6

Kane's Method in Robotics

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6.1 Introduction

In 1961, Professor Thomas Kane of Stanford University published a paper entitled "Dynamics of Nonholonomic Systems" [1] that described a method for formulating equations of motion for complex dynamical systems that was equally applicable to either holonomic or nonholonomic systems. As opposed to the use of Lagrange's equations, this new method allowed dynamical equations to be generated without the differentiation of kinetic and potential energy functions and, for nonholonomic systems, without the need to introduce Lagrange multipliers. This method was later complemented by the publication in 1965 of a paper by Kane and Wang entitled "On the Derivation of Equations of Motion" [2] that described the use of motion variables (later called generalized speeds) that could be any combinations of the time derivatives of the generalized coordinates that describe the configuration of a system. These two papers laid the foundation for what has since become known as Kane's method for formulating dynamical equations.

The origins of Kane's method can be found in Kane's undergraduate dynamics texts entitled *Analytical Elements of Mechanics* volumes 1 and 2 [3, 4] published in 1959 and 1961, respectively. In particular, in Section 4.5.6 of [4], Kane states a "law of motion" containing a term referred to as the *activity in R* (a reference frame) *of the gravitational and contact forces on P* (a particular particle of interest). Kane's focus on the *activity* of a set of forces was a significant step in the development of his more general dynamical method, as is elaborated in Section 6.2. Also important to Kane's approach to formulating dynamical equations was his desire to avoid what he viewed as the vagaries of the Principle of Virtual Work, particularly when applied to the analysis of systems undergoing three-dimensional rotational motions. Kane's response to the need to clarify the process of formulating equations of motion using the Principle of Virtual Work was one of the key factors that led to the development of his own approach to the generation of dynamical equations.

Although the application of Kane's method has clear advantages over other methods of formulating dynamical equations [5], the importance of Kane's method only became widely recognized as the space industry of the 1960s and 1970s drove the need to model and simulate ever more complex dynamical systems and as the capabilities of digital computers increased geometrically while computational costs concomitantly decreased. In the 1980s and early 1990s, a number of algorithms were developed for the dynamic analysis of multibody systems (references [6–9] provide comprehensive overviews of various forms of these dynamical methods), based on variations of the dynamical principles developed by Newton, Euler, Lagrange, and Kane. During this same time, a number of algorithms lead to commercially successful computer programs [such as ADAMS (Automatic Dynamic Analysis of Mechanisms) [10], DADS (Dynamic Analysis and Design of Systems) [11], NEWEUL [12], SD/FAST [13], AUTOLEV [14], Pro/MECHANICA MOTION, and Working Model [15], to name just a few], many of which are still on the market today. As elaborated in Section 6.7, many of the most successful of these programs were either directly or indirectly influenced by Kane and his approach to dynamics.

The widespread attention given to efficient dynamical methods and the development of commercially successful multibody dynamics programs set the stage for the application of Kane's method to complex robotic mechanisms. Since the early 1980s, numerous papers have been written on the use of Kane's method in analyzing the dynamics of various robots and robotic devices (see Section 6.6 for brief summaries of selected articles). These robots have incorporated revolute joints, prismatic joints, closed-loops, flexible links, transmission mechanisms, gear backlash, joint clearance, nonholonomic constraints, and other characteristics of mechanical devices that have important dynamical consequences. As evidenced by the range of articles described in Section 6.6, Kane's method is often the method of choice when analyzing robots with various forms and functions.

The broad goal of this chapter is to provide an introduction to the application of Kane's method to robots and robotic devices. It is essentially tutorial while also providing a limited survey of articles that address robot analysis using Kane's method as well as descriptions of multipurpose dynamical analysis software packages that are either directly or indirectly related to Kane's approach to dynamics. Although a brief description of the fundamental basis for Kane's method and its relationship to Lagrange's equations is given in Section 6.2, the purpose of this chapter is not to enter into a prolonged discussion of the relationship between Kane's method and other similar dynamical methods, such as the "orthogonal complement method" (the interested reader is referred to references [16,17] for detailed commentary) or Jourdain's principle (see "Kane's equations or Jourdain's principle?" by Piedboeuf [18] for further information and a discussion of Jourdain's original 1909 work entitled "Note on an analogue of Gauss' principle of least constraint" in which he established the principle of virtual power) or Gibbs-Appell equations [interested readers are referred to a lively debate on the subject that appeared in volumes 10 (numbers 1 and 6), 12(1), and 13(2) of the *Journal of Guidance, Control, and Dynamics* from 1987 to 1990]. The majority of this chapter (Section 6.3, Section 6.4, and Section 6.5) is in fact devoted to providing a tutorial illustration of the application of Kane's method to the dynamic analysis of two relatively simple robots: a two-degree-of-freedom planar robot with two revolute joints and a two-degree-of-freedom planar robot with one revolute joint and one prismatic joint. Extentions and modifications of these analyses that are facilitated by the use of Kane's method are also discussed as are special issues in the use of Kane's method, such as formulating linearized equations, generating equations of motion for systems subject to constraints, and developing equations of motion for systems with continuous elastic elements, leading to a detailed analysis of the two-degree-of-freedom planar robot with one revolute joint and one prismatic joint when the element traversing the prismatic joint is regarded as elastic.

Following these tutorial sections, a brief summary of the range of applications of Kane's method in robotics is presented. Although not meant to be an exhaustive list of publications involving the use of Kane's method in robotics, an indication of the popularity and widespread use of Kane's method in robotics is provided. The evolution of modern commercially available dynamical analysis computer software is also briefly described as is the relationship that various programs have to either Kane's method or more general work that Kane has contributed to the dynamics and robotics literature. Through this chapter, it is hoped that readers previously unfamiliar with Kane's method will gain at least a flavor of its application. Readers already acquainted with Kane's method will hopefully gain new insights into the method as well as have the opportunity to recognize the large number of robotic problems in which Kane's method can be used.

6.2 The Essence of Kane's Method

Kane's contributions to dynamics have been not only to the development of "Kane's method" and "Kane's equations" but also to the clarity with which one can deal with basic kinematical principles (including the explicit and careful accounting for the reference frames in which kinematical and dynamical relationships are developed), the definition of basic kinematical, and dynamical quantities (see, for example, Kane's paper entitled "Teaching of Mechanics to Undergraduates" [19]), the careful deductive way in which he derives all equations from basic principles, and the algorithmic approach he prescribes for the development of dynamical equations of motion for complex systems. These are in addition to the fundamental elements inherent in Kane's method, which allow for a clear and convenient separation of kinematical and dynamical considerations, the exclusion of nonworking forces, the use of generalized speeds to describe motion, the systematic way in which constraints can be incorporated into an analysis, and the ease and confidence with which linearized of equations of motion can be developed.

Before considering examples of the use of Kane's method in robotics, a simple consideration of the essential basis for the method may be illuminating. Those who have read and studied *DYNAMICS: Theory and Applications* [20] will recognize that the details of Kane's approach to dynamics and to Kane's method can obscure the fundamental concepts on which Kane's method is based. In Section 5.8 of the first edition of *DYNAMICS* [21] (note that an equivalent section is not contained in *DYNAMICS: Theory and Applications*), a brief discussion is given of the basis for "Lagrange's form of D'Alembert's principle" [Equation (6.1) in [21] and Equation (6.1) of Chapter 6 in [20] where it is referred to as *Kane's dynamical equations*]. This section of *DYNAMICS* offers comments that are meant to "shed light" on Kane's equations "by reference to analogies between these equations and other, perhaps more familiar, relationships." Section 5.8 of *DYNAMICS* is entitled "The Activity and Activity-Energy Principles" and considers the development of equations of motion for a single particle. While the analysis of a single particle does not give full insight into the advantages (and potential disadvantages) inherent in the use of Kane's method, it does provide at least a starting point for further discussion and for understanding the origins of Kane's method.

For a single particle P for which $\mathbf{F}$ is the resultant of all contact and body forces acting on P and for which $\mathbf{F}^*$ is the inertia force for P in an inertial reference frame R (note that for a single particle, $\mathbf{F}^*$ is simply equal to $-m\mathbf{a}$, where m is the mass of P and $\mathbf{a}$ is the acceleration of P in R), D'Alembert's principle states that

$$\mathbf{F} + \mathbf{F}^* = 0 \tag{6.1}$$

When this equation is dot-multiplied with the velocity $\mathbf{v}$ of P in R, one obtains

$$\mathbf{v} \cdot \mathbf{F} + \mathbf{v} \cdot \mathbf{F}^* = 0 \tag{6.2}$$

Kane goes on in Section 5.8 of *DYNAMICS* to define two scalar quantities A and A^* such that

$$A = \mathbf{v} \cdot \mathbf{F} \tag{6.3}$$

$$A^* = \mathbf{v} \cdot \mathbf{F}^* \tag{6.4}$$

and then presents

$$A + A^* = 0 \tag{6.5}$$

as a statement of the *activity principle* for a single particle P for which A and A^* are called the *activity* of force $\mathbf{F}$ and the *inertia activity* of the inertia force $\mathbf{F}^*$, respectively (note that Kane refers to A^* as the *activity* of the force $\mathbf{F}^*$; here A^* is referred to the *inertia activity* to distinguish it from the *activity* A).

Kane points out that Equation (6.5) is a scalar equation, and thus it cannot "furnish sufficient information for the solution in which P has more than one degree of freedom." He continues by noting that Equation (6.5) is weaker than Equation (6.1), which is equivalent to three scalar equations. Equation (6.5) does, however, possess one advantage over Equation (6.1). If $\mathbf{F}$ contains contributions from (unknown) constraint forces, these forces will appear in Equation (6.1) and then need to be eliminated from the final dynamical equation(s) of motion; whereas, in cases in which the components of $\mathbf{F}$ corresponding to constraint directions are ultimately not of interest, they are automatically eliminated from Equation (6.5) by the dot multiplication needed to produce A and A^* as given in Equation (6.3) and Equation (6.4).

The essence of Kane's method is thus to arrive at a procedure for formulating dynamical equations of motion that, on the one hand, contain sufficient information for the solution of problems in which P has more than one degree of freedom, and on the other hand, automatically eliminate unknown constraint forces. To that end, Kane noted that one may replace Equation (6.2) with

$$\mathbf{v}_r \cdot \mathbf{F} + \mathbf{v}_r \cdot \mathbf{F}^* = 0 \tag{6.6}$$

where $\mathbf{v}_r (r = 1, \ldots, n)$ are the *partial velocities* (see Section 6.3 for a definition of partial velocities) of P in R and n is the number of degrees of freedom of P in R (note that the $\mathbf{v}_r$ form a set of independent quantities). Furthermore, if F_r and F_r^* are defined as

$$F_r = \mathbf{v}_r \cdot \mathbf{F}, \qquad F_r^* = \mathbf{v}_r \cdot \mathbf{F}^* \tag{6.7}$$

one can then write

$$F_r + F_r^* = 0 \quad (r = 1, \ldots, n) \tag{6.8}$$

where F_r and F_r^* are referred to as the rth *generalized active force* and the rth *generalized inertia force* for P in R. Although referred to in Kane's earlier works, including [21], as *Lagrange's form of D'Alembert's principle*, Equation (6.9) has in recent years come to be known as *Kane's equations.*

Using the expression for the generalized inertia force given in Equation (6.8) as a point of departure, the relationship between Kane's equations and Lagrange's equations can also be investigated. From Equation (6.8),

$$F_r^* = \mathbf{v}_r \cdot \mathbf{F}^* = \mathbf{v}_r \cdot (-m\mathbf{a}) = -m\mathbf{v}_r \cdot \frac{d\mathbf{v}}{dt} = -\frac{m}{2}\left(\frac{d}{dt}\frac{\partial \mathbf{v}^2}{\partial \dot{q}_r} - \frac{\partial \mathbf{v}^2}{\partial q_r}\right) \quad [20, \text{p } 50] \tag{6.9}$$

$$= -\frac{d}{dt}\frac{\partial}{\partial \dot{q}_r}\left(\frac{m\mathbf{v}^2}{2}\right) + \frac{\partial}{\partial q_r}\left(\frac{m\mathbf{v}^2}{2}\right) = -\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_r}\frac{\partial K}{\partial q_r} \tag{6.10}$$

where K is the kinetic energy of P in R. Substituting Equation (6.10) into Equation (6.8) gives

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_r} + \frac{\partial K}{\partial q_r} = F_r \quad (r = 1, \ldots, n) \tag{6.11}$$

which can be recognized as *Lagrange's equations of motion of the first kind.*

6.3 Two DOF Planar Robot with Two Revolute Joints

In order to provide brief tutorials on the use of Kane's method in deriving equations of motion and to illustrate the steps that make up the application of Kane's method, in this and the following section, the dynamical equations of motion for two simple robotic systems are developed. The first system is a

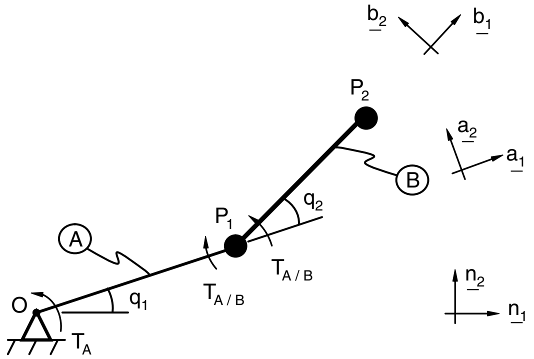

FIGURE 6.1 Two DOF planar robot with two revolute joints.

two-degree-of-freedom robot with two revolute joints moving in a vertical plane. The second is a two-degree-of-freedom robot with one revolute and one prismatic joint moving in a horizontal plane. Both of these robots have been chosen so as to be simple enough to give a clear illustration of the details of Kane's method without obscuring key points with excessive complexity.

As mentioned at the beginning of Section 6.2, Kane's method is very algorithmic and as a result is easily broken down into discrete general steps. These steps are listed below:

- Definition of preliminary information
- Introduction of generalized coordinates and speeds
- Development of requisite velocities and angular velocities
- Determination of partial velocities and partial angular velocities
- Development of requisite accelerations and angular accelerations
- Formulation of generalized inertia forces
- Formulation of generalized active forces
- Formulation of dynamical equations of motion by means of Kane's equations

These steps will now be applied to the system shown in Figure 6.1. This system represents a very simple model of a two-degree-of-freedom robot moving in a vertical plane. To simplify the system as much as possible, the mass of each of the links of the robot has been modeled as being lumped into a single particle.

6.3.1 Preliminaries

The first step in formulating equations of motion for any system is to introduce symbols for bodies, points, constants, variables, unit vectors, and generalized coordinates. The robot of Figure 6.1 consists of two massless rigid link A and B, whose motions are confined to parallel vertical planes, and two particles P_1 and P_2, each modeled as being of mass m. Particle P_1 is located at the distal end of body A and P_2 is located at the distal end of body B. Body A rotates about a fixed horizontal axis through point O, while body B rotates about a horizonal axis fixed in A and passing through P_1. A constant needed in the description of the robot is L, which represents the lengths of both links A and B. Variables for the system are the torque T_A, applied to link A by an inertially fixed actuator, and the torque $T_{A/B}$, applied to link B by an actuator attached to A. Unit vectors needed for the description of the motion of the system are $\mathbf{n}_i$, $\mathbf{a}_i$, and $\mathbf{b}_i$ $(i = 1, 2, 3)$. Unit vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are fixed in an inertial reference frame N, $\mathbf{a}_1$, and $\mathbf{a}_2$ are fixed in A, and $\mathbf{b}_1$ and $\mathbf{b}_2$ are fixed in B, as shown. The third vector of each triad is perpendicular to the plane formed by the other two such that each triad forms a right-handed set.

6.3.2 Generalized Coordinates and Speeds

While even for simple systems there is an infinite number of possible choices for the generalized coordinates that describe a system's configuration, generalized coordinates are usually selected based on physical relevance and analytical convenience. Generalized coordinates for the system of Figure 6.1 that are both relevant and convenient are the angles q_1 and q_2. The quantity q_1 measures the angle between an inertially fixed horizontal line and a line fixed in A, and q_2 measures the angle between a line fixed in A and a line fixed in B, both as shown.

Within the context of Kane's method, a complete specification of the kinematics of a system requires the introduction of quantities known as *generalized speeds*. Generalized speeds are defined as any (invertible) linear combination of the time derivatives of the generalized coordinates and describe the *motion* of a system in a way analogous to the way that generalized coordinates describe the *configuration* of a system. While, as for the generalized coordinates, there is an infinite number of possibilities for the generalized speeds describing the motion of a system, for the system at hand, reasonable generalized speeds (that will ultimately lead to equations of motion based on a "joint space" description of the robot) are defined as

$$u_1 = \dot{q}_1 \tag{6.12}$$

$$u_2 = \dot{q}_2 \tag{6.13}$$

An alternate, and equally acceptable, choice for generalized speeds could have been the $\mathbf{n}_1$ and $\mathbf{n}_2$ components of the velocity of P_2 (this choice would lead to equations of motion in "operational space"). For a comprehensive discussion of guidelines for the selection of generalized speeds that lead to "exceptionally efficient" dynamical equations for a large class of systems frequently encountered in robotics, see [22].

6.3.3 Velocities

The angular and translational velocities required for the development of the equations of motion for the robot of Figure 6.1 are the angular velocities of bodies A and B as measured in reference frame N and the translational velocities of particles P_1 and P_2 in N. With the choice of generalized speeds given above, expressions for the angular velocities are

$$\boldsymbol{\omega}^A = u_1 \mathbf{a}_3 \tag{6.14}$$

$$\boldsymbol{\omega}^B = (u_1 + u_2)\mathbf{b}_3 \tag{6.15}$$

Expressions for the translational velocities can be developed either directly from Figure 6.1 or from a straightforward application of the kinematical formula for relating the velocities of two points fixed on a single rigid body [20, p. 30]. From inspection of Figure 6.1, the velocities of P_1 and P_2 are

$$\mathbf{v}^{P_1} = L u_1 \mathbf{a}_2 \tag{6.16}$$

$$\mathbf{v}^{P_2} = L u_1 \mathbf{a}_2 + L(u_1 + u_2)\mathbf{b}_2 \tag{6.17}$$

6.3.4 Partial Velocities

With all the requisite velocity expressions in hand, Kane's method requires the identification of *partial velocities*. Partial velocities must be identified from the angular velocities of all nonmassless bodies and of bodies acted upon by torques that ultimately contribute to the equations of motion (i.e., bodies acted upon by nonworking torques and nonworking sets of torques need not be considered) as well as from the translational velocities of all nonmassless particles and of points acted upon by forces that ultimately contribute to the equations of motion. These partial velocities are easily identified and are simply the coefficients of the generalized speeds in expressions for the angular and translational velocities. For the system of Figure 6.1, the partial velocities are determined by inspection from Equation (6.14) through (6.17). The resulting partial velocities are listed in Table 6.1, where $\boldsymbol{\omega}_r^A$ is the rth partial angular velocity of A in N, $\mathbf{v}_r^{P_1}$ is the rth partial translational velocity of P_1 in N, etc.

TABLE 6.1 Partial Velocities for Robot of Figure 6.1

	$r = 1$	$r = 2$
$\boldsymbol{\omega}_r^A$	$\mathbf{a}_3$	0
$\boldsymbol{\omega}_r^B$	$\mathbf{b}_3$	$\mathbf{b}_3$
$\mathbf{v}_r^{P_1}$	$L\mathbf{a}_2$	0
$\mathbf{v}_r^{P_2}$	$L(\mathbf{a}_2 + \mathbf{b}_2)$	$L\mathbf{b}_2$

6.3.5 Accelerations

In order to complete the development of kinematical quantities governing the motion of the robot, one must develop expressions for the translational accelerations of particles P_1 and P_2 in N. The translational accelerations can be determined either by direct differentiation or by applying the basic kinematical formula for relating the accelerations of two points fixed on the same rigid body [20, p. 30]. For the problem at hand, the latter approach is more convenient and yields

$$\mathbf{a}^{P_1} = -Lu_1^2\mathbf{a}_1 + L\dot{u}_1\mathbf{a}_2 \tag{6.18}$$

$$\mathbf{a}^{P_2} = -Lu_1^2\mathbf{a}_1 + L\dot{u}_1\mathbf{a}_2 - L(u_1 + u_2)^2\mathbf{b}_1 + L(\dot{u}_1 + \dot{u}_2)\mathbf{b}_2 \tag{6.19}$$

where $\mathbf{a}^{P_1}$ and $\mathbf{a}^{P_2}$ are the translational accelerations of P_1 and P_2 in N.

6.3.6 Generalized Inertia Forces

In general, the generalized inertia forces F_r^* $(i = 1, \ldots, n)$, for a system of particles with n degrees of freedom in a reference frame N can be expressed as

$$F_r^* = -\sum_{i=1}^{n} \mathbf{v}_r^{P_i} \cdot m_i \mathbf{a}^{P_i} \qquad (r = 1, \ldots, n) \tag{6.20}$$

where $\mathbf{v}_r^{P_i}$ is the rth partial velocity of particle P_i in N, $\mathbf{a}^{P_i}$ is the acceleration of P_i in N, and m_i is the mass of particle P_i. For the robot of Figure 6.1, by making use of the partial velocities in Table 6.1 and the accelerations in Equation (6.18) and Equation (6.19), the general expression of Equation (6.20) yields

$$F_1^* = -mL^2\left\{(3 + 2c_2)\dot{u}_1 + (1 + c_2)\dot{u}_2 + s_2\left[u_1^2 - (u_1 + u_2)^2\right]\right\} \tag{6.21}$$

$$F_2^* = -mL^2\left[(1 + c_2)\dot{u}_1 + \dot{u}_2 + s_2u_2^2\right] \tag{6.22}$$

where s_2 and c_2 are defined as the sine and cosine of q_2, respectively.

6.3.7 Generalized Active Forces

Nonworking forces (or torques), or sets of forces (or sets of torques), make no net contribution to the generalized active forces. To determine the generalized active forces for the robot of Figure 6.1, therefore, one need only consider the gravitational force acting on each of the two particles and the two torques T_A and $T_{A/B}$. The fact that only these forces and torques contribute to the generalized active forces highlights one of the principal advantages of Kane's method. Forces such as contact forces exerted on parts of a system across smooth surfaces of rigid bodies, contact and body forces exerted by parts of a rigid body on one another, and forces that are exerted between two bodies that are in rolling contact make no net contribution to generalized active forces. Indeed, the fact that all such forces do not contribute to the generalized active forces is one of the principal motivations for introducing the concept of generalized active forces.

In general, generalized active forces are constructed by dot multiplying all contributing forces and torques with the partial translational velocities and partial angular velocities of the points and bodies to which they are applied. For the system at hand, therefore, one can write the generalized active forces F_r as

$$F_r = \mathbf{v}_r^{P_1} \cdot (-mg\mathbf{n}_2) + \mathbf{v}_r^{P_2} \cdot (-mg\mathbf{n}_2) + \boldsymbol{\omega}_r^A \cdot (T_A\mathbf{a}_3 - T_{A/B}\mathbf{b}_3) + \boldsymbol{\omega}_r^B \cdot T_{A/B}\mathbf{b}_3 \; (r = 1, 2) \tag{6.23}$$

Substituting from Table 6.1 into Equation (6.23) produces

$$F_1 = T_A + mgL(2s_2 + s_{23}) \tag{6.24}$$

$$F_2 = T_{A/B} + mgLs_{23} \tag{6.25}$$

where s_{23} is the sine of $q_2 + q_3$.

6.3.8 Equations of Motion

Finally, now that all generalized active and inertia forces have been determined, the equations of motion for the robot can be formed by substituting from Equation (6.21), Equation (6.22), Equation (6.24), and Equation (6.25) into Kane's equations:

$$F_r + F_r^* = 0 \qquad (r = 1, 2) \tag{6.26}$$

Equation (6.26) provides a complete description of the dynamics of the simple robotic system of Figure 6.1.

6.3.9 Additional Considerations

Although two of the primary advantages of Kane's method are the ability to introduce motion variables (generalized speeds) as freely as configuration variables (generalized coordinates) and the elimination of nonworking forces, Kane's method also facilitates modifications to a system once equations of motion have already been formulated. For example, to consider the consequence to the equations of motion of applying an external force $F_x^E\mathbf{n}_1 + F_y^E\mathbf{n}_2$ to the distal end of the robot (at the location of P_2), one simply determines additional contributions F_r^{Ext} to the generalized active forces given by

$$F_r^{Ext} = \mathbf{v}_r^{P_2} \cdot \left(F_x^E\mathbf{n}_1 + F_y^E\mathbf{n}_2\right) \qquad (r = 1, 2) \tag{6.27}$$

and adds these contributions to the equations of motion in Equation (6.26). One could similarly consider the effect of viscous damping torques at the joints by adding contributions to the generalized active forces given by

$$F_r^{Damp} = \boldsymbol{\omega}_r^A \cdot [-b_{t1}u_1\mathbf{a}_3 + b_{t2}(u_1 + u_2)\mathbf{b}_3] + \boldsymbol{\omega}_r^B \cdot [-b_{t2}(u_1 + u_2)\mathbf{b}_3] \qquad (r = 1, 2) \tag{6.28}$$

where b_{t1} and b_{t2} are viscous damping coefficients at the first and second joints.

Another consideration that often arises in robotics is the relationship between formulations of equations of motion in joint space and operational space. As mentioned at the point at which generalized speeds were defined, the above derivation could easily have produced equations corresponding to operational space simply by defining the generalized speeds to be the $\mathbf{n}_1$ and $\mathbf{n}_2$ components of the velocity of P_2 and then using this other set of generalized speeds to define partial velocities analogous to those appearing in Table 6.1. A systematic approach to directly converting between the joint space equations in Equation (6.26) and corresponding operational space equations is described in Section 6.5.2.

6.4 Two-DOF Planar Robot with One Revolute Joint and One Prismatic Joint

The steps outlined at the beginning of the previous section will now be applied to the system shown in Figure 6.2. This system represents a simple model of a two-degree-of-freedom robot with one revolute and one prismatic joint moving in a horizontal plane. While still relatively simple, this system is significantly more complicated than the one analyzed in the preceding section and gives a fuller understanding of issues

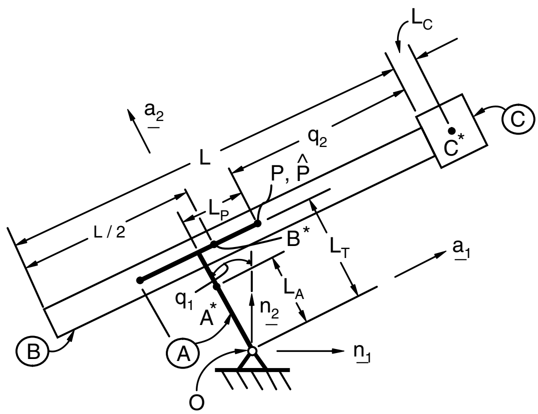

FIGURE 6.2 Two DOF robot with one revolute joint and one prismatic joint.

related to the application of Kane's method. Moreover, as is discussed in greater detail in Section 6.5.3, this robot represents a restricted model of the Stanford Arm [23] and thus provides insights into the formulation of equations of motion for Stanford-like manipulators.

6.4.1 Preliminaries

As in the preceding section, the first step in formulating equations of motion for the system now under consideration is to introduce symbols for bodies, points, constants, variables, unit vectors, and generalized coordinates. The robot of Figure 6.2 consists of three rigid bodies A, B, and C, whose motions are confined to parallel horizontal planes. Body A rotates about a fixed vertical axis while body B translates along an axis fixed in A. Body C is rigidly attached to one end of B, as shown. The masses of A, B, and C are denoted m_A, m_B, and m_C, respectively, and the moments of inertia of the three bodies about vertical lines passing through their mass centers are denoted I_A, I_B, and I_C, respectively. Points of interest are O, P, $\widehat{P}$, $\overline{A}^{B^*}$, and the mass centers of bodies A, B, and C. Point O lies on the axis about which body A rotates, P is a point of body A at that location where the force that drives the motion of B relative to A is applied, $\widehat{P}$ is a point of body B whose location instantaneously coincides with P, and $\overline{A}^{B^*}$ is a point regarded as fixed on body A but whose location is instantaneously coincident with the mass center of B. The mass centers of bodies A, B, and C are denoted A^*, B^*, and C^* and have locations as indicated in Figure 6.2. Constants needed in the description of the robot are L_A, L_T, L_P, L, and L_C. These constants all denote lengths as indicated in the figure. Variables in the analysis are a torque T_A applied to body A by an actuator fixed in an inertial reference frame N and a force $F_{A/B}$ applied to B at P by an actuator fixed in A. Unit vectors needed for the description of the motion of the system are $\mathbf{n}_i$ and $\mathbf{a}_i$ $(i = 1, 2, 3)$. Unit vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are fixed in N, and unit vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ are fixed in body A, as shown. The third vector of each triad is perpendicular to the plane formed by the other two such that each triad forms a right-handed set.

6.4.2 Generalized Coordinates and Speeds

A reasonable choice for generalized coordinates for this robot are q_1 and q_2, as shown in Figure 6.2. The coordinate q_1 measures the angle between an inertially fixed line and a line fixed in body A, while q_2 measures the distance between point P and the end of B attached to C. For the problem at hand, generalized speeds that are physically relevant and analytically convenient can be defined as

$$u_1 = \boldsymbol{\omega}^A \cdot \mathbf{a}_3 \tag{6.29}$$

$$u_2 = {}^A\mathbf{v}^{C^*} \cdot \mathbf{a}_1 \tag{6.30}$$

where $\boldsymbol{\omega}^A$ is the angular velocity of A in reference frame N and ${}^A\mathbf{v}^{C^*}$ is the velocity of C^* as measured in body A.

6.4.3 Velocities

The angular and translational velocities required for the development of equations of motion for the robot are the angular velocities of A, B, and C in N, and the velocities of points A^*, P, $\widehat{P}$, B^*, and C^* in N. With the choice of generalized speeds given above, expressions for the angular velocities are

$$\boldsymbol{\omega}^A = u_1\mathbf{a}_3 \tag{6.31}$$

$$\boldsymbol{\omega}^B = u_1\mathbf{a}_3 \tag{6.32}$$

$$\boldsymbol{\omega}^C = u_1\mathbf{a}_3 \tag{6.33}$$

An expression for the translational velocity of A^* in N can be developed directly by inspection of Figure 6.2, which yields

$$\mathbf{v}^{A^*} = -L_A u_1\mathbf{a}_1 \tag{6.34}$$

The velocity of point P is determined by making use of the facts that the velocity of point O, fixed on the axis of A, which is zero and the formula for the velocities of two points fixed on a rigid body. Specifically,

$$\mathbf{v}^P = \mathbf{v}^O + \boldsymbol{\omega}^A \times \mathbf{p}^{OP} \tag{6.35}$$

where $\mathbf{p}^{OP}$ is the position vector from O to P given by

$$\mathbf{p}^{OP} = L_P\mathbf{a}_1 + L_T\mathbf{a}_2 \tag{6.36}$$

Evaluating Equation (6.35) with the aid of Equation (6.31) and Equation (6.36) yields

$$\mathbf{v}^P = -L_T u_1\mathbf{a}_1 + L_P u_1\mathbf{a}_2 \tag{6.37}$$

The velocity of $\widehat{P}$ is determined from the formula for the velocity of a single point moving on a rigid body [20, p. 32]. For $\widehat{P}$, this formula is expressed as

$$\mathbf{v}^{\widehat{P}} = \mathbf{v}^{\overline{A}^{\widehat{P}}} + {}^A\mathbf{v}^{\widehat{P}} \tag{6.38}$$

where $\mathbf{v}^{\overline{A}^{\widehat{P}}}$ is the velocity of that point of body A whose location is instantaneously coincident with $\widehat{P}$, and ${}^A\mathbf{v}^{\widehat{P}}$ is the velocity of $\widehat{P}$ in A. The velocity of $\mathbf{v}^{\overline{A}^{\widehat{P}}}$, for the problem at hand, is simply equal to $\mathbf{v}^P$. The velocity of $\widehat{P}$ in A is determined with reference to Figure 6.2, as well as the definition of u_2 given in Equation (6.30), and is

$${}^A\mathbf{v}^{\widehat{P}} = u_2\mathbf{a}_1 \tag{6.39}$$

Substituting from Equation (6.37) and Equation (6.39) into Equation (6.38), therefore, produces

$$\mathbf{v}^{\widehat{P}} = (-L_T u_1 + u_2)\mathbf{a}_1 - L_P u_1\mathbf{a}_2 \tag{6.40}$$

The velocity of B^* is determined in a manner similar to that used to find the velocity of $\widehat{P}$, which produces

$$\mathbf{v}^{B^*} = (-L_T u_1 + u_2)\mathbf{a}_1 + (L_P + q_2 - L/2)u_1\mathbf{a}_2 \tag{6.41}$$

Since C^* is rigidly connected to B, the velocity of C^* can be related to the velocity of B^* by making use of the formula for two points fixed on a rigid body, which yields

$$\mathbf{v}^{C^*} = (-L_T u_1 + u_2)\mathbf{a}_1 + (L_P + q_2 + L_C)u_1\mathbf{a}_2 \tag{6.42}$$

TABLE 6.2 Partial Velocities for Robot of Figure 6.2

	$r = 1$	$r = 2$
$\boldsymbol{\omega}_r^A$	$\mathbf{a}_3$	0
$\boldsymbol{\omega}_r^B$	$\mathbf{a}_3$	0
$\boldsymbol{\omega}_r^C$	$\mathbf{a}_3$	0
$\mathbf{v}_r^{A^*}$	$-L_A\mathbf{a}_1$	0
$\mathbf{v}_r^P$	$-L_T\mathbf{a}_1 + L_P\mathbf{a}_2$	0
$\mathbf{v}_r^{\hat{P}}$	$-L_T\mathbf{a}_1 + L_P\mathbf{a}_2$	$\mathbf{a}_1$
$\mathbf{v}_r^{B^*}$	$-L_T\mathbf{a}_1 + (L_P + q_2 - L/2)\mathbf{a}_2$	$\mathbf{a}_1$
$\mathbf{v}_r^{C^*}$	$-L_T\mathbf{a}_1 + (L_P + q_2 + L_C)\mathbf{a}_2$	$\mathbf{a}_1$

6.4.4 Partial Velocities

As explained in the previous section, partial velocities are simply the coefficients of the generalized speeds in the expressions for the angular and translational velocities and here are determined by inspection from Equations (6.31) through (6.34), (6.37), and (6.40) through (6.42). The resulting partial velocities are listed in Table 6.2, where $\boldsymbol{\omega}_r^A$ is the rth partial angular velocity of A in N, $\mathbf{v}_r^{A^*}$ is the rth partial linear velocity of A^* in N, etc.

6.4.5 Accelerations

In order to complete the development of requisite kinematical quantities governing the motion of the robot, one must develop expressions for the angular accelerations of bodies A, B, and C in N as well as for the translational accelerations of A^*, B^*, and C^* in N. The angular accelerations can be determined by differentiating Equations (6.31) through (6.33) in N. Since the unit vector $\mathbf{a}_3$ is fixed in N, this is straightforward and produces

$$\boldsymbol{\alpha}^A = \dot{u}_1\mathbf{a}_3 \tag{6.43}$$

$$\boldsymbol{\alpha}^B = \dot{u}_1\mathbf{a}_3 \tag{6.44}$$

$$\boldsymbol{\alpha}^C = \dot{u}_1\mathbf{a}_3 \tag{6.45}$$

where $\boldsymbol{\alpha}^A$, $\boldsymbol{\alpha}^B$, and $\boldsymbol{\alpha}^C$ are the angular acceleration of A, B, and C in N.

The translational accelerations can be determined by direct differentiation of Equation (6.34), Equation (6.41), and Equation (6.42). The acceleration of A^* in N is obtained from

$$\mathbf{a}^{A^*} = \frac{{}^N d\mathbf{v}^{A^*}}{dt} = \frac{{}^A d\mathbf{v}^{A^*}}{dt} + \boldsymbol{\omega}^A \times \mathbf{v}^{A^*} \quad [20,\ \text{p. } 23] \tag{6.46}$$

where $\frac{{}^N d\mathbf{v}^{A^*}}{dt}$ and $\frac{{}^A d\mathbf{v}^{A^*}}{dt}$ are the derivatives of $\mathbf{v}^{A^*}$ in reference frames A and N, respectively. This equation takes advantage of the fact the velocity of A^* is written in terms of unit vectors fixed in A and expresses the derivative of $\mathbf{v}^{A^*}$ in N in terms of its derivative in A plus terms that account for the rotation of A relative to N. Evaluation of Equation (6.46) produces

$$\mathbf{a}^{A^*} = -L_A\dot{u}_1\mathbf{a}_1 - L_A u_1^2\mathbf{a}_2 \tag{6.47}$$

The accelerations of B^* and C^* can be obtained in a similar manner and are

$$\mathbf{a}^{B^*} = \left[-L_T\dot{u}_1 + \dot{u}_2 - (L_P + q_2 - L/2)u_1^2\right]\mathbf{a}_1 + \left[(L_P + q_2 - L/2)\dot{u}_1 - L_T u_1^2 + 2u_1u_2\right]\mathbf{a}_2 \tag{6.48}$$

$$\mathbf{a}^{C^*} = \left[-L_T\dot{u}_1 + \dot{u}_2 - (L_P + q_2 + L_C)u_1^2\right]\mathbf{a}_1 + \left[(L_P + q_2 + L_C)\dot{u}_1 + -L_T u_1^2 + 2u_1u_2\right]\mathbf{a}_2 \tag{6.49}$$

6.4.6 Generalized Inertia Forces

In general, the generalized inertia forces F_r^* in a reference frame N for a rigid body B that is part of a system with n degrees of freedom are given by

$$(F_r^*)_B = \mathbf{v}_r^* \cdot \mathbf{R}^* + \boldsymbol{\omega}_r \cdot \mathbf{T}^* \qquad (r = 1, \ldots, n) \tag{6.50}$$

where $\mathbf{v}_r^*$ is the rth partial velocity of the mass center of B in N, $\boldsymbol{\omega}_r$ is the rth partial angular velocity of B in N, and $\mathbf{R}^*$ and $\mathbf{T}^*$ are the inertia force for B in N and the inertia torque for B in N, respectively. The inertia force for a body B is simply

$$\mathbf{R}^* = -M\mathbf{a}^* \tag{6.51}$$

where M is the total mass of B and $\mathbf{a}^*$ is the acceleration of the mass center of B in N. In its most general form, the inertia torque for B is given by

$$\mathbf{T}^* = -\boldsymbol{\alpha} \cdot \mathbf{I} - \boldsymbol{\omega} \times \mathbf{I} \cdot \boldsymbol{\omega} \tag{6.52}$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\omega}$ are, respectively, the angular acceleration of B in N and the angular velocity of B in N, and $\mathbf{I}$ is the central inertia dyadic of B.

For the problem at hand, generalized inertia forces are most easily formed by first formulating them individually for each of the bodies A, B, and C and then substituting the individual results into

$$F_r^* = (F_r^*)_A + (F_r^*)_B + (F_r^*)_C \qquad (r = 1, 2) \tag{6.53}$$

where $(F_r^*)_A$, $(F_r^*)_B$, and $(F_r^*)_C$ are the generalized inertia forces for bodies A, B, and C, respectively. To generate the generalized inertia forces for A, one must first develop expressions for its inertia force and inertia torque. Making use of Equations (6.31), Equation (6.43), and Equation (6.47), in accordance with Equation (6.51) and Equation (6.52), one obtains

$$\mathbf{R}_A^* = -m_A\left(-L_A\dot{u}_1\mathbf{a}_1 - L_A u_1^2\mathbf{a}_2\right) \tag{6.54}$$

$$\mathbf{T}_A^* = -I_A\dot{u}_1\mathbf{a}_3 \tag{6.55}$$

The resulting generalized inertia forces for A, formulated with reference to Equation (6.50), Equation (6.54), and Equation (6.55), as well as the partial velocities of Table 6.2, are

$$(F_1^*)_A = -\left(m_A L_A^2 + I_A\right)\dot{u}_1 \tag{6.56}$$

$$(F_2^*)_A = 0 \tag{6.57}$$

Similarly, for bodies B and C,

$$(F_1^*)_B = -\left\{m_B\left[(L_P + q_2 - L/2)^2 + L_T^2\right] + I_B\right\}\dot{u}_1 + m_B L_T\dot{u}_2 - 2m_B u_1 u_2 \tag{6.58}$$

$$(F_2^*)_B = m_B L_T\dot{u}_1 - m_B\dot{u}_2 + m_B(L_P + q_2 - L/2)u_1^2 \tag{6.59}$$

and

$$(F_1^*)_C = -\left\{m_C\left[(L_P + q_2 + L_C)^2 + L_T^2\right] + I_C\right\}\dot{u}_1 + m_C L_T\dot{u}_2 - 2m_C u_1 u_2 \tag{6.60}$$

$$(F_2^*)_C = m_C L_T\dot{u}_1 - m_C\dot{u}_2 + m_C(L_P + q_2 - L_C)u_1^2 \tag{6.61}$$

Substituting from Equations (6.56) through (6.61) into Equation (6.53) yields the generalized inertia forces for the entire system.

6.4.7 Generalized Active Forces

Since nonworking forces, or sets of forces, make no net contribution to the generalized active forces, one need only consider the torque T_A and the force $F_{A/B}$ to determine the generalized active forces for the robot of Figure 6.2. One can, therefore, write the generalized active forces F_r as

$$F_r = \boldsymbol{\omega}_r^A \cdot T_A \mathbf{a}_3 + \mathbf{v}_r^{\hat{P}} \cdot F_{A/B}\mathbf{a}_1 + \mathbf{v}_r^P \cdot (-F_{A/B}\mathbf{a}_1) \qquad (r = 1, 2) \tag{6.62}$$

Substituting from Table 6.2 into Equation (6.62) produces

$$F_1 = T_A \tag{6.63}$$

$$F_2 = F_{A/B} \tag{6.64}$$

6.4.8 Equations of Motion

The equations of motion for the robot can now be formulated by substituting from Equation (6.53), Equation (6.63), and Equation (6.64) into Kane's equations:

$$F_r + F_r^* = 0 \qquad (r = 1, 2) \tag{6.65}$$

Kane's method can, of course, be applied to much more complicated robotic systems than the two simple illustrative systems analyzed in this and the preceding section. Section 6.6 describes a number of analyses of robotic devices that have been performed using Kane's method over the last two decades. These studies and the commercially available software packages related to the use of Kane's method described in Section 6.7, as well as studies of the efficacy of Kane's method in the analysis of robotic mechanisms such as in [24, 25], have shown that Kane's method is both analytically convenient for hand analyses as well as computationally efficient when used as the basis for general purpose or specialized robotic system simulation programs.

6.5 Special Issues in Kane's Method

Kane's method is applicable to a wide range of robotic and nonrobotic systems. In this section, attention is focused on ways in which Kane's method can be applied to systems that have specific characteristics or for which equations of motion in a particular form are desired. Specifically, described below are approaches that can be utilized when linearized dynamical equations of motion are to be developed, when equations are sought for systems that are subject to kinematical constraints, and when systems that have continuous elastic elements are analyzed.

6.5.1 Linearized Equations

As discussed in Section 6.4 of [20], dynamical equations of motion that have been linearized in all or some of the configuration or motion variables (i.e., generalized coordinates or generalized speeds) are often useful either for the study of the stability of motion or for the development of linear control schemes. Moreover, linearized differential equations have the advantage of being easier to solve than nonlinear ones while still yielding information that may be useful for restricted classes of motion of a system. In situations in which fully nonlinear equations for a system are already in hand, one develops linearized equations simply by expanding in a Taylor series all terms containing the variables in which linearization is to be performed and then eliminating all nonlinear contributions. However, in situations in which linearized dynamical equations are to be formulated directly without first developing fully nonlinear ones, or in situations in which fully nonlinear equations *cannot* be formulated, one can efficiently generate linear dynamical equations with Kane's method by proceeding as follows: First, as was done for the illustrative systems of Section 6.3 and Section 6.4, develop fully nonlinear expressions for the requisite angular and translational velocities of the particles and rigid bodies comprising the system under consideration.

These nonlinear expressions are then used to determine nonlinear partial angular velocities and partial translational velocities by inspection. Once the nonlinear partial velocities have been identified, however, they are no longer needed in their nonlinear form and these partial velocities can be linearized. Moreover, with the correct linearized partial velocities available, the previously determined nonlinear angular and translational velocities can also be linearized and then used to construct linearized angular and translational accelerations. These linearized expressions can then be used in the procedure outlined in Section 6.3 for formulating Kane's equations of motion while only retaining linearized terms in each expression throughout the process. The significant advantage of this approach is that the transition from nonlinear expressions to completely linearized ones can be made at the very early stages of an analysis, thus avoiding the need to retain terms that ultimately make no contribution to linearized equations of motion. While this is important for any system for which linearized equations are desired, it is particularly relevant to continuous systems for which fully nonlinear equations cannot be formulated in closed form (such as for the system described later in Section 6.5.3).

A specific example of the process of developing linearized equations of motion using Kane's method is given in Section 6.4 of [20]. Another example of issues associated with developing linearized dynamical equations using Kane's method is given below in Section 6.5.3 on continuous systems. Although a relatively complicated example, it demonstrates the systematic approach of Kane's method that guarantees that all the terms that should appear in linearized equations actually do. The ability to confidently and efficiently formulate linearized equations of motion for continuous systems is essential to ensure (as discussed at length in works such as [26,27]) that linear phenomena, such as "centrifugal stiffening" in rotating beam systems, are corrected and taken into account.

6.5.2 Systems Subject to Constraints

Another special case in which Kane's method can be used to particular advantage is in systems subject to constraints. This case is useful when equations of motion have already been formulated, and new equations of motion reflecting the presence of additional constraints are needed, and allows the new equations to be written as a recombination of terms comprising the original equations. This approach avoids the need to introduce the constraints as kinematical equations at an early stage of the analysis or to increase the number of equations through the introduction of unknown forces. Introducing unknown constraint forces is disadvantageous unless the constraint forces themselves are of interest, and the early introduction of kinematical constraint equations typically unnecessarily complicates the development of the dynamical equations. This approach is also useful in situations after equations of motion have been formulated and additional constraints are applied to a system, for example, when design objectives change, when a system's topology changes during its motion, or when a system is replaced with a simpler one as a means of checking a numerical simulation. In such situations, premature introduction of constraints deprives one of the opportunity to make maximum use of expressions developed in connection with the unconstrained system. The approach described below, which provides a general statement of how dynamical equations governing constrained systems can be generated, is based on the work of Wampler et al. [28].

In general, if a system described by Equation (6.9) is subjected to m independent constraints such that the number of degrees of freedom decreases from n to $n - m$, the independent generalized speeds for the system $u_1, \ldots, u_n$ must be replaced by a new set of independent generalized speeds $u_1, \ldots, u_{n-m}$. The equations of motion for the constrained system can then be generated by considering the problem as a completely new one, or alternatively, by making use of the following, one can make use of many of the expressions that were generated in forming the original set of equations.

Given an n degree-of-freedom system possessing n independent partial velocities, n generalized inertia forces F_r^*, and n generalized active forces F_r, each associated with the n independent generalized speeds $u_1, \ldots, u_n$ that are subject to m linearly independent constraints that can be written in the form

$$u_k = \sum_{l=1}^{n-m} \alpha_{kl} u_l + \beta_k \qquad (k = n - m + 1, \ldots, n) \tag{6.66}$$

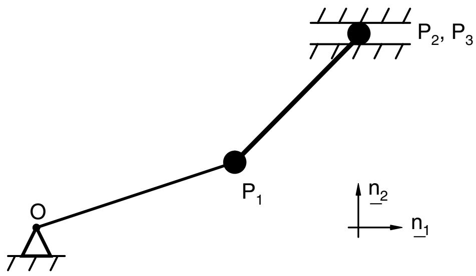

FIGURE 6.3 Two-DOF planar robot "grasping" an object.

where the u_l $(l = 1, \dots, n - m)$ are a set of independent generalized speeds governing the constrained system [α_{kl} and β_k $(l = 1, \dots, n - m; k = n - m + 1, \dots, n)$ are functions solely of the generalized coordinates and time], the equations of motion for the constrained system are written as

$$F_r + F_r^* + \sum_{k=n-m+1}^{n} \alpha_{kr}(F_k + F_k^*) = 0 \qquad (r = 1, \dots, n - m) \tag{6.67}$$

As can be readily observed from the above two equations, formulating equations of motion for the constrained system simply involves identifying the α_{kl} coefficients appearing in constraint equations expressed in the form of Equation (6.66) and then recombining the terms appearing in the unconstrained equations in accordance with Equation (6.67).

As an example of the above procedure, consider again the two-degree-of-freedom system of Figure 6.1. If this robot were to grasp a particle P_3 that slides in a frictionless horizontal slot, as shown in Figure 6.3, the system of the robot and particle, which when unconnected would have a total of three degrees of freedom, is reduced to one having only one degree of freedom. The two constraints arise as a result of the fact that when P_3 is grasped, the velocity of P_3 is equal to the velocity of P_1. If the velocity of P_3 before being grasped is given by

$$\mathbf{v}^{P_3} = u_3\mathbf{n}_1 \tag{6.68}$$

where u_3 is a generalized speed chosen to describe the motion of P_3, then the two constraint equations that are in force after grasping can be expressed as

$$-Ls_1u_1 - Ls_{12}(u_1 + u_2) = u_3 \tag{6.69}$$

$$Lc_1u_1 + Lc_{12}(u_1 + u_2) = 0 \tag{6.70}$$

where s_1, c_1, s_{12}, and c_{12} are equal to $\sin q_1, \cos q_1, \sin(q_1 + q_2)$, and $\cos(q_1 + q_2)$, respectively. Choosing u_1 as the independent generalized speed for the constrained system, expressions for the dependent generalized speeds u_2 and u_3 are written as

$$u_2 = -\frac{c_1 + c_{12}}{c_{12}}\, u_1 \tag{6.71}$$

$$u_3 = L\,\frac{s_2}{c_{12}}\, u_1 \tag{6.72}$$

where s_2 is equal to $\sin q_2$ and where the coefficients of u_1 in Equation (6.71) and Equation (6.72) correspond to the terms α_{21} and α_{31} defined in the general form for constraint equations in Equation (6.66). Noting that the generalized inertia and active forces for P_3 before the constraints are applied are given by

$$F_3^* = -m_3\dot{u}_3 \tag{6.73}$$

$$F_3 = 0 \tag{6.74}$$

where m_3 is the mass of particle P_3, the single equation of motion governing the constrained system is

$$F_1 + F_1^* + \alpha_{21}(F_2 + F_2^*) + \alpha_{31}(F_3 + F_3^*) = 0 \tag{6.75}$$

where F_r and F_r^* $(r = 1, 2)$ are the generalized inertia and active forces appearing in Equation (6.21), Equation (6.22), Equation (6.24), and Equation (6.25).

There are several observations about this approach to formulating equations of motion for constrained systems that are worthy of consideration. The first is that it makes maximal use of terms that were developed for the *unconstrained* system in producing equations of motion for the *constrained* system. Second, this approach retains all the advantages inherent in the use of Kane's method, most significantly the abilities to use generalized speeds to describe motion and to disregard in an analysis nonworking forces that ultimately do not contribute to the final equations. Additionally, this approach can be applied to either literal or numerical developments of the equations of motion, and while particularly useful when constraints are added to a system once equations have been developed for the unconstrained system, it is also often a convenient approach to formulating equations of motion for constrained system from the outset of an analysis. Three particular categories of systems for which this approach is particularly appropriate from the outset of an analysis are (1) those whose topologies and number of degrees of freedom change during motion, (2) systems that are so complex that the early introduction of constraints unnecessarily encumbers the formulation of dynamical equations, and (3) those for which the introduction of "fictitious constraints" affords a means of checking numerical solutions of dynamical equations (detailed descriptions of these three categories are given in [28]).

6.5.3 Continuous Systems

Another particular class of problems to which Kane's method can be applied is nonrigid body problems. There have been many such studies in the literature (for example, see [17,26,29–32]), but to give a specific example of the process by which this is done, outlined below are the steps taken to construct by means of Kane's method the equations of motion for the system of Figure 6.2 (considered previously in Section 6.3) when the translating link is regarded as elastic rather than rigid. The system is modeled as consisting of a uniform elastic beam connected at one end to a rigid block and capable of moving longitudinally over supports attached to a rotating base. Deformations of the beam are assumed to be both "small" and adequately described by Bernoulli-Euler beam theory (i.e., shear deformations and rotatory inertia are neglected), axial deformations are neglected, and all motions are confined to a single horizontal plane. The equations are formulated by treating the supports of the translating link as kinematical constraints imposed on an unrestrained elastic beam and by discretizing the beam by means of the assumed modes method (details of this approach can be found in [31,33]). This example, though relatively complex, thus represents not only an illustration of the application of Kane's method to continuous systems but also an opportunity to consider issues associated with developing linearized dynamical equations (discussed in Section 6.5.1) as well as issues arising when formulating equations of motion for complex systems regarded from the outset as subject to kinematical constraints (following the procedure outlined in Section 6.5.2).

6.5.3.1 Preliminaries

For the purposes at hand, the system can be represented schematically as shown in Figure 6.4. This system consists of a rigid T-shaped base A that rotates about a vertical axis and that supports a nonrigid beam B at two distinct points $\widehat{P}$ and $\widehat{Q}$. The beam is capable of longitudinal motions over the supports and is connected at one end to a rigid block C. Mutually perpendicular lines N_1 and N_2, fixed in an inertial reference frame N and intersecting at point O, serve as references to which the position and orientation of the components of the system can be related. The base A rotates about an axis fixed in N and passing through O so that the orientation of A in N can be described by the single angle θ_1 between N_2 and a line fixed in A, as shown. The mass center of A, denoted A^*, is a distance L_A from O, and the distance from O to the line A_1, fixed in A, is L_T. The two points $\widehat{P}$ and $\widehat{Q}$ at which B is supported are fixed in A. The distance from the line connecting O and A^* to $\widehat{P}$ is L_P, and the distance between $\widehat{P}$ and $\widehat{Q}$ is L_D. The

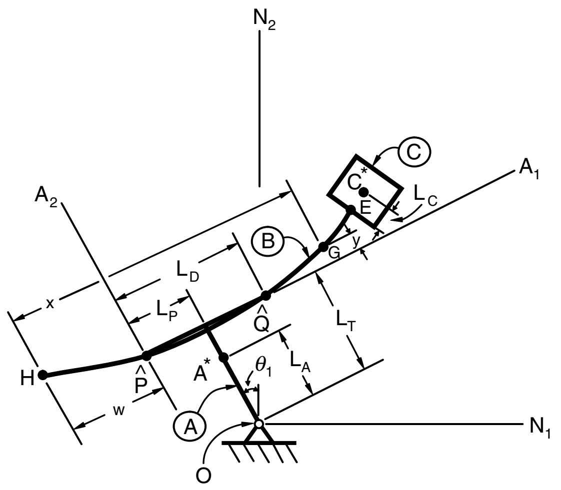

FIGURE 6.4 Robot of Figure 6.2 with continuously elastic translating link.

distance from $\widehat{P}$ to H, one endpoint of B, is w, and the distance from H to a generic point G of B, as measured along A_1, is x. The displacement of G from A_1 is y, as measured along the line A_2 that is fixed in A and intersects A_1 at $\widehat{P}$. The block C is attached to the endpoint E of B. The distance from E to the mass center C^* of C is L_C.

To take full advantage of the methods of Section 6.5.2, the system is again depicted in Figure 6.5, in which the beam B is now shown released from its supports and in a general configuration in the A_1-A_2 plane. As is described in [33], this representation facilitates the formulation of equations of motion and permits the formulation of a set of equations that can be numerically integrated in a particularly efficient manner. To this end, one can introduce an auxiliary reference frame R defined by the mutually perpendicular lines R_1 and R_2 intersecting at R^* and lying in the A_1-A_2 plane. The position of R^* relative to $\widehat{P}$, as measured

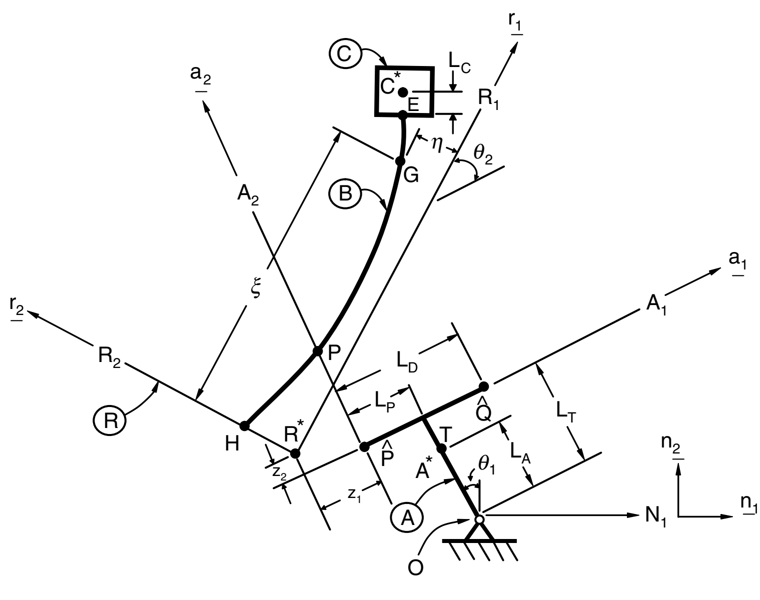

FIGURE 6.5 Robot of Figure 6.4 with translating link released from supports.

along A_1 and A_2, is characterized by z_1 and z_2, respectively, and the orientation of R in A is described by the angle θ_2 between A_1 and R_1. The R_1-coordinate and R_2-coordinate of G are ξ and η, respectively, while the angle (not shown) between R_1 and the tangent to B at G is α. Finally, P is a point on B lying on A_2, and a similar point Q (not shown) is on B lying on a line parallel to A_2 passing through $\widehat{Q}$.

6.5.3.2 Kinematics

By making use of the assumed modes method, one can express η in terms of modal functions ϕ_i as

$$\eta(s, t) = \sum_{i=1}^{\nu} \phi_i(s)\, q_i(t) \tag{6.76}$$

where the ϕ_i $(i = 1, \ldots, \nu)$ are functions of s, the arc length as measured along B from H to G, and ν is a positive integer indicating the number of modes to be used in the analysis. The quantities ξ and α can be directly related to ϕ_i and q_i, and thus the $\nu + 4$ quantities q_i $(i = 1, \ldots, \nu)$, θ_1, z_1, z_2, and θ_2 suffice to fully describe the configuration of the complete system (the system has $\nu + 4$ degrees of freedom). Generalized speeds u_i $(i = 1, \ldots, \nu + 4)$ describing the motion of the system can be introduced as

$$\begin{aligned} u_i &= \dot{q}_i \quad (i = 1, \ldots, \nu) \\ u_{\nu+1} &= \boldsymbol{\omega}^A \cdot \mathbf{a}_3 = \dot{\theta}_1 \\ u_{\nu+2} &= {}^A\mathbf{v}^{R^*} \cdot \mathbf{a}_1 = -\dot{z}_1 \\ u_{\nu+3} &= {}^A\mathbf{v}^{R^*} \cdot \mathbf{a}_2 = \dot{z}_2 \\ u_{\nu+4} &= {}^A\boldsymbol{\omega}^R \cdot \mathbf{a}_3 = \dot{\theta}_2 \end{aligned} \tag{6.77}$$

where $\boldsymbol{\omega}^A$ is the angular velocity of A in N, ${}^A\mathbf{v}^{R^*}$ is the velocity of R^* in A, ${}^A\boldsymbol{\omega}^R$ is the angular velocity of R in A, and $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ are elements of a dextral set of unit vectors such that $\mathbf{a}_1$ and $\mathbf{a}_2$ are directed as shown in Figure 6.5 and $\mathbf{a}_3 = \mathbf{a}_1 \times \mathbf{a}_2$.

The angular and translational velocities essential to the development of equations of motion are the angular velocity of A in N, the velocity of A^* in N, the velocity of G in N, the angular velocity of C in N, and the velocity of C^* in N. These are given by

$$\boldsymbol{\omega}^A = u_{\nu+1}\, \mathbf{a}_3 \tag{6.78}$$

$$\mathbf{v}^{A^*} = -L_A\, u_{\nu+1}\, \mathbf{a}_1 \tag{6.79}$$

$$\begin{aligned} \mathbf{v}^G = &\left\{ -\sum_{i=1}^{\nu} \left[\int_0^s \phi_i'(\sigma) \tan\alpha(\sigma, t)\, d\sigma \right] u_i - [(L_T + z_2)\, c_2 + (L_p + z_1) s_2 + \eta] u_{\nu+1} \right. \\ &\left. + c_2\, u_{\nu+2} + s_2\, u_{\nu+3} - \eta u_{\nu+4} \right\} \mathbf{r}_1 \\ &+ \left\{ \sum_{i=1}^{\nu} \phi_i\, u_i + [(L_T + z_2) s_2 - (L_P + z_1) c_2 + \xi] u_{\nu+1} - s_2\, u_{\nu+2} + c_2\, u_{\nu+3} + \xi\, u_{\nu+4} \right\} \mathbf{r}_2 \end{aligned} \tag{6.80}$$

$$\boldsymbol{\omega}^C = \left[\frac{1}{\cos\alpha^E} \sum_{i=1}^{\nu} \phi_i'(L) u_i + u_{\nu+1} + u_{\nu+4} \right] \mathbf{r}_3 \tag{6.81}$$

$$\begin{aligned} \mathbf{v}^{C^*} = \mathbf{v}^G \Big|_{s=L} &- L_C \sin\alpha^E \left[\frac{1}{\cos\alpha^E} \sum_{i=1}^{\nu} \phi_i'(L) u_i + u_{\nu+1} + u_{\nu+4} \right] \mathbf{r}_1 \\ &+ L_C \cos\alpha^E \left[\frac{1}{\cos\alpha^E} \sum_{i=1}^{\nu} \phi_i'(L) u_i + u_{\nu+1} + u_{\nu+4} \right] \mathbf{r}_2 \end{aligned} \tag{6.82}$$

where s_2 and c_2 are defined as $\sin\theta_2$ and $\cos\theta_2$, respectively, $\mathbf{r}_1$ and $\mathbf{r}_2$ are unit vectors directed as shown in Figure 6.5, α^E is the value of α corresponding to the position of point E, and $\mathbf{v}^G|_{s=L}$ is the velocity of G evaluated at $s = L$, where L is the undeformed length of B.

It should be noted that, so far, all expressions have been written in their full nonlinear form. This is in accordance with the requirements of Kane's method described in Section 6.5.2 for developing linearized equations of motion. These requirements mandate that completely nonlinear expressions for angular and translational velocities be formulated, and thereafter completely nonlinear partial angular velocities and partial translational velocities, before any kinematical expressions can be linearized. From Equation (6.78) through Equation (6.82), the requisite nonlinear partial velocities are identified as simply the coefficients of the generalized speeds u_i $(i = 1, \ldots, \nu + 4)$.

With all nonlinear partial velocities identified, Equation (6.76) and Equation (6.80) through Equation (6.82) may now be linearized in the quantities q_i $(i = 1, \ldots, \nu)$, z_2, θ_2, u_i $(i = 1, \ldots, \nu)$, $u_{\nu+3}$, and $u_{\nu+4}$ to produce equations governing "small" displacements of B from the A_1 axis. Linearization yields

$$\eta(\xi, t) = \sum_{i=1}^{\nu} \phi_i(\xi)\, q_i(t) \tag{6.83}$$

$$\begin{aligned}\mathbf{v}^G = &\left\{-\left[L_T + z_2 + (L_P + z_1)\theta_2 + \sum_{i=1}^{\nu}\phi_i\, q_i\right] u_{\nu+1} + u_{\nu+2}\right\}\mathbf{r}_1 \\ &+ \left[\sum_{i=1}^{\nu}\phi_i\, u_i + (L_T\theta_2 - L_P - z_1 + \xi)u_{\nu+1} - \theta_2\, u_{\nu+2} + u_{\nu+3} + \xi\, u_{\nu+4}\right]\mathbf{r}_2\end{aligned} \tag{6.84}$$

$$\boldsymbol{\omega}^C = \left[\sum_{i=1}^{\nu}\phi_i'(L)u_i + u_{\nu+1} + u_{\nu+4}\right]\mathbf{r}_3 \tag{6.85}$$

$$\mathbf{v}^{C^*} = \mathbf{v}^G\Big|_{\xi=L} - L_C\, u_{\nu+1}\sum_{i=1}^{\nu}\phi_i'(L)q_i\,\mathbf{r}_1 + L_C\left[\sum_{i=1}^{\nu}\phi_i'(L)u_i + u_{\nu+1} + u_{\nu+4}\right]\mathbf{r}_2 \tag{6.86}$$

while leaving Equation (6.78) and Equation (6.79) unchanged. Linearization of the partial velocities obtained from Equation (6.78) through Equation (6.82) yields the expressions recorded in Table 6.3, where

$$\epsilon_{ij}(\xi) = \int_0^{\xi}\phi_i'(\sigma)\phi_j'(\sigma)d\sigma \qquad (i, j = 1, \ldots, \nu) \tag{6.87}$$

Note that if linearization had been performed prematurely, and partial velocities had been obtained from the linearized velocity expressions given in Equation (6.84) through Equation (6.86), terms involving ϵ_{ij} would have been lost. These terms must appear in the equations of motion in order to correctly account for the coupling between longitudinal accelerations and transverse deformations of the beam.

As was done in Section 6.4, the generalized inertia forces F_r^* for the system depicted in Figure 6.5 can be expressed as

$$F_r^* = (F_r^*)_A + (F_r^*)_B + (F_r^*)_C \qquad (r = 1, \ldots, \nu + 4) \tag{6.88}$$

where $(F_r^*)_A$, $(F_r^*)_B$, and $(F_r^*)_C$ are the generalized inertia forces for bodies A, B, and C, respectively. The generalized inertia forces for A are generated by summing the dot product between the partial angular velocity of A and the inertia torque of A with the dot product between the partial translational velocity of A^* and the inertia force of A. This can be written as

$$(F_r^*)_A = \boldsymbol{\omega}_r^A \cdot \left(-I_3^A\,\dot{u}_{\nu+1}\,\mathbf{a}_3\right) + \mathbf{v}_r^{A^*}\cdot(-m_A\mathbf{a}^{A^*}) \qquad (r = 1, \ldots, \nu + 4) \tag{6.89}$$

where $\boldsymbol{\omega}_r^A$ and $\mathbf{v}_r^{A^*}$ are the partial velocities for body A given in Table 6.4, I_3^A is the moment of inertia of A about a line parallel to $\mathbf{a}_3$ and passing through A^*, m_A is the mass of A, and $\mathbf{a}^{A^*}$ is the acceleration of A^*.

TABLE 6.3 Linearized Partial Velocities for Robot of Figure 6.5

r	$\boldsymbol{\omega}_r^A$	$\mathbf{v}_r^{A^*}$	$\mathbf{v}_r^G$	$\boldsymbol{\omega}_r^C$	$\mathbf{v}_r^{C^*}$
$j = 1, \ldots, \nu$	0	0	$-\sum_{i=1}^{\nu} \epsilon_{ij} q_i \, \mathbf{r}_1 + \phi_j \, \mathbf{r}_2$	$\phi_j'(L)\mathbf{r}_3$	$\mathbf{v}_j^G \big\vert_{\xi=L} - L_C \sum_{i=1}^{\nu} \phi_i'(L)\phi_j'(L) q_i \, \mathbf{r}_1 + L_C \phi_j'(L) \, \mathbf{r}_2$
$\nu + 1$	$\mathbf{a}_3$	$-L_A \, \mathbf{a}_1$	$-[L_T + z_2 + (L_P + z_1)\theta_2 + \sum_{i=1}^{\nu} \phi_i q_i]\mathbf{r}_1 + (L_T \, \theta_2 - L_P - z_1 + \xi)\mathbf{r}_2$	$\mathbf{r}_3$	$\mathbf{v}_{\nu+1}^G \big\vert_{\xi=L} - L_C \sum_{i=1}^{\nu} \phi_i'(L) q_i \, \mathbf{r}_1 + L_C \, \mathbf{r}_2$
$\nu + 2$	0	0	$\mathbf{r}_1 - \theta_2 \, \mathbf{r}_2$	0	$\mathbf{v}_{\nu+2}^G \big\vert_{\xi+L}$
$\nu + 3$	0	0	$\theta_2 \, \mathbf{r}_1 + \mathbf{r}_2$	0	$\mathbf{v}_{\nu+3}^G \big\vert_{\xi=L}$
$\nu + 4$	0	0	$-\sum_{i=1}^{\nu} \phi_i \, q_i \, \mathbf{r}_1 + \xi \, \mathbf{r}_2$	$\mathbf{r}_3$	$\mathbf{v}_{\nu+4}^G \big\vert_{\xi=L} - L_C \sum_{i=1}^{\nu} \phi_i'(L) q_i \, \mathbf{r}_1 + L_C \, \mathbf{r}_2$

The generalized inertia forces for B are developed from

$$(F_r^*)_B = \int_0^L \mathbf{v}_r^G \cdot (-\rho \, \mathbf{a}^G) d\xi \qquad (r = 1, \ldots, \nu + 4) \tag{6.90}$$

where $\mathbf{v}_r^G$ $(r = 1, \ldots, \nu + 4)$ and $\mathbf{a}^G$ are the partial velocities and the acceleration of a differential element of B at G, respectively, and ρ is the mass per unit length of B. An expression for the generalized inertia forces for the block C is produced in a manner similar to that used to generate Equation (6.89) and has the form

$$\mathbf{F}_C^* = \boldsymbol{\omega}_r^C \cdot \left\{ I_3^C \left[\sum_{i=1}^{\nu} \phi_i' \, (L) \dot{u}_i + \dot{u}_{\nu+1} + \dot{u}_{\nu+4} \right] \mathbf{r}_3 \right\} + \mathbf{v}_r^{C^*} \cdot (-\, m_C \, \mathbf{a}^{C^*}) \qquad (r = 1, \ldots, \nu + 4) \tag{6.91}$$

where $\boldsymbol{\omega}_r^C$ and $\mathbf{v}_r^{C^*}$ are the partial velocities for body C, I_3^C is the moment of inertia of C about a line parallel to $\mathbf{a}_3$ and passing through C^*, m_C is the mass of C, and $\mathbf{a}^{C^*}$ is the acceleration of C^*.

For the sake of brevity, explicit expressions for the generalized inertia forces F_r^* $(r = 1, \ldots, \nu + 4)$ are not listed here. Once linearized expressions for the accelerations $\mathbf{a}^{A^*}$, $\mathbf{a}^G$, and $\mathbf{a}^{C^*}$ are constructed by differentiating Equation (6.79), Equation (6.84), and Equation (6.86) with respect to time in N, however, all the necessary information is available to develop the generalized inertia forces by performing the operations indicated above in Equations (6.88) through (6.91). The interested reader is referred to [33] for complete details.

As in the development of the generalized inertia forces, it is convenient to consider contributions to the generalized active forces F_r from bodies A, B, and C separately and write

$$F_r = (F_r)_A + (F_r)_B + (F_r)_C \qquad (r = 1, \ldots, \nu + 4) \tag{6.92}$$

The only nonzero contribution to the generalized active forces for A is due to an actuator torque T_A so that the generalized active forces for A are given by

$$(F_r)_A = \boldsymbol{\omega}_r^A \cdot (T_A \, \mathbf{a}_3) \qquad (r = 1, \ldots, \nu + 4) \tag{6.93}$$

Using a Bernoulli-Euler beam model to characterize deformations, the generalized active forces for B are determined from

$$(F_r)_B = \int_0^L \mathbf{v}_r^G \cdot \left(-EI \frac{\partial^4 \eta}{\partial \xi^4} \mathbf{r}_2 \right) d\xi \qquad (r = 1, \ldots, \nu + 4) \tag{6.94}$$

where E and I are the modulus of elasticity and the cross-sectional area moment of inertia of B, respectively. Nonzero contributions to the generalized active forces for C arise from the bending moment and shear force exerted by B on C so that $(F_r)_C$ is given by

$$(F_r)_C = \boldsymbol{\omega}_r^C \cdot \left(-EI \frac{\partial^2 \eta}{\partial \xi^2}\bigg|_{\xi=L} \mathbf{r}_3 \right) + \mathbf{v}_r^G \bigg|_{\xi=L} \cdot \left(EI \frac{\partial^3 \eta}{\partial \xi^3}\bigg|_{\xi=L} \mathbf{r}_2 \right) \qquad (r = 1, \ldots, \nu + 4) \tag{6.95}$$

Final expressions for the generalized active forces for the entire system are produced by substituting Equations (6.93) through (6.95) into Equation (6.92).

It is important to remember at this point that the generalized inertia and active forces of Equation (6.88) and Equation (6.92) govern *unrestrained* motions of the beam B. The equations of interest, however, are those that govern motions of B when it is constrained to remain in contact with the supports at $\widehat{P}$ and $\widehat{Q}$. These equations can be formulated by first developing constraint equations expressed explicitly in terms of the generalized speeds of Equation (6.77) and by then making use of the procedure for formulating equations of motion for constrained systems described in Section 6.5.2.

One begins the process by identifying the constraints on the position of the points P and Q of the beam B. These are

$$\frac{d}{dt}(\mathbf{p}^{\widehat{P}P} \cdot \mathbf{a}_2) = 0, \qquad \frac{d}{dt}(\mathbf{p}^{\widehat{Q}Q} \cdot \mathbf{a}_2) = 0 \tag{6.96}$$

where $\mathbf{p}^{\widehat{P}P}$ and $\mathbf{p}^{\widehat{Q}Q}$ are the position vectors of P and Q relative to $\widehat{P}$ and $\widehat{Q}$, respectively. These equations must next be expressed in terms of the generalized speeds of Equation (6.77). Doing so yields the constraint equations

$$u_{\nu+3} - (s_2\eta^P - c_2\xi^P)u_{\nu+4} + s_2\frac{d\xi^P}{dt} + c_2\frac{d\eta^P}{dt} = 0 \tag{6.97}$$

$$u_{\nu+3} - (s_2\eta^Q - c_2\xi^Q)u_{\nu+4} + s_2\frac{d\xi^Q}{dt} + c_2\frac{d\eta^Q}{dt} = 0 \tag{6.98}$$

where s^P and s^Q are the arc lengths, measured along B, from H to P and from H to Q, respectively. Note that since P and Q are not fixed on B, s^P and s^Q are not independent variables, but rather are functions of time.

One is now in a position to express Equation (6.97) and Equation (6.98) in a form such that the generalized speeds u_i $(i = 1, \ldots, \nu + 4)$ appear explicitly. Doing so produces

$$\sum_{i=1}^{\nu} \left[\sin \alpha^P \int_0^{s^P} \phi_i'(\sigma) \tan \alpha(\sigma, t) d\sigma + \phi_i(s^P) \cos \alpha^P \right] u_i - (s_2 \cos \alpha^P + c_2 \sin \alpha^P) u_{\nu+2}$$
$$+ (c_2 \cos \alpha^P - s_2 \sin \alpha^P) u_{\nu+3} + (\eta^P \sin \alpha^P + \xi^P \cos \alpha^P) u_{\nu+4} = 0 \tag{6.99}$$

and

$$\sum_{i=1}^{\nu} \left[\sin \alpha^Q \int_0^{s^Q} \phi_i'(\sigma) \tan \alpha(\sigma, t) d\sigma + \phi_i(s^Q) \cos \alpha^Q \right] u_i - (s_2 \cos \alpha^Q + c_2 \sin \alpha^Q) u_{\nu+2}$$
$$+ (c_2 \cos \alpha^Q - s_2 \sin \alpha^Q) u_{\nu+3} + (\eta^Q \sin \alpha^Q + \xi^Q \cos \alpha^Q) u_{\nu+4} = 0 \tag{6.100}$$

where α^P and α^Q denote the values of $\alpha(s, t)$ evaluated with $s = s^P$ and $s = s^Q$, respectively.

Since the ultimate goal is to develop equations of motion governing only small deformations of B, it is again appropriate to now linearize the coefficients of u_i $(i = 1, \ldots, \nu+4)$ in Equation (6.99) and Equation (6.100) in the quantities q_i $(i = 1, \ldots, \nu)$, z_2, θ_2, u_i $(i = 1, \ldots, \nu)$, $u_{\nu+3}$, and $u_{\nu+4}$. This is analogous to the linearization procedure used previously in developing linearized partial velocities. Note that linearization may not be performed prior to the formulation of Equation (6.99) and Equation (6.100) in their full nonlinear form. The consequence of premature linearization would be the loss of *linear* terms in the equations of motion reflecting the coupling between longitudinal and transverse motions of B.

Before the beam B is constrained to remain in contact with the supports at $\widehat{P}$ and $\widehat{Q}$, contributions to the generalized active forces from contact forces acting between bodies A and B cannot be accommodated in a consistent manner. Once the constraints are introduced, however, these forces must be taken into account. The only such forces that make nonzero contributions are $\widetilde{\mathbf{F}}_{\overline{P}}$ and $\widetilde{\mathbf{F}}_{\hat{P}}$. The force $\widetilde{\mathbf{F}}_{\overline{P}}$ drives B longitudinally over $\widehat{P}$ and $\widehat{Q}$ and is regarded as being applied tangentially to B at $\overline{P}$, where $\overline{P}$ is that point fixed on B whose location instantaneously coincides with that of P. The force $\widetilde{\mathbf{F}}_{\hat{P}}$ is equal in magnitude and opposite in direction to $\widetilde{\mathbf{F}}_{\overline{P}}$ and is applied to A at $\widehat{P}$. Taking advantage of the facts that $\widetilde{\mathbf{F}}_{\hat{P}}$ equals $-\widetilde{\mathbf{F}}_{\overline{P}}$ and that the velocity of $\overline{P}$ equals $\mathbf{v}^G|_{\xi=z_1}$, one can express the additional generalized forces $\widetilde{F}_r$ as

$$\widetilde{F}_r = \left(\mathbf{v}_r^G \bigg|_{\xi=z_1} - \mathbf{v}_r^{\hat{P}} \right) \cdot \left\{ F_{A/B} \left[\mathbf{r}_1 + \sum_{i=1}^{\nu} \phi_i'(z_1)\, q_i\, \mathbf{r}_2 \right] \right\} \qquad (r = 1, \ldots, \nu+4) \tag{6.101}$$

where the $\mathbf{v}_r^{\hat{P}}$ are the linearized partial velocities associated with the velocity of $\widehat{P}$ in N and $F_{A/B}$ is the component of $\widetilde{F}_{\overline{P}}$ tangent to B at P. The partial velocities needed in Equation (16.101) can be obtained from the expression for the velocity of $\widehat{P}$ in N given by

$$\mathbf{v}^{\widehat{P}} = -\, u_{\nu+1}[(L_T\, c_2 + L_P\, s_2)\,\mathbf{r}_1 \;+\; (L_P\, c_2 - L_T\, s_2)\,\mathbf{r}_2] \tag{6.102}$$

With all the necessary generalized inertia forces, generalized active forces, and constraint equations in hand, one is in a position to construct the complete dynamical equations governing the motion of the system of Figure 6.4. Without stating them explicitly, these are given by

$$\underline{\alpha}^{\mathrm{T}}(\underline{F}^* + \underline{F} + \underline{\widetilde{F}}) = 0 \tag{6.103}$$

where $\underline{F}^*$ is a $(\nu + 4)\times 1$ matrix whose elements are the generalized inertial forces of Equation (6.88), $\underline{F}$ and $\underline{\widetilde{F}}$ are $(\nu+4)\times 1$ matrices that contain the generalized active forces of Equation (6.92) and Equation (6.101), respectively, and $\underline{\alpha}^T$ is the transpose of a matrix of coefficients of the independent generalized speeds appearing in the constraint equations of Equation (6.99) and Equation (6.100).

One particular point to reinforce about the use of Kane's method in formulating linearized equations of motion for this and other systems with nonrigid elements is the fact that it automatically and correctly accounts for the phenomenon known as "centrifugal stiffening." This is guaranteed when using Kane's method so long as fully nonlinear kinematical expressions are used up through the formulation of partial velocities. This fact ensures that terms resulting from the coupling between longitudinal inertia forces, due either to longitudinal motions of the beam or to rotational motions of the base, and transverse deflections are correctly taken into account. The benefit of having these terms in the resulting equations of motion is that the equations are valid for any magnitude of longitudinal accelerations of the beam or of rotational speeds of the base.

6.6 Kane's Equations in the Robotics Literature

The application of Kane's method in robotics has been cited in numerous articles in the robotics literature since approximately 1982. For example, a search of the COMPENDEX database using the words "Kane" AND "robot" OR "robotic" produced 79 unique references. Moreover, a search of citations of the "The Use of Kane's Dynamical Equations in Robotics" by Kane and Levinson [24] in the Web of Science database produced 76 citations (some, but not all, of which are the same as those found in the COMPENDEX database). Kane's method has also been addressed in recent dynamics textbooks such as *Analytical Dynamics* by Baruh [34] and *Dynamics of Mechanical Systems* by Josephs and Huston [35]. These articles and books discuss not only the development of equations of motion for specific robotic devices but also general methods of formulating equations for robotic-like systems.

Although a comprehensive survey of all articles in the robotics literature that focus on the use of Kane's method is not possible within the limits of this chapter, an attempt to indicate the range of applications of Kane's method is given below. The listed articles were published between 1983 and the present. In some cases, the articles represent seminal works that are widely known in the dynamics and robotics literature. The others, although perhaps less well known, give an indication of the many individuals around the world who are knowledgeable about the use of Kane's method and the many types of systems to which it can be applied. In each case, the title, authors, and year of publication are given (a full citation can be found in the list of references at the end of this chapter) as well as an abbreviated version of the abstract of the article.

- "The use of Kane's dynamical equations in robotics" by Kane and Levinson (1983) [24]: Focuses on the improvements in computational efficiency that can be achieved using Kane's dynamical equations to formulate explicit equations of motion for robotic devices as compared to the use of general-purpose, multibody-dynamics computer programs for the numerical formulation and solution of equations of motion. Presents a detailed analysis of the Stanford Arm so that each step in the analysis serves as an illustrative example for a general method of analysis of problems in robot dynamics. Simulation results are reported and form the basis for discussing questions of computational efficiency.
- "Simulation der dynamik von robotern nach dem verfahren von Kane" by Wloka and Blug (1985) [36]: Discusses the need for modeling robot dynamics when simulating the trajectory of a robot arm. Highlights the inefficiency of implementing the well-known methods of Newton-Euler or Lagrange. Develops recursive methods, based on Kane's equations, that reduce computational costs dramatically and provide an efficient way to calculate the equations of motion of mechanical systems.
- "Dynamique de mecanismes flexibles" by Gallay et al. (1986) [37]: Describes an approximate method suitable for the automatic generation of equation of motion of complex flexible mechanical systems, such as large space structures, robotic manipulators, etc. Equations are constructed based on Kane's method. Rigid-body kinematics use relative coordinates and a method of modal synthesis is employed for the treatment of deformable bodies. The method minimizes the number of equations, while allowing complex topologies (closed loops), and leads to the development of a computer code, called ADAM, for dynamic analysis of mechanical assemblies.
- "Algorithm for the inverse dynamics of n-axis general manipulators using Kane's equations" by Angeles et al. (1989) [38]: Presents an algorithm for the numerical solution of the inverse dynamics of serial-type, but otherwise arbitrary, robotic manipulators. The algorithm is applicable to manipulators containing n rotational or prismatic joints. For a given set of Denavit-Hartenberg and inertial parameters, as well as for a given trajectory in the space of joint coordinates, the algorithm produces the time histories of the n torques or forces required to drive the manipulator through the prescribed trajectory. The algorithm is based on Kane's dynamical equations. Moreover, the complexity of the presented algorithm is lower than that of the most efficient inverse-dynamics algorithm reported in the literature. The applicability of the algorithm is illustrated with two fully solved examples.
- "The use of Kane's method in the modeling and simulation of robotic systems" by Huston (1989) [39]: Discusses the use of Euler parameters, partial velocities, partial angular velocities, and generalized speeds. Explicit expressions for the coefficients of the governing equations are obtained using these methods in forms ideally suited for conversion into computer algorithms. Two applications are discussed: flexible systems and redundant systems. For flexible systems, it is shown that Kane's method using generalized speeds leads to equations that are decoupled in the joint force and

moment components. For redundant systems, with the desired end-effector motion, it is shown that Kane's method uses generalized constraint forces that may be directly related to the constraint equation coefficients, leading directly to matrix methods for solving the governing equations.

- "Dynamics of elastic manipulators with prismatic joints" by Buffinton (1992) [31]: Investigates the formulation of equations of motion for flexible robots containing translationally moving elastic members that traverse a finite number of distinct support points. Specifically studies a two-degree-of-freedom manipulator whose configuration is similar to that of the Stanford Arm and whose translational member is regarded as an elastic beam. Equations of motion are formulated by treating the beam's supports as kinematical constraints imposed on an unrestrained beam, by discretizing the beam by means of the assumed modes technique, and by applying an alternative form of Kane's method which is particularly well suited for systems subject to constraints. The resulting equations are programmed and are used to simulate the system's response when it performs tracking maneuvers. Results provide insights into some of the issues and problems involved in the dynamics and control of manipulators containing highly elastic members connected by prismatic joints. (The formulation of equations of motion presented in this article is summarized in Section 6.5.3 above.)
- "Explicit matrix formulation of the dynamical equations for flexible multibody systems: a recursive approach" by Amirouche and Xie (1993) [32]: Develops a recursive formulation based on the finite element method where all terms are presented in a matrix form. The methodology permits one to identify the coupling between rigid and flexible body motion and build the necessary arrays for the application at hand. The equations of motion are based on Kane's equation and the general matrix representation for n bodies of its partial velocities and partial angular velocities. The algorithm developed is applied to a single two-link robot manipulator and the subsequent explicit equations of motion are presented.
- "Developing algorithms for efficient simulation of flexible space manipulator operations" by Van Woerkom et al. (1995) [17]: Briefly describes the European robot arm (ERA) and its intended application for assembly, maintenance, and servicing of the Russian segment of the International Space Station Alpha. A short review is presented of efficient Order-N**3 and Order-N algorithms for the simulation of manipulator dynamics. The link between Kane's method, the natural orthogonal complement method, and Jourdain's principle is described as indicated. An algorithm of the Order-N type is then developed for use in the ERA simulation facility. Numerical results are presented, displaying accuracy as well as computational efficiency.
- "Dynamic model of an underwater vehicle with a robotic manipulator using Kane's method" by Tarn et al. (1996) [40]: Develops a dynamic model for an underwater vehicle with an n-axis robot arm based on Kane's method. The presented technique provides a direct method for incorporating external environmental forces into the model. The model includes four major hydrodynamic forces: added mass, profile drag, fluid acceleration, and buoyancy. The derived model provides a closed form solution that is easily utilized in modern model-based control schemes.
- "Ease of dynamic modeling of wheeled mobile robots (WMRs) using Kane's approach" by Thanjavur (1997) [41]: Illustrates the ease of modeling the dynamics of WMRs using Kane's approach for nonholonomic systems. For a control engineer, Kane's method is shown to offer several unique advantages over the Newton-Euler and Lagrangian approaches used in the available literature. Kane's method provides physical insight into the nature of nonholonomic systems by incorporating the motion constraints directly into the derivation of equations of motion. The presented approach focuses on the number of degrees of freedom and not on the configuration, and thus eliminates redundancy. Explicit expressions to compute the dynamic wheel loads needed for tire friction models are derived, and a procedure to deduce the dynamics of a differentially driven WMR with suspended loads and operating on various terrains is developed. Kane's approach provides a systematic modeling scheme, and the method proposed in the paper is easily generalized to model WMRs with various wheel types and configurations and for various loading conditions. The resulting dynamic model is mathematically simple and is suited for real time control applications.

- “Haptic feedback for virtual assembly” by Luecke and Zafer (1998) [42]: Uses a commercial assembly robot as a source of haptic feedback for assembly tasks, such as a peg-hole insertion task. Kane’s method is used to derive the dynamics of the peg and the contact motions between the peg and the hole. A handle modeled as a cylindrical peg is attached to the end-effector of a PUMA 560 robotic arm equipped with a six-axis force/torque transducer. The user grabs the handle and the user-applied forces are recorded. Computed torque control is employed to feed the full dynamics of the task back to the user’s hand. Visual feedback is also provided. Experimental results are presented to show several contact configurations.
- “Method for the analysis of robot dynamics” by Xu et al. (1998) [43]: Presents a recursive algorithm for robot dynamics based on Kane’s dynamical equations. Differs from other algorithms, such as those based on Lagrange’s equations or the Newton-Euler formulation, in that it can be used for solving the dynamics of robots containing closed-chains without cutting the closed-chain open. It is also well suited for computer implementation because of its recursive form.
- “Dynamic modeling and motion planning for a hybrid legged-wheeled mobile vehicle” by Lee and Dissanayake (2001) [44]: Presents the dynamic modeling and energy-optimal motion planning of a hybrid legged-wheeled mobile vehicle. Kane’s method and the AUTOLEV symbolic manipulation package are used in the dynamic modeling, and a state parameterization method is used to synthesize the leg trajectories. The resulting energy-optimal gait is experimentally evaluated to show the effectiveness and feasibility of the path planned.
- “Kane’s approach to modeling mobile manipulators” by Tanner and Kyriakopoulos (2002) [45]: Develops a detailed methodology for dynamic modeling of wheeled nonholonomic mobile manipulators using Kane’s dynamic equations. The resulting model is obtained in closed form, is computationally efficient, and provides physical insight as to what forces really influence the system dynamics. Allows nonholonomic constraints to be directly incorporated into the model without introducing multipliers. The specific model constructed in the paper includes constraints for no slipping, no skidding, and no tipover. The dynamic model and the constraint relations provide a framework for developing control strategies for mobile manipulators.

Beyond the applications of Kane’s method to specific problems in robotics, there are a large number of papers that either apply Kane’s method to completely general multibody dynamic systems or develop from it related algorithms applicable to various subclasses of multibody systems. A small sampling of these articles is [13–15,46,47]. The algorithms described in [13–15] have, moreover, lead directly to the commercially successful software packages discussed below. A discussion of these packages is presented in the context of this chapter to provide those working in the field of robotics with an indication of the options available for simulating the motion of complex robotic devices without the need to develop the equations of motion and associated simulation programs on their own.

6.7 Commercially Available Software Packages Related to Kane’s Method

There are many commercially available software packages that are based, either directly or indirectly, on Kane’s method. The most successful and widely used are listed below:

- SD/FAST
- Pro/ENGINEER Simulation (formerly Pro/MECHANICA)
- AUTOLEV
- ADAMS
- Working Model and VisualNASTRAN

The general characteristics of each of these packages, as well as some of their advantages and disadvantages, are discussed below.

6.7.1 SD/FAST

SD/FAST is based on an extended version of Kane's method and was originally developed by Dan Rosenthal and Michael Sherman (founders of Symbolic Dynamics, Inc, and currently chief technical officer and executive vice president of Research and Development, respectively, at Protein Mechanics <www.proteinmechanics.com>) while students at Stanford University [13]. It was first offered commercially directly by Rosenthal and Sherman and later through Mitchell-Gauthier. SD/FAST is still available and since January 2001 has been distributed by Parametric Technologies Corporation, which also markets Pro/ENGINEER Simulation software (formerly Pro/MECHANICA; see below).

As described on the SD/FAST website <www.sdfast.com>:

> SD/FAST provides physically based simulation of mechanical systems by taking a short description of an articulated system of rigid bodies (bodies connected by joints) and deriving the full nonlinear equations of motion for that system. The equations are then output as C or Fortran source code, which can be compiled and linked into any simulation or animation environment. The symbolic derivation of the equations provides the fastest possible simulations. Many substantial systems can be executed in real time on modest computers.

The SD/FAST website also states that SD/FAST is "in daily use at thousands of sites worldwide." Typical areas of application mentioned include physically based animation, mechanism simulation, aerospace simulation, robotics, real-time hardware-in-the-loop and man-in-the-loop simulation, biomechanical simulation, and games. Also mentioned is that SD/FAST is the "behind-the-scenes dynamics engine" for both Pro/ENGINEER Mechanism Dynamics from Parametric Technologies Corporation and SIMM (Software for Interactive Musculoskeletal Modeling) from MusculoGraphics, Inc.

6.7.2 Pro/ENGINEER Simulation (formerly Pro/MECHANICA)

Pro/ENGINEER Simulation is a suite of packages "powered by MECHANICA technology" <www.ptc.com> for structural, thermal, durability, and dynamic analysis of mechanical systems. These packages are fully integrated with Pro/ENGINEER and the entire suite currently consists of

- Pro/ENGINEER Advanced Structural & Thermal Simulation
- Pro/ENGINEER Structural and Thermal Simulation
- Pro/ENGINEER Fatigue Advisor
- Pro/ENGINEER Mechanism Dynamics
- Pro/ENGINEER Behavioral Modeling

As mentioned above, Pro/ENGINEER Mechanism Dynamics is based on the dynamics engine originally developed by Rosenthal and Sherman. The evolution of the Mechanism Dynamics package progressed from SD/FAST to a package offered by Rasna Corporation and then to Pro/MECHANICA MOTION after Rasna was purchased in 1996 by Parametric Technology Corporation. In its latest version, Pro/ENGINEER Mechanism Dynamics allows dynamics analyses to be fully integrated with models constructed in the solids modeling portion of Pro/ENGINEER. Mechanisms can be created using a variety of joint and constraint conditions, input loads and drivers, contact regions, coordinate systems, two-dimensional cam and slot connections, and initial conditions. Various analyses can be performed on the resulting model including static, velocity, assembly, and full motion analyses for the determination of the position, velocity, and acceleration of any point or body of interest and of the reaction forces and torques exerted between bodies and ground. Mechanism Dynamics can also be used to perform design studies to optimize dimensions and other design parameters as well as to check for interferences. Results can be viewed as data reports or graphs or as a complete motion animation.

6.7.3 AUTOLEV

AUTOLEV was originally developed by David Schaechter and David Levinson at Lockheed Palo Alto Research Laboratory and was first released for commercial distribution in the summer of 1988 [14]. Levinson and Schaechter were later joined by Prof. Kane himself and Paul Mitiguy (a former Ph.D. student of Kane's) in forming a company built around AUTOLEV, OnLine Dynamics, Inc <www.autolev.com>. AUTOLEV is based on a symbolic construction of the equations of motion, and permits a user to formulate exact, literal motion equations. It is not restricted to any particular system topology and can be applied to the analysis of *any* holonomic or nonholonomic discrete dynamic system of particles and rigid bodies and allows a user to choose any convenient set of variables for the description of configuration and motion. Within the context of robotics, robots made up of a collection of rigid bodies containing revolute joints, prismatic joints, and closed kinematical loops can be analyzed in the most physically meaningful and computationally efficient terms possible.

While the use of AUTOLEV requires a user who is well-versed in dynamical and kinematical principles, the philosophy that has governed its development and evolution seeks to achieve a favorable balance between the tasks performed by an analyst and those relegated to a computer. Human analysts are best at creative activities that involve the integration of an individual's skills and experiences. A computer, on the other hand, is best suited to well-defined, repetitive operations. With AUTOLEV, an analyst is required to define only the number of degrees of freedom, the symbols with which quantities of interest are to be represented, and the *steps* that he would ordinarily follow in a manual derivation of the equations of motion. The computer is left to perform the mathematical operations, bookkeeping, and computer coding required to actually produce equations of motion and a computer program for their numerical solution. This approach necessitates that a user be skilled in dynamical principles but allows virtually any rigid multibody problem to be addressed. AUTOLEV thus does not necessarily address the needs of the widest possible class of users but rather the widest possible class of problems.

Beyond simply providing a convenient means of implementing Kane's method, AUTOLEV can also be used to symbolically evaluate mathematical expressions of many types, thus allowing it to be used to formulate equations of motion based on any variant of Newton's second law ($\mathbf{F} = m\mathbf{a}$). AUTOLEV is similar to other symbolic manipulation programs, such as MATLAB, Mathematica, and Maple, although it was specifically designed for motion analysis. It is thus a useful tool for solving problems in mechanics dealing with kinematics, mass and inertia properties, statics, kinetic energies, linear, or angular momentum, etc. and can perform either symbolic or numerical operations on scalars, matrices, vectors, and dyadics. It can also be used to automatically formulate linearized dynamical equations of motion, for example, for control system design, and can be used to produce ready-to-compile-link-and-run C and FORTRAN programs for numerical solving equations of motion. A complete list of the features and properties can be found at <www.autolev.com>.

6.7.4 ADAMS

The theory upon which ADAMS is based was originally developed by Dr. Nick Orlandea [10] at the University of Michigan and was subsequently transformed into the ADAMS computer program, marketed, and supported by Mechanical Dynamics, Inc., which was purchased in 2002 by MSC.Software. Although ADAMS was not originally developed using Kane's method, the evolution of ADAMS has been strongly influenced by Kane's approach to dynamics. Dr. Robert Ryan, a former Ph.D. student of Prof. Kane as well as a former professor at the University of Michigan, was directly involved in the development of ADAMS at Mechanical Dynamics, starting in 1988 as Vice President of Product Development, later as chief operating officer and executive vice president, and finally as president. Since the acquisition of Mechanical Dynamics by MSC.Software in 2002, Dr. Ryan has been executive vice president-products at MSC. One of Kane's method on ADAMS was through Dr. Ryan's early work on formulating dynamical equations for systems containing flexible members [29].

The broad features of the ADAMS "MSC.ADAMS suite of software packages" <www.mscsoftware.com> are described as including a number of modules that focus on various areas of application, such as ADAMS/3D Road (for the creation of roadways such as parking structures, racetracks, and highways), ADAMS/Aircraft (for the creation of aircraft virtual prototypes), ADAMS/Car (for the creation of virtual prototypes of complete vehicles combining mathematical models of the chassis subsystems, engine and driveline subsystems, steering subsystems, controls, and vehicle body), ADAMS/Controls (for integrating motion simulation and control system design), ADAMS/Exchange (for transferring data between MSC.ADAMS products and other CAD/CAM/CAE software), ADAMS/Flex (for studying the influence of flexible parts interacting in a fullly mechanical system), ADAMS/Linear (for linearizing or simplifying nonlinear MSC.ADAMS equations), ADAMS/Tire (for simulating virtual vehicles with the ADAMS/Tire product portfolio), ADAMS/Vibration (for the study of forced vibrations using frequency domain analysis), and MECHANISM/Pro (for studying the performance of mechanical simulations with Pro/ENGINEER data and geometry).

6.7.5 Working Model

As with ADAMS, Working Model is not directly based on Kane's method but was strongly influenced by Kane's overall approach to dynamics through the fact that one of the primary developers of Working Model is a former student of Kane, Dr. Paul Mitiguy (currently Director of Educational Products at MSC.Software). The dynamics engine behind Working Model is based on the use of the *constraint force algorithm* [15]. The constraint force algorithm circumvents the large set of equations produced with traditional Newton-Euler formulations by breaking the problem into two parts. The first part "uses linear algebra techniques to isolate and solve a relatively small set of equations for the constraint forces only" and then "with all the forces on each body known, it solves for the accelerations, without solving *any* additional linear equations" [15]. The constraint force algorithm is described by Dr. Mitiguy as the "inverse of Kane's method" in that Kane's method solves first for accelerations and then separately for forces. Although originally developed as an outgrowth of the software package Interactive Physics developed at Knowledge Revolution <www.krev.com>, Knowledge Revolution was acquired by MSC.Software in 1998 and Working Model is now sold and supported through MSC.Software (as is ADAMS, as described above). Working Model is described at <www.krev.com> as the "best selling motion simulation product in the world," and features of the program include the ability to

- analyze designs by measuring force, torque, acceleration, or other mechanical parameters acting on an object or set of objects;
- view output as vectors or in numbers and graphs in English or metric units;
- import 2D CAD drawings in DXF format;
- simulate nonlinear or user events using a built-in formula language;
- design linkages with pin joints, slots, motors, springs, and dampers;
- create bodies and specify properties such as mass properties, initial velocity, electrostatic charge, etc;
- simulate contact, collisions, and friction;
- analyze structures with flexible beams and shear and bending moment diagrams; and
- record simulation data and create graphs or AVI video files.

Working Model is a particularly attractive simulation package in an educational environment due to its graphical user interface, which allows it to be easily learned and mechanisms to be easily animated. Another positive aspect of Working Model is its approach to modeling. Modeling mechanisms in Working Model is performed "free-hand" and is noncoordinate based, thus allowing models to be created without the difficulties associated with developing complex geometries. The graphical user interface facilitates visualization of the motion of mechanisms and thus output is not restricted to simply numerical data or the analysis of a small number of mechanism configurations. Working model also allows the analysis of an array of relatively sophisticated problems, including those dealing with impacts, Coulomb friction, and gearing.

A three-dimensional version of working model was developed and marketed by Knowledge Revolution in the 1990s, but after the acquisition of Knowledge Revolution by MSC.Software, working model 3D became the basis for the creation of MSC.VisualNASTRAN 4D <www.krev.com>. VisualNASTRAN 4D integrates computer-aided-design, motion analysis, finite-element-analysis, and controls technologies into a single modeling system. Data can be imported from a number of different graphics packages, including SolidWorks, Solid Edge, Mechanical Desktop, Inventor, and Pro/ENGINEER, and then used to perform both motion simulations and finite element analyses. The interface of VisualNASTRAN 4D is similar to that of Working Model, and thus relatively complex models can be created easily and then tested and refined. In addition to the features of Working Model, VisualNASTRAN allows stress and strain to be calculated at every time step of a simulation using a linear finite element solver. Normal modes and buckling finite element analyses are also possible, as is the possibility of adding animation effects, such as photorealistic rendering, texture mapping, lights, cameras, and keyframing.

6.8 Summary

From all of the foregoing, the wide range of applications for which Kane's method is suitable is clear. It has been applied over the years to a variety of robotic and other mechanical systems and has been found to be both analytically convenient and computationally efficient. Beyond the systematic approach inherent in the use of Kane's method for formulating dynamical equations, there are two particular features of the method that deserve to be reiterated. The first is that Kane's method automatically eliminates "nonworking" forces in such a way that they may be neglected from an analysis at the outset. This feature was one of the fundamental bases for Kane's development of his approach and saves an analyst the need to introduce, and subsequently eliminate, forces that ultimately make no contribution to the equations of motion. The second is that Kane's method allows the use of motion variables (generalized speeds) that permit the selection of these variables as not only individual time-derivatives of the generalized coordinates but in fact any convenient linear combinations of them. These two features taken together mean that Kane's method not only allows dynamical equations of motion to be formulated with a minimum of effort but also enables one to express the equations in the simplest possible form.

The intent of this chapter has been to illustrate a number of issues associated with Kane's method. Brief historical information on the development of Kane's method has been presented, the fundamental essence of Kane's method and its relationship to Lagrange's equations has been discussed, and two simple examples have been given in a step-by-step format to illustrate the application of Kane's method to typical robotic systems. In addition, the special features and advantages of Kane's method have been described for cases related to formulating linearized dynamical equations, equations of motion for systems subject to constraints, and equations of motion for systems with continuous elastic elements. Brief summaries of a number of articles in the robotics literature have also been given to illustrate the popularity of Kane's method in robotics and the types of problems to which it can be applied. Finally, the influence of Kane and Kane's method on commercially available dynamics software packages has been described and some of the features of the most widely used of those packages presented. This chapter has hopefully provided those not familiar with Kane's method, as well as those who are, with a broad understanding of the fundamental issues related to Kane's method as well as a clear indication of the ways in which it has become the basis for formulating, either by hand or with readily available software, a vast array of dynamics problems associated with robotic devices.

References

[1] Kane, T.R., Dynamics of nonholonomic systems, *J. App. Mech.,* 28, 574, 1961.

[2] Kane, T.R. and Wang, C.F., On the derivation of equations of motion, *J. Soc. Ind. App. Math.,* 13, 487, 1965.

[3] Kane, T.R., *Analytical Elements of Mechanics,* vol. 1, Academic Press, New York, 1959.
[4] Kane, T.R., *Analytical Elements of Mechanics,* vol. 2, Academic Press, New York, 1961.
[5] Kane, T.R. and Levinson, D.A., Formulation of equations of motion for complex spacecraft, *J. Guid. Contr.,* 3, 99, 1980.
[6] Roberson, R.E. and Schwertassek, R., *Dynamics of Multibody Systems,* Springer-Verlag, Berlin, 1988.
[7] Shabana, A.A., *Dynamics of Multibody Systems,* John Wiley & Sons, New York, 1989.
[8] Schiehlen, W. (ed.), *Multibody Systems Handbook,* Springer-Verlag, Berlin, 1990.
[9] Amirouche, F.M.L., *Computational Methods in Multibody Dynamics,* Prentice Hall, Englewood Cliffs, New Jersey, 1992.
[10] Orlandea, N., Chace, M.A., and Calahan, D.A., A sparsity-oriented approach to the dynamic analysis and design of mechanical systems, part I and II, *J. Eng. Ind.,* 99, 773, 1977.
[11] Haug, E.J., Nikravesh, P.E., Sohoni, V.N., and Wehage, R.A., Computer aided analysis of large scale, constrained, mechanical systems, *Proc. 4th Int. Symp. on Large Engrg. Systems,* Calgary, Alberta, 1982, 3.
[12] Schiehlen, W.O. and Kreuzer, E.J., Symbolic computational derivations of equations of motion, *Proc. IUTAM Symp. on Dyn. of Multibody Sys.,* Munich, 1977, 290.
[13] Rosenthal, D.E. and Sherman, M.A., High performance multibody simulations via symbolic equation manipulation and Kane's method, *J. Astro. Sci.,* 34, 223, 1986.
[14] Schaechter, D.B. and Levinson, D.A., Interactive computerized symbolic dynamics for the dynamicist, *Proc. Amer. Contr. Conf.,* 1988, 177.
[15] Mitiguy, P.C. and Banerjee, A.K., A constraint force algorithm for formulating equations of motion, *Proc. First Asian Conf. on Multibody Dynamics,* 2002, 606.
[16] Lesser, M., A geometrical interpretation of Kane equations, *Proc. Royal Soc. London Ser. A - Math., Phy., Eng. Sci.,* Royal Soc. London, London, 1992, 69.
[17] Van Woerkom, P.Th.L.M., De Boer, A., Ellenbroek, M.H.M., and Wijker, J.J., Developing algorithms for efficient simulation of flexible space manipulator operations, *Acta Astronautica,* 36, 297, 1995.
[18] Piedboeuf, J.-C., Kane's equations or Jourdain's principle? *Proc. 36th Midwest Symp. on Circuits and Sys.,* IEEE, 1993, 1471.
[19] Kane, T.R., Teaching of mechanics to undergraduates, *Int. J. Mech. Eng. Ed.,* 6, 286, 1978.
[20] Kane, T.R. and Levinson, D.A., *DYNAMICS: Theory and Applications,* McGraw-Hill, New York, 1985.
[21] Kane, T.R., *DYNAMICS,* Holt, Rinehart and Winston, New York, 1968.
[22] Mitiguy, P.C and Kane, T.R., Motion variables leading to efficient equations of motion, *Int. J. Robotics Res.,* 15, 522, 1996.
[23] Scheinman, V.D., Design of a computer controlled manipulator, Engineer thesis, Stanford University, Stanford, 1969.
[24] Kane, T.R. and Levinson, D.A., The use of Kane's dynamical equations in robotics, *Int. J. Robotics Res.,* 2, 3, 1983.
[25] Wampler, C.W., A comparative study of computer methods in manipulator kinematics, dynamics, and control, Ph.D. thesis, Stanford University, Stanford, 1985.
[26] Kane, T.R., Ryan, R.R., and Banerjee, A.K., Dynamics of a cantilever beam attached to a moving base, *J. Guid., Contr., Dyn.,* 10, 139, 1987.
[27] Buffinton, K.W., Dynamics of beams moving over supports, Ph.D. thesis, Stanford University, Stanford, 1985.
[28] Wampler, C.W., Buffinton, K.W., and Jia, S.H., Formulation of equations of motion for systems subject to constraints, *J. Appl. Mech.,* 52, 465, 1985.
[29] Ryan, R.R., Flexibility modeling methods in multibody dynamics, Ph.D. thesis, Stanford University, Stanford, 1986.

[30] Huston, R.L., Multibody dynamics formulations via Kane's equations, in *Mechanics and Control of Large Flexible Structures,* Jenkins, J.L., ed., American Institute of Aeronautics and Astronautics (AIAA), Reston, Virginia, 1990, 71.

[31] Buffinton, K.W., Dynamics of elastic manipulators with prismatic joints, *J. Dyn. Sys., Meas., Contr.,* 114, 41, 1992.

[32] Amirouche, F.M.L. and Xie, M., Explicit matrix formulation of the dynamical equations for flexible multibody systems: a recursive approach, *Comp. and Struc.,* 46, 311, 1993.

[33] Buffinton, K.W. and Kane, T.R., Dynamics of a beam moving over supports, *Int. J. Solids Structures,* 21, 617, 1985.

[34] Baruh, H., *Analytical Dynamics,* McGraw-Hill, Boston, 1999.

[35] Josephs, H. and Huston, R.L., *Dynamics of Mechanical Systems,* CRC Press, Boca Raton, FL, 2002.

[36] Wloka, D. and Blug, K., Simulation der dynamik von robotern nach dem verfahren von Kane, *Robotersysteme,* 1, 211, 1985.

[37] Gallay, G., Girard, F., Auclair, J.-M., and Gerbeaux, F.-X., Dynamique de mecanismes flexibles, *Mecanique-Materiaux-Electricite,* 416, 17, 1986.

[38] Angeles, J., Ma, O., and Rojas, A., Algorithm for the inverse dynamics of n-axis general manipulators using Kane's equations, *Comp. Math. App.,* 17, 1545, 1989.

[39] Huston, R.L., The use of Kane's method in the modeling and simulation of robotic systems, *Proc. IMACS Symp.,* North-Holland, Amsterdam, 1989, 221.

[40] Tarn, T.J., Shoults, G.A., and Yang, S.P., Dynamic model of an underwater vehicle with a robotic manipulator using Kane's method, *Autonomous Robots,* 3, 269, 1996.

[41] Thanjavur, K., Ease of dynamic modeling of wheeled mobile robots (WMRs) using Kane's approach, *Proc. 1997 Int. Conf. on Robotics and Automation,* IEEE, 1997, 2926.

[42] Luecke, G.R. and Zafer, N., Haptic feedback for virtual assembly, *Proc. 1998 Conf. on Telemanipulator and Telepresence Technologies,* SPIE, 1998, 115.

[43] Xu, X.-R., Chung, W.-J., Choi, Y.-H., and Ma, X.-F., Method for the analysis of robot dynamics, *Modelling, Meas. Contr.,* 66, 59, 1998.

[44] Lee, K.Y. and Dissanayake, M.W.M.G., Dynamic modeling and motion planning for a hybrid legged-wheeled mobile vehicle, *J. Mech. Eng. Sci.,* 215, 7, 2001.

[45] Tanner, H.G. and Kyriakopoulos, K.J., Kane's approach to modeling mobile manipulators, *Advanced Robotics,* 16, 57, 2002.

[46] Kane, T.R. and Levinson, D.A., Multibody Dynamics, *J. Appl. Mech.,* 50, 1071, 1983.

[47] Huston, R.L., Multibody dynamics: modeling and analysis methods, *Appl. Mech. Rev.,* 44, 109, 1991.