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Machine learning over encrypted data

Rener Oliveira
Rodrigo Targino

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Supervised machine learning setup:

Data

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What if Data is sensitive?

- Medical records
- Financial data
- Genetic data

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Encrypted Data

%$*#!7E% - Encrypted predictors
$&h%$8 - Encrypted target vector

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Just encrypt it ;-)

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Maybe :)

Privacy Homomorphisms, 1978

ON DATA BANKS AND PRIVACY HOMOMORPHISMS

Ronald L. Rivest
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Michael L. Dertouzos

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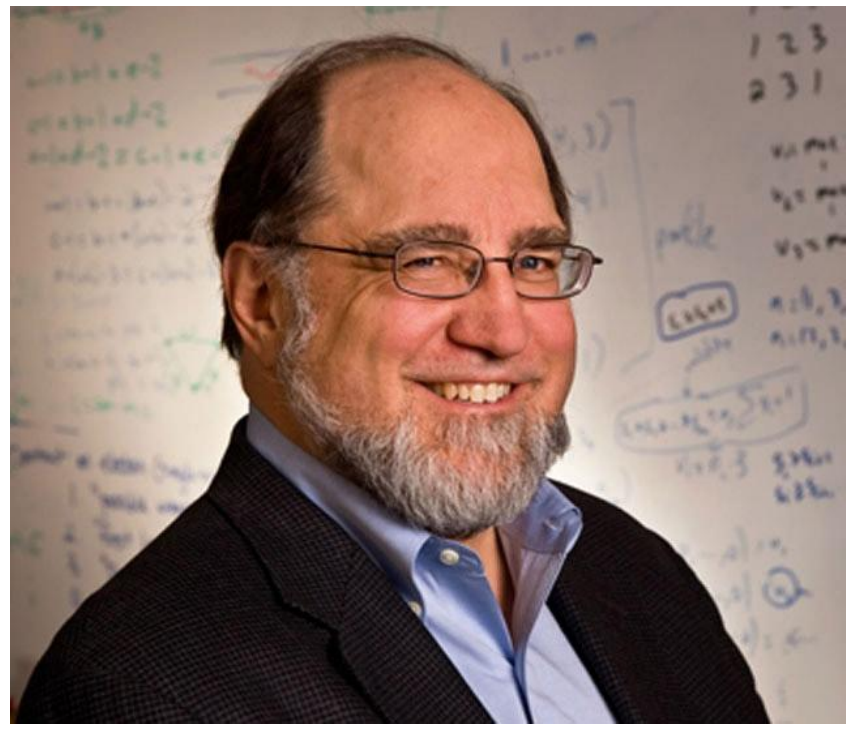

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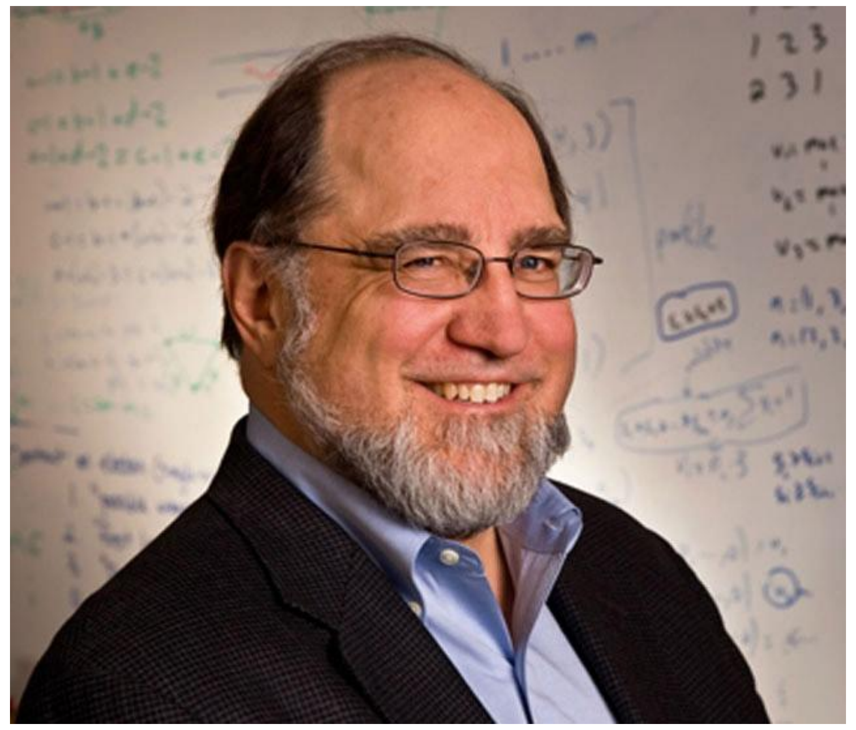

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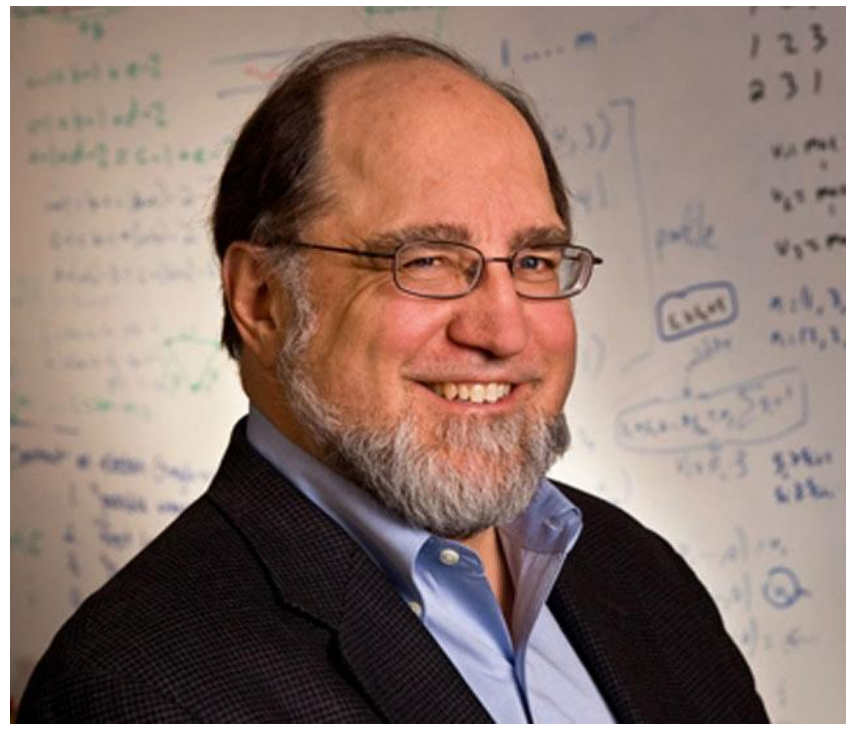

limitations on what can be accomplished, we shall see that it appears likely that there exist encryption functions which permit encrypted data to be operated on without preliminary decryption of the operands, for many sets of interesting operations. These special encryption functions we call "privacy homomorphisms"; they form an interesting subset of arbitrary encryption schemes

Fully Homomorphic Encryption

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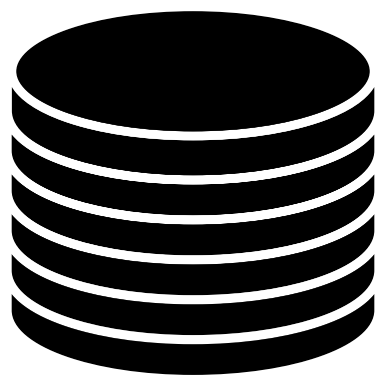
Fully Homomorphic Encryption

Message space:
(m1, m2, ...)



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How to securely compute f(m1,m2,...) ?

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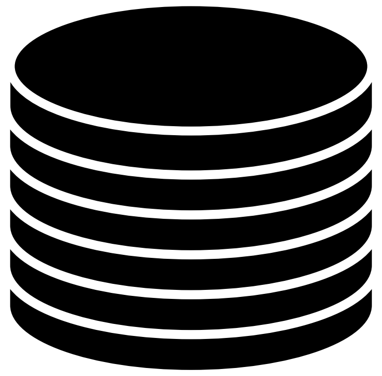

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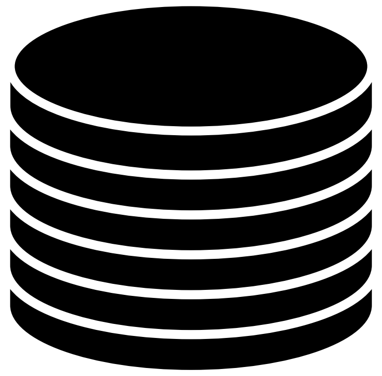

pk = public key

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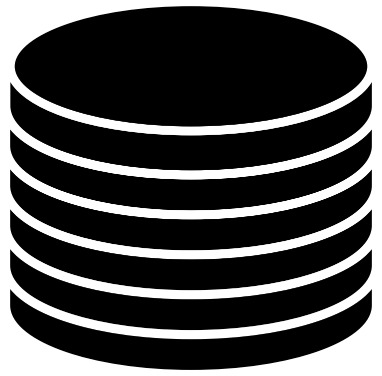

c1 = Encrypt(pk,m1)

pk = public key

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How to securely compute f(m1,m2,...) ?

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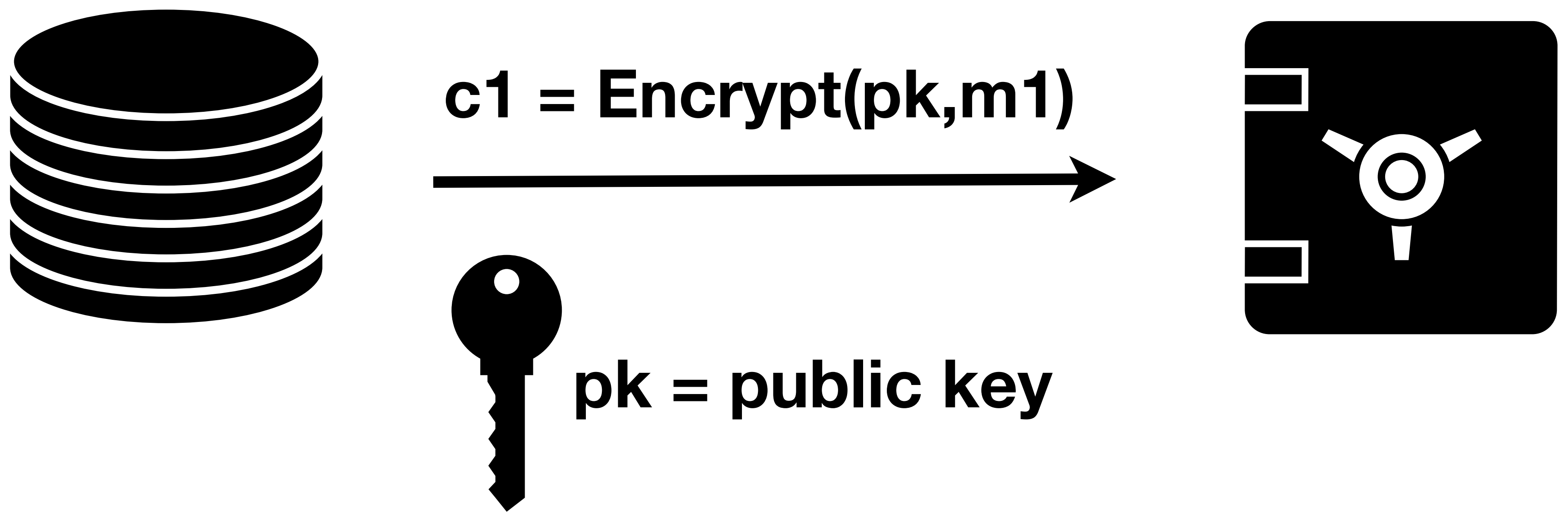
Cyphertext space:
(c1, c2, ...)



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How to securely compute f(m1,m2,...) ?

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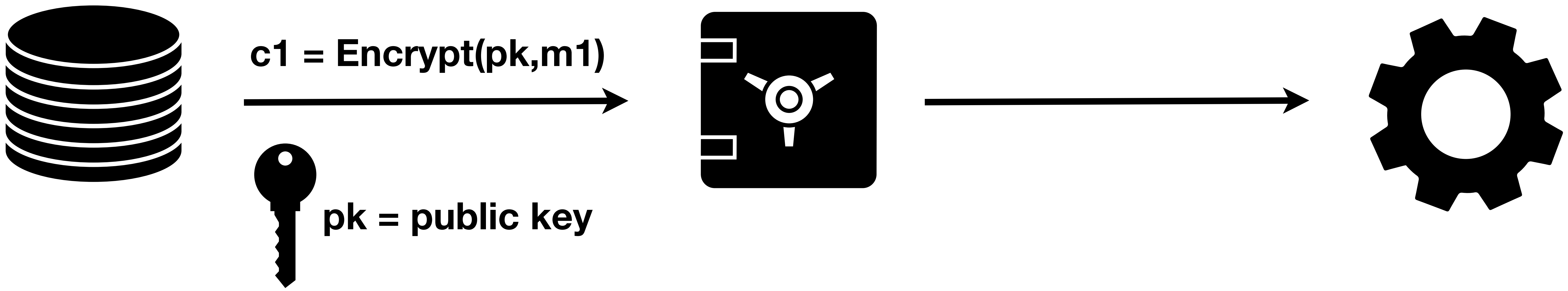
Cyphertext space: (c1, c2, ...)

Encrypted evaluation F = f(c1, c2, ...)



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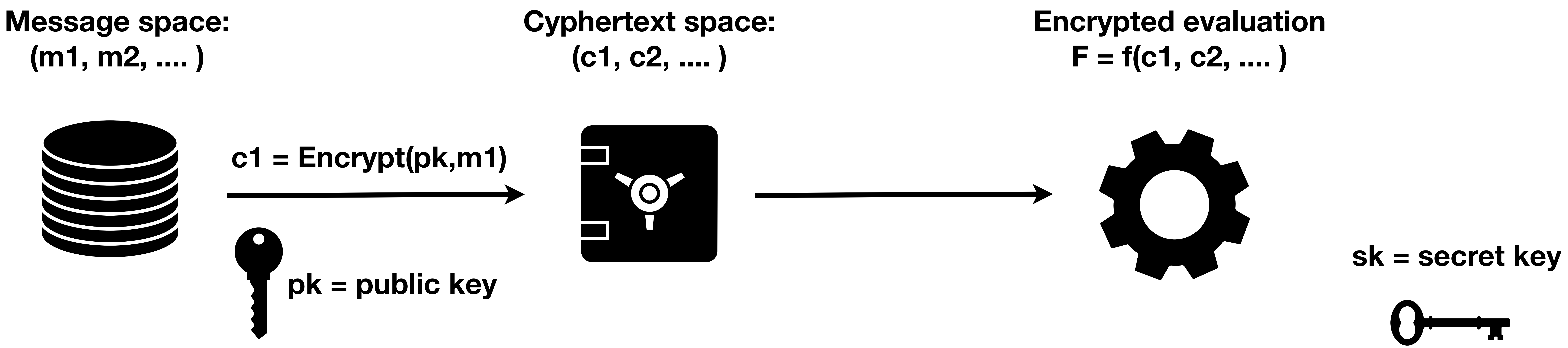

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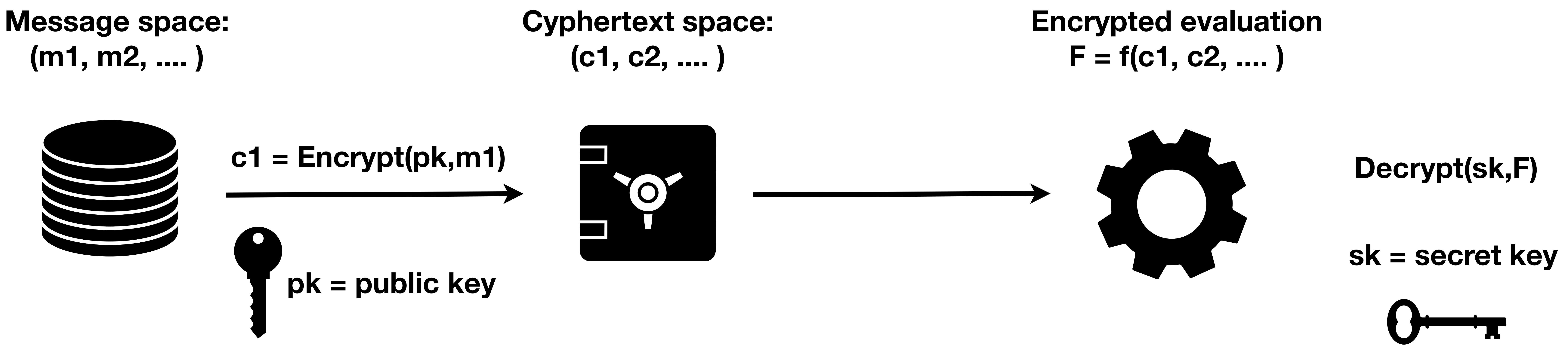

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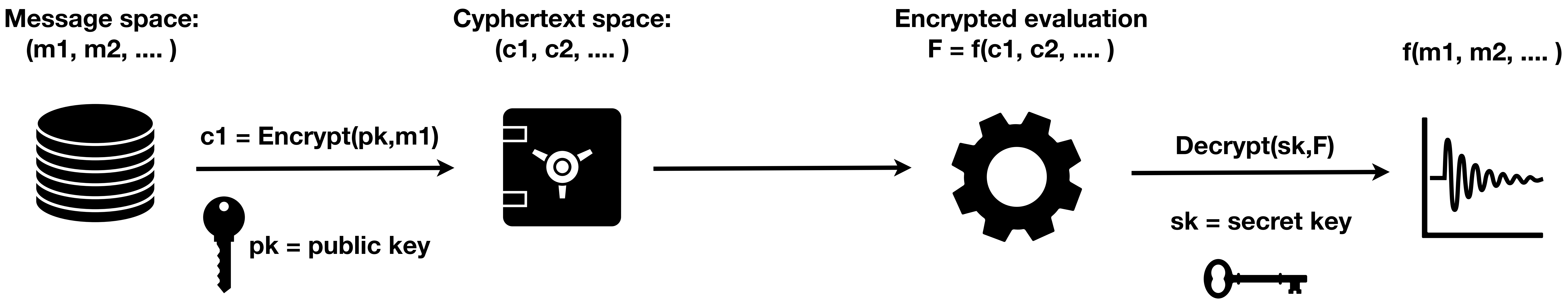
Fully Homomorphic Encryption

How to <u>securely</u> compute f(m1,m2,...) ?



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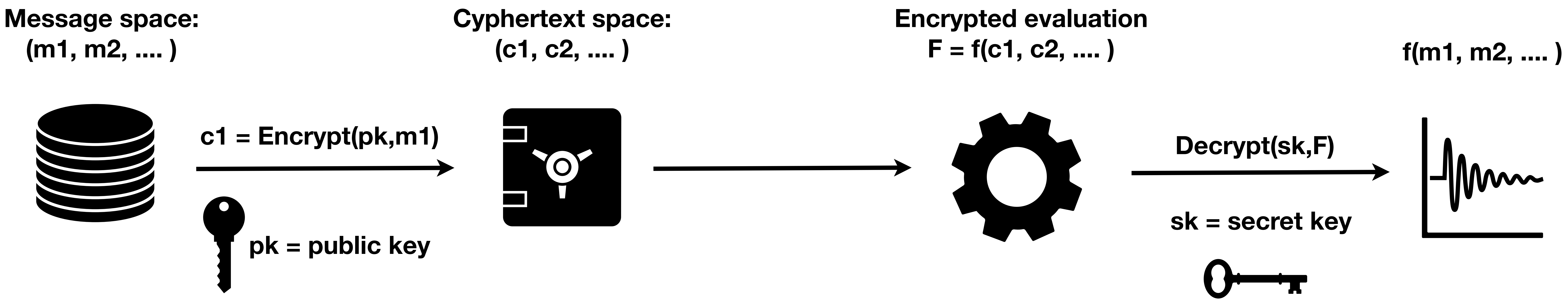

An encryption scheme is called Fully Homomorphic, if it can handle any desired function f !

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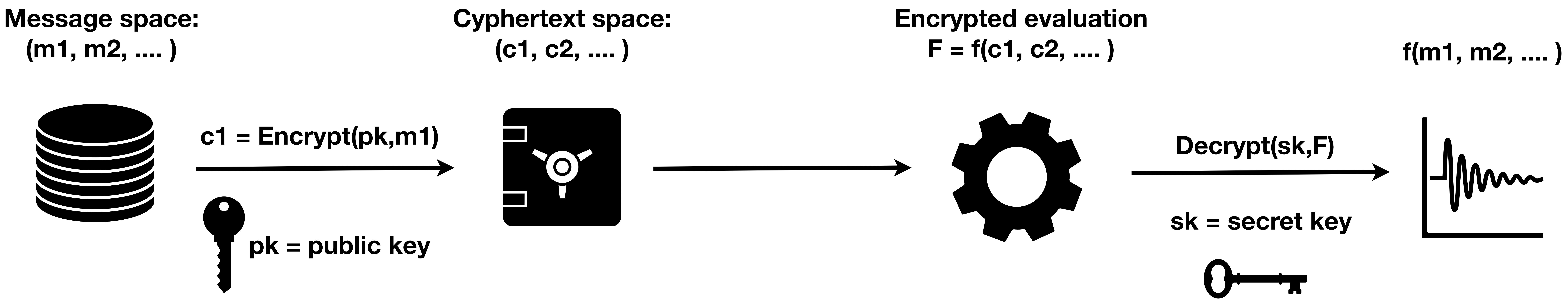
Fully Homomorphic Encryption

How to <u>securely</u> compute f(m1,m2,...) ?



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Fully Homomorphic Encryption

Craig Gentry built the first FHE, on his Ph.D. Thesis in 2009

Message space: {0,1}
Permitted functions: additions and multiplications

Semantically secure encryption

- Encrypt the same message 2+ times must generate different cyphertexts
- Encryption must be random

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Encrypt(pk,m) = c

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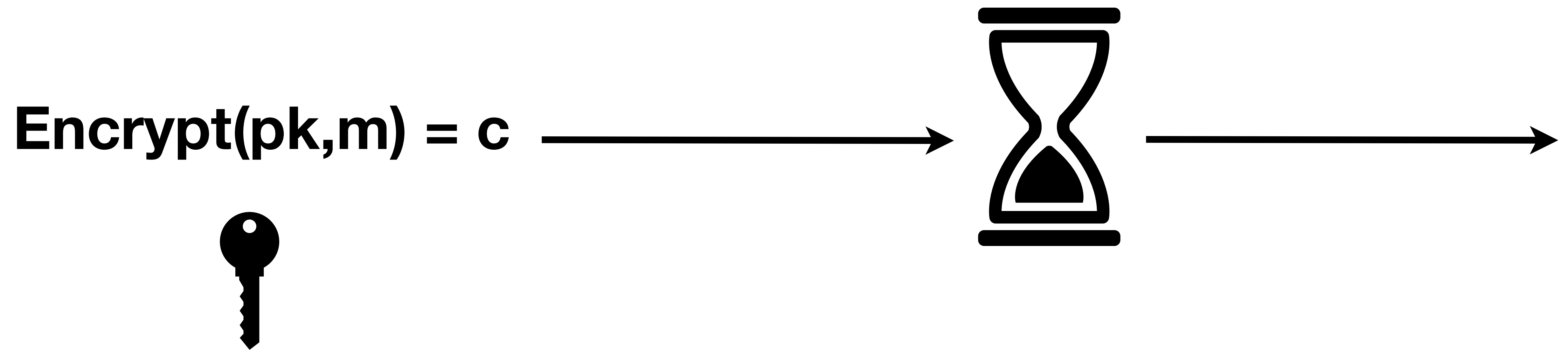

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Semantically secure encryption

- Encrypt the same message 2+ times must generate different cyphertexts
- Encryption must be random

Encrypt(pk,m) = c $\longrightarrow$  $\longrightarrow$ Encrypt(pk,m) = c*

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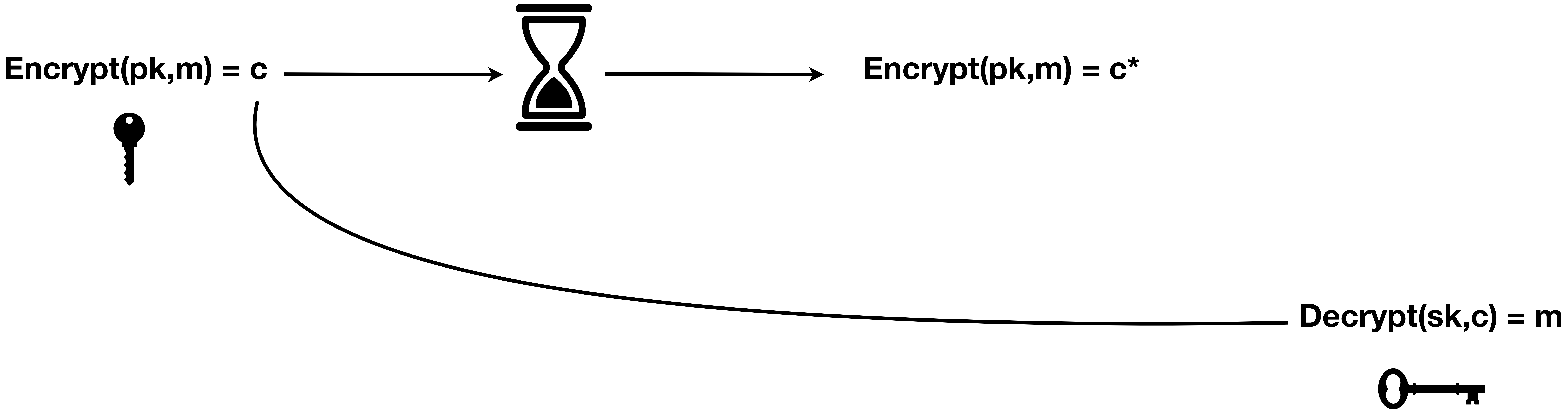

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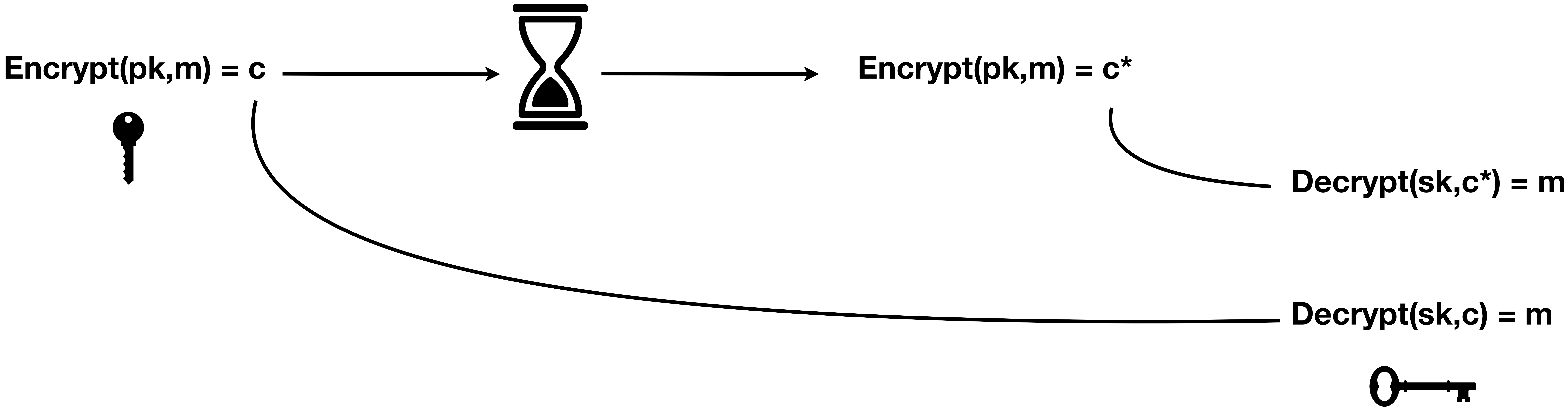

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Somewhat Homomorphic Encryption (SHE)

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A SHE scheme can homomorphically
evaluate functions f with a limited depth

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A SHE scheme can homomorphically evaluate functions f with a limited depth

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Message space:
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Key:
p, odd integer

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Encryption function:
$c = pq + 2r + m$

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Decryption function:
(c mod p) mod 2

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Encryption function: $c = pq + 2r + m$
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Decryption function: $(c \bmod p) \bmod 2$

If $2r < p$, the decryption is correct:

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$$= ((pq + 2r + m) \bmod p) \bmod 2$$

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&= m
\end{aligned}$$

If we add/multiply two cyphertext c1 and c2, the noise will be added/multiplied too

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Bootstrapping

Let's suppose we have a cyphertext c
of a message m, and want to reduce its noise

c

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Bootstrapping

Let's suppose we have a cyphertext c
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$$c \xrightarrow[\text{Dec}(sk,\ .)]{} m$$

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Let's suppose we have a cyphertext c
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The Decryption function completely removes any noise !!
But reveals the message and uses the private key :(

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Bootstrapping

Let's suppose we have a cyphertext c
of a message m, and want to reduce its noise

$$
\begin{tikzcd}
c^* & \\
c \arrow[u, "\mathrm{Enc}(pk2,\ .)"] \arrow[r, "\mathrm{Dec}(sk,\ .)"'] & m
\end{tikzcd}
$$

The Decryption function completely removes any noise !!
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c* is a double encryption of m

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We can homomorphically apply permitted functions f to c*

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Let's suppose we have a cyphertext c
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$$
\begin{tikzcd}
c^* \arrow[r, "f"'] & {} \\
c \arrow[u, "{Enc(pk2,\ .)}"] \arrow[r, "{Dec(sk,\ .)}"'] & m
\end{tikzcd}
$$

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Bootstrapping

Let's suppose we have a cyphertext c
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$$\begin{tikzcd}
c^* \arrow[r, "\mathrm{Dec}(sk^*,\, .)"'] & c2 \\
c \arrow[u, "\mathrm{Enc}(pk2,\, .)"] \arrow[r, "\mathrm{Dec}(sk,\, .)"'] & m
\end{tikzcd}$$

pk2 = second public key
sk* = Enc(pk2,sk)

The Decryption function completely removes any noise !!
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c* is a double encryption of m

We can homomorphically apply permitted functions f to c*

So, apply f(.) = Dec(sk*, .)

Bootstrapping

Let's suppose we have a cyphertext c
of a message m, and want to reduce its noise

$$
\begin{tikzcd}
c^* \arrow[r, "\text{Dec}(sk^*, .)"'] & c2 \arrow[d, "\text{Dec}(sk, .)"'] \\
c \arrow[u, "\text{Enc}(pk2, .)"] \arrow[r, "\text{Dec}(sk, .)"'] & m
\end{tikzcd}
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c* is a double encryption of m

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So, apply f(.) = Dec(sk*, .)

c2 is a new cyphertext that encrypts m, with less noise!

pk2 = second public key
sk* = Enc(pk2,sk)

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c^* \arrow[rr, "\mathrm{Dec}(sk^*,\ .)"'] & & c2 \arrow[d, "\mathrm{Dec}(sk,\ .)"'] \\
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\end{tikzcd}
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A SHE scheme is called <u>bootstrappable</u>,
if the Decryption function is a permitted function

Literature overview

4 Generations of FHE

1st Gen.	C. Gentry, 2009	FHE Using Ideal Lattices	Solved 1978 Rivest Problem
1st Gen.	V. Dijk, C. Gentry, S. Halevi, V. Vaikuntanathan (DGHV, 2010)	FHE Over the Integers	Simplification of 2009 scheme
2nd Gen.	Z. Brakerski, Fan, Vercauteren (BFV, 2012)	Somewhat Practical FHE	More eficient SHE
3rd Gen.	Gentry, A. Sahai, and B. Waters (GSW, 2013)	HE from LWE: Conceptually-Simpler, Asymptotically-Faster	Faster bootstrapping, more efficient multiplications
4th Gen.	Cheon, Kim, Kim and Song (CKKS, 2016)	HE for arithmetic of approximate numbers	Built over complex/real numbers

Calendar

MONTH	TASK
Mar - Jun	Mapped existing papers and studied Gentry's work (1st gen.)
July	Study next generation main papers/Map existing library's/packages
Ago - Set	Implementation of private regression models, tests and benchmarks
Out - Nov	Write

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References

Rivest, 1978 - ON DATA BANKS AND PRIVACY HOMOMORPHISMS
Gentry, 2009 - Fully Homomorphic Encryption using Ideal Lattices
Gentry, 2010 - Fully Homomorphic Encryption over the Integers