t) = 1$, and $f(T-t) = \dfrac{(T-t)^{H-(1/2)}}{\Gamma\left(H + \frac{1}{2}\right)}$

We have **Fractional Brownian Motion:**

$$\boxed{B^H(T) = \frac{1}{\Gamma\left(H + \frac{1}{2}\right)} \int_0^T (T-t)^{H-(1/2)} dB(t)}$$

Riemann-Liouville representation

H is the Hurst Exponent

$0 < H < ½$: *Subdiffusion*

$½ < H < 1$: *Superdiffusion*

$H = ½$: *Normal Diffusion*

Paths parametrized as:

$$x(T) = x_0 + Fluctuations$$

$$= x_0 + g(T) \int_0^T \underbrace{f(T-t)}_{\textbf{\textit{Memory Function}}} h(t)\, \underbrace{dB(t)}_{\textbf{\textit{Ordinary Brownian Motion}}}$$

initial value at time $t = 0$

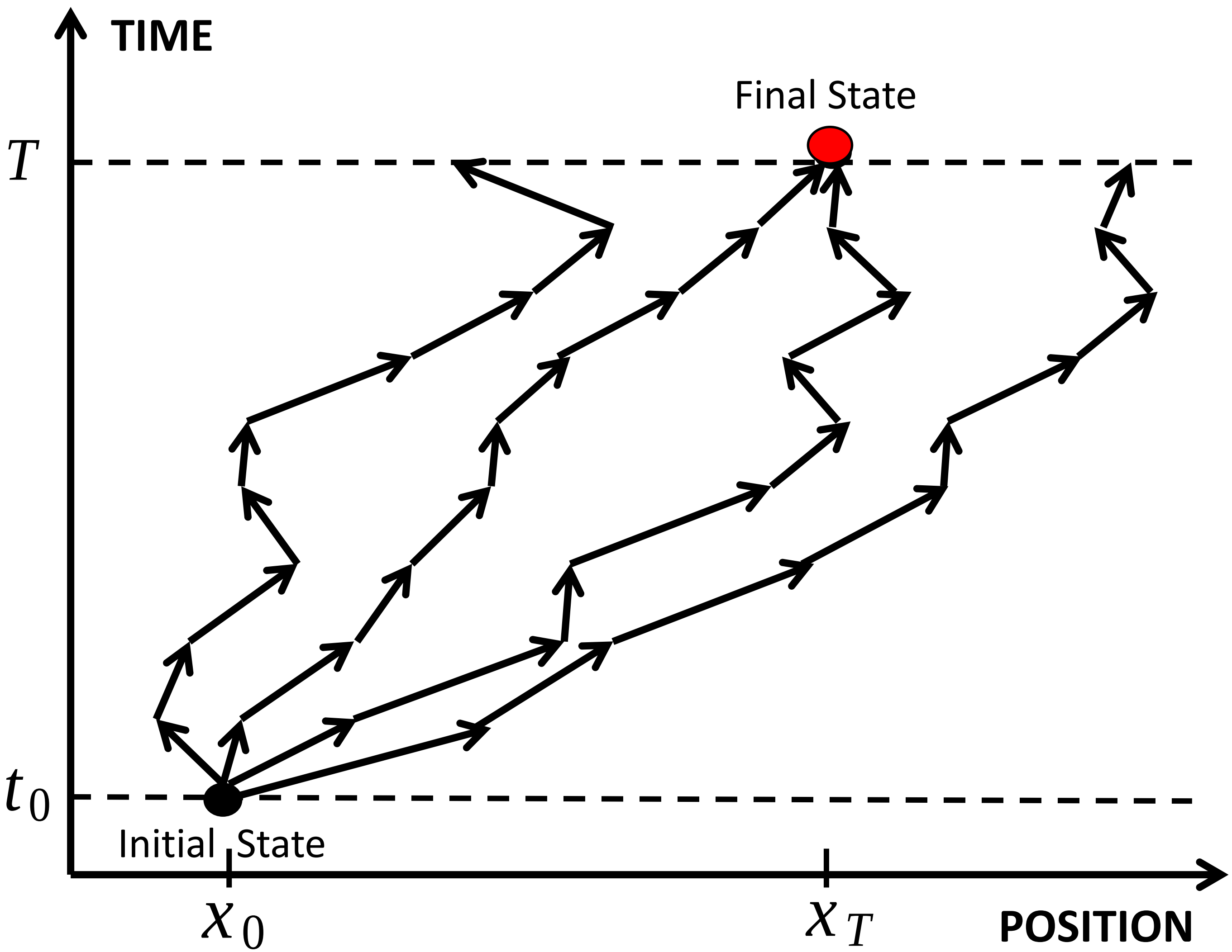
Memory Function

Ordinary Brownian Motion

How do we solve for the Probability Density Function $P(x_T, T; x_0, 0)$?

NOTE: Initial point x_0 is fixed, but **endpoint** could be ***anywhere*** !

$$x(T) = x_0 + g(T) \int_0^T f(T-t)\, h(t)\, dB(t)$$



To pin down the endpoint at $x(T) = x_T$, at time $t = T$, we consider the delta function constraint:

$$\delta(x(T) - x_T) = \delta\left(\underbrace{x_0 + g(T)\int_0^T f(T-t)\,h(t)\,dB(t)}_{x(T)} - x_T\right)$$

Note:

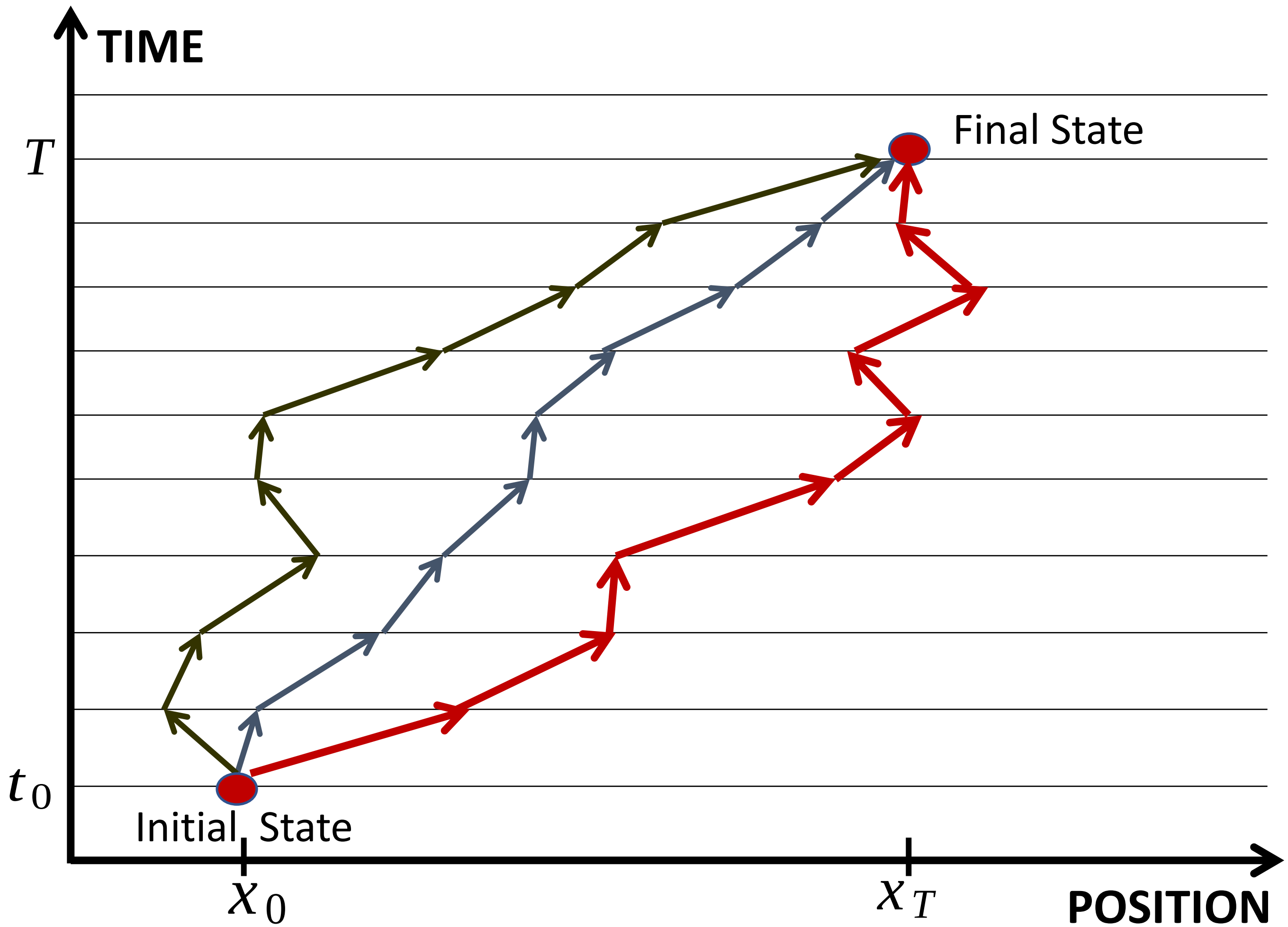
$$\delta(x(T) - x_T) = \delta\left(x_0 + g(T)\int_0^T f(T-t)\,h(t)\,\omega(t)dt - x_T\right)$$

$dB(t)$

*Fluctuating paths **not** ending at x_T will vanish*:

$$\boxed{\delta(x) = 0 \quad , \quad if \;\; x \neq 0\,.}$$

$$\delta(x(T) - x_T)$$



Summing-Over-All Paths

What is the Expectation Value of $\delta(x(T) - x_T)$?

The Probability Density Function (PDF), $P(x_T, T; x_0, 0)$, is given by the Expectation Value of $\delta(x(T) - x_T)$:

$$P(x_T, T; x_0, 0) = E\Big(\delta(x(T) - x_T)\Big)$$

$$= \int \delta(x(T) - x_T)\, d\mu$$

$$x(T) = x_0 + g(T) \int_0^T f(T-t)\, h(t)\, \omega(t) dt$$

NOTE: $\delta(x(T) - x_T) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} exp[ik(x(T) - x_T)]\, dk$

Fourier representation of the delta function

We have,

$$P(x_T, T; x_0, 0) = \int \delta(x(T) - x_T)\, d\mu \quad = \frac{1}{2\pi} \int \int_{-\infty}^{+\infty} exp[ik(\underline{x(T)} - x_T)]\, dk\ \, d\mu$$

Writing $x(T)$ explicitly,

$$P(x_T, T; x_0, 0) = \frac{1}{2\pi} \int \int_{-\infty}^{+\infty} exp\left[ik\left(\overbrace{x_0 + g(T) \int_0^T f(T-t)\, h(t)\, \omega(t) dt}^{x(T)} - x_T\right)\right] dk\ \, d\mu$$

Rearranging,

$$P(x_T, T; x_0, 0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk\, exp[ik(x_0 - x_T)]$$

$$\times \boxed{\int exp\left[ikg(T) \int_0^T f(T-t)\, h(t)\, \omega(t) dt\right] d\mu}$$

Integration over $d\mu$

Integration over $d\mu$: $\int exp\left[ikg(T)\int_0^T f(T-t)\,h(t)\,\omega(t)dt\right]d\mu$

Recall: Characteristic Functional $C(\xi)$,

$$C(\xi) = \int exp(i\langle\omega,\xi\rangle)\,d\mu(\omega)$$

$$= exp\left(-\frac{1}{2}\int_0^\tau \xi^2\,dt\right)$$

Let:

$\xi = k\,g(T)f(T-t)\,h(t)$

$$P(x_T, T; x_0, 0)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} exp\left\{ik(x_0 - x_T) - \frac{1}{2}k^2 g(T)^2 \int_0^T [f(T-t)\,h(t)]^2 dt\right\} \underline{dk}$$

T. Hida, H. H. Kuo, J. Potthoff, and L. Streit, *White Noise. An Infinite Dimensional Calculus* (Kluwer,1993) ;

C. C. Bernido and M. V. Carpio-Bernido, *Methods and Applications of White Noise Analysis in Interdisciplinary Sciences* (World Scientific, 2014).

The Probability Density Function (PDF) becomes:

$$P(x_T, T; x_0, 0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} exp\left\{ik(x_0 - x_T) - \frac{1}{2}k^2 g(T)^2 \int_0^T [f(T-t)\, h(t)]^2 dt\right\} dk$$

The integral over dk is a Gaussian integral:

$$\int_{-\infty}^{+\infty} exp(\pm qk - p^2 k^2)\, dk = \frac{\sqrt{\pi}}{p} exp(q^2/4p^2)$$

$$(for\ p > 0)$$

Equation (3.323.2) of
I. S. Gradshteyn and I. M. Ryzhik,
Table of Integrals, Series and Products
5th ed. (Academic Press, 1994).

PDF for a wide class of memory behavior:

$$P(x_T, T; x_0, 0) = \left(2\pi g(T)^2 \int_0^T [f(T-t)\, h(t)]^2 dt\right)^{-\frac{1}{2}}$$

$$\times\, exp\left(-\left[g(T)^2 \int_0^T [f(T-t)\, h(t)]^2 dt\right]^{-1} \frac{(x_T - x_0)^2}{2}\right)$$

C. C. Bernido and M. V. Carpio-Bernido, *Methods and Applications of White Noise Analysis in Interdisciplinary Sciences* (World Scientific, 2014).

Probability Density Function for a wide class of memory behavior:

$$P(x_T, T; x_0, 0) = \left(2\pi g(T)^2 \int_0^T [f(T-t)\, h(t)]^2 dt\right)^{-\frac{1}{2}}$$

$$\times\, exp\left(-\left[g(T)^2 \int_0^T [f(T-t)\, h(t)]^2 dt\right]^{-1} \frac{(x_T - x_0)^2}{2}\right)$$

Depending on the *Big Data* we can determine $g(T)$**,** $f(T-t)$ **and** $h(t)$**.**

$$x(T) = x_0 + g(T) \int_0^T f(T-t)\, h(t)\, dB(t)$$

Memory Function $f(T-t)$		$h(t)$
(1)	$f = \frac{(T-t)^{H-1/2}}{\Gamma(H+1/2)}$	$h = 1$
(2)	$f = \sin^{\frac{1}{2}}(T-t)$	$h = \sqrt{J_0(t)}$ J_ν is a Bessel function.
(3)	$f = \cos^{\frac{1}{2}}(T-t)$	$h = \sqrt{J_0(t)}$
(4)	$f = (T-t)^{\frac{\mu-1}{2}}$	$h = \frac{e^{-\beta/2t}}{t^{(\mu+1)/2}}$
(5)	$f = (T-t)^{\frac{\mu-1}{2}}$	$h = \frac{e^{-\beta/2t}}{t^{(1-\nu)/2}}$
(6)	$f = (T-t)^{\frac{\mu-1}{2}}$	$h = \frac{e^{-\beta/2t}}{t^{\mu}}$
(7)	$f = (T-t)^{\frac{\mu-1}{2}}$	$h = \frac{e^{\beta t/2}}{t^{(1-\nu)/2}}$

Memory Function $f(T-t)$		$h(t)$
(8)	$f = (T-t)^{\frac{\mu-1}{2}}$	$h = \dfrac{\cos^{\frac{1}{2}}(at)}{t^{\frac{1-\mu}{2}}}$
(9)	$f = (T-t)^{\frac{\mu-1}{2}}$	$h = \dfrac{\sin^{\frac{1}{2}}(at)}{t^{\frac{1-\mu}{2}}}$
(10)	$f = (T-t)^{\frac{\mu-1}{2}}$	$h = (t^2+\beta^2)^{\frac{\nu}{2}} / t^{\frac{1-\lambda}{2}}$
(11)	$f = (T-t)^{-\frac{\nu}{2}}$	$h = \sqrt{\dfrac{(t-a)^{\nu}}{(t-c)}}$
(12)	$f = (T-t)^{\nu/2}$	$h = e^{-\mu t/2}$
(13)	$f = \sqrt{J_{1-\nu}(T-t)}$	$h = \sqrt{J_{\nu}(t)}$
(14)	$f = \sqrt{J_{\nu}(T-t)}$	$h = \sqrt{t^{-1} J_{\mu}(t)}$

PDF for a wide class of memory behavior:

$$P(x_T, T; x_0, 0) = \left(2\pi g(T)^2 \int_0^T [f(T-t)\, h(t)]^2 dt\right)^{-\frac{1}{2}}$$

$$\times\, exp\left(-\left[g(T)^2 \int_0^T [f(T-t)\, h(t)]^2 dt\right]^{-1} \frac{(x_T - x_0)^2}{2}\right)$$

Example 1: Choosing, $g(T) = h(t) = 1$, and $f = \sqrt{2D} = constant$,

We get the PDF for the Wiener process:

$$P(x_T, T; x_0, 0) = \frac{1}{\sqrt{4\pi DT}} exp\left(\frac{-(x_0 - x_T)^2}{4DT}\right)$$

PDF for a wide class of memory behavior:

$$P(x_T, T; x_0, 0) = \left(2\pi g(T)^2 \int_0^T [f(T-t)\, h(t)]^2 dt\right)^{-\frac{1}{2}}$$

$$\times\, exp\left(-\left[g(T)^2 \int_0^T [f(T-t)\, h(t)]^2 dt\right]^{-1} \frac{(x_T - x_0)^2}{2}\right)$$

Example 2: Choosing, $g(T) = h(t) = 1$, and $f(T-t) = \dfrac{(T-t)^{H-(1/2)}}{\Gamma\left(H + \frac{1}{2}\right)}$.

We get the PDF for **fractional Brownian motion**:

$$P(x_T, T; x_0, 0) = \sqrt{\frac{H\, \Gamma^2\big(H + (1/2)\big)}{\pi\, T^{2H}}}\; exp\left\{-\frac{H\, \Gamma^2\big(H + (1/2)\big)(x_0 - x_T)^2}{T^{2H}}\right\}$$

Riemann-Liouville representation

APPLICATIONS

Biophysical Journal
Article

Damped White Noise Diffusion with Memory for Diffusing Microprobes in Ageing Fibrin Gels

Rev R. L. Aure,[1,2] Christopher C. Bernido,[3,4,*] M. Victoria Carpio-Bernido,[3,4] and Rommel G. Bacabac[1]
[1]Medical Biophysics Group, Department of Physics, University of San Carlos, Cebu City, Philippines; [2]Department of Mathematics and Physics, Visayas State University, Baybay City, Leyte, Philippines; [3]Theoretical and Computational Sciences and Engineering Group, Department of Physics, University of San Carlos, Cebu City, Philippines; and [4]Research Center for Theoretical Physics, Central Visayan Institute Foundation, Jagna, Bohol, Philippines

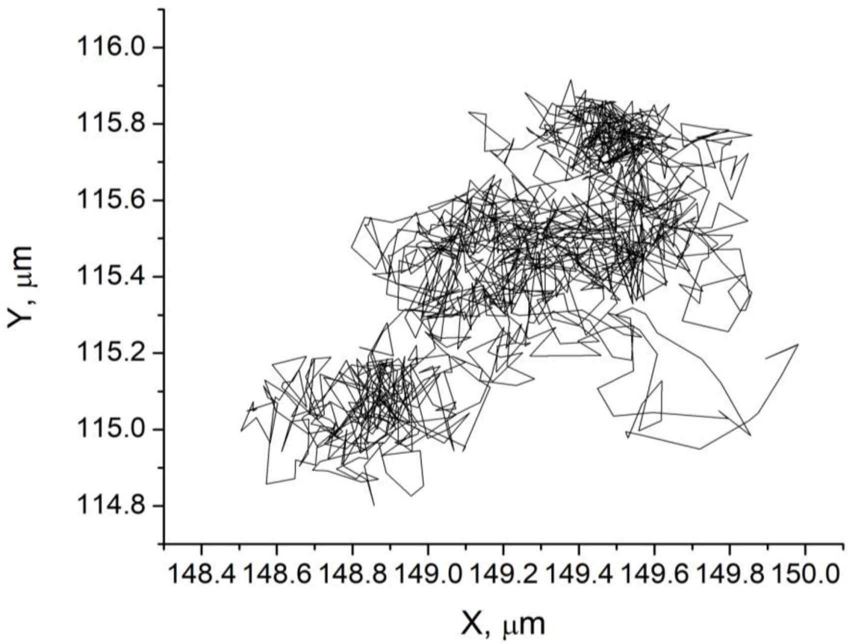
ABSTRACT From observations of colloidal tracer particles in fibrin undergoing gelation, we introduce an analytical framework that allows the determination of the probability density function for a stochastic process beyond fractional Brownian motion. Using passive microrheology via videomicroscopy, mean square displacements of tracer particles suspended in fibrin at different ageing times are obtained. The anomalous diffusion is then described by a damped white noise process with memory, with analytical results closely matching experimental plots of mean square displacements and probability density function. We further show that the white noise functional stochastic approach applied to passive microrheology reveals the existence of a gelation parameter μ which elucidates the dynamics of constrained tracer particles embedded in a time-dependent soft material. In addi-

R. Aure, C. C. Bernido, M. V. Carpio-Bernido, R. G. Bacabac
Biophysical Journal **117** (2019)

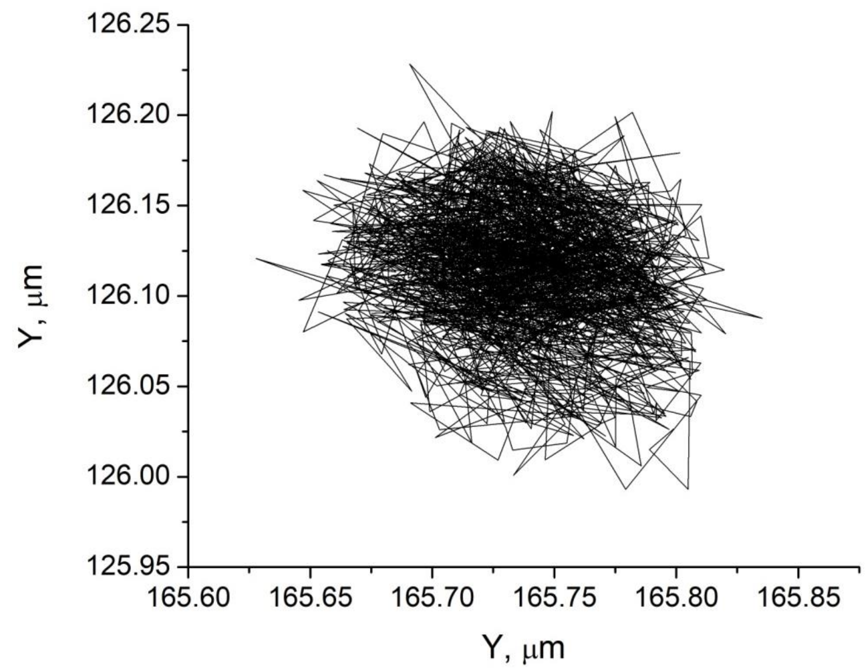
Fibrin Gelation

Fibrin is important in wound healing, tissue regeneration, and bioengineering.

Trajectory of Probe Particle embedded in Fibrin.



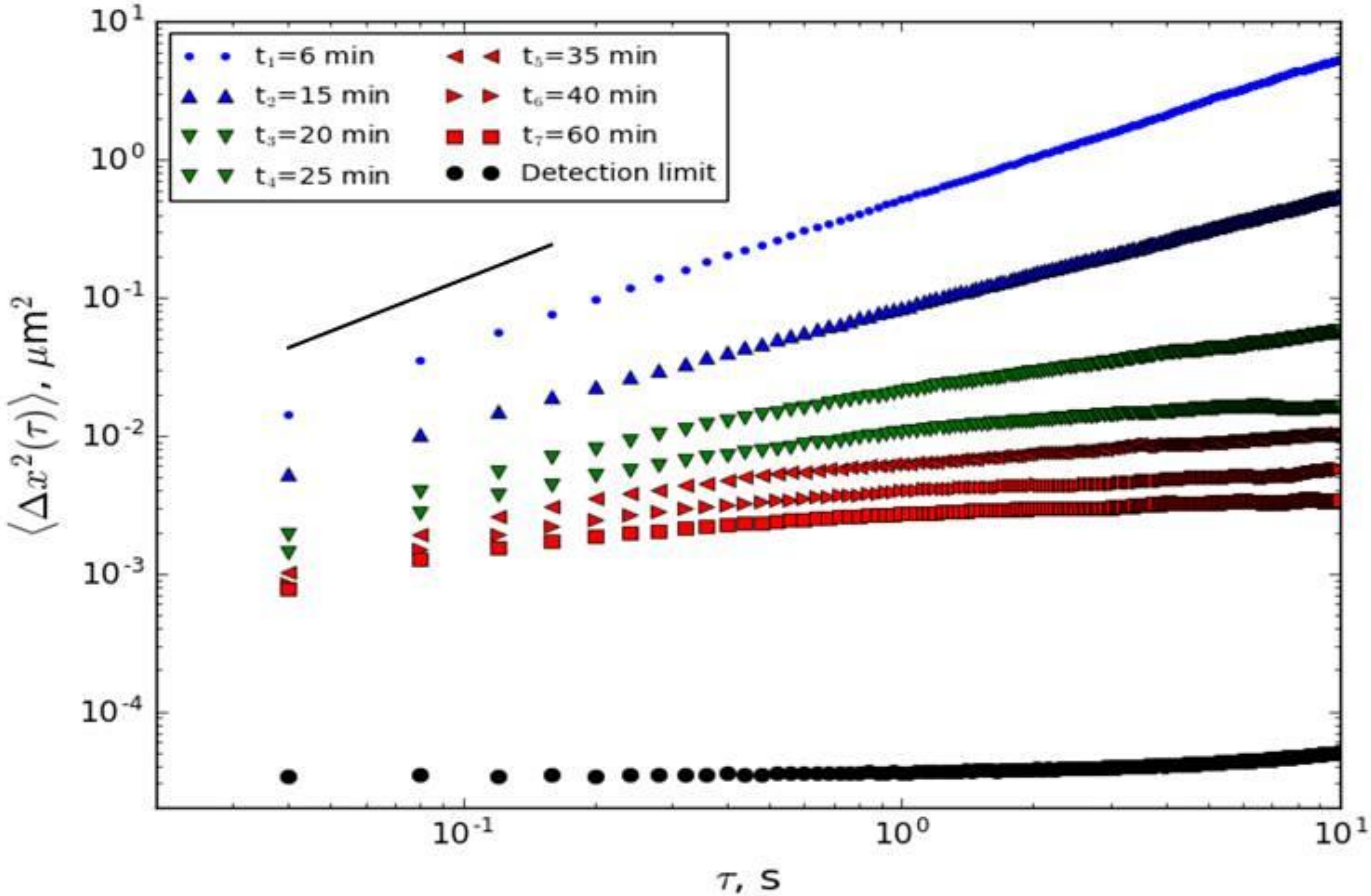
20 minutes



60 minutes

R. Aure, C. C. Bernido, M. V. Carpio-Bernido, and R. G. Bacabac, *Biophys. Jour.* **117** (2019)

MSD of tracer particles embedded in fibrin:



Modelling the fluctuating positions of probe particles in fibrin at different ageing times:

$$x(T) = x_0 + \text{ Fluctuations}$$

Ordinary Brownian Motion

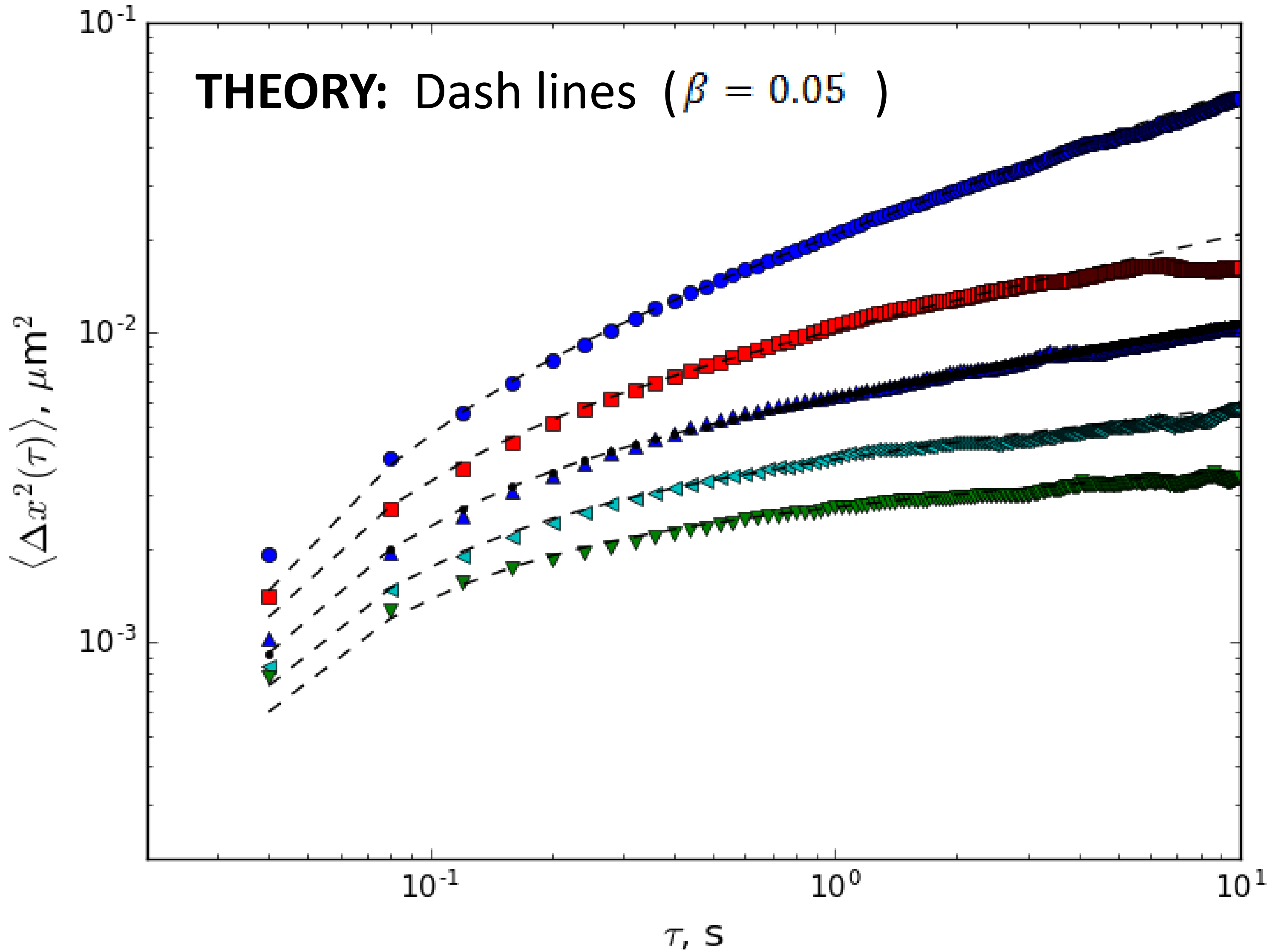
$$x(T) = x_0 + \int_0^T \underbrace{(T-\tau)^{(\mu-1)/2}}_{\textit{Memory Function}} \frac{exp(-\beta/2\tau)}{\tau^{(\mu+1)/2}}\, dB(\tau)$$

Probability Density Function:

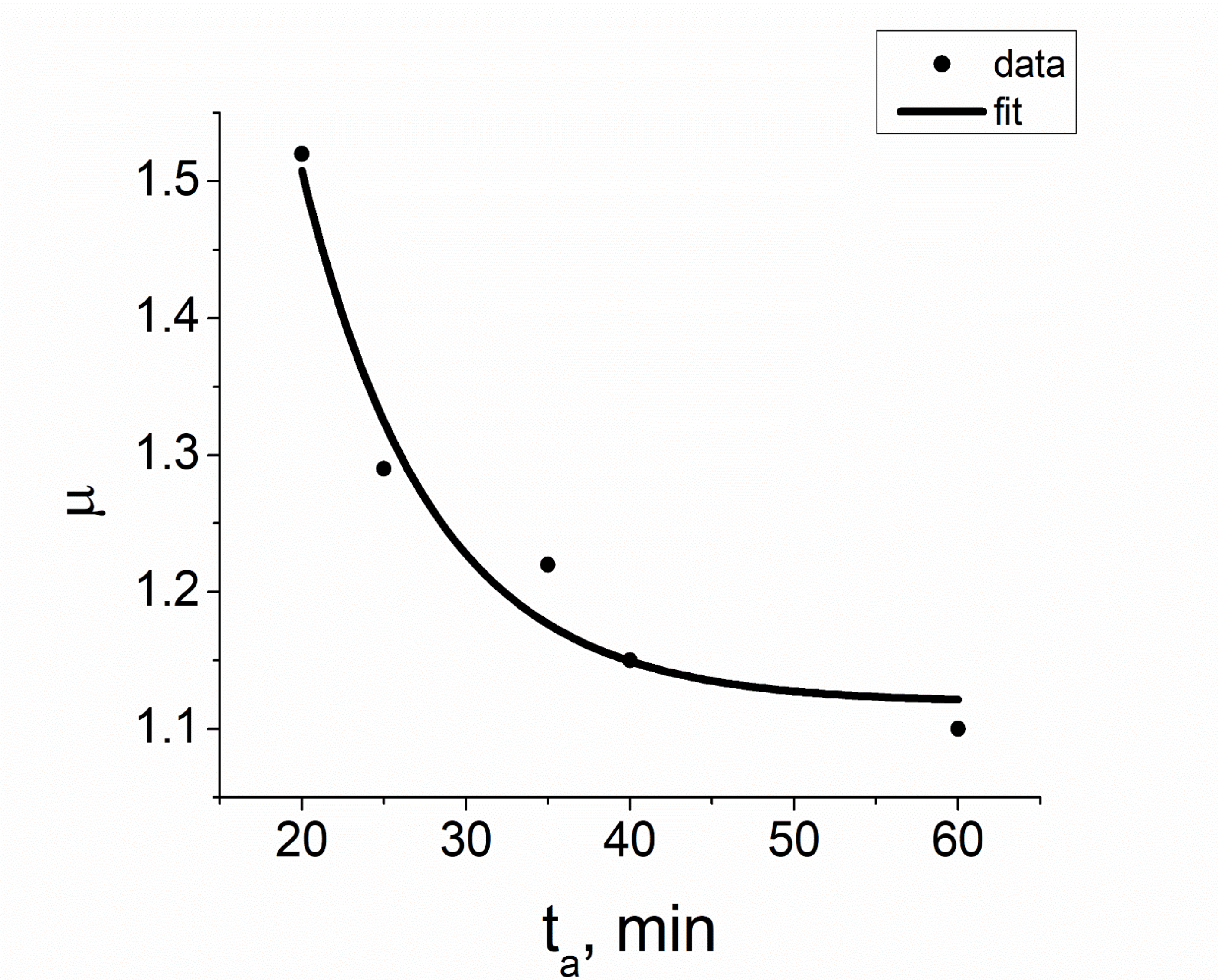
$$P(x_T, T; x_0, 0) = \frac{1}{\sqrt{2\pi\Gamma(\mu)\beta^{-\mu}T^{\mu-1}e^{-\beta/T}}}\, exp\left[\frac{-(x_T - x_0)^2}{2\Gamma(\mu)\beta^{-\mu}T^{\mu-1}e^{-\beta/T}}\right]$$

Mean square deviation: $\mathrm{MSD} = \langle \Delta x^2(\tau) \rangle$

$$\boxed{\mathrm{MSD} = \Gamma(\mu)\beta^{-\mu}T^{\mu-1}exp(-\beta/T)}$$



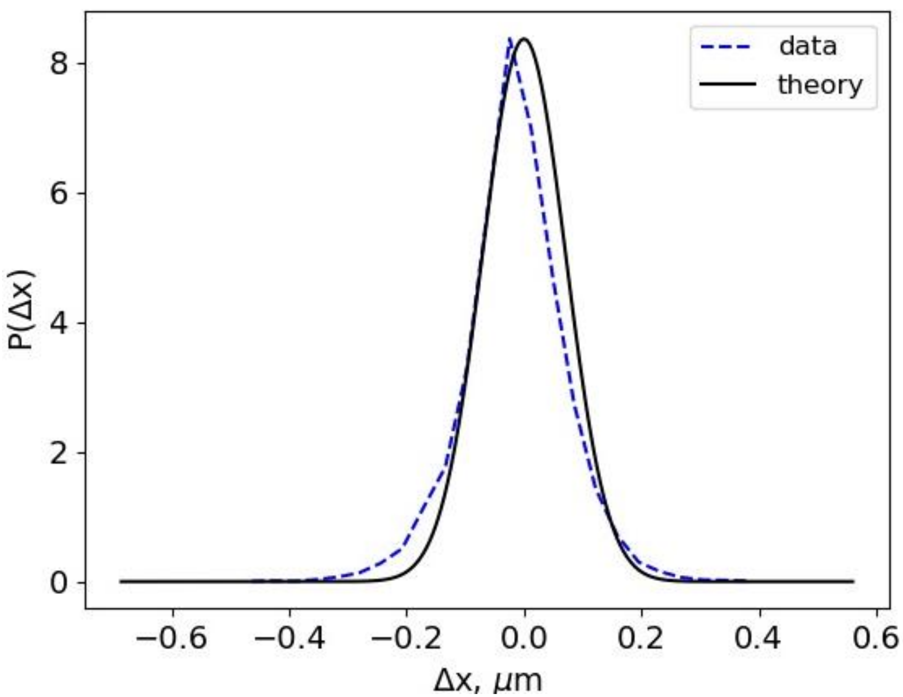
20 minutes (circle); 25 minutes (square); 35 minutes (triangle marker up); 40 minutes (triangle marker left); and 60 minutes (triangle marker down).



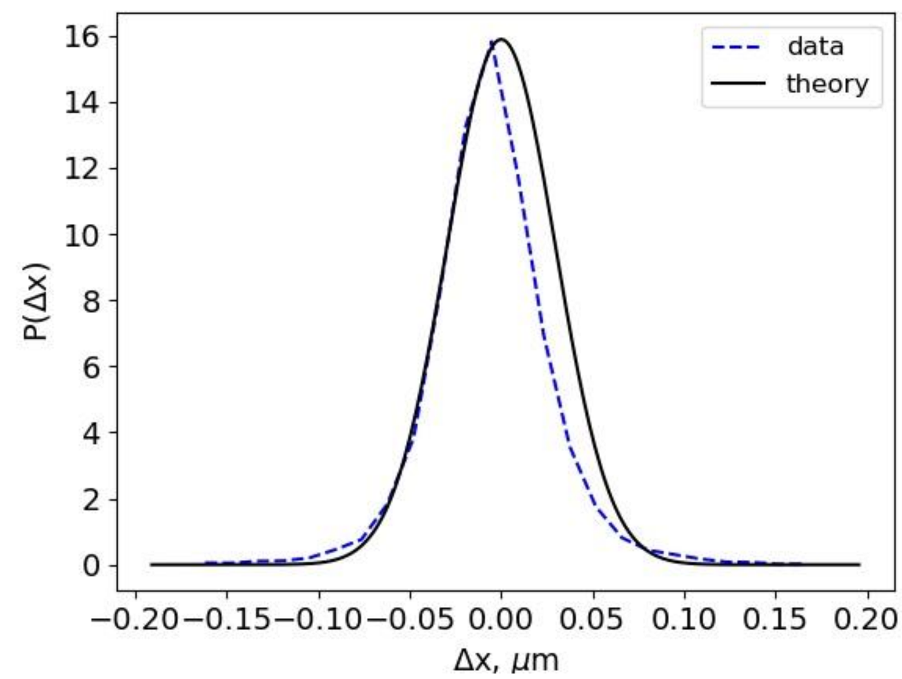
μ may be identified as an ageing parameter.

$$P(x_T, T; x_0, 0) = \frac{1}{\sqrt{2\pi\Gamma(\mu)\beta^{-\mu}T^{\mu-1}e^{-\beta/T}}} exp\left[\frac{-(x_T - x_0)^2}{2\Gamma(\mu)\beta^{-\mu}T^{\mu-1}e^{-\beta/T}}\right]$$

Probability Distribution of Δx

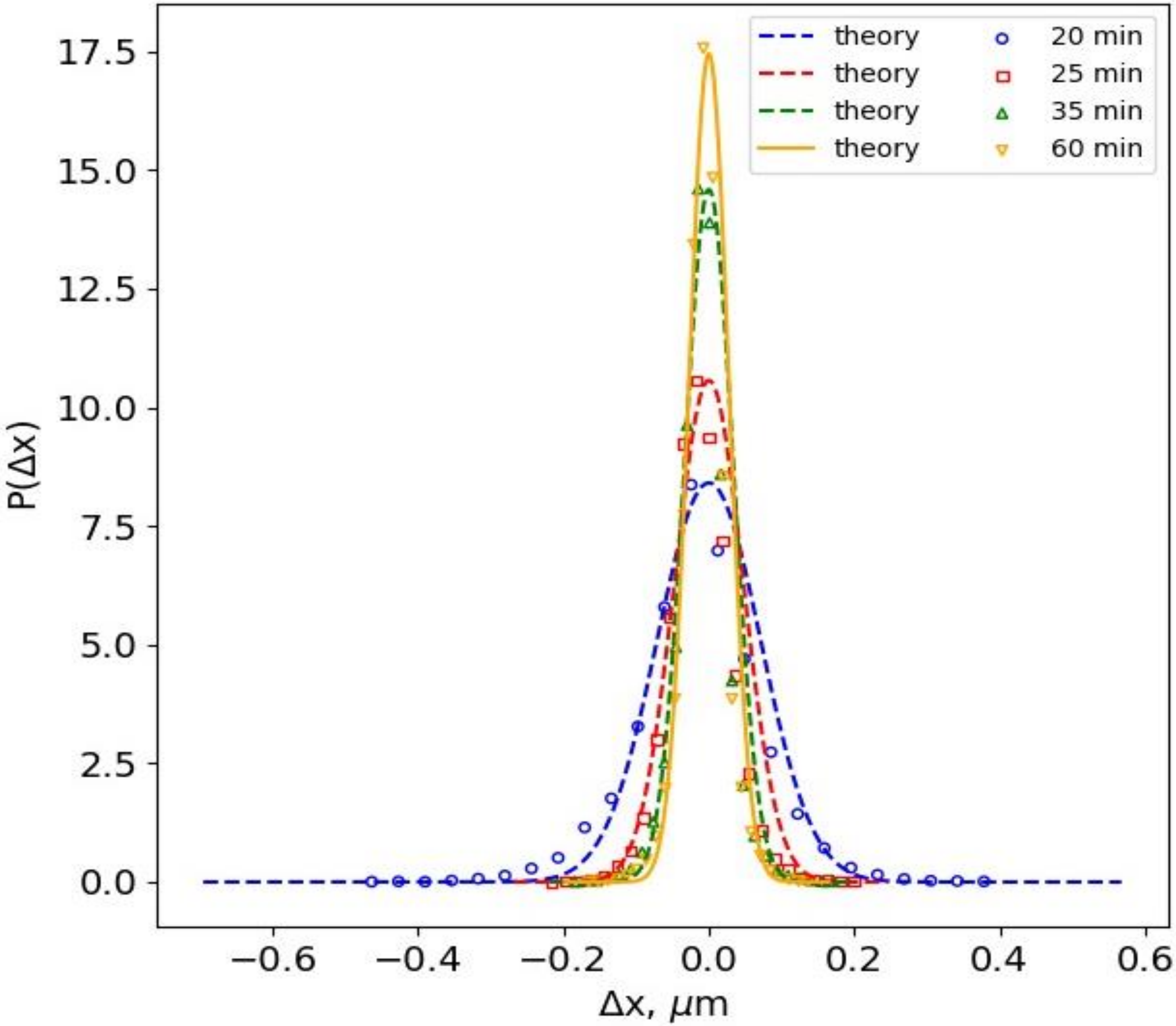


20 minutes



40 minutes

$$P(x_T, T; x_0, 0) = \frac{1}{\sqrt{2\pi\Gamma(\mu)\beta^{-\mu}T^{\mu-1}e^{-\beta/T}}} exp\left[\frac{-(x_T - x_0)^2}{2\Gamma(\mu)\beta^{-\mu}T^{\mu-1}e^{-\beta/T}}\right]$$



Example 2 **:** DNA Sequence Synechococcus elongatus PCC 7942

Is there a mathematical framework that describes the sequence of nucleotides?

TTAAAAAAGAGATCCGATCTAGTAATCGCACCGTCAAAAATCCTTGTGGAAGAGCAGTGCTACG
ATGCTTTGGCAAGATTGCGATCAAAGGCTCGGGCAGCCTCCCCCCATGAAGTTGGTCTGTCGCC
AAAACGAACTGAATACCAGTCTGTCGCTCGTTAGCCGTGCTGTACCTTCCCGCCCGAATCATCCG
GTGCTTGCCAACGTGCTTCTAGCGGCTGATGCCGGTACTCAACGACTGAGCTTAACCGCCTTTGA
TCTCAGCCTCGGAATTCAAACCAGCTTTGCCGCGCAAGTCGAACGGAGCGGTGCCATCACTCTA
CCGGCCAAGCTGCTCAACGATATCGTTTCGCGCCTCCCCAACGACAGCGACGTCACGCTGGAAG
ACAACGATGCAGCGGCGACTCTGTCAGTGGGGTCTGGCCAGTACCAAATGCGGGGGATCAGTG
CCGATGAATTTCCTGAGTTGCCTCTGGTTCAGAGCCAAGAAGCTCTACAACTCTCCGCCAGCGCT
TTGATCGAAGGGCTGCGTGGCACCCTGTTTGCGACTAGTGGGGATGAAACCAAGCAAATCCTG
ACGGGCGTCCACCTCAAGGTACAACCCGATGGCCTAGAGTTTGCAGCGACGGACGGTCACCGC
TTGGCTGTGGTTAAAACCGAGAATGCCGCAGCAACTCCAGCTACTGAGTTTGCTGTGACGGTGC
CTTCGCGGGCCTTACGGGACCTAGAGCGGATGATCGCGATTCGCGGCAGTGACGAGGCGATCG
CGCTTTATCATGACCAAGGTCAGACTGTCTTCCAGTGGGGCGACCAGTACCTAACGAGCCGGAC
ATTGGATGGCCAATATCCCAACTACGGGCAGCTGATTCCGCGGGAGTTCAATCGCAACGTTGCC
GTCGATCGCAAACGCCTGCTGGCCGCGCTGGAGCGGATTGCAGTGTTGGCGGATCAGCAAAA
....

IOP Publishing Physica Scripta
Phys. Scr. **94** (2019) 125006 (13pp) https://doi.org/10.1088/1402-4896/ab3739

White noise functional integral for exponentially decaying memory: nucleotide distribution in bacterial genomes

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[2] Research Center for Theoretical Physics, Central Visayan Institute Foundation, Jagna, Bohol 6308, Philippines

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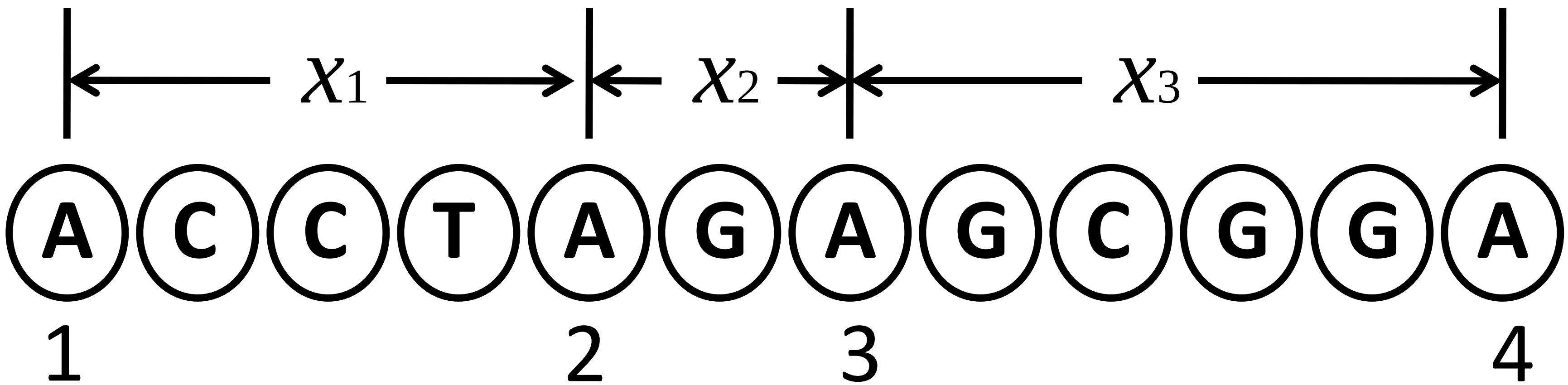
Published 11 September 2019

CrossMark

Abstract
We utilize a stochastic functional integral approach that forms a natural framework for analyzing ubiquitous complex sequences of fluctuations with underlying non-Markovian stochastic process beyond fractional Brownian motion. We demonstrate how Hida white noise calculus, guided by mean square deviation (MSD) analysis of empirical data, allows derivation of single nucleotide

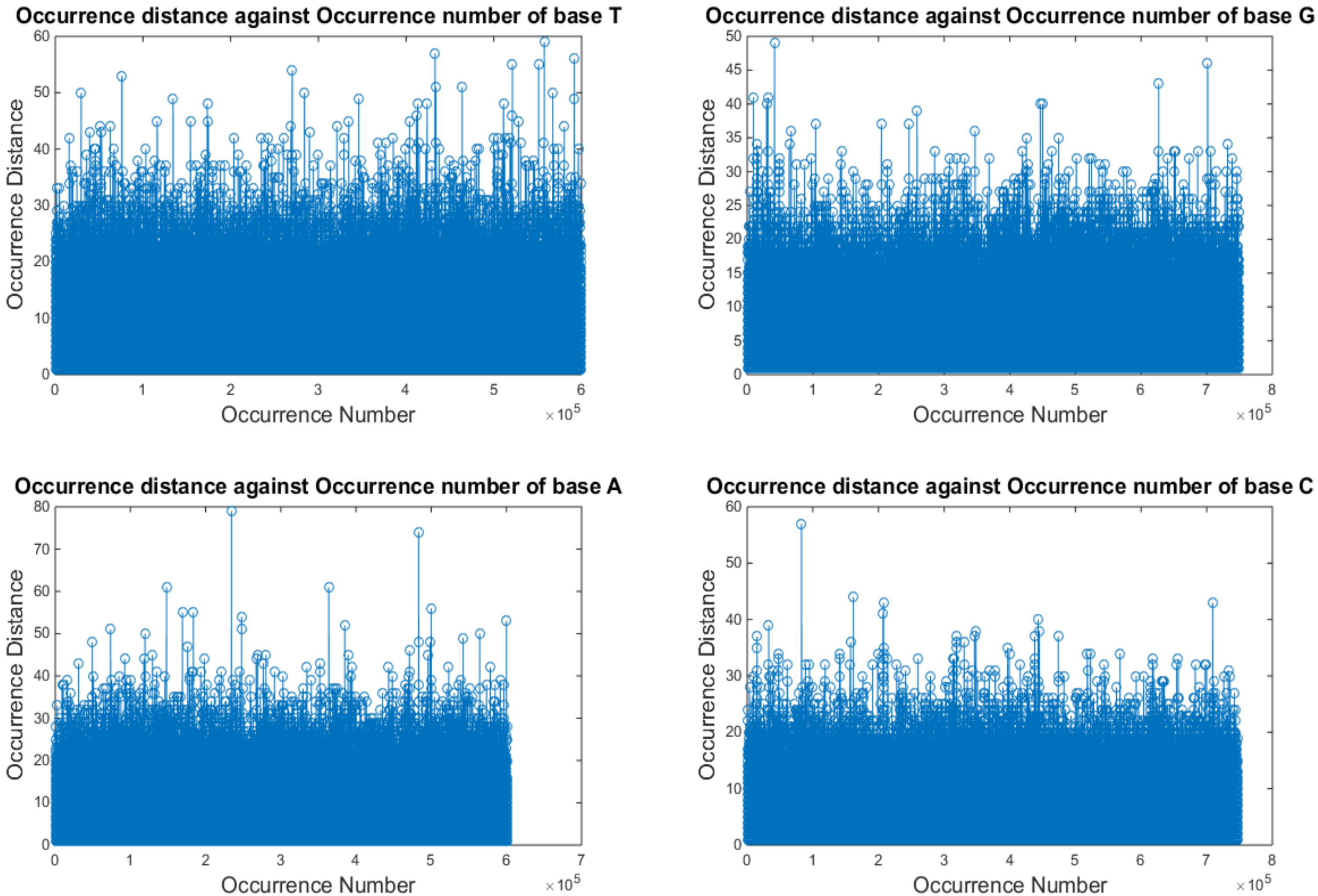
R. R. Violanda, C. C. Bernido and M. V. Carpio-Bernido, *Physica Scripta* **94** (2019)

DNA Sequence

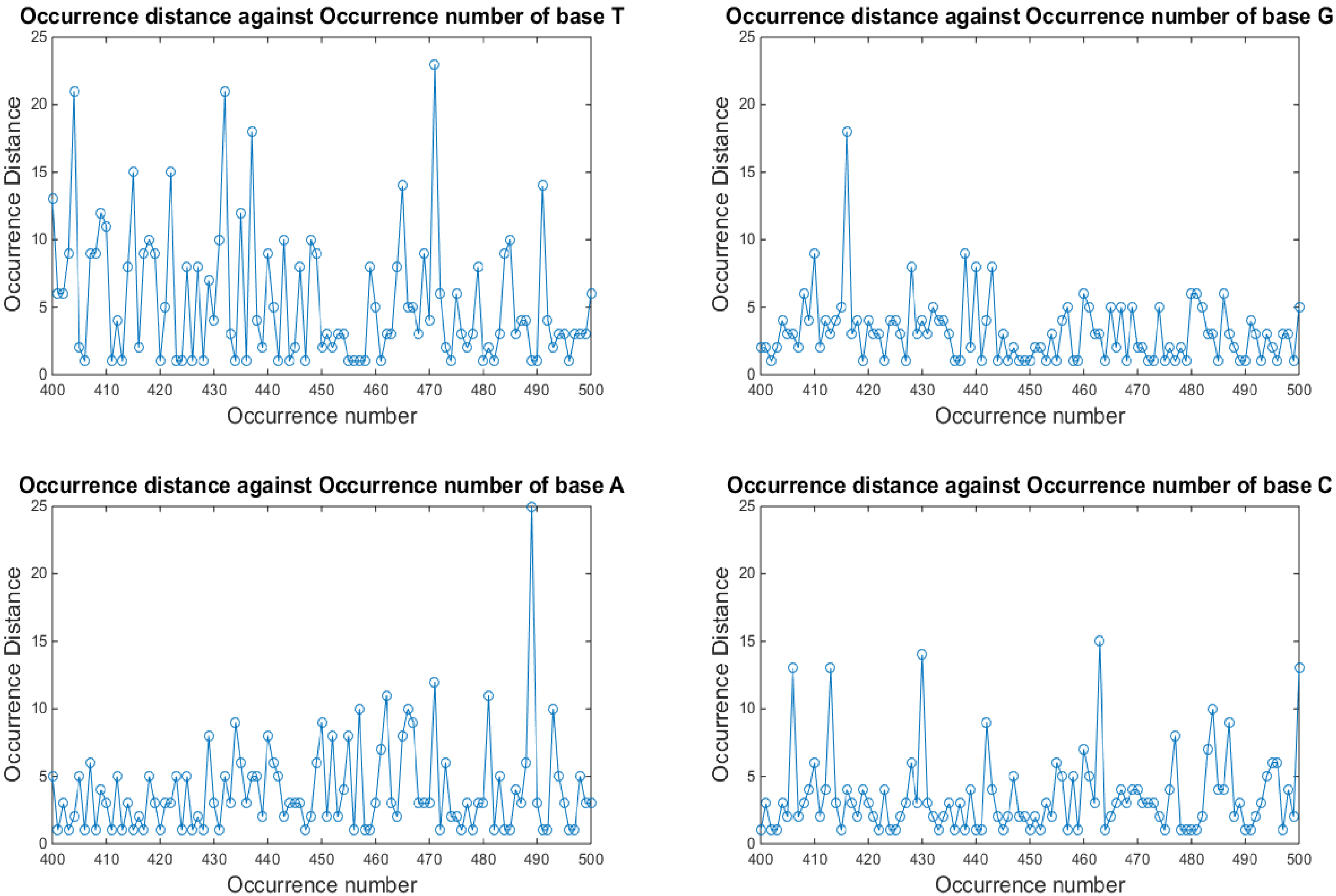


Distances between adjacent adenine (A) nucleotides are marked x_i where $i = 1,2,\ldots, n - 1$. Hence: $x_1 = 4$, $x_2 = 2$, and $x_3 = 5$.

R. Renante, C. C. Bernido, and M. V. Carpio-Bernido, *Physica Scripta* **94** (2019).

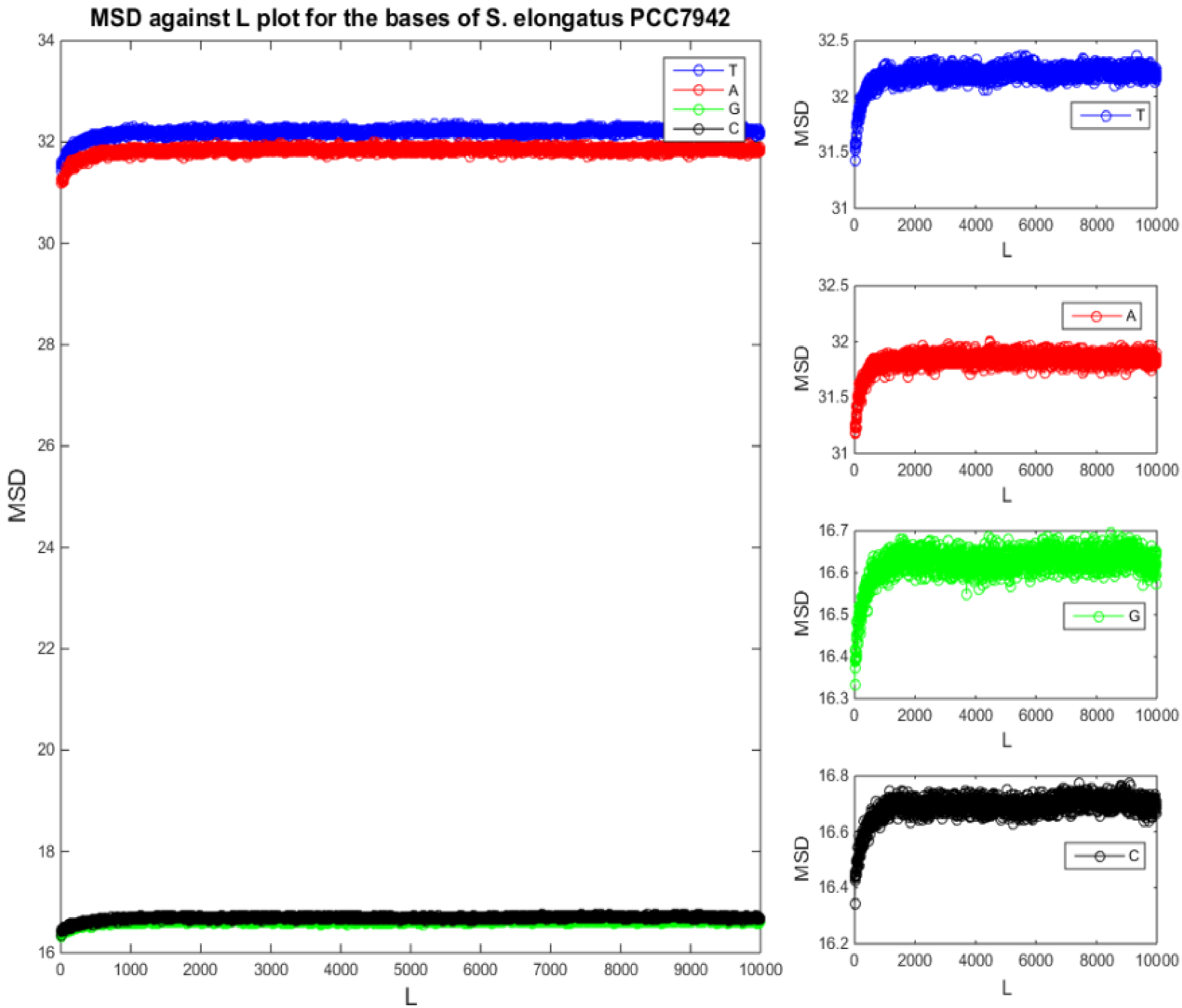


Separation distances between a nucleotide base and a similar nucleotide immediately preceding it for *Synechococcus elongatus* PCC7942.



Magnified view for the 400th up to the 500th nucleotide of the same type occurring in the DNA sequence.

Mean Square Displacement (MSD) of fluctuating separation distances



Can we model mathematically the fluctuating distances?:

$x(L) = x_0 +$ Fluctuating Separation Distances

initial value

$$x(L) = x_0 + g(L) \int_0^L \underbrace{f(L-s)}_{\textbf{\textit{Memory Function}}} h(s)\, dB(s)$$

We use, $g(L) = h(s) = 1$, and a memory function

$f(L-s) = \sqrt{bc}\ exp\left[-\frac{b}{2}(L-s)\right]$, with b and c constants:

$$\boxed{x(L) = x_0 + \sqrt{bc} \int_0^L exp\left[-\frac{b}{2}(L-s)\right] dB(s)}$$

Fluctuation:

$$x(L) = x_0 + \sqrt{bc}\int_0^L exp\left[-\frac{b}{2}(L-s)\right]dB(s)$$

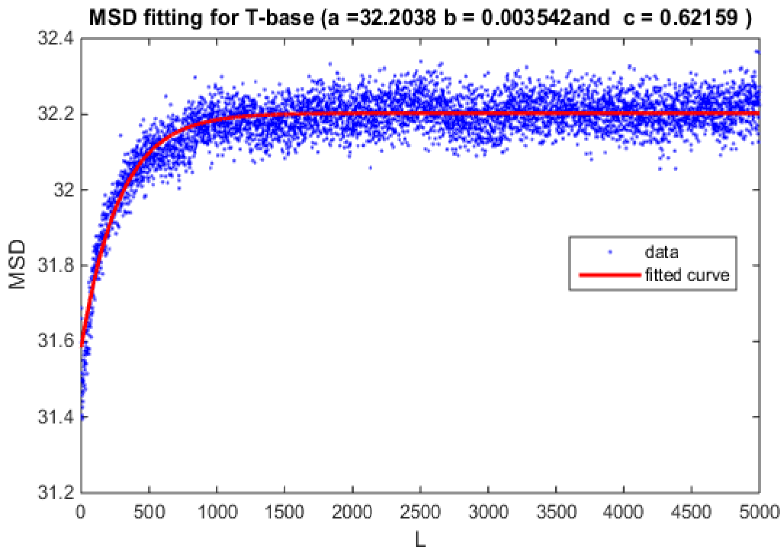
Probability Density Function:

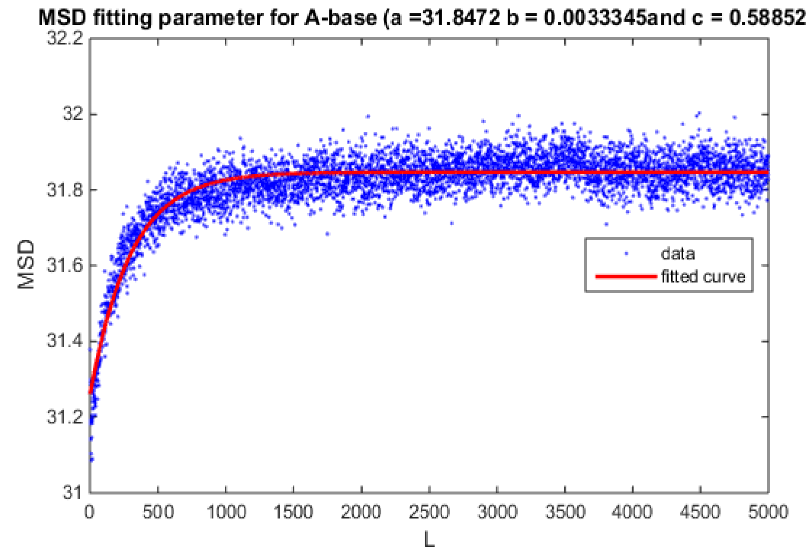
$$P(x_L, L; x_0, 0) = \frac{1}{\sqrt{2\pi c[1 - exp(-bL)]}}exp\left\{\frac{-(x_L - x_0)^2}{2c[1 - exp(-bL)]}\right\}$$

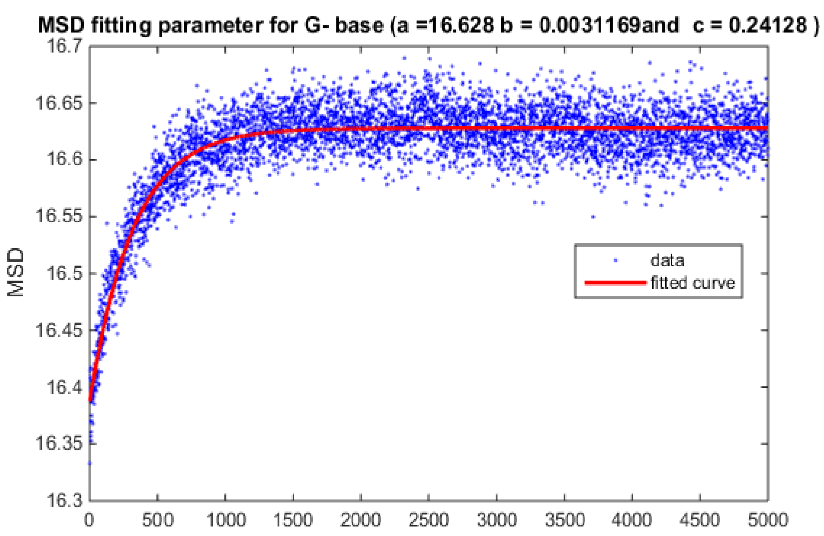
Theoretical Mean Square Deviation, shifted by (a - c) :

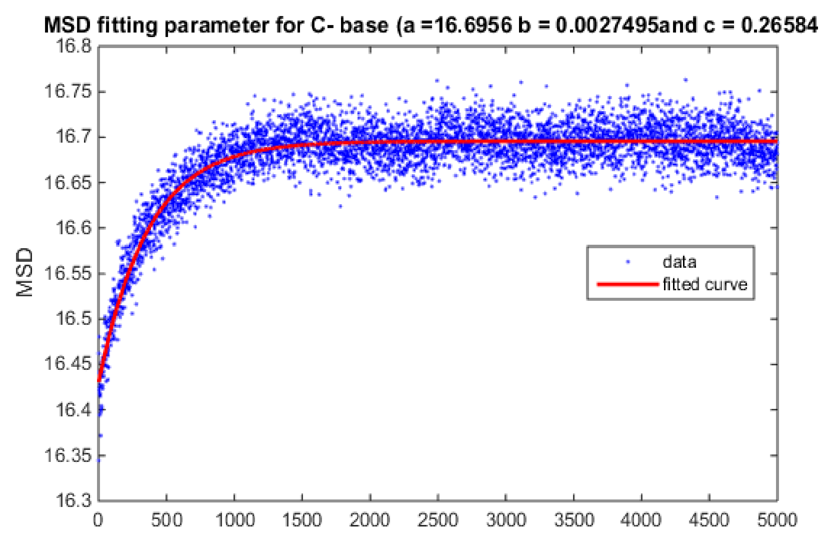
$$\mathrm{MSD}_a = a - c\, e^{-bL}$$

Mean Square Deviation (MSD): Blue dots (Empirical, NCBI); red curve (Theoretical)

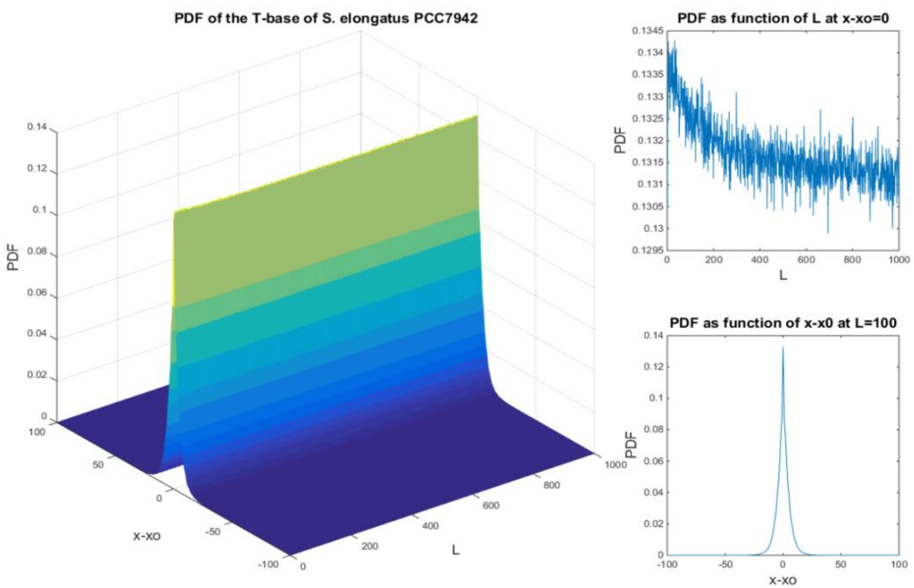






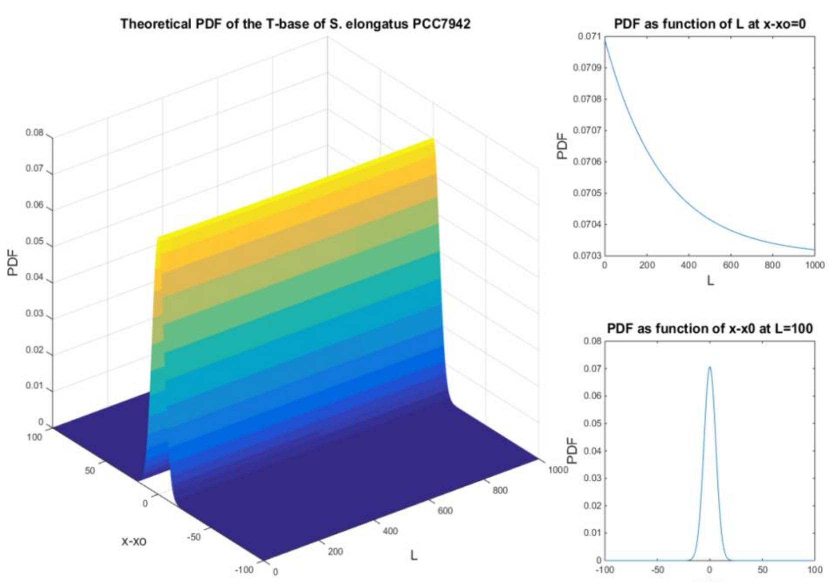


Probability Density Function:



Empirical

Synechococcus elongatus **PCC7942**



Theoretical

Similar results were obtained for the whole genome of the following species:

- *Synechococcus elongatus* PCC7942
- *Staphylococcus aureus* subsp. aureus NCTC 8325
- *Staphylococcus aureus* ILRI Eymole1/1
- *Prochlorococcus marinus* subsp. marinus str. CCMP1375

Example 3:

Climate Dynamics
https://doi.org/10.1007/s00382-021-05831-8

Great Barrier Reef degradation, sea surface temperatures, and atmospheric CO_2 levels collectively exhibit a stochastic process with memory

Allan R. B. Elnar[1,3] · Christianlly B. Cena[1] · Christopher C. Bernido[1,2] · M. Victoria Carpio-Bernido[1,2]

Received: 1 May 2018 / Accepted: 27 May 2021

A.R. Elnar, C.B. Cena, C.C. Bernido and M.V. Carpio-Bernido, *Climate Dynamics* (2021).

The past 30 years witnessed the loss of half the coral cover of the Great Barrier Reef due to elevated sea surface temperature, ocean acidification, and typhoons, among others.

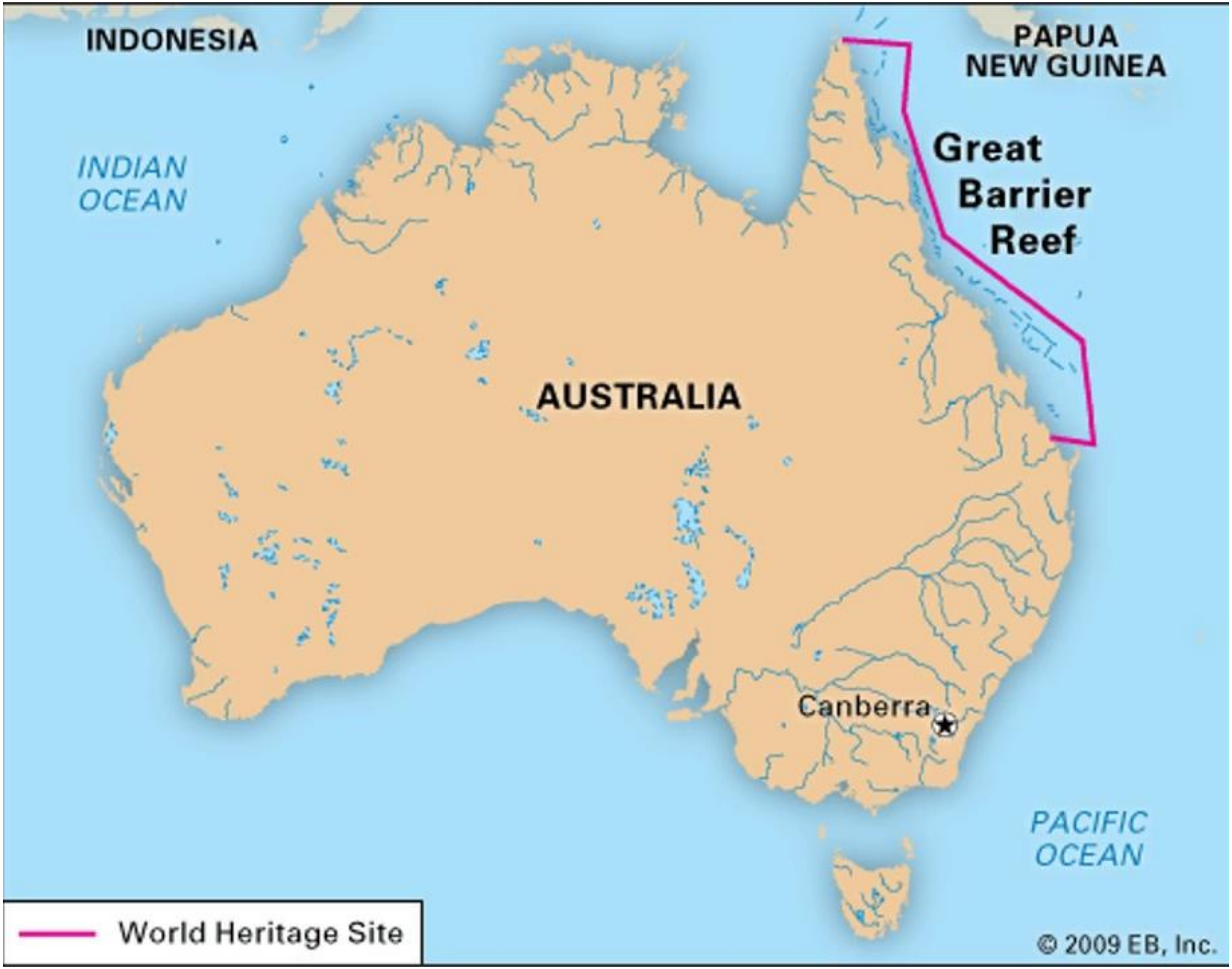
http://www.aims.gov.au/documents/30301/2107350/Acidification.pdf/4224fe9f-efd2-4f91-a7b2-604137a87f2d

https://thumbnails.trvl-media.com/zcvYGoswHQ7jvcxujhBOVsW2jTI=/768x432/images.trvl-media.com/media/content/shared/images/travelguides/destination/889/Great-Barrier-Reef-29303.jpg

https://www.vox.com/science-and-health/2016/11/29/13781434/great-barrier-reef-coral-dead

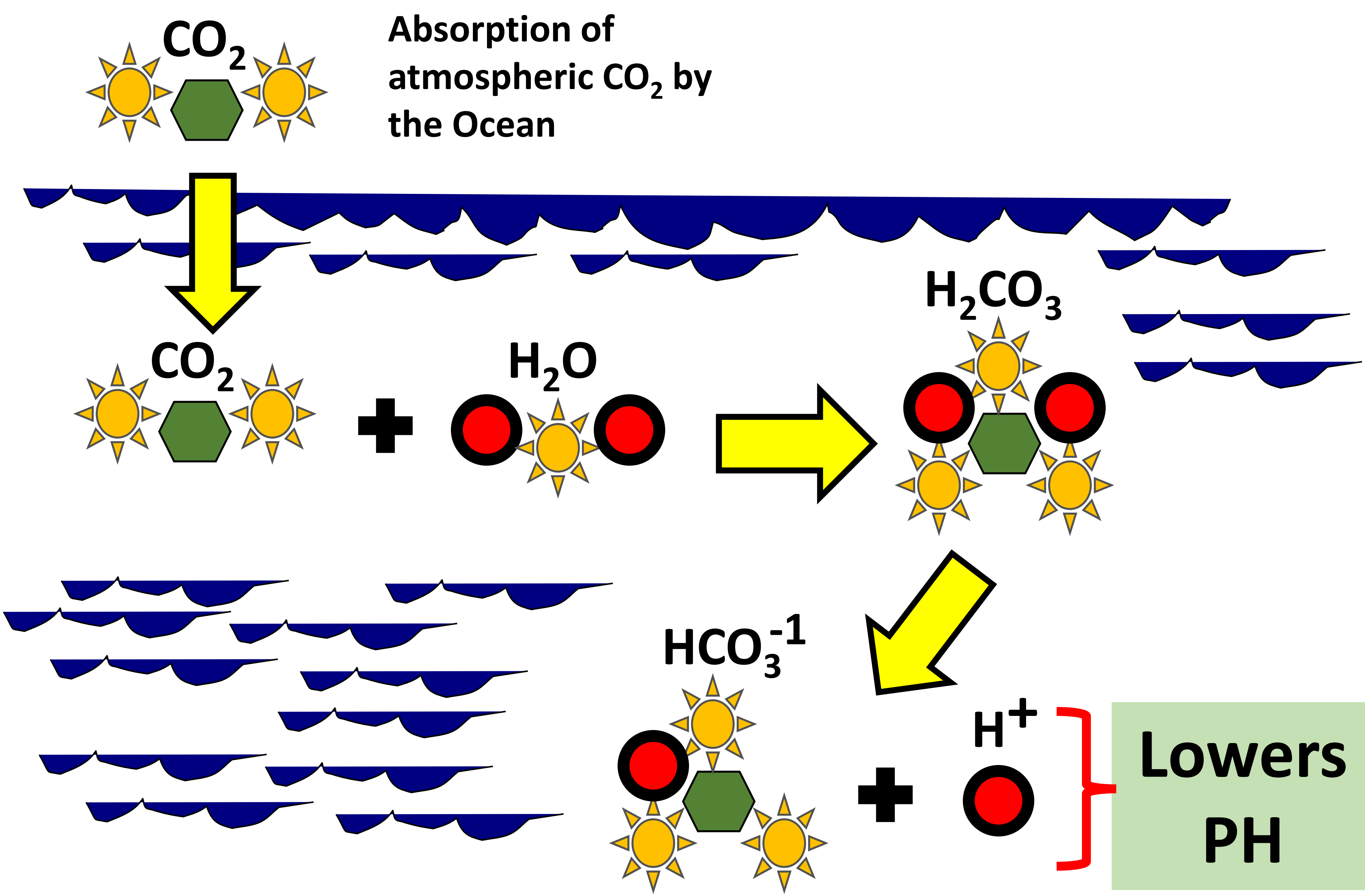
Great Barrier Reef: largest coral ecosystem in the world

Photo credit: http://fallenscoop.com/
https://divezone.net/best-great-barrier-reef-australia-liveaboard-reviews-2013



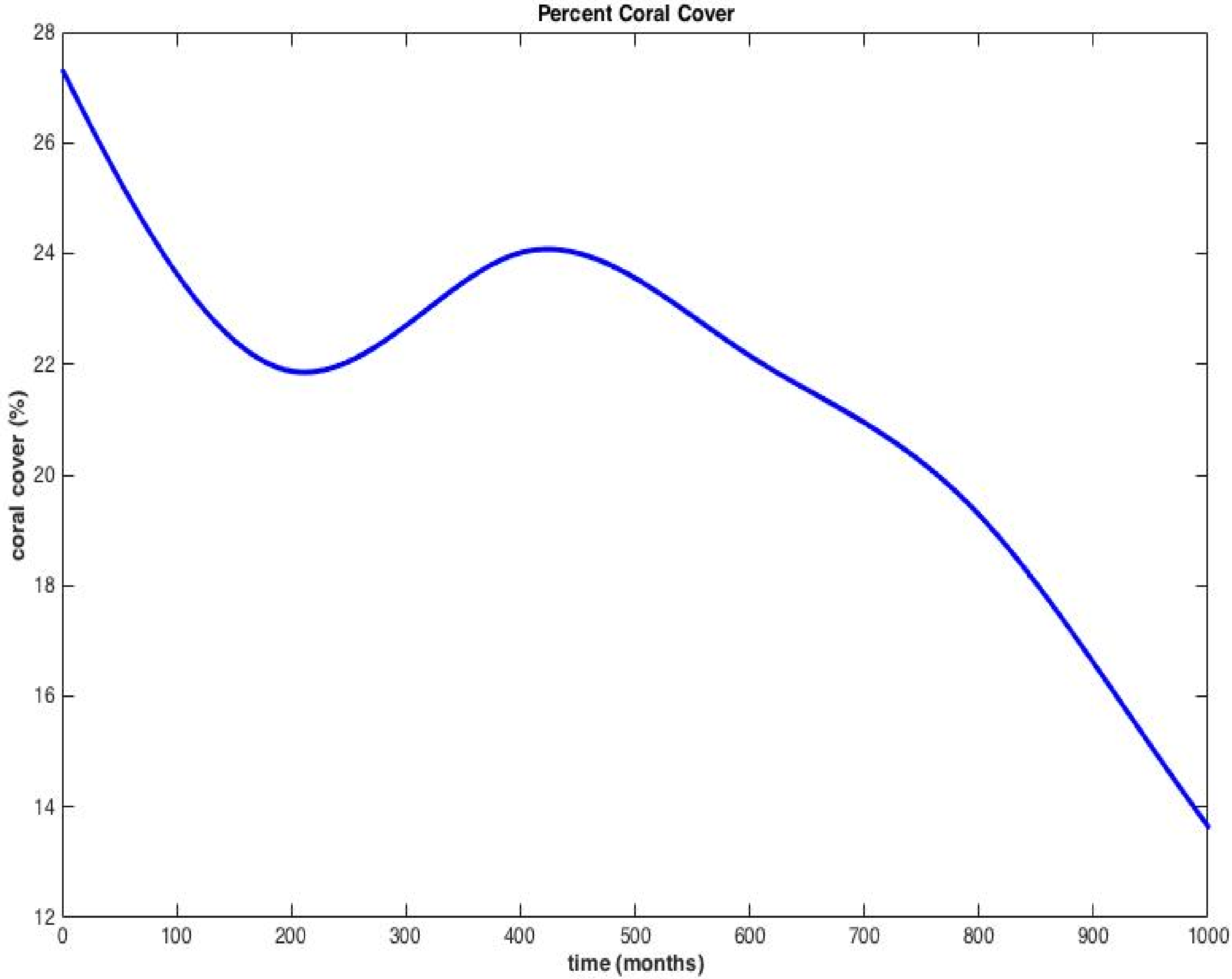
https://i.ytimg.com/vi/iZo7d8H8WzA/maxresdefault.jpg

Ocean Acidification of the Great Barrier Reef



https://thumbnails.trvl-media.com/zcvYGoswHQ7jvcxujhBOVsW2jTI=/768x432/images.trvl-media.com/media/content/shared/images/travelguides/destination/889/Great-Barrier-Reef-29303.jpg

https://www.vox.com/science-and-health/2016/11/29/13781434/great-barrier-reef-coral-dead



GREAT BARRIER REEF PERCENT CORAL COVER

Fluctuation:

$$x(T) = x_0 + \int_0^T (T-t)^{(\mu-1)/2} t^{(\mu-1)/2} \sqrt{cos(\nu t)}\, dB(t)$$

Probability Density Function:

$$P(x_T, T; x_0, 0) = \left(\frac{\pi^{-\frac{3}{2}} T^{\frac{1}{2}-\mu} \nu^{\mu-\frac{1}{2}}}{2\,\Gamma(\mu) cos\left(\frac{\nu T}{2}\right) J_{\mu-\frac{1}{2}}\left(\frac{\nu T}{2}\right)} \right)^{\frac{1}{2}} exp\left\{ \frac{-T^{\frac{1}{2}-\mu} \nu^{\mu-\frac{1}{2}} (x-x_0)^2}{2\sqrt{\pi}\,\Gamma(\mu) cos\left(\frac{\nu T}{2}\right) J_{\mu-\frac{1}{2}}\left(\frac{\nu T}{2}\right)} \right\}$$

Theoretical Mean Square Deviation :

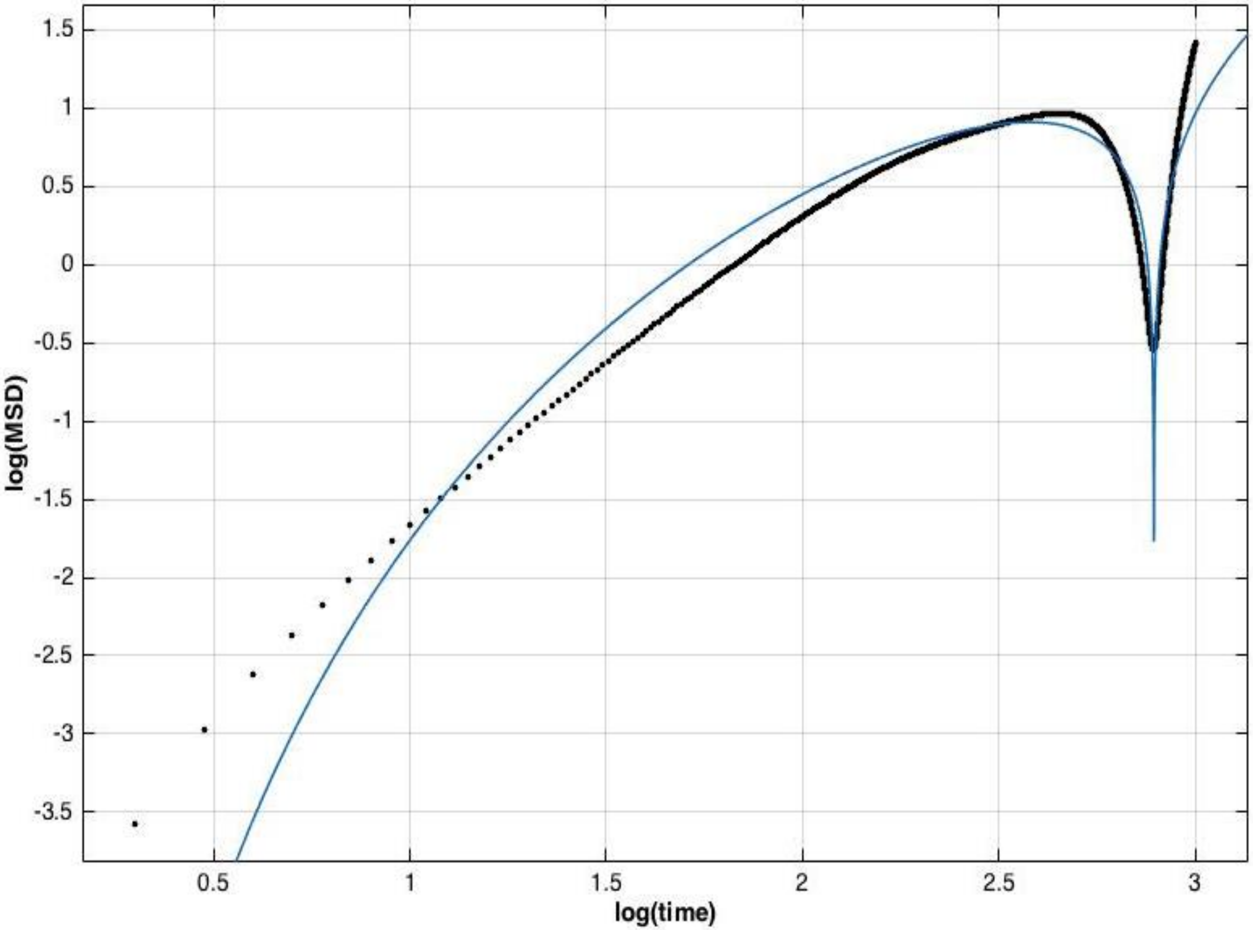
$$\mathrm{MSD} = \frac{\Gamma(\mu) cos(\nu T/2)\, J_{\mu-\frac{1}{2}}(\nu T/2)}{\pi^{-\frac{1}{2}} T^{\frac{1}{2}-\mu} \nu^{\mu-\frac{1}{2}}}$$

$$\mathrm{MSD} = \frac{\Gamma(\mu) cos(\nu T/2) J_{\mu-\frac{1}{2}}(\nu T/2)}{\pi^{-\frac{1}{2}} T^{\frac{1}{2}-\mu} \nu^{\mu-\frac{1}{2}}}$$

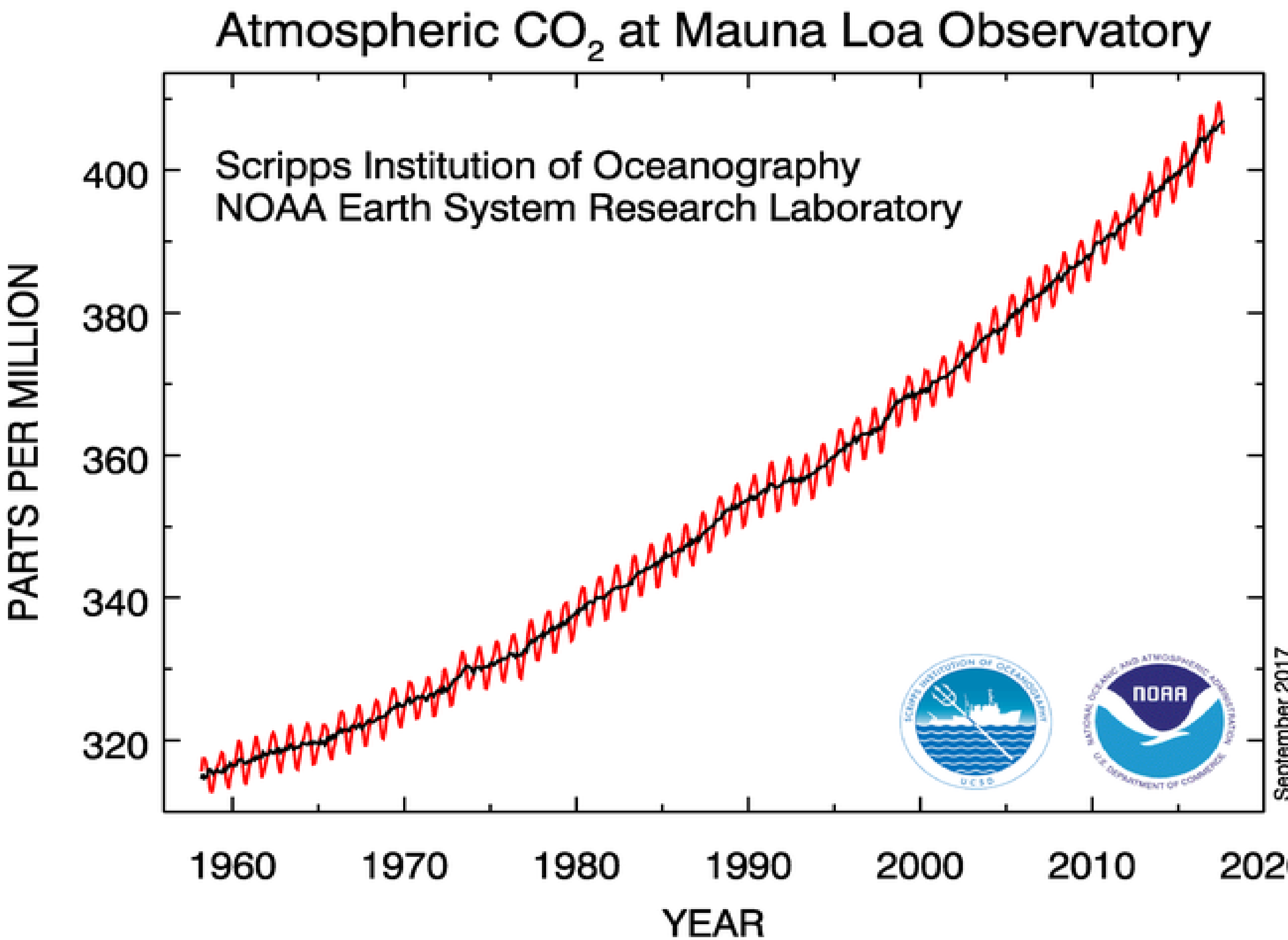
Theoretical MSD: Blue line

Empirical MSD: Black Dots

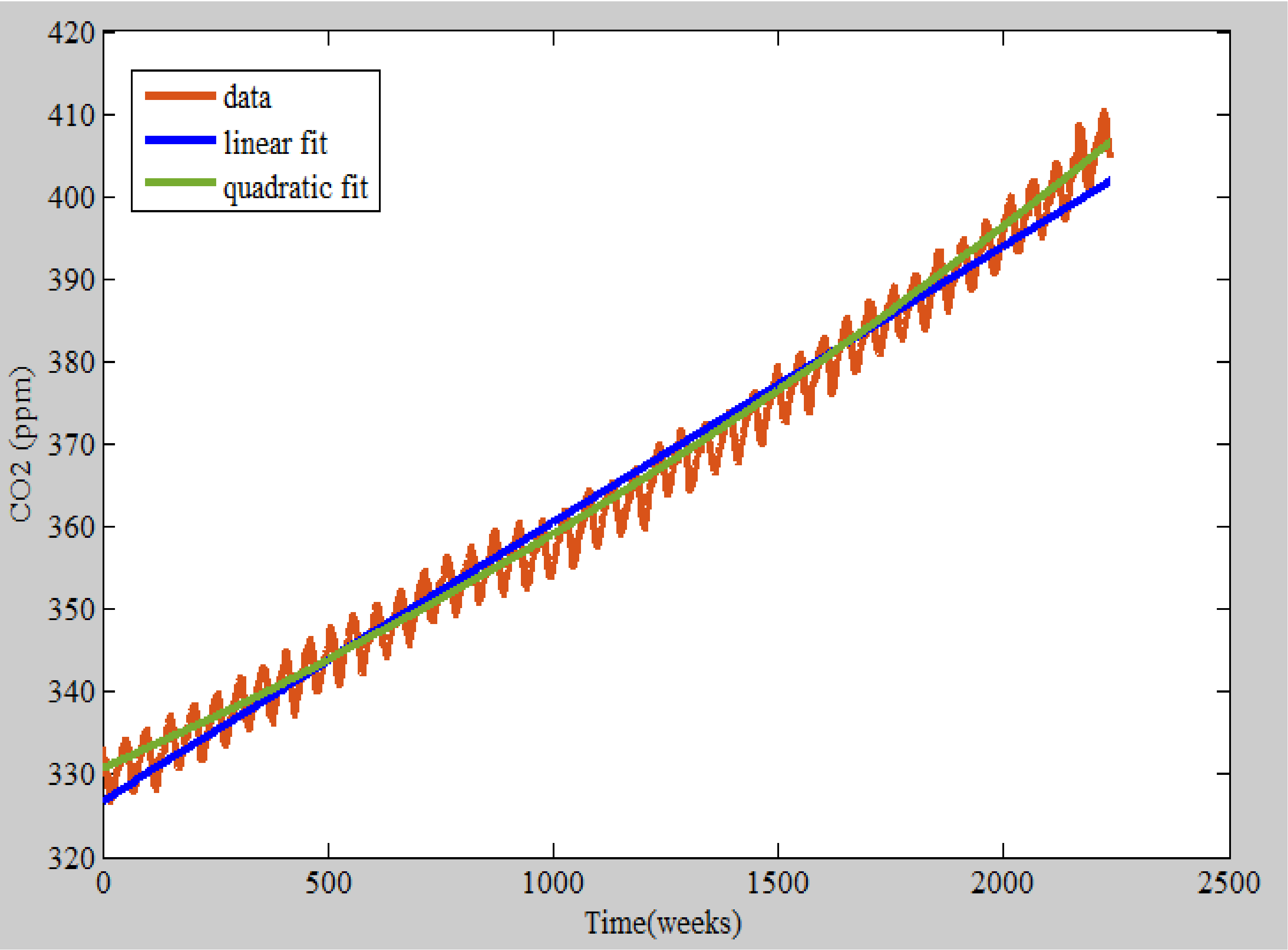
(**Great Barrier Reef**)



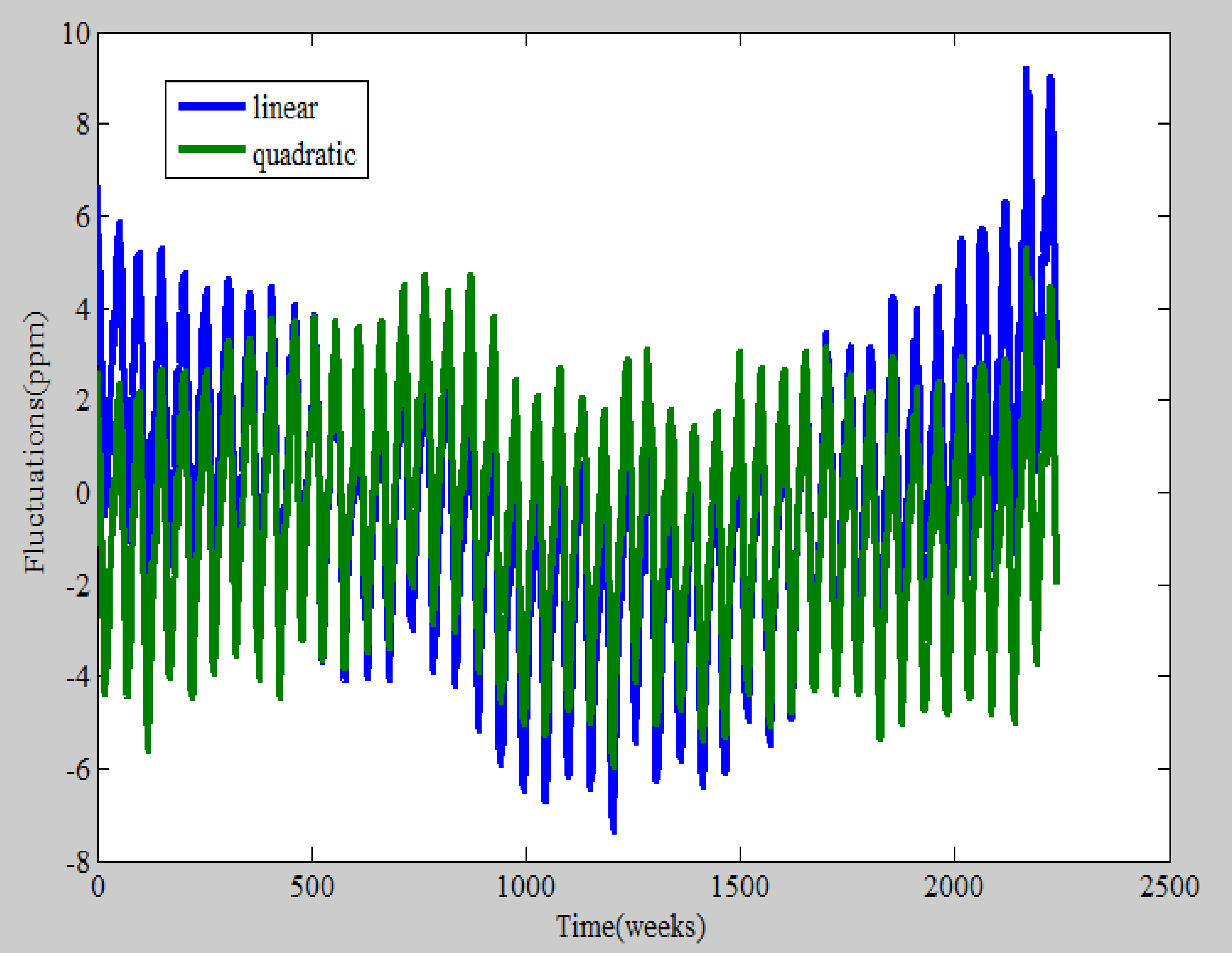
KEELING CURVE: *Cornerstone of Modern Climate Science*



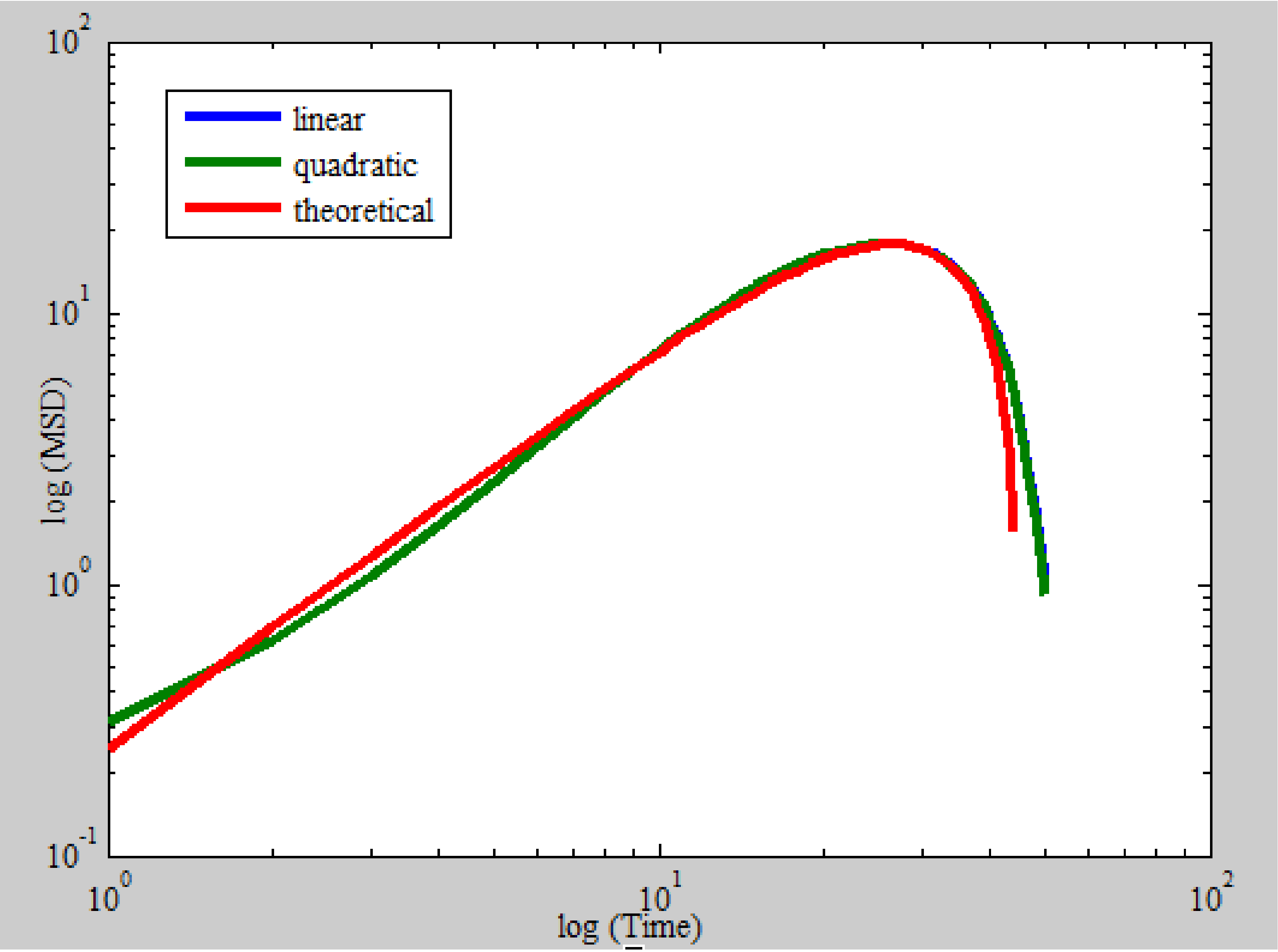
Keeling Curve is the daily record (red) of atmospheric CO_2 levels.



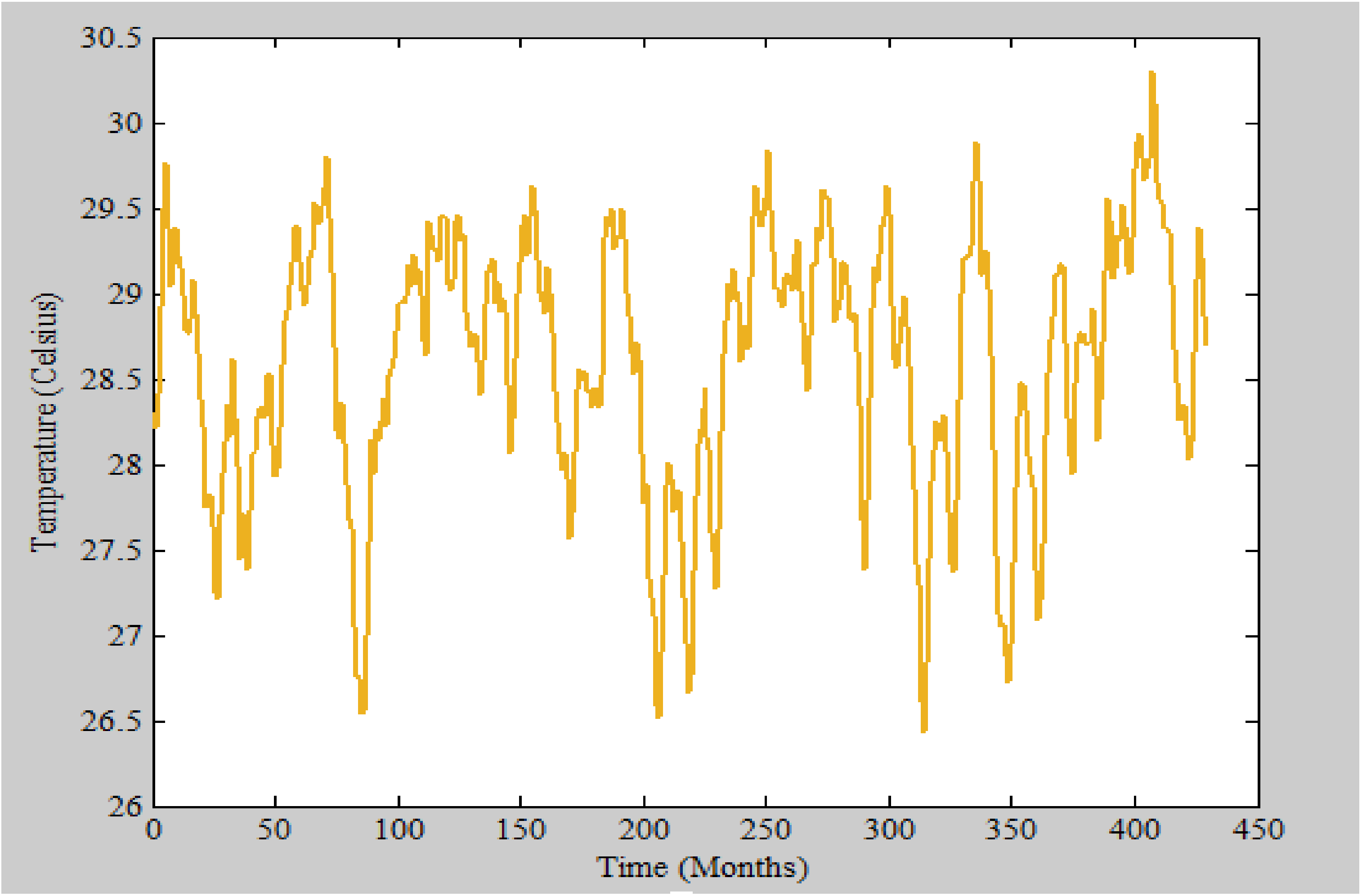
Fluctuations of CO_2 levels from the linear (or quadratic fit) are obtained by subtracting the data points from the linear (or quadratic) fit.



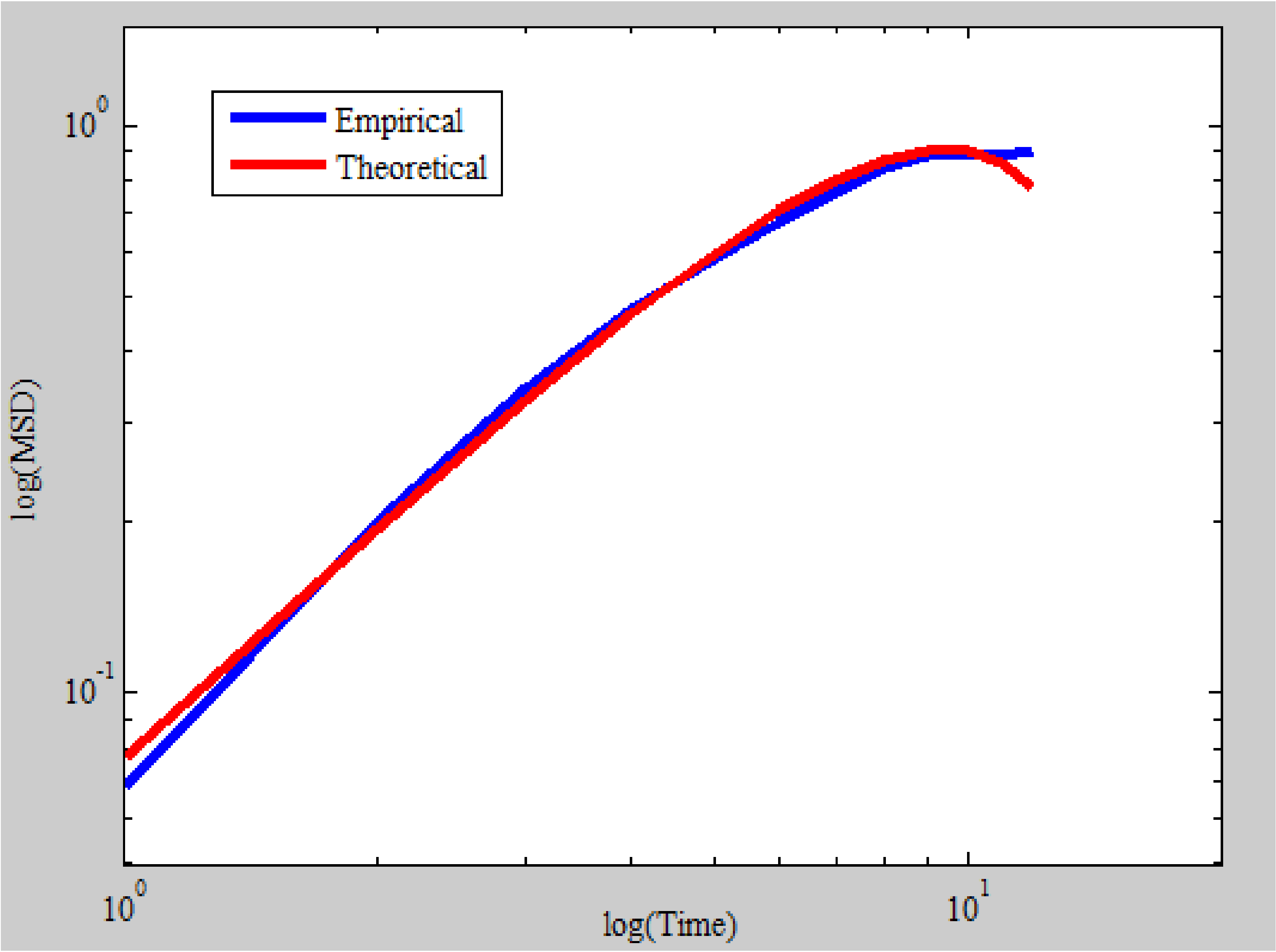
Fluctuations of CO_2 levels generated from the difference of the data from the linear fit (blue) and the quadratic fit (green).



MSD for fluctuations in **CO_2**. Empirical (blue/green line); theoretical (red line)



Sea surface temperatures at the equatorial Pacific Ocean (region: Niño 4).



MSD for fluctuations in **Sea Surface Temperatures** up to intervals of 12 months. Empirical (blue); theoretical (red).

SYSTEM	MEMORY PARAMETER	CHARACTERISTIC FREQUENCY
Great Barrier Reef Degradation	$\mu = 4.64$	$\nu = 0.99$
CO_2 Levels	$\mu = 1.25$	$\nu = 0.07$
Sea Surface Temperatures	$\mu = 1.18$	$\nu = 0.19$

Example 4:

Fluctuations in Diffusion Coefficients for Proteins of Variable Length

W.I. Barredo, C.C. Bernido, M.V. Carpio-Bernido, and J.B. Bornales, *Mathematical Biosciences* **297** (2018).

Mathematical Biosciences 297 (2018) 27–31

Contents lists available at ScienceDirect

Mathematical Biosciences

journal homepage: www.elsevier.com/locate/mbs

Modelling non-Markovian fluctuations in intracellular biomolecular transport

Wilson I. Barredo[a,b], Christopher C. Bernido[*,a,c,d], M. Victoria Carpio-Bernido[a,c,d], Jinky B. Bornales[a]

[a] Physics Department, MSU-Iligan Institute of Technology, Tibanga, Iligan City, 9200, Philippines
[b] Physics Department, Mindanao State University, Marawi City, 9700, Philippines
[c] Research Center for Theoretical Physics, Central Visayan Institute Foundation, Jagna, Bohol, 6308, Philippines
[d] Physics Department, University of San Carlos, Cebu City, 6000, Philippines

A R T I C L E I N F O

Keywords:
Sinusoidal Brownian motion
Fluctuations with memory
Diffusion of proteins

A B S T R A C T

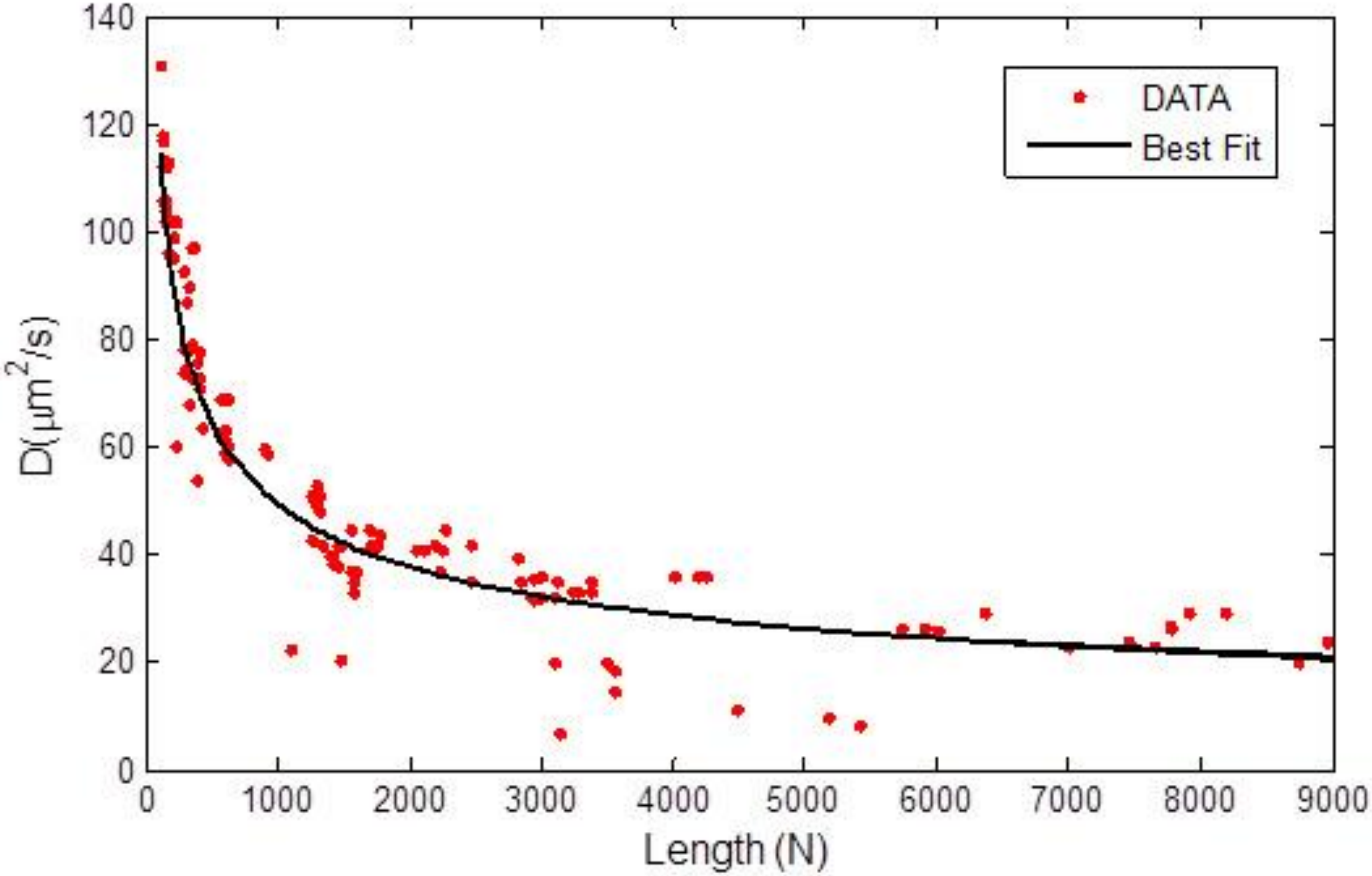
To model non-Markovian fluctuations arising in biomolecular transport, we introduce a stochastic process with memory where Brownian motion is modulated sinusoidally. The probability density function and moments of this non-Markovian process are evaluated analytically as Hida stochastic functional integrals. Comparison of graphs of computed variance vis-à-vis empirical data for protein diffusion coefficients closely match with both exhibiting emergent superdiffusive then subdiffusive behavior for longer proteins.

1. Introduction

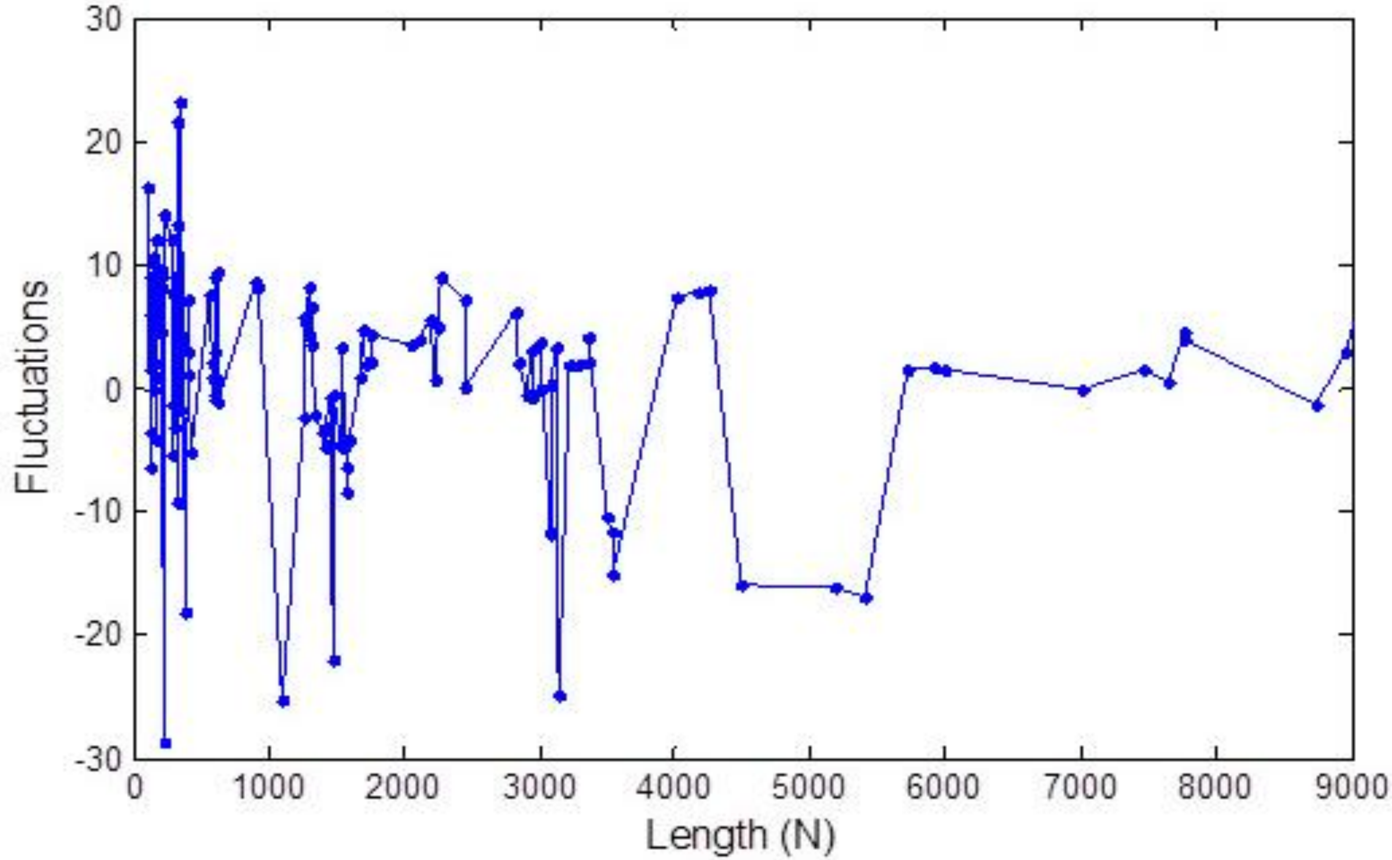
We present a Hida stochastic functional integral approach to infer non-Markovian structure of fluctuations generating nonlinear mean square deviations (MSD) of measured diffusion coefficients from estimated values for proteins of varying numbers of component amino

cellular membranes involve Markov models [8]. However, when memory or correlation between events is involved, it is necessary to go beyond Markov models. Fractional Brownian motion (fBm) has been used to describe anomalous diffusion in various phenomena including biological processes [9–11]. Nevertheless, this is still insufficient for the study of measured protein diffusion coefficients [2–4], since fBm may

Values of diffusion coefficients as protein length increases
(N is the number of amino acids)



K. A. Dill, K. Ghosh, and Jeremy D. Schmit, Physical limits of cells and proteomes, *Proc. Natl. Acad. Sci. USA* **108** (2011) 17876--17882.



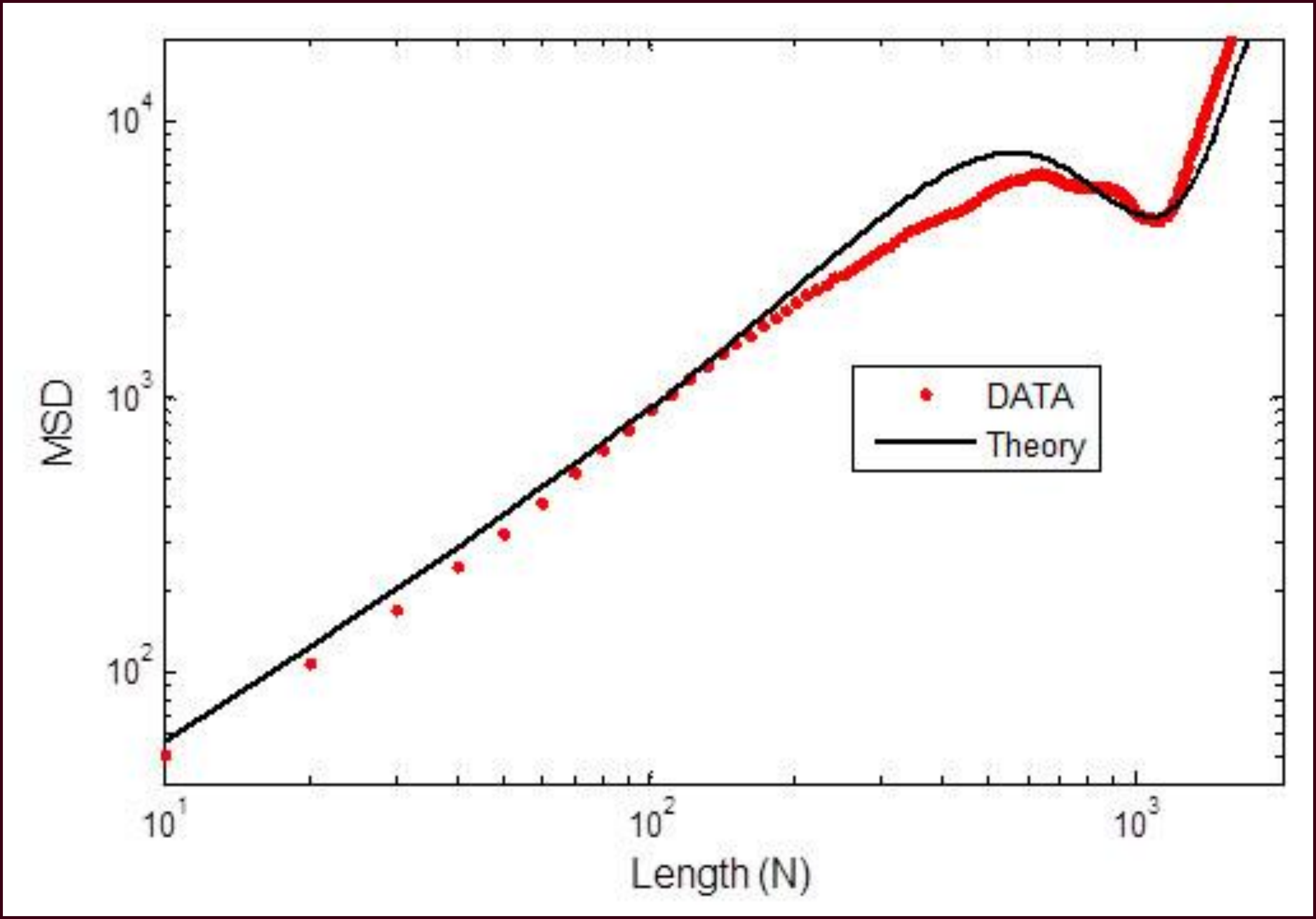
Displacements ξ of diffusion coefficients from the best fit curve vs. protein length.

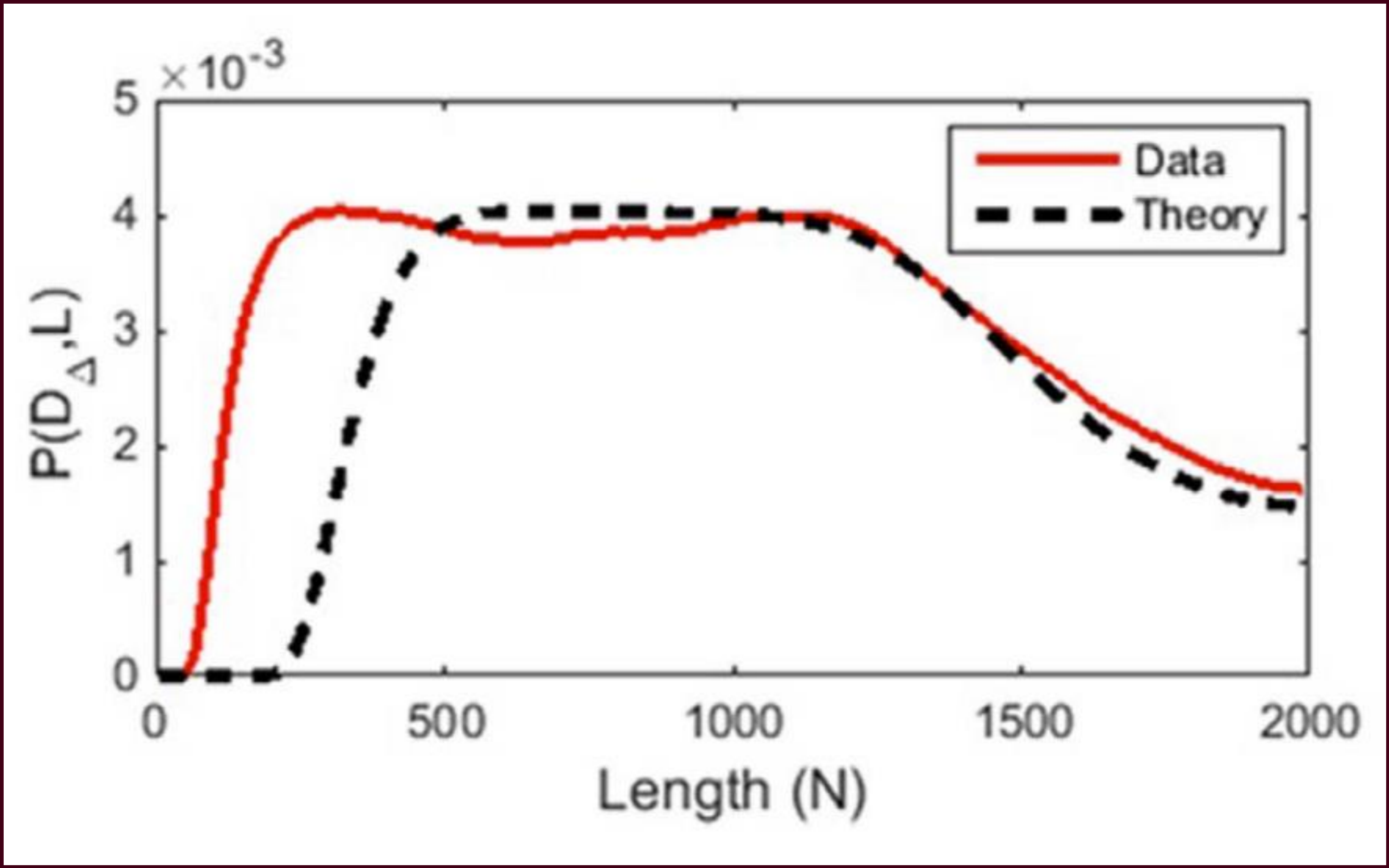
$$\xi(L) = \xi_0 + B^{SM}(L),$$

where:

$$B^{SM}(L) = \exp[b\sin(cL)] \int_0^L (L-s)^{(\nu-1)/2} \frac{\sin^{\frac{1}{2}}(as)}{s^{(1-\nu)/2}} dB(s).$$

$$\text{MSD}=\sqrt{\pi}\,\Gamma(\nu)\left(\frac{L}{a}\right)^{\nu-\frac{1}{2}} e^{2b\sin(cL)}\sin\left(\frac{aL}{2}\right) J_{\nu-\frac{1}{2}}\left(\frac{aL}{2}\right)$$





Empirical data (solid red line); Theoretical PDF (dotted line).