Physics 305 Demo Notebook 1: Simulating Stochastic Processes

Part 1: Simulating Brownian Motion

Brownian Motion: $B(t)$

Increment of the particle:

$$dx = \mu(t)dt + \sigma(t)dB(t),$$

where $\mu(t)$ is the time-dependent drift coefficient and $\sigma^2(t) = 2D$, where D is the diffusion coefficent.

Setting $\mu(t) = 0$ for simplicity (no drift), we have:

$$dx = \sigma(t)dB(t)$$

The increment has the Gaussian (normal) distribution $N(0, \sigma^2 dt)$, where $N(a, b)$ is the normal distribution with mean a and variance b.

Setting $\sigma(t) = 1$ and the initial position $x_0 = 0$ for simplicity, we get the *standard Brownian motion*. Note that this corresponds to $D = 1/2$.

We can numerically simulate Brownian motion by generating random numbers drawn from a normal distribution and getting their cumulative sum.

In [2]:
```
# import libraries
import numpy as np
from scipy.stats import norm
import matplotlib.pyplot as plt
%matplotlib inline
```

In [122]:
```
# Set random seed
np.random.seed(seed=17)

# Set parameters
n = int(1e6)  # no. of timesteps
dt = 1.  # size of time step
sd = np.sqrt(dt) # standard deviation

print("Number of timesteps n: %d" % n)
print("Time step dt: %.4f" % dt)
print("Standard Deviation sd: %.4f" % sd)
```

```
Number of timesteps n: 1000000
Time step dt: 1.0000
Standard Deviation sd: 1.0000
```

In [123]:
```
# Generate random numbers from Gaussian distribution centered at 0 and s
tandard deviation sd
rnd = norm.rvs(size = n, scale = sd)
```

In [124]:
```
len(rnd)
```

Out[124]:
```
1000000
```

In [125]:
```
np.min(rnd), np.max(rnd)
```

Out[125]:
```
(-4.964134609007479, 4.857362284140047)
```

In [126]:
```
(np.min(rnd), np.max(rnd))/sd
```

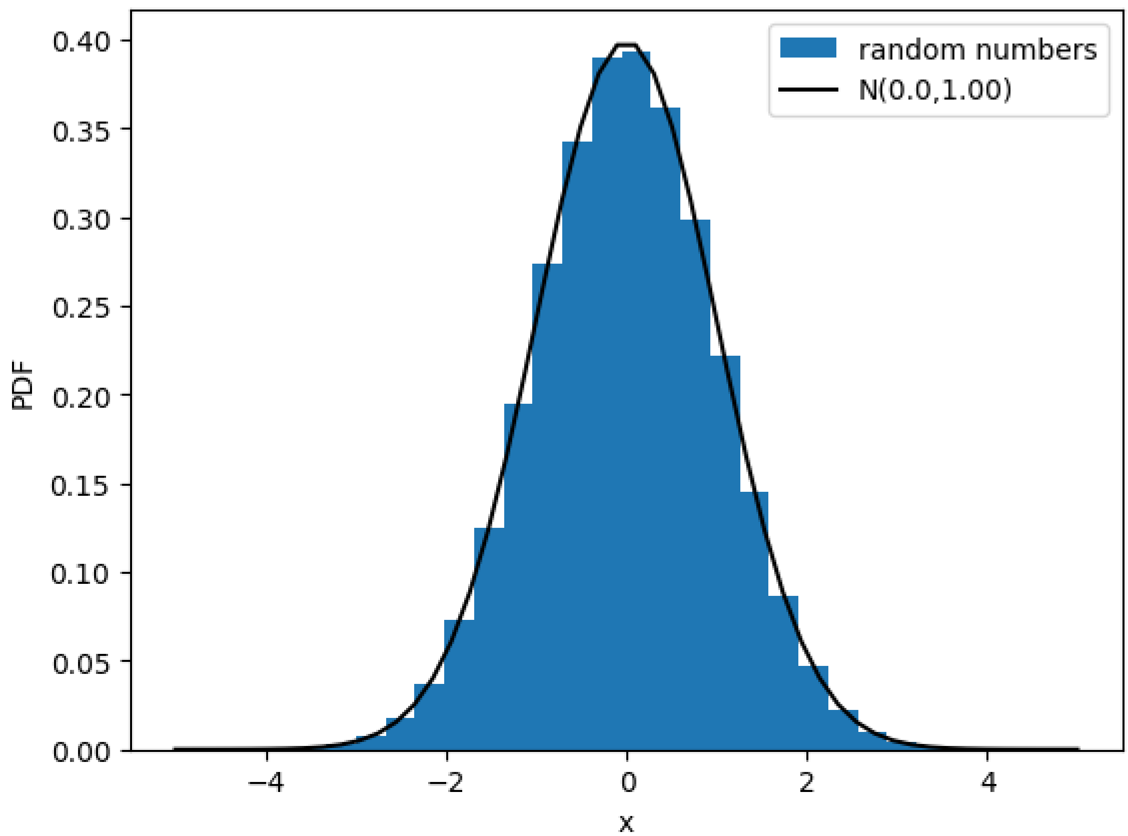
Out[126]:
```
array([-4.96413461,  4.85736228])
```

In [127]:
```
# Generate Gaussian distribution
xx = np.linspace(-5, 5)*sd # gridded points from -5 to 5 in units of sd
xx_mean = 0.
xx_sd = sd
yy = norm.pdf(xx, xx_mean, xx_sd)

# Plot histogram of random numbers
n_bins = 30
plt.hist(rnd, density=True, bins=n_bins, label="random numbers")

# Overlay normal distribution
plt.plot(xx, yy, 'k-', label="N(%.1f,%.2f)" % (xx_mean, xx_sd))
plt.ylabel("PDF")
plt.xlabel("x")
plt.legend(loc="best")
```

Out[127]:
```
<matplotlib.legend.Legend at 0x79c9027b29b0>
```



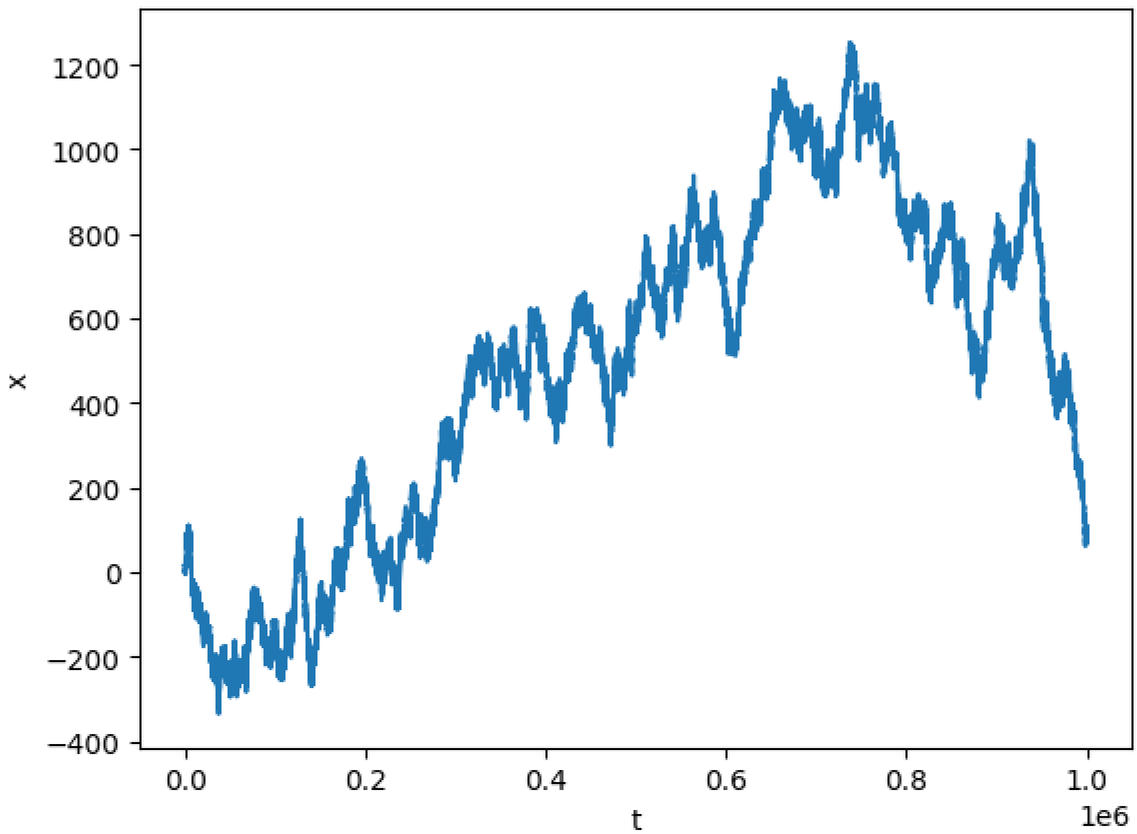
In [128]:
```
# Get cumulative sum of the elements of the array of random numbers
x = np.cumsum(rnd)
```

In [129]:
```
rnd[0:10], x[0:10]
```

Out[129]:
```
(array([ 0.27626589, -1.85462808,  0.62390111,  1.14531129,  1.0371904
7,
         1.88663893, -0.11169829, -0.36210134,  0.14867505, -0.4377831
5]),
 array([ 0.27626589, -1.57836219, -0.95446108,  0.19085021,  1.2280406
8,
         3.11467961,  3.00298132,  2.64087998,  2.78955503,  2.3517718
7]))
```

In [130]:
```
# Plot the Brownian motion
t = dt*np.arange(n)
plt.plot(t, x)
plt.xlabel("t")
plt.ylabel("x")
```

Out[130]:
```
Text(0, 0.5, 'x')
```



Part 2: Simulating Fractional Brownian Motion

Fractional Brownian Motion: $B^H(t)$

Here, H is the Hurst parameter. The different regimes depend on the value of H:

- 0 < H < 1/2: subdiffusion
- 1/2 < H < 1: superdiffusion
- H = 1/2: normal diffusion (Brownian motion)

We can numerically simulate fractional Brownian motion using the Wood-Chan or circulant method discussed in Georgiy Shevchenko's lecture notes from the 7th Jagna International Workshop: "Fractional Motion in a Nutshell" - [https://arxiv.org/pdf/1406.1956.pdf (https://arxiv.org/pdf/1406.1956.pdf)](https://arxiv.org/pdf/1406.1956.pdf) (Section 6). The following code is translated from the original Matlab code.

For this course, we can consider the above supplementary reading.

In [113]:
```
# import libraries
import numpy as np
import random
import matplotlib.pyplot as plt
%matplotlib inline
```

In [114]:
```
# define functions
def fbc(n,G):
    return ((n+1)**G + np.abs(n-1)**G - 2*n**G)/2.

def lambda_func(H,N):
    M = 2*N - 2
    C = np.zeros(M)
    G = 2*H
    for i in np.arange(N):
        C[i] = fbc(i,G) # fill in first N out of M values of C (i=0 to N
-1)
    C[N:] = C[1:N-1][::-1]
    return np.real(np.fft.fft(C))**0.5
```

In [115]:
```
# set process parameters
q = 11
N = 2**q + 1
M = 2*N - 2
delta = 0.002
print("q: %d, N: %d, M: %d" % (q, N, M))
```

```
q: 11, N: 2049, M: 4096
```

In [116]:
```
# initialize
noise = np.random.normal(size=(M))
fGnsamples = np.zeros((5,N))

# generate fractional Gaussian noise samples
for i in np.arange(5):
    H = 0.2*(i+1) - 0.1
    lambda_res = lambda_func(H,N)
    a = np.fft.ifft(noise)*lambda_res
    b = np.real(np.fft.fft(a))
    fGnsamples[i,0] = 15-3*(i+1)          # apply offset in starting v
alue for visualization
    fGnsamples[i,1:N] = delta**H*b[0:N-1]

# take cumulative sums to get the fractional Brownian motion samples
fBmsamples = np.transpose(np.cumsum(fGnsamples,axis=1)) # cumulative sum
over each row
```

In [117]:
```
np.shape(fBmsamples)
```

Out[117]:
```
(2049, 5)
```

In [118]:
```
# get values of H
Hval = (np.arange(5) + 1)*0.2 - 0.1
print(Hval)
```

```
[0.1 0.3 0.5 0.7 0.9]
```

In [121]:
```
# plot fractional Brownian motion samples
for i in np.arange(len(Hval)):
  plt.plot(fBmsamples[:,i], label="H=%.1f" % Hval[i])
plt.ylim(-5, 20)
plt.legend(loc="upper left", ncol=3)
plt.xlabel("t")
plt.ylabel("x")
```

Out[121]:
```
Text(0, 0.5, 'x')
```